

Wellposedness, spectral analysis and asymptotic stability of a multilayered heat-wave-wave system

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Abstract

In this work we consider a multilayered heat-wave system where a 3-D heat equation is coupled with a 3-D wave equation via a 2-D interface whose dynamics is described by a 2-D wave equation. This system can be viewed as a simplification of a certain fluid-structure interaction (FSI) PDE model where the structure is of composite-type; namely it consists of a “thin” layer and a “thick” layer. We associate the wellposedness of the system with a strongly continuous semigroup and establish its asymptotic decay.

Our first result is semigroup well-posedness for the (FSI) PDE dynamics. Utilizing here a Lumer-Phillips approach, we show that the fluid-structure system generates a C_0 -semigroup on a chosen finite energy space of data. As our second result, we prove that the solution to the (FSI) dynamics generated by the C_0 -semigroup tends asymptotically to the zero state for all initial data. That is, the semigroup of the (FSI) system is strongly stable. For this stability work, we analyze the spectrum of the generator $\mathbf{A}$ and show that the spectrum of $\mathbf{A}$ does not intersect the imaginary axis.

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1. Introduction

1.1. Motivation and literature

This work is motivated by a longstanding interest in the analysis of fluid-structure interaction (FSI) partial differential equation (PDE) dynamics. Such FSI problems deal with multi-physics systems consisting of fluid and structure PDE components. These systems are ubiquitous in nature and have many applications, e.g., in biomedicine [10] and aeroelasticity [22]. However, the resulting PDE systems are very complicated (due to nonlinearities, moving boundary phenomena and hyperbolic-parabolic coupling) and despite extensive research activity in last 20 years, the comprehensive analytic theory for such systems is still not available. Accordingly, by way of obtaining a better understanding of FSI dynamics, it would seem natural to consider those FSI PDE models, which although constitute a simplification of sorts, yet retain their crucial novelties and intrinsic difficulties. For example, in the past, coupled heat-wave PDE systems (and variations thereof) have been considered for study: the heat equation component is regarded as a simplification of the fluid flow component of the FSI dynamics; the wave equation component is regarded as a simplification of the structural (elastic) component; see e.g., [[30], Section 9] and [38]. See also the works [20,2,8,15,19], in which the fluid PDE component of fluid-structure interactions is governed by Stokes or Navier-Stokes flow.

Here we consider a multilayered version of such heat-wave system; where the coupling of the 3-D heat and the 3-D wave equations is realized via an additional 2-D wave equation on the boundary interface. This is a simplified (yet physically relevant) version of a benchmark fluid-component structure PDE model which was introduced in [37]. This particular FSI problem was principally motivated by the mathematical modeling of vascular blood flow: such modeling PDE dynamics will account for the fact that the blood-transporting vessels are generally composed of several layers, each with different mechanical properties and are moreover separated by the thin elastic laminae (see [13] for more details). In order to mathematically model these biological features, the multilayered structural component of such FSI dynamics is governed by a 3-D wave-2-D wave PDE system. For the physical interpretation and derivation of such coupled “thick-thin” structure models we refer reader to [17, Chapter 2] and references within.

As we said, although the present multilayered heat-wave-wave system constitutes a simplification somewhat of the FSI model in [37] – in particular, the 2-D wave equation takes the place of a fourth order plate or shell PDE – our results remain valid if we replace the 2-D wave equation with the corresponding linear fourth order equation. Within the context of the present multilayered heat-wave-wave coupled system, we are interested in asymptotic behavior of the solutions, and regularization effects of the fluid dissipation and coupling via the elastic interface, inasmuch as there is a dissipation of the natural energy of the heat-wave-wave PDE system – with this dissipation coming strictly from the heat component of the FSI dynamics – it is a reasonable objective to determine if this thermal dissipation actually gives rise to asymptotic decay (at least) to all three PDE solution components: That is, we seek to ascertain longtime decay of both 3-D and 2-D wave solution components, as well as the heat solution component. Such a strong stability can be seen as a measure of the “strength” of the coupling condition. For the classical heat-wave system (without the 2-D wave equation on the interface) this question is by now rather well understood and precise decay rates are well known (see [3,9] and references within). (We should emphasize that the high-frequency oscillations in the structure are not efficiently dissipated and therefore there is no exponential decay of the energy.)

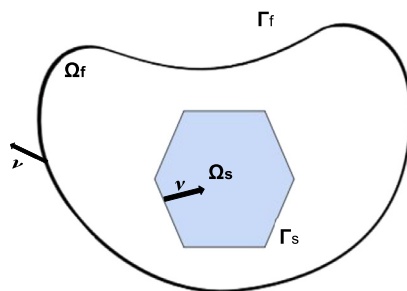


Fig. 1. Geometry of the FSI Domain.

Our present investigation into the multilayered wave-heat systems is motivated in part by [37] which considered a nonlinear FSI comprised by 2-D (thick layer) wave equation and 1-D wave equation (thin layer) coupled to a 2-D fluid PDE across a boundary interface. For these dynamics, wellposedness was established in [37], in part by exploiting an underlying regularity which was available by the presence of said wave equation. (Such regularizing effects were observed numerically in [13] and precisely quantified in the sense of Sobolev for a 1-D FSI system in [36]. For similar regularizing effects in the context of hyperbolic-hyperbolic PDE couplings, we refer to [29,32,33].) By way of gaining a better qualitative understanding of FSI systems, such as those in [37], we here embark upon an investigation of the aforesaid 3-D heat-2-D wave-3-D wave coupled PDE system; in particular, we will establish the semigroup wellposedness and asymptotic decay to zero of the underlying energy of this FSI. These objectives of wellposedness and decay will entail a precise understanding of the role played by the coupling mechanisms on the elastic interface and by the fluid dissipation. In future work, we will investigate possible regularizing effects, at least for certain polygonal configurations of the boundary interface.

We finish this section by giving a brief literature review, in addition to the ones mentioned above. FSI models have been very active and broad area of research in the last two decades and therefore here we avoid presenting a full literature review: we merely mention here a few recent monographs and review works [10,11,14,21,31,39], where interested reader can find further references. The study of various simplified FSI models which manifest parabolic-hyperbolic coupling has a long history going back at least to [[30], Section 9], where the Navier-Stokes equations are coupled with the wave equation along a fixed interface. However, even in the linear case the presence of the pressure term gives rise to significant mathematical challenges in developing the semigroup wellposedness theory [4]. Thus, the heat-wave system has been extensively studied in last decade as a suitable simplified model for stability analysis of parabolic-hyperbolic coupling occurring in FSI systems, see e.g. [1,5,23,28,41] and references within. To the best of our knowledge there are still no results about strong stability of FSI systems with an elastic interface.

1.2. PDE model

Let the fluid geometry $\Omega_f \subseteq \mathbb{R}^3$ be a Lipschitz, bounded domain. The structure domain $\Omega_s \subseteq \mathbb{R}^3$ will be “completely immersed” in Ω_f ; with Ω_s being a convex polyhedral domain.

In Fig. 1, Γ_f is the part of boundary of $\partial\Omega_f$ which does not come into contact with Ω_s ; $\Gamma_s = \partial\Omega_s$ is the boundary interface between Ω_f and Ω_s wherein the coupling between the two distinct fluid and elastic dynamics occurs. (And so, $\partial\Omega_f = \Gamma_s \cup \Gamma_f$.) We have that

$$\Gamma_s = \cup_{j=1}^K \bar{\Gamma}_j, \quad (1)$$

where $\Gamma_i \cap \Gamma_j = \emptyset$, for $i \neq j$. It is further assumed that each Γ_j is an open polygonal domain.

Moreover, n_j will denote the unit normal vector which is exterior to $\partial\Gamma_j$, $1 \leq j \leq K$. With respect to this geometry, the $\mathbb{R}^3$ wave– $\mathbb{R}^2$ wave– $\mathbb{R}^3$ heat interaction PDE model is given as follows: For $i \leq j \leq K$,

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0, T) \times \Omega_f \\ u|_{\Gamma_f} = 0 & \text{on } (0, T) \times \Gamma_f; \end{cases} \quad (2)$$

$$\begin{cases} \frac{\partial^2}{\partial t^2} h_j - \Delta h_j + h_j = \frac{\partial w}{\partial \nu}|_{\Gamma_j} - \frac{\partial u}{\partial \nu}|_{\Gamma_j} & \text{on } (0, T) \times \Gamma_j \\ h_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_l|_{\partial\Gamma_j \cap \partial\Gamma_l} & \text{on } (0, T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \\ \frac{\partial h_j}{\partial n_j} \Big|_{\partial\Gamma_j \cap \partial\Gamma_l} = - \frac{\partial h_l}{\partial n_l} \Big|_{\partial\Gamma_j \cap \partial\Gamma_l} & \text{on } (0, T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \\ \text{such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset. \end{cases} \quad (3)$$

$$\begin{cases} w_t - \Delta w = 0 & \text{on } (0, T) \times \Omega_s \\ w_t|_{\Gamma_j} = \frac{\partial}{\partial t} h_j = u|_{\Gamma_j} & \text{on } (0, T) \times \Gamma_j, \text{ for } j = 1, \dots, K \end{cases} \quad (4)$$

$$[u(0), h_1(0), \frac{\partial}{\partial t} h_1(0), \dots, h_K(0), \frac{\partial}{\partial t} h_K(0), w(0), w_t(0)] = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1]. \quad (5)$$

Equation (3)₁ is the dynamic coupling condition and represents a balance of forces on Γ_j . The left-hand side comes from the inertia and elastic energy of the thin structure, while the right-hand side accounts for the contact forces coming from the 3-D structure and the fluid, respectively. The last term of the left-hand side is added to ensure the uniqueness of the solution and physically means that the structure is anchored and therefore the displacement does not have a translational component. The coupling conditions (3)₂ and (3)₃ represent continuity of the displacement and contact force along the interface between sides Γ_i and Γ_l , respectively. Equation (4)₂ is a kinematic coupling condition and accounts for continuity of the velocity across the interface Γ_j . It corresponds to the no-slip boundary condition in fluid mechanics. Note that the boundary condition in (4) implies that for $t > 0$,

$$w(t)|_{\Gamma_j} - h_j(t) = w(0)|_{\Gamma_j} - h_j(0), \quad \text{for } j = 1, \dots, K.$$

Accordingly, the associated space of initial data $\mathbf{H}$ incorporates a compatibility condition. Namely,

$$\begin{aligned} \mathbf{H} = \{ & [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in L^2(\Omega_f) \times H^1(\Gamma_1) \times L^2(\Gamma_1) \times \dots \\ & \times H^1(\Gamma_K) \times L^2(\Gamma_K) \times H^1(\Omega_s) \times L^2(\Omega_s), \text{ such that for each } 1 \leq j \leq K : \\ & \text{(i) } w_0|_{\Gamma_j} = h_{0j}; \\ & \text{(ii) } h_{0j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{0l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \}. \end{aligned} \quad (6)$$

Because of the given boundary interface compatibility condition, $\mathbf{H}$ is a Hilbert space with the inner product

$$\begin{aligned}
(\Phi_0, \tilde{\Phi}_0)_{\mathbf{H}} = & (u_0, \tilde{u}_0)_{\Omega_f} + \sum_{j=1}^K (\nabla h_{0j}, \nabla \tilde{h}_{0j})_{\Gamma_j} + \sum_{j=1}^K (h_{0j}, \tilde{h}_{0j})_{\Gamma_j} \\
& + \sum_{j=1}^K (h_{1j}, \tilde{h}_{1j})_{\Gamma_j} + (\nabla w_0, \nabla \tilde{w}_0)_{\Omega_s} + (w_1, \tilde{w}_1)_{\Omega_s},
\end{aligned} \quad (7)$$

where

$$\Phi_0 = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H}; \quad \tilde{\Phi}_0 = [\tilde{u}_0, \tilde{h}_{01}, \tilde{h}_{11}, \dots, \tilde{h}_{0K}, \tilde{h}_{1K}, \tilde{w}_0, \tilde{w}_1] \in \mathbf{H}. \quad (8)$$

1.3. Novelty and challenges

The novelty of this work is that we consider an FSI model in which the interface is elastic and has mass. This is the simplest model 3-D of the interaction of the fluid with the composite structure which retains basic mathematical properties of the physical model. To the best of our knowledge this is the first result about asymptotic behavior of solution to such problems. We work in setting where the structure domain is polyhedron and dynamics of each polygon side of the boundary is governed by the 2-D linear wave equation. The wave equations are coupled via dynamic and kinematic coupling conditions over the common boundaries. We choose this setting because it will directly translate to numerical analysis of the problem. This work is an important first step to a finer analysis of the asymptotic decay (e.g. decay rates) and regularity properties of the solutions, and to better understanding of the influence of the elastic interface with mass to the qualitative properties of the solutions.

By way of establishing the semigroup wellposedness of the multilayered FSI model (2)-(5) – i.e., Theorem 1 below – we will show that the associated generator $\mathbf{A}$, defined by (10) and (A.i)-(A.iv) below, is maximal dissipative, and so generates a C_0 -semigroup of contractions on the natural Hilbert space of finite energy (21). The presence of the “thin layer” wave equation on Γ_j , $1 \leq j \leq K$, complicates this wellposedness work, vis-à-vis the situation which prevails for the previous 3-D heat-3-D wave models in [3,5,23,38,41] for which a relatively straight invocation of the Lax-Milgram Theorem suffices to establish the maximality of the associated FSI generator. In the present work, we will likewise apply Lax-Milgram in order to ultimately show the condition $\text{Range}(\lambda I - \mathbf{A}) = \mathbf{H}$ – where $\lambda > 0$ positive; in particular, Lax-Milgram will be applied for the solvability of a certain variational equation, relative to elements in a certain subspace of $H^1(\Omega_f) \times H^1(\Gamma_1) \times \dots \times H^1(\Gamma_K) \times H^1(\Omega_s)$. (See (24) below.) This variational equation of course reflects the presence of the thin wave components h_j in (2)-(5). The complications arise in the subsequent justification that the solutions of said variational equation give rise to solutions of the resolvent equation (in (15) below) which are indeed in $D(\mathbf{A})$. In particular, we must proceed delicately to show that the obtained thin layer solution components of resolvent relation (15) satisfy the continuity conditions (3)₂ and (3)₃.

Having established the existence of a C_0 -semigroup of contractions $\{e^{\mathbf{A}t}\}_{t \geq 0} \subset \mathcal{L}(\mathbf{H})$ which models the multilayer FSI PDE dynamics (2)-(5), we will subsequently show the strong decay of this semigroup; this is Theorem 2 below. Inasmuch as our analysis of the regularizing effects of the resolvent operator $\mathcal{R}(\lambda; \mathbf{A})$ is to be undertaken in future work – assuming there be such underlying smoothness, at least for some geometrical configurations of the polygonal boundary

segments; see Remark 5 below – the compactness of $D(\mathbf{A})$ is generally questionable. Accordingly, in order to establish asymptotic decay of solutions to the FSI PDE dynamics (2)–(5), we will work to satisfy the conditions of the wellknown [1]; see also [34]. In particular, we will show below that $\sigma(\mathbf{A}) \cap i\mathbb{R} = \emptyset$. (In our future work on discerning uniform decay properties of solutions to the multilayered FSI system (2)–(5), the spectral information in Theorem 2 is also requisite; see e.g., the resolvent criteria in [27] and [12].) In showing the nonpresence of $\sigma(\mathbf{A})$ on the imaginary axis – in particular, to handle the continuous spectrum of $\mathbf{A}$ – we will proceed in a manner somewhat analogous to what was undertaken in [7] (in which another coupled PDE system, with the coupling accomplished across a boundary interface, is analyzed with a view towards stability). However, the thin layer wave equation in (3) again gives rise to complications: In the course of eliminating the possibility of approximate spectrum of $\mathbf{A}$ on $i\mathbb{R}$, we find it necessary to invoke the wave multipliers which are used in PDE control theory for *uniform* stabilization of boundary controlled waves: namely, inasmuch as each h_j -wave equation in (3) carries the difference of the 3-D wave and heat fluxes as a forcing term, we cannot immediately control the thick wave trace $\frac{\partial w}{\partial \nu}|_{\Gamma_s}$ in $H^{-\frac{1}{2}}(\Gamma_s)$ -norm, this control being needed for strong decay. (This issue absolutely does not appear for the previously considered 3-D heat-3-D wave FSI models of [23] and the other mentioned works, since therein we have only the difference of heat and wave fluxes as a coupling boundary condition, which immediately leads to a decent $H^{-\frac{1}{2}}(\Gamma_s)$ estimate of the wave normal derivative, owing to the thermal dissipation.) Consequently, we must invoke static versions of the wave identities in [14], [40] and [6], by way of estimating the normal derivative of (a component of) the 3-D wave solution variable w in (4); see relation (74) below.

1.4. Notation

For the remainder of the text norms $\|\cdot\|$ are taken to be $L^2(D)$ for the domain D . Inner products in $L^2(D)$ is written $(\cdot, \cdot)$, while inner products $L_2(\partial D)$ are written $\langle \cdot, \cdot \rangle$. The space $H^s(D)$ will denote the Sobolev space of order s , defined on a domain D , and $H_0^s(D)$ denotes the closure of $C_0^\infty(D)$ in the $H^s(D)$ norm which we denote by $\|\cdot\|_{H^s(D)}$ or $\|\cdot\|_{s,D}$. We make use of the standard notation for the trace of functions defined on a Lipschitz domain D , i.e. for a scalar function $\phi \in H^1(D)$, we denote $\gamma(w)$ to be the trace mapping from $H^1(D)$ to $H^{1/2}(\partial D)$. We will also denote pertinent duality pairings as $(\cdot, \cdot)_{X \times X'}$.

2. Main results

2.1. The thick wave-thin wave-heat generator

With respect to the above setting, the PDE system given in (2)–(5) can be recast as an ODE in Hilbert space $\mathbf{H}$. That is, if $\Phi(t) = [u, h_1, \frac{\partial}{\partial t} h_1, \dots, h_K, \frac{\partial}{\partial t} h_K, w, w_t] \in C([0, T]; \mathbf{H})$ solves (2)–(5) for $\Phi_0 \in \mathbf{H}$, then there is a modeling operator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ such that $\Phi(\cdot)$ satisfies,

$$\frac{d}{dt} \Phi(t) = \mathbf{A} \Phi(t); \quad \Phi(0) = \Phi_0. \quad (9)$$

In fact, this operator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ is defined as follows:

$$\mathbf{A} = \begin{bmatrix} \Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial \nu}|_{\Gamma_1} & (\Delta - I) & 0 & \cdots & 0 & 0 & \frac{\partial}{\partial \nu}|_{\Gamma_1} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 \\ -\frac{\partial}{\partial \nu}|_{\Gamma_K} & 0 & 0 & \cdots & (\Delta - I) & 0 & \frac{\partial}{\partial \nu}|_{\Gamma_K} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Delta & 0 \end{bmatrix}; \quad (10)$$

$D(\mathbf{A}) = \{[u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H} :$

(A.i) $u_0 \in H^1(\Omega_f)$, $h_{1j} \in H^1(\Gamma_j)$ for $1 \leq j \leq K$, $w_1 \in H^1(\Omega_s)$;

(A.ii) (a) $\Delta u_0 \in L^2(\Omega_f)$, $\Delta w_0 \in L^2(\Omega_s)$, (b) $\Delta h_{0j} - \frac{\partial u_0}{\partial \nu}|_{\Gamma_j} + \frac{\partial w_0}{\partial \nu}|_{\Gamma_j} \in L^2(\Gamma_j)$ for $1 \leq j \leq K$;

(c) $\frac{\partial h_{0j}}{\partial n_j} \Big|_{\partial \Gamma_j} \in H^{-\frac{1}{2}}(\partial \Gamma_j)$, for $1 \leq j \leq K$;

(A.iii) $u_0|_{\Gamma_f} = 0$, $u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j}$, for $1 \leq j \leq K$;

(A.iv) For $1 \leq j \leq K$:

(a) $h_{1j}|_{\partial \Gamma_j \cap \partial \Gamma_l} = h_{1l}|_{\partial \Gamma_j \cap \partial \Gamma_l}$ on $\partial \Gamma_j \cap \partial \Gamma_l$, for all $1 \leq l \leq K$ such that $\partial \Gamma_j \cap \partial \Gamma_l \neq \emptyset$;

(b) $\frac{\partial h_{0j}}{\partial n_j} \Big|_{\partial \Gamma_j \cap \partial \Gamma_l} = -\frac{\partial h_{0l}}{\partial n_l} \Big|_{\partial \Gamma_j \cap \partial \Gamma_l}$ on $\partial \Gamma_j \cap \partial \Gamma_l$, for all $1 \leq l \leq K$ such that $\partial \Gamma_j \cap \partial \Gamma_l \neq \emptyset$ }.

(11)

Now, in our first result, we provide a semigroup wellposedness for $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$. This is given in the following theorem:

Theorem 1. *The operator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$, defined in (10)–(11), generates a C_0 -semigroup of contractions. Consequently, the solution $\Phi(t) = [u, h_1, \frac{\partial}{\partial t} h_1, \dots, h_K, \frac{\partial}{\partial t} h_K, w, w_t]$ of (2)–(5), or equivalently (9), is given by*

$$\Phi(t) = e^{\mathbf{A}t} \Phi_0 \in C([0, T]; \mathbf{H}),$$

where $\Phi_0 = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H}$.

After proving the existence and uniqueness of the solution, in our second result, we investigate the long term analysis of this solution. Our main goal here is to show that the solution to the system (2)–(5) is strongly stable, which is given as follows:

Theorem 2. *For the modeling generator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ of (2)–(5), one has $\sigma(\mathbf{A}) \cap i\mathbb{R}$. Consequently, the C_0 -semigroup $\{e^{\mathbf{A}t}\}_{t \geq 0}$ given in Theorem 1 is strongly stable. That is, the solution $\Phi(t)$ of the PDE (2)–(5) tends asymptotically to the zero state for all initial data $\Phi_0 \in \mathbf{H}$.*

Remark 3. The wellposedness and stability statements Theorems 1 and 2 are equally valid in the lower dimensional setting $n = 2$; i.e., for multilayered 2D heat – 1D wave – 2D wave coupled PDE systems (2)–(5), in which interface Γ_s is the boundary of a convex polygonal domain Ω_s (and so each segment Γ_j is a line segment). (Also, analogous to the present 3D setting, Ω_f is a Lipschitz domain with $\partial \Omega_f = \Gamma_s \cup \Gamma_f$, with $\overline{\Gamma_s} \cap \overline{\Gamma_f} = \emptyset$).

Remark 4. Inasmuch as we wish in future to turn our attention to the numerical analysis and simulation of solutions of the multilayered PDE system (2)–(5), the boundary interface is taken here to be polyhedral, with each polygonal boundary segment Γ_j having its own wave equation IC-BVP in variable h_j . Alternatively, the Theorems 1 and 2 will also hold true in the case that boundary interface Γ_s is smooth: in this case, the “thin” wave equation – in solution variable h , say – will have its spatial displacements described by the Laplace Beltrami operator Δ' . That is, for the multilayered FSI model on a smooth boundary interface Γ_s , the thin wave PDE component in (3) is replaced with

$$h_{tt} - \Delta' h + h = \frac{\partial w}{\partial \nu} \Big|_{\Gamma_s} - \frac{\partial u}{\partial \nu} \Big|_{\Gamma_s} \quad \text{on } (0, T) \times \Gamma_s,$$

with the matching velocity B.C.’s

$$w_t|_{\Gamma_s} = h_t = u|_{\Gamma_s} \quad \text{on } (0, T) \times \Gamma_s.$$

The heat and thick wave PDE components in (2) and (4) respectively are unchanged. In addition, there are the initial conditions

$$[u(0), h(0), h_t(0), w(0), w_t(0)] = [u_0, h_0, h_1, w_0, w_1] \in L^2(\Omega_f) \times H^1(\Gamma_s) \times L^2(\Gamma_s) \times H^1(\Omega_s) \times L^2(\Omega_s).$$

Also, the initial conditions satisfy the compatibility conditions $w_0|_{\Gamma_s} = h_0$.

Remark 5. In line with what is observed in [31] and [32], it seems possible – at least for certain configurations of the polygonal segments Γ_j , $j = 1, \dots, K$ – that the domain $D(\mathbf{A})$ of the multilayer FSI generator (as prescribed in (A.i)–(A.iv) above) manifests a regularity higher than that of finite energy; i.e., $D(\mathbf{A}) \subset H^1(\Omega_f) \times H^{1+\rho_1}(\Gamma_1) \times H^1(\Gamma_1) \times \dots \times H^{1+\rho_1}(\Gamma_K) \times H^1(\Gamma_K) \times H^{1+\rho_2}(\Omega_s) \times H^1(\Omega_s)$, where parameters $\rho_1, \rho_2 > 0$. In the course of our future work – e.g., an analysis of uniform decay properties of the FSI model (2)–(5) – this higher regularity will be fleshed out. We should note that in the case of a smooth boundary interface Γ_s (see Remark 4), smoothness of the associated FSI semigroup generator domain comes directly from classic elliptic regularity. In dimension $n = 2$ (see Remark 3), smoothness of the semigroup generator domain can be inferred by the work of P. Grisvard; see e.g., [26], Theorem 2.4.3 of p. 57, along with Remarks 2.4.5 and 2.4.6 therein.

3. Wellposedness–proof of Theorem 1

This section is devoted to prove the Hadamard well-posedness of the coupled system given in (2)–(5). Our proof hinges on the application of the Lumer Phillips Theorem which assures the existence of a C_0 -semigroup of contractions $\{e^{\mathbf{A}t}\}_{t \geq 0}$ once we establish that $\mathbf{A}$ is maximal dissipative.

Proof of Theorem 1. In order to prove the maximal dissipativity of $\mathbf{A}$, we will follow a few steps:

Step 1 (Dissipativity of $\mathbf{A}$) Given data Φ_0 in (8) to be in $D(\mathbf{A})$,

$$\begin{aligned}
 (\mathbf{A}\Phi_0, \Phi_0)_{\mathbf{H}} &= (\Delta u_0, u_0)_{\Omega_f} + \sum_{j=1}^K (\nabla h_{1j}, \nabla h_{0j})_{\Gamma_j} \\
 &\quad + \sum_{j=1}^K (h_{1j}, h_{0j})_{\Gamma_j} + \sum_{j=1}^K ([\Delta - I]h_{0j}, h_{1j})_{\Gamma_j} \\
 &\quad + \sum_{j=1}^K \left\langle \frac{\partial w_0}{\partial \nu}, h_{1j} \right\rangle_{\Gamma_j} - \sum_{j=1}^K \left\langle \frac{\partial u_0}{\partial \nu}, h_{1j} \right\rangle_{\Gamma_j} \\
 &\quad + (\nabla w_1, \nabla w_0)_{\Omega_s} + (\Delta w_0, w_1)_{\Omega_s} \\
 &= -(\nabla u_0, \nabla u_0)_{\Omega_f} + \left\langle \frac{\partial}{\partial \nu} u_0, u_0 \right\rangle_{\Gamma_s} \\
 &\quad + \sum_{j=1}^K (\nabla h_{1j}, \nabla h_{0j})_{\Gamma_j} + \sum_{j=1}^K (h_{1j}, h_{0j})_{\Gamma_j} \\
 &\quad - \sum_{j=1}^K \overline{(\nabla h_{1j}, \nabla h_{0j})_{\Gamma_j}} - \sum_{j=1}^K \overline{(h_{1j}, h_{0j})_{\Gamma_j}} + \sum_{j=1}^K \left(\frac{\partial h_{0j}}{\partial n_j}, h_{1j} \right)_{\partial \Gamma_j} \\
 &\quad + \sum_{j=1}^k \left(\frac{\partial w_0}{\partial \nu}, h_{1j} \right)_{\Gamma_j} - \sum_{j=1}^k \left\langle \frac{\partial u_0}{\partial \nu}, h_{1j} \right\rangle_{\Gamma_j} \\
 &\quad + (\nabla w_1, \nabla w_0)_{\Omega_s} - \overline{(\nabla w_1, \nabla w_0)_{\Omega_s}} - \left\langle \frac{\partial w_0}{\partial \nu}, w_1 \right\rangle_{\Gamma_s}. \tag{12}
 \end{aligned}$$

(In the last expression, we are implicitly using the fact the unit normal vector ν is *interior* with respect to Γ_s .) Note now via domain criterion (A.iv), we have for fixed index j , $1 \leq j \leq K$,

$$\left(\frac{\partial h_{0j}}{\partial n_j}, h_{1j} \right)_{\partial \Gamma_j} = \sum_{\substack{1 \leq l \leq K \\ \partial \Gamma_j \cap \partial \Gamma_l \neq \emptyset}} - \left(\frac{\partial h_{0l}}{\partial n_l}, h_{1l} \right)_{\partial \Gamma_j \cap \partial \Gamma_l}.$$

Such relation gives then the inference

$$\sum_{j=1}^K \left(\frac{\partial h_{0j}}{\partial n_j}, h_{1j} \right)_{\partial \Gamma_j} = 0. \tag{13}$$

Applying this relation and domain criterion (A.iii) to (12), we then have

$$(\mathbf{A}\Phi_0, \Phi_0)_{\mathbf{H}} = -\|\nabla u_0\|_{\Omega_f}^2$$

$$\begin{aligned}
& + 2i \sum_{j=1}^K \operatorname{Im}(\nabla h_{1j}, \nabla h_{0j})_{\Gamma_j} + 2i \sum_{j=1}^K \operatorname{Im}(h_{1j}, h_{0j})_{\Gamma_j} \\
& + 2i \operatorname{Im}(\nabla w_1, \nabla w_0)_{\Omega_s},
\end{aligned} \tag{14}$$

which gives

$$\operatorname{Re}(\mathbf{A}\Phi, \Phi)_{\mathbf{H}} \leq 0.$$

Step 2 (The Maximality of $\mathbf{A}$) Given parameter $\lambda > 0$, suppose $\Phi = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ is a solution of the equation

$$(\lambda I - \mathbf{A})\Phi = \Phi^*, \tag{15}$$

where $\Phi^* = [u_0^*, h_{01}^*, h_{11}^*, \dots, h_{0K}^*, h_{1K}^*, w_0^*, w_1^*] \in \mathbf{H}$. Then in PDE terms, the abstract equation (15) becomes

$$\begin{cases} \lambda u_0 - \Delta u_0 = u_0^* & \text{in } \Omega_f \\ u_0|_{\Gamma_f} = 0 & \text{on } \Gamma_f; \end{cases} \tag{16}$$

and for $1 \leq j \leq K$,

$$\begin{cases} \lambda h_{0j} - h_{1j} = h_{0j}^* & \text{in } \Gamma_j \\ \lambda h_{1j} - \Delta h_{0j} + h_{0j} - \frac{\partial w_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu} = h_{1j}^* & \text{in } \Gamma_j \\ u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j} & \text{in } \Gamma_j \\ h_{0j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{0l}|_{\partial\Gamma_j \cap \partial\Gamma_l} & \text{on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \\ \frac{\partial h_{0j}}{\partial n_j} \Big|_{\partial\Gamma_j \cap \partial\Gamma_l} = - \frac{\partial h_{0l}}{\partial n_l} \Big|_{\partial\Gamma_j \cap \partial\Gamma_l} & \text{on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset; \end{cases} \tag{17}$$

and also

$$\begin{cases} \lambda w_0 - w_1 = w_0^* & \text{in } \Omega_s \\ \lambda w_1 - \Delta w_0 = w_1^* & \text{in } \Omega_s. \end{cases} \tag{18}$$

With respect to this static PDE system, we multiply the heat equation in (16) by test function $\varphi \in H_{\Gamma_f}^1(\Omega_f)$, where

$$H_{\Gamma_f}^1(\Omega_f) = \left\{ \zeta \in H^1(\Omega_f) : \zeta|_{\Gamma_f} = 0 \right\}.$$

Upon integrating and invoking Green's Theorem, then solution component u_0 satisfies the variational relation,

$$\lambda(u_0, \varphi)_{\Omega_f} + (\nabla u_0, \nabla \varphi)_{\Omega_f} - \left\langle \frac{\partial u_0}{\partial \nu}, \varphi \right\rangle_{\Gamma_s} = (u_0^*, \varphi)_{\Omega_f} \text{ for } \varphi \in H_{\Gamma_f}^1(\Omega_f). \tag{19}$$

In addition, define Hilbert space $\mathcal{V}$ by

$$\mathcal{V} = \left\{ [\psi_1, \dots, \psi_K] \in H^1(\Gamma_1) \times \dots \times H^1(\Gamma_K) : \text{For all } 1 \leq j \leq K, \right. \\ \left. \psi_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = \psi_l|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \right\} \quad (20)$$

Therewith, we multiply both sides of the h_{0j} -wave equation in (17) by component ψ_j of $\psi \in \mathcal{V}$, for $1 \leq j \leq K$. Upon integration we have for $\psi \in \mathcal{V}$,

$$\begin{bmatrix} \lambda(h_{11}, \psi_1)_{\Gamma_1} - (\Delta h_{01}, \psi_1)_{\Gamma_1} + (h_{01}, \psi_1)_{\Gamma_1} - \left(\frac{\partial}{\partial \nu} w_0, \psi_1\right)_{\Gamma_1} + \left(\frac{\partial}{\partial \nu} u_0, \psi_1\right)_{\Gamma_1} \\ \vdots \\ \lambda(h_{1K}, \psi_K)_{\Gamma_K} - (\Delta h_{0K}, \psi_K)_{\Gamma_K} + (h_{0K}, \psi_K)_{\Gamma_K} - \left(\frac{\partial}{\partial \nu} w_0, \psi_K\right)_{\Gamma_K} + \left(\frac{\partial}{\partial \nu} u_0, \psi_K\right)_{\Gamma_K} \end{bmatrix} \\ = \begin{bmatrix} (h_{11}^*, \psi_1)_{\Gamma_1} \\ \vdots \\ (h_{1K}^*, \psi_K)_{\Gamma_K} \end{bmatrix}$$

For each vector component, we subsequently integrate by parts while invoking the resolvent relations in (17) (and using the domain criterion (A.iv.b)). Summing up the components of the resulting vectors, we see that the solution component $[h_{11}, \dots, h_{1K}] \in \mathcal{V}$ of (15) satisfies

$$\begin{aligned} & \sum_{j=1}^K \left[\lambda(h_{1j}, \psi_j)_{\Gamma_j} + \frac{1}{\lambda} (\nabla h_{1j}, \nabla \psi_j)_{\Gamma_j} + \frac{1}{\lambda} (h_{1j}, \psi_j)_{\Gamma_j} + \left(\frac{\partial}{\partial \nu} u_0 - \frac{\partial}{\partial \nu} w_0, \psi_j\right)_{\Gamma_j} \right] \\ &= \sum_{j=1}^K \left[(h_{1j}^*, \psi_j)_{\Gamma_j} - \frac{1}{\lambda} (h_{0j}^*, \psi_j)_{\Gamma_j} - \frac{1}{\lambda} (\nabla h_{0j}^*, \nabla \psi_j)_{\Gamma_j} \right], \text{ for } \psi \in \mathcal{V}. \end{aligned} \quad (21)$$

Moreover, multiplying the both sides of the wave equation in (18) by $\xi \in H^1(\Omega_s)$, and integrating by parts – while using the resolvent relations in (18) – we see that the solution component w_1 of (15) satisfies

$$\lambda(w_1, \xi)_{\Omega_s} + \frac{1}{\lambda} (\nabla w_1, \nabla \xi)_{\Omega_s} + \left(\frac{\partial}{\partial \nu} w_0, \xi\right)_{\Gamma_s} = (w_1^*, \xi)_{\Omega_s} - \frac{1}{\lambda} (\nabla w_0^*, \nabla \xi)_{\Omega_s}, \text{ for } \xi \in H^1(\Omega_s). \quad (22)$$

Set now

$$\mathbf{W} \equiv \left\{ [\varphi, \psi_1, \dots, \psi_K, \xi] \in H_{\Gamma_f}^1(\Omega_f) \times \mathcal{V} \times H^1(\Omega_s) : \varphi|_{\Gamma_j} = \psi_j = \xi|_{\Gamma_j}, \text{ for } 1 \leq j \leq K \right\}; \\ \|\varphi, \psi_1, \dots, \psi_K, \xi\|_{\mathbf{W}}^2 = \|\nabla \varphi\|_{\Omega_f}^2 + \sum_{j=1}^K \left[\|\nabla \psi_j\|_{\Gamma_j}^2 + \|\psi_j\|_{\Gamma_j}^2 \right] + \|\nabla \xi\|_{\Omega_s}^2. \quad (23)$$

With respect to this Hilbert space, we have the following conclusion, upon adding (19), (21) and (22): if $\Phi = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ solves (15), then necessarily its solution components $[u_0, h_{11}, \dots, h_{1K}, w_1] \in \mathbf{W}$ satisfy for $[\varphi, \psi, \xi] \in \mathbf{W}$,

$$\begin{aligned} & \lambda(u_0, \varphi)_{\Omega_f} + (\nabla u_0, \nabla \varphi)_{\Omega_f} + \lambda(w_1, \xi)_{\Omega_s} + \frac{1}{\lambda}(\nabla w_1, \nabla \xi)_{\Omega_s} \\ & + \sum_{j=1}^K [\lambda(h_{1j}, \psi_j)_{\Gamma_j} + \frac{1}{\lambda}(\nabla h_{1j}, \nabla \psi_j)_{\Gamma_j} + \frac{1}{\lambda}(h_{1j}, \psi_j)_{\Gamma_j}] = \mathbf{F}_\lambda \left(\begin{bmatrix} \varphi \\ \psi \\ \xi \end{bmatrix} \right); \end{aligned} \quad (24)$$

where

$$\begin{aligned} \mathbf{F}_\lambda \left(\begin{bmatrix} \varphi \\ \psi \\ \xi \end{bmatrix} \right) &= (u_0^*, \varphi)_{\Omega_f} + \sum_{j=1}^K \left[(h_{1j}^*, \psi_j)_{\Gamma_K} - \frac{1}{\lambda}(h_{0j}^*, \psi_j)_{\Gamma_j} - \frac{1}{\lambda}(\nabla h_{0j}^*, \nabla \psi_j)_{\Gamma_j} \right] \\ &+ (w_1^*, \xi)_{\Omega_s} - \frac{1}{\lambda}(\nabla w_0^*, \nabla \xi)_{\Omega_s}. \end{aligned} \quad (25)$$

In sum, in order to recover the solution $\Phi = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ to (15), one can straightaway apply the Lax-Milgram Theorem to the operator $\mathbf{B} \in \mathcal{L}(\mathbf{W}, \mathbf{W}^*)$, given by

$$\begin{aligned} \left\langle \mathbf{B} \begin{bmatrix} \varphi \\ \psi_1 \\ \vdots \\ \psi_k \\ \xi \end{bmatrix}, \begin{bmatrix} \tilde{\varphi} \\ \tilde{\psi}_1 \\ \vdots \\ \tilde{\psi}_k \\ \tilde{\xi} \end{bmatrix} \right\rangle_{\mathbf{W}^* \times \mathbf{W}} &= \lambda(\varphi, \tilde{\varphi})_{\Omega_f} + (\nabla \varphi, \nabla \tilde{\varphi})_{\Omega_f} + \lambda(\xi, \tilde{\xi})_{\Omega_s} + \frac{1}{\lambda}(\nabla \xi, \nabla \tilde{\xi})_{\Omega_s} \\ &+ \sum_{j=1}^K \left[\lambda(\psi_j, \tilde{\psi}_j)_{\Gamma_j} + \frac{1}{\lambda}(\nabla \psi_j, \nabla \tilde{\psi}_j)_{\Gamma_j} + \frac{1}{\lambda}(\psi_j, \tilde{\psi}_j)_{\Gamma_j} \right]. \end{aligned}$$

It is clear that $\mathbf{B} \in \mathcal{L}(\mathbf{W}, \mathbf{W}^*)$ is $\mathbf{W}$ -elliptic; so by the Lax-Milgram Theorem, the equation (24) has a unique solution

$$[u_0, h_{11}, \dots, h_{1K}, w_1] \in \mathbf{W}. \quad (26)$$

Subsequently, we set

$$\begin{cases} h_{0j} = \frac{h_{1j} + h_{0j}^*}{\lambda}, \text{ for } 1 \leq j \leq K, \\ w_0 = \frac{w_1 + w_0^*}{\lambda}. \end{cases} \quad (27)$$

In particular, since the data $[u_0^*, h_{01}^*, h_{11}^*, \dots, h_{0k}^*, h_{1k}^*, w_0^*, w_1^*] \in \mathbf{H}$, then the relations in (27) give that

$$w_0|_{\Gamma_j} = h_{0j}, \quad 1 \leq j \leq K. \quad (28)$$

We further show that the dependent variable $\Phi = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1]$, given by the solution of (24) and (27), is an element of $D(\mathbf{A})$: If we take $[\varphi, 0, \dots, 0, 0] \in \mathbf{W}$ in (24), where $\varphi \in \mathcal{D}(\Omega_f)$, then we have

$$\lambda(u_0, \varphi)_{\Omega_f} - (\Delta u_0, \varphi)_{\Omega_f} = (u_0^*, \varphi)_{\Omega_f} \quad \forall \varphi \in \mathcal{D}(\Omega_f),$$

whence

$$\lambda u_0 - \Delta u_0 = u_0^* \text{ in } L^2(\Omega_f). \quad (29)$$

Subsequently, the fact that $\{\Delta u_0, u_0\} \in L^2(\Omega_f) \times H^1(\Omega_f)$ gives

$$\frac{\partial u_0}{\partial \nu}|_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s). \quad (30)$$

In turn, using the relations in (27), if we take $[0, 0, \dots, 0, \xi] \in \mathbf{W}$, where $\xi \in \mathcal{D}(\Omega_s)$, then upon integrating by parts, we have

$$\lambda(w_1, \xi)_{\Omega_s} - (\Delta w_0, \xi)_{\Omega_s} = (w_1^*, \xi)_{\Omega_s} \quad \forall \xi \in \mathcal{D}(\Omega_s),$$

and so

$$\lambda w_1 - \Delta w_0 = w_1^* \text{ in } L^2(\Omega_s), \quad (31)$$

which gives that $\{\Delta w_0, w_0\} \in L^2(\Omega_s) \times H^1(\Omega_s)$. A subsequent integration by parts yields that

$$\frac{\partial w_0}{\partial \nu}|_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s). \quad (32)$$

Moreover, let $\gamma_s^+ \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H^1(\Omega_s))$ be the right continuous inverse for the Sobolev trace map $\gamma_s \in \mathcal{L}(H^1(\Omega_s), H^{\frac{1}{2}}(\Gamma_s))$; viz.,

$$\gamma_s(f) = f|_{\Gamma_s} \text{ for } f \in C^\infty(\overline{\Omega_s}).$$

Likewise, let $\gamma_f^+ \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H_{\Gamma_f}^1(\Omega_f))$ denote the right inverse for the Sobolev trace map $\gamma_f \in \mathcal{L}(H_{\Gamma_f}^1(\Omega_f), H^{\frac{1}{2}}(\Gamma_s))$. Also, for given $\psi_j \in H_0^1(\Gamma_j)$, $1 \leq j \leq K$, let

$$(\psi_j)_{ext}(x) \equiv \begin{cases} \psi_j, & x \in \Gamma_j \\ 0, & x \in \Gamma_s \setminus \Gamma_j. \end{cases} \quad (33)$$

Then $(\psi_j)_{ext} \in H^{\frac{1}{2}}(\Gamma_s)$ for all $1 \leq j \leq K$. We now specify test function $[\varphi, \psi_1, \dots, \psi_K, \xi] \in \mathbf{W}$ in (24): namely, $\psi_j \in H_0^1(\Gamma_j)$, $1 \leq j \leq K$, and

$$\varphi \equiv \gamma_f^+ \left[\sum_{j=1}^K (\psi_j)_{ext} \right], \quad \xi \equiv \gamma_s^+ \left[\sum_{j=1}^K (\psi_j)_{ext} \right]. \quad (34)$$

Therewith we have verbatim from (24),

$$\begin{aligned}
& \lambda(u_0, \varphi)_{\Omega_f} + (\nabla u_0, \nabla \varphi)_{\Omega_f} \\
& + \sum_{j=1}^K \left[\lambda(h_{1j}, \psi_j)_{\Gamma_j} + \frac{1}{\lambda} (\nabla h_{1j}, \nabla \psi_j)_{\Gamma_j} + \frac{1}{\lambda} (h_{1j}, \psi_j)_{\Gamma_j} \right] \\
& + \lambda(w_1, \xi)_{\Omega_s} + \frac{1}{\lambda} (\nabla w_1, \nabla \xi)_{\Omega_s} \\
& = (u_0^*, \varphi)_{\Omega_f} + \sum_{j=1}^k \left[(h_{1j}^*, \psi_j)_{\Gamma_j} - \frac{1}{\lambda} (\nabla h_{0j}^*, \nabla \psi_j)_{\Gamma_j} - \frac{1}{\lambda} (h_{0j}^*, \psi_j)_{\Gamma_j} \right] \\
& + (w_1^*, \xi)_{\Omega_s} - \frac{1}{\lambda} (\nabla w_0^*, \nabla \xi)_{\Omega_s}.
\end{aligned}$$

Upon integrating by parts, and invoking the relations in (27), as well as (29)–(32), we get

$$\left\langle \frac{\partial u_0}{\partial \nu}, \varphi \right\rangle_{\Gamma_s} + \sum_{j=1}^K [\lambda(h_{1j}, \psi_j)_{\Gamma_j} - (\Delta h_{0j}, \psi_j)_{\Gamma_j} + (h_{0j}, \psi_j)_{\Gamma_j}] - \left\langle \frac{\partial w_0}{\partial \nu}, \xi \right\rangle_{\Gamma_s} = \sum_{j=1}^K (h_{1j}^*, \psi_j)_{\Gamma_j}. \quad (35)$$

Since each test function component $\psi_j \in H_0^1(\Gamma_j)$ is arbitrary, we then deduce from this relation and (33)–(34) that each h_{0j} solves

$$\lambda h_{1j} - \Delta h_{0j} + h_{0j} - \frac{\partial w_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu} = h_{1j}^* \quad \text{in } \Gamma_j, \quad 1 \leq j \leq K. \quad (36)$$

In addition, we have from (36), (26), (30), and (32) that $\{\Delta h_{0j}, h_{0j}\} \in [H^1(\Gamma_j)]' \times H^1(\Gamma_j)$, for $1 \leq j \leq K$. Consequently, an integration by parts gives that

$$\frac{\partial h_{0j}}{\partial n_j} \in H^{-\frac{1}{2}}(\partial \Gamma_j), \quad \text{for } 1 \leq j \leq K. \quad (37)$$

Finally: Let given indices $j^*, l^*, 1 \leq j^*, l^* \leq K$, satisfy $\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \neq \emptyset$. Let g be a given element in $H_0^{\frac{1}{2}+\epsilon}(\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*})$. Then one has that $\tilde{g}_{j^*} \in H^{\frac{1}{2}+\epsilon}(\partial \Gamma_{j^*})$ and $\tilde{g}_{l^*} \in H^{\frac{1}{2}+\epsilon}(\partial \Gamma_{l^*})$, where

$$\tilde{g}_{j^*}(x) \equiv \begin{cases} g(x), & x \in \partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \\ 0, & x \in \partial \Gamma_{j^*} \setminus (\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}) \end{cases}; \quad \tilde{g}_{l^*}(x) \equiv \begin{cases} g(x), & x \in \partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \\ 0, & x \in \partial \Gamma_{l^*} \setminus (\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}) \end{cases}$$

(see e.g., Theorem 3.33, p. 95 of [35]). Subsequently, by the (limited) surjectivity of the Sobolev Trace Map on Lipschitz domains— see e.g., Theorem 3.38, p. 102 of [35] — there exists $\psi_{j^*} \in H^{1+\epsilon}(\Gamma_{j^*})$ and $\psi_{l^*} \in H^{1+\epsilon}(\Gamma_{l^*})$ such that

$$\psi_{j^*}|_{\partial \Gamma_{j^*}} = \tilde{g}_{j^*}, \quad \text{and} \quad \psi_{l^*}|_{\partial \Gamma_{l^*}} = \tilde{g}_{l^*}. \quad (38)$$

In turn, by the Sobolev Embedding Theorem, if we define, on $\overline{\Gamma_s}$ the function

$$\Upsilon(x) \equiv \begin{cases} \psi_{j^*}(x), & \text{for } x \in \overline{\Gamma}_{j^*} \\ \psi_{l^*}(x), & \text{for } x \in \overline{\Gamma}_{l^*} \\ 0, & \text{for } x \in \overline{\Gamma}_s \setminus (\overline{\Gamma}_{j^*} \cup \overline{\Gamma}_{l^*}), \end{cases} \quad (39)$$

then $\Upsilon(x) \in C(\overline{\Gamma}_s)$. Since also $\psi_{j^*} \in H^1(\Gamma_{j^*})$ and $\psi_{l^*} \in H^1(\Gamma_{l^*})$, we eventually deduce via an integration by parts that $\Upsilon \in H^1(\Gamma_s)$. (See e.g., the proof of Theorem 2, p. 36 of [18].) With this H^1 -function in hand, and with aforesaid continuous right inverses $\gamma_s^+ \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H^1(\Omega_s))$ and $\gamma_f^+ \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H_{\Gamma_f}^1(\Omega_f))$, we specify the vector

$$[\varphi, \psi, \xi] \equiv \left[\gamma_f^+(\Upsilon), 0, \dots, \psi_{j^*}, 0, \dots, 0, \psi_{l^*}, \dots, 0, \gamma_s^+(\Upsilon) \right] \in \mathbf{W}, \quad (40)$$

where again, space $\mathbf{W}$ is given in (23). With this vector in hand, we consider the thin wave equation in (36): With respect to the two fixed indices $1 \leq j^*, l^* \leq K$, we have via (36)

$$\begin{aligned} & \lambda (h_{1j^*}, \psi_{j^*})_{\Gamma_{j^*}} - (\Delta h_{0j^*}, \psi_{j^*})_{\Gamma_{j^*}} + (h_{0j^*}, \psi_{j^*})_{\Gamma_{j^*}} \\ & - \left(\frac{\partial w_0}{\partial \nu} - \frac{\partial u_0}{\partial \nu}, \psi_{j^*} \right)_{\Gamma_{j^*}} + \lambda (h_{1l^*}, \psi_{l^*})_{\Gamma_{l^*}} - (\Delta h_{0l^*}, \psi_{l^*})_{\Gamma_{l^*}} \\ & + (h_{0l^*}, \psi_{l^*})_{\Gamma_{l^*}} - \left(\frac{\partial w_0}{\partial \nu} - \frac{\partial u_0}{\partial \nu}, \psi_{l^*} \right)_{\Gamma_{l^*}} = (h_{1j^*}^*, \psi_{j^*})_{\Gamma_{j^*}} + (h_{1l^*}^*, \psi_{l^*})_{\Gamma_{l^*}}. \end{aligned}$$

A subsequent integration by parts, with (40) in mind, subsequently yields

$$\begin{aligned} & \lambda (h_{1j^*}, \psi_{j^*})_{\Gamma_{j^*}} + (\nabla h_{0j^*}, \nabla \psi_{j^*})_{\Gamma_{j^*}} - \left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}} + (h_{0j^*}, \psi_{j^*})_{\Gamma_{j^*}} \\ & + \lambda (h_{1l^*}, \psi_{l^*})_{\Gamma_{l^*}} + (\nabla h_{0l^*}, \nabla \psi_{l^*})_{\Gamma_{l^*}} - \left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}} + (h_{0l^*}, \psi_{l^*})_{\Gamma_{l^*}} \\ & + (\nabla w_0, \nabla \xi)_{\Omega_s} + (\Delta w_0, \xi)_{\Omega_s} + (\nabla u_0, \nabla \varphi)_{\Omega_f} + (\Delta u_0, \varphi)_{\Omega_f} \\ & = (h_{1j^*}^*, \psi_{j^*})_{\Gamma_{j^*}} + (h_{1l^*}^*, \psi_{l^*})_{\Gamma_{l^*}}. \end{aligned}$$

Invoking (29) and (31), we then have

$$\begin{aligned} & - \left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}} - \left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}} + (h_{0j^*}, \psi_{j^*})_{\Gamma_{j^*}} + (h_{0l^*}, \psi_{l^*})_{\Gamma_{l^*}} \\ & + \lambda (h_{1j^*}, \psi_{j^*})_{\Gamma_{j^*}} + (\nabla h_{0j^*}, \nabla \psi_{j^*})_{\Gamma_{j^*}} + \lambda (h_{1l^*}, \psi_{l^*})_{\Gamma_{l^*}} + (\nabla h_{0l^*}, \nabla \psi_{l^*})_{\Gamma_{l^*}} \\ & + (\nabla w_0, \nabla \xi)_{\Omega_s} + \lambda (w_1, \xi)_{\Omega_s} - (w_1^*, \xi)_{\Omega_s} + (\nabla u_0, \nabla \varphi)_{\Omega_f} + \lambda (u_0, \varphi)_{\Omega_f} - (u_0^*, \varphi)_{\Omega_f} \\ & = (h_{1j^*}^*, \psi_{j^*})_{\Gamma_{j^*}} + (h_{1l^*}^*, \psi_{l^*})_{\Gamma_{l^*}}. \end{aligned}$$

Invoking the relations in (27) and the variational equation (24), which is satisfied by $[u_0, h_{11}, \dots, h_{1K}, w_1]$ (where again vector $[\varphi, \psi, \xi]$ is given by (40)), we have the relation

$$\left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}} = - \left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}}, \quad \text{for all } g \in H_0^{\frac{1}{2}+\epsilon}(\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}).$$

Since $H_0^{\frac{1}{2}+\epsilon}(\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*})$ is dense in $H^{\frac{1}{2}}(\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*})$, we deduce now that

$$\frac{\partial h_{0j^*}}{\partial n_{j^*}} = -\frac{\partial h_{0l^*}}{\partial n_{l^*}}, \text{ for } \partial\Gamma_{j^*} \cap \partial\Gamma_{l^*} \neq \emptyset. \quad (41)$$

Collecting (26)–(32) and (36), (37) and (41), we have that the obtained variable

$$[u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A}),$$

and solves the resolvent equation (15). This concludes the proof of Theorem 1, upon application of the Lumer-Phillips Theorem.

4. Strong stability-proof of Theorem 2

In this section, our main aim is to address the issue of asymptotic behavior of the solution that we stated in Section 2. In this regard, we show that the system given in (2)–(5) is strongly stable. Our proof will be independent of the compactness or noncompactness of the resolvent of $\mathbf{A}$ (see Remark 3). It will hinge on an ultimate appeal to the following well known result:

Theorem 6. ([1]) *Let $T(t)_{t \geq 0}$ be a bounded C_0 -semigroup on a reflexive Banach space X , with generator $\mathbf{A}$. Assume that $\sigma_p(\mathbf{A}) \cap i\mathbb{R} = \emptyset$, where $\sigma_p(\mathbf{A})$ is the point spectrum of $\mathbf{A}$. If $\sigma(\mathbf{A}) \cap i\mathbb{R}$ is countable then $T(t)_{t \geq 0}$ is strongly stable.*

The proof of this theorem entails the elimination of all three parts of the spectrum of the generator $\mathbf{A}$ from the imaginary axis. For this, we will give the necessary analysis on the spectrum in the following subsection.

4.1. Spectral analysis on the generator $\mathbf{A}$

Since we wish to satisfy the conditions of Theorem 6, we will prove that $\sigma(\mathbf{A}) \cap i\mathbb{R} = \emptyset$ which is equivalent to show that

$$i\mathbb{R} \subset \rho(\mathbf{A}).$$

To do this, we start with the following Proposition:

Proposition 7. *With generator $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ given in (10)–(11), the point $0 \in \rho(\mathbf{A})$. That is, $\mathbf{A}$ is boundedly invertible.*

Proof. Given $\Phi^* = [u_0^*, h_{01}^*, h_{11}^*, \dots, h_{0K}^*, h_{1K}^*, w_0^*, w_1^*] \in \mathbf{H}$, we take up the task of finding $\Phi = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ which solves

$$\mathbf{A}\Phi = \Phi^*, \quad (42)$$

or

$$\begin{bmatrix} \Delta u_0 \\ h_{11} \\ -\frac{\partial u_0}{\partial \nu}|_{\Gamma_1} + (\Delta - I)h_{01} + \frac{\partial w_0}{\partial \nu}|_{\Gamma_1} \\ \vdots \\ h_{1K} \\ -\frac{\partial u_0}{\partial \nu}|_{\Gamma_K} + (\Delta - I)h_{0K} + \frac{\partial w_0}{\partial \nu}|_{\Gamma_K} \\ w_1 \\ \Delta w_0 \end{bmatrix} = \begin{bmatrix} u_0^* \\ h_{01}^* \\ h_{11}^* \\ \vdots \\ h_{0K}^* \\ h_{1K}^* \\ w_0^* \\ w_1^* \end{bmatrix}. \quad (43)$$

From the thin and thick wave component of this equation we see that

$$w_1 = w_0^* \in H^1(\Omega_s) \quad (44)$$

$$h_{1j} = h_{0j}^* \in H^1(\Gamma_j), \quad \text{for } 1 \leq j \leq K \quad (45)$$

Moreover, from the heat and thick wave components of (43), and the domain criterion (A.iii), we have that the solution component u_0 should satisfy the following BVP:

$$\begin{cases} \Delta u_0 = u_0^* & \text{in } \Omega_f \\ u_0|_{\Gamma_f} = 0 \\ u_0|_{\Gamma_s} = w_0^*|_{\Gamma_s} \end{cases} \quad (46)$$

Solving this BVP, and estimating its solution, in part by the Sobolev Trace Theorem, we have

$$\|u_0\|_{H_{\Gamma_f}^1(\Omega_f)} + \|\Delta u_0\|_{\Omega_f} \leq C \left[\|u_0^*\|_{\Omega_f} + \|w_0^*\|_{H^1(\Omega_s)} \right]. \quad (47)$$

In turn, the use of this estimate in an integration by parts gives

$$\left\| \frac{\partial u_0}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Omega_f)} \leq C \left[\|u_0^*\|_{\Omega_f} + \|w_0^*\|_{H^1(\Omega_s)} \right]. \quad (48)$$

In addition, with the space $\mathcal{V}$ as in (20), we set

$$\chi \equiv \left\{ [\psi, \xi] \in \mathcal{V} \times H^1(\Omega_s) : \psi_j = \xi|_{\Gamma_j} \quad \text{for } 1 \leq j \leq K \right\}. \quad (49)$$

With this space in hand, and with the thin-wave and thick-wave components of equation (43) in mind, we consider the variational relation

$$\begin{aligned} & (\nabla w_0, \nabla \xi)_{\Omega_s} \\ & + \sum_{j=1}^K [(\nabla h_{0j}, \nabla \psi_j)_{\Gamma_j} + (h_{0j}, \psi_j)_{\Gamma_j}] \\ & = -(w_1^*, \xi)_{\Omega_s} \\ & - \sum_{j=1}^K \left[(h_{1j}^*, \psi_j)_{\Gamma_j} + \left(\frac{\partial u_0}{\partial \nu}, \psi_j \right)_{\Gamma_j} \right], \end{aligned} \quad (50)$$

for every $[\psi, \xi] \in \chi$ where the term $\frac{\partial u_0}{\partial \nu}|_{\Gamma_s}$ is from (48). Since the bilinear form $b(\cdot, \cdot) : \chi \rightarrow \mathbb{R}$, given by

$$b([\psi, \xi], [\tilde{\psi}, \tilde{\xi}]) = (\nabla \xi, \nabla \tilde{\xi})_{\Omega_s} + \sum_{j=1}^K [(\nabla \psi_j, \nabla \tilde{\psi}_j)_{\Gamma_j} + (\psi_j, \tilde{\psi}_j)_{\Gamma_j}] \quad (51)$$

for every $[\psi, \xi], [\tilde{\psi}, \tilde{\xi}] \in \chi$, is continuous and χ -elliptic, then by Lax-Milgram, there exists a unique solution

$$\phi = [(h_{01}, h_{02}, \dots, h_{0K}), w_0] \in \chi \quad (52)$$

to the variational relation (50). To show that the obtained $[u_0, [h_{01}, h_{11}, \dots, h_{0K}, h_{1K}], w_0, w_1] \in \mathbf{H}$ is in $D(\mathbf{A})$ and satisfies the equation (43):

Proceeding very much as we did in the proof of Theorem 1, we take in (50)

$$[\psi, \xi] = [[0, 0, \dots, 0], \varphi],$$

where $\varphi \in \mathcal{D}(\Omega_s)$. This gives

$$(\nabla w_0, \nabla \xi)_{\Omega_s} = -(w_1^*, \xi)_{\Omega_s},$$

whence we obtain

$$-\Delta w_0 = -w_1^* \quad \text{in } \Omega_s, \quad (53)$$

with

$$\begin{aligned} \|\Delta w_0\|_{\Omega_s} + \left\| \frac{\partial w_0}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} &\leq C \left[\|w_1^*\|_{\Omega_s} + \|w_0\|_{H^1(\Omega_s)} \right] \\ &\leq C \| [u_0^*, [h_{01}^*, h_{11}^*, \dots, h_{0K}^*, h_{1K}^*], w_0^*, w_1^*] \|_{\mathbf{H}}, \end{aligned} \quad (54)$$

after using (52). In turn, using aforesaid right continuous inverse $\gamma_s^+ \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H^1(\Omega_s))$, let in (50), test function

$$[\psi, \xi] = \left[[(\psi_1)_{ext}, \dots, (\psi_K)_{ext}], \gamma_s^+ \left(\sum_{j=1}^K (\psi_j)_{ext} \right) \right] \in \chi,$$

where each $\psi_j \in H_0^1(\Gamma_j)$ ($1 \leq j \leq K$), and each $(\psi_j)_{ext}$ is as in (33). Applying this function to (50), integrating by parts and invoking (53), we have

$$\begin{aligned}
& -(\Delta w_0, \xi)_{\Omega_s} - \left\langle \frac{\partial w_0}{\partial \nu}, \xi|_{\Gamma_s} \right\rangle_{\Gamma_s} \\
& + \sum_{j=1}^K [(\nabla h_{0j}, \nabla \psi_j)_{\Gamma_j} + (h_{0j}, \psi_j)_{\Gamma_j}] \\
& = - \sum_{j=1}^K \left[\left\langle \frac{\partial u_0}{\partial \nu}, \psi_j \right\rangle_{\Gamma_j} + (h_{1j}^*, \psi_j)_{\Gamma_j} \right] - (w_1^*, \xi)_{\Omega_s}.
\end{aligned}$$

Again, as each $\psi_j \in H_0^1(\Gamma_j)$ is arbitrary, we deduce that each h_{0j} solves the thin-wave equation

$$-\Delta h_{0j} + h_{0j} - \frac{\partial w_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu} = -h_{1j}^*, \quad \text{in } \Gamma_j, \quad 1 \leq j \leq K. \quad (55)$$

A subsequent integration by parts, and invocation of (48), (52) and (54), give for $1 \leq j \leq K$,

$$\begin{aligned}
& \|\Delta h_{0j}\|_{\Gamma_j} + \left\| \frac{\partial h_{0j}}{\partial n_j} \right\|_{H^{-\frac{1}{2}}(\partial \Gamma_j)} \\
& \leq C \|[u_0^*, [h_{01}^*, h_{11}^*, \dots, h_{0K}^*, h_{1K}^*], w_0^*, w_1^*]\|_{\mathbf{H}}.
\end{aligned} \quad (56)$$

Now, proceeding as in the final stage of the proof of Theorem 1: let fixed indices $j^*, l^*, 1 \leq j^*, l^* \leq K$, satisfy $\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \neq \emptyset$. Given function $g \in H_0^{\frac{1}{2}+\epsilon}(\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*})$, we invoke the associated functions $\psi_{j^*} \in H^{1+\epsilon}(\Gamma_{j^*})$ and $\psi_{l^*} \in H^{1+\epsilon}(\Gamma_{l^*})$ as in (38), also $\Upsilon \in H^1(\Gamma_s)$ as in (39). With these functions, and said continuous right inverse $\gamma_s^+ \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H^1(\Omega_s))$, we consider test function

$$[\psi, \xi] = [[0, \dots, \psi_{j^*}, 0, \dots, \psi_{l^*}, \dots, 0], \gamma_s^+(\Upsilon)] \in \mathcal{X}.$$

Applying this test function to the variational relation (50), and subsequently invoking (53), we obtain

$$\begin{aligned}
& - \left\langle \frac{\partial w_0}{\partial \nu}, \xi|_{\Gamma_s} \right\rangle_{\Gamma_s} + (\nabla h_{0j^*}, \nabla \psi_{j^*})_{\Gamma_{j^*}} + (h_{0j^*}, \psi_{j^*})_{\Gamma_{j^*}} \\
& + (\nabla h_{0l^*}, \nabla \psi_{l^*})_{\Gamma_{l^*}} + (h_{0l^*}, \psi_{l^*})_{\Gamma_{l^*}} \\
& = -(h_{1j^*}, \psi_{j^*})_{\Gamma_{j^*}} - \left\langle \frac{\partial u_0}{\partial \nu}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}^*} \\
& - (h_{1l^*}^*, \psi_{l^*})_{\Gamma_{l^*}^*} - \left\langle \frac{\partial u_0}{\partial \nu}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}^*}.
\end{aligned}$$

Integrating by parts with respect to the thin wave components, and invoking (55) and (38), we then have

$$\left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}} + \left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}} = 0.$$

Since $g \in H_0^{\frac{1}{2}+\epsilon}(\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*})$ is arbitrary, a density argument yields

$$\left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*}} = - \left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*}}, \quad \forall j^*, l^*, 1 \leq j^*, l^* \leq K \quad (57)$$

such that $\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*} \neq \emptyset$. Collecting (44), (45), (47), (48), (52), (53), (55)–(57), we have now that the obtained $[u_0, [h_{01}, h_{11}, \dots, h_{0K}, h_{1K}], w_0, w_1] \in D(\mathbf{A})$ satisfies the equation (42) for arbitrary $\Phi^* \in \mathbf{H}$. Since also $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ is dissipative (and so injective), we conclude that $\mathbf{A}$ is boundedly invertible. $\square$

In what follows, we will need the Hilbert space adjoint of $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ which can be readily computed:

Proposition 8. *The Hilbert space adjoint $\mathbf{A}^* : D(\mathbf{A}^*) \subset \mathbf{H} \rightarrow \mathbf{H}$ of the thick wave-thin wave-heat generator is given as,*

$$\mathbf{A}^* = \begin{bmatrix} \Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & \cdots & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial \nu}|_{\Gamma_1} & (I - \Delta) & 0 & \cdots & 0 & 0 & -\frac{\partial}{\partial \nu}|_{\Gamma_1} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -I & 0 & 0 \\ -\frac{\partial}{\partial \nu}|_{\Gamma_K} & 0 & 0 & \cdots & (I - \Delta) & 0 & -\frac{\partial}{\partial \nu}|_{\Gamma_1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\Delta & 0 \end{bmatrix};$$

where

$$D(\mathbf{A}^*) = \{[u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H} :$$

$$(A^*.i) u_0 \in H^1(\Omega_f), h_{1j} \in H^1(\Gamma_j) \text{ for } 1 \leq j \leq K, w_1 \in H^1(\Omega_s);$$

$$(A^*.ii) (a) \Delta u_0 \in L^2(\Omega_f), \Delta w_0 \in L^2(\Omega_s), (b) -\Delta h_{0j} - \frac{\partial u_0}{\partial \nu}|_{\Gamma_j} - \frac{\partial w_0}{\partial \nu}|_{\Gamma_j} \in L^2(\Gamma_j) \text{ for } 1 \leq j \leq K;$$

$$(c) \frac{\partial h_{0j}}{\partial n_j} \Big|_{\partial\Gamma_j} \in H^{-\frac{1}{2}}(\partial\Gamma_j), \text{ for } 1 \leq j \leq K;$$

$$(A^*.iii) u_0|_{\Gamma_f} = 0, u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j}, \text{ for } 1 \leq j \leq K;$$

$$(A^*.iv) \text{ For } 1 \leq j \leq K:$$

$$(a) h_{1j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{1l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset;$$

$$(b) \frac{\partial h_{0j}}{\partial n_j} \Big|_{\partial\Gamma_j \cap \partial\Gamma_l} = - \frac{\partial h_{0l}}{\partial n_l} \Big|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \}.$$

Now, we continue with analyzing the point and continuous spectra of the generator $\mathbf{A}$:

Lemma 9. *The point $\sigma_p(\mathbf{A})$ and continuous spectra $\sigma_c(\mathbf{A})$ of $\mathbf{A}$ have empty intersection with $i\mathbb{R}$.*

Proof. To prove this, it will be enough to show that $i\mathbb{R} \setminus \{0\}$ has empty intersection with the approximate spectrum of $\mathbf{A}$; see e.g., Theorem 2.27, pg. 128 of [24]. To this end, given $\beta \neq 0$, suppose that $i\beta$ is in the approximate spectrum of $\mathbf{A}$. Then there exist sequences

$$\{\Phi_n\} = \left\{ \begin{bmatrix} u_n \\ h_{1n} \\ \xi_{1n} \\ \vdots \\ h_{Kn} \\ \xi_{Kn} \\ w_{0n} \\ w_{1n} \end{bmatrix} \right\} \subseteq D(\mathbf{A}); \quad \{(i\beta I - \mathbf{A})\Phi_n\} = \left\{ \begin{bmatrix} u_n^* \\ \varphi_{1n}^* \\ \psi_{1n}^* \\ \vdots \\ \varphi_{Kn}^* \\ \psi_{Kn}^* \\ w_{0n}^* \\ w_{1n}^* \end{bmatrix} \right\} \subseteq \mathbf{H}, \quad (58)$$

which satisfy for $n = 1, 2, \dots$,

$$\|\Phi_n\|_{\mathbf{H}} = 1, \quad \|(i\beta I - \mathbf{A})\Phi_n\|_{\mathbf{H}} < \frac{1}{n}. \quad (59)$$

As such, each Φ_n solves the following static system:

$$\begin{cases} i\beta u_n - \Delta u_n = u_n^* & \text{in } \Omega_f \\ u_n|_{\Gamma_f} = 0 & \text{on } \Gamma_f \end{cases} \quad (60)$$

For $1 \leq j \leq K$,

$$\begin{cases} i\beta h_{jn} - \xi_{jn} = \varphi_{jn}^* & \text{in } \Gamma_j \\ -\beta^2 h_{jn} - \Delta h_{jn} + h_{jn} + \frac{\partial u_n}{\partial \nu} - \frac{\partial w_{0n}}{\partial \nu} = \psi_{jn}^* + i\beta \varphi_{jn}^* & \text{in } \Gamma_j \end{cases} \quad (61)$$

Also

$$\begin{cases} i\beta w_{0n} - w_{1n} = w_{0n}^* & \text{in } \Omega_s \\ -\beta^2 w_{0n} - \Delta w_{0n} = w_{1n}^* + i\beta w_{0n}^* & \text{in } \Omega_s \end{cases} \quad (62)$$

and again for $1 \leq j \leq K$,

$$\left\{ \frac{\partial h_{nj}}{\partial n_j} \right\}_{\partial\Gamma_j \cap \partial\Gamma_l} = - \left\{ \frac{\partial h_{nl}}{\partial n_l} \right\}_{\partial\Gamma_j \cap \partial\Gamma_l} \quad \begin{matrix} u_n|_{\Gamma_j} = \xi_{jn} = w_{1n}|_{\Gamma_j} \\ \text{for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset. \end{matrix} \quad (63)$$

Now the left part of the proof of Lemma 9 will be given in five steps:

STEP 1: (Estimating the heat component of Φ_n)

Proceeding as we did in establishing the dissipativity of $\mathbf{A} : D(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$, (see relations (12) and (14)), if we denote

$$\Phi_n^* = (i\beta I - \mathbf{A})\Phi_n$$

then from the relation

$$((i\beta I - \mathbf{A})\Phi_n, \Phi_n)_{\mathbf{H}} = (\Phi_n^*, \Phi_n)_{\mathbf{H}},$$

we obtain

$$\|\nabla u_n\|_{\Omega_f}^2 = \operatorname{Re}(\Phi_n^*, \Phi_n)_{\mathbf{H}}. \quad (64)$$

From (59), we then have

$$\lim_{n \rightarrow \infty} u_n = 0 \quad \text{in } H^1(\Omega_f). \quad (65)$$

In turn, via the thin wave resolvent condition in (61) and boundary conditions in (63), we have for $1 \leq j \leq K$

$$h_{jn} = -\frac{i}{\beta} u_n|_{\Gamma_j} - \frac{i}{\beta} \varphi_{jn}^* \quad \text{in } \Gamma_j.$$

From this relation, we can then invoke (65), the Sobolev Trace Map, and (59), to have

$$\lim_{n \rightarrow \infty} h_{jn} = 0 \quad \text{in } H^{\frac{1}{2}}(\Gamma_j) \quad (66)$$

for $1 \leq j \leq K$. Moreover, an integration by parts, with respect to the heat equation (60), gives the estimate

$$\begin{aligned} \left\| \frac{\partial u_n}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Omega_f)} &\leq C \left[\|\nabla u_n\|_{\Omega_f} + \|\Delta u_n\|_{\Omega_f} \right] \\ &\leq C \left[\|\nabla u_n\|_{\Omega_f} + \|i\beta u_n - u_n^*\|_{\Omega_f} \right]. \end{aligned}$$

Now, invoking (64) and (59) gives

$$\lim_{n \rightarrow \infty} \frac{\partial u_n}{\partial \nu} = 0 \quad \text{in } H^{-\frac{1}{2}}(\Gamma_j). \quad (67)$$

STEP 2: We start here by defining the “Dirichlet” map $D_s : L^2(\Gamma_s) \rightarrow L^2(\Omega_s)$ via

$$D_s g = f \iff \begin{cases} \Delta f = 0 & \text{in } \Omega_s \\ f|_{\Gamma_s} = g & \text{on } \Gamma_s. \end{cases}$$

We know by the Lax-Milgram Theorem

$$D_s \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H^1(\Omega_s)). \quad (68)$$

Therewith, considering the resolvent relations in (62), we set

$$z_n \equiv w_{0n} + \frac{i}{\beta} D_s [u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}], \quad (69)$$

and so from (62) z_n satisfies the following BVP:

$$\begin{cases} -\beta^2 z_n - \Delta z_n = w_{1n}^* + i\beta w_{0n}^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}] & \text{in } \Omega_s \\ z_n|_{\Gamma_s} = 0 & \text{on } \Gamma_s. \end{cases} \quad (70)$$

Since Ω_s is convex, then $z_n \in H^2(\Omega_s)$. See e.g., Theorem 3.2.1.2, pg. 147 of [25]. In consequence, we can apply the static version of the well-known wave identity which is often used in PDE control theory—[see (Proposition 7 (ii) of [6]), [16], [40]. To wit, let $m(x)$ be any $[C^2(\overline{\Omega_s})]^3$ -vector field with associated Jacobian matrix

$$[M(x)]_{ij} = \frac{\partial m_i(x)}{\partial x_j}, \quad 1 \leq i, j \leq 3$$

Therewith, we have

$$\begin{aligned} & \int_{\Omega_s} M \nabla z_n \cdot \nabla z_n d\Omega_s \\ &= -\operatorname{Re} \int_{\Gamma_s} \frac{\partial z_n}{\partial \nu} m \cdot \nabla \overline{z_n} d\Gamma_s \\ & \quad - \frac{\beta^2}{2} \int_{\Gamma_s} |z_n|^2 m \cdot \nu d\Gamma_s + \frac{1}{2} \int_{\Gamma_s} |\nabla z_n|^2 m \cdot \nu d\Gamma_s \\ & \quad + \frac{1}{2} \int_{\Omega_s} \{|\nabla z_n|^2 - \beta^2 |z_n|^2\} \operatorname{div}(m) d\Omega_s \\ & \quad + \operatorname{Re} \int_{\Omega_s} \left[F_\beta^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}] \right] m \cdot \nabla \overline{z_n} d\Omega_s, \end{aligned} \quad (71)$$

where

$$F_\beta^* = (\operatorname{Re} w_{1n}^* - \beta I_m w_{0n}^*) + i(I_m w_{1n}^* + \beta \operatorname{Re} w_{0n}^*). \quad (72)$$

Again, relation (71) holds for any C^2 -vector field $m(x)$. We now specify it to be the smooth vector field of Lemma 1.5.1.9, pg. 40 of [25]. Namely, for some $\delta > 0$, the C^∞ vector field $m(x)$ satisfies

$$-m(x) \cdot \nu \geq \delta \quad \text{a.e. on } \Gamma_s \quad (73)$$

Specifying this vector field in (71), and considering that $z_n|_{\Gamma_s} = 0$, we have then

$$-\frac{1}{2} \int_{\Gamma_s} \left| \frac{\partial z_n}{\partial \nu} \right|^2 m \cdot \nu d\Gamma_s$$

$$\begin{aligned}
&= \int_{\Omega_s} M \nabla z_n \cdot \nabla z_n d\Omega_s \\
&\quad + \frac{1}{2} \int_{\Omega_s} \{\beta^2 |z_n|^2 - |\nabla z_n|^2\} d\Omega_s \\
&\quad - \operatorname{Re} \int_{\Omega_s} \left[F_\beta^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}] \right] m \cdot \nabla \overline{z_n} d\Omega_s.
\end{aligned} \tag{74}$$

Estimating this relation via (59), ((65), (69), (68) and the Sobolev Trace map, we then have

$$\int_{\Gamma_s} \left| \frac{\partial z_n}{\partial \nu} \right|^2 d\Gamma_s \leq C_{\delta, \beta, m}, \tag{75}$$

where positive constant $C_{\delta, \beta, m}$ is independent of $n = 1, 2, \dots$

STEP 3: (An energy estimate for h_{jn})

We multiply both sides of the thin wave h_{jn} – equation (61) by h_{jn} , integrate and subsequently integrate by parts to have for $1 \leq j \leq K$,

$$\begin{aligned}
\int_{\Gamma_j} |\nabla h_{jn}|^2 d\Gamma_j &= \int_{\Gamma_j} \frac{\partial w_{0n}}{\partial \nu} h_{jn} d\Gamma_j \\
&\quad + (\beta^2 - 1) \int_{\Gamma_j} |h_{jn}|^2 d\Gamma_j - \int_{\Gamma_j} \frac{\partial u_n}{\partial \nu} h_{jn} d\Gamma_j \\
&\quad + \int_{\Gamma_j} (\psi_{jn}^* + i\beta \varphi_{jn}^*) h_{jn} d\Gamma_j
\end{aligned} \tag{76}$$

Here, we are also implicitly using $D(\mathbf{A})$ -criterion (A.iv). For the first term on RHS: we note that upon combining the regularity for D_s in (68) with an integration by parts, we have that

$$\frac{\partial}{\partial \nu} D_s \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_s), H^{-\frac{1}{2}}(\Omega_s)) \tag{77}$$

This gives the estimate, via the decomposition (69),

$$\left\| \frac{\partial w_{0n}}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} \leq C \left[\left\| \frac{\partial z_n}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} + \left\| i\beta \frac{\partial}{\partial \nu} D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}] \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} \right] \leq C_\beta, \tag{78}$$

after also using (59), (65), the Sobolev Trace Map, and (75). Applying this estimate to RHS of (76), along with (66), (67), and (59) we have

$$\lim_{n \rightarrow \infty} h_{jn} = 0 \text{ in } H^1(\Gamma_j), \quad 1 \leq j \leq K. \tag{79}$$

STEP 4:

We note from the previous step that the limit in (79) when applied to the equation

$$\frac{\partial w_{0n}}{\partial \nu}|_{\Gamma_j} = -\Delta h_{jn} + (1 - \beta^2)h_{jn} + \frac{\partial u_n}{\partial \nu} - (\psi_{jn}^* + i\beta\varphi_{jn}^*) \text{ in } \Gamma_j, \quad 1 \leq j \leq K,$$

gives

$$\lim_{n \rightarrow \infty} \frac{\partial w_{0n}}{\partial \nu}|_{\Gamma_j} = 0 \text{ in } H^{-1}(\Gamma_j). \quad (80)$$

In obtaining this limit, along with (79), we are also using (67) and (59). In turn, via an interpolation we have for $1 \leq j \leq K$,

$$\begin{aligned} \left\| \frac{\partial z_n}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)} &\leq C \left\| \frac{\partial z_n}{\partial \nu} \right\|_{H^{-1}(\Gamma_j)}^{\frac{1}{2}} \left\| \frac{\partial z_n}{\partial \nu} \right\|_{L^2(\Gamma_j)}^{\frac{1}{2}} \\ &= C \left\| \frac{\partial w_{0n}}{\partial \nu} + i\beta \frac{\partial}{\partial \nu} D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}] \right\|_{H^{-1}(\Gamma_s)}^{\frac{1}{2}} \left\| \frac{\partial z_n}{\partial \nu} \right\|_{L^2(\Gamma_j)}^{\frac{1}{2}} \end{aligned} \quad (81)$$

Applying (77), (59), (80) and (75) to RHS of (81), we have now (upon summing up over j),

$$\lim_{n \rightarrow \infty} \frac{\partial z_n}{\partial \nu} = 0 \text{ in } H^{-\frac{1}{2}}(\Gamma_s). \quad (82)$$

STEP 5: By (59) we have that $\{z_n\}$ of (69) converges weakly to, say, z in $H_0^1(\Omega_s)$. With this limit in mind, we multiply both sides of the wave equation in (70) by given $\eta \in H^1(\Omega_s)$. Integrating by parts we then have

$$\begin{aligned} &-\beta^2(z_n, \eta)_{\Omega_s} + (\nabla z_n, \nabla \eta)_{\Omega_s} + \left\langle \frac{\partial z_n}{\partial \nu}, \eta \right\rangle_{\Gamma_s} \\ &= (w_{1n}^* + i\beta w_{0n}^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}], \eta)_{\Omega_s}, \quad \forall \eta \in H^1(\Omega_s). \end{aligned}$$

Taking the limit of both sides of this equation, while taking into account (59), (65), (68), The Sobolev Trace Map, and (82), we obtain that $z \in H_0^1(\Omega_s)$ satisfies the variational problem

$$-\beta^2(z, \eta)_{\Omega_s} + (\nabla z, \nabla \eta)_{\Omega_s} = 0, \quad \forall \eta \in H^1(\Omega_s)$$

That is, z satisfies the overdetermined eigenvalue problem

$$\begin{cases} -\Delta z = \beta^2 z & \text{in } \Omega_s \\ z|_{\Gamma_s} = \frac{\partial z}{\partial \nu}|_{\Gamma_s} = 0 \end{cases}$$

which gives that

$$z = 0 \text{ in } \Omega_s$$

Combining this convergence with (69), (65), (59) and (68), we get

$$\lim_{n \rightarrow \infty} w_{0n} = 0 \quad \text{in } H^1(\Omega_s). \quad (83)$$

Completion of the Proof of Lemma 9

The resolvent relations in (61), (62) and the convergences (66), (83) give also

$$\begin{cases} \lim_{n \rightarrow \infty} \xi_{jn} = 0 \quad \text{in } L^2(\Gamma_j), \quad 1 \leq j \leq K \\ \lim_{n \rightarrow \infty} w_{1n} = 0 \quad \text{in } H^1(\Omega_s) \end{cases} \quad (84)$$

Collecting now, (65), (79), (83) and (84) we have

$$\lim_{n \rightarrow \infty} \Phi_n = 0 \quad \text{in } \mathbf{H},$$

which contradicts (59) and finishes the proof of Lemma 9.

Lastly, we give the following Corollary regarding the residual spectrum $\sigma_r(\mathbf{A})$:

Corollary 10. *The residual spectrum $\sigma_r(\mathbf{A})$ of $\mathbf{A}$ does not intersect the imaginary axis.*

Proof. Given the form of the adjoint operator $\mathbf{A}^* : \mathbf{H} \rightarrow \mathbf{H}$ in Proposition 8, then proceeding identically as in the proof of Lemma 9 we obtain

$$\sigma_p(\mathbf{A}^*) \cap i\mathbb{R} = \sigma_c(\mathbf{A}^*) \cap i\mathbb{R} = \emptyset$$

which finishes the proof of Corollary 10. $\square$

Now, having established the above results for the spectrum of $\mathbf{A}$, we are in a position to give the proof of Theorem 2:

Proof of Theorem 2. If we combine the above results Proposition 7, Lemma 9 and Corollary 10 and remember that $\{e^{At}\}_{t \geq 0}$ is a contraction semigroup, the strong stability result follows immediately from the application of Theorem 6.

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