GLOBAL SOLUTION OF THE INITIAL VALUE PROBLEM FOR THE FOCUSING DAVEY-STEWARTSON II SYSTEM

EVGENY LAKSHTANOV AND BORIS VAINBERG

Abstract. We consider the two dimensional focusing Davey-Stewartson II system and construct the global solution of the Cauchy problem for a dense in $L^2(\mathbb{C})$ set of initial data. We do not assume that the initial data is small. So, the solutions may have singularities. We show that the blow-up may occur only on a real analytic variety and the variety is bounded in each strip $t \le T$.

1. Introduction

Let $q_0(z),\ z = x + iy,\ (x, y) \in \mathbb{R}^2$, be a compactly supported (or fast decaying) sufficiently smooth function. Consider the two dimensional focusing Davey-Stewartson II (DSII) system of equations for unknown functions $q = q(z, t),\ \phi = \phi(z, t),\ (x, y) \in \mathbb{R}^2,\ t \ge 0$:

$$\begin{aligned} q_t &= 2iq_{xy} - 4q(\overline{\varphi} - \varphi), \\ \partial\varphi &= \overline{\partial}|q|^2, \\ q(z, 0) &= q_0(z). \end{aligned} \tag{1.1}$$

A smooth, decaying in z at infinity solution of (1.1) exists for all $t > 0$ if q_0 is small enough, [1, 20, 21, 22, 3]. We will call this solution classical. If q_0 is not small, the solution was constructed locally in our previous work [14] via the IST (inverse scattering transform) using the $\overline{\partial}$-method that has been generalized in [16], [12], [13] to the case when Faddeev type exceptional points may be present. The solution was obtained in a neighbourhood of any point (z_0, t_0) for generic initial data q_0 that depend on the point. The main objectives of this article are to obtain the solution for an arbitrary initial data from a specific set and to get the global solution defined in the whole space, including a description of the set where the solution blows up. It will be shown that the latter set is a real analytic variety that is bounded in every strip $0 \le t \le T$.

Let us recall that the focusing DSII equation may have a finite time blow-up (e.g., [17]). While the uniqueness is known for smooth (in some sense) solutions, see [8], [9], one has to be careful with the definition of the solution that has singularities. We understand these solutions in the

2010 *Mathematics Subject Classification.* 37K15, 35G55.

Key words and phrases. Davey-Stewartson system, $\overline{\partial}$-method, Dirac equation, exceptional point, inverse problem.

following sense. Let us multiply the initial data q_0 by a positive parameter $a \in (0,1]$. We will show that the classical solution that exists when $a \ll 1$ allows an analytic continuation into a complex neighborhood of $(0,1]$, and this analytic continuation will be used to single out the solution with singularities when $a = 1$. The main statement of the present paper is given in Theorem 2.2 of Section 2.

Let us mention some recent articles on DSII: [11],[18],[19].

2. The solution of the Cauchy problem, main results

Let $q_0(z) \in L^2(\mathbb{C})$. Denote

$$Q_0(z) = \begin{pmatrix} 0 & q_0(z) \\ -\overline{q_0(z)} & 0 \end{pmatrix}, \quad z \in \mathbb{C}. \tag{2.1}$$

Let $\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, and let the 2×2 matrix $\psi(\cdot,k),\ k \in \mathbb{C}$, be a solution of the following problem for the Dirac equation in $\mathbb{C}$:

$$\frac{\partial \psi}{\partial \overline{z}} = Q_0\overline{\psi}, \quad \psi(z,k)e^{-i\overline{k}z/2} \to I, \quad z \to \infty. \tag{2.2}$$

The corresponding generalized Lippmann-Schwinger equation has the following form:

$$\psi(z,k) = e^{i\overline{k}z/2}I + \int_{\mathbb{C}} G(z-z',k)Q_0(z')\overline{\psi}(z',k)d\sigma_{z'}, \tag{2.3}$$

where $G(z,k) = \frac{1}{\pi}\frac{e^{i\overline{k}z/2}}{z}$, $d\sigma_{z'} = dx'dy'$. Here and below we use the same notation for functional spaces, irrespectively of whether those are the spaces of matrix-valued or scalar-valued functions. After the substitution,

$$\mu(z,k) = \psi(z,k)e^{-i\overline{k}z/2}, \qquad \mu(z,k) - I \to 0, \quad z \to \infty, \tag{2.4}$$

equation (2.3) takes the form

$$\mu(z,k) = I + \frac{1}{\pi}\int_{\mathbb{C}} \frac{e^{i\Re(\overline{k}z)}}{z-z'}Q_0(z')\overline{\mu}(z',k)d\sigma_{z'}, \tag{2.5}$$

and becomes Fredholm in $L^q(\mathbb{C}),\ q > 2$, after the additional substitution $\nu = \mu - I$ (see, e.g., [15, lemma 5.3]).

Solutions ψ of (2.3) are called the generalized *scattering solutions*, and the values of k such that the homogeneous equation (2.5) has a non-trivial solution are called *exceptional points*. The set of exceptional points will be denoted by $\mathcal{E}$. Thus the scattering solution may not exist if $k \in \mathcal{E}$. Note that the operator in equation (2.5) is not analytic in k, and $\mathcal{E} \subset \mathbb{C}$ may contain one-dimensional components. There are no exceptional points in a neighborhood of infinity (e.g., [20, lemma 2.8], [2, lemma C]). Let us choose $A \gg 1$ and $k_0 \in \mathbb{C}$ such that all the exceptional points are contained in the disk

$$D = \{k \in \mathbb{C} : 0 \le |k| < A\}, \tag{2.6}$$

and k_0 belongs to the same disc $\overline{D}$ and is not exceptional.

The generalized scattering data (an analogue of the scattering amplitude in the standard scattering problem) are defined by the following integral (when the integral converges)

$$h_0(\varsigma,k)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-i\overline{\varsigma}z/2}Q_0(z)\overline{\psi}(z,k)d\sigma_z,\ \varsigma\in\mathbb{C},\ k\in\mathbb{C}\backslash\mathcal{E}. \tag{2.7}$$

In fact, from the Green formula, it follows that h_0 can be determined without using the potential Q_0 or the solution ψ of the Dirac equation (2.2) if the Dirichlet data at $\partial\Omega$ are known for the solution of (2.2) in a bounded region Ω containing the support of Q_0:

$$h_0(\varsigma,k)=\frac{-i}{8\pi^2}\int_{\partial\mathcal{O}}e^{-i\overline{\varsigma}z/2}\overline{\psi}(z,k)dz,\ \varsigma\in\mathbb{C},\ k\notin\mathcal{E}. \tag{2.8}$$

Note that h_0 is continuous when $k\notin D$ under minimal assumptions on Q [20], [21], and moreover,

$$h_0=h_0(\varsigma,k)\in C^\infty\quad \text{when } |k|\geq A \tag{2.9}$$

if Q is bounded and decays faster then any power at infinity. This follows from the fact that (2.5) admits differentiation in z and k when $k\notin D$.

The inverse problem (recovery of Q when h_0 is given) was solved using $\overline{\partial}-$method in [1], [20, 21, 22] when the potential is small enough to guarantee the absence of exceptional points. When $\mathcal{E}\neq\varnothing$, the inverse problem was solved in a generic sense in [13]. The latter results were applied in [14] to construct solutions of the focusing DSII system. Let us recall some results obtained in [14].

Consider the space

$$\mathcal{B}^s=\left\{u\in L^s(\mathbb{C}\backslash D)\bigcap C(D)\right\},\quad s>2, \tag{2.10}$$

where $C(D)$ is the space of analytic functions in D with the norm $\|u\|=\sup_D|u|$. Here and below, we use the same space notation for matrices as for their entries.

Let operator $T_z:\mathcal{B}^s\to\mathcal{B}^s,\ s>2$, be defined as follows:

$$\begin{aligned}T_z\phi(k)&=\frac{1}{\pi}\int_{\mathbb{C}\backslash D}e^{i(\overline{\varsigma}z+\overline{z}\varsigma)/2}\overline{\phi}(\varsigma)\Pi^o h(\varsigma,\varsigma)\frac{d\sigma_\varsigma}{\varsigma-k}\\&+\frac{1}{2\pi i}\int_{\partial D}\frac{d\varsigma}{\varsigma-k}\int_{\partial D}[e^{i(\varsigma\overline{z}+\overline{\varsigma'}z)/2}\overline{\phi^-(\varsigma')}\Pi^o\\&+e^{i(\varsigma-\varsigma')\overline{z}/2}\phi^-(\varsigma')\Pi^d\mathbf{C}]\left[\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right]h(\varsigma',\varsigma)\overline{d\varsigma'},\end{aligned} \tag{2.11}$$

where $d\sigma_\varsigma=d\varsigma_R d\varsigma_I,\ z\in\mathbb{C},\ \phi\in\mathcal{B}^s,\ \phi^-$ is the boundary trace of ϕ from the interior of D, $\mathbf{C}$ is the operator of complex conjugation, Π^oM is the off-diagonal part of a matrix M, Π^dM is the diagonal part. Let us specify the logarithmic function in (2.12). Let us shift the coordinates in $\mathbb{C}$ and move the origin to the point $\overline{\varsigma'}\in\partial D$. Then we rotate the plane in such a way that the direction of the x-axis is defined by the vector from $\overline{\varsigma'}$ to $-\overline{\varsigma'}$. Then $|\arg(\overline{\varsigma'}-\overline{\varsigma})|<\pi/2,\ \varsigma'\neq\varsigma$, and $|\arg(\overline{\varsigma'}-\overline{k_0})|\leq\pi/2$, i.e.,

$$\left|\arg\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right|<\pi,\quad \varsigma',\varsigma\in\partial D,\ \varsigma'\neq\varsigma.$$

This defines the values of the logarithmic function uniquely.

It turns out that, after the substitution $w=v-I\in\mathcal{B}^s,\ s>2$, the equation

$$(I+T_z)v=I$$

becomes Fredholm in $\mathcal{B}^s$, and the potential q_0 can be expressed explicitly in terms of v (see [12, 13]).

In order to solve the DSII problem (1.1), we apply this reconstruction procedure to a specially chosen scattering data. We start with the scattering data h_0 defined by q_0 and extend it in time as follows:

$$h(\varsigma,k,t):=e^{-t(k^2-\overline{\varsigma}^2)/2}\Pi^o h_0(\varsigma,k)+e^{-t(\overline{k}^2-\overline{\varsigma}^2)/2}\Pi^d h_0(\varsigma,k), \tag{2.12}$$

where $\varsigma\in\mathbb{C},\ k\in\mathbb{C}\backslash\mathcal{E},\ t\geq 0$. For $t\geq 0$, we define the operator

$$\begin{aligned} T_{z,t}\phi(k)=&\frac{1}{\pi}\int_{\mathbb{C}\backslash D}e^{i(\overline{\varsigma}z+\overline{z}\varsigma)/2}\overline{\phi}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)\frac{d\sigma_\varsigma}{\varsigma-k}\\ &+\frac{1}{2\pi i}\int_{\partial D}\frac{d\varsigma}{\varsigma-k}\int_{\partial D}[e^{i(\varsigma\overline{z}+\overline{\varsigma'}z)/2}\overline{\phi^-(\varsigma')}\Pi^o\\ &+e^{i(\varsigma-\varsigma')\overline{z}/2}\phi^-(\varsigma')\Pi^d\mathbf{C}]\left[\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right]h(\varsigma',\varsigma,t)\overline{d\varsigma'}. \end{aligned} \tag{2.13}$$

Theorem 2.1. *([14]) Let $q_0(\cdot)$ be a function with compact support. Assume that it is 6 times differentiable in x and y. Alternatively, this condition can be replaced[1] by the superexponential decay of q_0:*

$$\lim_{z\to\infty}e^{\widetilde{A}|z|}\partial_x^i\partial_y^j q_0(z)=0\ \ for\ each\ \widetilde{A}>0,\ i+j\leq 6. \tag{2.14}$$

Then, for each $s>2$, the following statements are valid.

1) The operator $T_{z,t}$ is compact in $\mathcal{B}^s$ for all $z\in\mathbb{C},\ t\geq 0$, and depends continuously on z and $t\geq 0$. The same property holds for its first derivative in time and all the derivatives in x,y up to the third order, where the derivatives are defined in the norm convergence. The function $T_{z,t}I$ belongs to $\mathcal{B}^s$ for all $t\geq 0$.

2) Let the kernel of $I+T_{z,t}$ in the space $\mathcal{B}^s$ be trivial for (z,t) in an open or half open[2] set $\omega\subset\mathbb{R}^3$. Let $v_{z,t}=w_{z,t}+I$, where $w_{z,t}\in\mathcal{B}^s$ is the solution of the equation

$$(I+T_{z,t})w_{z,t}=-T_{z,t}I. \tag{2.15}$$

[1]This fact was not mentioned in the paper, but it can be easily checked

[2]We will say that a set ω of points (z,t) in $\mathbb{R}^3_+=\mathbb{R}^3\bigcap\{t\geq 0\}$ is half-open if ω contains points where $t=0$ and, for each point $(z_0,0)\in\omega$, there is a ball B_0 centered at this point such that $B_0\bigcap\{t\geq 0\}\subset\omega$.

Then functions q, φ defined by

$$\begin{pmatrix} \varphi(z,t) & q(z,t) \\ -q(z,t) & \varphi(z,t) \end{pmatrix} := \frac{-i}{2\pi}(\Pi^o + \overline{\partial}\Pi^d)\left(\int_{\mathbb{C}\setminus D} e^{i(\overline{\varsigma}z+\overline{z}\varsigma)/2}\overline{v_{z,t}}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)d\sigma_\varsigma\right.$$

$$-\frac{1}{2i}\int_{\partial D} d\varsigma \int_{\partial D} [e^{i(\varsigma\overline{z}+\overline{\varsigma'}z)/2}\overline{v^-_{z,t}(\varsigma')}\Pi^o$$

$$\left. -e^{i(\varsigma-\varsigma')\overline{z}/2}v^-_{z,t}(\varsigma')\Pi^d\mathbf{C}]\left[\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}\right]h(\varsigma',\varsigma,t)\overline{d\varsigma'}\right), \tag{2.16}$$

satisfy all the relations (1.1) in the classical sense when $(z,t) \in \omega$. In particular, $q(z,0) = q_0(z)$.

3) Consider a set of initial data $aq_0(z)$ that depend on $a \in (0,1]$. Then equation (2.15) with Q^0 replaced by aQ^0 (Q^0 is fixed) is uniquely solvable for almost every $(z,t,a) \in \mathbb{R}^2 \times \mathbb{R}^+ \times (0,1]$. Moreover,[3] for each (z,t), the solution of (2.15) is meromorphic in $a \in [0,1]$ and has at most a finite set of poles $a = a_j(z,t)$.

Remark. All the exceptional points are located in a disk whose radius depends only on the norm of aq_0. Hence D and k_0 can be chosen independently of $a \in [0,1]$ (see [13, Lemma 5.1]). From now on, we assume that the disk D is fixed and contains the exceptional points for all the potentials $aq_0,\ a \in [0,1]$.

In order to state the main results of the present paper, we need to recall the construction (e.g., [20]) of the global solution of (1.1) when q_0 is small. The latter expression ("q_0 is small") will be used below only for problem (1.1) with initial data aq_0 where q_0 is infinitely smooth and satisfies (2.14), and $0 < a \ll 1$. Let us recall that the scattering problem (2.2) and the Lippmann-Schwinger equation (2.3) are uniquely solvable for all k when q_0 is small, i.e., there are no exceptional points in this case and $h_0(k,k)$ is defined for all the values of k. Operator $T_{z,t}$ is needed only with $D = \emptyset$ if q_0 is small. Hence, only the first term is present in the right-hand side of (2.13). Moreover, $\|T_{z,t}\| < 1$ when q_0 is small, and therefore equation (2.15) is uniquely solvable for all $z \in \mathbb{C},\ t \geq 0$. Then (q,ϕ) given by (2.16) with $D = \emptyset$ is a smooth global solution of problem (1.1) with the small initial data. We will call this solution *classical*. It exists under a weaker assumption on the decay of q_0 than in Theorem 2.1.

We will consider analytic continuations of functions h_0, q_0, and we need some notation. Let $\gamma = (\gamma_1,\gamma_2) \in \mathbb{C}^2$ and let $A_\gamma : \mathbb{C} \to \mathbb{C}^2$ be the map defined by $A_\gamma z = A_\gamma(x+iy) = (x+\gamma_1, y+\gamma_2) \in \mathbb{C}^2$, i.e., the map A_γ shifts real points x,y into complex planes. If a function $f = f(z)$ is analytic in (x,y), then $B_\gamma f(z) := f(A_\gamma z)$ is the value of the analytic continuation of f at point $A_\gamma z$. We will use notation $A'_\sigma, B'_\sigma,\ \sigma \in \mathbb{C}^2$, for the same operations

[3] that statement can be found in [13, lemma 5.1]

applied to a function of $k \in \mathbb{C}$, and $A''_\eta, B''_\eta,\ \eta \in \mathbb{C}^2$, if they are applied to a function of $\varsigma \in \mathbb{C}$.

The main result of the paper is obtained under the following condition on the initial data that must hold for large enough R:

Condition $Q(R)$**.** The initial data q_0 admits analytic continuation in (x, y) and, for a given $R > 0$, there exist a $C = C(R)$ such that

$$|B_\gamma q_0(z)| \leq Ce^{-R|z|}, \quad z \in \mathbb{C}, \text{ when } |\gamma| \leq R.$$

Remark. Clearly, linear combinations of Gaussian functions satisfy Condition $Q(R)$ for all $R > 0$, and it was shown in [4, 5] that these combinations form a dense set in $L^2(\mathbb{C})$.

We will show that Condition $Q(R)$ implies a similar behavior of the scattering data, i.e., the validity of the following assumption.

Condition $H(R)$**.** For a given $R > 0$, the estimate

$$|h_0(\varsigma, \varsigma)| \leq e^{-R|\varsigma|} \quad \text{as} \quad |\varsigma| \to \infty$$

holds, and there exists $C = C(R)$ and $a_0(R)$ such that the scattering data $h_0(\varsigma, \varsigma)$ for the potential aq_0 with $a \in (0, a_0)$ admits analytic continuation $B''_\eta h_0(\varsigma, \varsigma),\ |\eta| \leq R$, with respect to variables ς_i, and

$$|B''_\eta h_0(\varsigma, \varsigma)| \leq C(R)e^{-R|\varsigma|}, \quad \varsigma \in \mathbb{C}, \text{ when } |\eta| \leq R.$$

We will show that there is a duality between these two conditions. To be more exact, the validity of $Q(R)$ implies the validity of $H(R - \varepsilon)$. Conversely, if the initial data is small, Condition $H(R)$ holds, and h_0 is extended in time according to (2.12), then the potential $q(z, t)$ that corresponds to the extended data $h(\varsigma, \varsigma, t)$ satisfies Condition $Q(R)$ with a smaller R that depends on t. These results will be obtained in the next section. Note that they are an analogue of similar results of L. Sung ([21, Cor. 4.16]) who established a duality of the non-linear Fourier transform in the Schwartz class. We need a refined result to study the more complicated form (2.13) of operator $T_{z,t}$ that appears in the presence of exceptional points.

Below is the main statement of the present paper.

Theorem 2.2. *Let us fix an arbitrary disk D containing all the exceptional points for the potentials $aq_0(z),\ a \in [0, 1]$. Let Condition $Q(R)$ hold with $R > (1 + 2T)A$, where A is the radius of the disk D. Then*

1) for each point $(z, t),\ 0 \leq t \leq T$, the classical solution (q, ϕ) of problem (1.1) with the initial data aq_0 and small enough $a > 0$ admits a meromorphic continuation with respect to a in a neighborhood of the segment $[0, 1]$. This meromorphic continuation is given by (2.16) with an arbitrary choice of the disk D and an arbitrary choice of point $k_0 \in \partial D$[4].

2) when $a = 1$, the analytic continuation of (q, ϕ) is infinitely smooth and satisfies (1.1) everywhere, except possibly a set S that is bounded in the strip

[4]all the exceptional points are inside D, i.e., all the points on ∂D are non-exceptional.

$0 \leq t \leq T,\ (x,y) \in R^2$, and is such that $S_t = S \bigcap \{t = const\}$ is a bounded 1D real analytic variety.

Remark. The theorem implies that the local solutions found in Theorem 2.1 are analytic continuations in a of the global classical solutions (under the assumption that condition $Q(R)$ holds). At the same time, the theorem does not prohibit the solution from blowing up at an arbitrarily small time $t > 0$ (see the recent paper [10] and citations there on instantaneous blow-ups). We can't say anything about relation between our global solution and local solutions found in [8].

Two important technical improvements of the previous results will be used in the proof of Theorem 2.2. First, we will show that the Hilbert space $\mathcal{B}^2$ can be used in Theorem 2.1 instead of the Banach space $\mathcal{B}^s,\ s > 2$. The space $\mathcal{B}^2$ is defined as follows:

$$\mathcal{B}^2 = \left\{ u \in \left(L^2(\mathbb{C}\backslash D) \oplus \mathbb{C}^1 \right) \bigcap L^2_+(\partial D) \right\}. \tag{2.17}$$

Here $\mathbb{C}^1$ is the one-dimensional space of functions of the form $\frac{c\beta(k)}{k}$, where c is a complex constant, $\beta \in C^\infty$ is a fixed function that vanishes in a neighbourhood of the disk D and equals one in a neighbourhood of infinity. By $L^2_+(\partial D)$ we denote the space of analytic functions $u = \sum_{n \geq 0} c_n z^n$ in D with the boundary values in $L^2(\partial D)$ and the norm

$$\|u\|_{L^2_+(\partial D)} = \left(\sum_{n \geq 0} A^{2n} |c_n|^2 \right)^{1/2},$$

where A is the radius of the disk D.

Secondly, we will simplify the form of the operator $T_{z,t}$ by writing the second term in (2.12) and (2.13) without the logarithmic factor. We also will allow k_0 to be on ∂D, and not necessarily in D, and show that formula (2.13) in the latter case can be written as

$$T_{z,t}\phi(k) = \frac{1}{\pi} \int_{\mathbb{C}\backslash D} e^{i(\overline{\varsigma} z + \overline{z}\varsigma)/2} \overline{\phi}(\varsigma) \Pi^o h(\varsigma, \varsigma, t) \frac{d\sigma_\varsigma}{\varsigma - k} - \tag{2.18}$$

$$i \int_{\partial D} \frac{d\varsigma}{\varsigma - k} \int_{\widehat{k_0, \varsigma}} [e^{i(\varsigma \overline{z} + \overline{\varsigma'} z)/2} \overline{\phi^-(\varsigma')} \Pi^o + e^{i(\varsigma - \varsigma')\overline{z}/2} \phi^-(\varsigma') \Pi^d \mathbf{C}] \left[h(\varsigma', \varsigma, t) \overline{d\varsigma'} \right],$$

where $\widehat{k_0, \varsigma}$ is the arc on ∂D between points k_0 and ς with the counter clock-wise direction on it.

The following two difficulties were resolved in the paper. We show that if one starts with a small potential q_0 and its scattering data $h_0(\varsigma, \varsigma)$, and extends $h_0(\varsigma, \varsigma)$ in time according to (2.12), then the solution $q(z,t)$ of the inverse scattering problem with the scattering data (2.12) decays exponentially at infinity, and the scattering data (2.7) for this potential $q(z,t)$ coincides with the scattering data $h(\varsigma, \varsigma, t)$ from which the potential was obtained (this will be done in the next section). Another difficulty concerns the proof of the invertibility of operator $I+T_{z,t}$ for large $|z|$ in spite of the exponential growth of the integrands in the second terms of (2.13) and (2.18) as $|z|\to\infty$ (see section 5).

3. Exponential decay of the scattering data and of $q(z,t)$

Lemma 3.1. *Let*

$$I(z)=\int_{\mathbb{C}}\frac{f(z_1)}{z-z_1}d\sigma_{z_1},\quad J(z)=\int_{\mathbb{C}}\frac{f(z_1)}{\overline{z}-\overline{z}_1}d\sigma_{z_1},\quad z\in\mathbb{C},$$

where $f(z)$ is analytic in (x,y), and

$$|f(A_\gamma z)|,|\nabla_\gamma f(A_\gamma z)|\le\frac{C(\gamma)}{1+x^2+y^2}.$$

Then $I(z),J(z)$ admit analytic continuation in (x,y), and

$$B_\gamma I(z)=\int_{\mathbb{C}}\frac{f(A_\gamma z_1)}{z-z_1}d\sigma_{z_1},\quad B_\gamma J(z)=\int_{\mathbb{C}}\frac{f(A_\gamma z_1)}{\overline{z}-\overline{z}_1}d\sigma_{z_1}.$$

Proof. Let us rewrite $I(z)$ in the form

$$I(z)=-\int_{\mathbb{C}}\frac{f(z+z_1)}{z_1}d\sigma_{z_1}.$$

This immediately implies that $I(z)$ is analytic in (x,y), and

$$B_\gamma I(z)=\int_{\mathbb{C}}\frac{B_\gamma f(z+z_1)}{-z_1}d\sigma_{z_1}=\int_{\mathbb{C}}\frac{f(x+\gamma_1+x_1,y+\gamma_2+y_2)}{-z_1}d\sigma_{z_1}$$
$$=\int_{\mathbb{C}}\frac{f(A_\gamma z_1)}{z-z_1}d\sigma_{z_1}.$$

The statement for J can be proved absolutely similarly.

□

Let us provide some examples of analytic continuations of functions from $\mathbb{C}$ into $\mathbb{C}^2$: (1) If $f(z)=z=x+iy$, then $B_\gamma f(z)=x+\gamma_1+i(y+\gamma_2)=z+\gamma'$, $\gamma'=\gamma_1+i\gamma_2\in\mathbb{C}$. (2) If $f(z)=\overline{z}=x-iy$, then $B_\gamma f(z)=x+\gamma_1-i(y+\gamma_2)=\overline{z}+\gamma''$, $\gamma''=\gamma_1-i\gamma_2\in\mathbb{C}$ (note that $\gamma'\neq\overline{\gamma''}$ since γ_i are complex.) (3) if $f(z)=\Re(k\overline{z})=k_1x+k_2y$, then $B_\gamma f(z)=\Re(k\overline{z})+k_1\gamma_1+k_2\gamma_2$ and $B_\sigma\Re(k\overline{z})=\Re(k\overline{z})+\sigma_1x+\sigma_2y$.

Lemma 3.2. *Let the potential be $aq_0(z)$ where q_0 satisfies Condition $Q(R)$ for some $R>0$. Then there exists $a_0=a_0(R)$ such that function $\overline{\mu}=\overline{\mu}(z,k)$ defined by (2.3) via the solution of the Lippmann-Schwinger equation with the potential aq_0, $a\in(0,a_0)$, admits analytic continuation to $\mathbb{C}^4$ with respect to variables x,y,k_1,k_2, and*

$$|B'_\sigma B_\gamma\overline{\mu}(z,k)|<C(R,\varepsilon)\quad\text{when } z,k\in\mathbb{C},\ |\sigma|,|\gamma|\le R-\varepsilon,\ a<a_0. \tag{3.1}$$

The statement remains valid if $a=1$, but $|k|\ge\rho(R)$ with large enough ρ.

Proof. We will prove the statement of the lemma for the component μ_{11} of the matrix μ. Other components can be treated similarly. Let us iterate equation (2.5). The following equation is valid for the first component:
(3.2)
$$\overline{\mu_{11}} = 1 + \frac{1}{\pi^2}\int_{\mathbb{C}} d\sigma_{z_1}\int_{\mathbb{C}} d\sigma_{z_2}\frac{e^{i\Re(k\overline{z}_1)}}{\overline{z}-\overline{z_1}}\overline{Q}_{12}(z_1)\frac{e^{-i\Re(k\overline{z}_2)}}{z_1-z_2}Q_{21}(z_2)\overline{\mu_{11}}(z_2,k),$$
where Q_{21} and Q_{12} are entries of the matrix Q_0. Denote $Q = Q_{12} = -Q_{21}$. Assume that the analytic continuation $B_\gamma\overline{\mu}_{11}$ exists. Then from Lemma 3.1, formula (3.2) and the relation
$$B_\gamma e^{\pm i\Re(k\overline{z})} = e^{\pm i\Re(k\overline{z})\pm i<k,\gamma>}$$
it follows that $B_\gamma\overline{\mu}_{11}$ is equal to
$$1 - \frac{1}{\pi^2}\int_{\mathbb{C}} d\sigma_{z_1}\frac{B_\gamma e^{i\Re(k\overline{z}_1)}}{\overline{z}-\overline{z_1}}B_\gamma\overline{Q}(z_1)B_\gamma\int_{\mathbb{C}} d\sigma_{z_2}\frac{e^{-i\Re(k\overline{z}_2)}}{z_1-z_2}Q(z_2)\overline{\mu_{11}}(z_2,k)$$
$$= 1 - \frac{1}{\pi^2}\int_{\mathbb{C}} d\sigma_{z_1}\frac{e^{i\Re(k\overline{z}_1)}}{\overline{z}-\overline{z_1}}B_\gamma\overline{Q}(z_1)\int_{\mathbb{C}}\frac{e^{-i\Re(k\overline{z}_2)}}{z_1-z_2}B_\gamma Q(z_2)B_\gamma\overline{\mu_{11}}(z_2,k)d\sigma_{z_2}.$$
Hence, if the analytic continuation $\Psi := B'_\sigma B_\gamma\overline{\mu}_{11}$ exists, then it satisfies the equation
$$\Psi(z,k) = 1 - \frac{1}{\pi^2}\int_{\mathbb{C}} d\sigma_{z_1}\frac{e^{i\Re(k\overline{z}_1)+i<\sigma,z_1>}}{\overline{z}-\overline{z_1}}B_\gamma\overline{Q}(z_1)\cdot$$
$$\cdot\int_{\mathbb{C}}\frac{e^{-i\Re(k\overline{z}_2)-i<\sigma,z_2>}}{z_1-z_2}B_\gamma Q(z_2)\Psi(z_2,k)d\sigma_{z_2}.$$
Denote by $K^\pm = K^\pm_{k,\sigma,\gamma}$ the integral operators given by the exterior and interior integrals above, respectively. Their norms in the space $L^\infty(\mathbb{C})$ can be estimated from above by the norms of the potential (see [20]):
$$\|K^-\| < C(\|e^{-i<\sigma,z_2>}B_\gamma Q(z_2)\|_{L^p(\mathbb{C})} + \|e^{-i<\sigma,z_2>}B_\gamma Q(z_2)\|_{L^q(\mathbb{C})}),$$
where $1 < p < 2 < q < \infty$. A similar estimate is valid for K^+. Thus the assumption $a_0 \ll 1$ and Condition Q imply that $\|K^\pm\| < 1$, Ψ exists, and
$$|\Psi| < C(R) \quad \text{when } |\Im\sigma|, |\gamma| \le R,\ a < a_0.$$
Moreover, the derivatives of $K^\pm$ with respect to complex variables σ_i, γ_j also have small norms, i.e., $\Psi = \Psi(z,k,\sigma,\gamma)$ is analytic in $(\sigma_1,\sigma_2,\gamma_1,\gamma_2)$. One can easily see that $\Psi = \Psi(z+\gamma, k+\sigma)$. Hence Ψ is the analytic continuation of $\overline{\mu}$. The proof of (3.1) is complete.

In order to prove the statement of Lemma 3.2 concerning $a = 1$, one needs only to show that operator $K := K^+K^-$ and its derivatives in σ_i, γ_j are small (less than one) as $|k| \to \infty$. This can be done by a standard procedure: one splits K into two terms $K = K_1 + K_2$, where K_1 is obtained by adding the factor $\alpha(\frac{z_1-z}{\varepsilon})\alpha(\frac{z_1-z_2}{\varepsilon})$ in the integral kernel of K. Here $\alpha = \alpha(z)$ is a cut-off function that is equal to one when $|z| < 1$ and vanishes when $|z|>2$. Then $\|K_1\|\to 0$ as $\varepsilon\to 0$, and $\|K_2\|=O(|k|^{-1})$ as $|k|\to\infty$. The latter can be shown by appropriate integration by parts in x_1,y_1.

$\square$

Theorem 3.3. *If Condition $Q(R)$ holds for some $R>0$, then Condition $H(R-\varepsilon)$ holds for each $\varepsilon>0$.*

Proof. Recall that

$$h_0(\varsigma,\varsigma)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}} e^{-i\Re(\overline{\varsigma}z)}Q_0(z)\overline{\mu}(z,\varsigma)d\sigma_z,\ \ k,\varsigma\in\mathbb{C}.$$

We shift the complex plane $\mathbb{C}$ in the integral above by vector $\gamma=-i\frac{(\varsigma_1,\varsigma_2)}{|\varsigma|}(R-\varepsilon)$, and then apply operator B_η. This leads to

$$|B''_\eta h_0|<\frac{1}{(2\pi)^2}\int_{\mathbb{C}}\left|e^{-i(<\eta,z>+<\varsigma,\gamma>+<\eta,\gamma>)}Q_0(A_\gamma z)B_\gamma B''_\eta\overline{\mu}(z,\varsigma)\right|d\sigma_z.$$

It remains to use Lemma 3.2 and Condition $Q(R)$.

$\square$

Theorem 3.4. *Let Condition $Q(R)$ hold for some $R>0$ and let the scattering data h_0 be defined by the potential aq_0, $0<a<a_0(R)$, where $a_0(R)$ is defined in Lemma 3.2. Then the time dependent scattering data $h(\varsigma,\varsigma,t)$, $0\le t\le T$, given by (2.12), admits an analytic continuation in $(\varsigma_1,\varsigma_2)$, and*

$$|B''_\eta h(\varsigma,\varsigma,t)|\le C(R,\varepsilon)e^{(-\frac{R}{1+2T}+\varepsilon)|\varsigma|},\quad |\eta|\le\frac{R}{1+2T}.$$

The statement remains valid if $a=1$, but $|\varsigma|>\rho$, where $\rho=\rho(R)$ is large enough.

Proof. The statement follows immediately from Theorem 3.3 and formula (2.12). One needs only to combine the upper bound $Ce^{(-R+\varepsilon)|\varsigma|}$ for the analytic continuation of h_0 obtained in Theorem 3.3 with the upper bound $Ce^{\frac{2TR}{1+2T}|\varsigma|}$ for the time-dependent factor in (2.12).

$\square$

Let us recall again the procedure to obtain the classical solution of the focusing DSII equation with initial data aq_0 and a very small a such that there are no exceptional points. As the first step, one needs to solve the equation $(I+T_{z,t})v=I$, where $T_{z,t}$ is given by (2.13) with $D=\emptyset$, i.e., the equation for $v=v_{z,t}$ has the form

$$v_{z,t}(k)+\frac{1}{\pi}\int_{\mathbb{C}} e^{i(\overline{\varsigma}z+\overline{z}\varsigma)/2}\overline{v_{z,t}}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)\frac{d\sigma_\varsigma}{\varsigma-k}=I, \tag{3.3}$$

where $w_{z,t}(\cdot)=v_{z,t}(\cdot)-I\in\mathcal{B}^s$. Then the solution of the focusing DSII equation with initial data aq_0 is given by (2.16). In particular,

$$q(z,t)=\frac{1}{2\pi i}\int_{\mathbb{C}} e^{i(\overline{\varsigma}z+\overline{z}\varsigma)/2}(\overline{v_{z,t}})_{11}(\varsigma)h_{12}(\varsigma,\varsigma,t)d\sigma_\varsigma. \tag{3.4}$$

Theorem 3.5. *Let Condition $Q(R)$ hold for q_0, and let the potential $q(z,t)$, $0 \leq t \leq T$, in (3.4) be constructed from the initial data $aq_0(z)$ with $0 < a < a_1 \ll 1$. Then there exists $a_1 = a_1(R,T)$ such that Condition $Q(\frac{R}{1+2T} - \varepsilon)$ holds for the potential (3.4) for all $t \in [0,T]$.*

Proof. There is a complete duality (e.g. [21, Th. 4.15]) between the nonlinear Fourier transform given by (2.5), (2.7) and the inverse transform (3.3), (3.4). Function h in (3.3) plays the role of the potential Q_0 in (2.5). Theorem 3.4 implies that the Condition $Q(R')$ holds for h with $R' = \frac{R}{1+2T} - \frac{\varepsilon}{2}$. From Lemma 3.2 applied to (3.3) instead of (2.5), it follows that v has the same properties as the properties of μ established in Lemma 3.2. One needs only to take a small enough to guarantee that the analogues of operators $K^{\pm}$ have norms that do not exceed one. Then

$$|B'_\sigma B_\gamma \overline{v}(z,k)| < C(R',\varepsilon) \quad \text{when } z,k \in \mathbb{C},\ |\sigma|,|\gamma| \leq R' - \frac{\varepsilon}{2},\ a \ll 1.$$

Then the statement of the theorem can be obtained similarly to the proof of Theorem 3.3, i.e., by using the shift of the complex plane $\mathcal{C}$ in (3.4) by the vector $\eta = i\frac{(x,y)}{|z|}(R' - \frac{\varepsilon}{2})$.

□

4. Proof of the first statement of Theorem 2.2

Consider problem (1.1) with q_0 replaced by aq_0, $a \in (0,1]$. Let D be a disk containing all the exceptional points for problems (2.2), (2.3) for all $a \in (0,1]$. Let $k_0 \in \partial D$ be a non-exceptional point for all $a \in (0,1]$. We will use notation v^1 for the solution of (2.15) and (q^1,φ^1) for the pair defined by (2.16) when the operator $T_{z,t}$ is defined using the disk D. We preserve the notations v, (q,φ) for the same objects when there are no exceptional points and $D = \emptyset$. Since q^1,φ^1 are meromorphic in a in a neighbourhood of $(0,1]$ (see Theorem 2.1), the first statement of Theorem 2.2 will be proved if we show that $(q^1,\varphi^1) = (q,\varphi)$ when $a > 0$ is small and $t > 0$.

From (2.4), (2.7) and Condition $Q(R)$ with $R > (1+2T)A > A$, it follows that the scattering data $h_0 = h_0(\varsigma,k)$ is defined for all the potentials aq_0 when $|\varsigma|,|k| \leq A$ (i.e., $\varsigma,k \in \overline{D}$) and also for all $\varsigma = k$. We define $h(\varsigma,k,t)$ (extension of h_0 in t) according to (2.12). Let $v = v_{z,t} = w_{z,t} + I$, where $w_{z,t} \in \mathcal{B}^s, s > 2$, is the solution of (2.15) with $T_{z,t}$ given by (2.13) with $D = \emptyset$ (i.e., the right-hand side in (2.13) contains only the first term, see equation (3.3)). Then (q,ϕ) given by (2.16) with $D = \emptyset$ solves the DSII equation (1.1) (see [7]), and

$$\psi = \psi(z,k,t) := \Pi^d \overline{v} e^{i\overline{k}z/2} + e^{-i\overline{z}k/2}\Pi^o v,\ \varsigma,k \in \mathbb{C},\ t \geq 0, \tag{4.1}$$

is the solution of the scattering problem (2.2) (and the Lippmann-Schwinger equation (2.3)) with the potential $Q_t(z) = \begin{pmatrix} 0 & q(z,t) \\ -q(z,t) & 0 \end{pmatrix}$ instead of Q_0.

Consider now the scattering data

$$\widehat{h}(\varsigma,k,t) := \frac{1}{(2\pi)^2}\int_{\mathbb{C}} e^{-i\overline{\varsigma}z/2}Q_t(z)\overline{\psi}(z,k,t)d\sigma_z \tag{4.2}$$

defined by the solution ψ of the Lippmann-Schwinger equation (2.3) with the potential $Q_t(z)$. If $0\le t\le T$, then from Theorem 3.5 (it is assumed there that $R>(1+2T)A$) it follows that integral (4.2) converges when $|\varsigma|,|k|\le A$ (i.e., $\varsigma,k\in\overline{D}$) and when $\varsigma=k$. Moreover, $\widehat{h}(\varsigma,k,t)=\widehat{h}(k+\alpha,k,t)$ is an anti-analytic continuation of $\widehat{h}(k,k,t)$ in α. We will prove that $\widehat{h}$ coincides with the scattering data $h(\varsigma,k,t)$ defined in (2.12). We also will prove that there exists an analytic in k function $\widehat{v}_1^+=\widehat{v}_1^+(k,t),\ k\in D$, such that

$$(v-\widehat{v}_1^+)|_{\varsigma\in\partial D} = \tag{4.3}$$

$$=\int_{\partial D}[e^{i/2(\varsigma\overline{z}+\overline{\varsigma'}z)}\overline{\widehat{v}_1^+(\varsigma')}\Pi^o - e^{i/2(\varsigma-\varsigma')\overline{z}}\widehat{v}_1^+(\varsigma')\Pi^d\mathbf{C}]\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0^1}}\widehat{h}_t(\varsigma',\varsigma)d\varsigma'.$$

From these two facts and the $\overline{\partial}$-equation (see [20])

$$\frac{\partial}{\partial\overline{k}}v(z,k,t)=e^{i(\overline{k}z+\overline{z}k)/2}\overline{v}(z,k,t)\Pi^o h(k,k,t),\quad k\in\mathbb{C}\backslash D, \tag{4.4}$$

it follows (see [13, Lemma 3.3]) that the function

$$v'(z,k):=\begin{cases} v(z,k), & k\in\mathbb{C}\backslash D,\\ \widehat{v}_1^+(z,k), & k\in D,\end{cases} \tag{4.5}$$

satisfies the integral equation (2.15), where operator $T_{z,t}$ is constructed using the scattering data $\widehat{h}$. Equation (2.15) has a unique solution when a is small enough. Under the assumption that $\widehat{h}=h$, we have $v^1\equiv v'$. Therefore $v^1(z,k)=v(z,k)$ when $k\in\mathbb{C}\backslash D$. Solution (q,ϕ) of the DSII equation can be determined via the asymptotics of v at large values of k (e.g., [20, (1.17)], [14, Lemma 3.3]). Hence $(q^1,\phi^1)=(q,\phi)$ for small a. Thus the first statement of the theorem will be proved as soon as we show that $\widehat{h}=h,\ t>0$, and that $\widehat{v}^+$ exists.

Justification of the equality $\widehat{h}=h,\ t>0$. Everywhere below, till the end of the section, we omit mentioning the parameter a and assume that the initial data q_0 is small. Let us recall (see [14, Lemmas 4.1, 4.2]) that the symmetry of the matrix Q_0 (see (2.1)) implies that $h_{11}=h_{22}, h_{12}=-h_{21}$, and the same relations hold for matrix v determined from the integral equation (2.15) and related to ψ by (4.1).

Let us introduce functions

$$\begin{pmatrix} a & b\\ -b & a\end{pmatrix}=h_0(k+\alpha,k),$$

and note that

$$b(\alpha,k)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}e^{-i(k\overline{z}+\overline{k}z)/2}q_0(z)v_{11}(z,k)d\sigma_z,$$
$$a(\alpha,k)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}q_0(z)\overline{v}_{12}(z,k)d\sigma_z.$$

Now define

$$\begin{pmatrix} a(\alpha,k,t) & b(\alpha,k,t)\\ -b(\alpha,k,t) & a(\alpha,k,t)\end{pmatrix}=h(k+\alpha,k,t),$$

where h is given by (2.12). Similar quantities $\widehat{a},\widehat{b}$ are defined via the solutions $v(\cdot,k,t)$:

$$\widehat{b}(\alpha,k,t):=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}e^{-i(k\overline{z}+\overline{k}z)/2}q(z,t)v_{11}(z,k,t)d\sigma_z,$$
$$\widehat{a}(\alpha,k,t):=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}e^{-\overline{\alpha}z/2}q(z,t)\overline{v}_{12}(z,k,t)d\sigma_z.$$

These quantities are well defined due to Theorem 3.5. Let

$$\widehat{h}=\begin{pmatrix} \widehat{a}(\alpha,k,t) & \widehat{b}(\alpha,k,t)\\ -\widehat{b}(\alpha,k,t) & \widehat{a}(\alpha,k,t)\end{pmatrix}.$$

Consider solution $\psi(z,k,t)$ of (2.2) with potential Q_0 replaced by Q_t, and let v be defined by (4.1). From (4.1) it follows that $v\to I$ uniformly on each compact with respect to the variable z when $k\to\infty$. Therefore, from Theorem 3.5 it follows that

$$\widehat{a}(\alpha,k,t)\to 0,\ \ k\to\infty. \tag{4.6}$$

Obviously (see (2.12)), the same relation holds for $a(\alpha,k,t)$.

The $\overline{\partial}$-equation (4.4) implies that the following rules are valid when $t=0$:

$$\frac{\partial b}{\partial\overline{k}}=\frac{\partial b}{\partial\overline{\alpha}}+ab_0,\quad \frac{\partial a}{\partial k}=b\overline{b_0},\quad \text{where}\quad b_0=b(0,k),\ |\alpha|\le A,\ k\in\mathbb{C}. \tag{4.7}$$

Due to Theorem 3.5, the same relations are valid for $\widehat{a}(\alpha,k,t),\widehat{b}(\alpha,k,t)$:

$$\frac{\partial\widehat{b}}{\partial\overline{k}}=\frac{\partial\widehat{b}}{\partial\overline{\alpha}}+\widehat{a}\widehat{b}_0,\quad \frac{\partial\widehat{a}}{\partial k}=\widehat{b}\overline{\widehat{b}_0},\ \widehat{b}_0=\widehat{b}(0,k,t),\quad |\alpha|\le A,\ k\in\mathbb{C},\ 0\le t\le T. \tag{4.8}$$

From (2.12), (4.7), and the obvious relations

$$e^{-t(\overline{k}^2-(\overline{k+\alpha})^2)/2}=e^{-t(k^2-(\overline{k+\alpha})^2)/2}\overline{e^{-i\Im k^2}},$$
$$\frac{\partial}{\partial\overline{\alpha}}e^{-t(k^2-(\overline{k+\alpha})^2)/2}=\frac{\partial}{\partial\overline{k}}e^{-t(k^2-(\overline{k+\alpha})^2)/2},$$

it follows that (4.8) holds for $a(\alpha,k,t),b(\alpha,k,t)$:

$$\frac{\partial b}{\partial\overline{k}}=\frac{\partial b}{\partial\overline{\alpha}}+ab_0,\quad \frac{\partial a}{\partial k}=b\overline{b_0},\quad b_0=b(0,k,t), \tag{4.9}$$

when $|\alpha|\le A,\ k\in\mathbb{C},\ 0\le t\le T.$

Now we note that $\widehat{b}(0,k,t)=b(0,k,t)$ (see [22, Theorem 5.3]). The second relations in (4.8), (4.9) with $\alpha=0$ imply that $(\widehat{a}-a)|_{\alpha=0}$ is anti-analytic in k. Then the maximum principle, together with (4.6) for both $\widehat{a}$ and a, imply that $\widehat{a}|_{\alpha=0}=a|_{\alpha=0}$. Now from the first relations in (4.8), (4.9), with $\alpha=0$, it follows that $\frac{\partial\widehat{b}}{\partial\overline{\alpha}}|_{\alpha=0}=\frac{\partial b}{\partial\overline{\alpha}}|_{\alpha=0}$. Then we differentiate the second relations in (4.8), (4.9) in $\overline{\alpha}$ and put $\alpha=0$ there. This leads to the anti-analyticity in k of $\frac{\partial\widehat{a}}{\partial\overline{\alpha}}|_{\alpha=0}-\frac{\partial a}{\partial\overline{\alpha}}|_{\alpha=0}$. The maximum principle with (4.6) imply that $\frac{\partial\widehat{a}}{\partial\overline{\alpha}}|_{\alpha=0}=\frac{\partial a}{\partial\overline{\alpha}}|_{\alpha=0}$. After the differentiation in $\overline{\alpha}$ of the first relations in (4.8), (4.9), we obtain that $\frac{\partial^2\widehat{b}}{\partial\overline{\alpha}^2}|_{\alpha=0}=\frac{\partial b^2}{\partial\overline{\alpha}^2}|_{\alpha=0}$, and so on. Hence all the derivatives in $\overline{\alpha}$ of the vectors $(\widehat{a},\widehat{b})$ and (a,b) coincide when $\alpha=0$. Since both vectors are anti-analytic in α, they are identical, i.e., $\widehat{h}=h,\ t>0$.

The *existence of* $\widehat{v}^+$ can be shown similarly to the proof of same statement in [13], where the potential was assumed to be compactly supported. Namely, consider the following analogue of the Lippmann-Schwinger equation with different values $k_0,k\in\overline{D}$ of the spectral parameter in the operator and in the free term of the equation:

$$\psi^+(z,k)=e^{i\frac{\overline{k}z}{2}}I+\int_{z\in\mathbb{C}}G(z-z',k_0)Q_t(z')\overline{\psi^+}(z',k)d\sigma_{z'}, \tag{4.10}$$

where $G(z,k)=\frac{1}{\pi}\frac{e^{i\overline{k}z/2}}{z}$. We substitute here $\psi^+=\mu^+e^{i\overline{k_0}z/2}$ and rewrite the equation in terms of

$$w^+=\mu^+(z,k)-e^{i(\overline{k-k_0})z/2}I\in L_z^\infty(L_k^p),\ p>1. \tag{4.11}$$

The equation takes the form

$$\begin{aligned} w^+(z,k)-\int_{z\in\mathbb{C}}\frac{e^{-i\Re(\overline{k_0}z')}}{z-z'}Q_t(z')\overline{w^+}(z',k)d\sigma_{z'}= \\ =\int_{z\in\mathbb{C}}\frac{e^{-i\Re(\overline{k_0}z')}}{z-z'}\left[Q_t(z')e^{-i\overline{z'}(k-k_0)/2}\right]d\sigma_{z'}. \end{aligned} \tag{4.12}$$

Theorem 3.5 implies that function $\left[Q_t(z')e^{-i\overline{z'}(k-k_0)/2}\right]$ decays exponentially as $z\to\infty$, and $|k|,|k_0|\le A$. The unique solvability of the problem (4.12) is obvious since the potential is small.

Function $\widehat{v}^+$ is defined by ψ^+ in the same way as v is defined by ψ in (4.1). The analyticity of $\widehat{v}^+$ and (4.3) are proved in Lemmas 3.1 and 3.5 of [13].

□

5. Proof of statement 2 of the Theorem 2.2.

Reduction to Theorem 5.2 and Lemma 5.3. Theorem 3.5 immediately implies that the operator $T_{z,t}:\mathcal{B}^s\to\mathcal{B}^s,\ s>2$, is analytic in x and y in a complex neighborhood of $\mathbb{R}^2$. In order to use the multidimensional analytic Fredholm theory ([24, Th. 4.11, 4.12] or [23]) and obtain a decay of operator norm $\|T^2_{z,t}\|$ as $|z|\to\infty$, we would like to consider this operator in the Hilbert space $\mathcal{B}^2$ instead of the Banach space $\mathcal{B}^s,\ s>2$. All the previous and new results mentioned in this paper remain valid if $s>2$ is replaced by $s=2$ (with the appropriate definition of the space $\mathcal{B}^2$ given in (2.17)). In order to justify the latter statement, one needs to show that the properties of the operator $T_{z,t}$ are preserved when $s>2$ is replaced by $s=2$. This will be done in Theorem 5.2 below (we will not discuss the properties that obviously are s-independent), but we will show that operator $T_{z,t}:\mathcal{B}^2\to\mathcal{B}^2, 0\le t\le T$, is compact, continuous in (z,t), and analytic in (x,y) in a complex neighborhood of $\mathbb{R}^2$. After that, we will show (Lemma 5.3) the invertibility of $I+T_{z,t}$ at large values of $|z|$. Then the second statement of the theorem will be a simple consequence of the first statement and the analytic Fredholm theory. Note that the invertibility of operator $I+T_{z,t}$ will be proved for z on each ray $\arg z=\psi=const, |z|\ge Z_0$, with ψ-independent Z_0 and with $T_{z,t}$ defined (see (2.13)) using a special value of $k_0=k_0(\psi)$. Since the solution (q,ϕ) of problem (1.1) does not depend on the choice of k_0 (see Theorem 2.2), it remains only to prove Theorem 5.2 and Lemma 5.3.

5.1. **Compactness of operator T.** We will need the following lemma.

Lemma 5.1. *Let operator $M:\mathcal{B}^2\to\mathcal{B}^2$ have the form*

$$(Mf)(k)=\int_{\mathbb{C}\backslash D}\frac{g(\varsigma)}{\varsigma-k}f(\varsigma)d\sigma_\varsigma,\quad k\in\mathbb{C},$$

where function $g_\delta=g(\varsigma)(1+|\varsigma|)^\delta$ has the following properties

$$|g_\delta|<a_1<\infty,\ g_\delta\to 0 \text{ as } \varsigma\to\infty, \text{ and } \|g_\delta\|_{L^2(\mathbb{C}\backslash D)}=a_2<\infty$$

for some $\delta>0$. Then M is compact and $\|M\|\le C(a_1+a_2)$.

Proof. Let P be the following operator in $\mathcal{B}^2$ of rank one:

$$Pf=-\frac{\beta(k)}{k}\int_{\mathbb{C}\backslash D}g(\varsigma)f(\varsigma)d\sigma_\varsigma, \tag{5.1}$$

where β is the function introduced in the definition of the space $\mathcal{B}^2$. Since $Pf=0$ in a neighborhood of D, and

$$\int_{\mathbb{C}\backslash D}g(\varsigma)f(\varsigma)d\sigma_\varsigma\le a_2\int_{\mathbb{C}\backslash D}|\frac{f(\varsigma)}{(1+|\varsigma|)^\delta}|^2d\sigma_\varsigma\le Ca_2\|f\|_{\mathcal{B}^2},$$

it is enough to prove the statement of the lemma for operator $M-P=M_1+M_2$, where

$$M_if=\int_{\mathbb{C}\backslash D}K_i(k,\varsigma)f(\varsigma)d\sigma_\varsigma,$$

$$K_1(k,\varsigma)=\frac{\alpha(\varsigma-k)}{\varsigma-k}g(\varsigma),\ K_2(k,\varsigma)=[\frac{\beta(\varsigma-k)}{\varsigma-k}+\frac{\beta(k)}{k}]g(\varsigma),$$

and $\alpha:=1-\beta$ is a cut-off function which is equal to one in a neighborhood of D.

Let M_i' be the operator defined by the same formulas as operators M_i, but considered as operators in $L^2(\mathbb{C})$. Let us show that operators M_i' are compact and their norms do not exceed $C(a_1+a_2)$.

Since $|g| \leq a_1$, we have

$$\sup_{k\in\mathbb{C}} \int_{\mathbb{C}} |K_1(k,\varsigma)| d\sigma_\varsigma + \sup_{\varsigma\in\mathbb{C}} \int_{\mathbb{C}} |K_1(k,\varsigma)| d\sigma_k \leq Ca_1.$$

Hence, from the Young theorem, it follows that $\|M_i'\| \leq Ca_1$. Similarly, using the decay of g_1 at infinity, we obtain that $M_1' = \lim_{R\to\infty} M_{1,R}'$, where $M_{1,R}'$ are operators in $L^2(\mathbb{C})$ with the integral kernels $K_1(k,\varsigma)\alpha(\varsigma/R)$. Operators $M_{1,R}$ are pseudo-differential operators of order -1 (they increase the smoothness of functions by one) defined in a bounded domain. Hence operators $M_{1,R}'$ and their limit M_1' are compact operators in $L^2(\mathbb{C})$.

The boundedness (with the upper bound Ca_2) and compactness of the operator M_2 will be proved if we show that

$$\int_{\mathbb{C}} \int_{\mathbb{C}} |K_2(k,\varsigma)|^2 d\sigma_k d\sigma_\varsigma \leq Ca_2.$$

We split the interior integral in two parts: over region $|k| < 2|\varsigma|$ and over region $|k| > 2|\varsigma|$, and estimate each of them separately. We have

$$\int_{|k|<2|\varsigma|} |K_2(k,\varsigma)|^2 d\sigma_k \leq 2|g(\varsigma)|^2 \int_{|k|<2|\varsigma|} [\frac{\beta^2(\varsigma-k)}{|\varsigma-k|^2} + \frac{\beta^2(k)}{|k|^2}] d\sigma_k$$
$$\leq C|g(\varsigma)|^2(1+|\varsigma|)^\delta.$$

A better estimate with a logarithmic factor is valid, but we do not need this accuracy. Next,

$$\int_{|k|>2|\varsigma|} |K_2(k,\varsigma)|^2 d\sigma_k = |g(\varsigma)|^2 \int_{|k|>2|\varsigma|} \frac{|k\beta(\varsigma-k)+(\varsigma-k)\beta(k)|^2}{|(\varsigma-k)k|^2} d\sigma_k.$$

The denominator of the integrand can be estimated from below by $\frac{1}{4}|k|^4$. The numerator, denoted by n, has the following properties. If $|\varsigma|$ is large enough, than both beta functions in n are equal to one, and $n = |\varsigma|^2$. The same is true if $|\varsigma|$ is bounded and $|k|$ is large. If both variables are bounded, than $|n|$ is bounded. Thus $|n| < (C+|\varsigma|)^2$, and the integrand above does not exceed $C\frac{1+|\varsigma|^2}{|k|^4}$. Obviously, the integrand vanishes when $|k|$ is small enough. Thus there is a constant $c > 0$ such that

$$\int_{|k|>2|\varsigma|} |K_2(k,\varsigma)|^2 d\sigma_k \leq C|g(\varsigma)|^2 \int_{|k|>\max(c,2|\varsigma|)} \frac{1+|\varsigma|^2}{|k|^4} d\sigma_k$$

$$\leq C|g(\varsigma)|^2 \int_{|k|>c} \frac{1}{|k|^4} d\sigma_k + C|g(\varsigma)|^2 \int_{|k|>2|\varsigma|} \frac{|\varsigma|^2}{|k|^4} d\sigma_k = C_1|g(\varsigma)|^2.$$

Hence

$$\int_{\mathbb{C}} \int_{\mathbb{C}} |K_2(k,\varsigma)|^2 d\sigma_k d\sigma_\varsigma \leq C \int_{\mathbb{C}} |g(\varsigma)|^2 (1+|\varsigma|)^\delta d\sigma_\varsigma \leq C' a_2.$$

Thus, operators $M_i' : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ are compact and $\|M_i'\| \le C(a_1 + a_2)$.

Denote by $M_i'' : \mathcal{B}^2 \to L^2(\mathbb{C})$ operators with the same integral kernels K_i as for operators M_i', but with the domain $\mathcal{B}^2$ instead of $L^2(\mathbb{C})$. Compactness of these operators will be proved if we show the boundedness of M_i' on the one-dimensional space of functions of the form $f_c(\varsigma) = c\frac{\beta(\varsigma)}{\varsigma}$, c =const. The upper estimate on $\|M_i'' f_c\|$ can be obtained by repeating the arguments above used to estimate $\|M_i'\|$. One needs only to replace f_c by the function $f = f_c/|\varsigma|^{\delta/2} \in L^2(\mathbb{C})$ and replace the kernel K_i by $K_i|\varsigma|^{\delta/2}$. Hence, operators M_i'' are compact and $\|M_i'\| \le C(a_1 + a_2)$.

Obviously, for each $f \in \mathcal{B}^2$, the function $(M_1 + M_2)f$ is analytic in D. Consider its trace on ∂D. Let $M_D : \mathcal{B}^2 \to L^2(\partial D)$ be the operator that maps each $f \in \mathcal{B}^2$ into the trace of $(M_1 + M_2)f$ on ∂D. In order to complete the proof of the lemma, it remains to show that operator M_D is well defined, compact, and $\|M_D\| \le C(a_1 + a_2)$. To prove these properties of M_D, we split the operator into two terms $M_D = M_D\phi + M_D(1-\phi)$, where ϕ is the operator of multiplication by the indicator function of a disk D_1 of a larger radius than the radius of D. Then $M(1-\phi)f$ is analytic in D_1, and

$$\|M(1-\phi)f\|_{L^2(D_1)} \le \|Mf\|_{L^2(\mathbb{C})} \le C(a_1 + a_2)\|f\|_{\mathcal{B}^2}.$$

From a priori estimates for elliptic operators, it follows that

$$\|M(1-\phi)f\|_{H^s(D)} \le C_s\|M(1-\phi)f\|_{L^2(D_1)} \le C_s(a_1 + a_2)\|f\|_{\mathcal{B}^2},$$

where H^s is the Sobolev space and s is arbitrary. Hence

$$\|M(1-\phi)f\|_{H^{s-1/2}(\partial D)} \le C(a_1 + a_2)\|f\|_{\mathcal{B}^2}.$$

This implies that operator $M_D(1-\phi)$ is compact and its norm does not exceed $C(a_1+a_2)$. We will take D_1 not very large, so that function β vanishes on D_1. Then $M\phi f$ is the convolution of $1/k$ and $\phi g f$, i.e., $M\phi f = \frac{1}{k} * (\phi g f)$. The latter expression is a pseudo differential operator of order -1 applied to the function $\phi g f$ with a compact support. Thus,

$$\|M\phi f\|_{H^1(D)} \le C\|\phi g f\|_{L^2(D_1)} \le Ca_1\|f\|_{\mathcal{B}^2},$$

and therefore $\|M_D\phi f\|_{H^{1/2}(D)} \le Ca_1\|f\|_{\mathcal{B}^2}$. Hence, operator $M_D\phi$ is compact and its norm does not exceed Ca_1.

□

Theorem 5.2. *Let conditions of Theorem 2.2 hold. Then operator $T_{z,t}$: $\mathcal{B}^2 \to \mathcal{B}^2, 0 \le t \le T$, is compact, continuous in (z,t), and analytic in (x,y) in a complex neighborhood of $\mathbb{R}^2$. The same properties are valid for derivatives of $T_{z,t}$ of any order in t, x, y.*

Remark. $T_{z,t}$ is analytic in x, y in the region $|\Im x|^2 + |\Im y|^2 \le R^2$.

Proof. The operator $T_{z,t}$ can be naturally split into two terms: $T_{z,t} = \mathcal{M} + \mathcal{D}$, where $\mathcal{M}$ involves integration over $\mathbb{C}\backslash D$ and $\mathcal{D}$ involves integration over ∂D. In particular,

$$\mathcal{M}\phi = \frac{1}{\pi}\int_{\mathbb{C}\backslash D} \frac{e^{i\Re(\varsigma \overline{z})}\overline{\phi}(\varsigma)\Pi^o h(\varsigma,\varsigma,t)}{\varsigma - k} d\sigma_\varsigma.$$

The statements of the theorem are valid for operator $\mathcal{M}$ due to (2.9), Lemma 5.1 and Theorem 3.4. Indeed, the compactness and continuity of M in (z,t) is proved in Lemma 5.1. The analyticity in (x,y) follows from the fast decay of h at infinity which is established in Theorem 3.4.

Let us show that the same properties are valid for $\mathcal{D}$. We write $\mathcal{D}$ in the form $D = I_1 I_2$, where operator $I_2 : L^2(\partial D) \to C^\alpha(\partial D)$ is defined by the interior integral in the expression for $\mathcal{D}$ in (2.13), and operator $I_1 : C^\alpha(\partial D) \to \mathcal{H}^s$ is defined by the exterior integral in the same expression. Here $C^\alpha(\partial D)$ is the Holder space and α is an arbitrary number in $(0, 1/2)$. The integral kernel of operator I_2 has a logarithmic singularity at $\varsigma = \varsigma'$, i.e., I_2 is a pseudo differential operator of order -1, and therefore I_2 is a bounded operator from $L^2(\partial D)$ into the Sobolev space $H^1(\partial D)$. Thus it is compact as operator from $C(\partial D)$ to $C^\alpha(\partial D), \alpha \in (0, 1/2)$, due to the Sobolev embedding theorem. Thus the compactness of $\mathcal{D}$ will be proved as soon as we show that I_1 is bounded.

For each $\phi \in C^\alpha(\partial D)$, function $I_1\phi$ is analytic outside of ∂D and vanishes at infinity. Due to the Sokhotski–Plemelj theorem, the limiting values $(I_1\phi)_\pm$ of $(I_1\phi)$ on ∂D from inside and outside of D, respectively, are equal to $\frac{\pm\phi}{2} + P.V.\frac{1}{2\pi i}\int_{\partial D}\frac{\phi(\varsigma)d\varsigma}{\varsigma - \lambda}$. Thus

$$\max_{\partial D} |(I_1\phi)_\pm| \le C\|\phi\|_{C^\alpha(\partial D)}.$$

From the maximum principle for analytic functions, it follows that the same estimate is valid for function $I_1\phi$ on the whole plane. Taking also into account that $I_2\phi$ has the following behavior at infinity $I_2\phi \sim c/k + O(|k|^2)$, we obtain that operator I_1 is bounded. Hence operator $\mathcal{D}$ is compact. Since h decays superexponentially at infinity, the arguments above allow one to obtain not only the compactness of $\mathcal{D}$, but also its smoothness in t, x, y and analyticity in (x, y).

□

5.2. **The invertibility of $I + T_{z,t}$ at large values of z.** We will prove the following lemma.

Lemma 5.3. *The following relation is valid for operator norm of $T_{z,t}^2$ in $\mathcal{B}^2$:*

$$\max_{0\le t\le T} \|T_{z,t}^2\| \to 0, \quad z \in \mathbb{C},\ z \to \infty.$$

Hence the operator $I + T_{z,t}$ is invertible when $z \in \mathbb{C}$, $|z| \gg 1$.

We split operator $T_{z,t}$ into two terms $T_{z,t} = \mathcal{M} + \mathcal{D}$ that correspond to the integration over $\mathcal{C}\backslash D$ and D, respectively, in (2.13). The entries

$M^{ij},\ D^{ij},\ i,j=1,2,$ of the matrix operators $\mathcal{M}$ and $\mathcal{D}$ are

$$M^{11}=M^{22}=0,$$

$$M^{12}\phi=-M^{21}\phi=\frac{1}{\pi}\int_{\mathbb{C}\setminus D}\frac{e^{i\Re(\varsigma\overline{z})-t(\varsigma^2-\overline{\varsigma}^2)/2}\overline{\phi}(\varsigma)h_{12}(\varsigma,\varsigma)}{\varsigma-k}d\sigma_\varsigma,$$

$$D^{11}\phi=D^{22}\phi=$$

$$=\frac{1}{2\pi i}\int_{\partial D}\frac{d\zeta}{\zeta-k}\int_{\partial D}\overline{\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}}h_{11}(\varsigma',\varsigma)e^{\frac{i}{2}(\varsigma-\varsigma')\overline{z}+\frac{t}{2}(\varsigma'^2-\varsigma^2)}\phi(\varsigma')d\zeta',$$

$$D^{12}\phi=-D^{21}\phi=$$

$$=\frac{1}{2\pi i}\int_{\partial D}\frac{d\zeta}{\zeta-k}\int_{\partial D}\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}h_{12}(\varsigma',\varsigma)e^{\frac{i}{2}(\varsigma\overline{z}+\overline{\varsigma'}z)+\frac{t}{2}(\overline{\varsigma'}^2-\varsigma^2)}\overline{\phi}(\varsigma')\overline{d\varsigma'}.$$

We used here the relations $h_{12}=-h_{12},h_{11}=h_{22}$ for the entries of h_0 that were established, for example, in [14, Lemma 4.1].

Lemma 5.1 implies the uniform boundedness of M^{21},M^{12} when $0\le t\le T,\ z\in\mathbb{C}$. Thus Lemma 5.3 will be proved if we show that operator norms of $M^{21}\overline{M^{12}}$ and $D^{ij},i,j=1,2,$ vanish as $z\to\infty$. Let us prove the statement about D^{ij}.

Lemma 5.4. *For each $T>0$, there exists a constant C_T such that*

$$\|D\varphi\|_{\mathcal{B}^2}\le\frac{C_{\alpha,T}}{1+|z|^{1/4}}\|\varphi\|_{\mathcal{B}^2},\quad z\in\mathbb{C},\ 0\le t\le T,$$

if k_0 in the definition of operator $\mathcal{D}$ is chosen to belong to ∂D and equal to $k_0=-iAe^{i\psi}$, where $\psi=\arg z$ and A is the radius of the disk D.

Proof. We will prove the estimate for the component D^{12} of the matrix D. Other components of D can be estimated similarly. Consider the interior integral in D^{12}:

$$R^{12}\phi=\int_{\partial D}\mathrm{Ln}\frac{\overline{\varsigma'}-\overline{\varsigma}}{\overline{\varsigma'}-\overline{k_0}}h_{12}(\varsigma',\varsigma)e^{\frac{i}{2}(\varsigma\overline{z}+\overline{\varsigma'}z)+\frac{t}{2}(\overline{\varsigma'}^2-\varsigma^2)}\overline{\phi}(\varsigma')\overline{d\varsigma'},\tag{5.2}$$

where $\varsigma\in\partial D,\ \phi\in\mathcal{B}^2$. Our goal is to show that

$$\|R^{12}\phi\|_{L^\infty(\partial D)}\le\frac{C_T}{1+|z|^{1/4}}\|\phi\|_{L^2(\partial D)},\ \phi\in\mathcal{B}^2.\tag{5.3}$$

The integrand in (5.2) is anti-holomorphic in $\varsigma'\in D$ with logarithmic branching points at k_0 and ς. If k_0 is strictly inside D, then the integration over ∂D in (5.2) can be replaced by the integration over two sides of the segment $[k_0,\varsigma]$, which are passed in the counter clock-wise direction. The values of the logarithm on these sides differ by the constant 2π. This leads to an alternative form of the operator $\mathcal{D}$:

$$D^{12}\phi=-D^{21}\phi=i\int_{\partial D}\frac{d\zeta}{\zeta-k}\int_{[k_0,\varsigma]}h_{12}(\varsigma',\varsigma)e^{\frac{i}{2}(\varsigma\overline{z}+\overline{\varsigma'}z)+\frac{t}{2}(\overline{\varsigma'}^2-\varsigma^2)}\overline{\phi}(\varsigma')\overline{d\varsigma'}.$$

If $k_0 \in \partial D$, the contour of integration above can be replaced by $\mathrm{arc}[k_0, \varsigma]$. Thus

$$R^{12}\phi = i \int_{\widehat{k_0,\varsigma}} h_{12}(\varsigma', \varsigma) e^{\frac{i}{2}(\varsigma\overline{z}+\overline{\varsigma'}z)+\frac{t}{2}(\overline{\varsigma'}^2-\varsigma^2)} \phi(\varsigma') d\overline{\zeta'}, \quad \varsigma \in \partial D, \ \phi \in L^2(\partial D).$$

Consider the following function (from the exponent in the integrand above): $\Phi = \Re\left[\frac{i}{2}\varsigma\overline{z}\right]$. This function is linear in ς, and for each fixed $z = |z|e^{i\psi}, \psi \in [0, 2\pi)$, it has the unique global maximum on D. The maximum occurs on the boundary at the point $\varsigma_0 = -iAe^{i\psi}$, which depends only on the argument of z. Due to Theorem 2.2, point $k_0 \in \partial D$ can be chosen arbitrarily. We choose $k_0 = \varsigma_0 \in \partial D$, and we get that

$$|R^{12}\phi| \le C \left(\int_{\widehat{\varsigma_0,\varsigma}} \exp 2\left(\Phi(\varsigma) - \Phi(\varsigma')\right) |d\varsigma'| \right)^{1/2} \|\phi\|_{L^2}.$$

Let us estimate the integral above. Let $\varsigma = -iAe^{i(\psi+\varphi)}$, $|\varphi| \le \pi$. For $\varsigma' \in \widehat{\varsigma_0, \varsigma}$, we have

$$\Phi(\varsigma') = A|z|(\cos\varphi')/2, \quad \Phi(\varsigma) = A|z|(\cos\varphi)/2,$$

and the integral is equal to

$$\int_0^\varphi e^{A|z|(\cos\varphi - \cos\varphi')/2} d\varphi' = O(\frac{1}{\sqrt{|z|}}), \ z \to \infty.$$

This justifies (5.3).

□

Let us show now that the following statement holds.

Lemma 5.5.

$$\max_{0 \le t \le T} \|M^{21}\overline{M^{12}}\|_{\mathcal{B}^2} \to 0, \quad z \in \mathbb{C}, \ z \to \infty. \tag{5.4}$$

Proof. Kernels of M^{12}, M^{21} are smooth, see (2.9). From Theorem 3.4, it follows that the kernels and rapidly decaying functions in $\mathbb{C}$. Therefore, Lemma 5.1 implies that operators M^{12}, M^{21} can be approximated in $\mathcal{B}^2$ by operators with function h_{12} replaced by a compactly supported one. Therefore, without loss of the generality, we will assume below that the supports of h_{12}, h_{21} belong to a bounded domain $\mathcal{O}$.

We will use the notation P for the one-dimensional operator defined in (5.1) with the density $g = e^{i\Re(\varsigma\overline{z}) - t(\varsigma - \overline{\varsigma}^2)/2} h_{12}(\varsigma, \varsigma)$. Let $\widehat{M} := (M^{12} - P)\overline{(M^{21} - P)}$. We will prove that

$$\max_{0 \le t \le T} \|\widehat{M}\|_{\mathcal{B}^2} \to 0, \quad z \in \mathbb{C}, \ z \to \infty. \tag{5.5}$$

The other three terms $M^{12}\overline{(M^{21} - P)}$, $(M^{12} - P)\overline{M^{21}}$, and $P\overline{P}$ can be treated in the same way. We have

$$\widehat{M}\varphi = \frac{1}{\pi^2} \int_{\mathcal{O}\setminus D} A(z, \varsigma, \varsigma_2) \overline{h_{21}}(\varsigma_2, \varsigma_2) e^{-i\Re(\varsigma_2\overline{z}) + t(\varsigma_2 - \overline{\varsigma_2}^2)/2} \varphi(\varsigma_2) d\sigma_{\varsigma_2},$$

where $A(z,\varsigma,\varsigma_2)$ is given by the following integral
(5.6)

$$\int_{\mathcal{O}\backslash D} e^{i\Re(\varsigma_1\overline{z})-t(\varsigma_1-\overline{\varsigma_1}^2)/2} h_{12}(\varsigma_1,\varsigma_1)\left(\frac{1}{\varsigma_1-\varsigma}+\frac{\beta(\varsigma)}{\varsigma}\right)\overline{\left(\frac{1}{\varsigma_2-\varsigma_1}+\frac{\beta(\varsigma_1)}{\varsigma_1}\right)}d\sigma_{\varsigma_1}.$$

The Minkovsky inequality in the integral form implies the following two estimates, that are valid when $f\in\mathcal{B}^2$:

$$\|\widehat{M}f\|_{L^2(\mathbb{C}\backslash D)}\le\int_{\mathcal{O}\backslash D}\left[\int_{\mathcal{O}\backslash D}|A(z,\varsigma,\varsigma_2)|^2d\sigma_\varsigma\right]^{1/2}|h_{21}(\varsigma_2,\varsigma_2)f(\varsigma_2)|d\sigma_{\varsigma_2},$$

$$\|\widehat{M}f\|_{L^2(\partial D)}\le\int_{\mathcal{O}\backslash D}\left[\int_{\partial D}|A(z,\varsigma,\varsigma_2)|^2|d\varsigma|\right]^{1/2}|h_{21}(\varsigma_2,\varsigma_2)f(\varsigma_2)|d\sigma_{\varsigma_2}.$$

Since the norm of the operator $L^2(\mathbb{C}\backslash D)\to L^1(\mathbb{C}\backslash D)$ of multiplication by h_{21} can be estimated by a constant, the validity of (5.5) will follow from the estimates above if we show that the following relations hold as $z\to\infty$:

$$\sup_{\varsigma_2\in\mathbb{C}\backslash D}\int_{\mathcal{O}\backslash D}|A(z,\varsigma,\varsigma_2)|^2d\sigma_\varsigma\to 0,\qquad \sup_{\varsigma_2\in\mathbb{C}\backslash D}\int_{\partial D}|A(z,\varsigma,\varsigma_2)|^2|d\varsigma|\to 0.$$

We will prove only the first of them, since the second one can be proved similarly. Note that, uniformly in $\varsigma_2\in\mathcal{O}$,

$$\int_{\mathcal{O}\backslash D}|A(z,\varsigma,\varsigma_2)|^2d\sigma_\varsigma\le$$

$$\int_{\mathcal{O}\backslash D}\left|\int_{\mathcal{O}\backslash D}h_{12}(\varsigma_1,\varsigma_1)\left(\frac{1}{\varsigma_1-\varsigma}+\frac{\beta(\varsigma)}{\varsigma}\right)\overline{\left(\frac{1}{\varsigma_2-\varsigma_1}+\frac{\beta(\varsigma_1)}{\varsigma_1}\right)}d\sigma_{\varsigma_1}\right|^2d\sigma_\varsigma<C.$$

The boundedness follows from the fact that the internal integral is $O(\ln|\varsigma-\varsigma_2|),\varsigma-\varsigma_2\to 0$. Let A^s be given by (5.6) with the extra factor $\eta_s:=\eta(s|\varsigma-\varsigma_1|)\eta(s|\varsigma_1-\varsigma_2|)),s>0$, in the integrand, where $\eta\in C^\infty(\mathbb{R}),\ \eta=1$ outside of a neighborhood of the origin, and η vanishes in a smaller neighborhood of the origin.

For each ε, there exists $s=s_0(\varepsilon)$ such that

$$\int_{\mathcal{O}\backslash D}|A-A^{s_0}|^2\,d\sigma_\varsigma<\varepsilon$$

for all the values of $\varsigma_2\in\mathcal{O},z\in\mathbb{C}$. Denote by R^{s_0} the function A^{s_0} with the potential h_{12} replaced by its L_1-approximation $\widetilde{h}_{12}\in C_0^\infty(\mathbb{C}\backslash D)$. We can choose this approximation in such a way that

$$\int_{\mathcal{O}\backslash D}|A^{s_0}-R^{s_0}|^2\,d\sigma_\varsigma<\varepsilon$$

for all the values of ς_2,z. Now it is enough to show that

$$|R^{s_0}(\varsigma,\varsigma_2,z)|\to 0\quad\text{as}\quad z\to\infty$$

uniformly in $\varsigma, \varsigma_2 \in \mathcal{O}$. The latter can be obtained by integration by parts in $R^{s_0}(\varsigma, \varsigma_2, z)$, defined by integral (5.6) with h_{12} replaced by $(1-\eta_s)\widetilde{h_{12}}(\varsigma_1, \varsigma_1)$ (integrating $e^{i\Re(\varsigma_1 \overline{z})}$ and differentiating the complementary factor). This completes the proof of (5.4).

□

Acknowledgments. 1. The work of E. Lakshtanov was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications and the Portuguese Foundation for Science and Technology ("FCT–Fundção para a Ciência e a Tecnologia") within project UID/MAT/0416/2019.

2. The work of B. Vainberg was partially supported by the NSF grant DMS-1714402 and the Simons Foundation grant 527180.

References

[1] R. Beals and R.R. Coifman, *The spectral problem for the Davey- Stewartson and Ishimori hierarchies,* In Nonlinear evolution equations: Integrability and spectral methods, Manchester University Press, 1988, pp 15-23.

[2] R. M. Brown, G. A. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions,* Comm. in partial diff. equations, 22(5-6), (1997), 1009-1027.

[3] R. M. Brown, *Estimates for the scattering map associated with a two-dimensional first-order system*, J. Nonlinear Sci., 11(6), (2001), 459–471.

[4] C. Calcaterra, A. Boldt, A. *Approximating with Gaussians*, (2008), arXiv:0805.3795.

[5] C. Calcaterra, *Linear Combinations of Gaussians with a single variance are dense in L_2*, Proceedings of the World Congress on Engineering, Vol II, 2008.

[6] H. Cornille, *Solutions of the generalized nonlinear Schrodinger equation in two spatial dimensions.* J. Math. Phys., 20 (1979), 199-209.

[7] A. S. Fokas, L-Y. Sung, *On the solvability of the N-wave, Davey-Stewartson and Kadomtsev-Petviashvili equations*, Inverse problems 8.5 (1992), 673.

[8] J. M. Ghidaglia, J. C. Saut, *On the initial value problem for the Davey-Stewartson systems.* Nonlinearity, 3(2), (1990), 475.

[9] N. Hayashi, J. C. Saut, *Global existence of small solutions to the Davey-Stewartson and the Ishimori systems*, Differential and Integral Equations, 8(7), (1995), 1657-1675.

[10] M. Korpusov, S. Mikhailenko, *Instantaneous blow-up of classical solutions to the Cauchy problem for the Khokhlov–Zabolotskaya equation*, Computational Mathematics and Mathematical Physics 57(7), (2017), 1167-1172.

[11] C. Klein, J. C. Saut, *A numerical approach to blow-up issues for Davey-Stewartson II systems*, Communications on Pure and Applied Analysis, 14(4), (2015).

[12] E. Lakshtanov, R. Novikov, B. Vainberg, *A global Riemann-Hilbert problem for two-dimensional inverse scattering at fixed energy,* Rend. Inst. Mat. Univ. Trieste, 48, (2016), (Special issue in honor of G. Alessandrini's 60th birthday), 1-26.

[13] E. Lakshtanov, B. Vainberg, *On reconstruction of complex-valued once differentiable conductivities,* J. Spectr. Theory 6, 4 (Special issue in memory of Yuri Safarov), (2016), 881-902.

[14] E. Lakshtanov, B. Vainberg, *Solution of the initial value problem for the focusing Davey-Stewartson II system*, Differential equations, mathematical physics, and applications. Selim Grigorievich Krein Centennial, Contemp. Math., Amer. Math. Soc., Providence, RI, Vol. 734 (2019).

[15] M. Music, *The nonlinear Fourier transform for two-dimensional subcritical potentials,* Inverse Problems and Imaging, 8(4), (2014), 1151-1167.
[16] R. G. Novikov, *The Inverse Scattering Problem on a Fixed Energy Level for the Two-Dimensional Schrödinger Operator,* J. of Funct. Analysis, 103, (1992), 409-463.
[17] T. Ozawa, Exact blow-up solutions to the Cauchy problem for the Davey–Stewartson systems, Proc. Roy. Soc. London Ser. A 436 (1992), no. 1897, 345–349.
[18] P. Perry, *Global Well-Posedness and Long-time Asymptotics for the Defocussing Davey-Stewartson II Equation in* $H^{1,1}(\mathbb{C})$*. With an appendix by Michael Christ.,* J. Spectr. Theory 6 (2016), no. 3, 429-481.
[19] P. Perry, *Soliton Solutions and their (In) stability for the Focusing Davey-Stewartson II Equation*, (2016), arXiv:1611.00215.
[20] L. Y. Sung, *An Inverse Scattering Transform for the Davey-Stewartson-II Equations, I*, Journal of mathematical analysis and applications, 183(1), (1994), 121-154.
[21] L. Y. Sung, *An Inverse Scattering Transform for the Davey-Stewartson II Equations II,* Journal of Mathematical Analysis and Applications, 183(2), (1994), 289-325.
[22] L. Y. Sung, *An inverse scattering transform for the Davey-Stewartson II equations, III,* Journal of mathematical analysis and applications, 183(3), (1994), 477-494.
[23] Michael E. Taylor, Lectures. http://www.unc.edu/math/Faculty/met/fred.pdf
[24] M. Zaidenberg, S. Krein, P. Kuchment, A. Pankov, *Banach bundles and linear operators*, Russian Mathematical Surveys, 30(5), (1975), 115-175.

(E. Lakshtanov) Department of Mathematics, Aveiro University, Aveiro 3810, Portugal
Email address: lakshtanov@ua.pt

(B. Vainberg) Department of Mathematics and Statistics, University of North Carolina, Charlotte, NC 28223, USA
Email address: brvainbe@uncc.edu