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# On 2- and 3-factorizations of complete 3-uniform hypergraphs of order up to 9

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**Abstract**

A $k$-factorization of the complete $t$-uniform hypergraph $K_v^{(t)}$ is an $H$-decomposition of $K_v^{(t)}$ where $H$ is a $k$-regular spanning subhypergraph of $K_v^{(t)}$. For $v \leq 9$, we use nauty to generate the 2-regular and 3-regular spanning subhypergraphs of $K_v^{(3)}$ and investigate which of these subhypergraphs factorize $K_v^{(3)}$ or $K_v^{(3)} - I$, where $I$ is a 1-factor. We settle this question for all but two of these subhypergraphs.

## 1 Introduction

A commonly studied problem in combinatorics concerns decompositions of graphs into edge-disjoint subgraphs. A *decomposition* of a graph $K$ is a set $\Delta = \{G_1, G_2, \ldots,$

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$G_s\}$ of subgraphs of $K$ such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_s) = E(K)$ and $E(G_i) \cap E(G_j) = \varnothing$ for all $1 \leq i < j \leq s$. If each element of $\Delta$ is isomorphic to a fixed graph $G$, then $\Delta$ is called a *$G$-decomposition* of $K$ and in this case we may say that *$G$ decomposes $K$*. If $G$ is a spanning subgraph of $K$, then a $G$-decomposition of $K$ is also a *$G$-factorization* of $K$ and in this case we may say that *$G$ factorizes $K$*. If in addition, $G$ is $k$-regular, then a $G$-factorization of $K$ is also a *$k$-factorization* of $K$. A $G$-decomposition of $K_v$ is also known as a *$G$-design of order $v$*. The problem of determining all $v$ for which there exists a $G$-design of order $v$ is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to decompositions of uniform hypergraphs. A *hypergraph $H$* consists of a finite nonempty set $V$ of *vertices* and a set $E$ of nonempty subsets of $V$ called *hyperedges* or simply *edges*. If for each $e \in E$, we have $|e| = t$, then $H$ is said to be *$t$-uniform*. Thus graphs are 2-uniform hypergraphs. For integers $v \geq 1$ and $t \geq 2$, the *complete $t$-uniform hypergraph of order $v$*, denoted $K_v^{(t)}$, is the hypergraph with a vertex set $V$ of size $v$ and edge set the set of all $t$-element subsets of $V$. A *decomposition* of a hypergraph $K$ is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of $K$ such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$ and $E(H_i) \cap E(H_j) = \varnothing$ for all $1 \leq i < j \leq s$. If each element $H_i$ of $\Delta$ is isomorphic to a fixed hypergraph $H$, then $H_i$ is called an *$H$-block*, and $\Delta$ is called an *$H$-decomposition* of $K$. If $H$ is a spanning subhypergraph of $K$, then an $H$-decomposition of $K$ is also an *$H$-factorization* of $K$ and in this case we may say that *$H$ factorizes $K$*. If in addition, $H$ is $k$-regular, then an $H$-factorization of $K$ is also a *$k$-factorization* of $K$. An $H$-decomposition of $K_v^{(t)}$ is also known as an *$H$-design of order $v$*. The problem of determining all values of $v$ for which there exists an $H$-design of order $v$ is known as the *spectrum problem* for $H$.

A $K_k^{(t)}$-design of order $v$ is a generalization of Steiner systems and is equivalent to an $S(t, k, v)$-design. A summary of results on $S(t, k, v)$-designs appears in [11]. Keevash [18] has recently shown that for all $t$ and $k$ the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of $v$. Similar results were obtained by Glock, Kühn, Lo, and Osthus [13, 14] and extended to include the corresponding asymptotic results for $H$-designs of order $v$ for all uniform hypergraphs $H$. These results for $t$-uniform hypergraphs mirror the celebrated results of Wilson [26] for graphs. Although these asymptotic results assure the existence of $H$-designs for sufficiently large values of $v$ for any uniform hypergraph $H$, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on $G$-decompositions of $K_v$ where $G$ is a graph with a relatively small number of edges (see [1] and [12] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [7], the spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. In [21], Mathon and Street give necessary conditions for the existence of decompositions of $K_v^{(3)}$ into copies of the projective plane $PG(2, 2)$ and into copies of the affine plane $AG(2, 3)$. They give sufficient conditions for several infinite classes in both cases. More recently, the spectrum problem was settled in [8] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [8], they also settle the spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered $H$-designs where $H$ is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let $T$, $O$, and $I$ denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph $T$ is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [15]. In another paper [16], Hanani settled the spectrum problem for $O$-designs and gave necessary conditions for the existence of $I$-designs. Perhaps the best known general result on decompositions of complete $t$-uniform hypergraphs is Baranyai's result [6] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers $m$. There are, however, several articles on decompositions of complete $t$-uniform hypergraphs (see [5] and [23]) and of $t$-uniform $t$-partite hypergraphs (see [19] and [25]) into variations on the concept of a Hamilton cycle. For $t = 3$, most such decompositions correspond to 3-factorizations of $K_v^{(3)}$. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [17] and [20]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

For each order $v \leq 9$, we use the nauty [22] function in SageMath [24] to generate the 2-regular and 3-regular 3-uniform hypergraphs on $v$ vertices and investigate which of these hypergraphs factorize $K_v^{(3)}$, or $K_v^{(3)} - I$ where $I$ is a 1-factor. This work mirrors some of the graph factorization results by Anderson [4] and by Adams, Bryant and Khodkar [2]. It also generalizes the Oberwolfach problem which deals with isomorphic 2-factorizations of $K_v$ or $K_v - I$ (see [3] for example) and expands on Baranyai's 1-factorizations results [6]. Our results are summarized in the following theorem.

**Theorem 1.** *All but one of the eight non-isomorphic* 3*-uniform* 2*-regular hypergraphs of order* $v \leq 9$ *factorize* $K_v^{(3)}$. *All but one of the* 49 *non-isomorphic* 3*-uniform* 3*-regular hypergraphs of order* $v \leq 8$ *factorize* $K_v^{(3)}$ *or* $K_v^{(3)} - I$, *where* $I$ *is a* 1*-factor. At most two of the* 148 *non-isomorphic* 3*-uniform* 3*-regular hypergraphs of order* 9 *do not factorize* $K_9^{(3)} - I$, *where* $I$ *is a* 1*-factor.*

If $a$ and $b$ are integers with $a \leq b$, we define $[a, b]$ to be $\{a, a + 1, \ldots, b\}$. Let $\mathbb{Z}_n$ denote the group of integers modulo $n$. We will often describe our hypergraphs by giving their edge set only. Since the hypergraphs we consider will never contain isolated vertices, this is enough to uniquely define them. To save space, we will often list an edge $\{a, b, c\}$ as the string $abc$.

## 2 2-Factors of $K_v^{(3)}, v \leq 9$

If $H$ is a 2-regular spanning subhypergraph of $K_v^{(3)}$ on $x$ edges, then we must have $x = 2v/3$ and thus $v \equiv 0 \pmod 3$. Also, since $K_3^{(3)}$ consists of a single edge, we must have $v \geq 6$.

**Lemma 2.** *There are two non-isomorphic* 2*-regular spanning subhypergraphs of* $K_6^{(3)}$. *One of them factorizes* $K_6^{(3)}$ *and the other does not.*

*Proof.* For $k \in [1,2]$, let $H_{2,k}[0,1,2,3,4,5]$ denote the hypergraph $H_{2,k}$ with vertex set $[0,5]$ and edge sets $E(H_{2,1}) = \{012, 234, 450, 135\}$ and $E(H_{2,2}) = \{012, 123, 345, 045\}$. It is easy to see that these are the only non-isomorphic 2-regular spanning subhypergraphs of $K_6^{(3)}$. It is shown in [8] that $H_{2,1}$ does not factorize $K_6^{(3)}$. Let $V\left(K_6^{(3)}\right) = \mathbb{Z}_5 \cup \{\infty\}$ and let $B_{2,2} = \{H_{2,2}[2,0,1,3,\infty,4]\}$. Then an $H_{2,2}$-factorization of $K_6^{(3)}$ consists of the orbit of the $H_{2,2}$-block in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod 5$. □

**Lemma 3.** *There are six non-isomorphic* 2*-regular spanning subhypergraphs of* $K_9^{(3)}$. *All six factorize* $K_9^{(3)}$.

*Proof.* For $k \in [3,8]$, let $H_{2,k}[0,1,2,3,4,5,6,7,8]$ denote the hypergraph $H_{2,k}$ with vertex set $[0,8]$ and edge set as defined in Table 1.

| $k$ | Edge Set $E(H_{2,k})$ | $k$ | Edge Set $E(H_{2,k})$ |
|---|---|---|---|
| 3 | $\{014, 058, 123, 234, 567, 678\}$ | 6 | $\{012, 078, 156, 234, 345, 678\}$ |
| 4 | $\{012, 067, 138, 234, 456, 578\}$ | 7 | $\{012, 078, 123, 345, 456, 678\}$ |
| 5 | $\{012, 078, 168, 234, 345, 567\}$ | 8 | $\{012, 036, 147, 258, 345, 678\}$ |

Table 1: Edge sets for the six non-isomorphic 2-regular spanning subhypergraphs of $K_9^{(3)}$

Let $V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$
\begin{aligned}
B_{2,3} &= \{H_{2,3}[0,1,2,3,5,4,6,\infty_2,\infty_1], H_{2,3}[0,1,2,\infty_1,4,3,6,\infty_2,5]\},\\
B_{2,4} &= \{H_{2,4}[0,1,2,3,6,5,\infty_1,4,\infty_2], H_{2,4}[0,1,3,5,6,4,2,\infty_1,\infty_2]\},\\
B_{2,5} &= \{H_{2,5}[0,1,2,3,\infty_1,5,\infty_2,6,4], H_{2,5}[0,1,3,2,6,4,\infty_1,\infty_2,5]\},\\
B_{2,6} &= \{H_{2,6}[0,1,2,3,6,5,\infty_1,4,\infty_2], H_{2,6}[0,1,3,2,\infty_1,4,6,5,\infty_2]\},\\
B_{2,7} &= \{H_{2,7}[0,1,2,5,6,\infty_1,4,3,\infty_2], H_{2,7}[0,1,3,4,2,6,\infty_1,5,\infty_2]\},\\
B_{2,8} &= \{H_{2,8}[0,1,2,3,4,\infty_1,6,\infty_2,5], H_{2,8}[0,1,3,2,4,5,\infty_1,6,\infty_2]\}.
\end{aligned}
$$

Then for $k \in [3,8]$, an $H_{2,k}$-factorization of $K_9^{(3)}$ consists of the orbit of the $H_{2,k}$-block in $B_{2,k}$, under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 7$. □

## 3 3-Factorizations of $K_v^{(3)}, v \leq 9$

If $H$ is a 3-regular 3-uniform hypergraph on $v$ vertices and $x$ edges, then we must have $x = v$. If $\gcd(v,3) = 1$, then $v | \binom{v}{3}$ and hence a 3-factorization of $K_v^{(3)}$ is possible. On the other hand, if $\gcd(v,3) = 3$, then $v \nmid \binom{v}{3}$, and in this case, removing a 1-factor from $K_v^{(3)}$ yields the desired size divisibility condition.

**Lemma 4.** *There is a unique* 3*-regular spanning subhypergraph of* $K_4^{(3)}$ *and it trivially factorizes* $K_4^{(3)}$*. There is also a unique* 3*-regular spanning subhypergraph of* $K_5^{(3)}$ *and it factorizes* $K_5^{(3)}$*.*

*Proof.* The only 3-regular subgraph of $K_4^{(3)}$ is $H_{3,1} = K_4^{(3)}$ with a trivial 3-factorization. Similarly, there is only one 3-regular 3-uniform hypergraph on 5 vertices. Let $H_{3,2}[0,1,2,3,4]$ denote the hypergraph with vertex set $[0,4]$ and edge set $\{012, 123, 234, 340, 401\}$. Then $B_{3,2} = \{H_{3,2}[0,1,2,4,3], H_{3,2}[1,3,2,0,4]\}$ is an $H_{3,2}$-factorization of $K_5^{(3)}$. □

**Lemma 5.** *There are four non-isomorphic* 3*-regular spanning subhypergraphs of* $K_6^{(3)}$*. All four factorize* $K_6^{(3)} - I$*, where* $I$ *is a* 1*-factor.*

*Proof.* For $k \in [3,6]$, let $H_{3,k}[0,1,2,3,4,5]$ denote the hypergraph graph $H_{3,k}$ with vertex set $[0,5]$ and edge set as defined in Table 2. It can be checked that these are all of the non-isomorphic 3-regular spanning subhypergraphs of $K_6^{(3)}$.

| $k$ | Edge Set $E(H_{3,k})$ | $k$ | Edge Set $E(H_{3,k})$ |
|---|---|---|---|
| 3 | $\{012, 013, 045, 234, 235, 145\}$ | 5 | $\{012, 345, 013, 245, 023, 145\}$ |
| 4 | $\{012, 034, 135, 245, 023, 145\}$ | 6 | $\{012, 345, 013, 245, 024, 135\}$ |

Table 2: Edge sets for the four non-isomorphic 3-regular spanning subhypergraphs of $K_6^{(3)}$

Let $V\left(K_6^{(3)}\right) = \mathbb{Z}_3 \cup \{\infty_1, \infty_2, \infty_3\}$, let $I$ denote the 1-factor with edge set $\{012, \infty_1\infty_2\infty_3\}$. Let

$$B_{3,3} = \{H_3[0,\infty_1,1,\infty_2,2,\infty_3]\}, \qquad B_{3,4} = \{H_4[0,1,\infty_1,\infty_2,2,\infty_3]\},$$
$$B_{3,5} = \{H_5[0,1,\infty_1,\infty_2,2,\infty_3]\}, \qquad B_{3,6} = \{H_6[0,\infty_1,\infty_2,1,2,\infty_3]\}.$$

For $k \in [3,6]$, an $H_{3,k}$-factorization of $K_6^{(3)} - I$ consists of the orbit of the $H_{3,k}$-block in $B_{3,k}$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 3$. □

**Lemma 6.** *There are ten non-isomorphic* 3*-regular spanning subhypergraphs of* $K_7^{(3)}$*. All but one factorize* $K_7^{(3)}$*.*

*Proof.* For $k \in [7,16]$, let $H_{3,k}[0,1,2,3,4,5,6]$ denote the graph $H_{3,k}$ with vertex set $[0,6]$ and edge set as defined in Table 3.
Let $V\left(K_7^{(3)}\right) = \mathbb{Z}_5 \cup \{\infty_1, \infty_2\}$ and let

$$B_{3,7} = \{H_{3,7}[0,1,3,\infty_1,2,\infty_2,4]\}, \qquad B_{3,8} = \{H_{3,8}[0,1,2,3,\infty_1,\infty_2,4]\},$$
$$B_{3,9} = \{H_{3,9}[0,1,3,2,4,\infty_1,\infty_2]\}, \qquad B_{3,10} = \{H_{3,10}[0,1,2,3,\infty_1,4,\infty_2]\},$$
$$B_{3,11} = \{H_{3,11}[0,1,4,\infty_1,\infty_2,2,3]\}, \qquad B_{3,13} = \{H_{3,13}[0,1,4,2,\infty_1,\infty_2,3]\},$$
$$B_{3,14} = \{H_{3,14}[0,1,4,2,\infty_1,\infty_2,3]\}, \qquad B_{3,15} = \{H_{3,15}[0,1,2,3,\infty_1,4,\infty_2]\},$$
$$B_{3,16} = \{H_{3,16}[0,1,2,4,\infty_1,3,\infty_2]\}.$$

| $k$ | Edge Set, $E(H_{3,k})$ | $k$ | Edge Set, $E(H_{3,k})$ |
|---|---|---|---|
| 7 | $\{012, 013, 056, 123, 246, 345, 456\}$ | 12 | $\{012, 036, 045, 135, 146, 234, 256\}$ |
| 8 | $\{012, 035, 056, 126, 134, 234, 456\}$ | 13 | $\{012, 034, 056, 123, 135, 246, 456\}$ |
| 9 | $\{012, 013, 056, 146, 235, 246, 345\}$ | 14 | $\{012, 036, 045, 134, 156, 234, 256\}$ |
| 10 | $\{012, 013, 056, 134, 234, 256, 456\}$ | 15 | $\{012, 016, 023, 134, 256, 345, 456\}$ |
| 11 | $\{012, 034, 056, 123, 124, 356, 456\}$ | 16 | $\{012, 016, 023, 156, 234, 345, 456\}$ |

Table 3: Edge sets for the ten non-isomorphic 3-regular spanning subhypergraphs of $K_7^{(3)}$

We note that $H_{3,12}$ is a Steiner triple system of order 7. A decomposition of $K_v^{(3)}$ into Steiner triple systems of order $v$ is a known as a *large set* of Steiner triple systems of order $v$. It was shown by Cayley [9] that there does not exist a large set of Steiner triple systems of order 7 (see also [10]). For $k \in [7, 16] \setminus \{12\}$, an $H_{3,k}$-factorization of $K_7^{(3)}$ consists of the orbit of the $H_{3,k}$-block in $B_{3,k}$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod 5$. □

**Lemma 7.** *There are* 33 *non-isomorphic* 3*-regular spanning subhypergraphs of* $K_8^{(3)}$*. All of them factorize* $K_8^{(3)}$*.*

*Proof.* For $k \in [17, 49]$, let $H_{3,k}[0, 1, 2, 3, 4, 5, 6, 7]$ denote the graph $H_{3,k}$ with vertex set $[0, 7]$ and edge set as defined in Table 4.

| $k$ | Edge Set $E(H_{3,k})$ | $k$ | Edge Set $E(H_{3,k})$ |
|---|---|---|---|
| 17 | $\{012, 013, 023, 456, 457, 467, 123, 567\}$ | 34 | $\{012, 034, 567, 135, 267, 234, 045, 167\}$ |
| 18 | $\{012, 013, 045, 467, 567, 236, 123, 457\}$ | 35 | $\{012, 034, 567, 135, 267, 046, 124, 357\}$ |
| 19 | $\{012, 013, 045, 467, 567, 236, 145, 237\}$ | 36 | $\{012, 034, 567, 135, 267, 046, 234, 157\}$ |
| 20 | $\{012, 034, 056, 137, 257, 467, 145, 236\}$ | 37 | $\{012, 034, 567, 135, 267, 046, 127, 345\}$ |
| 21 | $\{012, 034, 156, 357, 246, 017, 234, 567\}$ | 38 | $\{012, 034, 567, 135, 267, 046, 347, 125\}$ |
| 22 | $\{012, 034, 156, 357, 246, 017, 235, 467\}$ | 39 | $\{012, 034, 567, 135, 267, 146, 237, 045\}$ |
| 23 | $\{012, 034, 156, 357, 246, 037, 567, 124\}$ | 40 | $\{012, 034, 567, 156, 234, 137, 027, 456\}$ |
| 24 | $\{012, 034, 156, 357, 246, 137, 045, 267\}$ | 41 | $\{012, 034, 567, 156, 234, 137, 457, 026\}$ |
| 25 | $\{012, 034, 156, 357, 246, 137, 457, 026\}$ | 42 | $\{012, 034, 567, 156, 234, 057, 126, 347\}$ |
| 26 | $\{012, 034, 156, 357, 246, 137, 467, 025\}$ | 43 | $\{012, 034, 567, 156, 234, 057, 267, 134\}$ |
| 27 | $\{012, 034, 156, 357, 246, 237, 045, 167\}$ | 44 | $\{012, 034, 567, 156, 234, 057, 367, 124\}$ |
| 28 | $\{012, 034, 156, 357, 246, 237, 057, 146\}$ | 45 | $\{012, 034, 567, 156, 347, 023, 125, 467\}$ |
| 29 | $\{012, 034, 156, 357, 246, 147, 025, 367\}$ | 46 | $\{012, 034, 567, 156, 347, 235, 124, 067\}$ |
| 30 | $\{012, 034, 156, 357, 246, 147, 036, 257\}$ | 47 | $\{012, 034, 567, 156, 347, 235, 027, 146\}$ |
| 31 | $\{012, 034, 156, 357, 246, 147, 067, 235\}$ | 48 | $\{012, 034, 567, 156, 347, 027, 123, 456\}$ |
| 32 | $\{012, 034, 567, 015, 267, 346, 125, 347\}$ | 49 | $\{012, 034, 567, 156, 347, 027, 134, 256\}$ |
| 33 | $\{012, 034, 567, 015, 367, 246, 124, 357\}$ | | |

Table 4: Edge sets for the 33 non-isomorphic 3-regular spanning subhypergraphs of $K_8^{(3)}$

Let $V\left(K_8^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty\}$ and let

$$\begin{aligned}
B_{3,17} &= \{H_{3,17}[0,1,2,4,3,5,6,\infty]\}, & B_{3,18} &= \{H_{3,18}[0,1,2,4,3,5,\infty,6]\},\\
B_{3,20} &= \{H_{3,20}[0,1,2,4,6,3,\infty,5]\}, & B_{3,21} &= \{H_{3,21}[0,1,2,4,6,\infty,5,3]\},\\
B_{3,22} &= \{H_{3,22}[0,1,4,2,3,6,5,\infty]\}, & B_{3,23} &= \{H_{3,23}[0,1,4,2,6,\infty,5,3]\},\\
B_{3,24} &= \{H_{3,24}[0,1,5,3,\infty,2,4,6]\}, & B_{3,25} &= \{H_{3,25}[0,1,5,2,\infty,6,4,3]\},\\
B_{3,26} &= \{H_{3,26}[0,1,3,\infty,2,6,4,5]\}, & B_{3,27} &= \{H_{3,27}[0,1,2,3,\infty,6,4,5]\},\\
B_{3,28} &= \{H_{3,28}[0,1,2,4,6,3,5,\infty]\}, & B_{3,29} &= \{H_{3,29}[0,1,2,4,5,3,6,\infty]\},\\
B_{3,30} &= \{H_{3,30}[0,1,3,4,2,6,\infty,5]\}, & B_{3,31} &= \{H_{3,31}[0,1,2,3,4,5,\infty,6]\},\\
B_{3,32} &= \{H_{3,32}[0,\infty,1,2,6,5,3,4]\}, & B_{3,33} &= \{H_{3,33}[0,1,2,3,4,5,6,\infty]\},\\
B_{3,34} &= \{H_{3,34}[0,1,6,5,2,\infty,3,4]\}, & B_{3,35} &= \{H_{3,35}[0,1,2,3,4,5,6,\infty]\},\\
B_{3,36} &= \{H_{3,36}[0,1,2,6,4,5,3,\infty]\}, & B_{3,37} &= \{H_{3,37}[0,1,2,6,3,\infty,5,4]\},\\
B_{3,38} &= \{H_{3,38}[0,1,5,6,2,\infty,4,3]\}, & B_{3,39} &= \{H_{3,39}[0,1,2,6,3,5,4,\infty]\},\\
B_{3,40} &= \{H_{3,40}[0,1,3,6,5,2,\infty,4]\}, & B_{3,41} &= \{H_{3,41}[0,1,3,5,6,2,\infty,4]\},\\
B_{3,42} &= \{H_{3,42}[0,1,2,5,\infty,3,6,4]\}, & B_{3,43} &= \{H_{3,43}[0,1,3,6,\infty,2,5,4]\},\\
B_{3,44} &= \{H_{3,44}[0,1,2,5,\infty,4,6,3]\}, & B_{3,46} &= \{H_{3,46}[0,1,4,\infty,2,3,5,6]\},\\
B_{3,47} &= \{H_{3,47}[0,1,2,3,5,\infty,4,6]\}, & B_{3,48} &= \{H_{3,48}[0,1,2,5,3,4,\infty,6]\}.
\end{aligned}$$

In addition, let

$$\begin{aligned}
B_{3,19} = \{&H_{3,19}[0,1,2,3,4,5,6,\infty], H_{3,19}[0,1,4,6,2,\infty,3,5],\\
&H_{3,19}[0,1,5,\infty,3,6,2,4], H_{3,19}[0,2,3,4,5,6,1,\infty],\\
&H_{3,19}[0,4,3,\infty,2,5,1,6], H_{3,19}[0,6,4,\infty,3,5,1,2],\\
&H_{3,19}[0,\infty,3,5,2,6,1,4]\},\\
B_{3,45} = \{&H_{3,45}[0,1,2,3,4,5,6,\infty], H_{3,45}[0,1,3,5,4,6,2,\infty],\\
&H_{3,45}[0,1,4,2,5,6,\infty,3], H_{3,45}[0,1,\infty,6,2,3,5,4],\\
&H_{3,45}[0,2,\infty,3,6,1,4,5], H_{3,45}[0,4,6,1,5,3,2,\infty],\\
&H_{3,45}[0,4,\infty,5,6,1,3,2]\},\\
B_{3,49} = \{&H_{3,49}[0,1,2,3,4,5,6,\infty], H_{3,49}[0,1,3,2,5,4,6,\infty],\\
&H_{3,49}[0,1,4,2,6,3,5,\infty], H_{3,49}[0,1,5,2,4,3,6,\infty],\\
&H_{3,49}[0,2,3,4,6,1,\infty,5], H_{3,49}[0,4,5,1,\infty,2,3,6],\\
&H_{3,49}[2,1,3,4,5,0,6,\infty]\}.
\end{aligned}$$

For $k \in [17,49] \setminus \{19,45,49\}$, an $H_{3,k}$-factorization of $K_8^{(3)}$ consists of the orbit of the $H_{3,k}$-block in $B_{3,k}$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod 7$. For $k \in \{19,45,49\}$, an $H_{3,k}$-factorization of $K_8^{(3)}$ consists of the $H_{3,k}$-blocks in $B_{3,k}$. □

**Lemma 8.** *There are* 148 *non-isomorphic* 3*-regular subhypergraphs of* $K_9^{(3)}$. *At most two of them do not factorize* $K_9^{(3)} - I$, *where* $I$ *is a* 1*-factor.*

*Proof.* For $k \in [50, 197]$, let $H_{3,k}[0,1,2,3,4,5,6,7,8]$ denote the hypergraph $H_{3,k}$ with vertex set $[0,8]$ and edge set as defined in Table 5.

| $k$ | Edge Set $E(H_{3,k})$ | $k$ | Edge Set $E(H_{3,k})$ |
|---|---|---|---|
| 50 | $\{012, 013, 024, 567, 358, 468, 257, 367, 148\}$ | 124 | $\{012, 013, 245, 267, 468, 358, 067, 578, 134\}$ |
| 51 | $\{012, 013, 024, 567, 358, 468, 457, 138, 267\}$ | 125 | $\{012, 013, 245, 267, 468, 358, 367, 047, 158\}$ |
| 52 | $\{012, 013, 024, 567, 358, 468, 457, 368, 127\}$ | 126 | $\{012, 013, 245, 267, 468, 358, 367, 057, 148\}$ |
| 53 | $\{012, 013, 024, 567, 358, 678, 246, 458, 137\}$ | 127 | $\{012, 013, 245, 267, 468, 578, 034, 137, 568\}$ |
| 54 | $\{012, 013, 024, 567, 358, 678, 456, 234, 178\}$ | 128 | $\{012, 034, 135, 067, 248, 568, 237, 156, 478\}$ |
| 55 | $\{012, 013, 024, 567, 358, 678, 456, 138, 247\}$ | 129 | $\{012, 034, 135, 067, 248, 568, 257, 134, 678\}$ |
| 56 | $\{012, 013, 045, 267, 348, 678, 456, 235, 178\}$ | 130 | $\{012, 034, 135, 067, 248, 568, 257, 146, 378\}$ |
| 57 | $\{012, 013, 045, 267, 458, 367, 468, 257, 138\}$ | 131 | $\{012, 034, 135, 067, 248, 568, 257, 147, 368\}$ |
| 58 | $\{012, 013, 045, 267, 368, 457, 278, 134, 568\}$ | 132 | $\{012, 034, 135, 067, 248, 568, 257, 347, 168\}$ |
| 59 | $\{012, 013, 045, 267, 368, 457, 378, 245, 168\}$ | 133 | $\{012, 034, 135, 067, 248, 568, 257, 348, 167\}$ |
| 60 | $\{012, 013, 045, 267, 368, 457, 378, 456, 128\}$ | 134 | $\{012, 034, 135, 067, 248, 568, 578, 123, 467\}$ |
| 61 | $\{012, 013, 045, 267, 468, 357, 238, 146, 578\}$ | 135 | $\{012, 034, 135, 067, 248, 568, 578, 126, 347\}$ |
| 62 | $\{012, 013, 045, 267, 468, 357, 168, 458, 237\}$ | 136 | $\{012, 034, 135, 067, 268, 457, 258, 467, 138\}$ |
| 63 | $\{012, 013, 045, 267, 468, 357, 368, 257, 148\}$ | 137 | $\{012, 034, 135, 067, 268, 457, 168, 348, 257\}$ |
| 64 | $\{012, 013, 045, 267, 468, 357, 568, 123, 478\}$ | 138 | $\{012, 034, 135, 067, 268, 457, 168, 478, 235\}$ |
| 65 | $\{012, 013, 045, 267, 468, 357, 568, 137, 248\}$ | 139 | $\{012, 034, 135, 067, 268, 458, 257, 146, 378\}$ |
| 66 | $\{012, 013, 045, 267, 468, 357, 478, 235, 168\}$ | 140 | $\{012, 034, 135, 067, 268, 458, 257, 147, 368\}$ |
| 67 | $\{012, 013, 045, 267, 468, 357, 478, 356, 128\}$ | 141 | $\{012, 034, 135, 067, 268, 458, 257, 367, 148\}$ |
| 68 | $\{012, 013, 045, 267, 468, 358, 257, 367, 148\}$ | 142 | $\{012, 034, 135, 067, 268, 458, 257, 138, 467\}$ |
| 69 | $\{012, 013, 045, 267, 468, 358, 257, 138, 467\}$ | 143 | $\{012, 034, 135, 067, 268, 458, 257, 348, 167\}$ |
| 70 | $\{012, 013, 045, 267, 468, 358, 367, 147, 258\}$ | 144 | $\{012, 034, 135, 067, 268, 458, 167, 345, 278\}$ |
| 71 | $\{012, 013, 045, 267, 468, 358, 567, 138, 247\}$ | 145 | $\{012, 034, 135, 067, 268, 458, 167, 478, 235\}$ |
| 72 | $\{012, 013, 045, 267, 468, 358, 567, 248, 137\}$ | 146 | $\{012, 034, 135, 067, 268, 458, 567, 123, 478\}$ |
| 73 | $\{012, 013, 045, 267, 468, 358, 278, 345, 167\}$ | 147 | $\{012, 034, 135, 067, 268, 458, 567, 134, 278\}$ |
| 74 | $\{012, 013, 045, 267, 468, 358, 278, 456, 137\}$ | 148 | $\{012, 034, 135, 067, 268, 458, 567, 237, 148\}$ |
| 75 | $\{012, 013, 045, 267, 468, 378, 457, 123, 568\}$ | 149 | $\{012, 034, 135, 067, 268, 458, 567, 147, 238\}$ |
| 76 | $\{012, 013, 045, 267, 468, 378, 457, 235, 168\}$ | 150 | $\{012, 034, 135, 067, 268, 458, 567, 247, 138\}$ |
| 77 | $\{012, 013, 045, 267, 468, 378, 457, 156, 238\}$ | 151 | $\{012, 034, 135, 067, 268, 458, 178, 256, 347\}$ |
| 78 | $\{012, 013, 045, 267, 468, 378, 457, 356, 128\}$ | 152 | $\{012, 034, 135, 067, 268, 458, 178, 237, 456\}$ |
| 79 | $\{012, 013, 045, 267, 468, 378, 568, 237, 145\}$ | 153 | $\{012, 034, 135, 067, 268, 458, 178, 247, 356\}$ |
| 80 | $\{012, 013, 045, 267, 468, 578, 367, 458, 123\}$ | 154 | $\{012, 034, 156, 078, 235, 478, 456, 178, 236\}$ |
| 81 | $\{012, 013, 045, 267, 468, 578, 368, 127, 345\}$ | 155 | $\{012, 034, 156, 078, 235, 478, 456, 378, 126\}$ |
| 82 | $\{012, 013, 045, 467, 238, 678, 458, 123, 567\}$ | 156 | $\{012, 034, 156, 078, 235, 478, 267, 358, 146\}$ |
| 83 | $\{012, 013, 045, 467, 238, 678, 458, 156, 237\}$ | 157 | $\{012, 034, 156, 078, 345, 278, 236, 457, 168\}$ |
| 84 | $\{012, 013, 045, 467, 238, 678, 458, 256, 137\}$ | 158 | $\{012, 034, 156, 078, 345, 278, 256, 178, 346\}$ |
| 85 | $\{012, 013, 045, 467, 238, 678, 568, 123, 457\}$ | 159 | $\{012, 034, 156, 078, 237, 456, 138, 567, 248\}$ |
| 86 | $\{012, 013, 045, 467, 258, 367, 268, 347, 158\}$ | 160 | $\{012, 034, 156, 078, 237, 456, 258, 346, 178\}$ |
| 87 | $\{012, 013, 045, 467, 258, 367, 268, 457, 138\}$ | 161 | $\{012, 034, 156, 078, 237, 456, 258, 138, 467\}$ |
| 88 | $\{012, 013, 045, 467, 258, 367, 568, 247, 138\}$ | 162 | $\{012, 034, 156, 078, 237, 456, 258, 478, 136\}$ |
| 89 | $\{012, 013, 045, 467, 568, 237, 258, 146, 378\}$ | 163 | $\{012, 034, 156, 078, 347, 568, 257, 123, 468\}$ |
| 90 | $\{012, 013, 045, 467, 568, 237, 258, 368, 147\}$ | 164 | $\{012, 034, 156, 078, 347, 568, 257, 134, 268\}$ |
| 91 | $\{012, 013, 045, 467, 568, 237, 578, 123, 468\}$ | 165 | $\{012, 034, 156, 078, 347, 568, 257, 234, 168\}$ |
| 92 | $\{012, 013, 045, 678, 246, 357, 238, 167, 458\}$ | 166 | $\{012, 034, 156, 078, 347, 568, 257, 346, 128\}$ |
| 93 | $\{012, 013, 045, 678, 246, 357, 368, 245, 178\}$ | 167 | $\{012, 034, 156, 078, 357, 248, 236, 457, 168\}$ |
| 94 | $\{012, 013, 045, 678, 246, 357, 368, 457, 128\}$ | 168 | $\{012, 034, 156, 078, 357, 248, 146, 678, 235\}$ |
| 95 | $\{012, 013, 045, 678, 246, 357, 468, 178, 235\}$ | 169 | $\{012, 034, 156, 078, 357, 248, 256, 347, 168\}$ |
| 96 | $\{012, 013, 045, 678, 246, 357, 568, 137, 248\}$ | 170 | $\{012, 034, 156, 078, 357, 248, 256, 367, 148\}$ |
| 97 | $\{012, 013, 045, 678, 456, 237, 268, 345, 178\}$ | 171 | $\{012, 034, 156, 078, 357, 248, 256, 348, 167\}$ |
| 98 | $\{012, 013, 045, 678, 267, 345, 238, 458, 167\}$ | 172 | $\{012, 034, 156, 078, 357, 248, 256, 468, 137\}$ |

| | | | |
|---|---|---|---|
| 99 | $\{012, 013, 045, 678, 267, 458, 136, 348, 257\}$ | 173 | $\{012, 034, 156, 078, 357, 248, 456, 137, 268\}$ |
| 100 | $\{012, 013, 045, 678, 267, 458, 368, 134, 257\}$ | 174 | $\{012, 034, 156, 078, 357, 248, 456, 128, 367\}$ |
| 101 | $\{012, 013, 045, 678, 267, 458, 368, 145, 237\}$ | 175 | $\{012, 034, 156, 078, 357, 248, 456, 278, 136\}$ |
| 102 | $\{012, 013, 045, 678, 267, 458, 368, 245, 137\}$ | 176 | $\{012, 034, 567, 158, 236, 478, 135, 047, 268\}$ |
| 103 | $\{012, 013, 045, 678, 267, 458, 368, 457, 123\}$ | 177 | $\{012, 034, 567, 158, 236, 478, 245, 017, 368\}$ |
| 104 | $\{012, 013, 045, 678, 467, 235, 568, 247, 138\}$ | 178 | $\{012, 034, 567, 158, 236, 478, 146, 023, 578\}$ |
| 105 | $\{012, 013, 045, 678, 467, 258, 356, 123, 478\}$ | 179 | $\{012, 034, 567, 158, 236, 478, 146, 035, 278\}$ |
| 106 | $\{012, 013, 045, 678, 467, 258, 356, 247, 138\}$ | 180 | $\{012, 034, 567, 158, 346, 278, 126, 348, 057\}$ |
| 107 | $\{012, 013, 045, 678, 467, 258, 358, 234, 167\}$ | 181 | $\{012, 034, 567, 158, 267, 348, 123, 058, 467\}$ |
| 108 | $\{012, 013, 045, 678, 467, 258, 368, 134, 257\}$ | 182 | $\{012, 034, 567, 158, 267, 348, 015, 238, 467\}$ |
| 109 | $\{012, 013, 045, 678, 467, 258, 368, 125, 347\}$ | 183 | $\{012, 034, 567, 158, 267, 348, 035, 248, 167\}$ |
| 110 | $\{012, 013, 045, 678, 467, 258, 368, 245, 137\}$ | 184 | $\{012, 034, 567, 158, 267, 348, 235, 467, 018\}$ |
| 111 | $\{012, 013, 045, 678, 467, 258, 368, 457, 123\}$ | 185 | $\{012, 034, 567, 568, 123, 478, 145, 023, 678\}$ |
| 112 | $\{012, 013, 245, 267, 458, 367, 038, 678, 145\}$ | 186 | $\{012, 034, 567, 568, 123, 478, 145, 036, 278\}$ |
| 113 | $\{012, 013, 245, 267, 458, 367, 048, 358, 167\}$ | 187 | $\{012, 034, 567, 568, 123, 478, 056, 178, 234\}$ |
| 114 | $\{012, 013, 245, 267, 458, 367, 048, 568, 137\}$ | 188 | $\{012, 034, 567, 568, 127, 348, 013, 257, 468\}$ |
| 115 | $\{012, 013, 245, 267, 458, 368, 067, 138, 457\}$ | 189 | $\{012, 034, 567, 568, 127, 348, 123, 057, 468\}$ |
| 116 | $\{012, 013, 245, 267, 458, 368, 378, 014, 567\}$ | 190 | $\{012, 034, 567, 568, 127, 348, 015, 237, 468\}$ |
| 117 | $\{012, 013, 245, 267, 468, 357, 038, 146, 578\}$ | 191 | $\{012, 034, 567, 568, 127, 348, 056, 347, 128\}$ |
| 118 | $\{012, 013, 245, 267, 468, 357, 348, 016, 578\}$ | 192 | $\{012, 034, 567, 568, 578, 123, 014, 678, 234\}$ |
| 119 | $\{012, 013, 245, 267, 468, 357, 348, 567, 018\}$ | 193 | $\{012, 345, 678, 036, 147, 258, 013, 246, 578\}$ |
| 120 | $\{012, 013, 245, 267, 468, 357, 058, 167, 348\}$ | 194 | $\{012, 345, 678, 036, 147, 258, 013, 256, 478\}$ |
| 121 | $\{012, 013, 245, 267, 468, 358, 347, 016, 578\}$ | 195 | $\{012, 345, 678, 036, 147, 258, 013, 268, 457\}$ |
| 122 | $\{012, 013, 245, 267, 468, 358, 067, 357, 148\}$ | 196 | $\{012, 345, 678, 036, 147, 258, 123, 047, 568\}$ |
| 123 | $\{012, 013, 245, 267, 468, 358, 067, 378, 145\}$ | 197 | $\{012, 345, 678, 036, 147, 258, 246, 057, 138\}$ |

Table 5: Edge sets for the 148 non-isomorphic 3-regular 3-uniform hypergraphs on 9 vertices

Let $V\left(K_9^{(3)}\right) = \mathbb{Z}_9$ and let $I$ denote the 1-factor with edge set $\{036, 147, 258\}$. Let

$$
\begin{aligned}
B_{3,50} &= \{H_{3,50}[0,1,3,5,2,6,4,8,7]\}, & B_{3,51} &= \{H_{3,51}[0,1,2,4,5,7,8,3,6]\},\\
B_{3,52} &= \{H_{3,52}[0,1,3,5,7,2,4,8,6]\}, & B_{3,53} &= \{H_{3,53}[0,1,2,4,5,6,7,3,8]\},\\
B_{3,54} &= \{H_{3,54}[0,1,2,4,7,3,6,5,8]\}, & B_{3,55} &= \{H_{3,55}[0,1,2,3,5,8,4,6,7]\},\\
B_{3,56} &= \{H_{3,56}[0,1,2,6,4,8,5,3,7]\}, & B_{3,57} &= \{H_{3,57}[0,1,2,5,6,4,7,8,3]\},\\
B_{3,58} &= \{H_{3,58}[0,3,2,5,8,7,1,4,6]\}, & B_{3,59} &= \{H_{3,59}[0,1,2,5,4,7,8,3,6]\},\\
B_{3,60} &= \{H_{3,60}[0,1,5,2,4,6,3,7,8]\}, & B_{3,61} &= \{H_{3,61}[0,1,6,2,7,5,3,4,8]\},\\
B_{3,62} &= \{H_{3,62}[0,1,2,4,3,7,6,5,8]\}, & B_{3,63} &= \{H_{3,63}[0,1,4,2,5,7,3,6,8]\},\\
B_{3,64} &= \{H_{3,64}[0,1,4,3,7,5,2,8,6]\}, & B_{3,65} &= \{H_{3,65}[0,1,3,5,8,6,4,7,2]\},\\
B_{3,66} &= \{H_{3,66}[0,1,2,5,6,4,8,7,3]\}, & B_{3,67} &= \{H_{3,67}[0,1,4,6,5,7,2,8,3]\},\\
B_{3,68} &= \{H_{3,68}[0,1,2,3,5,6,8,4,7]\}, & B_{3,69} &= \{H_{3,69}[0,1,2,3,4,5,8,7,6]\},\\
B_{3,70} &= \{H_{3,70}[0,1,4,5,8,7,3,6,2]\}, & B_{3,71} &= \{H_{3,71}[0,1,2,3,5,4,8,6,7]\},\\
B_{3,72} &= \{H_{3,72}[0,1,5,6,7,4,2,3,8]\}, & B_{3,73} &= \{H_{3,73}[0,1,2,4,6,8,7,5,3]\},\\
B_{3,74} &= \{H_{3,74}[0,1,2,3,8,5,4,7,6]\}, & B_{3,75} &= \{H_{3,75}[0,3,1,4,2,7,6,5,8]\},\\
B_{3,76} &= \{H_{3,76}[0,1,2,7,4,6,8,3,5]\}, & B_{3,77} &= \{H_{3,77}[0,1,4,5,6,8,3,2,7]\},\\
B_{3,78} &= \{H_{3,78}[0,1,3,7,4,5,2,8,6]\}, & B_{3,79} &= \{H_{3,79}[0,1,2,7,4,6,8,3,5]\},
\end{aligned}
$$

$$B_{3,80} = \{H_{3,80}[0,1,2,4,5,8,7,6,3]\}, \qquad B_{3,81} = \{H_{3,81}[0,1,2,3,7,4,5,6,8]\},$$
$$B_{3,82} = \{H_{3,82}[0,1,4,8,2,6,3,5,7]\}, \qquad B_{3,83} = \{H_{3,83}[0,1,2,5,3,7,8,6,4]\},$$
$$B_{3,84} = \{H_{3,84}[0,1,2,5,8,6,4,7,3]\}, \qquad B_{3,85} = \{H_{3,85}[0,1,2,7,5,3,4,8,6]\},$$
$$B_{3,86} = \{H_{3,86}[0,1,3,2,4,6,7,8,5]\}, \qquad B_{3,87} = \{H_{3,87}[0,1,4,6,7,2,8,5,3]\},$$
$$B_{3,88} = \{H_{3,88}[0,1,3,2,5,7,4,8,6]\}, \qquad B_{3,89} = \{H_{3,89}[0,1,2,3,4,5,8,6,7]\},$$
$$B_{3,90} = \{H_{3,90}[0,1,2,7,5,4,3,8,6]\}, \qquad B_{3,91} = \{H_{3,91}[0,1,3,7,2,4,6,8,5]\},$$
$$B_{3,92} = \{H_{3,92}[0,1,4,6,2,5,3,8,7]\}, \qquad B_{3,93} = \{H_{3,93}[0,1,2,4,6,8,5,3,7]\},$$
$$B_{3,94} = \{H_{3,94}[0,1,2,5,6,8,4,3,7]\}, \qquad B_{3,95} = \{H_{3,95}[0,1,2,5,7,3,8,6,4]\},$$
$$B_{3,96} = \{H_{3,96}[0,1,3,7,5,6,4,2,8]\}, \qquad B_{3,97} = \{H_{3,97}[0,1,3,4,2,5,7,8,6]\},$$
$$B_{3,98} = \{H_{3,98}[0,1,3,5,2,4,6,7,8]\}, \qquad B_{3,100} = \{H_{3,100}[0,1,2,4,3,7,6,5,8]\},$$
$$B_{3,101} = \{H_{3,101}[0,1,2,4,3,7,6,5,8]\}, \qquad B_{3,102} = \{H_{3,102}[0,1,4,8,2,5,3,6,7]\},$$
$$B_{3,103} = \{H_{3,103}[0,1,3,4,2,5,8,7,6]\}, \qquad B_{3,104} = \{H_{3,104}[0,1,2,4,5,7,8,6,3]\},$$
$$B_{3,105} = \{H_{3,105}[0,1,2,4,5,7,8,3,6]\}, \qquad B_{3,106} = \{H_{3,106}[0,1,2,7,5,4,8,3,6]\},$$
$$B_{3,108} = \{H_{3,108}[0,1,2,3,6,5,8,4,7]\}, \qquad B_{3,109} = \{H_{3,109}[0,1,3,2,6,5,4,7,8]\},$$
$$B_{3,110} = \{H_{3,110}[0,1,2,5,6,8,7,3,4]\}, \qquad B_{3,111} = \{H_{3,111}[0,1,2,6,3,8,7,5,4]\},$$
$$B_{3,112} = \{H_{3,112}[0,1,2,5,3,8,4,7,6]\}, \qquad B_{3,113} = \{H_{3,113}[0,3,1,2,7,6,4,8,5]\},$$
$$B_{3,114} = \{H_{3,114}[0,1,2,5,4,6,8,7,3]\}, \qquad B_{3,115} = \{H_{3,115}[0,1,2,6,3,5,4,8,7]\},$$
$$B_{3,116} = \{H_{3,116}[0,1,2,4,5,3,6,8,7]\}, \qquad B_{3,117} = \{H_{3,117}[0,1,2,6,3,7,8,4,5]\},$$
$$B_{3,118} = \{H_{3,118}[0,1,3,4,2,8,5,6,7]\}, \qquad B_{3,119} = \{H_{3,119}[0,1,3,4,8,5,7,2,6]\},$$
$$B_{3,120} = \{H_{3,120}[0,1,2,3,4,6,5,7,8]\}, \qquad B_{3,122} = \{H_{3,122}[0,1,2,3,6,5,4,8,7]\},$$
$$B_{3,123} = \{H_{3,123}[0,1,2,3,4,8,6,5,7]\}, \qquad B_{3,124} = \{H_{3,124}[0,1,2,3,7,4,5,6,8]\},$$
$$B_{3,125} = \{H_{3,125}[0,1,2,3,5,6,7,4,8]\}, \qquad B_{3,126} = \{H_{3,126}[0,1,2,6,7,3,4,8,5]\},$$
$$B_{3,127} = \{H_{3,127}[0,1,2,4,7,6,3,8,5]\}, \qquad B_{3,128} = \{H_{3,128}[0,1,3,7,8,5,2,6,4]\},$$
$$B_{3,129} = \{H_{3,129}[0,1,2,8,3,5,4,6,7]\}, \qquad B_{3,130} = \{H_{3,130}[0,1,2,5,7,4,6,8,3]\},$$
$$B_{3,131} = \{H_{3,131}[0,1,2,3,5,8,7,6,4]\}, \qquad B_{3,132} = \{H_{3,132}[0,1,2,5,7,4,3,8,6]\},$$
$$B_{3,133} = \{H_{3,133}[0,1,2,8,5,6,3,7,4]\}, \qquad B_{3,134} = \{H_{3,134}[0,1,2,4,6,5,3,8,7]\},$$
$$B_{3,135} = \{H_{3,135}[0,1,3,5,8,7,2,4,6]\}, \qquad B_{3,136} = \{H_{3,136}[0,1,2,5,4,3,6,7,8]\},$$
$$B_{3,137} = \{H_{3,137}[0,1,2,6,8,7,5,4,3]\}, \qquad B_{3,138} = \{H_{3,138}[0,1,2,5,3,4,6,7,8]\},$$
$$B_{3,139} = \{H_{3,139}[0,1,2,4,8,5,3,7,6]\}, \qquad B_{3,140} = \{H_{3,140}[0,1,2,3,7,5,8,6,4]\},$$
$$B_{3,141} = \{H_{3,141}[0,1,2,4,8,5,6,7,3]\}, \qquad B_{3,142} = \{H_{3,142}[0,1,2,3,8,5,7,4,6]\},$$
$$B_{3,143} = \{H_{3,143}[0,1,3,2,7,8,5,6,4]\}, \qquad B_{3,144} = \{H_{3,144}[0,1,2,8,3,7,4,6,5]\},$$
$$B_{3,145} = \{H_{3,145}[0,1,2,3,8,5,6,4,7]\}, \qquad B_{3,146} = \{H_{3,146}[0,1,2,6,7,8,5,3,4]\},$$
$$B_{3,147} = \{H_{3,147}[0,1,2,5,3,4,6,7,8]\}, \qquad B_{3,148} = \{H_{3,148}[0,1,2,4,5,3,7,6,8]\},$$
$$B_{3,149} = \{H_{3,149}[0,1,2,8,4,7,3,5,6]\}, \qquad B_{3,150} = \{H_{3,150}[0,1,2,7,6,5,8,4,3]\},$$
$$B_{3,151} = \{H_{3,151}[0,1,2,5,3,4,6,8,7]\}, \qquad B_{3,152} = \{H_{3,152}[0,1,3,5,7,8,4,6,2]\},$$
$$B_{3,153} = \{H_{3,153}[0,1,4,5,3,7,8,2,6]\}, \qquad B_{3,155} = \{H_{3,155}[0,1,2,3,4,7,5,6,8]\},$$
$$B_{3,156} = \{H_{3,156}[0,1,2,4,5,7,3,6,8]\}, \qquad B_{3,157} = \{H_{3,157}[0,1,2,4,7,5,6,8,3]\},$$

$B_{3,158} = \{H_{3,158}[0,1,4,3,7,5,6,2,8]\}$, $B_{3,159} = \{H_{3,159}[0,1,2,3,4,5,8,7,6]\}$,
$B_{3,160} = \{H_{3,160}[0,1,2,5,8,4,3,7,6]\}$, $B_{3,161} = \{H_{3,161}[0,1,2,4,3,8,7,5,6]\}$,
$B_{3,162} = \{H_{3,162}[0,1,2,3,4,8,5,7,6]\}$, $B_{3,163} = \{H_{3,163}[0,1,2,4,7,6,5,8,3]\}$,
$B_{3,164} = \{H_{3,164}[0,1,2,4,8,5,3,7,6]\}$, $B_{3,165} = \{H_{3,165}[0,1,3,2,7,5,4,8,6]\}$,
$B_{3,166} = \{H_{3,166}[0,1,2,6,7,8,3,4,5]\}$, $B_{3,167} = \{H_{3,167}[0,1,2,3,4,8,7,6,5]\}$,
$B_{3,168} = \{H_{3,168}[0,1,2,4,8,7,6,5,3]\}$, $B_{3,169} = \{H_{3,169}[0,1,2,3,8,5,4,7,6]\}$,
$B_{3,171} = \{H_{3,171}[0,1,2,7,4,3,6,8,5]\}$, $B_{3,172} = \{H_{3,172}[0,1,2,4,7,5,3,8,6]\}$,
$B_{3,173} = \{H_{3,173}[0,1,4,3,2,6,5,8,7]\}$, $B_{3,174} = \{H_{3,174}[0,1,2,5,6,3,7,8,4]\}$,
$B_{3,175} = \{H_{3,175}[0,1,2,3,4,5,8,6,7]\}$, $B_{3,176} = \{H_{3,176}[0,1,4,5,3,7,6,2,8]\}$,
$B_{3,177} = \{H_{3,177}[0,1,6,8,4,2,5,7,3]\}$, $B_{3,178} = \{H_{3,178}[0,1,3,7,2,4,8,5,6]\}$,
$B_{3,179} = \{H_{3,179}[0,1,2,3,7,5,8,4,6]\}$, $B_{3,181} = \{H_{3,181}[0,1,2,4,8,5,3,6,7]\}$,
$B_{3,182} = \{H_{3,182}[0,1,2,6,8,5,4,7,3]\}$, $B_{3,183} = \{H_{3,183}[0,1,2,4,3,8,5,7,6]\}$,
$B_{3,184} = \{H_{3,184}[0,1,2,4,5,6,3,8,7]\}$, $B_{3,185} = \{H_{3,185}[0,1,2,7,6,4,8,3,5]\}$,
$B_{3,186} = \{H_{3,186}[0,1,2,4,5,3,6,7,8]\}$, $B_{3,187} = \{H_{3,187}[0,1,2,7,4,5,6,3,8]\}$,
$B_{3,188} = \{H_{3,188}[0,1,2,3,5,6,4,7,8]\}$, $B_{3,189} = \{H_{3,189}[0,1,2,5,3,6,4,7,8]\}$,
$B_{3,190} = \{H_{3,190}[0,1,2,3,5,4,6,7,8]\}$, $B_{3,193} = \{H_{3,193}[0,1,3,4,5,2,7,6,8]\}$,
$B_{3,194} = \{H_{3,194}[0,1,2,3,5,6,7,8,4]\}$, $B_{3,195} = \{H_{3,195}[0,1,2,6,3,4,5,8,7]\}$,
$B_{3,196} = \{H_{3,196}[0,1,2,6,3,5,4,8,7]\}$.

In addition, let

$$\begin{aligned} B_{3,99} = \{&H_{3,99}[0,1,7,2,4,3,5,8,6], H_{3,99}[0,1,4,3,7,5,2,8,6],\\ &H_{3,99}[0,1,5,8,7,4,2,3,6], H_{3,99}[0,7,2,3,8,6,4,5,1],\\ &H_{3,99}[0,7,6,8,2,4,1,5,3], H_{3,99}[0,8,2,3,1,6,5,7,4],\\ &H_{3,99}[1,2,4,7,3,6,8,0,5], H_{3,99}[1,8,5,4,7,3,6,0,2],\\ &H_{3,99}[7,1,5,6,8,4,2,0,3]\}, \end{aligned}$$

$$\begin{aligned} B_{3,107} = \{&H_{3,107}[0,1,2,3,6,4,7,5,8], H_{3,107}[0,1,6,4,2,7,3,5,8],\\ &H_{3,107}[0,1,7,5,2,3,6,8,4], H_{3,107}[0,2,6,4,1,8,3,7,5],\\ &H_{3,107}[1,2,7,5,4,3,0,8,6], H_{3,107}[1,5,6,8,3,2,0,4,7],\\ &H_{3,107}[1,8,2,4,7,6,0,3,5], H_{3,107}[3,8,1,2,6,4,0,7,5],\\ &H_{3,107}[4,2,1,3,6,7,0,5,8]\}, \end{aligned}$$

$$\begin{aligned} B_{3,121} = \{&H_{3,121}[0,1,4,7,3,5,2,8,6], H_{3,121}[0,1,3,5,4,7,6,8,2],\\ &H_{3,121}[0,4,7,3,1,5,8,6,2], H_{3,121}[0,4,5,2,7,8,6,1,3],\\ &H_{3,121}[0,7,3,2,4,8,5,1,6], H_{3,121}[0,5,3,8,1,4,2,7,6],\\ &H_{3,121}[0,2,3,8,7,5,6,4,1], H_{3,121}[0,8,1,7,6,4,3,2,5],\\ &H_{3,121}[0,6,5,7,1,2,8,3,4]\}, \end{aligned}$$

$$\begin{aligned} B_{3,154} = \{&H_{3,154}[0,1,3,4,2,7,5,8,6], H_{3,154}[0,1,4,3,7,2,8,5,6],\\ &H_{3,154}[0,1,2,3,4,7,6,5,8], H_{3,154}[0,1,7,6,2,3,8,4,5], \end{aligned}$$

$$
\begin{aligned}
&H_{3,154}[1,3,6,2,4,7,5,0,8], H_{3,154}[4,0,6,8,1,3,5,2,7],\\
&H_{3,154}[4,3,1,6,5,0,2,7,8], H_{3,154}[4,8,2,1,6,3,5,0,7],\\
&H_{3,154}[2,3,5,0,8,1,7,4,6]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{3,170} = \{&H_{3,170}[0,6,2,8,1,3,7,5,4], H_{3,170}[0,6,8,2,1,3,4,5,7],\\
&H_{3,170}[0,6,1,2,8,5,4,7,3], H_{3,170}[0,6,7,8,5,2,1,4,3],\\
&H_{3,170}[0,6,5,2,4,1,3,8,7], H_{3,170}[0,6,4,1,3,8,7,5,2],\\
&H_{3,170}[0,2,7,8,3,6,5,4,1], H_{3,170}[0,8,4,1,7,6,2,5,3],\\
&H_{3,170}[0,1,5,2,3,6,8,7,4]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{3,180} = \{&H_{3,180}[4,2,3,8,7,0,1,6,5], H_{3,180}[0,1,3,2,4,6,5,8,7],\\
&H_{3,180}[0,1,2,3,4,6,5,7,8], H_{3,180}[0,1,4,3,2,7,6,8,5],\\
&H_{3,180}[0,3,8,2,7,4,6,5,1], H_{3,180}[0,3,7,2,6,8,1,4,5],\\
&H_{3,180}[3,4,6,1,8,7,2,5,0], H_{3,180}[4,1,5,2,8,7,6,3,0],\\
&H_{3,180}[8,3,2,0,5,6,7,4,1],
\end{aligned}
$$

$$
\begin{aligned}
B_{3,192} = \{&H_{3,192}[0,3,1,4,2,7,5,8,6], H_{3,192}[0,3,4,7,5,1,2,8,6],\\
&H_{3,192}[0,3,7,1,8,4,2,5,6], H_{3,192}[0,1,4,8,6,3,2,7,5],\\
&H_{3,192}[0,4,7,2,6,3,1,5,8], H_{3,192}[0,7,1,5,6,3,4,2,8],\\
&H_{3,192}[3,2,1,7,6,0,4,5,8], H_{3,192}[3,5,4,1,6,0,2,7,8],\\
&H_{3,192}[3,8,7,4,6,0,1,2,5]\}.
\end{aligned}
$$

For $k \in [50, 197] \setminus \{99, 107, 121, 154, 170, 180, 191, 192, 197\}$, an $H_{3,k}$-factorization of $K_9^{(3)} - I$ consists of the orbit of the $H_{3,k}$-block in $B_{3,k}$ under the action of the map $j \mapsto j + 1 \pmod 9$. For $k \in \{99, 107, 121, 154, 170, 180, 192\}$, an $H_{3,k}$-factorization of $K_9^{(3)} - I$ consists of the $H_{3,k}$-blocks in $B_{3,k}$. For $k \in \{191, 197\}$, we are uncertain about the existence of the factorization. □

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