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Structural form-finding of Auxetic Materials using Graphic Statics

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Abstract
Auxetic materials are structural systems with negative Poisson's ratio. Such materials show unexpected behavior when subjected to uni-axial compression or tension forces. For instance, they expand perpendicular to the direction of an applied compressive force. This behavior is the result of their internal structural geometry. These materials, with their unique behavior, have recently found many applications in the fields of sensors, medical devices, sport wears, and aerospace. Thus, there is a lot of relevant research in the artificial design of auxetic metamaterials and the prediction of their behavior [1]. Since the behavior of these materials heavily relies on the geometry of their internal structure, the geometry-based methods of structural design, known as graphic statics, are very well suited to derive their geometry or describe their behavior. Nevertheless, graphic statics has never been used in the design of such materials. For the first time, this paper provides an introduction to the use of graphic statics in the design and form-finding of auxetic metamaterials. The paper explains multiple equilibrium states of various auxetic structures using algebraic formulations of 2d/3d graphic statics [2, 3]. Moreover, it sheds light on the geometric behavior of auxetic materials by changing the force diagram of graphic statics. Therefore, it suggests a novel approach in predicting the changes in the geometry of the material under various loading conditions by controlling the force equilibrium geometrically.

1 Introduction Metamaterials are artificially structured materials with unusual properties. Recent breakthroughs in advanced micro- and nanofabrications allowed the construction of such previously–inaccessible structures. These materials can be used to control and manipulate light, sound, and many other physical phenomena for a variety of applications including the construction of ultra-strong materials [4, 5], shock-absorbing materials [6, 7, 8] and materials with negative Poisson-ratio [9, 10, 11, 12, 13, 14, 15]. Even though many applications have been found, currently, there is no systematic approach to design the geometry of metamaterials (Figure 1).

In this paper, we propose a methodology to construct 2-dimensional metamaterials using geometry-based structural design method based on 2-dimensional algebraic graphic statics proposed by Maxwell and Rankine [16, 17, 18]. In graphic statics, the geometry of the structure and its equilibrium are represented by the *form* and the *force* diagrams. The form diagram incorporates the location of the supports and the applied loads, while the force diagram represents the equilibrium and the magnitude of the forces. These two diagrams are *reciprocal*: nodes of one diagram correspond to faces of the other, and the edges of one diagram correspond to geometrically perpendicular edges of the other [17].

Recently, an algebraic formulation using constrained equations was provided to explore the topological and geometrical relationships between the reciprocal diagrams of 2D graphics statics (2DGS) [2, 20].

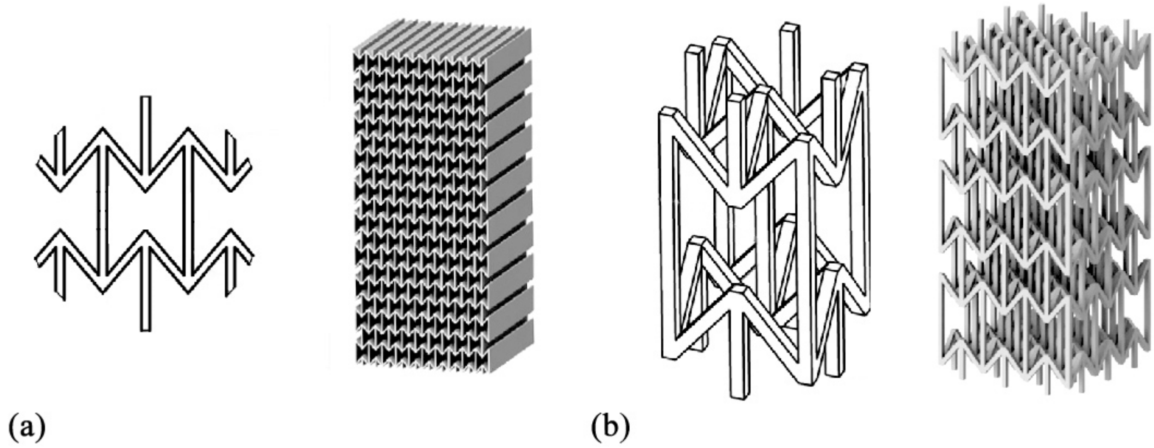

Figure 1: (a) a 2D re-entrant structure based on honeycomb geometry; (b) a 3D structure made of similar geometry [19].

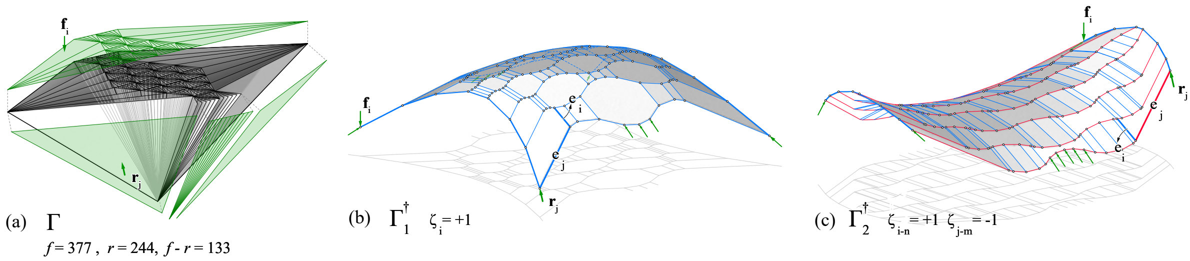

Figure 2: (a) A force diagram consisting of convex polyhedral cells with 377 number of faces resulting in (b) a (synclastic) compression-only form with 133 degrees of freedom; (c) a different (anticlastic) shell with both tension and compression members by assigning both negative and positive values to the edges of the form (b) [22].

This construction allows the users to study and manipulate the diagrams interactively. This formulation was later generalized to 3-dimensional reciprocal systems as well provided an interactive tool in the 3-dimensional case [21, 3].

Previously, the authors of this paper defined the notion of Geometric Degrees of Freedom (GDoF) describing the dimension of the family of possible form diagram corresponding to a given reciprocal diagram [22]. Therefore, the GDoF describes in how many ways a structure can be manipulated so that the reciprocal dual diagram remains the same. This idea was used in the form-finding of an anticlastic shell-structure with non-convex polygonal faces (Figure 2). The non-convex faces as a result of this study are very similar to the non-convex geometry of auxetic systems suggested in [23, 19] (See Figure 1). This similarity suggests the potentials of using graphic statics in generating the geometry of auxetic systems that have not been explored before.

1.1 Problem statement and Objectives Even though many applications of metamaterials have been found, currently, there is no systematic approach to design the geometry of metamaterials. The goal of the paper is to develop a systematic approach to design metamaterials with a negative Poisson's ratio from space-filling polygonal diagrams using reciprocal diagrams of 2D graphic statics. This research is novel since the reciprocal diagrams and the methods of algebraic graphic statics have not been used in the design and generation of the geometry of metamaterials. In this paper, we transform a doubly-periodic polygonal system with convex faces into a network with non-convex, non-self-intersecting faces by utilizing the Geometric Degrees of Freedom (GDoF) of the network. Explicitly, using GDoF, we change the direction of the parallel edges into the opposite direction without changing the equilibrium in the system which results in a network with re-entrant faces for the same force diagram. In this method, there are several challenges: (a) not any network can be transformed into a non-convex, non-self-intersecting network and a proper methodology is needed to adjust the topological properties of the network; (b) the directions along which the network can be manipulated should be identified; and, (c) transformation parameters should be controlled such that the faces of the non-convex network do not self-intersect.

1.2 Contributions In this paper, we propose a generalizable, unconventional approach in the design of auxetic metamaterials using geometry-based structural design method based on 2-dimensional algebraic graphic statics. This research uses a unique approach in design. It utilizes the topological modeling technique which uses the dual space of the actual structural geometry as a medium for design and transformation. With this method, we transform the input geometry of any known convex tessellation of the two-dimensional space into an auxetic configuration: a re-entrant, non-convex geometry with negative Poisson's ratio.

2 Methodology In general, convex polyhedral systems, and space packing systems have been explored extensively. However, non-convex aggregations have not been addressed adequately in the literature. In this research, we transform convex tessellations into non-convex, non-self-intersecting polygonal systems that can be used as the geometry of auxetic materials. Our methodology uses an unconventional, unique topological modeling method, we utilize the dual space of the input geometry to transform it into an auxetic configuration. This method is a two-step process. First, we utilize the dual space to transform a convex tessellation into a tessellation on which a trapezoid grid can be imposed (Section 2.6). Second, we use the Geometric Degrees of Freedom (GDoF) of this system to convert the convex tessellation into a re-entrant system by flipping certain parallel edges (Sections 2.4, 2.5).

2.1 Overview of 2DGS In 2D graphic statics, the geometry of a structure and its equilibrium are represented by two perpendicular reciprocal diagrams referred to as the *form* and *force* diagrams. In this context, we call the starting diagram the *primal*, and the reciprocal diagram the *dual* diagram. These two diagrams are topologically dual, meaning that vertices (v), edges (e) and faces (f) of the primal diagram correspond to faces ($f^\dagger$), edges ($e^\dagger$) and vertices ($v^\dagger$) of the dual diagram (in this order). Moreover, each edge of the primal diagram is perpendicular to its corresponding edge in the dual diagram (see Figure 3).

Recently, an algebraic formulation was provided to construct the dual diagram in 2-dimensional space using the primal as an input [2, 20]. In this formulation, each vertex (v_i) and its connected edges (e_j) of the primal diagram provides equations for the construction of the dual diagram. The vertex v_i corresponds to a closed polygon ($f_i^\dagger$) in the dual diagram, hence the sum of the edge vectors $\mathbf{e}_j{}^\dagger$ of $f_i^\dagger$ has to be the zero vector. Thus, each vertex v_i provides a vector equation

$$\sum_{e_j^\dagger} \mathbf{u}_j{}^\dagger q_j = \mathbf{0}$$

where the sum runs over the edges $e_j^\dagger$ of the face $f_i^\dagger$, $\mathbf{u}_j^\dagger$ denotes the unit edge vector of the edge $e_j^\dagger$, and the q_j are the variables representing the lengths of the edges $e_j^\dagger$ in the dual diagram. Writing these equations around every face $f_i^\dagger$ provides a linear equation system with the variable being the vector of

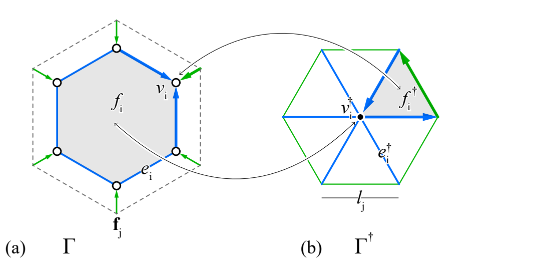

Figure 3: The form diagram (a) and the force diagram (b) and their dual reciprocal elements.

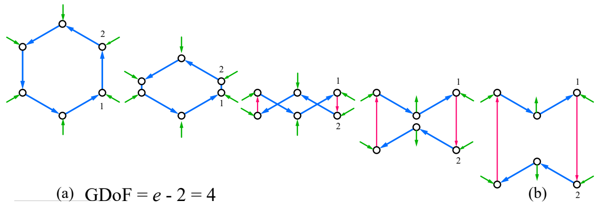

Figure 4: (a) Initial form diagram with the GDoF of 4 where the lengths of two parallel edges have been changed from positive value to negative value (b).

edge lengths $\mathbf{q}$. The equation system can be described by a $[2v \times e]$ matrix, called the *equilibrium matrix* $\mathbf{A}$:

$$\mathbf{A}\mathbf{q} = \mathbf{0}. \tag{1}$$

The solutions of the equilibrium matrix provide the edge lengths of the dual diagram, from which the diagram can be constructed.

2.2 Geometric Degrees of Freedom (GDoF) The Geometric Degrees of Freedom (GDoF) of one diagram is the number of edge lengths that can be chosen independently to find a unique solution of the equilibrium equations. For instance, the geometric degrees of freedom of the force diagram can be defined as the number of edge lengths that can be independently chosen to construct the force diagram with the same topology and parallel edges to its initial configuration. In the case of the form diagram, the GDoF is the dimension of the family of networks with parallel edges but different edge lengths that are in equilibrium. In other words, this number represents the number of ways the geometry can be manipulated without breaking the perpendicular reciprocity with the dual diagram.

In 2D graphic statics, the GDoF of the *dual* diagram equals $e - r$ where e is the number of edges of the *primal* diagram and r is the rank of the equilibrium matrix $\mathbf{A}$. If the GDoF is one, scaling the diagram is the only possible manipulation without breaking the reciprocity. If the GDoF is more than one, the diagram can be modified in many significantly different ways within the equilibrium state. This observation is illustrated in Figure 4 where the force diagram is a hexagon. The GDoF of the hexagon is $e - 2 = 6 - 2 = 4$, thus, the hexagon can be manipulated in many significantly different ways. In particular, by changing the lengths of its vertical edges from positive to negative. In fact, any value can be given to the length of the vertical edges. More importantly, the edges may flip into their negative edge lengths without breaking the equilibrium or the reciprocity with its dual diagram.

Comparing the geometry of a hexagon with the non-convex geometry of the Figure 4 suggests a method by which the convex geometry can be turned into a non-convex (re-entrant) geometry for the design of auxetic structures. However, not any polygon can be transformed into a non-convex geometry. In the next sections, we provide a technique to check whether a network can be transformed into an auxetic configuration using GDoF.

2.3 2D Trapezoid Arrangements with a Direction (TAD) We say that an aggregation of polygons in the two-dimensional space is a *two-dimensional Trapezoid Arrangement with a Direction* (TAD) if every polygon is a trapezoid, whose two parallel sides are parallel to a given direction $\mathbf{d}$. We call a convex quadrilateral with at least one pair of parallel sides a trapezoid, thus a rectangle is also considered a trapezoid in this definition. Moreover, we require from a TAD that every interior vertex has valency four, and every interior edge is the edge of exactly two neighboring trapezoids (Figure 5a provides an example of a TAD with a vertical direction). This direction **d** represents the parallel lines of the trapezoids in the grid. In this paper, we require the TAD to be periodic in two perpendicular directions: in the direction of **d** and $\mathbf{d}^\perp$ (Figure 5a is an example of a doubly-periodic TAD which is periodic both in the horizontal and the vertical directions). In this section, we show that the dual diagram of a TAD is also a TAD with the direction given by $\mathbf{d}^\perp$ which is perpendicular to **d**.

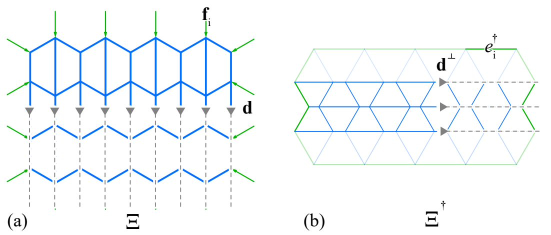

Figure 5: (a) A TAD, Ξ, as a form diagram and (b) its reciprocal force diagram $\Xi^\dagger$.

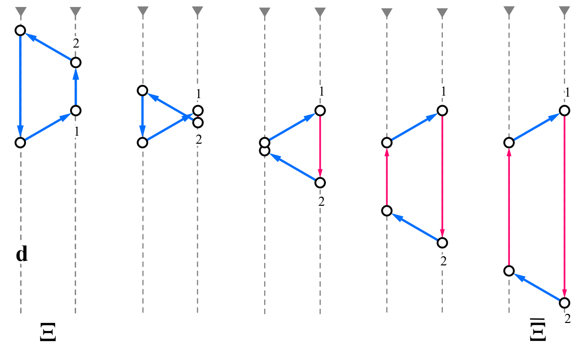

Figure 6: A single trapezoid with positive edge lengths and its transformation into another trapezoid with two negative edge lengths.

Every vertex v_i of a TAD, Ξ, has valency four, hence, the corresponding face $f_i^\dagger$ of $\Xi^\dagger$ is a quadrilateral (Figure 5b). Moreover, two of the four edges connected to v_i are parallel to the direction **d**; thus two of the four sides of $f_i^\dagger$ are parallel to the direction perpendicular to **d**. Similarly, every face f_j of Ξ is a trapezoid, hence the corresponding vertices $v_j^\dagger$ have valency four and two of the edges connected to $v_j^\dagger$ are perpendicular to **d**. As a result, $\Xi^\dagger$ is a TAD with direction, $\mathbf{d}^\perp$. This phenomenon is illustrated by Figure 5b, the interior network of the reciprocal diagram (excluding the triangles) is a TAD with a horizontal direction.

The doubly-periodic dual diagram, $\Xi^\dagger$ can be constructed in two steps. First, by constructing the dual diagram of the *fundamental domain* of Ξ. The fundamental domain is a subset of Ξ which generates Ξ under the double periodicity. As a result, $\Xi^\dagger$ can be generated from the dual of the fundamental domain which is our second step. In this way, we can ensure that the dual doubly-periodic TAD, $\Xi^\dagger$ has no self-intersecting faces. Indeed, since $\Xi^\dagger$ is generated by a small part of it (the dual of the fundamental domain), it is only needed to ensure that the faces in this part are non-self-intersecting. This can be guaranteed by changing the length of the parallel sides (Figure 6). We summarize the discussion of this section with the following Lemma.

Lemma 2.1 *For a doubly-periodic TAD, Ξ, there exists a dual diagram, $\Xi^\dagger$, which is a doubly-periodic TAD without self-intersecting faces.* □

2.4 Using GDoF in transforming a TAD Consider a doubly-periodic space-filling TAD, Ξ, and its dual TAD, $\Xi^\dagger$. The key observation of this paper is that Ξ can be transformed along its direction **d** without breaking the reciprocity with $\Xi^\dagger$. In particular, there exists a *parallel* TAD, $\overline{\Xi}$, dual to the TAD $\Xi^\dagger$, whose edges are parallel to the edges of Ξ, but the edges parallel to the direction **d** of the TAD are flipped. In other words, in the construction of $\overline{\Xi}$ we assign negative values to the signed edge lengths parallel to **d**.

The construction of $\overline{\Xi}$ can be done in a sequential process. First, we consider a single trapezoid cell f_1 of Ξ. The GDoF of the trapezoid is $e-2=4-2=2$, therefore, the trapezoid can be manipulated in a non-trivial way: by changing the lengths of its parallel edges (Figure 6). In fact, any value can be given to the length of any of the parallel edges; importantly, these edges can be flipped without breaking the duality with its dual diagram. In this way, we obtain a non-self-intersecting trapezoid cell with flipped edges. Then, we take the next trapezoid cell, f_2, connected to f_1. We flip the parallel edges of f_2 obtaining a non-self-intersecting trapezoid. We continue this process until we flip the edges parallel to $\mathbf{d}$ of all trapezoids in the fundamental domain of Ξ. We extend the construction of $\overline{\Xi}$ using the doubly-periodic property of the network.

2.5 Re-entrant arrangements using TAD's In this section, we show our main methodology to construct re-entrant geometries from convex, doubly-periodic network Γ. We assume that Γ has an overlaying TAD structure, meaning that there exists a doubly-periodic TAD, Ξ, so that all edges of Γ can be covered by the edges of Ξ. In other words, by adding extra edges to Ξ parallel to the direction $\mathbf{d}$ we obtain a TAD, Ξ. Figure 7a provides an example of the hexagonal tessellation with an overlaying TAD structure. The extra edges are highlighted with dashed arrows. Note that the overlaying TAD is the one in Figure 5a.

Consider the overlaying TAD of Γ, Ξ. Using Section 2.4, we construct a *parallel* network $\overline{\Xi}$ such that the edges of Ξ parallel to $\mathbf{d}$ are flipped in $\overline{\Xi}$. Since Γ is a sub-network of Ξ, in this process, along Ξ, we manipulated the diagram Γ as well. The signed edges of Γ covered by the edges of Ξ parallel to $\mathbf{d}$ are assigned negative values in this process. The process described above significantly changes the initial geometry. A convex tessellation with an overlaying TAD structure can be transformed into an auxetic configuration with re-entrant faces.

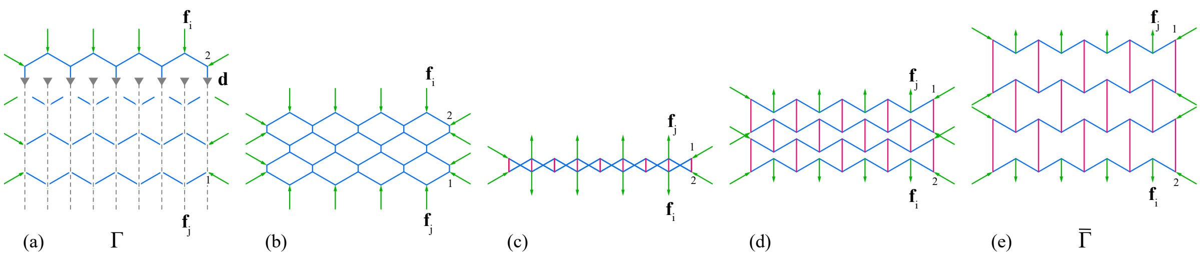

Figure 7: (a) The hexagonal tessellation with an overlaying TAD structure, (b)-(e) the transformation of the hexagonal tesselation into an auxetic configuration.

2.6 Construction of overlaying TADs for general networks In general, not every doubly-periodic tessellation Γ has an overlaying TAD. If a general network does not have an overlaying TAD, then the method previously described cannot be used to transform it into a non-convex doubly-periodic tessellation. In the following section, we briefly describe a framework to transform a general initial tessellation into a tessellation with an overlaying TAD structure.

Consider the network Γ of Figure 8a. Given the vertical direction, $\mathbf{d}$, we create a grid parallel to $\mathbf{d}$ overlaying Γ covering all the vertices of the network. This grid cuts the faces of Γ into triangles and quadrilaterals (Figure 9a). The GDoF of a triangle is $e-2=1$ meaning that the only possible manipulation of the triangle without breaking the duality is scaling. As a result, we cannot manipulate the triangle into a non-convex geometry. Thus, the methodology described in the previous sections cannot be applied immediately, and the triangles of the network need to be replaced by trapezoidal cells.

To construct these trapezoid cells, the dual diagram $\Gamma^{\dagger}$ of Γ is modified (see Figure 8a-b). In order to eliminate the triangles given by the parallel grid (Figure 9a), extra edges (showed by dashed lines) are added to $\Gamma^{\dagger}$ perpendicular to $\mathbf{d}$ obtaining a new dual geometry (Figure 9b). Consequently, the primal diagram corresponding to the new dual diagram can have an overlaying TAD with direction $\mathbf{d}$ (Figure 10a). This process changes the geometry of the doubly-periodic network Γ by transforming its topology into a network that can have an overlaying TAD structure. Changing the signed edge lengths of the new network in the direction of $\mathbf{d}$ transforms the network into a re-entrant aggregation with auxetic properties based on the method proposed in Section 2.5 (See Figure 12a,b). Figure 11 and Figure 12 show the initial network, its dual diagram and its re-entrant version and the corresponding dual diagram with additional edges.

Figure 13a-e show multiple examples of doubly-periodic convex tessellations and their related non-convex, auxetic geometry. In particular, the most studied auxetic material in the literature as shown in Figure 13a1 can be constructed with our methodology ([24, 25, 26, 27]). The input of Figure 13b1 is the very same geometry of Figure 10. Figure 13c1 is the subdivided version of the sample of Figure 13b1 where the force diagram is subdivided into smaller internal polygons [28]. Figure 13d1 is a network consisting of octagons and pentagons while Figure 13e1 is a network consisting of quadrilaterals, pentagons, hexagons, and octagons. This example is special in the sense that the overlaying TAD does not have parallel edges, meaning that in the construction of the auxetic geometry we flip all the edges which are close enough to be vertical. This property will be investigated thoroughly in the near future and its versatility will help to generalize the methodology of this paper to almost all space-filling periodic networks.

The auxetic behavior of the systems is illustrated in column (3) of Figure 13. Obviously, rigorous quantitative approaches such as the numerical homogenization method should be used to precisely measure the negative Poisson's ratio for each sample. And, a further thorough investigation is needed to precisely report their auxetic properties and behavior. We will address such characteristics in a future research.

3 Future directions The current methodology provides a framework for creating re-entrant networks from doubly-periodic convex polygonal systems. However, the explicit auxetic properties are not investigated in this paper and will be explored in future research. A robust computational implementation to deal with any 2D periodic input geometry should be developed in a future research. This will include transformation and modification of the dual diagram of the network and all the necessary adjustments to turn the network into a subset of a doubly-periodic TAD.

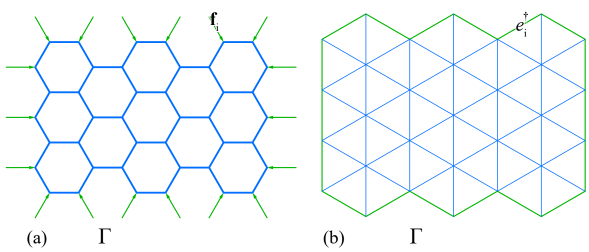

Figure 8: The hexagonal tessellation (a) with its dual diagram (b).

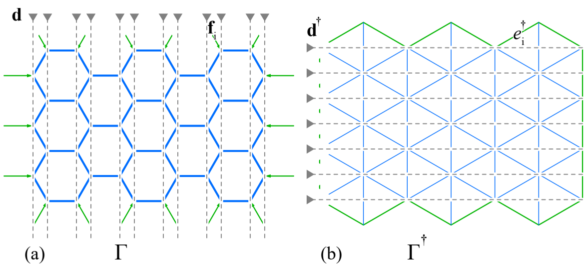

Figure 9: (a) Hexagonal grid with an imposed vertical grid on it, (b) the dual diagram with an imposed horizontal grid on it.

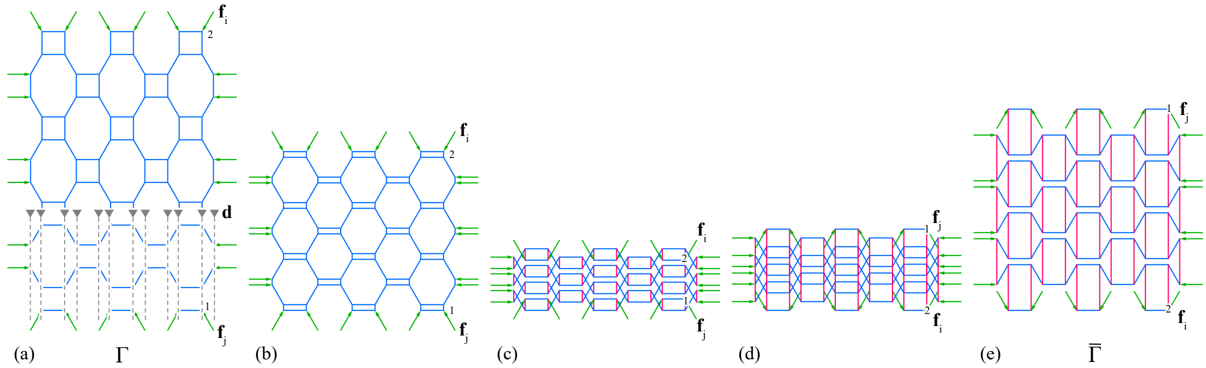

Figure 10: (a) A tessellation of squares and octagons; (b)-(e) the transformation of this network into an auxetic system.

Another interesting future direction is the generalization of the current framework to three dimensions. There are multiple directions to extend the current methodology into three dimensions. For instance, reciprocal polyhedral diagrams suggested by Rankine can be used in three-dimensions [16, 29]. The methodology described in Section 2.5 can be applied to polyhedral networks on which a network of trapezoidal polyhedrons can be imposed. In this way, we can obtain a re-entrant 3D structure similar to Figure 1b. Another possibility is to consider a 2D network in a plane of a three-dimensional space and add an extra vertex in the space to which every vertex of the 2D network is connected. Figure 2a provides such an example. The methodology of Section 2.5 can be applied to the dual diagram providing a methodology to create anticlastic surfaces from synclastic forms (Figure 2b-c).

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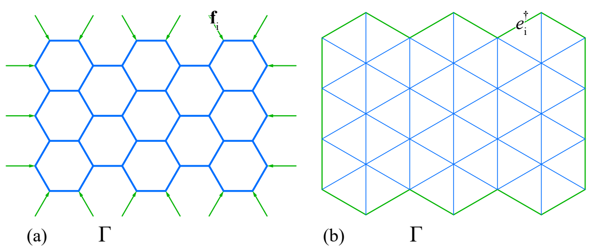

Figure 11: The hexagonal tessellation (a) with its dual diagram (b).

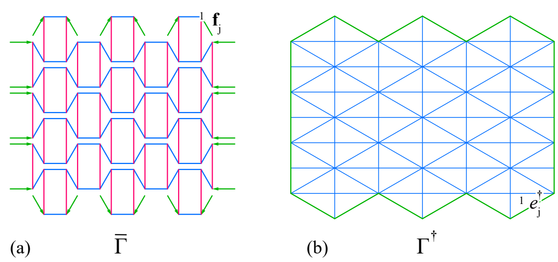

Figure 12: (a) The auxetic network of rectangles and octagons with (b) its dual diagram.

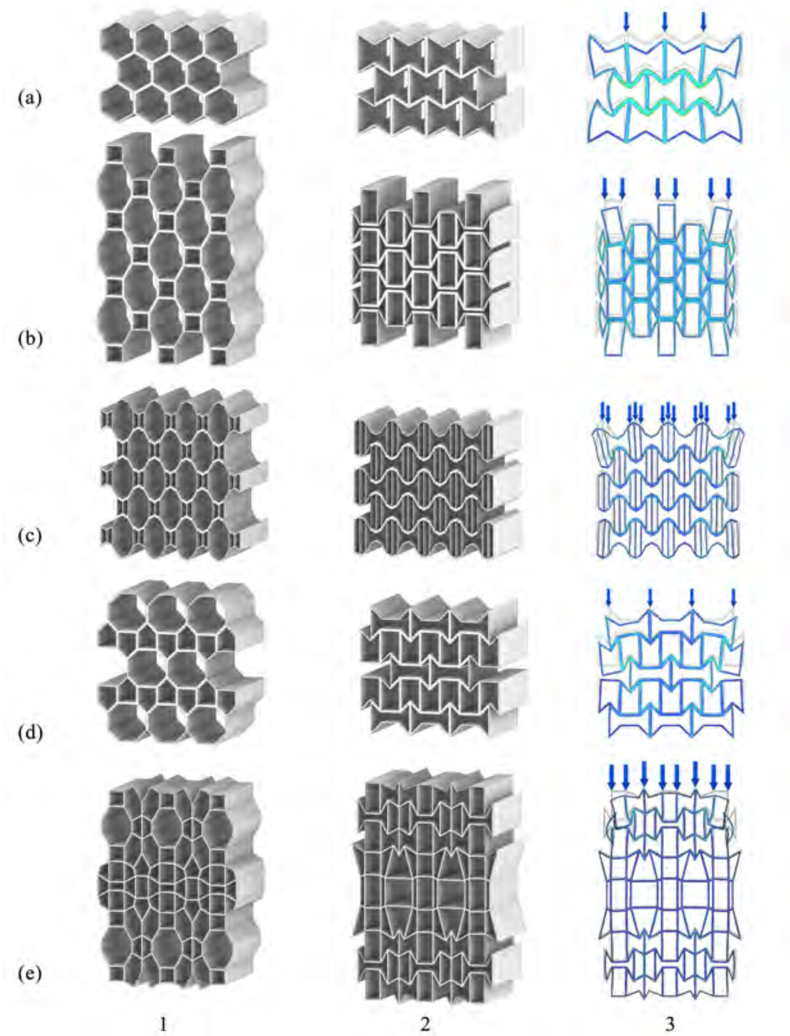

Figure 13: The diagrams in column (1) are examples of convex tessellations, which are transformed into auxetic systems in column (2); a preliminary FE analysis shows their auxetic behavior as the systems contract on its sides if compressive forces are applied on the top edges.

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