Feedforward boundary control of 2×2 nonlinear hyperbolic systems with application to Saint-Venant equations

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Abstract

Because they represent physical systems with propagation delays, hyperbolic systems are well suited for feedforward control. This is especially true when the delay between a disturbance and the output is larger than the control delay. In this paper, we address the design of feedforward controllers for a general class of 2×2 hyperbolic systems with a single disturbance input located at one boundary and a single control actuation at the other boundary. The goal is to design a feedforward control that makes the system output insensitive to the measured disturbance input. We show that, for this class of systems, there exists an efficient ideal feedforward controller which is causal and stable. The problem is first stated and studied in the frequency domain for a simple linear system. Then, our main contribution is to show how the theory can be extended, in the time domain, to general nonlinear hyperbolic systems. The method is illustrated with an application to the control of an open channel represented by Saint-Venant equations where the objective is to make the output water level insensitive to the variations of the input flow rate. Finally, we address a more complex application to a cascade of pools where a blind application of perfect feedforward control can lead to detrimental oscillations. A pragmatic way of modifying the control law to solve this problem is proposed and validated with a simulation experiment.

Keywords: Feedforward control, Hyperbolic systems, Saint-Venant equations.

1 Introduction

Feedforward control is a technique which is of interest when the system to be controlled is subject to a significant input disturbance that can be measured and compensated before it affects the system output. This control technique is used in many control engineering applications, especially in the industrial process sector (e.g. [30, Chapter 15]). In ideal situations, feedforward control is an open-loop technique which is theoretically able to achieve perfect control by anticipating adequately the effect of the perturbations. This is in contrast with closed-loop feedback control where corrective actions take place necessarily only after the effect of the disturbances has been detected at the output. However ideal feedforward controllers, which are based on some sort of process model inversion, may not

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be physically realizable because they can be non causal and/or unstable (e.g. [12]). In such situations, it is common practice to design approximate low order realizable feedforward controllers, possibly in combination with feedback control (e.g. [15], [16]). Furthermore, it is also well known that, in some instances, feedforward control may be a simple and low cost way of avoiding loss of stability due to actuator saturations in feedback loops (e.g. [8], [21]). For finite dimensional linear systems, the theory of feedforward control is well established and the basics can be found, for instance, in the classical textbooks [27] and [30].

In this paper, we are concerned with the application of the feedforward technique to the boundary control of 1-D hyperbolic systems. Our purpose is to address the design of feedforward controllers for a general class of 2×2 hyperbolic systems with a single disturbance input located at one boundary and a single control actuation at the other boundary. This class of systems includes many potential interesting applications, including those that are listed in the book [3, Chapter 1] for example.

Hyperbolic systems generally represent physical phenomena with important propagation delays. For that reason, they are particularly suitable for the implementation of feedforward control, especially when the input/output disturbance delay is larger than the control delay. In that case, as we shall see in this paper, it is indeed possible to design efficient ideal feedforward controllers that are causal and stable, and significantly improve the system performance.

For hyperbolic systems with **unmeasurable** disturbance inputs produced by a so-called exogenous "signal system", the asymptotic *closed-loop* rejection of disturbances by feedback of measurable outputs was extensively considered in the literature in the recent years, especially in the backstepping framework. Significant contributions on this topic have been published, among others, by Ole Morten Aamo (e.g.[1], [2]) and Joachim Deutsher (e.g.[7], [9]) and their collaborators. It is worth noting that the viewpoint adopted in the present paper is rather different, since we consider systems having an arbitrary and **measurable** disturbance input for which it is desired to design an *open-loop* control of an output variable which is not measured. We show that, for the class of systems considered in this paper, there exists an ideal causal controller that achieves perfect control with stability.

The feedforward control problem considered in this paper is defined and presented in the next Section 2. The physical system to be controlled is described by a 2×2 quasi-linear hyperbolic system with a density H and a flow density Q as state variables. The goal is to design a *feedforward control law* that makes the system output insensitive to the measured disturbance input. The overall control system is represented by the simple block diagram shown in Figure 1.

In Section 3, we first examine the simplest linear case, i.e. a physical system of two linear conservation laws with constant characteristic velocities. The reason for beginning in this way is that it allows an explicit and complete mathematical analysis of the feedforward control design in the frequency domain. Furthermore it also allows to derive an expression of the control law in the time domain that can then be used to justify the feedforward control design in the general nonlinear case.

Section 4 is then devoted to a theoretical analysis of the feedforward control design in the general nonlinear case. It is first shown that there exists an ideal causal feedforward dynamic controller that achieves perfect control. In a second step, sufficient conditions are given under which the controller, in addition to being causal, ensures the stability of the overall control system.

Applications are then presented. First, in Section 5, generalizing the previous results of [5] and [24, Section 9], it is shown how the theory can be directly applied to the control of an open channel whose dynamics are represented by the Saint-Venant equations. The control action is provided by a hydraulic gate at the downstream side of the channel. The control objective is to make the output water level insensitive to the variations of the input flow rate at the upstream side. The method is illustrated with a realistic simulation experiment.

Then in Section 6, we address the more complex application of the control of a long canal made up of a cascade of a large number of successive pools, as it is the case in navigable rivers for instance. It is then shown that, in this case, a blind application of perfect feedforward controllers leads to oscillations in the downstream direction that can be detrimental in practice. A pragmatic and efficient way of modifying the control design to solve this problem is proposed and validated with simulation results.

Remark 1. This paper is devoted to the disturbance compensation by feedforward control with the objective of output regulation. We are in a situation where the control input and the control output are collocated at one boundary, while the disturbance input is anti-collocated at the other boundary. In this framework, it is trivial to extend our design to output trajectory planning (see Remark 2), i.e. for the generation of an input control signal which achieves the tracking of a time varying output reference signal. However we do not address that issue in this paper so as not to burden the presentation. It is important to notice here that the trajectory planning problem is radically different when the control input and the control output are anti-collocated at the two boundaries. In that case, the issue is much more complex and typically addressed using differential flatness. To our knowledge, it is a topic which remains still relatively unexplored in the literature. Typical contributions have been published in specific applications to shallow water equations [23] and to tubular chemical reactors [34]. Motion planning with distributed control for hyperbolic models of water-tank systems is also discussed in [28]. Finally, let us mention that feedforward control for trajectory planning is treated in [29] using a parabolic Hayami model of an open channel.

2 The feedforward control problem

Let us consider a physical system represented by a general 2×2 nonlinear hyperbolic system of the form

$$H_t + Q_x = 0, \tag{1}$$

$$Q_t + (f(H,Q))_x + g(H,Q) = 0, \tag{2}$$

where :

- t and x are the two independent variables: a time variable $t \in [0, +\infty)$ and a space variable $x \in [0, L]$ on a finite interval;
- $(H, Q) : [0, +\infty) \times [0, L] \to \mathbb{R}^2$ is the vector of the two dependent variables (i.e. $H(t, x)$ and $Q(t, x)$ are the two states of the system);
- $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth functions.

The first equation (1) can be interpreted as a mass conservation law with H the density and Q the flux density. The second equation (2) can then be interpreted as a momentum balance law.

We are concerned with the solutions of the Cauchy problem for the system (1)–(2) over $[0, +\infty) \times [0, L]$ under an initial condition:

$$\big(H(0, x), Q(0, x)\big) \qquad x \in [0, L] \tag{3}$$

and two local boundary conditions of the form:

$$\alpha(H(t, 0), Q(t, 0)) = D(t), \quad t \in [0, +\infty), \tag{4}$$

$$\beta(H(t, L), Q(t, L)) = U(t), \quad t \in [0, +\infty), \tag{5}$$

where $\alpha : \mathbb{R}^2 \to \mathbb{R}$, $\beta : \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth functions.

At the left boundary (i.e. $x = 0$), the function $D(t)$ is supposed to be a bounded measurable time-varying disturbance. At the right boundary (i.e. $x = L$), $U(t)$ is a control function that can be freely selected by the operator.

The control objective is to keep the output density $H(t, L)$ insensitive to the variations of the disturbance $D(t)$. More precisely, we consider the problem of finding a *feedforward control law* $U(t)$, function of the measured disturbance $D(t)$, such that the output density $H(t, L)$ is identically equal to a desired value H_L^* (called 'set point'), i.e. $H(t, L) \equiv H_L^*$ $\forall t$. Equivalently it is required that the output function $Y(t) = H(t, L) - H_L^*$ is identically zero along the solutions of the Cauchy problem. In less technical terms, we want a control law which exactly cancels the influence of the left boundary disturbance $D(t)$ on the right boundary state $H(t, L)$. The overall control system configuration is illustrated in Fig.1. As we can see in this figure, we thus have a series interconnection of two dynamical systems: the "Physical System" with output $Y(t)$ and inputs $D(t), U(t)$, and the feedforward controller with output $U(t)$ and input $D(t)$.

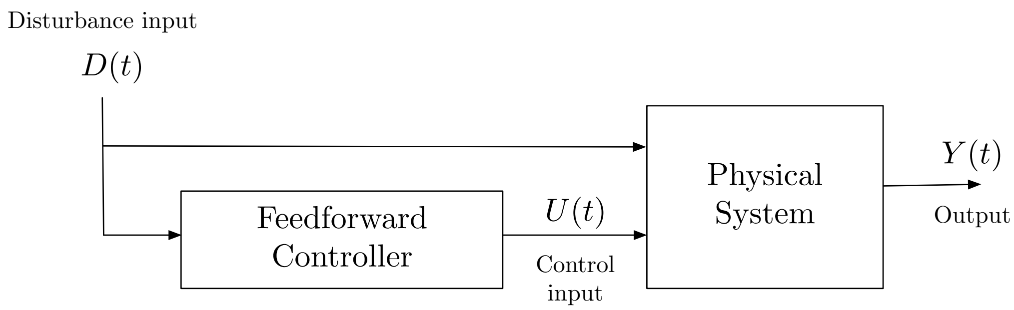


Figure 1: Configuration of the control system with feedforward.

3 A preliminary simple linear case

Let us first examine the special case where the physical system is a simple 2×2 linear hyperbolic system of the form

$$\begin{aligned} H_t + Q_x &= 0 \\ Q_t + \big(aH + bQ\big)_x &= 0, \end{aligned} \tag{6}$$

where a and b are two real positive constants, with boundary conditions

$$Q(t, 0) = D(t), \qquad Q(t, L) - \gamma H(t, L) = U(t), \tag{7}$$

where γ is a real constant.

The system is hyperbolic with one positive and one negative characteristic velocity which are defined as

$$\lambda_1 = \frac{b + \sqrt{b^2 + 4a}}{2} \quad \text{and} \quad -\lambda_2 = \frac{b - \sqrt{b^2 + 4a}}{2}. \tag{8}$$

The reason for beginning in this way is that the simple linear system (6), (7) allows an explicit and complete mathematical analysis of the feedforward control design. It is therefore an excellent starting point before to address the general nonlinear case for which less explicit and more complicated solutions will be discussed later on.

3.1 Feedforward control design in the frequency domain

In order to solve the feedforward control problem for the linear system (6), (7), we introduce the *Riemann coordinates* defined by the following change of coordinates:

$$\begin{aligned} R_1(t,x) &= Q(t,x) - D(0) + \lambda_2(H(t,x) - H_L^*), \\ R_2(t,x) &= Q(t,x) - D(0) - \lambda_1(H(t,x) - H_L^*), \end{aligned} \tag{9}$$

where, as mentioned in the introduction, H_L^* denotes the set point for the output variable $H(t, L)$.

This change of coordinates is inverted as follows:

$$\begin{aligned} H(t,x) - H_L^* &= \frac{R_1(t,x) - R_2(t,x)}{\lambda_1 + \lambda_2}, \\ Q(t,x) - D(0) &= \frac{\lambda_1 R_1(t,x) + \lambda_2 R_2(t,x)}{\lambda_1 + \lambda_2}. \end{aligned} \tag{10}$$

With these Riemann coordinates, the system (6) is rewritten in characteristic form as a set of two transport equations:

$$\begin{aligned} \partial_t R_1(t,x) + \lambda_1 \partial_x R_1(t,x) &= 0, \\ \partial_t R_2(t,x) - \lambda_2 \partial_x R_2(t,x) &= 0, \end{aligned} \tag{11}$$

or, equivalently, as the following two delay equations:

$$\begin{aligned} R_1(t,L) &= R_1(t-\tau_1, 0) \;\; \text{with} \;\; \tau_1 = L/\lambda_1, \\ R_2(t,0) &= R_2(t-\tau_2, L) \;\; \text{with} \;\; \tau_2 = L/\lambda_2. \end{aligned} \tag{12}$$

Taking the Laplace transform of (12), the system is written as follows in the frequency domain (with "s" the Laplace complex variable):

$$\begin{aligned} R_1(s,L) &= e^{-s\tau_1} R_1(s,0), \\ R_2(s,0) &= e^{-s\tau_2} R_2(s,L). \end{aligned} \tag{13}$$

Moreover, the inputs $D(t)$ and $U(t)$ are represented in the frequency domain by the following Laplace transforms:

$$\widetilde{U}(s) = \mathcal{L}\big(U(t) - D(0) + \gamma H_L^*\big), \qquad \widetilde{D}(s) = \mathcal{L}\big(D(t) - D(0)\big), \tag{14}$$

where $\mathcal{L}$ denotes the Laplace transform operator.

Then, using the boundary conditions (7) with the change of coordinates (10), the system equations (13) and the definitions (14), the input-output system dynamics are computed as follows :

$$Y(s) = P_o(s)\widetilde{U}(s) + P_d(s)\widetilde{D}(s), \tag{15}$$

with, for $\tau = \tau_1 + \tau_2$, the two transfer functions:

$$P_o(s) = -\frac{\lambda_1 + \lambda_2 e^{-s\tau}}{\lambda_1(\gamma + \lambda_2) + \lambda_2(\gamma - \lambda_1)e^{-s\tau}} \ , \tag{16}$$

$$P_d(s) = \frac{(\lambda_1 + \lambda_2)e^{-s\tau_1}}{\lambda_1(\gamma + \lambda_2) + \lambda_2(\gamma - \lambda_1)e^{-s\tau}} \ . \tag{17}$$

Let us now assume uniform initial conditions at time $t = 0$:

$$H(0, x) = H_L^*, \qquad Q(0, x) = D(0), \qquad \text{for all } x \in [0, L]. \tag{18}$$

Then it follows from (15) that, in order to satisfy the control objective $H(t, L) = H_L^*$ $\forall t > 0$, or equivalently in Laplace coordinates $Y(s) = 0 \ \forall s$, we must select the feedforward boundary control law $\widetilde{U}(s)$ in function of the boundary disturbance $\widetilde{D}(s)$ such that

$$\widetilde{U}(s) = P_c(s)\, \widetilde{D}(s) \quad \text{with} \quad P_c(s) = -P_o^{-1}(s)\, P_d(s) = \frac{(\lambda_1 + \lambda_2)e^{-s\tau_1}}{\lambda_1 + \lambda_2 e^{-s\tau}}. \tag{19}$$

or, in the time domain, as:

$$U(t) = -\frac{\lambda_2}{\lambda_1}U(t - \tau) + \left(1 + \frac{\lambda_2}{\lambda_1}\right) D(t - \tau_1) - \gamma \left(1 + \frac{\lambda_2}{\lambda_1}\right) H_L^*. \tag{20}$$

Hence, we can see that the feedforward controller $P_c(s)$ seems to achieve the desired purpose. There is however an important limitation: the result is obtained under the assumption that the initial condition (18) is uniform and that the initial output density $H(0, L)$ is already at the set point H_L^*. If this assumption is not verified, then initial transients may appear. Obviously, such transients will vanish exponentially only if the system is exponentially stable. Hence, the practical implementation of the feedforward controller clearly requires the stability of the transfer functions of the system. From (16), (17) we can see that the transfer functions $P_o(s)$ and $P_d(s)$ have the same poles that are stable if and only if λ_1, λ_2 and γ satisfy the following inequality

$$\left| \frac{\gamma - \lambda_1}{\gamma + \lambda_2} \right| < \frac{\lambda_1}{\lambda_2}. \tag{21}$$

Moreover the transfer function $P_c(s)$ of the controller has stable poles if and only if

$$\frac{\lambda_2}{\lambda_1} < 1. \tag{22}$$

These conditions imply in particular that, starting from any arbitrary initial condition, the states of the physical system and the controller are bounded and that $H(t, L)$ asymptotically converges to the set point

$$\lim_{t \to \infty} H(t, L) = H_L^*, \tag{23}$$

such that the feedforward control objective is achieved as soon as the initial transients have vanished. Remark that in the case where Condition (21) is not satisfied, the feedforward controller can be combined with a feedback controller in order to ensure the global system stability. However, for the sake of simplicity and clarity, we will not study this issue in more detail in this article.

3.2 Feedforward control design in the time domain

Let us now show that the same feedforward control law can also be computed in another way, as the output of the following copy of the system (1)-(2):

$$
\begin{aligned}
&\widehat{H}_t + \widehat{Q}_x = 0, \\
&\widehat{Q}_t + \big(a\widehat{H} + b\widehat{Q}\big)_x = 0,
\end{aligned}
\tag{24}
$$

with the boundary conditions:

$$
\begin{aligned}
&\widehat{Q}(t,0) = D(t), \\
&\widehat{H}(t,L) = H_L^*.
\end{aligned}
\tag{25}
$$

We are going to show that the control $U(t)$ given by (20) is equivalently given by:

$$
U(t) = \widehat{Q}(t,L) - \gamma H_L^*. \tag{26}
$$

Again we use the Riemann coordinates

$$
\begin{aligned}
\widehat{R}_1(t,x) &= \widehat{Q}(t,x) - D(0) + \lambda_2(\widehat{H}(t,x) - H_L^*), \\
\widehat{R}_2(t,x) &= \widehat{Q}(t,x) - D(0) - \lambda_1(\widehat{H}(t,x) - H_L^*),
\end{aligned}
\tag{27}
$$

which are inverted as

$$
\begin{aligned}
\widehat{H}(t,x) - H_L^* &= \frac{\widehat{R}_1(t,x) - \widehat{R}_2(t,x)}{\lambda_1 + \lambda_2}, \\
\widehat{Q}(t,x) - D(0) &= \frac{\lambda_1\widehat{R}_1(t,x) + \lambda_2\widehat{R}_2(t,x)}{\lambda_1 + \lambda_2}.
\end{aligned}
\tag{28}
$$

In these coordinates, the system (24) is written as follows:

$$
\begin{aligned}
&\partial_t\widehat{R}_1(t,x) + \lambda_1\partial_x\widehat{R}_1(t,x) = 0, \\
&\partial_t\widehat{R}_2(t,x) - \lambda_2\partial_x\widehat{R}_2(t,x) = 0,
\end{aligned}
\tag{29}
$$

or equivalently as the delay system:

$$
\widehat{R}_1(t,L) = \widehat{R}_1(t-\tau_1,0), \tag{30a}
$$

$$
\widehat{R}_2(t,0) = \widehat{R}_2(t-\tau_2,L). \tag{30b}
$$

In the Riemann coordinates, the boundary conditions (25) are written:

$$
\widehat{R}_1(t,0) = -\frac{\lambda_2}{\lambda_1}\widehat{R}_2(t,0) + \left(1 + \frac{\lambda_2}{\lambda_1}\right)(D(t) - D(0)), \tag{31a}
$$

$$
\widehat{R}_2(t,L) = \widehat{R}_1(t,L). \tag{31b}
$$

From (28), (30a) and (31b), we have:

$$
\begin{aligned}
\widehat{Q}(t,L) = \frac{\lambda_1\widehat{R}_1(t,L) + \lambda_2\widehat{R}_2(t,L)}{\lambda_1 + \lambda_2} + D(0) &= \widehat{R}_1(t,L) + D(0) \\
&= \widehat{R}_1(t-\tau_1,0) + D(0).
\end{aligned}
\tag{32}
$$

Since the boundary condition (31a) is obviously valid at any time instant, and using (32), we have:

$$\begin{aligned}\widehat{Q}(t,L) &= \widehat{R}_1(t-\tau_1,0)+D(0)\\ &= -\frac{\lambda_2}{\lambda_1}\widehat{R}_2(t-\tau_1,0)+\left(1+\frac{\lambda_2}{\lambda_1}\right)\Big(D(t-\tau_1)-D(0)\Big)+D(0).\end{aligned} \tag{33}$$

Moreover, using successively (30b), (31b) and (32), we have:

$$\widehat{R}_2(t-\tau_1,0)=\widehat{R}_2(t-\tau,L)=\widehat{R}_1(t-\tau,L)=\widehat{Q}(t-\tau,L)-D(0). \tag{34}$$

Then, combining (33) and (34), we get:

$$\widehat{Q}(t,L)=-\frac{\lambda_2}{\lambda_1}\widehat{Q}(t-\tau,L)+\left(1+\frac{\lambda_2}{\lambda_1}\right)D(t-\tau_1). \tag{35}$$

Comparing this equation with (20), we finally conclude that, as announced, the control law is given by:

$$U(t)=\widehat{Q}(t,L)-\gamma H_L^*. \tag{36}$$

This expression of the feedforward control law in the time domain is of special interest to motivate its extension to nonlinear systems as we shall see in the next section.

4 The general nonlinear case

In this section, we now address the feedforward control design problem stated in Section 2 for the general nonlinear physical system

$$\begin{aligned}H_t+Q_x&=0,\\ Q_t+(f(H,Q))_x+g(H,Q)&=0,\end{aligned} \tag{37}$$

$$\begin{aligned}\alpha(H(t,0),Q(t,0))&=D(t),\\ \beta(H(t,L),Q(t,L))&=U(t).\end{aligned} \tag{38}$$

The objective is to keep the output density $H(t,L)$ at the set point H_L^* despite the disturbance $D(t)$.

We assume that all the required conditions are met for this system to be well posed and have a unique solution in the domain of interest. The existence and uniqueness of solutions is a topic which is the subject of numerous publications. We do not address this issue in this article but we refer the reader to the paper [32] by Zhiqiang Wang (and the references therein) where explicit conditions for hyperbolic systems of the form (37), (38) are given.

4.1 Feedforward control design

Obviously the frequency method is not relevant for a nonlinear system such as (37), (38). However, from our analysis of the linear case in the previous section, a natural way to generalize the control design to nonlinear systems in the time domain is as follows. We use a copy of the system (37):

$$\begin{aligned}\widehat{H}_t+\widehat{Q}_x&=0,\\ \widehat{Q}_t+(f(\widehat{H},\widehat{Q}))_x+g(\widehat{H},\widehat{Q})&=0,\end{aligned} \tag{39}$$

with the boundary conditions:

$$\begin{aligned}\alpha(\widehat{H}(t,0),\widehat{Q}(t,0)) &= D(t),\\ \widehat{H}(t,L) &= H_L^*,\end{aligned} \tag{40}$$

and with the feedforward control defined as:

$$U(t) = \beta(H_L^*, \widehat{Q}(t,L)). \tag{41}$$

In the next theorem, it is shown that this feedforward controller (39), (40), (41) achieves the desired purpose.

Theorem 1. Assume that both systems (37), (38) and (39), (40) are interconnected with the control law (41) (see Figure 1) and have the same initial condition

$$H(0,x) = \widehat{H}(0,x), \quad Q(0,x) = \widehat{Q}(0,x), \quad \text{for all } x \in [0,L], \tag{42}$$

with $H(0,L) = \widehat{H}(0,L) = H_L^*$. Then, for all positive t it holds that $H(t,L) = H_L^*$.

Proof. Let us first observe that the condition (42) implies not only that the two systems have the same initial condition but also that they have identical boundary conditions at the initial time $t = 0$:

$$D(0) = \alpha(H(0,0), Q(0,0)) = \alpha(\widehat{H}(0,0), \widehat{Q}(0,0)), \tag{43}$$

$$U(0) = \beta(H_L^*, Q(0,L)) = \beta(H_L^*, \widehat{Q}(0,L)). \tag{44}$$

Then it is immediately clear that the solution $\widehat{H}(t,x), \widehat{Q}(t,x)$ (with $\widehat{H}(t,L) = H_L^* \; \forall t$) of the system (39), (40) is also a possible solution of the system (37), (38), i.e $H(t,x) = \widehat{H}(t,x)$ and $Q(t,x)) = \widehat{Q}(t,x)$ for all t and x. Since the solution of the system (37), (38) is unique, the result follows. $\square$

This theorem shows however that the stability issue mentioned in the linear case is still present here. The feedforward control leads to an exact cancellation of the disturbance for all $t \geqslant 0$ only if the physical system and the feedforward controller have exactly identical initial conditions. Otherwise some sort of stability of the solutions is required to guarantee an asymptotic decay of the initial transients. As already pointed out by T. Glad [12] for finite dimensional nonlinear systems, Lyapunov theory helps to discuss this stability issue as we shall see in the next section.

Remark 2. In this paper, for simplicity and clarity, we limit ourselves to the use of feedforward control for exact disturbance compensation. It is however worth noting that the feedforward control design, as formulated above, can trivially be extended to trajectory planning. Indeed, to generate a control input that achieves the tracking of a time varying reference output signal $H_L^{\text{ref}}(t)$, we just have to consider in (40) a boundary condition of the form $\widehat{H}(t,L) = H_L^{\text{ref}}(t)$.

4.2 Stability conditions

Our purpose in this section is to derive sufficient stability conditions for the overall control system (37), (38), (39), (40), (41). In particular, we give conditions that guarantee the robustness of the control with respect to errors in the initial condition (see Theorem 2).

A steady state is a system solution that does not change over time. We assume that, for any set point H^*_L and any given constant disturbance input $D(t) = D^*$ for all t, the system has a unique well-defined steady state $H(t,x) = \widehat{H}(t,x) = H^*(x)$, $Q(t,x) = \widehat{Q}(t,x) = Q^*$ for all t. The steady state flux density Q^* is uniform on the domain $[0,L]$ and the steady state density function $H^*(x)$ is a solution of the ordinary differential equation

$$\big(f(H^*, Q^*)\big)_x + g(H^*, Q^*) = 0, \qquad H^*(L) = H^*_L, \qquad x \in [0,L]. \tag{45}$$

In order to linearize the system, we define the deviations of the disturbance input $D(t)$ and the states $H(t,x)$, $\widehat{H}(t,x)$, $Q(t,x)$, $\widehat{Q}(t,x)$ with respect to the steady states D^*, $H^*(x)$ and Q^*:

$$\begin{aligned} &d(t) = D(t) - D^*, \\ &h(t,x) = H(t,x) - H^*(x), \qquad q(t,x) = Q(t,x) - Q^*, \\ &\hat{h}(t,x) = \widehat{H}(t,x) - H^*(x), \qquad \hat{q}(t,x) = \widehat{Q}(t,x) - Q^*. \end{aligned} \tag{46}$$

With these notations the linearization of the physical system (37), (38) about the steady state is

$$\begin{aligned} &h_t + q_x = 0, \\ &q_t + a(x)h_x + b(x)q_x + \big(a_x(x) + \tilde{a}(x)\big)h + \big(b_x(x) + \tilde{b}(x)\big)q = 0, \end{aligned} \tag{47}$$

with the boundary conditions

$$\begin{aligned} &\alpha_h h(t,0) + \alpha_q q(t,0) = d(t), \\ &\beta_h h(t,L) + \beta_q q(t,L) = \beta_q \hat{q}(t,L). \end{aligned} \tag{48}$$

In these equations, we use the following notations:

$$\begin{aligned} &a(x) = \frac{\partial f}{\partial H}(H^*(x), Q^*), \qquad b(x) = \frac{\partial f}{\partial Q}(H^*(x), Q^*), \\ &\tilde{a}(x) = \frac{\partial g}{\partial H}(H^*(x), Q^*), \qquad \tilde{b}(x) = \frac{\partial g}{\partial Q}(H^*(x), Q^*), \\ &\alpha_h = \frac{\partial \alpha}{\partial H}(H^*(0), Q^*), \qquad \alpha_q = \frac{\partial \alpha}{\partial Q}(H^*(0), Q^*) \\ &\beta_h = \frac{\partial \beta}{\partial H}(H^*_L, Q^*), \qquad \beta_q = \frac{\partial \beta}{\partial Q}(H^*_L, Q^*). \end{aligned} \tag{49}$$

Similarly, the linearization of the controller (39), (40), (41) about the steady state is

$$\begin{aligned} &\hat{h}_t + \hat{q}_x = 0, \\ &\hat{q}_t + a(x)\hat{h}_x + b(x)\hat{q}_x + \big(a_x(x) + \tilde{a}(x)\big)\hat{h} + \big(b_x(x) + \tilde{b}(x)\big)\hat{q} = 0, \end{aligned} \tag{50}$$

with the boundary conditions

$$\begin{aligned} &\alpha_h \hat{h}(t,0) + \alpha_q \hat{q}(t,0) = d(t), \\ &\hat{h}(t,L) = 0. \end{aligned} \tag{51}$$

Let us now introduce the following notations for the deviations between the states of the physical system and the controller:

$$\begin{aligned}\tilde{h}(t,x) &= H(t,x) - \widehat{H}(t,x) = h(t,x) - \hat{h}(t,x),\\ \tilde{q}(t,x) &= Q(t,x) - \widehat{Q}(t,x) = q(t,x) - \hat{q}(t,x).\end{aligned} \tag{52}$$

Then, from (47), (48) we have the following linear 'error' system

$$\begin{aligned}&\tilde{h}_t + \tilde{q}_x = 0,\\ &\tilde{q}_t + a(x)\tilde{h}_x + b(x)\tilde{q}_x + \big(a_x(x) + \tilde{a}(x)\big)\tilde{h} + \big(b_x(x) + \tilde{b}(x)\big)\tilde{q} = 0,\end{aligned} \tag{53}$$

with the boundary conditions

$$\begin{aligned}\alpha_h\tilde{h}(t,0) + \alpha_q\tilde{q}(t,0) = 0,\\ \beta_h\tilde{h}(t,L) + \beta_q\tilde{q}(t,L) = 0.\end{aligned} \tag{54}$$

This error system has clearly a unique uniform steady-state $\tilde{h}(t,x) \equiv 0$, $\tilde{q}(t,x) \equiv 0$. With the definitions

$$\begin{aligned}&\tilde{z} = \begin{pmatrix}\tilde{h}\\ \tilde{q}\end{pmatrix}, \quad A(x) = \begin{pmatrix}0 & 1\\ a(x) & b(x)\end{pmatrix},\\ &B(x) = \begin{pmatrix}0 & 0\\ a_x(x) + \tilde{a}(x) & b_x(x) + \tilde{b}(x)\end{pmatrix},\end{aligned} \tag{55}$$

the system (53) is rewritten in matrix form:

$$\tilde{z}_t + A(x)\tilde{z}_x + B(x)\tilde{z} = 0. \tag{56}$$

Since the system is supposed to be hyperbolic, it is assumed that the matrix $A(x)$ has two real distinct eigenvalues

$$\lambda_1(x) = \frac{b(x) + \sqrt{b^2(x) + 4a(x)}}{2} \quad \text{and} \quad -\lambda_2(x) = \frac{b(x) - \sqrt{b^2(x) + 4a(x)}}{2}, \tag{57}$$

with

$$b^2(x) + 4a(x) > 0 \ \text{ for all } x \in [0,L]. \tag{58}$$

Remark that

$$\lambda_1(x) > 0, \ \ \lambda_2(x) > 0, \tag{59}$$

$$a(x) = \lambda_1(x)\lambda_2(x), \ \ b(x) = \lambda_1(x) - \lambda_2(x), \tag{60}$$

$$b^2(x) + 4a(x) = \big(\lambda_1(x) + \lambda_2(x)\big)^2. \tag{61}$$

Therefore, for all $x \in [0,L]$, the matrix $A(x)$ can be diagonalized with the invertible matrix $N(x)$ defined as

$$N(x) = \begin{pmatrix}\lambda_2(x) & 1\\ -\lambda_1(x) & 1\end{pmatrix} \tag{62}$$

such that

$$N(x)A(x) = \Lambda(x)N(x) \quad \text{with} \quad \Lambda(x) = \begin{pmatrix}\lambda_1(x) & 0\\ 0 & -\lambda_2(x)\end{pmatrix}. \tag{63}$$

In order to address the stability of the linear system (53), (54), we introduce the following basic quadratic Lyapunov function candidate:

$$\mathbf{V} = \int_0^L \big(\tilde{z}^T P(x)\tilde{z}\big)dx \tag{64}$$

with $P(x)$ a symmetric positive definite matrix of the form

$$P(x) = N^T(x)\Delta(x)N(x), \;\; \Delta(x) = \begin{pmatrix} p_1(x) & 0 \\ 0 & p_2(x) \end{pmatrix} \tag{65}$$

where $p_i : [0, L] \to \mathbb{R}_+$ $(i = 1, 2)$ are two real positive functions to be determined.

We compute the time derivative of $\mathbf{V}$ along the C^1-solutions of the system (53), (54):

$$\begin{aligned} \frac{d\mathbf{V}}{dt} &= \int_0^L \big(\tilde{z}^T P(x)\tilde{z}_t + \tilde{z}_t^T P(x)\tilde{z}\big)dx \\ &= -\int_0^L \Big(\tilde{z}^T P(x)\big(A(x)\tilde{z}_x + B(x)\tilde{z}\big) + \big(\tilde{z}_x^T A^T(x) + \tilde{z}^T B^T(x)\big)P(x)\tilde{z}\Big)dx. \end{aligned} \tag{66}$$

Using (63) and (65), we see that the matrix $M(x) = P(x)A(x)$ is symmetric:

$$M(x) = P(x)A(x) = A^T(x)P(x) = N^T(x)\Delta(x)\Lambda(x)N(x). \tag{67}$$

Then, from (66), (67) and using integration by parts, we have

$$\begin{aligned} \frac{d\mathbf{V}}{dt} &= \tilde{z}^T(t,0)M(0)\tilde{z}(t,0) - \tilde{z}^T(t,L)M(L)\tilde{z}(t,L) \\ &\quad - \int_0^L \tilde{z}^T\Big(-M_x(x) + B^T(x)P(x) + P(x)B(x)\Big)\tilde{z}\,dx. \end{aligned} \tag{68}$$

Under the boundary conditions (54), it can be checked that $\tilde{z}^T(t,0)M(0)\tilde{z}(t,0) < 0$ if

(a1) $$\left(\frac{\alpha_h - \alpha_q\lambda_2(0)}{\alpha_h + \alpha_q\lambda_1(0)}\right)^2 < \frac{p_2(0)}{p_1(0)}\frac{\lambda_2(0)}{\lambda_1(0)},$$

and that $-\tilde{z}^T(t,L)M(L)\tilde{z}(t,L) < 0$ if

(a2) $$\left(\frac{\beta_h + \beta_q\lambda_1(L)}{\beta_h - \beta_q\lambda_2(L)}\right)^2 < \frac{p_1(L)}{p_2(L)}\frac{\lambda_1(L)}{\lambda_2(L)}.$$

Hence, from (68), it follows that if the two positive functions $p_i \in C^1([0,L],(0,+\infty))$ $(i = 1, 2)$ can be selected such that conditions (a1) and (a2) are satisfied and

(b) the matrix $-M_x(x) + B^T(x)P(x) + P(x)B(x)$ is positive definite for all $x \in [0, L]$,

then $d\mathbf{V}/dt$ is a negative definite function along the solutions of the system (53), (54), which induces the following stability property because $\mathbf{V}$ is equivalent to a L^2 norm for $\tilde{z}(t,.) \in L^2([0,L],\mathbb{R}^2)$.

Theorem 2. If there exist two functions $p_i \in C^1([0,L],(0,+\infty))$ $(i = 1, 2)$ such that conditions (a1), (a2) and (b) are satisfied, then the system (53), (54) is L^2-exponentially stable, that is there exist two positive constants C and ν such that, from any initial condition $\tilde{z}(0,.) \in L^2([0,L],\mathbb{R}^2)$, the system solution satisfies the inequality

$$\|\tilde{z}(t,.)\|_{L^2} \leqslant Ce^{-\nu t}\|\tilde{z}(0,.)\|_{L^2}, \qquad t \in [0,+\infty). \tag{69}$$

□

This theorem tells us that the solution of the (linearized) physical system asymptotically tracks the solution of the (linearized) controller system, independently of the disturbance. In particular the theorem implies that the output $H(t,L)$ asymptotically converges to the set point:

$$\lim_{t\to\infty} H(t,L) = H_L^* \tag{70}$$

whatever the size and the shape of the disturbance. This can be viewed as a generalization of the condition (21) which was obtained in the simple linear case addressed in Section 3. However, this is not sufficient to conclude that the feedforward control objective is achieved because we are not yet guaranteed that all the initial transients of the overall system will exponentially vanish under the conditions stated in Theorem 2. Indeed, we have to verify in addition that the solutions of the controller itself are not unstable.

For that purpose, let us thus consider the linearized controller system (50) with the boundary conditions (51). It can be observed that the system dynamics are very similar, but not equal, to those of the error system. The only difference lies in the boundary conditions. We can therefore use the same Lyapunov function candidate

$$\hat{\mathbf{V}} = \int_0^L \big(\hat{z}^T P(x)\hat{z}\big)dx \qquad \Big(\text{where } \hat{z} := (\hat{h},\hat{q})^T\Big) \tag{71}$$

for which we have

$$\begin{aligned}\frac{d\hat{\mathbf{V}}}{dt} &= \hat{z}^T(t,0)M(0)\hat{z}(t,0) - \hat{z}^T(t,L)M(L)\hat{z}(t,L)\\ &\quad - \int_0^L \hat{z}^T\Big(-M_x(x) + B^T(x)P(x) + P(x)B(x)\Big)\hat{z}\,dx.\end{aligned} \tag{72}$$

Under the boundary condition (51), we have

$$-\hat{z}^T(t,L)M(L)\hat{z}(t,L) = -\big(p_1(L)\lambda_1(L) - p_2(L)\lambda_2(L)\big)\hat{q}^2(t,L) \tag{73}$$

which is negative if and only if

(a3) $$\frac{p_2(L)\lambda_2(L)}{p_1(L)\lambda_1(L)} < 1.$$

Remark that this condition can be viewed as a generalization of condition (22) which was obtained from a frequency domain approach for the simple linear example of Section 3.

Let us now assume that $\alpha_q \neq 0$ (the case $\alpha_q = 0$ will be considered next). Then, under the boundary condition (51), we have

$$\hat{z}^T(t,0)M(0)\hat{z}(t,0) = -\gamma_0\hat{h}^2(t,0) + \gamma_1\hat{h}(t,0)d(t) + \gamma_2 d^2(t) \tag{74}$$

with

$$\gamma_0 = -p_1(0)\lambda_1(0)\left(\lambda_2(0) - \frac{\alpha_h}{\alpha_q}\right)^2 + p_2(0)\lambda_2(0)\left(\lambda_1(0) + \frac{\alpha_h}{\alpha_q}\right)^2, \tag{75}$$

$$\gamma_1 = \frac{2}{\alpha_q}\Big[p_1(0)\lambda_1(0)\left(\lambda_2(0) - \frac{\alpha_h}{\alpha_q}\right) + p_2(0)\lambda_2(0)\left(\lambda_1(0) + \frac{\alpha_h}{\alpha_q}\right)\Big], \tag{76}$$

$$\gamma_2 = \frac{1}{\alpha_q^2}\big(p_1(0)\lambda_1(0) - p_2(0)\lambda_2(0)\big). \tag{77}$$

In the case where $\alpha_q = 0$ and necessarily $\alpha_h \neq 0$[1], we have

$$\hat{z}^T(t,0)M(0)\hat{z}(t,0) = -\gamma_0\hat{q}^2(t,0) + \gamma_1\hat{q}(t,0)d(t) + \gamma_2 d^2(t) \tag{78}$$

with

$$\gamma_0 = -p_1(0)\lambda_1(0) + p_2(0)\lambda_2(0), \tag{79}$$

$$\gamma_1 = \frac{2\lambda_1(0)\lambda_2(0)}{\alpha_h}\big(p_1(0) + p_2(0)\big), \tag{80}$$

$$\gamma_2 = \frac{\lambda_1(0)\lambda_2(0)}{\alpha_h^2}\big(p_1(0)\lambda_2(0) - p_2(0)\lambda_1(0)\big). \tag{81}$$

In both cases, it can be verified that $\gamma_0 > 0$ if and only if condition (a1) is verified. From, (72), (73), (74), if conditions (a1), (a3) and (b) are satisfied, we can write

$$\frac{d\hat{\mathbf{V}}}{dt} \leqslant -\mu_0 \int_0^L \big(\hat{z}^T\hat{z}\big)dx - \gamma_0\hat{q}^2(t,0) + |\gamma_1||\hat{q}(t,0)||d(t)| + |\gamma_2|d^2(t). \tag{82}$$

where $\mu_0 > 0$ is the infimum over $[0,L]$ of the eigenvalues of the positive definite matrix $-M_x(x) + B^T(x)P(x) + P(x)B(x)$.

Let us now remark that

$$|\gamma_1||\hat{q}(t,0)||d(t)| \leqslant \frac{\gamma_0}{2}\hat{q}^2(t,0) + \frac{\gamma_1^2}{2\gamma_0}d^2(t). \tag{83}$$

Therefore:

$$\frac{d\hat{\mathbf{V}}}{dt} \leqslant -\mu_0 \int_0^L \big(\hat{z}^T\hat{z}\big)dx - \frac{\gamma_0}{2}\hat{q}^2(t,0) + \left(\frac{\gamma_1^2}{2\gamma_0} + |\gamma_2|\right)d^2(t) \tag{84}$$

$$\leqslant -\frac{\mu_0}{\mu_1}\hat{\mathbf{V}} + \left(\frac{\gamma_1^2}{2\gamma_0} + |\gamma_2|\right)d^2(t) \tag{85}$$

where μ_1 is the infimum over $[0,L]$ of the eigenvalues of the positive definite matrix $P(x)$.

Since $\hat{\mathbf{V}}$ is equivalent to the square of the L^2 norm for $\hat{z}(t,.) \in L^2([0,L],\mathbb{R}^2)$, if the disturbance input is bounded, this induces the input-to-state stability property stated in the following proposition.

Theorem 3. If there exist two functions $p_i \in C^1([0,L],(0,+\infty))$ $(i = 1,2)$ such that conditions (a1), (a3) and (b) are satisfied, then the system (53), (54) is L^2-input-to-state stable, that is there exist three positive constants C_1, C_2 and ν such that, from any initial condition $\hat{z}(0,.) \in L^2([0,L],\mathbb{R}^2)$, the system solution satisfies the inequality

$$\|\hat{z}(t,\cdot)\|_{L^2} \leqslant C_1\|\hat{z}(0,.)\|_{L^2}e^{-\nu t} + C_2 \sup_{t\geqslant 0}|d(t)|, \qquad t \in [0,+\infty). \tag{86}$$

□

Hence, if the input disturbance $d(t)$ is bounded, we can conclude from Theorems 2 and 3 that, starting from any arbitrary initial condition in L^2, the states of the (linearized)

[1] The case where both α_h and α_q would equal zero is pointless in the context of this paper because it would correspond to a system without disturbance.

physical system and the (linearized) controller are bounded in L^2 and that $H(t,L)$ asymptotically converges to the set point

$$\lim_{t\to\infty} H(t,L) = H_L^*, \tag{87}$$

such that the feedforward control objective is achieved as soon as the initial transients have vanished.

As it is justified in many recent publications (see e.g. [3, Theorem 6.6] and [6, Section 2.1]), it is also worth noting that the conditions (a1), (a2), (a3) and (b) for the L^2-stability of the linearized system, may also be sufficient to establish the H^2-stability of the overall *nonlinear* system in a neighbourhood of the steady-state. In the nonlinear case, various smallness assumptions have to be made, in particular on the initial conditions and on the disturbances. A rigorous detailed analysis of this generalization is however delicate and would go far beyond the scope of this article. We will limit ourselves here to a more pragmatic approach which consists in checking the applicability and the effectiveness of the method in the realistic nonlinear applications considered in the rest of the paper. In these nonlinear applications, it will be seen that the domain of stability that emerges empirically from the simulations may be fairly large.

To conclude this section, let us also mention that sufficient ISS conditions can also be established for the C^1-norm but the analysis is still more intricate. The interested reader is referred, among others, to the recent publications [22, Section 9.4], [11], [33], [6] and [26, Sections 5.3 and 5.4]. However, it must be said that in many 2×2 physical control systems of practical interest, the ISS conditions are equivalent for the H^2 and C^1 norms. This will appear for instance in the next section where we apply our theory to the example of level control in an open channel.

5 Application to level control in an open channel

In the field of hydraulics, the flow in open channels is generally represented by the Saint-Venant equations which are a typical example of a 2×2 nonlinear hyperbolic system.

We consider the special case of a pool of an open channel as represented in Figure 2. We assume that the channel is horizontal and prismatic with a constant rectangular section.

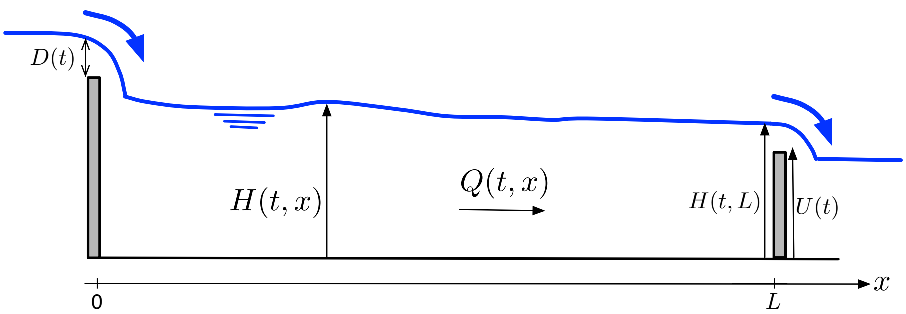


Figure 2: A pool of an open channel with overshot gates at the upstream and downstream sides.

The flow dynamics are described by the Saint-Venant equations

$$\begin{aligned} &H_t + Q_x = 0, \\ &Q_t + \left(\frac{Q^2}{H} + \mathrm{g}\frac{H^2}{2}\right)_x + c_f\frac{Q^2}{H^2} = 0. \end{aligned} \tag{88}$$

where $H(t,x)$ represents the water level and $Q(t,x)$ the water flow rate per unit of width in the pool while g denotes the gravitation constant and c_f is an adimensional friction coefficient. This system is in the form (1), (2) with

$$f(H,Q) = \frac{Q^2}{H} + \mathrm{g}\frac{H^2}{2} \ \text{ and } \ g(H,Q) = c_f\frac{Q^2}{H^2}. \tag{89}$$

The system is subject to the following boundary conditions:

$$\begin{aligned} Q(t,0) &= c_g\sqrt{\left[D(t)\right]^3}, \\ Q(t,L) &= c_g\sqrt{\left[H(t,L) - U(t)\right]^3}. \end{aligned} \tag{90}$$

These boundary conditions are given by standard hydraulic models of overshot gates (see Fig.2). The first boundary condition imposes the value of the canal inflow rate $Q(t,0)$ as a function of the water head above the gate $D(t)$ which is the measurable input disturbance. The second boundary condition corresponds to the control overshot gate at the downstream side of the canal. The control action is the vertical elevation $U(t)$ of the gate. In both models, c_g is a constant discharge coefficient. Let us also remark that these boundary conditions are written in the form (38) as follows:

$$\begin{aligned} \alpha(H(t,0),Q(t,0)) &= \left(c_g^{-1}Q(t,0)\right)^{2/3} = D(t), \\ \beta(H(t,L),Q(t,L)) &= H(t,L) - \left(c_g^{-1}Q(t,L)\right)^{2/3} = U(t). \end{aligned} \tag{91}$$

For a constant gate position $U(t) = U^* > 0 \ \forall t$ and a constant inflow rate $Q(t,0) = Q^* > 0 \ \forall t$, a steady state is a time-invariant solution $H^*(x), Q^*$ given by:

$$H(t,x) = H^*(x) \ \text{ and } \ Q(t,x) = Q^* \quad \forall t, \ \ x \in [0,L], \tag{92a}$$

$$H^*(L) = U^* + (c_g^{-1}Q^*)^{2/3}, \tag{92b}$$

$$H^*(x) \text{ solution of } (\mathrm{g}H^{*3}(x) - Q^{*2})H^*_x(x) + c_f Q^{*2} = 0 \tag{92c}$$

The existence of a solution to (92c) requires that $\mathrm{g}H^{*3}(L) \neq Q^{*2}$. If $\mathrm{g}H^{*3}(L) > Q^{*2}$, then (92c) has a solution (note that $H^*(x)$ is then a decreasing function of x over $[0,L]$) and the steady state flow is subcritical (or fluvial). In such case, according to the physical evidence, $H^*(x)$ is positive :

$$H^*(x) > 0 \ \text{ for all } x \in [0,L], \tag{93}$$

and satisfies the following inequality:

$$0 < \mathrm{g}H^{*3}(x) - Q^{*2}, \quad \forall x \in [0,L]. \tag{94}$$

In the case where $\mathrm{g}H^{*3}(L) < Q^{*2}$, the steady state, if it exists, is said to be supercritical (or torrential). We do not consider that case in the present article.

The control objective is to regulate the level $H(t,L)$ at the set point H_L^*, by acting on the gate position $U(t)$. More precisely, it is requested to adjust the control $U(t)$ in order to have $H(t,L) = H_L^*$ $\forall t$ in spite of the variations of the disturbing inflow rate measured by the signal $D(t)$.

To solve this control problem, the design of linear feedforward controllers based on simplified linear models of open channels with uniform steady states, as we have introduced in Section 3, was addressed previously in [5], [24, Chapters 9 and 10] and [25]. In this article, we extend these results to the general case of open channels with non linear Saint-Venant dynamics and non uniform steady-states.

On the basis of our previous discussions, the feedforward control law is defined as follows:

$$U(t) = H_L^* - (c_g^{-1}\widehat{Q}(t,L))^{2/3} \tag{95}$$

where $\widehat{Q}(t,L)$ is computed with the auxiliary system dynamics as in (39), (40):

$$\begin{gathered} \widehat{H}_t + \widehat{Q}_x = 0, \\ \widehat{Q}_t + \left(\frac{\widehat{Q}^2}{\widehat{H}} + \mathrm{g}\frac{\widehat{H}^2}{2}\right)_x + c_f\frac{\widehat{Q}^2}{\widehat{H}^2} = 0, \end{gathered} \tag{96}$$

$$\begin{gathered} \widehat{Q}(t,0) = c_g\sqrt{\left[D(t)\right]^3}, \\ \widehat{H}(t,L) = H_L^*, \end{gathered} \tag{97}$$

To simplify the notations, we define the steady state water velocity

$$V^*(x) = \frac{Q^*}{H^*(x)} > 0 \ \ \forall x \in [0,L]. \tag{98}$$

With this notation, the subcritical condition (94) is written:

$$\mathrm{g}H^*(x) - V^{*2}(x) > 0 \quad \forall x \in [0,L]. \tag{99}$$

Now, from the linearization of the control system (88), (90), (95), (96), (97), we have, in this application, the following matrices $A(x)$ and $B(x)$:

$$A(x) = \begin{pmatrix} 0 & 1 \\ \mathrm{g}H^*(x) - V^{*2}(x) & 2V^*(x) \end{pmatrix}, \tag{100}$$

$$B(x) = \begin{pmatrix} 0 & 0 \\ -3\dfrac{\mathrm{g}H^*}{V^*}V_x^*(x) & 2\dfrac{\mathrm{g}H^*}{V^{*2}}V_x^*(x) \end{pmatrix}. \tag{101}$$

The eigenvalues of the matrix $A(x)$ are

$$\lambda_1(x) = V^* + \sqrt{gH^*(x)} \quad \text{and} \quad -\lambda_2(x) = V^* - \sqrt{gH^*(x)}. \tag{102}$$

Using these eigenvalues in the matrix $N(x)$ defined in (63), the next step is to select the functions $p_1(x)$ and $p_2(x)$ of the matrix $P(x)$ defined in (65) to build the Lyapunov function candidate (64).

In this application we shall see that it is sufficient to take $p_1 = p_2 =$ constant. With $p_1 = p_2 = \frac{1}{2}$, the matrix $P(x)$ is

$$P(x) = \begin{pmatrix} \mathrm{g}H^*(x) + V^{*2}(x) & -V^*(x) \\ -V^*(x) & 1 \end{pmatrix}. \tag{103}$$

It is readily checked that, for all $x \in [0, L]$, this matrix is positive definite (since $\det P(x) = \mathrm{g}H^*(x)$). It follows that[2]

$$M(x) = P(x)A(x) = \begin{pmatrix} -(\mathrm{g}H^* - V^{*2})V^* & \mathrm{g}H^* - V^{*2} \\ \mathrm{g}H^* - V^{*2} & V^* \end{pmatrix}. \tag{104}$$

Moreover, we have

$$P(x)B(x) + B^T(x)P(x) = \begin{pmatrix} 6gH^*V_x^* & -5\dfrac{gH^*}{V^*}V_x^* \\ -5\dfrac{gH^*}{V^*}V_x^* & 4\dfrac{gH^*}{V^{*2}}V_x^* \end{pmatrix}, \tag{105}$$

while the matrix $-M_x(x)$ is as follows:

$$-M_x(x) = \begin{pmatrix} -3V^{*2}V_x^* & \dfrac{\mathrm{g}H^* + 2V^{*2}}{V^*}V_x^* \\ \dfrac{\mathrm{g}H^* + 2V^{*2}}{V^*}V_x^* & -V_x^* \end{pmatrix}. \tag{106}$$

Then we have

$$\begin{aligned} &- M_x(x) + P(x)B(x) + B^T(x)P(x) = \\ &\qquad\qquad \begin{pmatrix} (6\mathrm{g}H^* - 3V^{*2})V_x^* & \dfrac{-4\mathrm{g}H^* + 2V^{*2}}{V^*}V_x^* \\ \dfrac{-4\mathrm{g}H^* + 2V^{*2}}{V^*}V_x^* & \left(\dfrac{4\mathrm{g}H^* - V^{*2}}{V^{*2}}\right)V_x^* \end{pmatrix}. \end{aligned} \tag{107}$$

Under the subcritical condition (99), this matrix is positive definite for all $x \in [0, L]$ because $V_x^* > 0$ and the determinant is

$$\frac{V_x^*}{V^{*2}}\Big[4(\mathrm{g}H^*)^2 + V^{*2}(2\mathrm{g}H^* - V^{*2}) + 4\mathrm{g}H^*(\mathrm{g}H^* - V^{*2})\Big] > 0. \tag{108}$$

Therefore the stability condition (b) is satisfied. Let us now address the boundary stability conditions relative to the boundaries. For that purpose, from (90), (91), (97), we derive the boundary conditions of the linear error system which are

$$\begin{aligned} &\tilde{q}(t, 0) = 0 \quad \text{(i.e. } \alpha_h = 0, \alpha_q \neq 0), \\ &\tilde{q}(t, L) = \beta_L \tilde{h}(t, L) \ \text{ with } \ \beta_L = \frac{3}{2}\Big(c_g^2 Q^*\Big)^{1/3} \quad \text{(i.e. } \beta_q \neq 0, \beta_h = -\beta_L\beta_q). \end{aligned} \tag{109}$$

[2]From now on, when it does not lead to confusion, we often drop the argument x to simplify the notations.

We can check that the stability conditions (a1), (a2) and (a3) are verified. Indeed we have:

$$\text{(a1)} \quad \Longleftrightarrow \quad \frac{\lambda_2(0)}{\lambda_1(0)} = \frac{\sqrt{gH^*(0)} - V^*(0)}{\sqrt{gH^*(0)} + V^*(0)} < 1.$$

$$\begin{aligned}
\text{(a2)} \quad &\Longleftrightarrow \quad \left(\frac{\lambda_1(L) - \beta_L}{\lambda_2(L) + \beta_L}\right)^2 < \frac{\lambda_1(L)}{\lambda_2(L)} \\
&\Longleftrightarrow \quad \left(\frac{\sqrt{gH^*(L)} + V^*(L) - \beta_L}{\sqrt{gH^*(L)} - V^*(L) + \beta_L}\right)^2 < \frac{\sqrt{gH^*(L)} + V^*(L)}{\sqrt{gH^*(L)} - V^*(L)} \\
&\Longleftrightarrow \quad -V^*(L)\beta_L^2 - \big(gH^*(L) - V^{*2}(L)\big)(2\beta_L - V^*(L)) > 0.
\end{aligned}$$

This inequality is satisfied if $2\beta_L > V^*(L)$.

$$\text{(a3)} \quad \Longleftrightarrow \quad \frac{\lambda_2(L)}{\lambda_1(L)} = \frac{\sqrt{gH^*(L)} - V^*(L)}{\sqrt{gH^*(L)} + V^*(L)} < 1.$$

Furthermore, in this special case of Saint-Venant equations, it is worth noting that conditions (a1), (a2), (a3) and (b) ensure exponential stability not only in L^2 but also in C^0 for the linearized system (and thus locally in C^1 for the nonlinear system). This property follows directly from Theorem 3.2 and Corollary 1 in [18] (see also [17]). This means that, if the disturbance $D(t)$ is bounded, then all the internal signals of the control system are also guaranteed to be bounded.

In this example of an open channel, we thus see that the feedforward control can completely remove the effect of the disturbance while maintaining the system stability. This feedforward control analysis can be extended to the case of a channel with a space varying slope by using the Lyapunov function proposed for instance in [19] and [20]. It is however important to notice that, in real life applications, feedforward control is most often not used alone but in combination with feedback control. Indeed feedback control may be required to reduce the effect of unmeasured disturbances and modelling uncertainties that are always present in any real process. For a channel represented by Saint-Venant equations, with arbitrary friction and slope, the combination of feedforward control and PI feedback control is addressed in reference [19]. In this reference, using a local dissipative entropy, explicit stability conditions on the PI parameters are given that are independent of the slope, the friction and the length of the channel and which achieve the asymptotic compensation of input disturbances by a feedforward action.

Let us now illustrate this feedforward control design with a numerical simulation. The simulation is done with the 'hpde' solver [31]. We consider a pool with the following parameters:

$$\begin{aligned}
&\text{length: } L = 5000 \text{ (meters)}, \\
&\text{friction coefficient: } c_f = 0.01, \\
&\text{discharge coefficient: } c_g = 2 \text{ m}^{1/2}\text{ s}^{-1}.
\end{aligned}$$

At the initial time ($t = 0$), the system is at steady state with a constant flow rate per unit of width $Q^* = 2$ m^2/s and a boundary water level $H(0, L) = 5$ m. The initial steady state profile $H^*(x)$ of the water level is shown in Figure 3 (dotted red curve).

The system is subject to an isolated input disturbance which occurs around $t = 15$ minutes and is shown in Figure 4. The inflow rate per unit of width $Q(t, 0)$ is increased

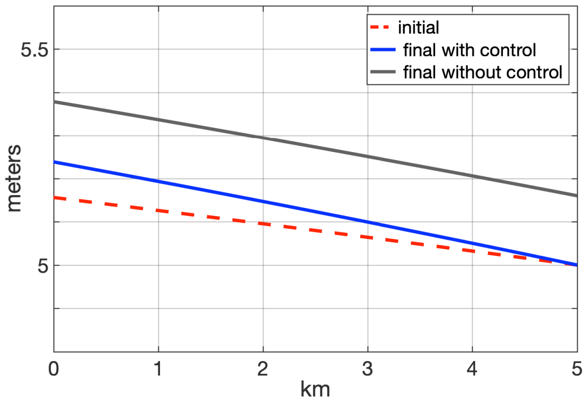


Figure 3: Initial water level profile and final water level profiles with and without control.

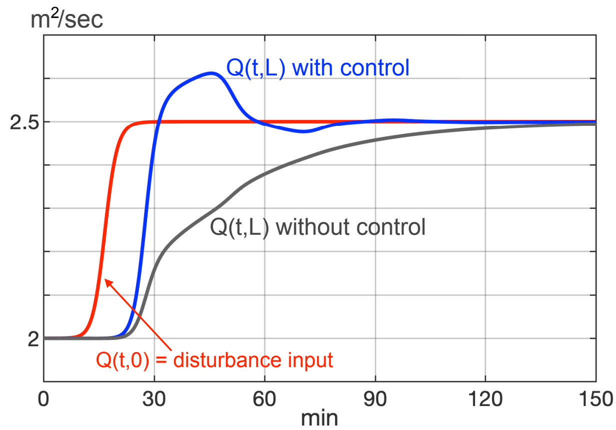


Figure 4: Input and output water flow rates per unit of width

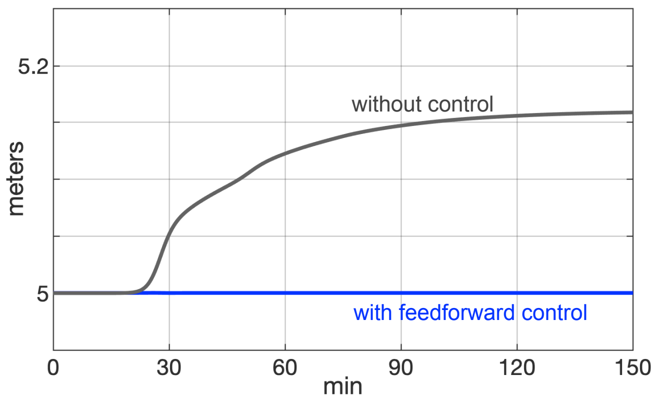


Figure 5: Time evolution of output water level $H(t, L)$

by about 25 %, from 2 to 2.5 m^2/s (red curve). In this figure, we can also see the time evolution of the output flow rate $Q(t, L)$ with and without the feedforward control.

The control result is shown in Figure 5. With the feedforward control we see that the water level $H(t, L)$ (blue curve) remains constant at the set point $H_L^* = 5$ m despite the inflow disturbance. In contrast, without control, the same disturbance leads to an output level increase of about 16 cm (grey curve). The final steady state profile reached after the passage of the disturbance is illustrated in Figure 3.

Finally, let us also compute the parameter β_L for this example. For the initial steady-state, we have:

$$Q^* = 2 \ \mathrm{m}^2/\mathrm{s}, \quad H_L^* = 5 \ \mathrm{m}, \quad V_L^* = 0.4 \ \mathrm{m/s} \ \text{ and } \ \beta_L = \frac{3}{2}\Big(c_g^2 Q^*\Big)^{1/3} = 3\,. \tag{110}$$

For the final steady-state, we have:

$$Q^* = 2.5 \ \mathrm{m}^2/\mathrm{s}, \quad H_L^* = 5 \ \mathrm{m}, \quad V_L^* = 0.5 \ \mathrm{m/s} \ \text{ and } \ \beta_L = \frac{3}{2}\Big(c_g^2 Q^*\Big)^{1/3} = 3.23\,. \tag{111}$$

In both cases, we see that the stability condition $2\beta_L > V_L^*$ holds.

The above simulation is for an ideal situation where the initial condition is supposed to be perfectly known such that the feedforward controller can be exactly initialized (which is obviously unrealistic in real life implementations). Moreover, the disturbance drives the system from one steady-state to another one.

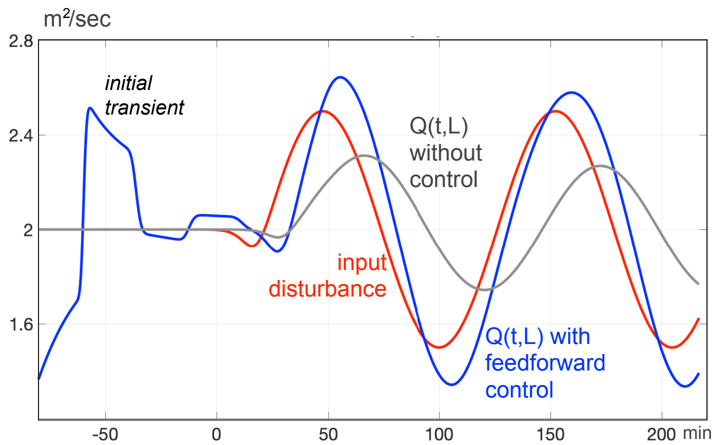


Figure 6: Input and output water flow rates per unit of width.

These limitations are removed in the next simulation whose results are shown in Figures 6 and 7. Here we assume that the control starts at time $t =$ -75 minutes and that, for $t \leqslant 0$, the physical system is initially at steady-state but the initial condition is unknown so that there is an initial error in the control law, and consequently an initial transient occurs. Moreover, we assume that, from $t = 0$, a persistent oscillating disturbance is applied to the system (red curve in Figure 6). The performance of the control can be appreciated in Figure 7. In particular, we can see that the convergence of the output water level $H(t, L)$ to the set point (blue curve in Figure 7) is achieved as soon as the initial transient has vanished, independently of the big permanent oscillating input disturbance shown in Figure 6 (red curve). In contrast, without control, the same disturbance produces an oscillation of the output level with an amplitude of about 10 cm (grey curve in Figure 7).

Let us also remark that systematic simulations of this kind would show that the domain of stability that emerges empirically is fairly large and that the feedforward control works with quite big disturbances and initial errors.

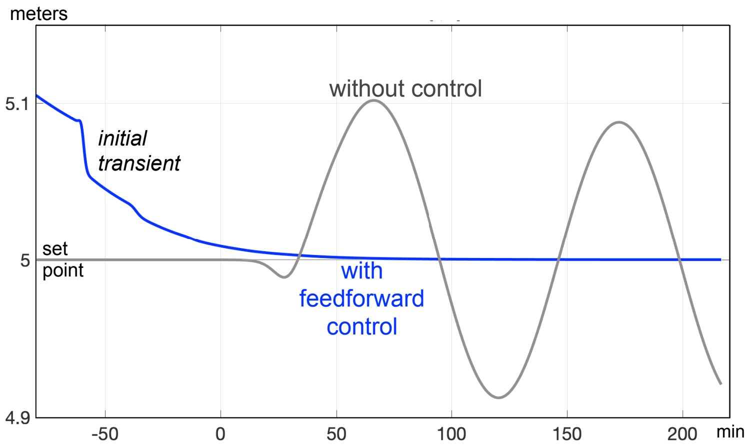


Figure 7: Time evolution of output water level $H(t, L)$.

6 Feedforward control in navigable rivers

In navigable rivers the water is transported along the channel under the power of gravity through successive pools separated by control gates used for the control of the water level as illustrated in Figure 8.

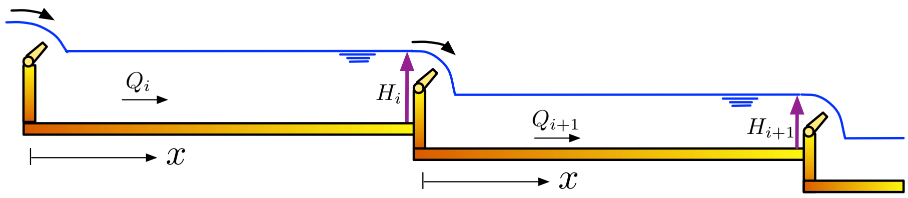


Figure 8: Navigable river

In this section, for simplicity, we consider the ideal case of a sequence of n identical pools having the same length L and the same rectangular cross section. The channel dynamics are described by the following set of Saint-Venant equations

$$\begin{aligned} &\partial_t H_i + \partial_x Q_i = 0, \\ &\partial_t Q_i + \partial_x \left(\frac{Q_i^2}{H_i} + \mathrm{g}\frac{H_i^2}{2} \right) + c_f Q_i^2 = 0. \end{aligned} \qquad i = 1, \ldots, n, \tag{112}$$

and the following set of boundary conditions

$$\begin{aligned} Q_1(t,0) &= c_g \sqrt{\left[D_\mathrm{o}(t)\right]^3}, \\ Q_i(t,0) &= Q_{i-1}(t,L) \qquad i = 2, \ldots, n, \\ Q_i(t,L) &= c_g \sqrt{\left[H_i(t,L) - U_i(t)\right]^3} \qquad i = 1, \ldots, n, \end{aligned} \tag{113}$$

where H_i and Q_i denote the water level and the flow rate per unit of width in the i-th pool, U_i is the position of the i-th gate which is used as control action, c_f and c_g are constant friction and gate shape coefficients respectively, $D_\mathrm{o}(t)$ is the water head above the input gate considered here as the external measurable disturbance. Furthermore it is assumed that the water levels $H_i(t, L)$ are measurable at the downstream side of the pools.

Let us now assume that the objective is to find a set of *feedforward control laws* $U_i(t)$, function of the measured disturbance $D_0(t)$ and the measurable levels $H_i(t,L)$, such that each output $Y_i(t) = H_i(t,L) - H_L^*$ is identically zero.

In this framework, each pool can be considered as a dynamical system with a controlled output $Y_i(t) = H_i(t,L) - H_L^*$ and a disturbance input $D_{i-1}(t)$. It is then natural to design the feedforward control laws U_i for each pool on the pattern of the control which was derived in the previous section for a single pool, as follows:

$$
\begin{aligned}
&\text{For } i = 1, \dots, n, \\
&\partial_t \widehat{H}_i + \partial_x \widehat{Q}_i = 0, && \text{(114a)} \\
&\partial_t \widehat{Q}_i + \partial_x \left(\frac{\widehat{Q}_i^2}{\widehat{H}_i} + \mathrm{g}\frac{\widehat{H}_i^2}{2} \right) + c_f \frac{\widehat{Q}_i^2}{\widehat{H}_i^2} = 0, && \text{(114b)} \\
&\widehat{H}_i(t,L) = H_L^*, && \text{(114c)} \\
&\widehat{Q}_i(t,0) = c_g \sqrt{\left[D_{i-1}(t)\right]^3}, && \text{(114d)} \\
&D_i(t) = H_i(t,L) - U_i(t), && \text{(114e)} \\
&U_i(t) = H_L^* - \left(c_g^{-1}\widehat{Q}_i(t,L)\right)^{2/3}. && \text{(114f)}
\end{aligned}
$$

Theorem 4. For the control system (112), (113), (114), for all $i = 1, \dots, n$, assume that the initial conditions satisfy

$$
H_i(0,x) = \widehat{H}_i(0,x), \quad Q_i(0,x) = \widehat{Q}_i(0,x), \quad \forall x \in [0,L], \tag{115}
$$

with $H_i(0,L) = \widehat{H}_i(0,L) = H_L^*$. Then, for all positive t and for all $i = 1, \dots, n$, it holds that $Y_i(t) = H_i(t,L) - H_L^* = 0$. □

The proof of this theorem is clearly an immediate consequence of the proof of Theorem 1.

In order to discuss the system stability we introduce the following notations:

$$
\begin{aligned}
&\text{For } i = 1, \dots, n, \\
&h_i(t,x) = H_i(t,x) - H^*(x), \qquad q_i(t,x) = Q_i(t,x) - Q^*, \\
&\hat{h}_i(t,x) = \widehat{H}_i(t,x) - H^*(x), \qquad \hat{q}_i(t,x) = \widehat{Q}_i(t,x) - Q^*, \\
&\tilde{h}_i(t,x) = h_i(t,x) - \hat{h}_i(t,x), \qquad \tilde{q}_i(t,x) = q_i(t,x) - \hat{q}_i(t,x), \\
&\tilde{z}_i = \begin{pmatrix} \tilde{h}_i \\ \tilde{q}_i \end{pmatrix}, \quad \hat{z}_i = \begin{pmatrix} \hat{h}_i \\ \hat{q}_i \end{pmatrix}.
\end{aligned} \tag{116}
$$

The linear error system is written

$$
\partial_t \tilde{z}_i + A(x)\partial_x \tilde{z}_i + B(x)\tilde{z}_i = 0, \qquad i = 1, ..., n, \tag{117}
$$

with the boundary conditions

$$
\left.\begin{aligned}
&\tilde{q}_i(t,0) = 0, \\
&\tilde{q}_i(t,L) = \beta_L \tilde{h}_i(t,L),
\end{aligned}\right. \qquad i = 1, ..., n. \tag{118}
$$

Here we see that the error subsystems corresponding to each pool are decoupled. Therefore the stability of the global error system (117), (118) directly results from the stability of the error system for a single pool which was established in Section 5.

On the other hand, the linearization of the controller system is written

$$\partial_t \hat{z}_i + A(x)\partial_x \hat{z}_i + B(x)\hat{z}_i = 0, \qquad i = 1, ..., n, \tag{119}$$

with the boundary conditions

$$\begin{aligned} &\hat{q}_1(t,0) = \alpha_0 d(t), \\ &\hat{q}_i(t,L) = \hat{q}_{i+1}(t,0) - \tilde{q}_i(t,L), \quad i = 1, ..., n-1, \\ &\hat{h}_i(t,L) = 0, \quad i = 1, ..., n. \end{aligned} \tag{120}$$

Here, we remark that the subsystems corresponding to each pool are interconnected through the boundary conditions and their respective stabilities cannot be considered separately. Therefore we introduce the following Lyapunov function candidate

$$\mathbf{V} = \sum_{i=1}^{n} \int_0^L \omega_i \big(\hat{z}_i^T P(x) \hat{z}_i\big) dx \tag{121}$$

where ω_i are positive coefficients to be determined.

The time derivative of this Lyapunov function along the system solutions is then as follows:

$$\frac{d\mathbf{V}}{dt} = \mathcal{I}(t) + \mathcal{B}(t) \tag{122}$$

with the "internal" term

$$\mathcal{I}(t) = -\sum_{i=1}^{n} \omega_i \int_0^L \hat{z}_i^T \Big(- M_x(x) + B^T(x)P(x) + P(x)B(x) \Big) \hat{z}_i \, dx \tag{123}$$

and the "boundary" term

$$\mathcal{B}(t) = -\sum_{i=1}^{n} \omega_i \Big[z_i^T M(x) z_i \Big]_0^L. \tag{124}$$

The matrices $M(x)$ and $-M_x(x)+B^T(x)P(x)+P(x)B(x)$ are those defined in the previous section by equations (104) and (107) respectively. It follows directly that the interior term is negative : $\mathcal{I}(t) < 0$. Moreover, using the definition of $M(x)$ and the boundary conditions (120), the boundary term $\mathcal{B}(t)$ may be written as follows:

$$\begin{aligned} \mathcal{B}(t) = &- \omega_1 V^*(0)(gH^*(0) - V^{*2}(0))h_1^2(t,0) - \omega_n V^*(L)\hat{q}_n^2(t,L) \\ &- \sum_{i=1}^{n-1} \big(\hat{q}_i(t,L) \quad \hat{h}_{i+1}(t,0)\big)\, \Omega_i \begin{pmatrix} \hat{q}_i(t,L) \\ \hat{h}_{i+1}(t,0) \end{pmatrix} \\ &+ 2\,\omega_1 \alpha_0 (gH^*(0) - V^{*2}(0))\hat{h}_1(t,0)d(t) + \omega_1 \alpha_0^2 V^*(0) d^2(t) \\ &+ \sum_{i=1}^{n-1} \omega_{i+1} V^*(0) \Big[-2\hat{q}_i(t,L)\tilde{q}_i(t,L) + V^*(0)\tilde{q}_i^2(t,L) \Big] \end{aligned} \tag{125}$$

where for $i = 1, \ldots, n-1$

$$\Omega_i = \begin{pmatrix} \omega_i V^*(L) - \omega_{i+1} V^*(0) & -\omega_{i+1}(gH^*(0) - V^{*2}(0)) \\ -\omega_{i+1}(gH^*(0) - V^{*2}(0)) & \omega_{i+1}(gH^*(0) - V^{*2}(0))V^*(0) \end{pmatrix}. \tag{126}$$

Assume that the positive coefficients ω_i are selected according to

$$\frac{\omega_{i+1}}{\omega_i} = \varepsilon \tag{127}$$

where ε is a positive constant to be determined. Then

$$\Omega_i = \omega_i \begin{pmatrix} V^*(L) - \varepsilon V^*(0) & -\varepsilon(gH^*(0) - V^{*2}(0)) \\ -\varepsilon(gH^*(0) - V^{*2}(0)) & \varepsilon(gH^*(0) - V^{*2}(0))V^*(0) \end{pmatrix}. \tag{128}$$

Using this expression, it can be seen that, under the subcritical condition (99), each matrix Ω_i is positive definite provided ε is selected such that

$$\varepsilon < \frac{V^*(0)V^*(L)}{gH^*(0)}. \tag{129}$$

Hence it follows that the (linearized) controller system (119), (120), with inputs $d(t)$ and $\tilde{q}_i(t, L)$, is L^2-input-to-state stable with an estimate of the form

$$\|\hat{z}(t,\cdot)\|_{L^2} \leqslant C_1 \|\hat{z}(t,0)\|_{L^2} e^{-\nu t} + C_2 \sup_{t \geqslant 0} \left[|d(t)| + \sum_{i=1}^{n-1} |\tilde{q}_i(t,L)| \right] \tag{130}$$

where ν and C_i $(i = 1, 2)$ are positive constants. Thus, here again, we can conclude that the state of the system is bounded and that the feedforward control objective is achieved. Let us now illustrate the control performance through simulation experiments.

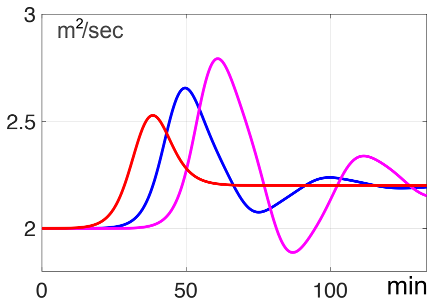
Simulation experiments

We consider a channel with two identical successive pools as shown in Figure 8, with the following parameters:

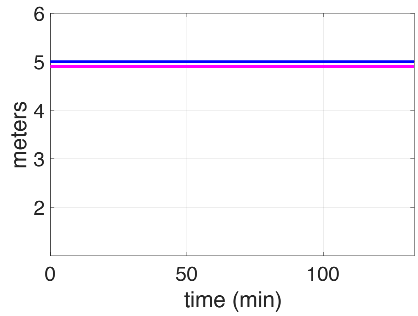
length: $L = 5000$ (meters),
friction coefficient: $c_f = 0.008$,
discharge coefficient: $c_g = 2\ \mathrm{m}^{1/2}\ \mathrm{s}^{-1}$.

The results of a simulation of the feedforward control is given in Figure 9 with set points $H_L^* = 5$ meters in the first pool and $H_L^* = 4.9$ meters in the second pool.

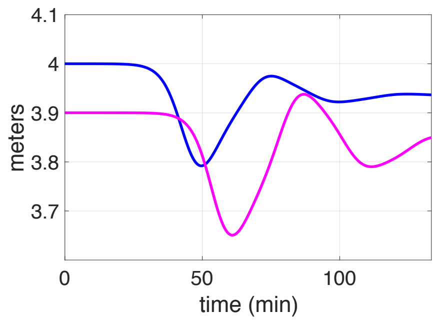
At the initial time ($t = 0$), the system is at steady state with a constant flow rate per unit of width $Q^* = 2\ \mathrm{m}^2/\mathrm{s}$ and boundary water levels $H_1(0, L) = 5$ m and $H_2(0, L) = 4.9$ m respectively. The system is subject to an input disturbance which occurs around $t = 15$ minutes and is shown in Figure 9a (red curve). This disturbance takes the form of a pulse starting from the steady-state value 2 m^2/s, then peaking at 2.5 m^2/s (i.e. an increase of about 25 %), and finally stabilizing at 2.2 m^2/s. In this figure, we can also see the time evolution of the flow rates per unit of width $Q(t, L)$ in the two pools under the feedforward control (blue and magenta curves). The control actions computed by the two feedforward controllers are shown in Figure 9c. Obviously, as expected, we can see in Figure 9b that



(a) Flow rate per unit of width $Q(t, L)$



(b) Water level $H(t, L)$



(c) Control actions $U(t)$

Figure 9: Simulation of feedforward control in a channel with two successive pools: ■ = disturbance input, ■ = first pool, ■ = second pool.

the feedforward control is perfectly efficient and that the water levels $H(t, L)$ in the two pools are totally insensitive to the disturbance.

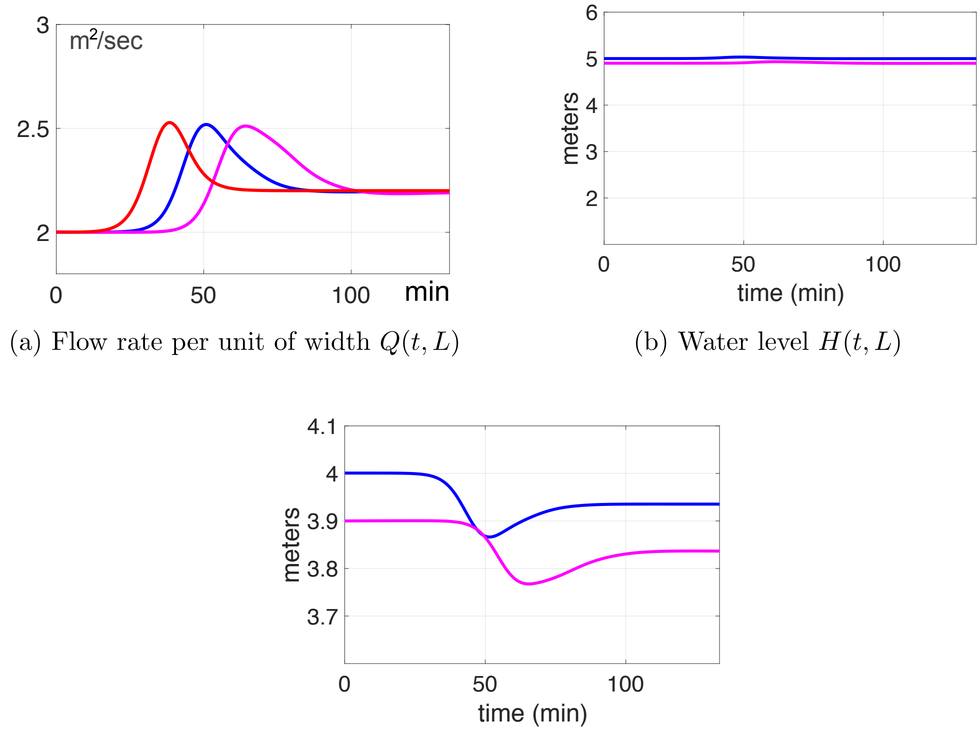
The simulation is done with a friction coefficient which is in the range of usual values for natural channels and rivers (for a sufficiently wide chanel and a water depth of about 5 meters, the friction coefficient $c_f = 0.008$ is equivalent to a Manning-Strickler coefficient $K_s \simeq 27 \text{ m}^{1/3}/\text{sec}$). In this case, there is however a drawback, that is very visible in Figure 9a, under the form of an amplification of the oscillations of the flow rates in the downstream direction. This can be detrimental in some practical applications. In order to mitigate this phenomenon, some filtering of the control must be applied.

A simple very natural and efficient way to implement such filtering is to fictitiously increase the value of the friction coefficient in the feedforward controller. This strategy is illustrated in Figure 10 where the control laws (114) are implemented with a fake overestimated value of the friction $\hat{c}_f = 0.024$. The nice performance of this control can be appreciated in Figures 10a and 10b. Indeed, it can be observed that, in this case, the flow rates are no longer amplified in the downstream direction while the water levels remain nevertheless rather insensitive to the disturbance effect.

7 Conclusions

In this paper, we have addressed the design of feedforward controllers for a general class of 2×2 hyperbolic systems with a disturbance input located at one boundary and a control actuation at the other boundary. The goal is to design a feedforward control that makes the system output insensitive to a measured disturbance input.

The problem was first stated and studied in the frequency domain for a simple linear



(a) Flow rate per unit of width $Q(t, L)$

(b) Water level $H(t, L)$

(c) Control actions $U(t)$

Figure 10: Reduced amplification of the flow oscillations with a modified feedforward controller: ■ = disturbance input, ■ = first pool, ■ = second pool.

system. Then, our main contribution was to extend the theory, in the time domain, to general nonlinear hyperbolic systems. First it has been shown that there exists an ideal causal feedforward dynamic controller that achieves perfect control. In a second step, sufficient conditions have been given under which the controller, in addition to being causal, ensures the stability of the overall control system.

The method has been illustrated with an application to the control of an open channel represented by Saint-Venant equations where the objective is to make the output water level insensitive to the variations of the input flow rate. In the last section, we have discussed a more complex application to a cascade of pools where a blind application of perfect feedforward control can lead to detrimental oscillations. A pragmatic way of modifying the control law to solve this problem has been proposed and validated with a simulation experiment.

Finally, we would also like to mention that the application to the Saint-Venant equations with hydraulic gates can be transposed to gas pipelines with compressors described by the isentropic Euler equations. The interested reader can consult the references [13], [10], [14] and [4].

Acknowledgments

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