IDENTIFICATION OF LINEAR DYNAMICAL SYSTEMS AND MACHINE LEARNING

dedicated to Umberto Mosco, for his 80th birthday

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1 INTRODUCTION

The topic of identification of dynamic systems, has been at the core of modern control , following the fundamental works of Kalman. A good state of the art for linear dynamic systems can be found in the references [2], [6], see also [1] and [3]. Realization Theory has been one of the major outcomes in this domain, with the possibility of identifying a dynamic system from an input-output relationship. The recent development of machine learning concepts has rejuvanated interest for identification. In this paper, we review briefly the results of realization theory, and develop some methods inspired by Machine Learning concepts. We have been inspired by papers [8], [10] and [11].

The interaction between system-control theory and signal processing on the one hand and machine learning and more generally data science on the other hand has been steadily increasing in recent years.

Given that all these disciplines may be viewed as part of the activity of solving inverse problems, this interaction is both inevitable and inexorable. The papers [8, 10, 11] provide compelling instances of this interaction The paper [8] argues this interplay persuasively. Similarly, [11] discusses the naturality and intervention of stochastic control and Hamilton-Jacobi theory in the Entropy-Stochastic Gradient Descent in the study of deep neural networks, amongst many more such examples in deep learning. [10] provides a unified approach for kernels on dynamical systems used in machine learning inspired by the behavioural framework in system theory.

2 REALIZATION THEORY

2.1 BASIC PROBLEM

The basic problem is to go from an input-output relationship to a dynamical system with state observation and partial observation of the state

$$x_{t+1} = Ax_t + Bv_t \tag{2.1}$$

$$y_t = Cx_t$$

The function v_t is the input and the function y_t is the out-put. We have $v_t \in R^m$,$t = 1, \cdots$ and $y_t \in R^p$, $t = 1, \cdots$. The map $v \to y$ is the input-output relationship. If this map can be written as (2.1) then we say that the input-output relationship has an internal state realization, denoted by (A, B, C). The function $x_t \in R^n, t = 1, \cdots$ is the state of the system. The number n is called the model order. The identification consists in finding three matrices A, B, C such that (2.1) holds , given the input-output relationship. We can write the observation y_t as

$$y_{t+1} = CA^t x_1 + \sum_{s=0}^{t} G_{t-s} v_{s+1}, \ t \geq 0 \tag{2.2}$$

where

$$G_t = CA^{t-1}B, \ t \geq 1, \ G_0 = 0 \tag{2.3}$$

are the Markov parameters. We set $\mathcal{G} = (G_0, G_1, \cdots)$, called the impulse response of the system.

2.2 MINIMUM REALIZATION THEORY

The problem solved in classical dynamic systems theory consists in finding matrices A, B, C which satisfy (2.3) for a large number of t. This research topic has raised a huge amount of work. It supposes to know the impulse response $\mathcal{G}$ of the dynamic system. Beautiful results have been obtained to characterize impulse responses for which there exists an internal state realization, and the issue of uniqueness. The basic tool is the block Hankel matrix

$$
\mathcal{H}_{r,r'}(\mathcal{G}) = \begin{bmatrix}
G_1 & G_2 & G_3 & \cdots & G_{r'} \\
G_2 & G_3 & G_4 & \ldots & G_{r'+1} \\
G_3 & G_4 & G_5 & \cdots & G_{r'+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
G_r & G_{r+1} & G_{r+2} & \cdots & G_{r+r'-1}
\end{bmatrix}
\tag{2.4}
$$

There exists an internal state realization if the Block Hankel matrix can be written as follows

$$
\mathcal{H}_{r,r'}(\mathcal{G}) = \mathcal{O}_r(C, A)\mathcal{C}_{r'}(A, B), \ \forall r, r'
\tag{2.5}
$$

with

$$
\mathcal{O}_r(C, A) = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{r-1}
\end{bmatrix}
\tag{2.6}
$$

$$
\mathcal{C}_{r'}(A, B) = \begin{bmatrix} B & AB & \cdots & A^{r'-1}B \end{bmatrix}
\tag{2.7}
$$

The matrix $\mathcal{O}_r(C, A)$ is the observability matrix and the matrix $\mathcal{C}_{r'}(A, B)$ is the controllability matrix. The pair A, C is said observable if the observability matrix has full rank. The pair A, B is controllable if the controllability matrix has full rank. If an internal state realization exists , then it is minimal if the model order is minimal . Kalman proved the important result [5]: A realization (A, B, C) is minimal if and only if the pair (A, B) is controllable and the pair (A, C) is observable. A minimal realization is unique up to a change of basis of the state space. Silverman [9] proved the following characterization: An impulse response $\mathcal{G}$ has a realization if and only if there exist positive integers r, r' and ρ such that

$$
\text{rank} \mathcal{H}_{r,r'}(\mathcal{G}) = \text{rank } \mathcal{H}_{r+1,r'+j}(\mathcal{G}) = \rho
\tag{2.8}
$$

for $j = 1, 2, \cdots$. The integer ρ is the minimal order of the system.

In the sequel , we will consider the dynamic system

$$x_{t+1} = Ax_t, \; t \geq 1 \tag{2.9}$$

$$x_1 = x$$

with observation

$$y_t = Cx_t \tag{2.10}$$

with $x_t \in R^n$, $y_t \in R^p$. To simplify we have taken an input $v_t = 0$, so there is no way we can learn about a potential matrix B. Because there is no input, the only way to stir the system is to have a non zero initial state x. To simplify further , we assume that x and the matrix C are known. The number n is the model order, which is fixed. So the only unknown is the matrix A.

3 OBSERVATION OF THE STATE

We assume here that $C = I$, identity, so the state of the system x_t is observable, but the $n \times n$ matrix A is unknown and must be identified.

3.1 LEAST SQUARE APPROACH

If we stack

$$X_T = \begin{bmatrix} x_2^* \\ \vdots \\ x_T^* \end{bmatrix} \in \mathcal{L}(R^n; R^{T-1}), \; Z_T = \begin{bmatrix} x_1^* \\ \vdots \\ x_{T-1}^* \end{bmatrix} \in \mathcal{L}(R^n; R^{T-1})$$

we can write

$$X_T = Z_T A^*$$

and A^* can be recovered by

$$A^* = (Z_T^* Z_T)^{-1} Z_T^* X_T$$

provided $Z_T^* Z_T \in \mathcal{L}(R^n; R^n)$ is invertible. So

$$A = \sum_{t=1}^{T-1} x_{t+1} x_t^* (\sum_{t=1}^{T-1} x_t x_t^*)^{-1} \tag{3.1}$$

3.2 MACHINE LEARNING APPROACH

The basic idea is to complete the least square function with a penalty term. We thus define the function

$$J_\gamma(A) = \frac{1}{2} \text{tr}(AA^*) + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1} - Ax_t|^2 \tag{3.2}$$

in which the vectors x_t are known. The solution that we get by this approach is different from (3.1). However, it coincides when $\gamma = +\infty$.

Proposition 1. *The solution of problem (3.2) is given by formula*

$$A^\gamma = \sum_{t=1}^{T-1} x_{t+1} x_t^* (\frac{I}{\gamma} + \sum_{t=1}^{T-1} x_t x_t^*)^{-1} \tag{3.3}$$

Proof. The function $J_\gamma(A)$ is convex quadratic. The result is obtained easily from computing the gradient of $J_\gamma(A)$.∎ □

3.3 OTHER FORMULATIONS

We introduce the vector $p_{t+1}, t = 1, \cdots, T-1$ by the formula

$$x_{t+1} - A_\gamma x_t = -\frac{1}{\gamma} p_{t+1} \tag{3.4}$$

then a simple calculation shows that

$$A^\gamma = -\sum_{t=1}^{T-1} p_{t+1} x_t^* \tag{3.5}$$

So A_γ appears as a linear combination of the vectors x_t. So also, combining (3.4) and (3.5) we obtain

$$\frac{p_{t+1}}{\gamma} + \sum_{s=1}^{T-1} x_t . x_s p_{s+1} = -x_{t+1} \tag{3.6}$$

which defines uniquely the coefficients $p_2, \cdots, p_T$ entering in formula (3.5).

3.4 DUAL PROBLEM

The system (3.6) can be interpreted as a necessary and sufficient condition of optimality for a different problem, called the dual problem. The decision is a control $q_1, \cdots, q_{T-1}$ where $q_t \in R^n$. We define the payoff

$$K_\gamma(q) = \frac{1}{2\gamma} \sum_{t=1}^{T-1} |q_t|^2 + \frac{1}{2} \sum_{t,s=1}^{T-1} x_t.x_s q_s.q_t + \sum_{t=1}^{T-1} x_{t+1}.q_t \tag{3.7}$$

and the optimal $q = (q_1, \cdots, q_{T-1})$ is the control $(p_2, \cdots, p_T)$ solution of the system (3.6).

3.5 GRADIENT DESCENT ALGORITHM

Consider the payoff $J(A) = J_\gamma(A)$, we drop the index γ for simplicity. We can compute the gradient $DJ(A)$ which is a matrix

$$DJ(A) = A(I + \gamma \sum_{t=1}^{T-1} x_t x_t *) - \gamma \sum_{t=1}^{T-1} x_{t+1} x_t^* \tag{3.8}$$

The optimal value of A, noted A^γ satisfies $DJ(A^\gamma) = 0$. A gradient descent algorithm is defined by the sequence

$$A^{n+1} = A^n - \rho DJ(A^n) \tag{3.9}$$

where ρ is a positive number to be chosen conveniently. We use

$$\frac{d}{d\theta} J(A^n - \rho\theta DJ(A^n)) = -\rho \mathrm{tr} DJ(A^n - \rho\theta DJ(A^n))(DJ(A^n))^*$$

So

$$J(A^{n+1}) - J(A^n) = -\rho \mathrm{tr} DJ(A^n)(DJ(A^n))^* - \rho \int_0^1 \mathrm{tr}\ (DJ(A^n - \rho\theta DJ(A^n)) - DJ(A^n))(DJ(A^n))^* d\theta \tag{3.10}$$

$$= -\rho \mathrm{tr} DJ(A^n)(DJ(A^n))^* + \rho^2 \int_0^1 \theta \mathrm{tr}\ (DJ(A^n)(I + \gamma \sum_{t=1}^{T-1} x_t x_t *)(DJ(A^n))^*) d\theta =$$

$$= (-\rho + \frac{\rho^2}{2}) \mathrm{tr} DJ(A^n)(DJ(A^n))^* + \frac{\rho^2}{2} \gamma \mathrm{tr}\ \left(DJ(A^n) \sum_{t=1}^{T-1} x_t x_t * (DJ(A^n))^* \right) = \tag{3.11}$$

$$\leq \rho(-1 + \frac{\rho}{2}(1 + \gamma \sum_{t=1}^{T-1} |x_t|^2)) \mathrm{tr} DJ(A^n)(DJ(A^n))^*$$

We obtain the

Proposition 2. *Asume that*

$$2 < \rho < \frac{2}{1 + \gamma \sum_{t=1}^{T-1} |x_t|^2} \tag{3.12}$$

then $A^n \to A^\gamma$ given by formula (3.3) which satisfies $DJ(A^\gamma) = 0$.

Proof. From the assumption (3.12) , we have $-1 + \frac{\rho}{2}(1 + \gamma \sum_{t=1}^{T-1} |x_t|^2) < 0$, hence the sequence $J(A^n)$ is decreasing, thus converging since it is bounded below. From (3.2) it is clear that the sequence A^n is bounded. We first note that $J(A^{n+1}) - J(A^n) \to 0$. Moreover, we can extract from A^n a subsequence, still denoted A^n which converges to some A. From (3.11) we can immediately write

$$(1 - \frac{\rho}{2}) \mathrm{tr} DJ(A)(DJ(A))^* = \frac{\rho}{2}\gamma \mathrm{tr}\left(DJ(A) \sum_{t=1}^{T-1} x_t x_t * (DJ(A))^*\right)$$

From the condition on ρ the left hand side is negative and the right hand side positive. Necessarily $DJ(A) = 0$, hence $A = A^\gamma$. Since the limit will be the same for any converging subsequence, the full sequence converges, which completes the proof. ∎

3.6   RECURSIVITY

We emphasize here the dependence of A^γ with respect to T. So we shall write $A^T = A^\gamma$ and we want to calculate A^{T+1}. We first introduce

$$B^T = (\frac{I}{\gamma} + \sum_{t=1}^{T-1} x_t x_t^*)^{-1} \tag{3.13}$$

then clearly

$$(B^{T+1})^{-1} = (B^T)^{-1} + x_T x_T^* \tag{3.14}$$

and we can see that

$$A^{T+1} = A^T + (x_{T+1} - A^T x_T) x_T^* B^{T+1} \tag{3.15}$$

In this way, we can compute A^T recursively.

3.7 ASYMPTOTIC ANALYSIS

We can check easily that the matrix A^γ converges as $\gamma \to +\infty$ towards the solution of the least square problem (3.1). In fact we can write the asymptotic expapansion

$$A^\gamma = \sum_{t=1}^{T-1} x_{t+1} x_t^* (\sum_{t=1}^{T-1} x_t x_t^*)^{-1} \left(I + \right.$$

(3.16)

$$\left. \sum_{j=1}^{+\infty} \frac{(-1)^j}{\gamma^j} (\sum_{t=1}^{T-1} x_t x_t^*)^{-j} \right)$$

This result requires the invertibility of the matrix $\sum_{t=1}^{T-1} x_t x_t^*$. If this is not true, we can state a weaker result . Since the observation x_t is not arbitrary, we may assume that there exists a matrix $\bar{A}$ such that

$$x_{t+1} = \bar{A} x_t, \ t = 1, \cdots T - 1$$

(3.17)

We can state the

Proposition 3. *Assume the existence of matrices $\bar{A}$ such that (3.17) holds. Then the matrix A^γ converges as $\gamma \to +\infty$ towards the matrix $\bar{A}$ of minimum norm.*

Proof. From (3.2) we can write

$$\frac{1}{2} \mathrm{tr}(A^\gamma (A^\gamma)^*) + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1} - A^\gamma x_t|^2 \leq \frac{1}{2} \mathrm{tr}(\bar{A}(\bar{A})^*)$$

(3.18)

from which it follows immediately that

$$A^\gamma \text{ is bounded}, \ \sum_{t=1}^{T-1} |x_{t+1} - A^\gamma x_t|^2 \to 0, \text{ as } \gamma \to +\infty$$

So , it is clear that any converging subsequence will tend towards one matrix $\bar{A}$ satisfying (3.17). Thanks to (3.18) in which the right hand side refers to any matrix $\bar{A}$ satisfying (3.17), it is clear that the limit point is unique and is the matrix $\bar{A}$ satisfying (3.17) of minimum norm. This completes the proof of the result. ∎ □

4 PARTIALLY OBSERVABLE SYSTEM

4.1 THE MODEL

We extend the identification problem above to the case of partially observable systems. So we have

$$x_{t+1} = Ax_t \tag{4.1}$$

$$x_1 = x \tag{4.2}$$

and

$$y_t = Cx_t \tag{4.2}$$

with $C \in \mathcal{L}(R^n; R^d)$. In the model (4.1), (4.2) we suppose that we know the matrix C and the initial condition x. We want to find the unknown matrix A. This problem generalizes the problem considered in the previous sections, which is recovered when $C = I$.

4.2 A NATURAL APPROACH

Let us assume that the rows of C are linearly independent, which implies

$$CC^* \text{is invertible} \tag{4.3}$$

then the vector $C^*(CC^*)^{-1}y_t$ is solution of (4.2) and is the solution with minimum norm. So we can naturally consider that the state $x_t, t \geq 2$ is in fact reasonably estimated by $C^*(CC^*)^{-1}y_t$ and we are back in the situation of fully observable systems . So we can estimate A by the formula

$$A_\gamma = \sum_{t=1}^{T-1} \hat{x}_{t+1}(\hat{x}_t)^* (\frac{I}{\gamma} + \sum_{t=1}^{T-1} \hat{x}_t(\hat{x}_t)^*)^{-1} \tag{4.4}$$

with

$$\hat{x}_1 = x, \ \hat{x}_t = C^*(CC^*)^{-1}y_t, t = 2, \cdots T \tag{4.5}$$

and we can proceed with similar considerations as above

4.3 MACHINE LEARNING APPROACH

A machine learning approach in the spirit of section 3.2 would be to look for A and vectors $x_t, t = 2, \cdots T$ to minimize the functional

$$J(A, x(.)) = \frac{1}{2}\text{tr}(AA^*) + \frac{\gamma}{2}\sum_{t=1}^{T-1}|x_{t+1} - Ax_t|^2 + \frac{\mu}{2}\sum_{t=2}^{T}|y_t - Cx_t|^2 \qquad (4.6)$$

with $x_1 = x$. In this payoff $x_t, t = 2, \cdots T$ are decision variables, unlike in the above sections. We note the introduction of the parameter μ. The case $\mu = +\infty$ corresponds to the situation of section 4.2. This problem leads surprisingly to considerable difficulties. The reason is because the functional $J(A, x(.))$ is not convex in the pair of arguments $A, x(.)$. It is convenient to make a change of arguments. We replace $x(.)$ by $v(.)$, $v_1, \cdots, v_{T-1}$ and define the state x_t by the relations

$$x_{t+1} - Ax_t = v_t, \ t = 1, \cdots, T-1 \qquad (4.7)$$

$$x_1 = x$$

So we define

$$J(A, v(.)) = \frac{1}{2}\text{tr}(AA^*) + \frac{\gamma}{2}\sum_{t=1}^{T-1}|v_t|^2 + \frac{\mu}{2}\sum_{t=2}^{T}|y_t - Cx_t|^2 \qquad (4.8)$$

with x_t defined by (4.7). Since the values of y_t are not arbitrary, we shall assume that there exists $\bar{A}$ such that , setting

$$\bar{x}_{t+1} = \bar{A}\bar{x}_t, \ t = 1, \cdots, T-1 \qquad (4.9)$$

$$\bar{x}_1 = x$$

$$y_t = C\bar{x}_t$$

so we have the inequality

$$\inf_{A, v(.)} J(A, v(.)) \leq \frac{1}{2}\text{tr}(\bar{A}\bar{A}^*) \qquad (4.10)$$

However, this bound is nor really known, since $\bar{A}$ is not known. A more practical bound will be

$$\inf_{A, v(.)} J(A, v(.)) \leq \frac{\mu}{2}\sum_{t=2}^{T}|y_t|^2 \qquad (4.11)$$

This bound depends on the parameter μ,and will not be useful when we let $\mu \to +\infty$.

To simplify notation , we shall write $Z = (A, v(.))$. The space of vectors Z is called $\mathcal{Z}$ and define the norm in $\mathcal{Z}$ by

$$||Z||^2 = \text{tr}(AA^*) + \sum_{t=1}^{T-1} |v_t|^2 \tag{4.12}$$

We shall compute the gradient $DJ(Z)$. For that , we introduce the sequences of vectors $p_t, t = 1, \cdots T$ defined by

$$p_t = A^* p_{t+1} - \mu C^*(y_t - Cx_t), \ t = 1, \cdots T - 1 \tag{4.13}$$

$$p_T = -\mu C^*(y_T - Cx_T)$$

We have the

Lemma 4. *The gradient of the function $J(A, v(.))$ is given by the formulas*

$$DJ(Z) = \left| \begin{array}{l} A + \sum_{t=1}^{T-1} p_{t+1} x_t^* \\ \\ \gamma v_t + p_{t+1}, \ t = 1, \cdots T - 1 \end{array} \right. \tag{4.14}$$

with x_t given by (4.7) and p_t given by (4.13).

Proof. A simple calculation yields

$$\frac{d}{d\theta} J(Z + \theta \tilde{Z})|_{\theta=0} = \text{tr } A(\tilde{A})^* + \gamma \sum_{t=1}^{T-1} v_t . \tilde{v}_t - \mu \sum_{t=2}^{T}(y_t - Cx_t).C\tilde{x}_t \tag{4.15}$$

with

$$\tilde{x}_{t+1} = A\tilde{x}_t + \tilde{A}x_t + \gamma v_t + p_{t+1}, t = 1, \cdots T - 1$$

$$\tilde{x}_1 = 0$$

Using (4.13) we get easily

$$\frac{d}{d\theta} J(Z + \theta \tilde{Z})|_{\theta=0} = \text{tr } (A + \sum_{t=1}^{T-1} p_{t+1} x_t^*)(\tilde{A})^* +$$

$$+ \sum_{t=1}^{T-1} (\gamma v_t + p_{t+1}) \tilde{v}_t$$

and the result follows. ∎ □

4.4 NECESSARY CONDITIONS OF OPTIMALITY

A minimum point (or a local minimum point) $\hat{z} = (\hat{A}, \hat{v}_t, t = 1, \cdots T - 1)$ will satisfy the equations $DJ(\hat{Z}) = 0$. Therefore

$$\hat{A} + \sum_{t=1}^{T-1} \hat{p}_{t+1}(\hat{x}_t)^* = 0 \tag{4.16}$$

$$\gamma \hat{v}_t + \hat{p}_{t+1} = 0$$

$$\hat{x}_{t+1} - \hat{A}\hat{x}_t + \frac{\hat{p}_{t+1}}{\gamma} = 0, \ t = 1, \cdots T - 1, \ \hat{x}_1 = x \tag{4.17}$$

$$\hat{p}_t = (\hat{A})^* \hat{p}_{t+1} - \mu C^* (y_t - C\hat{x}_t), t = 1, \cdots T - 1, \ \hat{p}_T = -\mu C^* (y_T - C\hat{x}_T)$$

We claim

Proposition 5. *We assume (4.9). The set of miminimum of the function $J(Z)$ is not empty and thus the set of triple $\hat{A}$, $\hat{x}_t, \hat{p}_t$ satisfying (4.16), (4.17) is not empty.*

Proof. In view of (4.9) , (4.10) holds. Therefore minimizing sequences remain bounded . Since $J(Z)$ is continuous , the result follows. □

4.5 GRADIENT DESCENT ALGORITHM

We first show that the function $J(Z)$ has a second derivative $D^2 J(Z) \in \mathcal{L}(\mathcal{Z}; \mathcal{Z})$. Indeed from (4.14) we can easily obtain

$$D^2 J(Z)\tilde{Z} = \left| \begin{array}{l} \tilde{A} + \sum_{t=1}^{T-1} \tilde{p}_{t+1} x_t^* + \sum_{t=1}^{T-1} p_{t+1}(\tilde{x}_t)^* \\ \\ \gamma \tilde{v}_t + \tilde{p}_{t+1}, \ t = 1, \cdots, T - 1 \end{array} \right. \tag{4.18}$$

where $\tilde{Z} = (\tilde{A}, \tilde{v}(.))$ and

$$\tilde{x}_{t+1} = A\tilde{x}_t + \tilde{A}x_t + \tilde{v}_t, \ \tilde{x}_1 = 0, t = 1, \cdots, T - 1 \tag{4.19}$$

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$$\tilde{p}_t = A^* \tilde{p}_{t+1} + p_{t+1}(\tilde{A})^* + \mu C^* C \tilde{x}_t, t = 1, \cdots, T-1$$

$$\tilde{p}_T = \mu C^* C \tilde{x}_T$$

We also state

Lemma 6. *We have the formula*

$$< D^2 J(Z)\tilde{Z}, \tilde{Z} >= tr\,(\tilde{A}(\tilde{A})^*) + 2\sum_{t-1}^{T-1} p_{t+1}.\tilde{A}\tilde{x}_t + \gamma \sum_{t-1}^{T-1} |\tilde{v}_t|^2 + \mu \sum_{t=2}^{T} |C\tilde{x}_t|^2 \tag{4.20}$$

Proof. From (4.18) we get

$$< D^2 J(Z)\tilde{Z}, \tilde{Z} >= tr\,\left(\tilde{A}(\tilde{A})^* + \sum_{t=1}^{T-1} \tilde{p}_{t+1} x_t^* (\tilde{A})^* + \sum_{t=1}^{T-1} p_{t+1}(\tilde{x}_t)^* (\tilde{A})^* \right) +$$

$$+\gamma \sum_{t-1}^{T-1} |\tilde{v}_t|^2 + \sum_{t=1}^{T-1} \tilde{p}_{t+1}.\tilde{v}_t$$

Using the system (4.19), we can compute the term $\sum_{t=1}^{T-1} \tilde{p}_{t+1}.\tilde{v}_t$ and after some rearrangements we derive formula (4.20) where $\tilde{p}_{t+1}$ is absent. ∎

In the sequel we shall use the properties

$$| < D^2 J(Z)\tilde{Z}, \tilde{Z} > | \leq \varphi(||Z||)||\tilde{Z}||^2 \tag{4.21}$$

$$||DJ(Z)|| \leq \psi(||Z||) \tag{4.22}$$

where $\varphi(r), \psi(r)$ are continuous and monotone increasing functions. These properties are consequences of formulas (4.20) and (4.14) and technical calculations, which we do not detail. Since we are interested in minimizing $J(Z)$, we can from (4.8) and (4.11) consider the ball

$$||Z|| \leq M = \sqrt{\frac{\mu}{\min(1, \gamma)} \sum_{t=2}^{T} |y_t|^2} \tag{4.23}$$

The gradient descent algorithm is defined by

$$Z^{n+1} = Z^n - \rho DJ(Z^n) \tag{4.24}$$

$$J(Z^1) \leq \frac{\mu}{2} \sum_{t=2}^{T} |y_t|^2 \Rightarrow ||Z^1|| \leq M$$

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We can state the

Theorem 7. *We choose*

$$\rho < \min(\frac{2}{\varphi(M + \psi(M))}, 1) \tag{4.25}$$

then the sequence $J(Z^n)$ is decreasing, $\|Z^n\| \le M$ and $DJ(Z^n) \to 0$, as $n \to +\infty$. So the limit points of the sequence Z^n are solutions of $DJ(\hat{Z}) = 0$.

Proof. We use the formulas

$$J(Z^{n+1}) - J(Z^n) = -\rho \int_0^1 < DJ(Z^n - \rho\theta DJ(Z^n)), DJ(Z^n) > d\theta =$$

$$= -\rho\|DJ(Z^n)\|^2 + \rho^2 \int_0^1 \int_0^1 \theta < D^2J(Z^n - \rho\theta\lambda DJ(Z^n))DJ(Z^n), DJ(Z^n) > d\lambda d\theta \tag{4.26}$$

So

$$J(Z^n) - J(Z^{n+1}) = \rho\|DJ(Z^n)\|^2 - \rho^2 \int_0^1 \int_0^1 \theta < D^2J(Z^n - \rho\theta\lambda DJ(Z^n))DJ(Z^n), DJ(Z^n) > d\lambda d\theta$$

Suppose $J(Z^n) \le \frac{\mu}{2} \sum_{t=2}^T |y_t|^2 \Rightarrow \|Z^n\| < M$, then, from (4.22) we have $\|DJ(Z^n)\| \le \psi(M)$ and

$$\|Z^n - \rho\theta\lambda DJ(Z^n)\| \le M + \rho\psi(M) \le M + \psi(M)$$

Therefore, from (4.21) we get

$$| < D^2J(Z^n - \rho\theta\lambda DJ(Z^n))DJ(Z^n), DJ(Z^n) > | \le \varphi(M + \psi(M))\|DJ(Z^n)\|^2$$

So

$$J(Z^n) - J(Z^{n+1}) \ge (\rho - \frac{\rho^2}{2}\varphi(M + \psi(M)))\|DJ(Z^n)\|^2 \tag{4.27}$$

Choosing ρ as in (4.25) the number $(\rho - \frac{\rho^2}{2}\varphi(M + \psi(M))) > 0$. Therefore $J(Z^{n+1}) < J(Z^n) < \frac{\mu}{2} \sum_{t=2}^T |y_t|^2 \Rightarrow \|Z^{n+1}\| \le M$. We can iterate, and conclude that the sequence $J(Z^n)$ is monotone decreasing. It follows that $J(Z^n) < \frac{\mu}{2} \sum_{t=2}^T |y_t|^2, \forall n$ and $\|Z^n\| < M, \forall n$. Looking at the inequality (4.27) we get, from the convergence of the sequence $J(Z^n)$, that $J(Z^n) - J(Z^{n+1}) \to 0$ hence $\|DJ(Z^n)\| \to 0$. From the continuity of the gradient, the statement of the Theorem follows. ∎

We can detail the steepest gradient. Namely

$$A^{n+1} = A^n - \rho(A^n + + \sum_{t=1}^{T-1} p_{t+1}^n (x_t^n)^*) \tag{4.28}$$

$$v_t^{n+1} = v_t^n - \rho(\gamma v_t^n + p_{t+1}^n), \ t = 1, \cdots T - 1$$

with

$$x_{t+1}^n = A^n x_t^n + v_t^n, \ t = 1, \cdots, T - 1 \tag{4.29}$$

$$x_1^n = x$$

$$p_t^n = (A^n)^* p_{t+1}^n - \mu C^*(y_t - C x_t^n), \ t = 1, \cdots T - 1 \tag{4.30}$$

$$p_T^n = -\mu C^*(y_T - C x_T^n)$$

Remark 8. The algorithm (4.28),(4.29),(4.30) is the straightforward application of the gradient descent method to the function $J(Z)$. One of the difficulties is to estimate the bound (4.25).

5 SPECIFIC DESCENT METHOD

5.1 METHOD

We exploit here some specific aspects of our optimization problem. Turning to (4.16), (4.17) , we write also

$$\hat{A}(\frac{I}{\gamma} + \sum_{t=1}^{T-1} \hat{x}_t(\hat{x}_t)^*) = \sum_{t=1}^{T-1} \hat{x}_{t+1}(\hat{x}_t)^* \tag{5.1}$$

$$\hat{x}_{t+1} - \hat{A}\hat{x}_t + \frac{\hat{p}_{t+1}}{\gamma} = 0, \ t = 1, \cdots T - 1, \ \hat{x}_1 = x \tag{5.2}$$

$$\hat{p}_t = (\hat{A})^* \hat{p}_{t+1} - \mu C^*(y_t - C \hat{x}_t), t = 1, \cdots T - 1, \ \hat{p}_T = -\mu C^*(y_T - C \hat{x}_T)$$

Considering $\hat{A}$ given in the system (5.2) we obtain a unique pair $\hat{x}_t, \hat{p}_t$, since (5.2) is the Euler condition of a standard linear quadratic control problem. We can formulate it as a problem of calculus of variations

$$\min_{x_2,\cdots,x_T} K_x(\hat{A}, x_2, \cdots, x_T) \tag{5.3}$$

with

$$K_x(\hat{A}, x_2, \cdots, x_T) = \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1} - \hat{A}x_t|^2 + \frac{\mu}{2} \sum_{t=2}^{T} |y_t - Cx_t|^2, \ x_1 = x \tag{5.4}$$

On the other hand, when $\hat{x}_t$ is given , with $\hat{x}_1 = x$, then $\hat{A}$ defined by (5.1) minimizes the function

$$\min_A L(A, \hat{x}_2, \cdots, \hat{x}_T) \tag{5.5}$$

with

$$L(A, \hat{x}_2, \cdots, \hat{x}_T) = \frac{1}{2}\text{tr } AA^* + \frac{\gamma}{2} \sum_{t=1}^{T-1} |\hat{x}_{t+1} - A\hat{x}_t|^2, \ \hat{x}_1 = x \tag{5.6}$$

So $\hat{A}$ appears as the solution of a fixed point problem. We exploit this fact in designing the algorithm. We define a sequence A^n as follows . For A^n given, we define $x_t^n, t = 2, \cdots, T$ by minimizing $K_x(A^n, , x_2, \cdots, x_T)$ in $x_2, \cdots, x_T$. We then define A^{n+1},by minimizing a modification of $L(A, x_2^n, \cdots, x_T^n)$, namely

$$L_\rho(A, x_2^n, \cdots, x_T^n) = \frac{\rho+1}{2}\text{tr } AA^* - \rho\text{tr } A^n A^* + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^n - Ax_t^n|^2 \tag{5.7}$$

The parameter ρ is positive. Finally the sequence A^n is defined by

$$x_{t+1}^n - A^n x_t^n + \frac{p_{t+1}^n}{\gamma} = 0, \ t = 1, \cdots T - 1, \ x_1^n = x \tag{5.8}$$

$$p_t^n = (A^n)^* p_{t+1}^n - \mu C^*(y_t - Cx_t^n), \ p_T^n = -\mu C^*(y_T - Cx_T^n)$$

$$A^{n+1}(\frac{\rho+1}{\gamma}I + \sum_{t=1}^{T-1} x_t^n(x_t^n)^*) = \frac{\rho}{\gamma}A^n + \sum_{t=1}^{T-1} x_{t+1}^n(x_t^n)^* \tag{5.9}$$

5.2 CONVERGENCE

We have the following convergence result

Theorem 9. *Assume $\rho \geq 0$, then the sequence $J(A^n, x^n(.))$ (see (4.6)) is monotone decreasing. The sequence $A^n, x^n(.), p^n(.)$ is bounded , $A^{n+1} - A^n \to 0$ and limits of converging subesquences of $A^n, x^n(.), p^n(.)$ are solutions of (5.1), (5.2).*

Proof. We first compute $K_x(A^{n+1}, , x_2^n, \cdots, x_T^n) - K_x(A^{n+1}, , x_2^{n+1}, \cdots, x_T^{n+1}) > 0$, since $x_2^{n+1}, \cdots, x_T^{n+1}$ minimizes $K_x(A^{n+1}, , x_2, \cdots, x_T)$. Since it is a quadratic function, we get easily

$$K_x(A^{n+1}, , x_2^n, \cdots, x_T^n) - K_x(A^{n+1}, , x_2^{n+1}, \cdots, x_T^{n+1}) = \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^n - x_{t+1}^{n+1} - A^{n+1}(x_t^n - x_t^{n+1})|^2 + \quad (5.10)$$

$$+ \frac{\mu}{2} \sum_{t=2}^{T} |C(x_t^n - x_t^{n+1})|^2$$

Similarly

$$L_\rho(A^n, x_2^n, \cdots, x_T^n) - L_\rho(A^{n+1}, x_2^n, \cdots, x_T^n) = \frac{\rho+1}{2} \operatorname{tr}(A^{n+1} - A^n)(A^{n+1} - A^n)^* + \frac{\gamma}{2} \sum_{t=1}^{T-1} |(A^{n+1} - A^n)x_t^n|^2$$

$$(5.11)$$

The relation (5.11) yields

$$\frac{1}{2} \operatorname{tr} A^n (A^n)^* + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^n - A^n x_t^n|^2 = \frac{1}{2} \operatorname{tr} A^{n+1} (A^{n+1})^* + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^n - A^{n+1} x_t^n|^2 + \quad (5.12)$$

$$+ (\rho + \frac{1}{2}) \operatorname{tr}(A^{n+1} - A^n)(A^{n+1} - A^n)^*$$

and (5.10) yields

$$\frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^n - A^{n+1} x_t^n|^2 + \frac{\mu}{2} \sum_{t=2}^{T} |y_t - C x_t^n|^2 = \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^{n+1} - A^{n+1} x_t^{n+1}|^2 + \frac{\mu}{2} \sum_{t=2}^{T} |y_t - C x_t^{n+1}|^2 \quad (5.13)$$

$$+ \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^n - x_{t+1}^{n+1} - A^{n+1}(x_t^n - x_t^{n+1})|^2 + \frac{\mu}{2} \sum_{t=2}^{T} |C(x_t^n - x_t^{n+1})|^2$$

Adding (5.12) and (5.13) we obtain

$$\frac{1}{2} \operatorname{tr} A^n (A^n)^* + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^n - A^n x_t^n|^2 + \frac{\mu}{2} \sum_{t=2}^{T} |y_t - C x_t^n|^2 = \frac{1}{2} \operatorname{tr} A^{n+1} (A^{n+1})^* + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1}^{n+1} - A^{n+1} x_t^{n+1}|^2 +$$

$$(5.14)$$

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$$+\frac{\mu}{2}\sum_{t=2}^{T}|y_t - Cx_t^{n+1}|^2 + (\rho + \frac{1}{2})\text{tr}(A^{n+1} - A^n)(A^{n+1} - A^n)^* + \frac{\gamma}{2}\sum_{t=1}^{T-1}|x_{t+1}^n - x_{t+1}^{n+1} - A^{n+1}(x_t^n - x_t^{n+1})|^2 +$$

$$+\frac{\mu}{2}\sum_{t=2}^{T}|C(x_t^n - x_t^{n+1})|^2$$

It follows that the sequence $\frac{1}{2}\text{tr } A^n(A^n)^* + \frac{\gamma}{2}\sum_{t=1}^{T-1}|x_{t+1}^n - A^n x_t^n|^2 + \frac{\mu}{2}\sum_{t=2}^{T}|y_t - Cx_t^n|^2$ is decreasing and thus convergente. From (5.14) we get that $A^{n+1} - A^n \to 0$. Clearly the sequences A^n and x_t^n are bounded. From the second relation (5.8), the sequence p_t^n is also bounded. If we extract a converging subsequence, the limit is a soltion of the system (5.1), (5.2). This concludes the proof $\qquad\square$

5.3 DUALITY

In (4.17) we replace $\hat{A}$ by its value coming from (4.16). We obtain

$$\hat{x}_{t+1} + \sum_{s-1}^{T-1}\hat{p}_{s+1}\hat{x}_s.\hat{x}_t + \frac{\hat{p}_{t+1}}{\gamma} = 0, \ t = 1, \cdots T-1, \ \hat{x}_1 = x \tag{5.15}$$

$$\hat{p}_t = -\sum_{s=1}^{T-1}\hat{x}_s\hat{p}_{s+1}.\hat{p}_{t+1} - \mu C^*(y_t - C\hat{x}_t), t = 1, \cdots T-1, \ \hat{p}_T = -\mu C^*(y_T - C\hat{x}_T)$$

The unknowns are the pair $\hat{x}_t, \hat{p}_t, t = 1, \cdots T$. The first one is linear in $\hat{p}(.)$ and the second one is linear in $\hat{x}(.)$. We can interpret the first equation as the Euler equation for the the optimization of the functional

$$K(q(.)) = \frac{1}{2\gamma}\sum_{t=1}^{T-1}|q_t|^2 + \frac{1}{2}\sum_{t,s=1}^{T-1}\hat{x}_t.\hat{x}_s q_s.q_t + \sum_{t=1}^{T-1}\hat{x}_{t+1}.q_t \tag{5.16}$$

and $\hat{p}_{t+1}, t = 1, \cdots, T-1$ attains the minimal value of $K(q)$. Unfortunately, this observation is not very useful, since we do not know the vectors $\hat{x}_t$. One can think, of course, of using the linear system , described by the second equation (5.15) to obtain the vectors $\hat{x}_t$, but this system is not immediately well posed. So , it is not clear how to design an iteration for the pair of equations (5.15). Another possibility to introduce duality is to consider the dual problem of $K_x(\hat{A}, x_2, \cdots, x_T)$. It consists in considering $\hat{p}_t$ as a state and $\hat{x}_t$ as an adjoint state. We can consider indeed the following control problem.

The evolution of the system is described by the following backward dynamics: The control is a sequence $z_2, \cdots z_T$ of vectors in R^d, and we state

$$q_T = -\mu C^* y_T + C^* z_T \tag{5.17}$$

$$q_t = (\hat{A})^* q_{t+1} - \mu C^* y_t + C^* z_t, \ t = T - 1, \cdots 2$$

$$q_1 = (\hat{A})^* q_2 - \mu C^* y_2 + \mu C^* C x$$

and we minimize the functional

$$\mathcal{K}(z(.)) = -q_1.x + \frac{1}{2\gamma} \sum_{t=2}^{T} |q_t|^2 + \frac{1}{2\mu} \sum_{t=2}^{T} |z_t|^2 \tag{5.18}$$

then the solution is $z_t = \mu C \hat{x}_t$ and the optimal state is $\hat{p}_t$. We can then design the following algoritm . Assuming A^n known, we obtain $x_t^n, t = 2, \cdots, T$ by minimizing $K_x(A^n, , x_2, \cdots, x_T)$ in $x_2, \cdots, x_T$. We then obtain p_t^n by minimizing the functional $\mathcal{K}(A^n, z(.))$ defined by the following relations

$$q_T = -\mu C^* y_T + C^* z_T \tag{5.19}$$

$$q_t = (A^n)^* q_{t+1} - \mu C^* y_t + C^* z_t, \ t = T - 1, \cdots 2$$

$$q_1 = (A^n)^* q_2 - \mu C^* y_2 + \mu C^* C x$$

and

$$\mathcal{K}(A^n, z(.)) = -q_1.x + \frac{1}{2\gamma} \sum_{t=2}^{T} |q_t|^2 + \frac{1}{2\mu} \sum_{t=2}^{T} |z_t|^2 \tag{5.20}$$

Then , we can define A^{n+1} by the formula

$$A^{n+1} = -\sum_{t=1}^{T-1} p_{t+1}^n (x_t^n)^* \tag{5.21}$$

This algorithm is different from (5.9) (with $\rho = 0$). In fact, it corresponds to

$$A^{n+1} = -\gamma(x_{t+1}^n - A^n x_t^n)(x_t^n)^* \tag{5.22}$$

We do not claim convergence of this algorithm

5.4   RECURSIVITY

We consider now the dependence in T. We use the notation

$$A\left(\frac{I}{\gamma} + \sum_{t=1}^{T-1} x_t(x_t)^*\right) = \sum_{t=1}^{T-1} x_{t+1}(x_t)^* \tag{5.23}$$

$$x_{t+1} - Ax_t + \frac{p_{t+1}}{\gamma} = 0,\ t = 1, \cdots T-1,\ x_1 = x \tag{5.24}$$

$$p_t = (A)^* p_{t+1} - \mu C^*(y_t - Cx_t),\ t = 1, \cdots T-1,\ p_T = -\mu C^*(y_T - Cx_T)$$

The dependence in T can be emphasaized with the notation A^T, x_t^T, p_t^T. To obtain resursive formulas, it is essential to rely on classical results of control theory, which decouple the forward-backward system of equations (5.23),(5.24). In fact, a linear relation holds

$$x_t = r_t - \Sigma_t p_t \tag{5.25}$$

By well known calculations we have the formulas

$$\Sigma_{t+1} = A\Sigma_t A^* + \frac{I}{\gamma} - A\Sigma_t C^*\left(C\Sigma_t C^* + \frac{I}{\mu}\right)^{-1} C\Sigma_t A^* \tag{5.26}$$

$$\Sigma_1 = 0$$

$$r_{t+1} = Ar_t + A\Sigma_t C^*\left(C\Sigma_t C^* + \frac{I}{\mu}\right)^{-1}(y_t - Cr_t) \tag{5.27}$$

$$r_1 = x$$

and then the sequence p_t is defined by

$$p_t = (I + \mu C^* C\Sigma_t)^{-1}\left(A^* p_{t+1} - \mu C^*(y_t - Cr_t)\right) \tag{5.28}$$

$$p_T = -\mu(I + \mu C^* C\Sigma_T)^{-1} C^*(y_T - Cr_T)$$

In the calculations, we have used the fact that Σ_t is symmetric and we have the relation

$$(I + \mu C^* C\Sigma_t)^{-1} = I - C^*\left(C\Sigma_t C^* + \frac{I}{\mu}\right)^{-1} C\Sigma_t \tag{5.29}$$

The important point is that Σ_t, r_t do not depend on T. Reinstating the notation T, we have the formulas

$$p_t^T = (I + \mu C^* C \Sigma_t)^{-1} \left((A^T)^* p_{t+1}^T - \mu C^* (y_t - Cr_t) \right), t = 1, \cdots T - 1 \tag{5.30}$$

$$p_T^T = -\mu (I + \mu C^* C \Sigma_T)^{-1} C^* (y_T - Cr_T)$$

$$A^T = -\sum_{t=1}^{T-1} p_{t+1}^T (r_t - \Sigma_t p_t^T)^* \tag{5.31}$$

We write

$$A^{T,T+1} = A^{T+1} - A^T \tag{5.32}$$

$$p_t^{T,T+1} = p_t^{T+1} - p_t^T, t = 1, \cdots T$$

then , we get the formulas

$$A^{T,T+1} = -p_{T+1}^{T+1} r_T^* + p_{T+1}^{T+1} (p_T^{T+1})^* \Sigma_T + \tag{5.33}$$

$$-\sum_{t=1}^{T-1} p_t^{T,T+1} r_t^* + \sum_{t=1}^{T-1} p_{t+1}^{T,T+1} (p_t^T)^* \Sigma_t + \sum_{t=1}^{T-1} p_{t+1}^T (p_t^{T,T+1})^* \Sigma_t + \sum_{t=1}^{T-1} p_{t+1}^{T,T+1} (p_t^{T,T+1})^*$$

$$p_t^{T,T+1} = (I + \mu C^* C \Sigma_t)^{-1} \left((A^{T,T+1})^* p_{t+1}^T + (A^T)^* p_{t+1}^{T,T+1} + (A^{T,T+1})^* p_{t+1}^{T,T+1} \right), t = 1, \cdots T - 1 \tag{5.34}$$

$$p_T^{T,T+1} = (I + \mu C^* C \Sigma_T)^{-1} (A^T + A^{T,T+1})^* p_{T+1}^{T+1}$$

We obtain recursivity , but at the price of complex equations.

5.5 ASYMPTOTIC ANALYSIS

We take $\mu = \gamma$ and emphasize the dependence in γ as follows:

$$J_\gamma(A, x(.)) = \frac{1}{2} \mathrm{tr}(AA^*) + \frac{\gamma}{2} \sum_{t=1}^{T-1} |x_{t+1} - Ax_t|^2 + \frac{\gamma}{2} \sum_{t=2}^{T} |y_t - Cx_t|^2 \tag{5.35}$$

and the Euler necessary conditions of optimality

$$A^\gamma = -\sum_{t=1}^{T-1} p_{t+1}^\gamma (x_t^\gamma)^* = 0 \tag{5.36}$$

$$x_{t+1}^\gamma - A^\gamma x_t^\gamma + \frac{p_{t+1}^\gamma}{\gamma} = 0,\ t = 1, \cdots T-1,\ x_1^\gamma = x \tag{5.37}$$

$$p_t^\gamma = (A^\gamma)^* p_{t+1}^\gamma - \gamma C^*(y_t - Cx_t^\gamma), t = 1, \cdots T-1,\ p_t^\gamma = -\gamma C^*(y_T - Cx_T^\gamma)$$

We want to study the behavior of these quantities as $\gamma \to +\infty$. We assume the existence of a matrix $\bar{A}$ such that

$$y_t = C\bar{x}_t \tag{5.38}$$

$$\bar{x}_{t+1} = \bar{A}\bar{x}_t,\ \bar{x}_1 = x$$

We first state the

Proposition 10. *Assume (5.38). Let $A^\gamma, x^\gamma(.)$ be a minimum of $J_\gamma(A, x(.))$, then as $\gamma \to +\infty, A^\gamma$ converges towards the element $\bar{A}$ satisfying (5.38) of minimum norm.*

Proof. The proof is similar to that of Proposition 3. Necessarily

$$\frac{1}{2}\mathrm{tr}(A^\gamma (A^\gamma)^*) + \frac{\gamma}{2}\sum_{t=1}^{T-1} |x_{t+1}^\gamma - A^\gamma x_t^\gamma|^2 + \frac{\gamma}{2}\sum_{t=2}^{T} |y_t - Cx_t^\gamma|^2 \le \frac{1}{2}\mathrm{tr}(\bar{A}(\bar{A})^*)$$

Therefore the sequence A^γ is bounded. Hence also the sequence $x_t^\gamma, t = 2, \cdots T-1$ is bounded. If we consider a convergingnorm. subsequence, the limit satisfies necessarily (5.38) and has minimum. $\qquad \square$

We next consider the triple $A^\gamma, x_t^\gamma, p_t^\gamma$, $t = 1, \cdots T$ solution of (5.36), (5.37). We look for an asympotic expansion of the form

$$x_t^\gamma = \bar{x}_t + \sum_{j=1}^{+\infty} \frac{x_t^j}{\gamma^j} \tag{5.39}$$

$$p_t^\gamma = p_t^0 + \sum_{j=1}^{+\infty} \frac{p_t^j}{\gamma^j},\ A^\gamma = \bar{A} + \sum_{j=1}^{+\infty} \frac{A^j}{\gamma^j}$$

After easy but tedious calculations, we obtain the sequence of systems ,$j \ge 1$

$$x_{t+1}^j - \bar{A}x_t^j - A^j \bar{x}_t - \sum_{k=1}^{j-1} A^k x_t^{j-k} + p_{t+1}^{j-1} = 0,\ t = 1, \cdots T-1 \tag{5.40}$$

$$p_t^{j-1} = (\bar{A})^* p_{t+1}^{j-1} + \sum_{k=1}^{j-1} (A^k)^* p_{t+1}^{j-1-k} + C^* C x_t^j$$

$$x_1^j = 0, \ p_T^{j-1} = C^* C x_T^j$$

where the sum $\sum_{k=1}^{j-1}$ disappears for $j = 1$. We add the relations

$$\bar{A} = -\sum_{t=1}^{T-1} p_{t+1}^0 (\bar{x}_t)^* \tag{5.41}$$

$$A^j = -\sum_{t=1}^{T-1} p_{t+1}^j (\bar{x}_t)^* - \sum_{t=1}^{T-1} \sum_{k=0}^{j-1} p_{t+1}^k (x_t^{j-k})^*$$

In the system (5.40) the unknowns are the pair $x_t^j, p_t^{j-1}, t = 1, \cdots T$. The matrices $A^1, \cdots A^j$ are known, as well as the vectors $x_t^{j-k}, p_{t+1}^{j-1-k}$, for $k = 1, \cdots j - 1$. The first equation (5.41) is an equation for A^1 and the second equation (5.41) is an equation for A^{j+1}. These systems of equations are linear in the unknowns, although very complicated. If they have a solution then the expansion (5.39) is solution of (5.36), (5.37). We shall focus on the first one, which is generic for the following ones. Namely, we have to solve the system

$$x_{t+1}^1 - \bar{A} x_t^1 - A^1 \bar{x}_t + p_{t+1}^0 = 0, \ t = 1, \cdots T - 1 \tag{5.42}$$

$$p_t^0 = (\bar{A})^* p_{t+1}^0 + C^* C x_t^1$$

$$x_1^1 = 0, \cdots, p_T^0 = C^* C x_T^0$$

and

$$\bar{A} = -\sum_{t=1}^{T-1} p_{t+1}^0 (\bar{x}_t)^* \tag{5.43}$$

As said earlier, in the system (5.42), the unknowns are x_t^1 and p_t^0, and A^1 is a parameter. We define A^1 by solving the equation (5.43). We first decouple the system of forward backward equations (5.42). We write

$$x_t^1 = r_t^1 - \Sigma_t p_t^0 \tag{5.44}$$

and standard calculations lead to

$$\Sigma_{t+1} = \bar{A} \left(\Sigma_t - \Sigma_t C^* (C \Sigma_t C^* + I)^{-1} C \Sigma_t \right) (\bar{A})^* + I \tag{5.45}$$

$$\Sigma_1 = 0$$

$$r_{t+1}^1 = \bar{A}\left(I - \Sigma_t C^*(C\Sigma_t C^* + I)^{-1}C\right)r_t^1 + A_1\bar{x}_t \tag{5.46}$$

$$r_1^1 = 0$$

then using (5.44) in the second equation (5.42) leads to the following backward recursion for p_t^0

$$p_t^0 = \left(I - C^*(C\Sigma_t C^* + I)^{-1}C\Sigma_t\right)(\bar{A})^* p_{t+1}^0 + C^*(C\Sigma_t C^* + I)^{-1}Cr_t^1 \tag{5.47}$$

$$p_T^0 = C^*(C\Sigma_t C^* + I)^{-1}Cr_T^1$$

To simplify notation we define

$$\Gamma_t = \bar{A}\left(I - \Sigma_t C^*(C\Sigma_t C^* + I)^{-1}C\right) \tag{5.48}$$

$$\Lambda_t = C^*(C\Sigma_t C^* + I)^{-1}C$$

then we get the system

$$r_{t+1}^1 = \Gamma_t r_t^1 + A_1\bar{x}_t \tag{5.49}$$

$$p_t^0 = (\Gamma_t)^* p_{t+1}^0 + \Lambda_t r_t^1$$

$$r_1^1 = 0, \; p_T^0 = \Lambda_T r_T^1$$

If we use the notation

$$\Phi(t,s) = \Gamma_t \cdots \Gamma_s, \; s = 1, \cdots t \tag{5.50}$$

$$\Phi(t,t+1) = I$$

then we obtain

$$r_{t+1}^1 = \sum_{s=1}^{t} \Phi(t,s+1)A_1\bar{x}_s \tag{5.51}$$

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$$p_{t+1}^0 = \sum_{s=t}^{T-1} \Phi^*(s, t+1)\Lambda_{s+1}r_{s+1}^1 \tag{5.52}$$

and we can write the equation for A_1

$$\bar{A} = -\sum_{t=1}^{T-1}\sum_{\sigma=1}^{T-1}\left(\sum_{s=\max(\sigma,t)}\Phi^*(s,t+1)\Lambda_{s+1}\Phi(s,\sigma+1)\right)A_1\bar{x}_\sigma(x_t)^* \tag{5.53}$$

We can apply this formula in the scalar case, with the notation $\bar{A} = \bar{a}$, $A_1 = a_1$, $C = c$ and $T = 3$. We get $\Sigma_1 = 0$, $\Sigma_2 = 1$, $\Sigma_3 = \dfrac{(\bar{a})^2 + (1+c^2)}{1+c^2}$. Next $\Gamma_1 = \bar{a}$, $\Gamma_2 = \dfrac{\bar{a}}{1+c^2}$, $\Gamma_3 = \dfrac{\bar{a}(1+c^2)}{(1+c^2)^2 + c^2(\bar{a})^2}$. We next have $\Lambda_1 = c^2$, $\Lambda_2 = \dfrac{c^2}{1+c^2}$, $\Lambda_3 = \dfrac{c^2(1+c^2)}{(1+c^2)^2 + c^2(\bar{a})^2}$. Then equation (5.53) becomes

$$\bar{a} = -(\Lambda_2 + \Lambda_3(\Gamma_2 + \bar{a})^2)a_1x^2 \tag{5.54}$$

which gives the value of a_1.

6 CONCLUSION

The concepts and methods of machine learning are most meaningful when the system is already described by a state representation and the state has a physical meaning. Otherwise, if the system is decribed by an input-output linear map, it is probably better to look for the minimum realization, which can be obtained by the Ho algorithm [4]. For purely deterministic systems as decsribed here, the best is probably to try to obtain enough observation to be in the case (4.3), and apply methods of full observation. But , in general, there is a noise which affects the observation and we cannot reduce the problem to the full observation case. In this situation, the methods described above are perfectly applicable. It is clear that the penalty terms play a considerable role, and must be tuned adequately.

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