SURVEY ON GEOMETRIC LANGLANDS AND NON-ABELIAN HODGE THEORY IN CHARACTERISTIC p

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Abstract. In this report, we give a survey on new results on geometric Langlands correspondence and non-abelian Hodge theory in characteristic p.

1. Introduction

In this survey article we discuss new results on geometric Langlands correspondence in characteristic p and non-abelian Hodge theory for smooth projective curves in characteristic p. We also discuss applications of those results to quantization of Hitchin's integrable system. The main references are [BB, BT, CZ1, CZ2].

The article is organized as follows. In §2 we recall some facts about reductive groups, connections on torsors, D-modules in characteristic p, and Azumaya property of sheaf of crystalline differential operators in characteristic p. In §3 we discuss the construction of Hitchin maps, p-Hitchin maps, and abelian duality for Hitchin maps. In §4 we discuss results on non-abelian Hodge theory for algebraic curves in characteristic p. In §5 we discuss geometric Langlands correspondence in characteristic p. In §6 we discuss applications to quantization of Hitchin's integrable system.

2. Preliminaries

2.1. **Notations related to reductive groups.** Let G be a reductive algebraic group over k of rank l. We denote by $\check{G}$ its Langlands dual group over k. We denote by $\mathfrak{g}$ (resp. by $\check{\mathfrak{g}}$) the Lie algebra of G (resp. $\check{G}$). Let T denote the abstract Cartan of G with its Lie algebra $\mathfrak{t}$. The counterparts on the Langlands dual side are denoted by $\check{T}, \check{\mathfrak{t}}$. We denote by W the abstract Weyl group of G, which acts on T and $\check{T}$.

From now on, we assume that the char $k = p$ is zero or $p \nmid |\mathrm{W}|$. We fix a W-invariant non-degenerate bilinear form $(\, , \,) : \mathfrak{t} \times \mathfrak{t} \to k$ and identify $\mathfrak{t}$ with $\check{\mathfrak{t}}$ using $(\, , \,)$. This invariant form also determines a unique G-invariant non-degenerate bilinear form $\mathfrak{g} \times \mathfrak{g} \to k$, still denoted by $(\, , \,)$. Let $\mathfrak{g} \simeq \mathfrak{g}^*$ be the resulting G-equivariant isomorphism.

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2.2. **Notations related to stacks and Frobenius.** We fix a smooth projective curve C of genus at least two[1] over an algebraically closed field k of characteristic p. Let $\omega = \omega_C$ denote the canonical line bundle of C. Let S be a Noetherian scheme and $\mathscr{X} \to S$ be an algebraic stack over S. If $p > 0$, we denote by $Fr_S : S \to S$ be the absolute Frobenius map of S. We have the following commutative diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "F_{\mathscr{X}/S}"] \arrow[dr] & \mathscr{X}^{(S)} \arrow[r, "\pi_{\mathscr{X}/S}"] \arrow[d] & \mathscr{X} \arrow[d] \\ & S \arrow[r, "Fr_S"] & S \end{tikzcd}$$

where the square is Cartesian. We call $\mathscr{X}^{(S)}$ the Frobenius twist of $\mathscr{X}$ along S, and $F_{\mathscr{X}/S} : \mathscr{X} \to \mathscr{X}^{(S)}$ the relative Frobenius morphism. If the base scheme S is clear, $\mathscr{X}^{(S)}$ is also denoted by $\mathscr{X}'$ for simplicity and $F_{\mathscr{X}/S}$ is denoted by $F_{\mathscr{X}}$ or F.

Let $\mathscr{X} \to S$ be a smooth algebraic stack over S. We write $T(\mathscr{X}/S)$ for relative tangent stack on $\mathscr{X}$ and $T^*(\mathscr{X}/S)$ relative cotangent stack on $\mathscr{X}$. We denote by $T_{\mathscr{X}/S}$ and $\Omega^1_{\mathscr{X}/S}$ the sheaves of sections of $T(\mathscr{X}/S)$ and $T^*(\mathscr{X}/S)$ respectively, and $\mathcal{O}_{\mathscr{X}}$ the structure sheaf of $\mathscr{X}$.

2.3. D**-modules.** Let X be a smooth scheme over k. We denote by $\mathcal{D}_X$ the sheaf of crystalline differential operators. Recall $\mathcal{D}_X$ is the sheaf of algebra generated by $\mathcal{O}_X$ and T_X subject to the relations $v_1v_2 - v_2v_1 = [v_1, v_2]$ for $v_1, v_2 \in T_X$, and $vf - fv = v(f)$ for $v \in T_X$ and $f \in \mathcal{O}_X$. By definition, a D-module on X is a sheaf of $\mathcal{D}_X$-module which is quasi-coherent as $\mathcal{O}_X$-module. It follows from the definition of $\mathcal{D}_X$ that a D-module is just a quasi-coherent $\mathcal{O}_X$-module N equipped with a flat connection, that is, a k-linear map

$$\nabla : T_X \to \mathrm{End}_k(N)$$

satisfying $\nabla(f \cdot v) = v \otimes df + f\nabla(v)$ for $f \in \mathcal{O}_X, v \in T_X$ and $\nabla([v, w]) = [\nabla(v), \nabla(w)]$ for $v, w \in T_X$.

2.4. **Azumaya property of $\mathcal{D}_X$ in positive characteristic and p-curvatures.** If $\mathrm{char}k = 0$, $\mathcal{D}_X$ is simple, that is, it has no proper two-sided ideals. If $\mathrm{char}k > 0$, the center $Z(\mathcal{D}_X)$ of $\mathcal{D}_X$ (resp. the centralizer $Z_{\mathcal{O}_X}(\mathcal{D}_X)$ of $\mathcal{O}_X$ in $\mathcal{D}_X$) is freely generated by elements of the form f^p (resp. f), $f \in \mathcal{O}_X$ and $v^p - v^{[p]}$, $v \in T_X$, where $v^{[p]}$ is the p-th restricted power of the vector field v. The center $Z(\mathcal{D}_X)$ (resp. the centralizer $Z_{\mathcal{O}_X}(\mathcal{D}_X)$) is canonically isomorphic to the structure sheaf $\mathcal{O}_{T^*X'}$ (resp. $\mathcal{O}_{X \times_{X'} T^*X'}$). Thus $\mathcal{D}_X$ can be regraded as a quasi-coherent sheaf of algebra on T^*X'. We have the following fundamental observation: recall that an *Azumaya algebra* on a scheme is a locally free sheaf A of $\mathcal{O}_X$-algebra, such that the fiber of A at every geometric point is isomorphic to a matrix algebra.

Proposition 2.1. [BMR] *$\mathcal{D}_X$ is an Azumaya algebra of rank $p^{2\dim X}$ on T^*X'.*

[1]This assumption should not be essential. We impose it to avoid the DG structure on moduli spaces.

Since $Z_{\mathcal{O}_X}(\mathcal{D}_X) \simeq \mathcal{O}_{X\times_{X'}T^*X'}$, any D-module N carries an action of $\mathcal{O}_{X\times_{X'}T^*X'}$. Such an action is the same as a section $s \in \Gamma(X, Fr_X^*\Omega_X^1 \otimes \mathrm{End}_{\mathcal{O}_X}(N))$. As we noted before N can be considered as a quasi-coherent sheaf on X with a flat connection $\nabla : T_X \to \mathrm{End}_k(N)$ and the map $\Psi_\nabla : T_X \to (Fr_X)_*\mathrm{End}_{\mathcal{O}_X}(N)$ corresponding to s, via adjunction, is given by

$$\Psi_\nabla : T_X \to (Fr_X)_*\mathrm{End}_{\mathcal{O}_X}(N), v \to \nabla(v)^p - \nabla(v^{[p]}).$$

The map Ψ_∇ is known as the *p-curvature* of the connection.

2.5. **Connections on G-torsors.** Let E be a G-torsor on a smooth scheme X over k. We have the Atiyah sequence (or Atiyah algebroid)

$$0 \to \mathrm{ad}(E) \to \widetilde{T}_E \xrightarrow{\sigma} T_X \to 0$$

associated to E. Here $\mathrm{ad}(E)$ is the adjoint bundle for E and $\widetilde{T}_E$ is the Lie algebroid of infinitesimal symmetries of E consisting of pairs $(v, \tilde{v})$, where $v \in T_X$ is a vector field on X and $\tilde{v} \in T_E$ is a G-invariant vector field on E that lifts v.

A connection on E is a section $\nabla : T_X \to \widetilde{T}_E$ of σ. A connection ∇ is called flat, or integrable, if ∇ commutes with the natural Lie brackets on T_X and $\widetilde{T}_E$. The connections on E form a $\mathrm{Hom}(T_X, \mathrm{ad}(E)) = \Gamma(X, \mathrm{ad}(E) \otimes \Omega_X^1)$-torsor. Note that when $X = C$ is a smooth curve, all connections on a G-torsor is flat.

Example 2.2. Assume $G = GL_n(k)$. There is an equivalence between the category of G-torsors and rank n bundles on X given by $E \to N := E \times^G k^n$. Then under the equivalence above a connection ∇ on E corresponds to k-linear map

$$\nabla_N : T_X \to \mathrm{End}_k(N)$$

satisfying $\nabla_N(f \cdot v) = v \otimes df + f\nabla(v)$ for $f \in \mathcal{O}_X, v \in T_X$. If the connection is flat then we have $\nabla_N([v, w]) = [\nabla_N(v), \nabla_N(w)]$ for $v, w \in T_X$, hence by Example 2.3, a D-module structure on N. It is not hard to see that the assignment $(E, \nabla) \to (N, \nabla_N)$ defines an equivalence between the category of flat connections on GL_n-torsors and the category D-modules on X which are locally free (as $\mathcal{O}_X$-module) of rank n.

Example 2.3. Let ∇ be a connection on a G-torsor E. Choose a local trivialization of E, we can write the connection ∇ as a differential operator

$$d + A, \ A \in \mathfrak{g} \otimes \Omega_X^1.$$

Here A is called the connection matrix for ∇ with respect to the trivialization.

2.6. **p-curvatures of connections on G-torsors.** Assume $\mathrm{char}\, k = p > 0$. The p-curvature of a connection ∇ on G-bundle E is the following map

$$T_X \to \mathrm{ad}(E), v \to \Psi_\nabla(v) := \nabla(v)^p - \nabla(v^{[p]})$$

here $\cdot^p$ and $\cdot^{[p]}$ are the p-th restrict Lie algebra structures on $\widetilde{T}_E$ and T_X respectively. According to [Bo, K], the map Ψ_∇ is p-linear, that is $\Psi_\nabla(fv) = f^p\Psi_\nabla(v)$. Hence one can view the p-curvature as a $\mathcal{O}_X$-linear map

$$\Psi_\nabla : T_X \to (Fr_X)_*\mathrm{ad}(E).$$

By adjunction, the p-curvature gives rise to a section $s \in \Gamma(X, Fr_X^*\Omega_X^1 \otimes \mathrm{ad}(E))$.

Example 2.4. Assume $G = GL_n$ and $X = \mathrm{Spec}(k[z])$ is the affine line. Let $\nabla = d + A$ be a connection on the trivial G-bundle, where $A = A_z \otimes dz \in M_n(k[z]) \otimes dz$ is the connection matrix. Then we have

$$\Psi_\nabla(\partial_z) = (\partial_z + A_z)^p = (\partial_z + A_z)^{p-1} A_z \in M_n(k[z]).$$

The results in §2 can be generalized to the setting of smooth algebraic stacks, see [BB, BD, CZ2].

3. Hitchin map and p-Hitchin map

In this subsection, we recall the definition of Hitchin map and p-Hitchin map following [N, CZ1].

3.1. **Hitchin map.** Let $k[\mathfrak{g}]$ and $k[\mathfrak{t}]$ be the algebra of polynomial function on $\mathfrak{g}$ and $\mathfrak{t}$. By Chevalley's theorem, we have an isomorphism $k[\mathfrak{g}]^G \simeq k[\mathfrak{t}]^{\mathrm{W}}$. Moreover, $k[\mathfrak{t}]^{\mathrm{W}}$ is isomorphic to a polynomial ring of l variables $u_1, \dots, u_l$ and each u_i is homogeneous in degree e_i. Let $\mathfrak{c} = \mathrm{Spec}(k[\mathfrak{t}]^W)$. Let

$$\chi : \mathfrak{g} \to \mathfrak{c}$$

be the map induced by $k[\mathfrak{c}] \simeq k[\mathfrak{g}]^G \hookrightarrow k[\mathfrak{g}]$. This is $G \times \mathbb{G}_m$-equivariant map where G acts trivially on $\mathfrak{c}$, and $\mathbb{G}_m$ acts on $\mathfrak{c}$ through the gradings on $k[\mathfrak{t}]^{\mathrm{W}}$. Let $\mathcal{L}$ be an invertible sheaf on C and $\mathcal{L}^\times$ be the corresponding $\mathbb{G}_m$-torsor. Let $\mathfrak{g}_\mathcal{L} = \mathfrak{g} \times^{\mathbb{G}_m} \mathcal{L}^\times$ and $\mathfrak{c}_\mathcal{L} = \mathfrak{c} \times^{\mathbb{G}_m} \mathcal{L}^\times$ be the $\mathbb{G}_m$-twist of $\mathfrak{g}$ and $\mathfrak{c}$ with respect to the natural $\mathbb{G}_m$-action.

Let $\mathrm{Higgs}_{G,\mathcal{L}} = \mathrm{Sect}(C, [\mathfrak{g}_\mathcal{L}/G])$ be the stack of section of $[\mathfrak{g}_\mathcal{L}/G]$ over C, i.e., for each k-scheme S the groupoid $\mathrm{Higgs}_{G,\mathcal{L}}(S)$ consists of maps over C:

$$h_{E,\phi} : C \times S \to [\mathfrak{g}_\mathcal{L}/G].$$

Equivalently, $\mathrm{Higgs}_{G,\mathcal{L}}(S)$ consists of a pair (E, ϕ) (called a Higgs bundle), where E is a G-torsor over $C \times S$ and ϕ is an element in $\Gamma(C \times S, \mathrm{ad}(E) \otimes \mathcal{L})$.

Let $B_{G,\mathcal{L}} = \mathrm{Sect}(C, \mathfrak{c}_\mathcal{L})$ be the scheme of sections of $\mathfrak{c}_\mathcal{L}$ over C, i.e., for each k-scheme S, $B_{G,\mathcal{L}}(S)$ is the set of sections over C

$$b : C \times S \to \mathfrak{c}_\mathcal{L}.$$

This is called the Hitchin base of G.

The natural G-invariant projection $\chi : \mathfrak{g} \to \mathfrak{c}$ induces a map

$$[\chi_\mathcal{L}] : [\mathfrak{g}_\mathcal{L}/G] \to \mathfrak{c}_\mathcal{L},$$

which in turn induces a natural map

$$h_{G,\mathcal{L}} : \mathrm{Higgs}_{G,\mathcal{L}} = \mathrm{Sect}(C, [\mathfrak{g}_\mathcal{L}/G]) \to \mathrm{Sect}(C, \mathfrak{c}_\mathcal{L}) = B_{G,\mathcal{L}}.$$

We call $h_{G,\mathcal{L}} : \mathrm{Higgs}_{G,\mathcal{L}} \to B_{G,\mathcal{L}}$ the Hitchin map associated to $\mathcal{L}$. We denote by $\mathrm{Higgs}_{G,\mathcal{L},b} := h_{G,\mathcal{L}}^{-1}(b)$ the fiber of the Hitchin map over b.

If the group G is clear from the content, we simply write $\mathrm{Higgs}_{\mathcal{L}}$, $B_{\mathcal{L}}$, and $h_{\mathcal{L}}$ for $\mathrm{Higgs}_{G,\mathcal{L}}$, $B_{G,\mathcal{L}}$, and $h_{G,\mathcal{L}}$. We are mostly interested in the case $\mathcal{L} = \omega$. For simplicity, from now on we denote $B = B_{G,\omega}$, $\mathrm{Higgs} = \mathrm{Higgs}_{G,\omega}$ and $h = h_{G,\omega}$, etc.

For a choice of a square root $\mathcal{L}^{1/2}$, there is a section $\epsilon_{\mathcal{L}^{1/2}} : B_{\mathcal{L}} \to \mathrm{Higgs}_{\mathcal{L}}$ of $h_{\mathcal{L}} : \mathrm{Higgs}_{\mathcal{L}} \to B_{\mathcal{L}}$, induced by the Kostant section $kos : \mathfrak{c} \to \mathfrak{g}$. Sometimes, we also call $\epsilon_{\mathcal{L}^{1/2}}$ the Kostant section of the Hitchin fibration.

Example 3.1. Consider the case $G = GL_n$. The stack of Higgs bundles $\mathrm{Higgs}_{\mathcal{L}}$ consists of pair (E, ϕ), where E is a rank n bundle on C and ϕ is a $\mathcal{O}_C$-linear map $\phi : E \to E \otimes \mathcal{L}$. The Hitchin base is equal to $B_{\mathcal{L}} = \oplus_{i=1}^{n} \Gamma(C, \mathcal{L}^i)$ and the Hitchin map is given by taking the characteristic polynomial of ϕ

$$h_{\mathcal{L}} : \mathrm{Higgs}_{\mathcal{L}} \to B_{\mathcal{L}},\ (E, \phi) \to (b_1(\phi), ..., b_n(\phi)),$$

where $b_i(\phi) := \mathrm{Tr}(\wedge^i \phi) \in \Gamma(C, \mathcal{L}^i)$.

3.2. **The universal centralizer group schemes.** Consider the group scheme I over $\mathfrak{g}$ consisting of pairs

$$I = \{(g, x) \in G \times \mathfrak{g} \mid \mathrm{Ad}_g(x) = x\}.$$

We define $J = kos^* I$, where $kos : \mathfrak{c} \to \mathfrak{g}$ is the Kostant section. This is called the universal centralizer group scheme of $\mathfrak{g}$ (see Proposition 3.2).

The following proposition is proved in [N] (see also [DG]).

Proposition 3.2. [N, Lemma 2.1.1] *There is a unique morphism of group schemes $a : \chi^* J \to I \subset G \times \mathfrak{g}$, which extends the canonical isomorphism $\chi^* J|_{\mathfrak{g}^{reg}} \simeq I|_{\mathfrak{g}^{reg}}$.*

All the constructions above can be twisted. Namely, there are $\mathbb{G}_m$-actions on I, J. Moreover, the $\mathbb{G}_m$-action on I can be extended to a $G \times \mathbb{G}_m$-action given by $(h, t) \cdot (x, g) = (t \cdot hxh^{-1}, hgh^{-1})$. The natural morphisms $J \to \mathfrak{c}$ and $I \to \mathfrak{g}$ are $\mathbb{G}_m$-equivariant, and therefore we can twist everything by the $\mathbb{G}_m$-torsor $\mathcal{L}^\times$ to get $J_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$, $I_{\mathcal{L}} \to \mathfrak{g}_{\mathcal{L}}$ where $J_{\mathcal{L}} = J \times^{\mathbb{G}_m} \mathcal{L}^\times$ and $I_{\mathcal{L}} = I \times^{\mathbb{G}_m} \mathcal{L}^\times$. The group scheme $I_{\mathcal{L}}$ over $\mathfrak{g}_{\mathcal{L}}$ is equivariant under the G-action, hence it descends to a group scheme $[I_{\mathcal{L}}]$ over $[\mathfrak{g}_{\mathcal{L}}/G]$.

3.3. **Symmetries of Hitchin fibration.** Let $b : S \to B_{\mathcal{L}}$ be an S-point of $B_{\mathcal{L}}$, corresponding to a map $b : C \times S \to \mathfrak{c}_{\mathcal{L}}$. Pulling back $J_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$ along this map, we obtain a smooth group scheme $J_b = b^* J$ over $C \times S$.

Let $\mathscr{P}_{G,b}$ be the Picard category of J_b-torsors over $C \times S$. The assignment $b \to \mathscr{P}_b$ defines a Picard stack over B, denoted by $\mathscr{P}_{G,\mathcal{L}}$. Let us fix $b \in B_{\mathcal{L}}(S)$, and let $(E, \phi) \in \mathrm{Higgs}_{G,\mathcal{L},b}$ corresponding to the map $h_{E,\phi} : C \times S \to [\mathfrak{g}_{\mathcal{L}}/G]$. Observe that the morphism $\chi^* J \to I$ in Proposition 3.2 induces $[\chi_{\mathcal{L}}]^* J_{\mathcal{L}} \to [I_{\mathcal{L}}]$ of group schemes over $[\mathfrak{g}_{\mathcal{L}}/G]$. Pulling back to $C \times S$ using $h_{E,\phi}$, we get a map

$$a_{E,\phi} : J_b \to h_{E,\phi}^*[I] = \mathrm{Aut}(E, \phi) \subset \mathrm{Aut}(E), \tag{3.1}$$

which allows us to twist $(E, \phi) \in \mathrm{Higgs}_{G,\mathcal{L},b}$ by a J_b-torsor. This construction defines an action of $\mathscr{P}_{G,\mathcal{L}}$ on $\mathrm{Higgs}_{G,\mathcal{L}}$ over $B_{\mathcal{L}}$.

Let $\mathrm{Higgs}_{G,\mathcal{L}}^{reg}$ be the open stack of $\mathrm{Higgs}_{G,\mathcal{L}}$ consisting of $(E,\phi):C\to[\mathfrak{g}_{\mathcal{L}}/G]$ that factors through $C\to[(\mathfrak{g}^{reg})_{\mathcal{L}}/G]$. If $(E,\phi)\in\mathrm{Higgs}_{G,\mathcal{L}}^{reg}$, then $a_{E,\phi}$ above is an isomorphism. The Kostant section $\epsilon_{\mathcal{L}^{1/2}}:B_{\mathcal{L}}\to\mathrm{Higgs}_{G,\mathcal{L}}$ factors through $\epsilon_{\mathcal{L}^{1/2}}:B_{\mathcal{L}}\to\mathrm{Higgs}_{G,\mathcal{L}}^{reg}$. Following [N, §4], we define $B_{\mathcal{L}}^0$ as the open sub-scheme of $B_{\mathcal{L}}$ consisting of $b\in B_{\mathcal{L}}(k)$ such that the image of the map $b:C\to\mathfrak{c}_{\mathcal{L}}$ intersects the discriminant divisor transversally. The following proposition can be extracted from [DG, DP, N]:

Proposition 3.3. *We have the following:*

1. *The stack* $\mathrm{Higgs}_{G,\mathcal{L}}^{reg}$ *is a* $\mathscr{P}_{G,\mathcal{L}}$*-torsor, which can be trivialized by a choice of a Kostant section* $\epsilon_{\mathcal{L}^{1/2}}$*.*
2. *One has* $\mathrm{Higgs}_{G,\mathcal{L}}^{reg}\times_{B_{\mathcal{L}}}B_{\mathcal{L}}^0=\mathrm{Higgs}_{G,\mathcal{L}}\times_{B_{\mathcal{L}}}B_{\mathcal{L}}^0$ *and the Hitchin map* $h_{\mathcal{L}}:\mathrm{Higgs}_{\mathcal{L}}\to B_{\mathcal{L}}$ *is smooth over* $B_{\mathcal{L}}^0$*.*

Example 3.4. We give examples of fibers $\mathrm{Higgs}_{\mathcal{L},b}$ and $\mathscr{P}_{\mathcal{L},b}$ in the case $G=GL_n$. We keep the notation in Example 3.1. We first recall the construction of spectral curves. Let $\mathrm{Tot}(\mathcal{L}^i)$ be the total space of the line bundle $\mathcal{L}^i$. Consider the map

$$i_b:B_{\mathcal{L}}\times\mathrm{Tot}(\mathcal{L})\to\mathrm{Tot}(\mathcal{L}^n),\ (b_1,...,b_n,v)\to\sum_{i=1}^n(-1)^ib_i\otimes v^{n-1}.$$

Recall $B_{\mathcal{L}}=\oplus_{i=1}^n\Gamma(C,\mathcal{L}^i)$. The universal spectral curves, denoted by $\tilde{S}$, is the scheme-theoretic pre-image of the zero section in $\mathrm{Tot}(\mathcal{L}^n)$ under the map i_b. It is a closed subscheme S of $B_{\mathcal{L}}\times\mathrm{Tot}(\mathcal{L})$ and we have a natural projection map

$$S\to B_{\mathcal{L}}\times C$$

with is finite flat of degree n. For any $b\in B_{\mathcal{L}}$, we let S_b be the fiber over b under the projection map $S\to B_{\mathcal{L}}$. We call S_b the spectral curve associated to b. We have $b\in B_{\mathcal{L}}^0$ if and only if S_b is smooth. We have $\mathscr{P}_{\mathcal{L},b}\simeq\mathrm{Pic}(S_b)$ the Picard stack of line bundles on the spectral curve. For $b\in B_{\mathcal{L}}$ such that the corresponding spectral curve S_b is integral, we have $\mathrm{Higgs}_{\mathcal{L},b}\simeq\overline{\mathrm{Pic}}(S_b)$, the stack classifying torsion-free coherent sheaves that are generically of rank 1, and $\mathscr{P}_{\mathcal{L},b}\simeq\mathrm{Pic}(S_b)$ acts on $\mathrm{Higgs}_{\mathcal{L},b}$ by tensoring.

3.4. **Abelian duality.** Let $\mathscr{P}_{G,\mathcal{L}}$ and $\mathscr{P}_{\check{G},\mathcal{L}}$ be the Picard stacks in §3.3 for G and $\check{G}$ respectively. Let $(\mathscr{P}_{G,\mathcal{L}})^\vee:=\mathrm{Hom}(\mathscr{P}_{G,\mathcal{L}},B\mathbb{G}_m)$ be the dual Picard stack (see [CZ2, Appendix A] for more details about duality of Picard stacks). The W-invariant $(,)$ form on $\mathfrak{t}$ (fixed in the beginning of §2) induces isomorphisms $B_{G,\mathcal{L}}\simeq B_{\check{G},\mathcal{L}}$ and the corresponding open subset $B_{G,\mathcal{L}}^0\simeq B_{\check{G},\mathcal{L}}^0$. We have the following abelian duality theorem.

Theorem 3.5. [CZ2, DP] *Assume that the characteristic* $\mathrm{char}\,k=p$ *is zero or* $p>0$ *with* $p\nmid|\mathrm{W}|$*. There is a canonical isomorphism of Picard stack*

$$(\mathscr{P}_{G,\mathcal{L}})^\vee|_{B_{G,\mathcal{L}}^0}\simeq\mathscr{P}_{\check{G},\mathcal{L}}|_{B_{\check{G},\mathcal{L}}^0}.$$

Example 3.6. In the case of $G=GL_n$, we have $\mathscr{P}_{\mathcal{L}}|_{B_{\mathcal{L}}^0}\simeq\mathrm{Pic}(S/B_{\mathcal{L}}^0)$ and Theorem 3.5 amounts to the well-known autoduality $\mathrm{Pic}(S_b)^\vee\simeq\mathrm{Pic}(S_b)$ of Picard stack of line bundles on smooth spectral curves S_b.

Remark 3.7. In the case $p=0$, Theorem 3.5 is the main theorem of [DP] (for $G=SL_n$, see [HT]). As mentioned by the authors, transcendental arguments are used in [DP] in an essential way, and therefore cannot be applied directly to the case when $p>0$. The argument in [CZ2] works for any algebraically closed field k of characteristic zero or $p>0$ with $p\nmid |\mathrm{W}|$.

Remark 3.8. In [A], the author extends the autoduality in Example 3.6, hence Theorem 3.5 for GL_n, to integral spectral curves S_b.

3.5. ***p*-Hitchin map.** In this section we assume $p>0$. Let LocSys_G be the stack of G-local system on C, that is, for every scheme S over k, $\mathrm{LocSys}_G(S)$ is the groupoid of all G-torsors E on $C\times S$ together with a connection $\nabla: T_{C\times S/S}\to\widetilde{T}_E$. For any $(E,\nabla)\in\mathrm{LocSys}_G$ let

$$\Psi_\nabla: T_C\to Fr_*\mathrm{ad}(E),\quad v\to\nabla(v)^p-\nabla(v^p)$$

be its p-curvature (see §2.6). We regard Ψ_∇ as a section $\Psi_\nabla\in\Gamma(C,\mathrm{ad}(E)\otimes\omega^p)$ and call such a pair an F-Higgs field. The assignment $(E,\nabla)\to(E,\Psi_\nabla)$ defines a map $\Psi_G:\mathrm{LocSys}_G\to\mathrm{Higgs}_{G,\omega^p}$. Combining this map with h_{ω^p}, we get a morphism from LocSys_G to B_{ω^p}:

$$\tilde{h}_p:\mathrm{LocSys}_G\to B_{\omega^p}.$$

Observe that the pullback along $F_C: C\to C'$ induces a natural map $F^p: B'\to B_{\omega^p}$, where the superscript denotes the Frobenius twist.

Theorem 3.9. [CZ2, Theorem 3.1 and Proposition 3.5] *We have the following:*

(1) The p-curvature morphism $\tilde{h}_p:\mathrm{LocSys}_G\to B_{\omega^p}$ *factors through a unique morphism*

$$h_p:\mathrm{LocSys}_G\to B'.$$

We called this map the p-Hitchin map.

(2) There is a canonical action $\mathscr{P}'\times_{B'}\mathrm{LocSys}_G\to\mathrm{LocSys}_G$ *of the Picard stack* $\mathscr{P}'$ *on* LocSys_G.

Remark 3.10. In the case of $G=GL_n$, part (1) of Theorem 3.9 is the main theorem of [LP].

Example 3.11. Consider the case $G=GL_1$. Let $(\mathcal{L},\nabla)\in\mathrm{LocSys}_{GL_1}$, here $\mathcal{L}$ is a line bundle on C and $\nabla:\mathcal{L}\to\mathcal{L}\otimes\omega$ is a flat connection. The p-curvature of ∇ is a section $\Psi_\nabla\in\Gamma(C,\omega_C^p)$. By [K], the section is Ψ_∇ is flat with respect to the canonical connection coming from the Frobenius pullback $\omega_C^p\simeq F_C^*\omega_{C'}$, hence by Cartier descent, we have $\Psi_\nabla=F_C^*(\alpha)$ for a 1-form $\alpha\in\Gamma(C',\omega_{C'})$. The p-Hitchin map is given by

$$h_p:\mathrm{LocSys}_{GL_1}\to B'=\Gamma(C',\omega_{C'}),\ (\mathcal{L},\nabla)\to\alpha.$$

The action of $(\mathcal{L}')\in\mathscr{P}'_{GL_1,b'}\simeq\mathrm{Pic}(C')$ on $\mathrm{LocSys}_{GL_1,b'}$ is given by $(\mathcal{L},\nabla)\to(\mathcal{L}\otimes F_{C'}^*\mathcal{L}',\nabla\otimes\nabla_{can})$, where ∇_{can} is the canonical connection on $F_{C'}^*\mathcal{L}'$ coming from the Frobenius pullback.

4. Non-abelian Hodge theory in characteristic p.

Let G be a reductive group over an algebraically closed field k of positive characteristic p. Let C be a smooth projective curve over k. In a joint work with X. Zhu [CZ1], we use Hitchin map and p-Hitchin map to give a description of the moduli space of flat G-bundles in terms of the moduli space of G-Higgs bundles over the Frobenius twist C' of C. This extends the work of [BB] and [G] in the case of GL_n and can be regarded as the non-abelian Hodge theory for algebraic curves.

Let me describe the result in more detail. Let Higgs_G be the moduli stack of G-Higgs bundles on C and let

$$h : \mathrm{Higgs}_G \to B$$

denote the Hitchin fibration, on which the Picard stack $\mathscr{P} \to B$ of J-torsors acts. Here J is the group scheme of regular centralizer over $C \times B$. Let LocSys_G be the moduli stack of G-flat connections on C. Recall the p-Hitchin map

$$h_p : \mathrm{LocSys}_G \to B'$$

in §3.5. To relate Higgs_G to LocSys_G, we introduce a characteristic p analogue of moduli stack of harmonic bundles, which we denote by $\mathscr{H}_G$, as introduced by Simpson in [S]. By definition, the stack $\mathscr{H}_G$ classifies (relative) splittings of a certain J'-gerbe over B' (see [CZ1, §3.2]). We show that $\mathscr{H}_G$ is naturally a stack over B'

$$\mathscr{H}_G \to B'. \tag{4.1}$$

Moreover, there is a canonical action of $\mathscr{P}'_G$ on $\mathscr{H}_G$ and under this action the stack $\mathscr{H}_G$ becomes a $\mathscr{P}'_G$-torsor. We write $\mathscr{H}_{b'}$ for the fiber over $b' \in B'$. We will give examples of the stack $\mathscr{H}_{b'}$ (in the case when $G = GL_n$) in Example 4.5.

The following theorem is proved in [CZ2].

Theorem 4.1. [CZ2, Theorem 1.2] *Over B', there is a canonical isomorphism*

$$\mathfrak{C} : \mathscr{H}_G \times^{\mathscr{P}'_G} \mathrm{Higgs}'_G \to \mathrm{LocSys}_G .$$

Here $\mathfrak{C} : \mathscr{H}_G \times^{\mathscr{P}'_G} \mathrm{Higgs}'_G$ is the twist of Higgs'_G by the $\mathscr{P}'_G$-torsor $\mathscr{H}_G$.

Since the Picard stack $\mathscr{P}'_G$ is smooth, the $\mathscr{P}'_G$-torsor $\mathscr{H}_G$ can be trivialized étale locally and the theorem above implies the following

Corollary 4.2. *There is an étale cover U of B', such that $\mathrm{LocSys}_G \times_{B'} U$ is isomorpshic to $\mathrm{Higgs}'_G \times_{B'} U$.*

Remark 4.3. In the case $G = GL_n$, this corollary is one of the main theorems of [G], which extends a result of [BB] that establishes the above isomorphism over $(B^0)'$.

Remark 4.4. The relation between our work and the construction of A. Ogus and V. Vologodsky in [OV] (in the curve case) is as follows. The construction of [OV] amounts to the restriction of $\mathfrak{C}$ to $0 \in B'$. Namely, upon a choice of a lifting of C to $W_2(k)$, $\mathscr{H}$ admits a canonical trivialization, and thus $\mathfrak{C}$ induces an isomorphism between the moduli of *nilpotent* G-Higgs bundles $\mathrm{Higgs}'_{G,0}$ over C' and the moduli $\mathrm{LocSys}_{G,0}$ of flat G-local systems with *nilpotent* p-curvatures.

Example 4.5. We give a description of the isomorphism $\mathfrak{C}$ in the case when $G = \mathrm{GL}_n$. For simplicity, we fix $b' \in B'$. We first describe the fibers $\mathrm{Higgs}'_{G,b'}, \mathrm{LocSys}_{G,b'}$ and $\mathscr{H}_{b'}$ in this case.

Let $S'_{b'} \subset T^*C'$ be the spectral curve for b', and $S_{b^p} \subset T^*C' \times_{C'} C$ be the pullback of $S'_{b'}$ which fits into the following Cartesian diagram

$$\begin{array}{ccc} S_{b^p} & \xrightarrow{W} & S'_{b'} \\ {\scriptstyle \pi_{b^p}}\downarrow & & \downarrow \\ C & \longrightarrow & C'. \end{array}$$

Note that S_{b^p} is the spectral curve corresponding to $b^p = F^*b'$. The stack $\mathrm{Higgs}'_{G,b'}$ is the space of Higgs fields (E', ϕ') with the characteristic polynomial b'. By the classical BNR correspondence (see [BNR]), such a Higgs field defines a coherent sheaf $\mathcal{F}_{(E',\phi')}$ on $S'_{b'}$. The space $\mathrm{LocSys}_{G,b'}$ consist of rank n vector bundles with a connection (E, ∇) on C whose p-curvature Ψ_∇ has the characteristic polynomial b^p. Therefore, every object in $\mathrm{LocSys}_{G,b'}$ defines a coherent sheaf $\mathcal{F}_{(E,\Psi_\nabla)}$ on S_{b^p}. The stack $\mathscr{H}_{b'}$ is the moduli stack of splitting modules of the restriction to $S'_{b'}$ of the Azumaya algebra of the ring of crystalline differential operators $\mathcal{D}_C$, that is, we have

$$\mathscr{H}_{b'} = \{(V, i) | V \text{a rank } p \text{ bundle on } S'_{b'},\ i : \mathrm{End}(V) \simeq \mathcal{D}_C|_{S'_{b'}}\}.$$

The Picard stack $\mathrm{Pic}(S'_{b'})$ acts naturally on $\mathscr{H}_{b'}$ by tensoring and under this action $\mathscr{H}_{b'}$ becomes a $\mathrm{Pic}(S'_{b'})$-torsor. Note that $\mathscr{H}_{b'}$ is isomorphic to the open substack $\mathrm{LocSys}^{reg}_{G,b'}$ of $\mathrm{LocSys}_{G,b'}$ consisting of those (E, ∇), such that $\mathcal{L} := \mathcal{F}_{(E,\Psi_\nabla)}$ is an invertible sheaf on S_{b^p}. To see this, we observe that for such a pair (E, ∇), the direct image of $\mathcal{L}$ along $W : S_{b^p} \to S'_{b'}$ is locally free of rank p on $S'_{b'}$ and therefore is a splitting of $\mathcal{D}_C$ on $S'_{b'}$. This defines the desired isomorphism from $\mathrm{LocSys}^{reg}_{G,b'}$ to $\mathscr{H}_{b'}$. The isomorphism

$$\mathfrak{C}_{b'} : \mathscr{H}_{b'} \times^{\mathscr{P}'_{b'}} \mathrm{Higgs}'_{G,b'} \to \mathrm{LocSys}_{G,b'}$$

will send an object $(E', \phi') \in \mathrm{Higgs}'_{G,b'}$, regarded as a coherent sheaf $\mathcal{F}_{(E',\phi')}$ on $S'_{b'}$, and an object $\mathscr{H}_{b'}$, which is a splitting module (V, i) of $\mathcal{D}_C|_{S'_{b'}}$, to the tensor product $\mathcal{F}_{(E',\phi')} \otimes V$.

5. Geometric Langlands correspondence in characteristic p

Let C be a smooth projective curve over $\mathbb{C}$. Let G be a reductive algebraic group over $\mathbb{C}$ and let $\check{G}$ be its Langlands dual group. Geometric Langlands conjecture (GLC), proposed by Beilinson and Drinfeld, is a conjectural equivalence between derived category of quasi-coherent sheaves on $\mathrm{LocSys}_{\check{G}}$ and derived category of $\mathcal{D}$-modules on Bun_G. In the paper [BB], the authors proved a characteristic p version of geometric Langlands conjecture in the case $G = GL_n$. In the joint work with Xinwen Zhu [CZ1], we generalized the methods in [BB] to arbitrary semi-simple group and proved a characteristic p version of geometric Langlands conjecture for arbitrary semi-simple groups.

We now describe the result in more details. Let G be semi-simple group over an algebraically closed field k whose characteristic p is positive and does not divide the order of the Weyl group of G and let $\check{G}$ be its Langlands dual group, defined over k. Let C be a smooth projective curve over k and let ω_C denote the canonical bundle of C. We will identify the Hitchin base B and the corresponding open subset B^0 for G and $\check{G}$ using the W-invariant form $(,)$. We denote by

$$\mathrm{Higgs}_G^0 = \mathrm{Higgs}_G \times_B B^0, \quad \mathrm{LocSys}_{\check{G}}^0 = \mathrm{LocSys}_{\check{G}} \times_{B'} (B^0)',$$

the preimage of $(B^0)'$ along the Hitchin map and p-Hitchin map respectively. The open subset Higgs_G^0 (or rather its Frobenius twist) defines a localization

$$\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$$

of the category $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)$ of $\mathcal{D}$-modules on Bun_G, that is, $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ is the category of $\mathcal{D}^0_{\mathrm{Bun}_G}$-modules on $(\mathrm{Higgs}_G^0)'$, where $\mathcal{D}^0_{\mathrm{Bun}_G}$ is the restriction of $\mathcal{D}_{\mathrm{Bun}_G}$, the sheaf of Azumaya algebra of differential operators on $T^*\mathrm{Bun}_G' \cong \mathrm{Higgs}_G'$, to the open set $(\mathrm{Higgs}_G^0)'$.

The following theorem is proved in [CZ2], which can be viewed as a generic version of geometric Langlands correspondence in positive characteristic.

Theorem 5.1. [CZ2, Theorem 5.0.4] *For a choice of a square root κ of ω_C, we have a canonical equivalence of bounded derived categories*

$$\mathfrak{D}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^0)).$$

Remark 5.2. In the case $G = GL_n$, the theorem above was first proved in [BB]. In [G], the author extends the equivalence in [BB] over the locus of B' where the corresponding spectral curves are integral.

Theorem 5.1 follows from the non-abelian Hodge theory in Theorem 4.1, and the following twisted version of Theorem 3.5: let $\mathscr{H}_{\check{G}} \to B'$ be the $\mathscr{P}'_{\check{G}}$-torsor of harmonic bundles for the dual group $\check{G}$ introduced in (4.1) and we define $\mathscr{H}^0_{\check{G}} := \mathscr{H}_{\check{G}} \times_{B'} (B^0)'$. Let $i_\kappa : \mathscr{P}'_G|_{(B^0)'} \simeq (\mathrm{Higgs}_G^0)'$ be the isomorphism induced by the Kostant section ϵ_κ associated to κ (see §3.3). It is shown in [CZ1, §4] that the the Azumaya algebra $\mathcal{A} := i_\kappa^*(\mathcal{D}_{\mathrm{Bun}_G})$ on $\mathscr{P}'_G|_{(B^0)'}$ has a canonical multplicative structure compatible with the multiplication on the Picard stack $\mathscr{P}'_G|_{(B^0)'}$ and we define

$$\mathscr{S}_G^0 \to (B^0)'$$

to be the $(\mathscr{P}'_G)^\vee$-torsor classifying mutiplicative splittings of the Azumaya algebra $\mathcal{A}$ relatively over $(B^0)'$ (see [CZ2, Appendix A] for the notions of multiplicative structure on an Azumaya algebra over a Picard stack and multiplicative splittings).

Theorem 5.3. [CZ1, Theorem 1.2.2] *There is an isomorphism $(\mathscr{P}'_G)^\vee|_{(B^0)'} \simeq \check{\mathscr{P}}'_{\check{G}}|_{(B^0)'}$-torosrs*

$$\mathscr{S}_G^0 \simeq \mathscr{H}^0_{\check{G}}.$$

To complete the proof of Theorem 5.1, we first observe that Theorem 4.1 and the Kostant section $\epsilon_\kappa : B' \to \mathrm{Higgs}'_{\check{G}}$ induce an isomorphism

$$\mathscr{H}^0_{\check{G}} \simeq \mathscr{H}^0_{\check{G}} \times^{\mathscr{P}'_{\check{G}}} (\mathrm{Higgs}^0_{\check{G}})' \simeq \mathrm{LocSys}^0_{\check{G}} \tag{5.1}$$

over $(B^0)'$. On the other hand, a twisted version of Fourier-Mukai transform (see [CZ2, Appendix A]) implies

$$D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathscr{S}_G^0)). \tag{5.2}$$

All together we arrive the desired equivalence

$$D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \overset{(5.2)}{\simeq} D^b(\mathrm{QCoh}(\mathscr{S}_G^0)) \overset{\mathrm{Thm}\ 5.3}{\simeq} D^b(\mathrm{QCoh}(\mathscr{H}_{\check{G}}^0)) \overset{(5.1)}{\simeq} D^b(\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^0)).$$

6. Quantization of Hitchin's intergrable systems

In the last section we give a report on the recent work of R.Bezrukavnikov and R.Travkin [BT] on quantization of Hitchin's intergrable systems.

We assume $G = GL_n$. Let $\mathrm{Bun} = \mathrm{Bun}_G$ be the moduli stack of rank n-bundles on C and Bun^d be the component of rank n-bundles of degree d. Let ω_{Bun} be the canonical bundle on Bun. We fixed a square root $(\omega_{\mathrm{Bun}})^{1/2}$ of ω_{Bun} and we write $\mathcal{D}_{(\omega_{\mathrm{Bun}})^{1/2}}$ for the sheaf of $(\omega_{\mathrm{Bun}})^{1/2}$-twisted differential operators on Bun (see [BD, §2]). Let Op denote the space of *marked* opers. Recall that a marked oper consists of the data $(E, \nabla, \{E_i\}_{i=1,\dots,n}, \phi)$ where (E, ∇) is a rank n bundle on C with a flat connection, $0 \subset E_1 \subset \cdots \subset E_n = E$ is a complete flag, $\phi : E_1 \simeq \omega_C^{(n-1)/2}$, such that 1) $\nabla(E_i) \subset E_{i+1} \otimes \omega_C$ and 2) the induced map $gr(\nabla) : E_i/E_{i-1} \to E_{i+1}/E_i \otimes \omega_C$ is an isomorphism (see [BT, Appendix] or [B] for more details).

Theorem 6.1. [BT, Theorem 1] *Assume* $\mathrm{char}\, k = p$ *is zero or* $p \neq 2$.

(1) For every $d \in \mathbb{Z}$ *we have a canonical isomorphism* $\Gamma(\mathrm{Bun}^d, \mathcal{D}_{(\omega_{\mathrm{Bun}})^{1/2}}) \simeq \Gamma(\mathrm{Op}, \mathcal{O})$.
(2) Let $x \in \mathrm{Op}$ *be a marked oper with underling flat connection* (E, ∇). *Let* $f_x : \Gamma(\mathrm{Op}, \mathcal{O}) \to k$ *be the corresponding homomorphism of* k*-algebra. Then the D-module*

$$\mathrm{Aut}_x := (\mathcal{D}_{(\omega_{\mathrm{Bun}})^{1/2}} \otimes_{\Gamma(\mathrm{Op},\mathcal{O}),f_x} k) \otimes \omega_{\mathrm{Bun}}^{-1/2}$$

is a Hecke-eigen D-module on Bun *with respect to* (E, ∇) *(see* [BD, §5] *for the notion of Hecke-eigen D-modules).*

In the case when the characteristic of k is zero, Theorem 6.1 is the main result of the work of Beilinson-Drinfeld [BD] on quantization of Hitchin's integrable system and Hecke eigensheaves. The construction in [BD] used local to global argument and results on representation theory of affine Lie algebras at critical level. The proof of Theorem 6.1 in [BT] is quite interesting and different from the one in [BD]: they first establish the result when the characteristic of k is positive using the generic geometric Langlands correspondence in characteristic p in Theorem 5.1 and then formally deduce the characteristic zero case.

Remark 6.2. Since the stack Bun is not *good* in the sense of [BD, §1], one has to use a modified version of Bun^d in the statement of Theorem 6.1 (see [BT, §2] or [BB, §4.6]). We ignore this technical details here.

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