

Generalized $\mathbb{Z}$ -homotopy fixed points of C_n spectra with applications to norms of $MU_{\mathbb{R}}$

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ABSTRACT. We introduce a computationally tractable way to describe the $\mathbb{Z}$ -homotopy fixed points of a C_n -spectrum E , producing a genuine C_n spectrum $E^{hn\mathbb{Z}}$ whose fixed and homotopy fixed points agree and are the $\mathbb{Z}$ -homotopy fixed points of E . These form the bottom piece of a contravariant functor from the divisor poset of n to genuine C_n -spectra, and when E is an N_{∞} -ring spectrum, this functor lifts to a functor of N_{∞} -ring spectra.

For spectra like the Real Johnson–Wilson theories or the norms of Real bordism, the slice spectral sequence provides a way to easily compute the $RO(G)$ -graded homotopy groups of the spectrum $E^{hn\mathbb{Z}}$, giving the homotopy groups of the $\mathbb{Z}$ -homotopy fixed points. For the more general spectra in the contravariant functor, the slice spectral sequences interpolate between the one for the norm of Real bordism and the especially simple $\mathbb{Z}$ -homotopy fixed point case, giving us a family of new tools to simplify slice computations.

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1. Introduction

In describing the category of K -local spectra [8], Bousfield considered a “united” K -theory, combining the fixed point Mackey functor for the C_2 -spectrum of Atiyah’s Real K -theory [3] with Anderson and Green’s “self-conjugate” K -theory K_{SC} [1, 11]. Self-conjugate K -theory assigns to a space X the isomorphism classes of pairs (E, ϕ) , where E is a complex vector bundle on X and where ϕ is an isomorphism

$$E \xrightarrow{\cong} \bar{E}.$$

This can be described as the homotopy fixed points of KU with respect to the $\mathbb{Z}$ action given by having a generator act via complex conjugation. Since Real K -theory is cofree, this is equivalently the $\mathbb{Z}$ -homotopy fixed points of the C_2 -spectrum $K_{\mathbb{R}}$.

Atiyah’s Real K -theory is part of a bi-infinite family of genuine equivariant cohomology theories built out of the Fujii–Landweber spectrum of Real bordism [10, 23] and out of the norms of this to larger groups [17]. In the C_2 -equivariant context, foundational work of Hu–Kriz described Real analogues of many of the classical spectra encountered in chromatic homotopy [20], and Kitchloo–Wilson have used the Real Johnson–Wilson theories to describe new non-immersion results [22]. In chromatic height 2, the corresponding Real truncated Brown–Peterson spectrum can be taken to be the spectrum of (connective) topological modular forms with a $\Gamma_1(3)$ -structure [19].

The norm functor allows us to extend these classical chromatic constructions to larger groups G which contain C_2 , starting with the universal case of $MU_{\mathbb{R}}$ [17]. This gives the “hyperreal” bordism spectrum

$$MU^{((G))} = N_{C_2}^G MU_{\mathbb{R}},$$

and the localizations of quotients of it play a role in G -equivariant chromatic homotopy theory. For example, recent work of Hahn–Shi shows that if G is a finite subgroup of the Morava stabilizer group $\mathbb{S}_n$ which contains $C_2 = \{\pm 1\}$, then there is a G -equivariant orientation map

$$MU^{((G))} \rightarrow E_n,$$

a hyperreal orientation of E_n [14].

In all of these cases, we have a finite, cyclic group C_n acting on a spectrum. A choice of generator of that cyclic group gives a surjection $\mathbb{Z} \rightarrow C_n$, and this allows us to form the $\mathbb{Z}$ -homotopy fixed points for any C_n -spectrum.

Theorem 1.1. *For any C_n -spectrum E , there is a cofree C_n -spectrum $E^{hn\mathbb{Z}}$ such that for all subgroups $C_d \subset C_n$, we have*

$$(E^{hn\mathbb{Z}})^{C_d} \simeq E^{h(\frac{n}{d}\mathbb{Z})},$$

the homotopy fixed points of E with respect to the subgroups $\frac{n}{d}\mathbb{Z}$.

The reason we care about this generality is that it allows us to embed the $\mathbb{Z}$ -graded homotopy groups into more general $RO(G)$ -graded homotopy Mackey functors.

Corollary 1.2. *The homotopy groups of the $\mathbb{Z}$ -homotopy fixed points are the $\mathbb{Z}$ -graded piece of the $RO(C_n)$ -graded homotopy groups of $E^{hn\mathbb{Z}}$.*

These $RO(C_n)$ -graded homotopy groups give information about the orientability of representations with respect to the theory $E^{hn\mathbb{Z}}$ and about the image of the representation ring in the Picard group, as in [4].

Including the representation gradings simplifies many of the computations and it allows for an interesting generalization. The C_n -spectra $E^{hn\mathbb{Z}}$ have another interpretation that makes leverages this extended grading and provides intuition. Associated to any representation V of C_n for which $V^{C_n} = \{0\}$, we have an Euler class

$$a_V: S^0 \rightarrow S^V.$$

These Euler classes play a fundamental role in equivariant homotopy theory. Inverting a_V gives a model for the nullification which destroys any cells with stabilizer a subgroup H such that $V^H \neq \{0\}$, and varying the representation allows us to isolate particular conjugacy classes of stabilizers. These classes and some of their basic properties are reviewed in Section 2.

The C_n -spectrum $E^{hn\mathbb{Z}}$ is the quotient of E by $a_{\lambda(1)}$, where $\lambda(1)$ is the faithful, irreducible representation of C_n given by choosing a primitive n th root of unity. The representation $\lambda(1)$ is itself part of a family of representations $\lambda(k)$, indexed by the integers. If we instead quotient by other fixed-point free, orientable representations of C_n , then we produce other, interesting spectra.

Theorem 1.3. *We have a contravariant functor from the divisor poset of n to C_n -spectra, given by*

$$d \mapsto E/a_{\lambda(d)}.$$

Moreover, if E is an $\mathcal{O}$ -algebra for an N_∞ -operad $\mathcal{O}$, then for all $d \mid n$, $E/a_{\lambda(d)}$ is an $\mathcal{O}'$ -algebra for some N_∞ -operad $\mathcal{O}'$ under $\mathcal{O}$ that depends only on d , and the maps above out of $E/a_{\lambda(d)}$ are maps of $\mathcal{O}'$ -algebras.

The functor $E \mapsto E/a_{\lambda(d)}$ is a lax-monoidal functor, so the fact that this preserves operadic algebras is unsurprising. The interest part of the previous theorem is that the quotients $E/a_{\lambda(d)}$ become increasingly structured as we work down the divisor poset of d : even if E were a naive E_∞ -ring spectrum, $E/a_{\lambda(1)}$ is always a G - E_∞ -ring spectrum.

The second part of the paper shows the computational usefulness of this more general framework, especially when combined with the slice spectral sequence of [17]. In Section 4, we begin by describing the homotopy groups of the “self-conjugate” Johnson–Wilson theories $E_{\mathbb{R}}(n)^{h2\mathbb{Z}}$, using the fiber

sequences of Kitchloo–Wilson, showing that just as $K_{SC} = KO/\eta^2$, we have

$$E_{SC}(n) = EO(n)/x^2,$$

where x is the distinguished class identified by Kitchloo–Wilson.

We then generalize this to arbitrary heights and arbitrary cyclic 2-groups, exploring the universal example of the norms of $MU_{\mathbb{R}}$ as in [17]. The slice filtration of $MU^{((G))}$ gives us a spectral sequence computing the $RO(G)$ -graded homotopy Mackey functors of $MU^{((G))}/a_{\lambda(1)}$.

Theorem 1.4. *The slice spectral sequence for $MU^{((G))}/a_{\lambda(1)}$ collapses at E_2 with no extensions. For every orientable representation V , the associated orientation class u_V survives the slice spectral sequence, providing an orientation for the V -sphere.*

The collapse of the slice spectral sequence here is formal: everything will be concentrated in filtrations 0 and 1. The orientability with respect to $MU^{((G))}/a_{\lambda(1)}$ of all orientable representations comes from a more general analysis of the cofreeness of the spectra $E/a_{\lambda(d)}$.

The slice spectral sequences for $MU^{((G))}/a_{\lambda(d)}$ are all compatible in a natural way. Since the subgroups of C_{2^n} are nested, we have a tower of spectra with $(n+1)$ -layers:

$$MU^{((G))}/a_{\lambda(2^n)} \rightarrow MU^{((G))}/a_{\lambda(2^{n-1})} \rightarrow \cdots \rightarrow MU^{((G))}/a_{\lambda(1)}.$$

The slice filtration of $MU^{((G))}$ gives a spectral sequence computing the $RO(G)$ -graded homotopy groups of each of these spectra, and the maps in the tower produce maps of spectral sequences. The first spectrum is $MU^{((G))} \vee \Sigma^{-1}MU^{((G))}$, and for exposition, we may replace it with the unit copy of $MU^{((G))}$.

At one end, we have the ordinary slice spectral sequence for the $RO(G)$ -graded homotopy groups of $MU^{((G))}$. At the other end, we have the slice spectral sequence for the $RO(G)$ -graded homotopy of $MU^{((G))}/a_{\lambda(1)}$ considered in Section 4. In the first, we have all Euler classes a_V and no surviving orientation classes u_V . In the last, we have no Euler classes and all of the orientation classes u_V survive. As we pass from one spectral sequence to the next, we kill another Euler class and find that the corresponding orientation class (and all powers) survive. Since the complicated part of the slice spectral sequence is exactly the orientation classes (all other classes are permanent cycles), this procedure allows us to isolate each orientation class, one irreducible at a time. We conclude with several computations results which give some progress along this front.

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2. Representations and Euler classes

The irreducible representations of C_n over $\mathbb{R}$ are parameterized by conjugate pairs of n th roots of unity.

Notation 2.1. Let $\zeta = e^{2\pi i/n}$ be a fixed primitive n th root of unity.

Definition 2.2. For each k , let $\lambda(k)$ denote the real representation of $C_n = \langle \gamma \rangle$ given by complex numbers with

$$\gamma \cdot z = \zeta^k z.$$

The following are immediate consequences of the definition and elementary representation theory.

Proposition 2.3.

- (1) For each k , the identity gives an isomorphism $\lambda(k+n) \cong \lambda(k)$.
- (2) For each k , complex conjugation gives an isomorphism $\lambda(k) \cong \lambda(-k)$.
- (3) If $k = 0$, then $\lambda(k)$ is a sum of two copies of the trivial representation.
- (4) If n is odd, then for each $k \not\equiv 0 \pmod{n}$, $\lambda(k)$ is irreducible.
- (5) If n is even, then for each $k \not\equiv n/2 \pmod{n}$, $\lambda(k)$ is irreducible, and $\lambda(n/2)$ is two copies of the sign representation of C_n .

A basic object in equivariant stable homotopy is the Euler class of a fixed-point free representation, and for $k \not\equiv 0 \pmod{n}$, the representations $\lambda(k)$ are all fixed point free.

Definition 2.4. If V is a representation such that $V^{C_n} = \{0\}$, then let

$$a_V: S^0 \rightarrow S^V$$

be the inclusion of the origin and the point at infinity.

The classes a_V are all essential, but there are various interesting divisibility relations among them arising from our ability to use non-linear maps between spheres.

Lemma 2.5. For each k and ℓ , there is a class

$$a_{\lambda(k\ell)-\lambda(k)}: S^{\lambda(k)} \rightarrow S^{\lambda(k\ell)}$$

such that

$$a_{\lambda(k\ell)} = a_{(\lambda(k\ell)-\lambda(k))} a_{\lambda(k)}.$$

Proof. The map is given by the ℓ -power map $z \mapsto z^\ell$. □

Corollary 2.6. *If $\gcd(k, n) = \gcd(\ell, n)$, then there are classes*

$$a_{(\lambda(\ell)-\lambda(k))} \text{ and } a_{(\lambda(k)-\lambda(\ell))}$$

such that

$$a_{\lambda(\ell)} = a_{(\lambda(\ell)-\lambda(k))} a_{\lambda(k)} \text{ and } a_{\lambda(k)} = a_{(\lambda(k)-\lambda(\ell))} a_{\lambda(\ell)}.$$

Remark 2.7. We chose notation to match the feature of Euler classes:

$$a_{V \oplus W} = a_V \cdot a_W,$$

but even here, the notation here is slightly misleading, since in general

$$a_{(\lambda(\ell)-\lambda(k))} \cdot a_{(\lambda(k)-\lambda(\ell))} \neq a_0 = 1.$$

On fixed points, this previous corollary shows this is true. On underlying homotopy, this is some map of non-trivial degree which is prime to n . This failure to be an equivalence is detecting that $\pi_0(S^{\lambda(\ell)-\lambda(k)})$ is a non-trivial rank one projective Mackey functor. This distinction goes away for hyperreal spectra, however, since π_0 of the norms of $MU_{\mathbb{R}}$ is $\mathbb{Z}$ which has a trivial Picard group.

Proposition 2.8. *Let $\mathcal{S}p_{(n)}^{C_n}$ denote the category of C_n -spectra in which all primes not dividing n are inverted, and assume that $\gcd(\ell, n) = \gcd(k, n)$. Then the map $a_{(\lambda(\ell)-\lambda(k))}$ is a weak-equivalence.*

If we are working locally, we may therefore ignore any distinction between $\lambda(k)$ and $\lambda(\ell)$, provided $\gcd(k, n) = \gcd(\ell, n)$.

Corollary 2.9. *It suffices to consider $a_{\lambda(p^k)}$ for p a prime and p^k dividing n , and in this case, $a_{\lambda(p^{k-1})}$ divides $a_{\lambda(p^k)}$.*

3. Quotienting by $a_{\lambda(d)}$

3.1. Various forms of the quotients.

Definition 3.1. For any C_n -spectrum E , let $E/a_{\lambda(k)}$ denote the cofiber of

$$\Sigma^{-\lambda(k)} E \xrightarrow{a_{\lambda(k)}} E.$$

Remark 3.2. If $n|k$, then $a_{\lambda(k)}$ is the zero map, and the spectrum $E/a_{\lambda(k)}$ is just $E \vee \Sigma^{-1} E$.

Since the spectra $E/a_{\lambda(k)}$ are defined by a cofiber sequence, we have a natural long exact sequence given by maps into this.

Proposition 3.3. *For any virtual representation V and for any E , we have a natural long exact sequence*

$$\cdots \rightarrow \pi_{V+\lambda(k)}(E) \rightarrow \pi_V(E) \rightarrow \pi_V(E/a_{\lambda(k)}) \rightarrow \pi_{V+\lambda(k)-1}(E) \rightarrow \cdots$$

Remark 3.4. This gives us a way to interpret the exact sequence in Proposition 3.3: this is an odd primary analogue of the long exact sequence which connects the restriction to an index 2 subgroup, multiplication by a corresponding a_σ , and a signed transfer.

In general, these spectra are much better behaved than we might have originally expected as a cofiber, since we can reinterpret these as functions out of a space.

Proposition 3.5. *We have a natural weak equivalence*

$$E/a_{\lambda(k)} \simeq F(S(\lambda(k))_+, E).$$

Proof. This immediate from the cofiber sequence in pointed spaces

$$S(\lambda(k))_+ \rightarrow S^0 \rightarrow S^{\lambda(k)}. \quad \square$$

Corollary 3.6. *The functor $E \mapsto E/a_{\lambda(k)}$ is a lax monoidal functor via the composite*

$$E/a_{\lambda(k)} \wedge E'/a_{\lambda(k)} \rightarrow F(S(\lambda(k))_+ \wedge S(\lambda(k))_+, E \wedge E') \rightarrow F(S(\lambda(k))_+, E \wedge E'),$$

where the last map is induced by the diagonal.

Corollary 3.7. *The functor $E \mapsto E/a_{\lambda(d)}$ preserves operadic algebras.*

Remark 3.8. This is just the cotensoring of operadic algebras in G -spectra over G -spaces.

Corollary 3.9. *If R is a ring spectrum and M is an R -module, then $M/a_{\lambda(d)}$ is a module over $R/a_{\lambda(d)}$.*

Corollary 3.10. *Let $d = \gcd(k, n)$. Then for all ℓ such that $d|\ell$, multiplication by $a_{\lambda(\ell)}$ is zero in the $RO(G)$ -graded homotopy of $E/a_{\lambda(k)}$.*

Proof. By Corollary 3.9, it suffices to show this for $E = S^0$. By construction, the image of $a_{\lambda(\ell)}$ in $\pi_*(S^0/a_{\lambda(d)})$ is zero, and since this is a ring, we deduce that multiplication by $a_{\lambda(\ell)}$ is always zero. $\square$

The divisibilities between the $a_{\lambda(k)}$ produce a tower of spectra under E .

Proposition 3.11. *If $k|\ell$, then there is a natural map of C_n -spectra under E*

$$E/a_{\lambda(\ell)} \rightarrow E/a_{\lambda(k)}.$$

Proof. If $k|\ell$, then we have a map of cofiber sequences

$$\begin{array}{ccccc} S(\lambda(k))_+ & \longrightarrow & S^0 & \xrightarrow{a_{\lambda(k)}} & S^{\lambda(k)} \\ \downarrow z \mapsto z^{\ell/k} & & \downarrow & & \downarrow a_{\lambda(\ell)}/a_{\lambda(k)} \\ S(\lambda(\ell))_+ & \longrightarrow & S^0 & \xrightarrow{a_{\lambda(\ell)}} & S^{\lambda(\ell)}. \end{array}$$

Mapping out of this gives the desired result, together with the compatible maps from E . $\square$

Corollary 3.12. *The assignment*

$$d \mapsto E/a_{\lambda(d)}$$

extends naturally to a contravariant functor from the poset of divisors of n to C_n -spectra.

This has an equivariant algebraic geometry interpretation. Much of the recent work in equivariant algebraic geometry begins with the prime spectrum of the Burnside ring $\text{Spec}(A)$ of finite G -sets. There are distinguished prime ideals in the Burnside ring for A , namely the ideals $I_{\mathcal{F}}$ generated by those finite G -sets such that their stabilizer subgroups are in some fixed family of subgroups $\mathcal{F}$. These are obviously nested in the same way the corresponding families are nested. To each non-trivial, irreducible representation V , we have a family of subgroups

$$\mathcal{F}_V = \{H \mid V^H \neq \{0\}\},$$

and the effect on π_0 of the localization map $S^0 \rightarrow S^0[a_V^{-1}]$ is the reduction modulo $I_{\mathcal{F}}$ (see, for example, [15]). The quotient S^0/a_V is then instead isolating the ideal $I_{\mathcal{F}}$. These are those pieces built out of cells for which the stabilizer is in $\mathcal{F}$, and the functor above should be thought of as carving down to smaller and smaller ideals.

3.2. A genuine C_n -spectrum computing $\mathbb{Z}$ -homotopy fixed points. Our initial interest in the spectra $R/a_{\lambda(k)}$ came from thinking about $\mathbb{Z}$ -homotopy fixed points of a C_n -spectrum.

Definition 3.13. If E is a C_n -spectrum, then let

$$E^{h\mathbb{Z}} = F(S(\lambda(1))_+, E).$$

Proposition 3.5 connects this back to the earlier discussion.

Corollary 3.14. For any C_n -spectrum E , we have a natural equivalence

$$E^{h\mathbb{Z}} \simeq E/a_{\lambda(1)}.$$

The name comes from an analysis of the fixed points of the quotients by the Euler classes.

Proposition 3.15. For any C_n -spectrum E and for any divisor d of n , we have a natural weak equivalence

$$(E/a_{\lambda(d)})^{C_n} \simeq (E^{C_d})^{h\mathbb{Z}},$$

where the right-hand side is the ordinary $\mathbb{Z}$ homotopy fixed points, with $\mathbb{Z}$ acting via the image of the generator γ in the quotient group C_n/C_d .

Proof. We have a cofiber sequence of C_n -spectra

$$C_n/C_{d+} \xrightarrow{1-\gamma} C_n/C_{d+} \rightarrow S(\lambda(d))_+.$$

This means that we have another fiber sequence of C_n -spectra

$$E/a_{\lambda(d)} \rightarrow F(C_n/C_{d+}, E) \xrightarrow{1-\gamma} F(C_n/C_{d+}, E).$$

Since the fixed points of $F(C_n/C_{d+}, E)$ are E^{C_d} , the result follows. $\square$

Remark 3.16. If we apply the C_n -fixed points to the diagram of spectra indexed by the divisor poset of n of Corollary 3.12, then we recover the composite of the coefficient system of spectra arising from the fixed points with the $\mathbb{Z}$ -homotopy fixed points.

By restriction, our $n\mathbb{Z}$ homotopy fixed point spectrum also gives us the homotopy fixed points for any subgroup between $n\mathbb{Z}$ and $\mathbb{Z}$.

Corollary 3.17. *If $d|n$, then*

$$E^h(\frac{n}{d}\mathbb{Z}) = (E^{hn\mathbb{Z}})^{C_d}.$$

Recall that any G -spectrum E gives us a spectral Mackey functor via the functor

$$T \mapsto (F(T, E))^G,$$

which extends to a spectral enrichment of the Burnside category by work of Guillou-May [12]. We therefore have a spectral Mackey functor with

$$C_n/C_d \mapsto E^h(\frac{n}{d}\mathbb{Z}).$$

Remark 3.18. The quotient map

$$S(\lambda(1))_+ \rightarrow S^0$$

gives a canonical C_n -equivariant map

$$E \rightarrow E^{hn\mathbb{Z}}.$$

A choice of point in $S(\lambda(1))$ gives a retraction

$$i_e^* E^{hn\mathbb{Z}} \rightarrow i_e^* E.$$

These give us a way to connect the Mackey functor structure on $E^{hn\mathbb{Z}}$ to that of E itself.

3.3. Localness with respect to families. For k and n not relatively prime, the spectra $E/a_{\lambda(k)}$ are in general not cofree. They are, however, local with respect to a larger family $\mathcal{F}_k$ of subgroups in the sense that the natural map

$$E/a_{\lambda(k)} \rightarrow F(E\mathcal{F}_{k+}, E/a_{\lambda(k)})$$

is a weak equivalence.

Definition 3.19. Let $d|n$. Let

$$\mathcal{F}_d = \{H \mid H \subset C_d \subset C_n\}.$$

For k not dividing n , let

$$\mathcal{F}_k = \mathcal{F}_{\gcd(k, n)}.$$

Proposition 3.20. *The C_n -spectrum $E/a_{\lambda(k)}$ is local for the family $\mathcal{F}_k$: the natural map*

$$E/a_{\lambda(k)} \rightarrow F(E\mathcal{F}_{k+}, E/a_{\lambda(k)})$$

is a weak equivalence.

Proof. Let $d = \gcd(n, k)$. Since every element of C_d acts as the identity in $\lambda(k)$, the representation is the pullback of the representation $\lambda(k)$ for the group $C_n/C_d \cong C_{n/d}$. Similarly, the space $E\mathcal{F}_k$ is the pullback of the space $E(C_n/C_d)$ under the quotient map. Since the C_n/C_d -equivariant map

$$E(C_n/C_d) \times S(\lambda(k)) \rightarrow S(\lambda(k))$$

is a C_n/C_d -equivariant equivalence, the pullback

$$E\mathcal{F}_k \times S(\lambda(k)) \rightarrow S(\lambda(k))$$

is a C_n -equivariant equivalence. This means we have a natural equivalence for any C_n -spectrum E :

$$F\left(S(\lambda(k))_+, E\right) \xrightarrow{\cong} F\left(E\mathcal{F}_{k+} \wedge S(\lambda(k))_+, E\right) \cong F(E\mathcal{F}_{k+}, E/a_{\lambda(k)}),$$

as desired. $\square$

Remark 3.21. For cyclic p -groups, the linear ordering of the subgroups induces divisibility relations amongst the classes $a_{\lambda(k)}$. The argument above shows that if the p -adic valuations of k and ℓ agree, then $a_{\lambda(k)}$ and $a_{\lambda(\ell)}$ each divide the other. On the other hand, the class $a_{\lambda(p^j)}$ divides $a_{\lambda(p^{j+1})}$ (but not $a_{\lambda(p^{j-1})}$). This means that if we consider the commutative ring spectra

$$S^0/a_{\lambda(p^k)} = F\left(S(\lambda(p^k))_+, S^0\right),$$

then all of the classes $a_{\lambda(p^j)}$ are equal to zero, and hence the geometric fixed points for any subgroup containing $C_{p^{k-1}}$ are contractible.

3.4. Structured multiplications. Corollary 3.7 says that the functor

$$E \mapsto E/a_{\lambda(k)}$$

preserves operadic algebras. When the operad in question is an N_∞ operad, then these quotients actually have more commutativity than we might initially expect (and in general, they are more commutative than the original ring).

Recall that in genuine equivariant homotopy, there are a family of equivariant refinements of the classical E_∞ -operad. These N_∞ operads, defined in [5], have the property that algebras over them have not only a coherently commutative multiplication (so a “naive” E_∞ structure) but also coherently defined norm maps for so pairs of subgroups. This structure is described by an indexing system or equivalently an indexing category [5, 6].

Definition 3.22. Let $\mathcal{F}in^G$ denote the category of finite G -sets.

An indexing subcategory of $\mathcal{F}in^G$ is a wide, pullback stable subcategory of $\mathcal{F}in^G$ that is finite coproduct complete.

The collection of all indexing subcategories of $\mathcal{F}in^G$ becomes a poset under inclusion, and work of Blumberg–Hill and Gutierrez–White shows that there is an equivalence of categories between the homotopy category of N_∞ operads and (see also [7, 26]) this poset.

Theorem 3.23 ([5, Theorem 3.24], [13]). *There is an equivalence of categories from the homotopy category of N_∞ -operads to the poset of indexing subcategories:*

$$\mathcal{O} \mapsto \mathcal{F}in_{\mathcal{O}}^G.$$

Definition 3.24 ([5, Definition 6.14]). Let $\mathcal{O}$ be an N_∞ -operad for C_n and let $H \subset C_n$ be a subgroup. Define a new indexing subcategory $\mathcal{F}in_{N_H^G \mathcal{O}}^G$ by saying that a map $f: S \rightarrow T$ is in $\mathcal{F}in_{N_H^G \mathcal{O}}^G$ if and only if

$$G/K \times f: G/K \times S \rightarrow G/K \times T \in \mathcal{F}in_{\mathcal{O}}^G.$$

Proposition 3.25. *The assignment $\mathcal{O} \mapsto N_H^G \mathcal{O}$ gives an endofunctor of the homotopy category of N_∞ operads. We have a natural map*

$$\mathcal{O} \rightarrow N_H^G \mathcal{O}.$$

Theorem 3.26. *Let $H_k = \ker(C_n \xrightarrow{g \mapsto g^k} C_n)$. Then if R is an $\mathcal{O}$ -algebra, then $R/a_{\lambda(k)}$ is an algebra over $N_{H_k}^G \mathcal{O}$. Moreover, if $k|\ell$, then the map*

$$R/a_{\lambda(\ell)} \rightarrow R/a_{\lambda(k)}$$

is a map of $N_{H_\ell}^G \mathcal{O}$ -algebras.

Proof. By Proposition 3.20, the spectrum $R/a_{\lambda(k)}$ is local for the family of subgroups of $H_k = C_d$, where $d = \gcd(k, n)$. In particular, it is an algebra over the operad

$$F(E\mathcal{F}_{d+}, \mathcal{O}).$$

The proof of the special case that $d = 1$ [5, Proposition 6.25] goes through without change, showing that this localized operad is $N_{H_k}^G \mathcal{O}$. $\square$

Corollary 3.27. *If R is an $\mathcal{O}$ -algebra, then $R/a_{\lambda(1)}$ is a G - E_∞ -ring spectrum.*

Algebras over a G - E_∞ -operad have all norms in their homotopy Mackey functors. These constructions were first studied by Brun who showed that π_0 carried the structure of Tambara's TNR-functors [9], [27], and then in other degrees, this was described in [17] and in work of Angeltveit–Bohmann [2].

Corollary 3.28. *If R is an $\mathcal{O}$ -algebra, then $\pi_\star(R/a_{\lambda(1)})$ is an $\underline{RO}$ -graded Tambara functor in the sense of [2].*

Remark 3.29. There is an obvious generalization of Angeltveit–Bohmann to incomplete Tambara functors, and the analogue of Corollary 3.28 holds there.

3.5. A Greenlees–Tate approach. We now restrict attention to $n = p^m$ for p a prime. We can combine our quotienting by Euler classes with inverting others, giving an approach to computation. First note that the divisibility relations immediate give the following.

Proposition 3.30. *Inverting a class $a_{\lambda(p^k)}$ also inverts $a_{\lambda(p^{k-1})}$.*

In general, inverting the classes $a_{\lambda(d)}$ is closely connected to the geometric fixed points.

Proposition 3.31. *The infinite $\lambda(d)$ -sphere $S^{\infty\lambda(d)} = S^0[a_{\lambda(d)}^{-1}]$ is a model for $\tilde{E}\mathcal{F}_d$.*

Proof. The colimit of the unit spheres in $k\lambda(d)$ as $k \rightarrow \infty$ is a model for $E\mathcal{F}_d$, from which the result follows. $\square$

Corollary 3.32. *The C_{p^k} -geometric fixed points of E can be computed as*

$$\Phi^{C_{p^k}} E = (E[a_{\lambda(p^{k-1})}^{-1}])^{C_{p^k}}.$$

The C_{p^k} -fixed points of a C_{p^m} -spectrum are naturally a C_{p^m}/C_{p^k} -spectrum. Since the fixed points of

$$(E[a_{\lambda(p^{k-1})}^{-1}])^{C_{p^r}} \simeq *$$

for $r < k$, the C_{p^m} -spectrum $E[a_{\lambda(p^{k-1})}^{-1}]$ is completely determined by its C_{p^k} -fixed points.

Theorem 3.33. *For any C_{p^m} -spectrum E and for $\ell \geq k$, we have a natural weak equivalence*

$$\Phi^{C_{p^k}} (E/a_{\lambda(p^\ell)}) \simeq (\Phi^{C_{p^k}} E)/a_{\lambda(p^\ell)}.$$

Proof. A finite G -CW complex like $S(\lambda(p^\ell))$ is small, and hence

$$\begin{aligned} F\left(S(\lambda(p^\ell))_+, \varinjlim \Sigma^{s\lambda(p^{k-1})} E\right) &\simeq \varinjlim F\left(S(\lambda(p^\ell))_+, \Sigma^{s\lambda(p^{k-1})} E\right) \\ &\simeq \varinjlim \Sigma^{s\lambda(p^{k-1})} F\left(S(\lambda(p^\ell))_+, E\right). \end{aligned}$$

Since $\ell \geq k$, C_{p^k} acts trivially on $S(\lambda(p^\ell))$, and hence the C_{p^k} -fixed points pass through the function spectrum:

$$F\left(S(\lambda(p^\ell))_+, E[a_{\lambda(p^{k-1})}^{-1}]\right)^{C_{p^k}} \simeq F\left(S(\lambda(p^\ell))_+, E[a_{\lambda(p^{k-1})}^{-1}]^{C_{p^k}}\right),$$

giving the result. $\square$

4. The $\mathbb{Z}$ -homotopy fixed points of hyperreal bordism spectra

4.1. The Real Johnson–Wilson spectra. Building on work of Kitchloo–Wilson, we can quickly compute the homotopy groups of the $\mathbb{Z}$ -homotopy fixed points of any of the spectra $E_{\mathbb{R}}(n)$. Work of Hahn–Shi then also allows us to compute the $\mathbb{Z}$ -homotopy fixed points of any of the Lubin–Tate spectra, viewed as C_2 -spectra [14].

Definition 4.1. Let

$$E_{SC}(n) := E_{\mathbb{R}}(n)^{h\mathbb{Z}}$$

be the “self-conjugate Johnson–Wilson theory”.

For each chromatic height n , Kitchloo–Wilson identify a particular element

$$x_n \in \pi_{b_n} EO(n),$$

where

$$b_n = 2^{2n+1} - 2^{n+2} + 1,$$

such that we have a cofiber sequence

$$\Sigma^{b_n} EO(n) \xrightarrow{x_n} EO(n) \rightarrow E(n)$$

and the homotopy fixed point spectral sequence computing $\pi_* EO(n)$ can be recast as the Bockstein spectral sequence for this cofiber sequence [21]. We can reinterpret this cofiber sequence using the equivariant periodicity in the spectrum $E_{\mathbb{R}}(n)$, using slice spectral sequence names.

Proposition 4.2. *The class $u_{2\sigma}^{2^n}$ is a permanent cycle in the slice spectral sequence for $E_{\mathbb{R}}(n)$, and multiplication by $u_{2\sigma}^{2^n}$ gives an equivariant equivalence*

$$\Sigma^{2^{n+1}} E_{\mathbb{R}}(n) \rightarrow \Sigma^{2^{n+1}\sigma} E_{\mathbb{R}}(n).$$

Combined with the isomorphism given by multiplication by $\bar{v}_n$, we can rewrite x_n in terms of a_σ :

$$x_n = a_\sigma \bar{v}_n^{2^n-1} (u_{2\sigma}^{2^n})^{2^{n-1}-1}.$$

Corollary 4.3. *We have an equivalence*

$$E_{SC}(n) \simeq EO(n)/x_n^2.$$

An almost identical analysis can be applied to the work of Hahn–Shi on the homotopy groups of the Hopkins–Miller spectra EO_n . We leave this to the interested reader.

4.2. The slice spectral sequence for $MU^{((G))}$ -cohomology. From this point on, let $G = C_{2^n}$, and again, let

$$MU^{((G))} = N_{C_2}^G MU_{\mathbb{R}}$$

be the hyperreal bordism spectrum introduced in [17].

The slice tower for $MU^{((G))}$ gives for every G -space X a spectral sequence computing the $RO(G)$ -graded Mackey functor cohomology $\underline{MU}^{((G))^\star}(X)$. This spectral sequence arises from computing homotopy classes of maps from X to the slice tower for $MU^{((G))}$, and if X is finite, then it converges strongly. Moreover, this spectral sequence is a spectral sequence of $RO(G)$ -graded Tambara functors, though we will not need this structure.

To describe the $\underline{E}_2$ -term, we need some notation and elements from [17].

Notation 4.4. Let

$$\rho_{2^k} = \rho_{C_{2^k}} = \mathbb{R}[C_{2^k}]$$

be the regular representation of C_{2^k} .

Theorem 4.5 ([17, Proposition 5.27, Lemma 5.33]). *There are classes*

$$\bar{r}_i : S^{i\rho_2} \rightarrow i_{C_2}^* MU^{((G))}$$

such that if $G = \langle \gamma \rangle$ then

$$\pi_{*\rho_2} MU^{((G))} \cong \mathbb{Z}[\bar{r}_1, \gamma \bar{r}_1, \dots, \gamma^{2^{n-1}-1} \bar{r}_1, \bar{r}_2, \dots].$$

By the free-forget adjunction for associative rings, the classes $\bar{r}_i$ give associative ring maps

$$\mathbb{S}^0[\bar{r}_i] \rightarrow i_{C_2}^* MU^{((G))},$$

and smashing these together and norming up gives us a map of associative algebras

$$A = \bigwedge_{i \geq 1} N_{C_2}^G(\mathbb{S}^0[\bar{r}_i]) \rightarrow MU^{((G))}.$$

The underlying spectrum of A looks like a polynomial algebra in the classes $\bar{r}_i$, in that it is a wedge of [representation] spheres indexed by the monic monomials in the polynomial ring $\pi_* i_e^* MU^{((G))}$. The group G acts on the various monomials via the action on generators given by

$$\gamma \cdot \gamma^j \bar{r}_i = \begin{cases} \gamma^{j+1} \bar{r}_i & j < 2^{n-1} - 1 \\ (-1)^i \bar{r}_i & j = 2^{n-1} - 1. \end{cases}$$

The sign here is reflecting the degree of the action map on the sphere $S^{i\rho_2}$, and hence is accounted for via the representations. Our indexing set for the wedge can really be viewed as the monic monomials in the ring $\pi_* i_e^* MU^{((G))} \otimes \mathbb{Z}/2$.

Notation 4.6. If $\bar{p}$ is a monic monomial in

$$\mathbb{Z}[\bar{r}_1, \gamma \bar{r}_1, \dots, \gamma^{2^{n-1}-1} \bar{r}_1, \bar{r}_2, \dots],$$

then let $H_{\bar{p}}$ be the stabilizer of $\bar{p}$ modulo 2.

Let $|\bar{p}|$ be the underlying degree, i.e. the degree when we forget all equivariance, and let

$$||\bar{p}|| = \frac{|\bar{p}|}{|H_{\bar{p}}|} \rho_{H_{\bar{p}}}.$$

Finally, let $\mathcal{I}_n$ be the set of orbits of monic monomials in

$$\mathbb{Z}/2[\bar{r}_1, \gamma \bar{r}_1, \dots, \gamma^{2^{n-1}-1} \bar{r}_1, \bar{r}_2, \dots]$$

of underlying degree n , and let $\mathcal{I} = \bigcup \mathcal{I}_n$.

The distributive law for the norm over the wedge then gives us a description of A as a G -spectrum.

Proposition 4.7. *We have an identification*

$$A \simeq \bigvee_{\bar{p} \in \mathcal{I}} G_+ \wedge_H S^{||\bar{p}||}.$$

The algebra A gives the slice associated graded.

Theorem 4.8 ([17, Theorem 6.1]). *The slice associated graded for $MU^{((G))}$ is*

$$H\mathbb{Z}[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots] := H\mathbb{Z} \wedge A \simeq \bigvee_{\bar{p} \in \mathcal{I}} G_+ \wedge_H S^{||\bar{p}||} \wedge H\mathbb{Z}.$$

The odd slices are contractible, and the $(2n)^{\text{th}}$ slice are the wedge summands of $A \wedge H\mathbb{Z}$ corresponding to monomials of underlying degree $2n$.

Corollary 4.9. *For any finite G -space X , there is a strongly convergent spectral sequence of $RO(G)$ -graded Mackey functors*

$$E_2 = \underline{H}^*(X; \mathbb{Z})[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots] \Rightarrow \underline{MU}^{((G))^\star}(X).$$

Corollary 4.10. *There is a strongly convergent spectral sequence of $RO(G)$ -graded Mackey functors*

$$E_2 = \pi_\star(H\mathbb{Z}/a_{\lambda(k)})[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots] \Rightarrow \pi_\star(MU^{((G))}/a_{\lambda(k)}).$$

A remark on the notation here might be helpful to the reader. Consider a monic monomial $\bar{p}$. The gives a summand

$$G_+ \wedge_{H_{\bar{p}}} S^{||\bar{p}||} \wedge H\mathbb{Z}$$

of $A \wedge H\mathbb{Z}$, and hence for any X , a summand of

$$\underline{H}^*(X; \mathbb{Z})[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots].$$

By construction, the summand corresponding to $\bar{p}$ is

$$\uparrow_{H_{\bar{p}}}^G \underline{H}^{\star-||\bar{p}||}(i_H^* X; \mathbb{Z}),$$

where

$$\uparrow_{H_{\bar{p}}}^G : \text{Mackey}^{H_{\bar{p}}} \rightarrow \text{Mackey}^G$$

is induction, defined by $\uparrow_{H_{\bar{p}}}^G \underline{M}(T) = \underline{M}(i_{H_{\bar{p}}}^* T)$.

The bidegrees are determined by recalling that the spectral sequence is the Adams grading applied to the spectral sequence for a tower. In particular, we have

$$E_2^{s,V} = \pi_{V-s} F(X, P_{\dim V}^{\dim V}) \Rightarrow \pi_{V-s} F(X, MU^{((G))}) = \underline{MU}^{((G))^{s-V}}(X).$$

Note that this formula works for any $V \in RO(G)$. Because the odd slices are contractible, this formula is only non-zero when $\dim V$ is even and non-negative. Consider a virtual representation V with virtual dimension $2n$. Since

$$P_{2n}^{2n} = \bigvee_{\bar{p} \in \mathcal{I}_n} G_+ \wedge_{H_{\bar{p}}} S^{||\bar{p}||} \wedge H\mathbb{Z},$$

the E_2 term for this V is:

$$\begin{aligned} E_2^{s,V} &= \pi_{V-s} F \left(X, \bigvee_{\bar{p} \in \mathcal{I}_n} G_+ \wedge_{H_{\bar{p}}} S^{||\bar{p}||} \wedge H\mathbb{Z} \right) \\ &= \bigoplus_{\bar{p} \in \mathcal{I}_n} \uparrow_{H_{\bar{p}}}^G \pi_{i_{H_{\bar{p}}}^* V - s} F(X, \Sigma^{||\bar{p}||} H\mathbb{Z}) \\ &= \bigoplus_{\bar{p} \in \mathcal{I}_n} \uparrow_{H_{\bar{p}}}^G \pi_{(i_{H_{\bar{p}}}^* V - s - ||\bar{p}||)} F(X, H\mathbb{Z}) \\ &= \bigoplus_{\bar{p} \in \mathcal{I}_n} \uparrow_{H_{\bar{p}}}^G \underline{H}^{s+||\bar{p}||-i_H^* V}(X; \mathbb{Z}). \end{aligned}$$

Remark 4.11. The slice spectral sequence for computing homotopy groups as considered in [17] lies in Quadrants I and III, (and we note that the fact that s can be negative here is important for that). Since we are using the slice filtration of $MU^{((G))}$ to compute the $MU^{((G))}$ -cohomology of a space, we no longer have the usual simple vanishing lines or regions, since the cells of X can move us into Quadrants II and IV.

4.3. The $RO(G)$ -graded homotopy of $H\mathbb{Z}/a_{\lambda(1)}$. Recall that for any orientable representation V , there are orientation classes

$$u_V \in H_{\dim V}(S^V; \mathbb{Z}) \cong \mathbb{Z}.$$

These classes have the property that for any subgroup H ,

$$i_H^* u_V \in H_{\dim V}(S^{i_H^* V}; \mathbb{Z}) \cong \mathbb{Z}$$

is a generator. In particular, multiplication by u_V is always an underlying equivalence.

Proposition 4.12. *The elements*

$$u_V \in \pi_{\dim V - V}(H\mathbb{Z}/a_{\lambda(1)})(G/G)$$

are invertible.

Proof. Since $H\mathbb{Z}/a_{\lambda(1)}$ is cofree (Proposition 3.20), weak equivalences are equivariant maps detected on the underlying homotopy. In particular, multiplication by u_V is an invertible self-map. $\square$

Before we describe the $RO(G)$ -graded homotopy groups of $H\mathbb{Z}/a_{\lambda(1)}$, we need to single out some Mackey functors. For the readers convenience, we also include the “hieroglyphics” used in [18], which we will use in our spectral sequence charts.

Notation 4.13.

- (1) Let $\mathbb{Z}^*$ be the dual to the constant Mackey functor $\mathbb{Z}$, and we will use $\blacksquare$ to denote this.
- (2) Let $\mathbb{Z}_-$ be the fixed point Mackey functor for the integral sign representation, and we will use $\overline{\square}$ to denote this.
- (3) Let $\mathbb{Z}_-^*$ be the dual of $\mathbb{Z}_-$, and we will use $\blacksquare$ to denote this.
- (4) Let $\underline{B}$ be defined as the cokernel of the unique map $\mathbb{Z}^* \rightarrow \mathbb{Z}$ which is the identity when evaluated at G/e . We will use $\circ$ to denote this.
- (5) Let $\underline{B}_-$ and $\underline{G}$ be defined by the short exact sequence

$$0 \rightarrow \underline{B}_- \rightarrow \begin{matrix} \uparrow_{C_{2^{n-1}}}^{C_{2^n}} \\ \downarrow_{C_{2^{n-1}}}^{C_{2^n}} \end{matrix} \underline{B} \xrightarrow{Tr} \underline{B} \rightarrow \underline{G} \rightarrow 0,$$

where the map Tr is adjoint to the identity under the induction-restriction adjunction. We will use $\bar{\circ}$ to denote $\underline{B}_-$ and $\bullet$ to denote $\underline{G}$.

- (6) Let $\underline{T}$ be the unique non-trivial extension of Mackey functors

$$0 \rightarrow \underline{G} \rightarrow \underline{T} \rightarrow \mathbb{Z}_-^*,$$

where the transfer on $\mathbb{Z}_-^*(G/C_{2^{n-1}})$ hits the generator of $\underline{G}(G/G)$. We will use $\dot{\blacksquare}$ to denote this.

Remark 4.14. The Mackey functors $\underline{B}$ and $\underline{G}$ are so named because $\underline{B}$ shows up in the Borel homology of a point with coefficients in $\mathbb{Z}$, while $\underline{G}$ arises in the homotopy of the Geometric fixed points of $H\mathbb{Z}$.

The Eilenberg-Mac Lane spectra associated to many of these Mackey functors are actually $RO(C_{2^n})$ -graded suspensions of $H\mathbb{Z}$.

Proposition 4.15. *We have equivalences*

$$\begin{aligned} H\mathbb{Z}^* &\simeq \Sigma^{2-\lambda(1)} H\mathbb{Z} \\ H\mathbb{Z}_- &\simeq \Sigma^{1-\sigma} H\mathbb{Z} \\ HT &\simeq \Sigma^{3-\sigma-\lambda(1)} H\mathbb{Z}. \end{aligned}$$

Proof. These are direct consequences of the standard chain complexes computing Bredon homology for C_{2^n} . See, for example [15, 16, 17]. $\square$

Let $\underline{M}$ be the Mackey functor graded by $i + j\sigma$ depicted in Figure 1. Here the horizontal coordinate is i and the vertical coordinate is j .

	$\begin{array}{ c } \hline \blacksquare \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	
0	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \blacksquare \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$
	-2		0

FIGURE 1. The Mackey functor $\underline{M}$

Theorem 4.16. *The $RO(G)$ -graded homotopy groups of $H\mathbb{Z}/a_{\lambda(1)}$ are given by*

$$\underline{M}[u_{\lambda(1)}^{\pm 1}, u_{\lambda(2)}^{\pm 1}, u_{\lambda(4)}^{\pm 1}, \dots, u_{2\sigma}^{\pm 1}].$$

Proof. Since multiplication by u_V is invertible, for any orientable V , we have

$$\pi_V(H\mathbb{Z}/a_{\lambda(1)}) \cong \pi_{\dim V}(H\mathbb{Z}/a_{\lambda(1)}).$$

This means that

$$\pi_*(H\mathbb{Z}/a_{\lambda(1)}) \cong \left(\pi_*(H\mathbb{Z}/a_{\lambda(1)}) \oplus \pi_{*+\sigma}(H\mathbb{Z}/a_{\lambda(1)}) \right) [u_{\lambda(1)}^{\pm 1}, \dots, u_{\lambda(2^{n-1})}^{\pm 1}].$$

The statement of the Theorem is then that as graded Mackey functors,

$$\underline{M} \cong \pi_*(H\mathbb{Z}/a_{\lambda(1)}) \oplus \pi_{*+\sigma}(H\mathbb{Z}/a_{\lambda(1)}).$$

We apply Proposition 3.3 to compute these. For this, we need to determine the $\mathbb{Z}$ -graded homotopy groups of $\Sigma^{-\lambda(1)}H\mathbb{Z}$, of $\Sigma^{-\sigma}H\mathbb{Z}$, and of $\Sigma^{-\lambda(1)-\sigma}H\mathbb{Z}$. These are given by Proposition 4.15. $\square$

Remark 4.17. The image of the element a_σ is non-zero in $\pi_*^G(H\mathbb{Z}/a_{\lambda(1)})$: this is the generator of $\pi_{\sigma-2}^G(H\mathbb{Z}/a_{\lambda(1)})$, multiplied by the class $u_{2\sigma}$. That this is a transfer is a consequence of the fact that $a_\sigma^2 = 0$.

4.4. The $\mathbb{Z}$ -homotopy fixed points of $MU^{((G))}$. We can now put everything together to describe the $RO(G)$ -graded homotopy Mackey functors for the $\mathbb{Z}$ -homotopy fixed points of $MU^{((G))}$.

Corollary 4.18. *The elements $u_V^{\pm 1}$ and all polynomials in the $\bar{r}_i$ and their conjugates are all in slice filtration zero. The minus first homotopy Mackey functor $\underline{Z}^*$ of the zero slice is in bidegree $(-1, 1)$, while the Mackey functor including a_σ has bidegree $(-\sigma, 1)$. Finally, the Mackey functor $\underline{Z}_-$ is in bidegree $(1, 0)$.*

Theorem 4.19. *The slice spectral sequence collapses at E_2 with no extensions.*

Proof. The entire spectral sequence is concentrated in filtrations zero and one. This gives the collapse of the spectral sequence, and $E_2 = E_\infty$. Since the only torsion classes are in filtration 1, there are no possible additive extensions. $\square$

Remark 4.20. In fact, there are no extensions in the category of Mackey functors. This can be seen by computing the relevant Ext groups.

Corollary 4.21. *The $RO(G)$ -graded homotopy of $MU^{((G))}/a_{\lambda(1)}$ is given by*

$$\underline{M}[u_{\lambda(1)}^{\pm 1}, u_{\lambda(2)}^{\pm 1}, u_{\lambda(4)}^{\pm 1}, \dots, u_{2\sigma}^{\pm 1}][G \cdot \bar{r}_1, \dots].$$

Corollary 4.22. *For $MU^{((G))}/a_{\lambda(1)}$, all orientable representations are orientable: if V is an orientable representation, then multiplication by u_V gives an equivariant equivalence*

$$\Sigma^V MU^{((G))}/a_{\lambda(1)} \xrightarrow{u_V} \Sigma^{\dim V} MU^{((G))}/a_{\lambda(1)}$$

This means in particular, that the $RO(G)$ -graded homotopy groups are especially simple: we see only the $\mathbb{Z}$ -graded homotopy Mackey functors (corresponding to trivial representations) and the $\mathbb{Z}$ -graded homotopy Mackey functors of a single sign suspension. Put another way, the image of $RO(G)$ in $Pic(MU^{((G))}/a_{\lambda(1)})$ is fairly small.

Corollary 4.23. *The image of $RO(C_{2^n})$ in $Pic(MU^{((G))}/a_{\lambda(1)})$ is*

$$\mathbb{Z} \cdot \{1\} \oplus \mathbb{Z}/2 \cdot \{1 - \sigma\}.$$

Every orientable virtual representation acts as its virtual dimension.

The slice filtration of $MU^{((G))}/a_{\lambda(1)}$. The slice associated graded of $MU^{((G))}/a_{\lambda(1)}$ does not agree with maps from $S(\lambda(1))_+$ into the slice associated graded for $MU^{((G))}$. For convenience, we used the latter, and it also has extremely nice multiplicative properties guaranteed. We sketch some of the analysis of the slice filtration for $MU^{((G))}/a_{\lambda(1)}$.

Proposition 4.24. *The slice tower for $H\mathbb{Z}/a_{\lambda(1)}$ is given by*

$$\begin{array}{ccc} H\mathbb{Z} & \longrightarrow & H\mathbb{Z}/a_{\lambda(1)} \\ & & \downarrow \\ & & \Sigma^{-1}H\mathbb{Z}^* \xrightarrow{a_{\lambda(1)}} \Sigma^1 H\mathbb{Z}. \end{array}$$

Proof. Theorem 4.16 shows that the Postnikov tower for $H\mathbb{Z}/a_{\lambda(1)}$ is given by this small tower. Since the fibers are all slices in order, this is also the slice tower, by [17, Proposition 4.45]. $\square$

The k -invariant listed is actually $a_{\lambda(1)}$: the source is $\Sigma^{1-\lambda(1)}H\mathbb{Z}$.

Remark 4.25. The category of slice $\leq n$ spectra is cotensored over G -spaces: if E is slice $\leq n$, then for every G -space X , $F(X_+, E)$ is also slice $\leq n$. This is the dual of the statement that the category of slice $\geq (n+1)$ -spectra is tensored over G -spaces (and more generally, over (-1) -connected G -spectra).

The argument in [17, Section 6] then gives a way to determine the slice tower for $MU^{((G))}/a_{\lambda(1)}$. Since we do not need this for later arguments, we only sketch the results here. The basic insight is that the obvious monoidal ideals in the algebra A above give a filtration of $MU^{((G))}/a_{\lambda(1)}$. The bottom stage is

$$(MU^{((G))}/a_{\lambda(1)}) \wedge_A S^0 \simeq H\mathbb{Z}/a_{\lambda(1)},$$

which is a two-stage slice tower with a zero and -1 -slice. The rest of the argument goes through essentially unchanged now, showing that the filtration by these ideals coincides with a (slightly speeded up) version of the slice filtration.

Proposition 4.26. *The slice associated graded for $MU^{((G))}/a_{\lambda(1)}$ is*

$$Gr(MU^{((G))}/a_{\lambda(1)}) = A \wedge (\Sigma^{-1}H\mathbb{Z}^* \vee H\mathbb{Z}),$$

where the graded degree comes from the underlying degree of the pieces in A , together with a -1 for the $\Sigma^{-1}H\mathbb{Z}^*$ and a 0 for $H\mathbb{Z}$.

Nothing really changes in our argument above except the filtrations of some of the elements. The Mackey functor $\mathbb{Z}^*$ in filtration 1 and degree -1 moves to filtration zero, and the class whose transfer is a_σ now occurs in filtration zero.

4.5. The Hurewicz image. We can link the Hurewicz images of $MU^{((G))}$ and of $MU^{((G))}/a_{\lambda(1)}$ using the slice filtration. The ring map

$$MU^{((G))} \rightarrow MU^{((G))}/a_{\lambda(1)}$$

induces a map of slice spectral sequences.

Corollary 4.27. *The map of spectral sequences induced by $MU^{((G))} \rightarrow MU^{((G))}/a_{\lambda(1)}$ factors as the quotient by filtrations greater than or equal to two, followed by the inclusion induced by $H\mathbb{Z} \rightarrow H\mathbb{Z}/a_{\lambda(1)}$.*

This identification of the map of spectral sequences limits the possible size of the Hurewicz image.

Corollary 4.28. *If $x \in \pi_k^G(MU^{((G))})$ is in the Hurewicz image, then the Hurewicz image of x in $\pi_k^G(MU^{((G))}/a_{\lambda(1)})$ is non-zero if and only if the slice filtration of x is at most 1.*

Corollary 4.29. *The Hurewicz image of $E_{\mathbb{R}}(n)^{h\mathbb{Z}}$ is the same as that of*

$$K_{SC} = K_{\mathbb{R}}^{h\mathbb{Z}}.$$

For larger groups, the Hurewicz image does grow, although this is still a very harsh condition. The Hurewicz image is actually quite small, since in fact, we shall see this bounds the Adams–Novikov filtration of a class in the Hurewicz image by 1. For this, we recall a result of Ullman’s thesis.

Theorem 4.30 ([28]). *The slice and homotopy fixed point spectral sequences agree below the line of slope 1 through the origin.*

All of the classes we are considering are in this range, so it suffices to consider the homotopy fixed point statement. For this, recall that we have a comparison map between the Adams–Novikov and homotopy fixed point spectral sequences for hyperreal spectra.

Theorem 4.31 ([17, Section 11.3.3]). *There is a natural map of spectral sequences from the Adams–Novikov spectral sequence computing the homotopy groups of the homotopy fixed points to the homotopy fixed point spectral sequence.*

Putting this together, we deduce an upper bound on the Hurewicz image for any of the $\mathbb{Z}$ -homotopy fixed points of the norms of $MU_{\mathbb{R}}$.

Theorem 4.32. *The Hurewicz image of $(MU^{((G))})^{h\mathbb{Z}}$ factors through the homotopy of the connective image of j spectrum.*

Proof. Since a map of spectral sequences can only increase filtration, the Adams–Novikov filtration of any element in the Hurewicz image is bound above by its slice filtration, and this is at most 1. The result follows from [24] (See [25, Theorem 5.2.6(c,d)] for more detail). $\square$

5. Towards the homotopy Mackey functors of $MU^{((G))}/a_{\lambda(2^j)}$

One of the computationally exciting features of the spectra $MU^{((G))}/a_{\lambda(k)}$ is that we effectively isolate the contribution of a single Euler class. In the case where all orientable Euler classes were killed above, the putative orientation classes u_V all survived the slice spectral sequence, giving orientations for $MU^{((G))}/a_{\lambda(1)}$. Working more generally, not all of these will survive. We illustrate this with the first non-trivial examples.

5.1. Surviving orientation classes. While not all orientation classes survive the slice spectral sequence for $j > 0$, many still do.

Theorem 5.1. *For all $k > j$, the orientation classes $u_{\lambda(2^k)}$ survive the slice spectral sequence for $MU^{((G))}/a_{\lambda(2^j)}$.*

Proof. Since the classes $u_{\lambda(2^k)}$ are all invertible in the $RO(G)$ -graded homotopy of $H\mathbb{Z}/a_{\lambda(2^j)}$, the E_2 terms computing

$$\pi_0(MU^{((G))}/a_{\lambda(2^j)}) \text{ and } \pi_{(2-\lambda(2^k))}(MU^{((G))}/a_{\lambda(2^j)})$$

are isomorphic. By the vanishing lines in this slice spectral sequence, there are no possible targets for the differentials on the classes $u_{\lambda(2^k)}$ for $k > j$. $\square$

This in turn gives us periodicity results analogous to those we deduced for the reduction modulo $a_{\lambda(1)}$. Proposition 3.20 above shows that the spectrum $MU^{((G))}/a_{\lambda(2^j)}$ is local for the family of subgroups of C_{2^j} .

Lemma 5.2. *Let X and Y be finite G -CW complexes, and let*

$$f: (MU^{((G))}/a_{\lambda(2^j)}) \wedge X \rightarrow (MU^{((G))}/a_{\lambda(2^j)}) \wedge Y.$$

If the restriction $i_{C_{2^j}}^ f$ is an equivariant equivalence, then f is an equivariant equivalence.*

Proof. Since X and Y are assumed to be finite G -CW complexes, the smash products with $MU^{((G))}/a_{\lambda(2^j)}$ are again C_{2^j} -local. This is the statement of the lemma. $\square$

Corollary 5.3. *If V is any orientable representation of C_{2^n} such that $i_{C_{2^j}}^* V$ is trivial, then multiplication by u_V induces an equivariant equivalence*

$$\Sigma^V (MU^{((G))}/a_{\lambda(2^j)}) \xrightarrow{u_V} \Sigma^{\dim V} (MU^{((G))}/a_{\lambda(2^j)}).$$

Proof. By assumption, the irreducible summands of V are all of the form $\lambda(2^k r)$ for $k > j$. In particular, the corresponding class u_V is a permanent cycle, and hence homotopy class, by Theorem 5.1. Since the restriction of u_V to C_{2^j} is the element 1, Lemma 5.2 gives the result. $\square$

5.2. The remaining orientation classes. Unsurprisingly, the remaining orientation classes all support differentials in the slice spectral sequence, just as in the initial case of $MU^{((G))}$ itself.

Theorem 5.4. *Let V be a representation such that $V^{C_{2^j}} \neq V$. Then u_V supports a non-trivial differential in the slice spectral sequence.*

Proof. Let $H = C_{2^k}$ be the smallest subgroup of C_{2^n} such that $i_H^* V$ is not trivial. By assumption, the largest possible value for this group is C_{2^j} . By the definition of H , we must then have

$$i_H^* V = a\mathbb{R} \oplus 2^r b\sigma,$$

for some natural number a and natural numbers $r > 0$ and b odd, and hence

$$i_H^* u_V \cong (u_{2\sigma}^{2^{r-1}})^b.$$

Since $H \subset C_{2^j}$, we know that

$$i_H^* MU^{((G))}/a_{\lambda(2^j)} \simeq i_H^* MU^{((G))} \vee \Sigma^{-1} i_H^* MU^{((G))},$$

and the associated spectral sequence is just two copies of the slice spectral sequence for $i_H^* MU^{((G))}$. In particular, the class $(u_{2\sigma}^{2^{r-1}})^b$ supports a differential here [17, Theorem 9.9]:

$$d_{(2^r-1)(2^k-1)+2^r}((u_{2\sigma}^{2^{r-1}})^b) = b(u_{2\sigma}^{2^{r-1}})^{b-1} \left(N_{C_2}^{C_{2^k}} r_{2^r-1} a_{(2^r-1)\bar{\rho}_{2^k}} a_{\sigma}^{2^r} \right).$$

By naturality, we therefore deduce that u_V must have supported as differential of potentially smaller length, completing the result. $\square$

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