Vehicle Following Over a Closed Ring Road under Safety Constraint

Milad Pooladsanj Ketan Savla Petros Ioannou

***Abstract*— Increasing traffic volume with respect to physical space motivates explicit consideration of space constraint in traffic system analysis. We study dynamics of a system of homogeneous vehicles executing safe vehicle following on a closed single lane ring road. Dynamics of each vehicle is governed by a standard second order model and a two mode vehicle following controller. One mode is cruise control and the other is a constant time headway control for safety; switching between modes is determined by a linear combination of relative distance and speed. We show that there exists a threshold value for the number of vehicles at which the equilibria for inter-vehicle configurations transition from being infinite to being unique. We explicitly characterize the unique equilibrium in the latter case as well as the threshold value for transition in terms of system parameters (road length, constant time headway and free flow speed). We also show that, starting from any initial condition, the inter-vehicle configuration converges to an equilibrium. The threshold value for the number of vehicles is also shown to define the boundary of when the transfer function from external disturbance to error in relative spacing changes.**

I. Introduction

The proposed technologies for connected and autonomous vehicles (CAV) in urban traffic systems have been projected to potentially reduce congestion related societal costs, e.g., see [1]. The CAV paradigms enrich the possibilities for controlling traffic at the microscopic level, e.g., through vehicle following. The emergent manifestation of these innovations at the microscopic scale naturally also have implications on control modalities at the macroscopic level. It is therefore of interest to develop frameworks for performance evaluation of CAV, and to develop their macroscopic abstractions. Such frameworks, independent of the CAV paradigm, have to include relevant constraints. Explicit consideration of space constraint has received relatively little attention in microscopic analysis, especially in combination with hard safety constraints. This paper addresses this through a simple abstraction of homogeneous vehicles executing safe vehicle following on a closed single lane ring road.

Microscopic study of vehicle following has traditionally been performed using the *platooning* framework, where a lead vehicle follows a desired speed trajectory on an unbounded line and the remaining vehicles execute prescribed vehicle following controllers. Commonly studied control paradigms are the ones to achieve desired spacing, e.g., by adopting constant spacing or constant headway policy, and under different communication architecture among vehicles, e.g., leader following, leader and predecessor, and nearest neighbor [2]–[10]. The objectives of these studies is to analyze the impact of controller on the dynamics of inter-vehicle spacing. Such analysis yields guarantees on collision avoidance, ride comfort, and safe entrance/exit of traffic. However, the unbounded line setting precludes all of these studies to explicitly include space constraint.

Inspired in part by the well-known experiment in [11], there has recently been interest in vehicle following on a ring road for mixed-autonomy settings [12]–[15]. However, the focus of the analytical aspects of these works is on formation and dissipation of traffic jams using autonomous vehicles, without explicit consideration of collision avoidance. The impact of external disturbances, e.g., unmodeled dynamics, sensing, or actuation noise, is not considered either. The objective of this paper is to address these shortcomings. It is important to note that the ring road setup, with entry and exit points, has also been used as an abstraction for macroscopic study of highway network, e.g., see [16].

The ring road setup in this paper has no entry or exit. No inter-vehicle communication is considered, and the throttle and brake commands for a vehicle are determined by its own on-board control system which uses information from its on board sensors about relative distance and speed. This framework is consistent with that of an autonomous vehicle. In this paper, we adopt a vehicle dynamics model consisting of a validated second order model under a two-mode vehicle following controller. One mode is cruise control and the other is a constant time headway control for safety; switching between modes is determined by a linear combination of relative distance and speed. The resulting system level dynamics for inter-vehicle configurations is a switched system. We provide analysis for existence, uniqueness and stability of equilibria, as well as for attenuation of external disturbance.

We show that there exists a threshold value $n_{critical}$ for the number of vehicles n at which the equilibria for inter-vehicle configurations transition from being infinite to being unique. We explicitly characterize the unique equilibrium in the latter case as well as $n_{critical}$ in terms of system parameters (road length, constant time headway and free flow speed). In particular, if $n \geq n_{critical}$, then the equilibrium vehicle speed is shown to be inversely proportional to n, thereby quantifying the effect of space constraint. We also show that, starting from any initial condition, the inter-vehicle configuration converges to an equilibrium. We also observe a

Milad Pooladsanj and Petros Ioannou are with the Department of Electrical Engineering, University of Southern California, CA. {pooladsa, ioannou}@usc.edu. Ketan Savla is with the Department of Civil and Environmental Engineering, University of Southern California, CA. ksavla@usc.edu. This work was supported in part by NSF CMMI 1636377. K. Savla has financial interest in Xtelligent, Inc.

phase transition in the impact of external disturbance on error in relative spacing. If $n < n_{critical}$, then this effect mimics that of the classical platoon setup without space constraint, i.e., disturbance in a vehicle affects its predecessor only. On the other hand, if $n \geq n_{critical}$, then disturbance in one vehicle changes the reference speed of all vehicles.

The main contributions can be summarized as follows. First, we present a simple abstraction for analysis of vehicle following controller under explicit consideration of limited space and safety constraint. This setup has the potential to serve as a canonical abstraction for various scenarios involving mixed autonomy, V2X communication, and traffic network control. Second, for a validated vehicle dynamics and a well-known safe vehicle following controller, we provide comprehensive stability analysis for homogeneous vehicles. Our analysis suggests an intuitive phase transition in the set of equilibria with increase in vehicle density. Such a phase transition also plays a role in our third contribution where we analyze the impact of external disturbance. In particular, our results show that the effect of disturbance is more pronounced in a *circular* platoon, i.e., without a leader, than a *linear* platoon.

The rest of the paper is structured as follows. Section II formulates the problem and reviews the relevant results when space constraint is not considered. Section III presents results on existence and stability of equilibria. Section IV presents results on propagation of vehicle-to-vehicle spacing error under external disturbance. Simulations are presented in Section V, and concluding remarks and suggestions for future research are provided in Section VI.

II. Preliminaries and Problem Formulation

A. Problem Formulation

In this section we formulate the dynamics of n vehicles travelling on a ring road with perimeter P. We assign coordinates in $[0, P]$ to the road in the clock-wise direction. Vehicle i is the i^{th} vehicle from point 0 at $t = 0$, $i = 1, 2, \cdots, n$. The configuration for three vehicles is depicted in Figure 1.

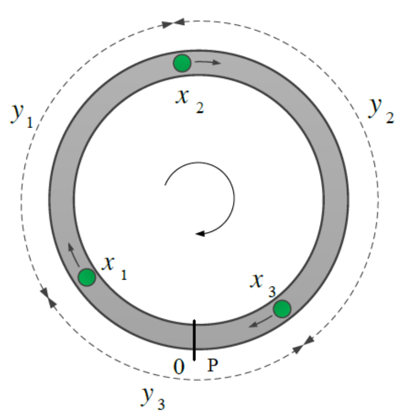

Fig. 1: Configuration of three vehicles moving on a closed ring road

We use the validated vehicle dynamics model from [17]:

$$\frac{d}{dt}x_i(t) = \dot{x}_i(t) = v_i, \tag{1a}$$

$$\frac{d}{dt}\dot{x}_i(t) = \ddot{x}_i(t) = -a(v_i - v_{i_{des}}) + b\bar{u}_i + d_i \tag{1b}$$

where $a > 0$, $b > 0$ are vehicle parameters, $x_i(t)$ is the distance travelled by the i^{th} vehicle at time t, $t \geq 0$, relative to point 0, v_i and $v_{i_{des}}$ are its actual speed and desired speed respectively, $\bar{u}_i(t)$ is the deviation of its throttle angle from the desired value, and d_i accounts for external disturbances. We use the following switched controller:

$$\bar{u}_i = \begin{cases} \frac{1}{b}(\frac{1}{h} - a)\dot{y}_i(t) - \frac{\alpha}{bh}(h\dot{x}_i(t) - y_i(t)) & \text{if } y_i(t) \leq \frac{-1}{\alpha}\dot{y}_i(t) + hV_f \\ -\frac{1}{b}(\alpha - a)(\dot{x}_i(t) - V_f) & \text{otherwise} \end{cases} \tag{2}$$

where $y_i(t) = x_{i+1}(t) - x_i(t)$, is the relative spacing, and $\dot{y}_i(t) = \dot{x}_{i+1}(t) - \dot{x}_i(t)$ is the relative speed for the i^{th} vehicle ($x_{n+1} := x_1 + P$). The space constraint is captured by the implicit constraint that $\sum_{i=1}^{n} y_i(t) = P$ for all $t \geq 0$.

The first mode of the controller (2) is the constant time headway control and the second is the cruise control mode. In these two modes, $v_{i_{des}}$ is equal to $\dot{x}_{i+1}$ and V_f, respectively, where V_f is a pre-specified free-flow speed. α is a design parameter and so is the time headway constant h. The time headway constant is chosen such that two consecutive vehicles do not collide under a worst-case stopping condition [18]. An instance of such a scenario is when vehicle $i+1$ is decelerating at maximum deceleration while its predecessor, i.e., vehicle i, is accelerating with the maximum acceleration, while the two vehicles are travelling with the same speed. Accordingly, the constant time headway control in (2) ensures no collision if $y_i(0) \geq h\dot{x}_i(0)$.

The switching criterion in (2) is inspired by [19]. The relative distance at which a vehicle switches between the modes is not constant; it also depends on its relative speed with respect to the vehicle in front. For instance, if vehicle i is moving slower than vehicle $i+1$, then it switches from the cruise control to the constant time headway control mode at a distance less than the threshold value hV_f. If α and h are chosen such that $h > \frac{1}{\alpha}$, then this switching is safe because the cruise control speed is no more than V_f. Finally, note that while the control input $\bar{u}_i$ is not continuous at the switching surface, substituting (2) into (1) ensures that the right hand side is Lipschitz continuous and hence (1) is well-posed.

The primary objective of this paper is to analyze the existence, uniqueness and stability of equilibria for (1)-(2), as well as the impact of external disturbance.

B. Results for No Space Constraint

We provide a brief summary of relevant results [18], when space constraint is not considered explicitly. Consider a platoon of n vehicles on an unbounded line where the lead vehicle, numbered n, is in the cruise control mode, i.e., the second mode in (2), and the rest of the vehicles, $i = 1, 2, \cdots, n-1$, are using constant time headway control, i.e., the first mode in (2). Let $\delta_i = y_i - h\dot{x}_i$ be the error in relative spacing for the i^{th} vehicle. Let $G(s)$ be the frequency-domain transfer function between δ_i and δ_{i+1}, and $g(t)$ be the impulse response. For desired steady-state performance, i.e., desired speed and relative spacing, the necessary and sufficient condition is that the poles of $G(s)$ lie in the open left half of the s-plane. For desired transient behavior, e.g., attenuation of δ_i through the platoon, a sufficient condition is $|G(j\omega)| < 1$, $\forall \omega > 0$, and $g(t) > 0$, $\forall t > 0$ [18]. For the constant time headway controller in (2), these conditions are equivalent to $h > 0$ and $\alpha > 0$. To the best of our knowledge, no such results are known under explicit consideration of space constraint.

III. Dynamical Analysis under No Disturbance

Let $d_i \equiv 0$, $i = 1, 2, \cdots, n$. Substituting (2) into (1) and subtracting equations for consecutive vehicles, the dynamics of relative spacing is:

$$\ddot{y}_i = \begin{cases} \frac{1}{h}(\dot{y}_{i+1} - \dot{y}_i) - \frac{\alpha}{h}(h\dot{y}_i - y_{i+1} + y_i) & \text{if } \begin{cases} y_{i+1} \le \frac{-1}{\alpha}\dot{y}_{i+1} + hV_f \\ y_i \le \frac{-1}{\alpha}\dot{y}_i + hV_f \end{cases} \\ \frac{1}{h}\dot{y}_{i+1} - \frac{\alpha}{h}(h\dot{y}_i - y_{i+1} + hV_f) & \text{if } \begin{cases} y_{i+1} \le \frac{-1}{\alpha}\dot{y}_{i+1} + hV_f \\ y_i > \frac{-1}{\alpha}\dot{y}_i + hV_f \end{cases} \\ \frac{-1}{h}\dot{y}_i - \frac{\alpha}{h}(h\dot{y}_i + y_i - hV_f) & \text{if } \begin{cases} y_{i+1} > \frac{-1}{\alpha}\dot{y}_{i+1} + hV_f \\ y_i \le \frac{-1}{\alpha}\dot{y}_i + hV_f \end{cases} \\ -\alpha\dot{y}_i & \text{if } \begin{cases} y_{i+1} > \frac{-1}{\alpha}\dot{y}_{i+1} + hV_f \\ y_i > \frac{-1}{\alpha}\dot{y}_i + hV_f \end{cases} \end{cases} \tag{3}$$

subject to $\sum_{i=1}^n y_i(t) = P$. Let $e_{2i-1} = y_i - hV_f$ and $e_{2i} = \dot{y}_i$, $i = 1, 2, \cdots, n$. The equations in (3) transform to

$$\begin{aligned} \dot{e}_{2i-1} &= e_{2i} \\ \dot{e}_{2i} &= \begin{cases} \frac{1}{h}(e_{2i+2} - e_{2i}) - \frac{\alpha}{h}(he_{2i} - e_{2i+1} + e_{2i-1}) & \text{if } \begin{cases} e_{2i+1} \le \frac{-1}{\alpha}e_{2i+2} \\ e_{2i-1} \le \frac{-1}{\alpha}e_{2i} \end{cases} \\ \frac{1}{h}e_{2i+2} - \frac{\alpha}{h}(he_{2i} - e_{2i+1}) & \text{if } \begin{cases} e_{2i+1} \le \frac{-1}{\alpha}e_{2i+2} \\ e_{2i-1} > \frac{-1}{\alpha}e_{2i} \end{cases} \\ \frac{-1}{h}e_{2i} - \frac{\alpha}{h}(he_{2i} + e_{2i-1}) & \text{if } \begin{cases} e_{2i+1} > \frac{-1}{\alpha}e_{2i+2} \\ e_{2i-1} \le \frac{-1}{\alpha}e_{2i} \end{cases} \\ -\alpha e_{2i} & \text{if } \begin{cases} e_{2i+1} > \frac{-1}{\alpha}e_{2i+2} \\ e_{2i-1} > \frac{-1}{\alpha}e_{2i} \end{cases} \end{cases} \end{aligned} \tag{4}$$

subject to $\sum_{i=1}^n e_{2i-1}(t) = P - nhV_f$. Note that $\dot{e}_{2i}$s are continuous at the switching surfaces. An equilibrium satisfies

$$\begin{aligned} e^*_{2i} &= 0 \\ e^*_{2i-1} &= \begin{cases} e^*_{2i+1} = c_i, \;\; \textit{for some } c_i \le 0 & \text{if } \begin{cases} e^*_{2i-1} \le 0 \\ e^*_{2i+1} \le 0 \end{cases} \\ c_i, \;\; \textit{for some } c_i > 0, e^*_{2i+1} = 0 & \text{if } \begin{cases} e^*_{2i-1} > 0 \\ e^*_{2i+1} \le 0 \end{cases} \\ 0 & \text{if } \begin{cases} e^*_{2i-1} \le 0 \\ e^*_{2i+1} > 0 \end{cases} \\ c_i, \;\; \textit{for some } c_i > 0 & \text{if } \begin{cases} e^*_{2i-1} > 0 \\ e^*_{2i+1} > 0 \end{cases} \end{cases} \end{aligned} \tag{5}$$

In general, there are infinite equilibria for relative spacing; however, the relative speed at all these equilibria is zero. Later in this section we show that for sufficiently large n, the relative spacing equilibria transition from being infinite to being unique.

Let us first consider the case when $e^*_{2i-1} \le 0$ for all $i = 1, 2, \cdots, n$, i.e., when all vehicles are using the constant time headway control at the steady state.

Proposition 1. *If $e^*_{2i-1} \le 0$ for all $i = 1, 2, \cdots, n$, then $e^*_{2i-1} = \frac{P - nhV_f}{n}$ for all $i = 1, 2, \cdots, n$.*

Proof: From (5), $e^*_{2i-1} \le 0$ for all $i = 1, 2, \cdots, n$ is equivalent to $e^*_{2i-1} = e^*_{2i+1}$ for all $i = 1, 2, \cdots, n$. From $\sum_{i=1}^n e^*_{2i-1} = P - nhV_f$, it follows that $e^*_{2i-1} = \frac{P - nhV_f}{n}$ for all $i = 1, 2, \cdots, n$.

Remark 1. *In the special case $e^*_{2i-1} = e^*_{2i+1} \le 0$ for all $i = 1, 2, \cdots, n$, we must have $P \le nhV_f$, which gives the critical number of vehicles on the ring road,*

$$n_{critical} = \lfloor \frac{P}{hV_f} \rfloor \tag{6}$$

We next characterize the dynamics of the quantity $A_i(t) := \alpha e_{2i-1}(t) + e_{2i}(t)$ associated with the switching surface.

Proposition 2. *If $A_k(0) = \alpha e_{2k-1}(0) + e_{2k}(0) \le 0$ for some $k \in \{1, 2, \cdots, n\}$ then: (i) $A_k(t) \le 0$, $\forall t > 0$; (ii) additionally, if $P > nhV_f$ then $\lim_{t\to\infty} A_k(t) = 0$.*

Proof:

$$\dot{A}_k(t) = \begin{cases} \frac{1}{h}(A_{k+1}(t) - A_k(t)) & \text{if } \begin{cases} A_k(t) \le 0 \\ A_{k+1}(t) \le 0 \end{cases} \\ \frac{1}{h}A_{k+1}(t) & \text{if } \begin{cases} A_k(t) > 0 \\ A_{k+1}(t) \le 0 \end{cases} \\ -\frac{1}{h}A_k(t) & \text{if } \begin{cases} A_k(t) \le 0 \\ A_{k+1}(t) > 0 \end{cases} \\ 0 & \text{if } \begin{cases} A_k(t) > 0 \\ A_{k+1}(t) > 0 \end{cases} \end{cases} \tag{7}$$

subject to $\sum_{i=1}^n A_i(t) = \alpha(P - nhV_f)$. Let $A_k(0) \le 0$. In order for $A_k(t)$ to become positive, it must pass through zero through either the first or third mode of (7). Let $A_k(t_0) = 0$ for some $t_0 \ge 0$. It can be seen from the first and third mode of (7) that $\dot{A}_k(t_0) \le 0$, which means that $A_k(t)$ stays non-positive for $t \ge t_0$. Therefore, if $A_k(0) \le 0$ then $A_k(t) \le 0$, $\forall t > 0$. In other words, if a vehicle switches to the constant time headway control mode, it remains in that mode.

Now assume that $P > nhV_f$. Since $\sum_{i=1}^n A_i(t) = \alpha(P - nhV_f) > 0$, there is at least one vehicle $j \in \{1, 2, \cdots, n\}$, $j \neq k$ for which $A_j(t) > 0$, $\forall t \ge 0$, i.e., vehicle j starts in the cruise control mode and remains in this mode for all future times. Since $A_j(t) > 0$, $\forall t \ge 0$, vehicle $j - 1$ starts and stays in either mode three or four of (7). The mode four can be analyzed using the following steps for vehicle $j - 1$ and its preceding vehicle. Hence, we assume that vehicle $j - 1$ operates in the mode three. This mode implies the exponential convergence of $A_{j-1}(t)$ to zero. Now consider vehicle $j - 2$. Since $A_{j-1}(t) \le 0$, $\forall t \ge 0$, vehicle $j - 2$ can only operate in the mode one or two of (7). Since $A_{j-1}(t)$

converges to zero exponentially fast, $A_{j-2}(t)$ also converges exponentially fast to some non-negative value. It converges to zero if vehicle $j-2$ ever operates in the mode one. We can repeat the above steps to show exponential convergence of all A_is, $i = 1, 2, \cdots, n$ to some non-negative values. Specifically, we can show that for any $k \in \{1, 2, \cdots, n\}$ such that $A_k(0) \leq 0$, $A_k(t)$ converges to zero exponentially fast.

Remark 2. *According to Proposition 2 and its proof, each vehicle will switch at most once from the cruise control to the constant time headway control mode. Furthermore, the time of the last switch, say t_0, is finite. This can be proved by using a similar argument to the proof of Proposition 2.*

Therefore, if $n < n_{critical}$, i.e., $P > nhV_f$, vehicles will eventually form one or multiple separated platoons. On the other hand, if $n > n_{critical}$, vehicles form a single doubly-connected platoon, i.e., there is no lead vehicle, after a finite time t_0.

We are now ready to prove that the states in system (4) converge to an equilibrium point in the set (5). This point is determined by the initial condition.

Proposition 3. *Let $\alpha > \frac{1}{4}$. For every initial condition, the solution to* (4) *converges to an equilibrium.*

Proof: From Remark 2, we know that there exists a finite $t_0 \geq 0$ after which there are no more switches.

Consider the function $V(e) = \sum_{i=1}^{n} \frac{1}{2} e_{2i}^2 + \frac{1}{h}(\alpha e_{2i-1} + e_{2i})^2$. It is easy to see that $V(e)$ is positive definite and radially unbounded. Let $I_i = 1$ only when $A_i \leq 0$ and be zero otherwise, $i = 1, 2, \cdots, n$.

$$\begin{aligned}\frac{d}{dt}V(e) = &-\alpha \sum_{i=1}^{n} e_{2i}^2 - \frac{1}{h}\sum_{i=1}^{n} e_{2i}(A_i I_i - A_{i+1} I_{i+1}) \\ &- \frac{2}{h^2}\sum_{i=1}^{n} A_i^2 I_i + \frac{2}{h^2}\sum_{i=1}^{n} A_i A_{i+1} I_{i+1}\end{aligned}$$

which simplifies to

$$\begin{aligned}\frac{d}{dt}V(e) = &-(\alpha - \frac{1}{4})\sum_{i=1}^{n} e_{2i}^2 \\ &- \sum_{i=1}^{n}(\frac{1}{2}e_{2i} + \frac{1}{h}A_i I_i - \frac{1}{h}A_{i+1} I_{i+1})^2 \\ &+ \frac{2}{h^2}\sum_{i=1}^{n}(A_i A_{i+1} I_{i+1} - A_i A_{i+1} I_i I_{i+1})\end{aligned}$$

The sign of $\frac{2}{h^2}\sum_{i=1}^{n}(A_i A_{i+1} I_{i+1} - A_i A_{i+1} I_i I_{i+1})$ can be shown to be non-positive by considering the following two cases. Let $F(A_i, A_{i+1}) = A_i A_{i+1} I_{i+1} - A_i A_{i+1} I_i I_{i+1}$. If $I_i = 0$, then $A_i > 0$ and hence $F(A_i, A_{i+1}) = A_i A_{i+1} I_{i+1} \leq 0$ since $A_{i+1} I_{i+1} \leq 0$. On the other hand if $I_i = 1$, then $F(A_i, A_{i+1}) = 0$ which proves our previous statement. Therefore, $\frac{d}{dt}V(e) \leq 0$. Moreover,

$$\frac{d}{dt}V(e) = 0 \Longrightarrow \begin{cases} e_{2i-1} I_i = e_{2i+1} I_{i+1}, \\ e_{2i} = 0 \end{cases}$$

which reduces to (5). LaSalle's invariance principle [20] then implies convergence to the set of equilibria. Note that since $V(e)$ is radially unbounded and $\frac{d}{dt}V(e) \leq 0$, the trajectory of (4) remains bounded for all $t \geq 0$. Moreover, since there is no switch after t_0, the system, for $t \geq t_0$, is linear time invariant with initial condition $(e_{2i-1}(t_0), e_{2i}(t_0))$, $i = 1, 2, \cdots, n$. As a result, the system trajectory converges exponentially fast to an equilibrium characterized by (5).

Remark 3. *Let $P < nhV_f$. From Remark 2 it follows that all vehicles switch to the constant time headway control mode after a finite time. This mode contains a unique equilibrium characterized by Proposition 1. Hence, from Proposition 3 it follows that, for all $i = 1, 2, \cdots, n$, (e_{2i-1}, e_{2i}) converges to $(\frac{P - nhV_f}{n}, 0)$ exponentially fast.*

Proposition 4. *For all $i = 1, 2, \cdots, n$, $\lim_{t\to\infty} \dot{x}_i(t) = \min(V_f, \frac{P}{hn})$.*

Proof: We know from Proposition 3 and the equilibrium set (5) that $\lim_{t\to\infty} \dot{y}_i(t) = 0$ and $\lim_{t\to\infty} y_i(t) = y_i^*$ for all $i = 1, 2, \cdots, n$. Note from Proposition 2 that if $P > nhV_f$ then at least one vehicle $j \in \{1, 2, \cdots, n\}$ is always on the cruise control mode, i.e., $\lim_{t\to\infty} \dot{x}_j(t) = V_f$. This combined with $\lim_{t\to\infty} \dot{y}_i(t) = 0$ for all $i = 1, 2, \cdots, n$, concludes $\lim_{t\to\infty} \dot{x}_i(t) = V_f$ for all $i = 1, 2, \cdots, n$.

On the other hand, if $P < nhV_f$, then according to Remark 3, e_{2i-1} for all $i = 1, 2, \cdots, n$, converges to $\frac{P - nhV_f}{n}$ exponentially fast. Therefore, y_i for all $i = 1, 2, \cdots, n$, converges to $\frac{P}{n}$ exponentially fast. It then follows from the first mode of (2) that $\lim_{t\to\infty} \dot{x}_i(t) = \frac{P}{hn}$ for all $i = 1, 2, \cdots, n$. The case $P = nhV_f$ is also trivial since it falls under one of the previous cases.

Remark 4. *Proposition 4 and its proof imply that $\lim_{t\to\infty} \ddot{x}_i(t) = 0$ for all $i = 1, 2, \cdots, n$.*

IV. Transient behavior and error propagation

A significant feature of the constant time headway controller is its ability to eliminate the error propagation and disturbance, e.g., sensing or actuation noise, amplification, in a platoon of vehicles moving on a line, without V2V communication [18]. We now analyze the transient behavior and the effect of external disturbance on the error in relative spacing in the ring road setup. As discussed in the previous section, if $n < n_{critical}$, vehicles will eventually form one or multiple separated platoons. As a result, the transient behavior and propagation of external disturbance, for each platoon on the ring road, is the same as on an unbounded line. Therefore, we now focus on $n \geq n_{critical}$, when there is only one circular platoon, i.e., without a leader.

For simplicity of notations, assume in (1) that $d_i \neq 0$ for some i and $d_j = 0$, $j \neq i$. Let $D_i(s)$ and $\Delta_i(s)$ denote the Laplace transforms of $d_i(t)$ and $\delta_i(t)$, respectively, $i = 1, 2, \cdots, n$. From (3) and noting the homogeneous platoon assumption, the following relationship relates the relative spacing errors of the i^{th} and $i+1^{th}$ vehicles,

$$\Delta_i(s) = G(s)\Delta_{i+1}(s) - \frac{h(1+hs)}{s+\alpha}G(s)D_i(s) \qquad (8)$$

where,

$$G(s) = \frac{s+\alpha}{hs^2 + (\alpha h + 1)s + \alpha}$$

Note from (8) and the discussion in Section II-B that for attenuation of errors in relative spacing, the design constants must be chosen such that $|G(j\omega)| < 1$, $\forall \omega > 0$, and $g(t) > 0, \forall t > 0$. The stability criteria for the system (4), i.e., $h > 0$, $\alpha > \frac{1}{4}$, satisfies both conditions. Therefore, the propagation of δ_i in the heavy traffic regime, i.e., $n \geq n_{critical}$, is the same as in the light traffic regime, i.e., $n < n_{critical}$.

Since it is assumed that all vehicles are using the constant time headway control,

$$\Delta_j(s) = G(s)\Delta_{j+1}(s), \;\; j \neq i-1, i \tag{9}$$

and,

$$\Delta_{i-1}(s) = G(s)\Delta_i(s) + \frac{h}{s+\alpha}G(s)D_i(s) \tag{10}$$

Combining (8), (9), and (10),

$$\Delta_i(s) = \frac{h(G^n(s) - (1+hs)G(s))}{(s+\alpha)(1-G^n(s))}D_i(s)$$

Therefore,

$$\left|\frac{h(G^n(j\omega) - (1+hj\omega)G(j\omega))}{(j\omega+\alpha)(1-G^n(j\omega))}\right| < 1 \quad \forall\omega > 0$$

is a necessary condition in order to eliminate the external disturbance amplification. Note that since $G(0) = 1$,

$$\lim_{s\to 0} \frac{h(G^n(s) - (1+hs)G(s))}{(s+\alpha)(1-G^n(s))} = \frac{-h}{\alpha}$$

Hence, constant disturbance such as unmodelled vehicle dynamics is not rejected with this controller. Additionally, disturbance in one vehicle propagates to all vehicles through equations (9), (10). As will be seen in the next section, constant disturbance can force all vehicles on the ring road to travel in an unsafe spacing at the steady state, i.e., $y_i^* < h\dot{x}_i^*$ for all $i = 1, 2, \cdots, n$.

V. Simulations

Let $P = 240\,m$, $h = 0.4\,s$, $\alpha = 4\,s^{-1}$, $V_f = 65\,mph \simeq 29\,m/s$, and assume that the initial condition for the vehicles is at rest. According to (6), $n_{critical} = \lfloor\frac{240}{0.4\times 29}\rfloor = 20$. Note that $h > \frac{1}{\alpha}$ satisfies the safety constraint.

1) Zero disturbance: Let $d_i = 0$, $i = 1, 2, \cdots, n$. Figures 2 - 7 show the evolution of relative spacing and speed for some vehicles for different values of n: 15, 21, and 25. It can be seen that for light traffic, i.e., Figures 2 and 3, and close-to-capacity traffic, i.e., Figures 4 and 5, vehicles reach the free-flow speed. However, the way they share the space differs in these two cases. As seen in Figure 2, some vehicles remain in the cruise control mode throughout the entire simulation, e.g., vehicle 10. On the other hand, all vehicles switch to the constant time headway controller in the close-to-capacity case and share the space equally, i.e., $y_i^* = \frac{P}{n}$, $i = 1, 2, \cdots, n$.

In the case of heavy traffic, i.e., Figures 6 and 7, vehicles share the space equally, but do not reach the free-flow speed because of the bounded space. Instead, they travel with $v_i^* = \frac{240}{0.4\times 25} = 24\,m/s$, $i = 1, 2, \cdots, 25$, at steady state.

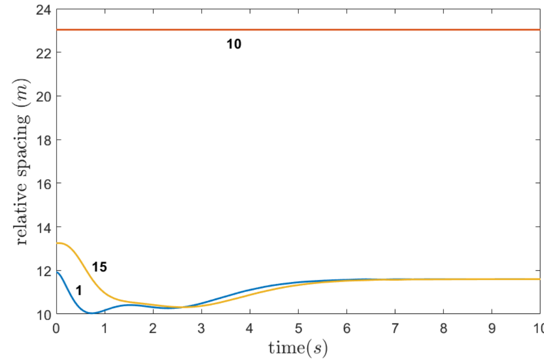

Fig. 2: Relative spacing profiles for sample vehicles when $n = 15$

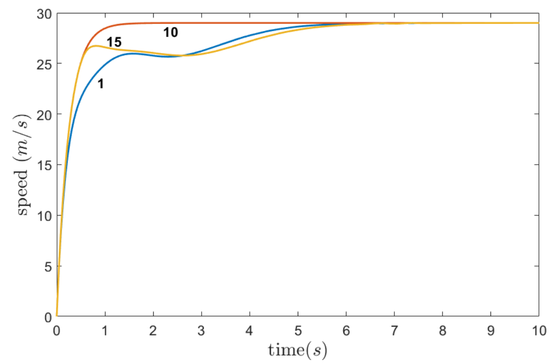

Fig. 3: Speed profiles for sample vehicles when $n = 15$

2) Constant non-zero disturbance: Let $d_i = 1\,m/s^2$, $i = 1, 2, \cdots, 25$. When $n = 25$ (heavy traffic), it is found that the steady-state spacing is $y_i^* = \frac{P}{n} = 9.6m$ while the steady-state speed is $v_i^* = 24.25\,m/s$, $i = 1, 2, \cdots, 25$. Note that $9.6 < 0.4 \times 24.25$, i.e., $y_i^* < hv_i^*$, $i = 1, 2, \cdots, 25$. In other words, the steady-state spacing does not follow the safety distance rule.

VI. Conclusion and Future Work

We studied dynamics of a system of homogeneous vehicles following a safe car following protocol, under explicit space constraint abstracted by a closed ring road. For standard second order vehicle dynamics under constant time headway control, we showed that the space constraint induces a phase transition, with increase in vehicle density, in the equilibrium of inter-vehicle configuration and in the propagation of external disturbance. The explicit and intuitive characterization of the threshold for this transition in terms of system parameters is useful for macroscopic traffic flow control.

It is naturally of interest to extend the analysis to include practical features such as acceleration limit and sensor/communication delay. Optimizing over the infinite equilibria, e.g., to minimize energy consumption, and realizing such optimal solutions under various coordination mechanisms between the vehicles is also relevant. Finally, the analysis presented in this paper and its various extensions need to be performed for other practical instances of vehicle

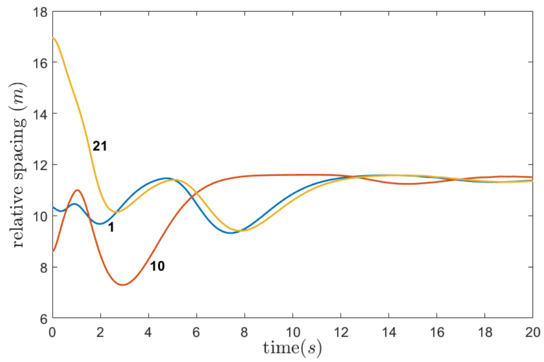

Fig. 4: Relative spacing profiles for sample vehicles when $n = 21$

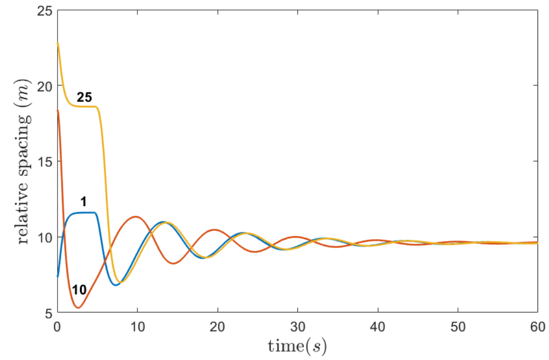

Fig. 6: Relative spacing profiles for sample vehicles when $n = 25$

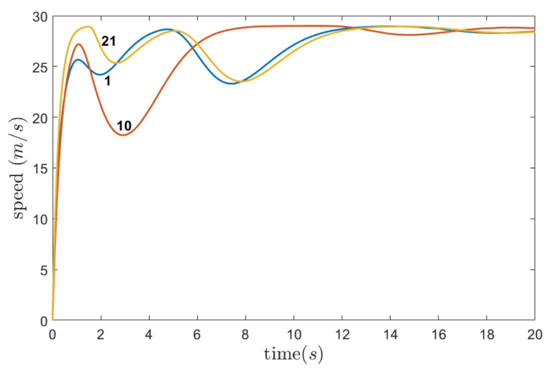

Fig. 5: Speed profiles for sample vehicles when $n = 21$

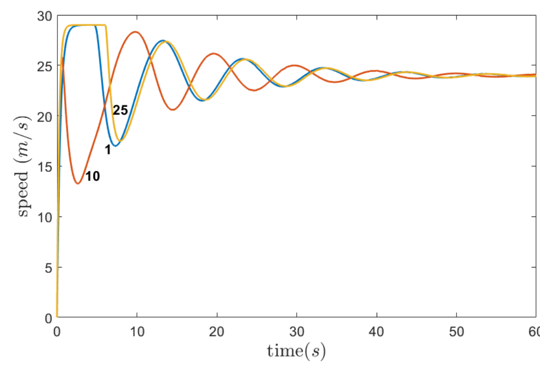

Fig. 7: Speed profiles for sample vehicles when $n = 25$

dynamics and control schemes, including mixture of human driven and autonomous vehicles, to develop comprehensive tools for performance evaluation of anticipated scenarios in urban traffic systems, as well as to inform their macroscopic control.

References

[1] J. Rios-Torres and A. A. Malikopoulos, "A survey on the coordination of connected and automated vehicles at intersections and merging at highway on-ramps," *IEEE Transactions on Intelligent Transportation Systems*, vol. 18, pp. 1066–1077, May 2017.

[2] H. J. Swaroop D.D., "Constant spacing strategies for platooning in automated highway systems," *ASME. J. Dyn. Sys., Meas., Control*, vol. 121, no. 3, pp. 462–470, 1999.

[3] E. Shaw and J. K. Hedrick, "String stability analysis for heterogeneous vehicle strings," in *2007 American Control Conference*, pp. 3118–3125, July 2007.

[4] H. Tan, R. Rajamani, and W. Zhang, "Demonstration of an automated highway platoon system," in *Proceedings of the 1998 American Control Conference. ACC (IEEE Cat. No.98CH36207)*, vol. 3, pp. 1823–1827 vol.3, June 1998.

[5] P. Seiler, A. Pant, and K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1835–1842, Oct 2004.

[6] P. Barooah, P. G. Mehta, and J. P. Hespanha, "Mistuning-based control design to improve closed-loop stability margin of vehicular platoons," *IEEE Transactions on Automatic Control*, vol. 54, pp. 2100–2113, Sep. 2009.

[7] Y. Zhang, B. Kosmatopoulos, P. A. Ioannou, and C. C. Chien, "Using front and back information for tight vehicle following maneuvers," *IEEE Transactions on Vehicular Technology*, vol. 48, pp. 319–328, Jan 1999.

[8] F. Lin, M. Fardad, and M. R. Jovanovic, "Optimal control of vehicular formations with nearest neighbor interactions," *IEEE Transactions on Automatic Control*, vol. 57, pp. 2203–2218, Sep. 2012.

[9] K.-c. Chu, "Decentralized control of high-speed vehicular strings," *Transportation Science*, vol. 8, no. 4, pp. 361–384, 1974.

[10] P. J. Seiler, "Coordinated control of unmanned aerial vehicles," 2001.

[11] Y. Sugiyama, M. Fukui, M. Kikuchi, K. Hasebe, A. Nakayama, K. Nishinari, S. ichi Tadaki, and S. Yukawa, "Traffic jams without bottlenecks—experimental evidence for the physical mechanism of the formation of a jam," 2008.

[12] S. Cui, B. Seibold, R. Stern, and D. B. Work, "Stabilizing traffic flow via a single autonomous vehicle: Possibilities and limitations," in *2017 IEEE Intelligent Vehicles Symposium (IV)*, pp. 1336–1341, June 2017.

[13] Y. Zheng, J. Wang, and K. Li, "Smoothing traffic flow via control of autonomous vehicles," *arXiv preprint arXiv:1812.09544*, 2018.

[14] R. E. Stern, S. Cui, M. L. Delle Monache, R. Bhadani, M. Bunting, M. Churchill, N. Hamilton, H. Pohlmann, F. Wu, B. Piccoli, *et al.*, "Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments," *Transportation Research Part C: Emerging Technologies*, vol. 89, pp. 205–221, 2018.

[15] A. R. Kreidieh, C. Wu, and A. M. Bayen, "Dissipating stop-and-go waves in closed and open networks via deep reinforcement learning," in *2018 21st International Conference on Intelligent Transportation Systems (ITSC)*, pp. 1475–1480, IEEE, 2018.

[16] C. F. Daganzo, "Urban gridlock: Macroscopic modeling and mitigation approaches," *Transportation Research Part B: Methodological*, vol. 41, no. 1, pp. 49 – 62, 2007.

[17] P. Ioannou and Z. Xu, "Throttle and brake control systems for automatic vehicle following," *I V H S Journal*, vol. 1, no. 4, pp. 345–377, 1994.

[18] P. Ioannou and C. Chien, "Autonomous intelligent cruise control," *IEEE TRANSACTIONS ON VEHICULAR TECHNOLOGY*, vol. 42, no. 4, pp. 657–672, 1993.

[19] P. Fancher, Z. Bareket, and G. Johnson, "Predictive analyses of the performance of a headway control system for heavy commercial vehicles," *Vehicle System Dynamics*, vol. 23, no. S1, pp. 128–141, 1994.

[20] H. K. Khalil and J. W. Grizzle, *Nonlinear systems*, vol. 3. Prentice hall Upper Saddle River, NJ, 2002.