A CRITERIA FOR THE REGULARITY OF ABSOLUTE MINIMIZERS INVOLVING HAMILTONIAN $H \in C^0(\mathbb{R}^n)$

PENG FA, CHANGYOU WANG AND YUAN ZHOU

Abstract. Let $n \geq 2$ and suppose that $H \in C^0(\mathbb{R}^n)$ is convex and $\liminf_{p\to\infty} H(p) = \infty$. The following are proved to be equivalent:

(i) H is not a constant in any line segment.

(ii) Any absolute minimizer for H in any domain $\Omega \subset \mathbb{R}^n$ enjoys the linear approximation property.

When $n = 2$, (i) is further proved to be equivalent to (iii) or (iv) below:

(iii) Any absolute minimizer for H in any domain $\Omega \subset \mathbb{R}^2$ enjoys $C^1(\Omega)$-regularity.

(iv) Any absolute minimizer for H in whole plane enjoying a linear growth at ∞ must be a linear function.

Contents

1. Introduction

Let $n \geq 2$ and suppose that $H \in C^0(\mathbb{R}^n)$ is a convex and coercive ($\liminf_{p\to\infty} H(p) = \infty$). Aronsson 1960's initiated the study of minimization problems for L^∞-functional

$$\mathcal{F}_H(u,\Omega) = \operatorname*{esssup}_{x\in\Omega} H(Du(x)) \text{ for any domain } \Omega \subset \mathbb{R}^n \text{ and function } u \in W^{1,\infty}_{\rm loc}(\Omega);$$

see [2, 3, 4, 5, 6]. Given a domain $\Omega \subset \mathbb{R}^n$, by Aronsson a function $u \in W^{1,\infty}_{\rm loc}(\Omega)$ is an absolute minimizer for H in Ω (write $u \in AM_H(\Omega)$ for short) if

$$\mathcal{F}_H(u,V) \leq \mathcal{F}_H(v,V) \text{ whenever } V \Subset \Omega,\ v \in W^{1,\infty}_{\rm loc}(V) \cap C(\overline{V}) \text{ and } u = v \text{ on } \partial V.$$

It turns out that the absolute minimizer is the correct notion of minimizers for such L^∞-functionals.

The main purpose of paper is to establish the following criteria for the regularity of absolute minimizers.

11a **Theorem 1.1.** *Let $n \geq 2$ and suppose that $H \in C^0(\mathbb{R}^n)$ is a convex and coercive. Then the following are equivalent:*

(i) *H is not a constant in any line segment.*

(ii) *Any absolute minimizer for H in any domain $\Omega \subset \mathbb{R}^n$ enjoys the linear approximation property.*

When $n = 2$, (i) is also equivalent to (iii) or (iv) below:

(iii) *Any absolute minimizer for H in any domain $\Omega \subset \mathbb{R}^2$ enjoys $C^1(\Omega)$-regularity.*

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(iv) *Any absolute minimizer for H in whole plane enjoying a linear growth at ∞ must be a linear function.*

We refer (iv) above as a Liouville property of absolute minimizer for H. A function $u \in C^0(\mathbb{R}^n)$ enjoys a linear growth at ∞ if $|u(x)| \le C(1+|x|)$ for all $x \in \mathbb{R}^n$, where C is a constant. Moreover, by Crandall-Evans [14] (see also [31] and [28]), a function $u \in C^{0,1}(\Omega)$ enjoys the linear approximation property if for any $x \in \Omega$ and any sequence $\{r_j\}_{j\in\mathbb{N}}$ which converges to 0, there exist a subsequence $\{r_{j_k}\}_{k\in\mathbb{N}}$ and a vector $e_{\{r_{j_k}\}_{k\in\mathbb{N}}}$ such that

$$\lim_{k\to\infty} \sup_{y\in B(0,1)} \left| \frac{u(x + r_{j_k} y) - u(x)}{r_{j_k}} - e_{\{r_{j_k}\}_{k\in\mathbb{N}}} \cdot y \right| = 0$$

and

$$H\left(e_{\{r_{j_k}\}_{k\in\mathbb{N}}}\right) = S u(x) := \lim_{r\to 0} \|H(Du)\|_{L^\infty(B(x,r))}.$$

Note that everywhere differentiability always implies the linear approximation property. But the converse is not necessarily correct due to Preiss' fucntion; in particular, Lipchitz functions do not have the linear approximation property necessarily.

We have the following interesting consequence of Theorem 1.1.

Corollary 1.2. *For any Banach norm $\|p\|$ in $\mathbb{R}^n$, the unit sphere $S^1_{\|\cdot\|} := \{p \in \mathbb{R}^n : \|p\| = 1\}$ does not contain any line-segment if and only if absolute minimizers for $\|p\|$ have the linear approximation property, or the C^1-regularity (when $n = 2$) or the Liouville property (when $n = 2$).*

In particular, let

$$|p|_t = \left(\sum_{i=1}^n p_i^t\right)^{1/t} \text{ when } t \ge 1 \text{ and } = \max_{1\le i\le n} |p_i| \text{ when } t = \infty \text{ for all } p \in \mathbb{R}^n.$$

For $1 < t < \infty$, absolute minimizers for $|p|_t$ always have the linear approximation property, the C^1-regularity (when $n = 2$) and the Liouville property (when $n = 2$); but for $t = 1$ and ∞, absolute minimizers do not have any of these properties necessarily.

Recall that if $H \in C^1(\mathbb{R}^n)$, Aronsson derived the Euler-Lagrange equations for absolute minimizers:

eq1.2

$$\mathscr{A}_H[u] := \langle D[H(Du)], D_p H(Du)\rangle = \sum_{i,j=1}^n H_{p_i}(Du) H_{p_j}(Du) u_{x_i x_j} = 0 \quad \text{in } \Omega, \tag{1.1}$$

which are highly degenerate nonlinear elliptic equations. In the special case $H(p) = \frac{1}{2}|p|^2$, (1.1) is the well-known ∞-Laplace equation

eq1.3

$$\Delta_\infty u := \frac{1}{2}\langle D|Du|^2, Du\rangle = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} = 0 \quad \text{in } \Omega. \tag{1.2}$$

By Crandall-Lions' theory [17], viscosity solutions to (1.1) and (1.2) are well-defined; viscosity solutions to (1.2) are called as ∞-harmonic functions. Jensen [23] identified ∞-harmonic functions with absolute minimizers for $\frac{1}{2}|p|^2$. If $H \in C^1(\mathbb{R}^n)$ is convex and coercive, by Crandall et al [18] and Yu [30] (see also [11, 13, 22, 9]) we know that absolute minimizers for H coincide with viscosity solutions to (1.1).

The existence and uniqueness of ∞-harmonic functions in bounded domains has been established by Jessen [23]; see also [10, 16, 1, 26] for other approaches for the uniqueness. If $H \in C^0(\mathbb{R}^n)$ is convex and coercive, we refer to [11, 8] for the existence of absolute minimizers. Assuming additionally that $H^{-1}(\min H)$ has empty interior, Armstrong et al [9] obtained their uniqueness; see also Jensen et al [24] when $H \in C^2(\mathbb{R}^n)$, and [8, 16] when H is a Banach norm. By [30, 24, 9], to get the uniqueness it is also necessary to assume $H^{-1}(\min H)$ having empty interior.

The regularity of absolute minimizers/viscosity solutions to (1.1)&(1.2) is the main issue in this field. By the definition, absolute minimizers are always locally Lipschitz, and hence differentiable almost everywhere.

Crandall-Evans [14] obtained the linear approximation property of ∞-harmonic functions. The interior C^1-regularity of planar ∞-harmonic functions and also a Liouville property as in Theorem 1.1 (iv) were proved by Savin [27], their interior $C^{1,\alpha}$-regularity for some $0 < \alpha < 1/3$ by Evans-Savin [19] and their boundary C^1-regularity by Wang-Yu [29]. When $n \geq 3$, Evans-Smart [20, 21] obtained the everywhere differentiability of ∞-harmonic functions.

If $H \in C^2(\mathbb{R}^n)$ is locally strongly convex, Wang-Yu [28] obtained the linear approximation property of absolute minimizers/viscosity solutions to (1.1), and moreover when $n = 2$, their interior C^1-regularity and a Liouville property as in Theorem 1.1 (iv). On the other hand, assuming $H \in C^1(\mathbb{R}^n)$, Katzourakis [25] showed that, to get C^1-regularity of all viscosity solutions to (1.1), it is necessary to assume that H is not a constant in any line-segment. Indeed, if $H \in C^1(\mathbb{R}^n)$ is a constant in some line-segment (say $[a, b]$), by [25] the following function $u_f \notin C^1(\mathbb{R}^n)$ is a viscosity solution to (1.1):

E0 (1.3) $$u_f(x) = \frac{b+a}{2} \cdot x + f\left(\frac{b-a}{2} \cdot x\right) \quad \forall x \in \mathbb{R}^n,$$

where $f \in C^{0,1}(\mathbb{R})$ with $\|f'\|_{L^\infty(\mathbb{R})} < 1$ but $f \notin C^1(\mathbb{R}^n)$. If $H \in C^0(\mathbb{R}^n)$ is convex and coercive, Theorem 1.1 indicates that, to get the linear approximation property, C^1-regularity (when $n = 2$) and the Liouville property (when $n = 2$) of absolute minimizers for H, it is sufficient and also necessary to assume that H is a not constant in any line-segment.

Now we turn to the proof of Theorem 1.1, and without loss of generality, under the following stronger assumption on H:

assh0 (1.4) $$H \in C^0(\mathbb{R}^n) \text{ is convex, } H(0) = \min_{p \in \mathbb{R}^n} H(p) = 0 \text{ and } \liminf_{p \to \infty} \frac{H(p)}{|p|} = \infty.$$

Indeed, if $H \in C^0(\mathbb{R}^n)$ is a convex and coercive, there exists a $p_0 \in \mathbb{R}^n$ such that $\min_{p \in \mathbb{R}^n} H(p) = H(p_0)$. Let $\widetilde{H}(p) = [H(p + p_0) - H(p_0)]^2$. Then $\widetilde{H}$ satisfies the assumption (1.4), and also, $\widetilde{H}$ satisfies (i) if and only if H satisfies (i). Note that $u \in AM_H(\Omega)$ if and only if $\widetilde{u}(x) \in AM_{\widetilde{H}}(\Omega)$, where $\widetilde{u}(x) = u(x) - p_0 \cdot x$ for all $x \in \Omega$. Moreover, u and $\widetilde{u}$ enjoy the same regularity.

To prove (ii), (iii) or $(iv) \Rightarrow (i)$ in Theorem 1.1, we show that if H is a constant in some line-segment $[a, b]$, then the function u_f given in (1.3) is an absolute minimizer for H in $\mathbb{R}^n$ whenever $\|f'\|_{L^\infty(\mathbb{R})} \leq 1$; see Lemma 4.2. By choosing suitable f, one easily see that u_f fails to have the linear approximation property and hence is not C^1-regular, and also that the Liouville theorem fails. Note that this proof also works in dimension $n \geq 2$. See Section 4 for Lemma 4.2 and the proof of $(ii) \Rightarrow (i)$ in Theorem 1.1. The proofs of (iii) or $(iv) \Rightarrow (i)$ in Theorem 1.1 is stated in Section 6.

The proof of $(i) \Rightarrow (ii)$ in Theorem 1.1 relies on the key Lemma 1.3.

LEMC4 **Lemma 1.3.** *Let H be as in* (1.4) *with $n = 2$ and satisfy (i). If $u \in C^{0,1}(\mathbb{R}^n)$ satisfies*

F1 (1.5) $$S_t^+ u(x), -S_t^- u(x) \leq k \text{ and } S_t^+ u(0) = -S_t^- u(0) = k \quad \forall x \in \mathbb{R}^n, t > 0$$

for some $0 \leq k < \infty$, then u is a linear function and $H(Du) \equiv k$, that is, there exists a vector $p_0 \in \mathbb{R}^n$ such that

$$u(x) = u(0) + p_0 \cdot x \quad \text{for all } x \in \mathbb{R}^n, \text{ and } H(p_0) \equiv k.$$

The notion $S_t^{\pm} u(x)$ above will be explained in Section 2, where we also recall the comparison property with cones and convex/concave criteria for absolute minimizers, which are given by Armstrong et al [9]. Lemma 1.3 and the convex/concave criteria allow us to get the linear approximation property of absolute minimizer in a standard way; see section 4 for details.

Lemma 1.3 was first proved by Cranial-Evans [14] when $H(p) = \frac{1}{2}|p|^2$ by using its Hilbert structure. When $H \in C^2(\mathbb{R}^2)$ is locally strongly convex, by using the C^1-regularity of cone functions, Wang-Yu [28] proved Lemma 1.3. When $H \in C^1(\mathbb{R}^n)$ is strictly convex and satisfies (1.4), observing and using the $C^1(\mathbb{R}^n)$-regularity and strict convexity of the conjugate L of H (see Theorem 3.1 (i) below), one could deduce Lemma 1.3 from the arguments of Yu [31], even where some stronger conditions on H were assumed. But when H satisfies (1.4) and Theorem 1.1 (i) (that is, H is not constant in any line-segment), the C^1-regularity of cone functions is unavailable, and the convex conjugate L of H does not necessarily have $C^1(\mathbb{R}^n)$-regularity and also is not necessarily strictly convex. This causes some essentially difficulty to prove Lemma 1.3; hence, new idea is needed to prove Lemma 1.3.

Indeed, in Section 3, we observe that, under (1.4) and Theorem 1.1 (i) for H, the sub-differential set of its convex conjugate L at any point must be a single point or be a line-segment; see Theorem 3.1 (where we also obtain a 2-dimensional analogue for the geometry of ∂L which may has its own interests). Using this, by some careful analysis on the analytic/geometric structures of Hamilton-Jacobi flows and also the subdifferential sets of H and L, we are able to determinant a unique vector $p_0 \in H^{-1}(k)$ so that $u(x) = u(0) + p_0 \cdot x$ and hence prove Lemma 1.3; see Section 4 for details.

To prove $(i) \Rightarrow (iii)\&(iv)$ in Theorem 1.1 we need the following crucial Proposition 1.4. Given any $x^0 \in \mathbb{R}^2$ and $r, \delta \in (0,1]$, for a function $u \in C^0(B(x^0, r))$ denote by $\mathscr{D}(u)(x^0; r; \delta)$ the collection of all vectors e such that

$$\sup_{B(x^0,1)} |u(x + x^0) - u(x^0) - e \cdot (x - x_0)| \le \delta r.$$

In other words, $\mathscr{D}(u)(x^0; r; \delta)$ collects all linear approximations at the scale δ of u in the ball $B(x^0, r)$.

xprop **Proposition 1.4.** *Suppose that H satisfies* (1.4) *with $n = 2$ and (i). Let $R \ge 1$. For any $\epsilon > 0$, there exist $\delta = \delta(H, R, \epsilon) > 0$ such that for any e_6 and $e_{0,6}$, if we can find a $u \in AM_H(B(0,6))$ satisfying*

xlin0 (1.6) $$\|H(Du)\|_{L^\infty(B(0,6))} \le R \text{ and } u \text{ is not linear in some neighborhood of } 0,$$

xlin1 (1.7) $$e_6 \in \mathscr{D}(u)(0; 6; \delta) \text{ and } 1 \le H(e_6) \le 2,$$

and

xlin2 (1.8) $$e_{0,6} \in \mathscr{D}(u)(0; 6r; \delta) \text{ for some } r \in (0, 1/2] \text{ and } H(e_{0,6}) = S\,u(0),$$

then we have

xlin3 (1.9) $$|e_6 - e_{0,6}| \le \epsilon.$$

By an argument essentially the same to those of [27, Theorem A,B&C] and [28, Theorem B,C&E], one could deduce the following Theorem 1.5 from Proposition 1.4 and Theorem 1.1 (ii), and then prove $(i) \Rightarrow (iii)$ and (iv) in Theorem 1.1 by using Theorem 1.5; we give the details in Section 6 for reader's convenience and also for the completeness of this paper, but the reader familiar with [27] and [28] may ignore Section 6.

c2 **Theorem 1.5.** *Let H be as in* (1.4) *with $n = 2$ and satisfy Theorem 1.1 (i).*

(i) *Given any domain $\Omega \subset \mathbb{R}^n$, if $u \in AM_H(\Omega)$, then $u \in C^1(\Omega)$.*
(ii) *For any $k > 0$, there exists an increasing continuous function ρ_k with $\rho_k(0) = 0$ such that*

rho (1.10) $$\sup_{x,y \in B(z,s)} |Du(x) - Du(y)| \le \rho_k(s/r) \text{ whenever } s < r,\ u \in AM_H(B(x, 2r)) \text{ and } \|H(Du)\|_{L^\infty(B(x,2r))} \le k.$$

So to get Theorem 1.1, we only need to prove Proposition 1.4. It was first proved by Savin [27, Proppsition 1] when $H(p) = \frac{1}{2}|p|^2$. and later by Wang-Yu [28, Propsotion 4.1] when $H \in C^2(\mathbb{R}^2)$ is locally strongly convex. In the two proofs, a planar topology observation by Savin and several properties of absolute minimizers (including comparison property with cones, comparison with linear function and linear approximation property) were used.

When H satisfies (1.4) and Theorem 1.1 (i), we will prove Proposition 1.4 by using Theorem 1.1 (ii) and some ideas of Savin [27] and Wang-Yu [28], more precisely, using the procedure by Wang-Yu [28, Propsotion 4.1]. Note that Savin (see the proof of [28, Proposition 4.1]) used the Hilbert structure of $\frac{1}{2}|p|^2$ and Wang-Yu (see the proof of [28, Proposition 4.1]) relied on the $C^2(\mathbb{R}^2)$-regularity and (locally) strong convexity of H. However, in Proposition 1.4, H is only supposed to satisfy (1.4) and the necessary (and hence minimal) assumption (i); none of Hilbert structure, C^1-regularity and (locally) strong convexity are available. To overcoming several essential difficulties caused by these, and then to get Proposition 1.4, new ideas/observations are definitely required.

Indeed, by using Theorem 1.1 (ii) and Savin's topology argument (see Lemma (5.1)) as in [28, Lemma 4.2], we get some auxiliary vector e. The proof of (1.9) is then reduced to $|e_6 - e| \le \epsilon/2$ (see (5.4)) and $|e_{0,6} - e| \le \epsilon/2$ (see (5.5)). To get $|e_6 - e| \le \epsilon/2$, the key observations are the analytic properties of H and L in Section 3, in particular, given in Theorems 3.2 and 3.3. These properties (in particular Theorem 3.3) allow us to build up a discrete flow and hence get the length estimate $H(e_6) \le H(e) + \eta$ as in Lemma 5.2, and also allows to get the angle estimates as in Lemma 5.3. From these and Theorem 3.2 we conclude the key inequality $|e_6 - e| \le \epsilon/2$. Similarly, we also have $|e_{0,6} - e| \le \epsilon/2$.

We emphasis that the angle estimate in Lemma 5.3 is more essential. Looking at u_f given (1.3), one easily see that $S(u_f)$ is always continuous (recalling that $H(Du_f) = S(u_f)$ almost everywhere) but can not expect any angle estimate similar to Lemma 5.3 (and also everywhere differentiability and C^1-regularity).

2. Preliminaries

The following comparison principle is established by [9]. It is known that linear functions are always absolute minimizers.

com-linear **Lemma 2.1.** *Suppose that H satisfies* (1.4) *and $H^{-1}(0)$ has empty interior. For any domain $U \subset \mathbb{R}^n$, and $u, v \in AM_H(U) \cap C^0(\overline{U})$ (in particular, v is any linear function), we have*

$$\max_{x \in U}[\pm u(x) - v(x)] \le \max_{x \in \partial U}[\pm u(x) - v(x)].$$

Next we recall the comparison property with cones. The cone functions for H are defined by

$$\mathscr{C}_a^H(x) = \sup_{H(p) \le a} p \cdot x \quad \forall a \ge 0,\ x \in \mathbb{R}^2.$$

It is evident that $\mathscr{C}_a^H \in C^{0,1}(\mathbb{R}^n)$ is convex, positively homogeneous, subadditive and $\mathscr{C}_a^H(x) > 0$ for every $a > 0$ and $x \ne 0$. See [9, Lemma 2.18] for the following lemma.

LEM7.11 **Lemma 2.2.** *Let H be as in* (1.4). *Let $U \subset \mathbb{R}^n$ be any domain, $u \in C^{0,1}(U)$ and $a \ge 0$. The following are equivalent:*

(i) *$H(Du) \le a$ almost everywhere in U;*
(ii) *$u(x) - u(y) \le \mathscr{C}_a^H(x - y)$ provided the line segment $[x, y] \subset U$.*

Below denote by usc (U) (resp. lsc (U)) the class of upper (resp. lower) semi-continuous functions in U.

Definition 2.3. Let H be as in (1.4).

(i) A function $u \in$ usc (U) satisfies the comparison property with cones for H from above in U if

$$\max_V\{u - \mathscr{C}_a^H(x - x_0)\} = \max_{\partial V}\{u - \mathscr{C}_a^H(x - x_0)\}$$

whenever $V \Subset \Omega$, $a \ge 0$ and $x_0 \in \mathbb{R}^n \setminus V$; for short, write $u \in CCA_H(U)$.

(ii) A function $u \in$ lsc (U) satisfies the comparison property with cones for H from below in U if and

$$\min_V\{u + C_a^H(x_0 - x)\} = \min_{\partial V}\{u + C_a^H(x_0 - x)\}$$

whenever $V \Subset \Omega$, $a \ge 0$ and $x_0 \in \mathbb{R}^n \setminus V$; for short, write $u \in CCB_H(U)$.

(iii) We say $u \in C^0(U)$ satisfies the comparison property with cones for H in U (for short, $u \in CC_H(U)$) if $u \in CCB_A(U) \cap CCB_H(U)$.

Denote by L the convex conjugate of H, that is,

$$L(q) = \sup_{p \in \mathbb{R}^2}[\langle p, q \rangle - H(p)].$$

Note that L is convex, $L(0) = \min_{p \in \mathbb{R}^n} L(q) = 0$ and $\liminf_{q \to \infty} L(q)/|q| = \infty$. Given any domain $U \subset \mathbb{R}^n$ and a bounded function $u \in C^0(U)$, the Hamilton-Jacobi flows are defined by

$$T^t u(x) = \sup_{y \in U}\left[u(y) - tL\left(\frac{y - x}{t}\right)\right] \text{ and } T_t u(x) = \inf_{y \in U}\left[u(y) + tL\left(\frac{x - y}{t}\right)\right] \quad \forall t > 0, x \in U$$

and $T^0 u(x) = u(x) = T_0 u(x)$ for all $x \in U$.

For any $r > 0$, we set $U_r := \{x \in U : \text{ dist } (x, \partial U) > r\}$.

Definition 2.4. Let H be as in (1.4). Let $U \subset \mathbb{R}^n$ be any domain.

(i) A bounded function $u \in C^0(U)$ enjoys a convex criteria if for any $r > 0$ there exists a $\delta_r > 0$ such that for all $x \in U_r$, the map $t \in [0, \delta_r) \to T^t u(x)$ is convex.

(ii) A bounded function $u \in C^0(U)$ enjoys a concave criteria if for any $r > 0$ there exists a $\delta_r > 0$ such that for all $x \in U_r$, the map $t \in [0, \delta_r) \to T_t u(x)$ is concave.

The following characterization of absolute minimizers follows from [9, Theorem 4.8].

LEM7.13 **Lemma 2.5.** *Let H be as in* (1.4) *and $H^{-1}(0)$ has empty interior point. For any domain $U \subset \mathbb{R}^2$, the following are equivalent:*

(i) *$u \in AM_H(U)$ is bounded;*

(ii) *$u \in CC_H(U)$ is bounded;*

(iii) *$u \in C^0(U)$ is bounded and enjoys the convex and concave criterion.*

Define the slope functions via Hamilton-Jacobi flows as below:

$$S_t^+ u(x) = \frac{1}{t}[T^t u(x) - u(x)] \text{ and } S_t^- u(x) = \frac{1}{t}[T_t u(x) - u(x)] \quad \forall x \in U, t > 0.$$

If $u \in AM_H(U)$ is bounded, then the maps $t \in (0, \delta_r] \to \pm S_t^\pm u(x)$ are increasing for all $x \in U_r$ and $r > 0$,

E-2 (2.1)
$$S u(x) := \lim_{r \to 0} \|H(Du)\|_{L^\infty(B(x,r))} = \lim_{t \to 0} \pm S_t^\pm u(x) \quad \text{for all } x \in U;$$

and $S u$ is upper semi-continuous in U; for details see [9, Lemmas 4.2 and 4.3].

Note that for $u \in C^{0,1}(\mathbb{R}^n)$ which is not necessary bounded, one may also define $T^t u, T_t u$ and hence $S_t^\pm u$ as above with $U = \mathbb{R}^n$. Since L satisfies (1.4), all of $T^t u, T_t u$ and hence $S_t^\pm u$ are finite. Indeed we have the following.

LEM3.3 **Lemma 2.6.** *Let H be as* (1.4)*. If $u \in C^{0,1}(\mathbb{R}^n)$ with $\|Du\|_{L^\infty(\mathbb{R}^n)} = K < \infty$, then there exists a constant $R_K > 0$ depending on k and L such that*

E3.1 (2.2)
$$\pm S_t^\pm u(x) = \sup_{y \in B(x, R_K t)} \left[\pm \frac{u(y) - u(x)}{t} - L\left(\pm \frac{y - x}{t}\right) \right] \quad \forall x \in \mathbb{R}^n, t > 0.$$

Proof. Note that $|u(y) - u(x)| \le Kt$ for all $x, y \in \mathbb{R}^n$. By the sup-linear growth of L, we there exist an increasing function $M : [0, \infty) \to [0, \infty)$ so that $M(R) \to \infty$ and $L(q) \ge M(R) R$ whenever $|q| \ge R$. If $M(R_K) > K$ and $|x - y| \ge R_K t$, we have

$$\pm \frac{u(y) - u(x)}{t} - L\left(\pm \frac{y - x}{t}\right) \le \left[K - M\left(\frac{|x - y|}{t} \right) \right] \frac{|x - y|}{t} \le 0,$$

which gives the desired identity (2.2). The proof of Lemma 2.6 is complete. □

We also recall the another slope function which are defined via cones as below:

$$\hat{S}_t^+ u(x) = \inf \left\{ k \ge 0, u(y) - u(x) \le \mathscr{C}_k^H(y - x) \quad \forall y \in B(0, t) \right\} \quad \forall x \in U, 0 < t < \text{ dist}(x, \partial U)$$

and

$$-\hat{S}_t^- u(x) = \inf \left\{ k \ge 0, u(x) - u(y) \le \mathscr{C}_k^H(y - x) \quad \forall y \in B(0, t) \right\} \quad \forall x \in U, 0 < t < \text{ dist}(x, \partial U).$$

Following the argument of [22, Proposiitons 3.1 and 3.3] line by line, we have the following Lemmas 2.7 which will be used later. Here we omit the details.

LEM **Lemma 2.7.** *Let H be as in* (1.4) *and satisfy Theorem 1.1 (i). Assume that $u \in CC_H(U)$ for $U \Subset \mathbb{R}^n$. Then for any $x \in U$, the functions $t \in (0, \text{ dist}(x, \partial U)) \to \pm \hat{S}_t^\pm u(x)$ are increasing, and $S u(x) = \lim_{t \to 0} \pm \hat{S}_t^\pm u(x)$*

eincreasing **Lemma 2.8.** *Let H be as in* (1.4) *and satisfy Theorem 1.1 (i). Assume that $u \in CC_H(U)$ for $U \Subset \mathbb{R}^n$. Then for any $x \in U$ and $0 < r <$ dist$(x, \partial U)$, if $u(y) - u(x) = \mathscr{C}^H_{\hat{S}^+_r u(x)}(y - x)$, then $S u(y) \geq \hat{S}^+_r u(x)$.*

3. Some geometric and analytic characterization of H and its convex conjugate L

Let H be as in (1.4) and L be its convex conjugate. Note that L also satisfies (1.4). For any $q \in \mathbb{R}^n$, denote by $\partial L(q)$ the sub-differential set of L at q, that is, $p \in \partial L(q)$ if

$$L(q') - L(q) \geq p \cdot (q' - q) \quad \forall q' \in \mathbb{R}^n.$$

Similarly, denote by $\partial H(p)$ the sub-differential set of L at $p \in \mathbb{R}^n$.

In Theorem 3.1, we obtain the following geometric characterization of ∂L when H is strictly convex or H is not a constant in any d-simplex with $d = 1, 2$. In general, given any $3 \leq d \leq n$, one may expect some analogue result as above when H is not a constant in any d-simplex, but we are not going that far in this paper. Recall that a 1-simplex is a line-segment, that is, the convex hull of 2 distinct points; for $2 \leq d \leq n$, a d-simplex is the convex hull of a $(d-1)$-simplex and a point, where the point is not contained in the $(d-1)$-dimensional hyperplane determined by the $(d-1)$-simplex. Theorem 3.1 and its analogue in d-simplex may have their own interests. In this paper, Theorem 3.1 (ii) plays a key role in the proof of Lemma 1.3.

LEMC3 **Theorem 3.1.** *Let H be as* (1.4).

- (i) *The following are equivalent:*
  - (i-a) *H is strictly convex*
  - (i-b) *for any $q \in \mathbb{R}^n$, $\partial L(q)$ contains a single point,*
  - (i-c) *$L \in C^1(\mathbb{R}^n)$.*
- (ii) *The following are equivalent:*
  - (ii-a) *H is not a constant in any line-segment*
  - (ii-b) *for any $q \in \mathbb{R}^n$, $\partial L(q)$ consists of either a single point, or a line-segment on which H is strictly monotone.*
- (iii) *The following are equivalent:*
  - (iii-a) *H is not a constant in any 2-simplex (convex hull of 3 points which are not in the same line)*
  - (iii-b) *for any $q \in \mathbb{R}^n$, $\partial L(q)$ must be one of the following:*
    - (iii-b-1) *a single point;*
    - (iii-b-2) *a bounded closed line-segment;*
    - (iii-b-3) *a bounded closed convex set contained in some 2-dimensional hyperplane whose boundary consists of 4 simple "curve" $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ oriented in order so that γ_0 (resp. γ_2) is either a single point or a line-segment on which H reaches its minimum (resp. maximum) in $\partial L(q)$, and along γ_1 (resp. γ_3), H is strictly increasing (resp. decreasing).*

In Theorems 3.2 and 3.3, we build up some analytic characterization of H when it is not a constant in any line-segment. Both of them will be used to prove Proposition 1.4; indeed, they play crucial roles there.

e-e6 **Theorem 3.2.** *Let H be as* (1.4). *Then the following are equivalent:*

- (i) *H is not a constant in any line-segment.*
- (ii) *For each $R \geq 1$ and each $\epsilon > 0$, there exist $\psi_R(\epsilon) \in (0, \epsilon)$ such that for any $v \in \overline{B(0,R)}$, if it satisfies*

e-e6angle
$$H(p+v) - H(p) \leq \psi_R(\epsilon) \text{ and } |\measuredangle(q, v) - \pi/2| \leq \psi_R(\epsilon) \tag{3.1}$$

for some $p \in \overline{B(0,R)}$, $q \in \partial H(p')$ and $p' \in \overline{B(p, \psi_R(\epsilon))}$, then $|v| \leq \epsilon$.

LEM4.5 **Theorem 3.3.** *Let H be as* (1.4). *Then the following are equivalent:*

- (i) *H is not a constant in any line-segment.*
- (ii) *For each $R \geq 1$ and each $\eta > 0$, we have*

$$\phi_R(\eta) = \inf_{e \in H^{-1}([0,R])} \min\left\{(p - e) \cdot \frac{q}{|q|} : H(p) = H(e), |p - e| \geq \eta, q \in \partial H(p)\right\} > 0$$

To prove above results, we recall the following basic properties; for reader's convenience, we give the details.

LEM3.1 **Lemma 3.4.** *Let H be as* (1.4).

(i) *For any $p, q \in \mathbb{R}^n$, we have*

$$q \in \partial H(p) \text{ if and only if } H(p) + L(q) = \langle p, q\rangle \text{ if and only if } p \in \partial L(q).$$

In particular, $0 \in \partial H(p)$ *if and only if* $H(p) = 0$ *and* $0 \in \partial L(q)$ *if and only if* $L(q) = 0$.

(ii) *If $p_1, p_2 \in \partial L(q)$ for some $q \in \mathbb{R}^n$, then*

$$\lambda p_1 + (1-\lambda) p_2 \in \partial L(q) \text{ and } H(\lambda p_1 + (1-\lambda) p_2) = \lambda H(p_1) + (1-\lambda) H(p_2) \text{ for all } \lambda \in [0,1].$$

(iii) *Assume that*

$$L(\lambda q_1 + (1-\lambda) q_2) = \lambda L(q_1) + (1-\lambda) L(q_2) \text{ for some } \lambda \in (0,1).$$

If $p \in \partial L(\lambda q_1 + (1-\lambda) q_2)$, then $p \in \partial L(q_1) \cap \partial L(q_2)$.

(iv) *The set $\partial L(q)$ is bounded locally uniformly in $q \in \mathbb{R}^n$. If $p_i \in \partial L(q_i)$ for all $i \in \mathbb{N}$ and $q_i \to q_0$ as $i \to \infty$, then up to some subsequence, $p_i \to p_0$ as $i \to \infty$ for some $p_0 \in \partial L(q_0)$. In particular, $\partial L(q)$ is always closed for any $q \in \mathbb{R}^n$.*

Proof. (i) Note that $q \in \partial H(p)$ if and only if

$$\langle p, q\rangle - H(p) \geq \langle q, p'\rangle - H(p') \, \forall p' \in \mathbb{R}^n,$$

Thus, $H(p) + L(q) = \langle p, q\rangle$ if and only if

$$\langle p, q\rangle - H(p) \geq \langle q, p'\rangle - H(p') \, \forall p' \in \mathbb{R}^n.$$

Thus, $q \in \partial H(p)$ if and only if $H(p) + L(q) = \langle p, q\rangle$. Similarly, $p \in \partial L(q)$ if and only if $H(p) + L(q) = \langle p, q\rangle$.

(ii) If $p_1, p_2 \in \partial L(q)$ for some $q \in \mathbb{R}^n$, by (i) one has

$$\begin{aligned} L(q) + H(\lambda p_1 + (1-\lambda) p_2) &\geq (\lambda p_1 + (1-\lambda) p_2) \cdot q \\ &= \lambda p_1 \cdot q + (1-\lambda) p_2 \cdot q \\ &= \lambda H(p_1) + (1-\lambda) H(p_2) + L(q) \qquad \forall \lambda \in [0,1], \end{aligned}$$

that is,

$$H(\lambda p_1 + (1-\lambda) p_2) \geq \lambda H(p_1) + (1-\lambda) H(p_2) \, \forall \lambda \in [0,1]$$

By the convexity of H,

$$H(\lambda p_1 + (1-\lambda) p_2) = \lambda H(p_1) + (1-\lambda) H(p_2) \, \forall \lambda \in [0,1],$$

and hence $(\lambda p_1 + (1-\lambda) p_2) \in \partial L(q)$.

(iii) If $L(\lambda q_1 + (1-\lambda) q_2) = \lambda L(q_1) + (1-\lambda) L(q_2)$ and $p \in \partial L(\lambda q_1 + (1-\lambda) q_2)$ for some $\lambda \in (0,1)$ and by (i) one gets

$$L(\lambda q_1 + (1-\lambda) q_2) = (\lambda q_1 + (1-\lambda) q_2) \cdot p - H(p) = \lambda [q_1 \cdot p - H(p)] + (1-\lambda) [q_2 \cdot p - H(p)].$$

Since $q_i \cdot p - H(p) \leq L(q_i)$ for $i = 1, 2$, by above and the assumption $L(\lambda q_1 + (1-\lambda) q_2) = \lambda L(q_1) + (1-\lambda) L(q_2)$ we have $q_i \cdot p - H(p) = L(q_i)$ for $i = 1, 2$.

(iv) Let $R \geq 1$. For any $|q| \leq R$ and $p \in \partial L(q)$, we have

$$H(p) = p \cdot q - L(q) \leq |q||p| - L(q) \leq C(R) + R|p|.$$

By the sup-linear growth of H, we know that $|p| \leq C(R)$ as desired.

Moreover, by $p_i \in \partial L(q_i)$ for $i \in \mathbb{N}$ and $q_i \to q_0$ as $i \to \infty$, one has that $p_i \cdot q_i = H(p_i) + L(q_i)$ for $i \in \mathbb{N}$, and p_i is bounded. Thus p_i converges to some p_0 as $i \to \infty$ (up to some subsequence). By the continuity of H and L we then have $p_0 \cdot q_0 = H(p_0) + L(q_0)$, by (i) which implies $p_0 \in \partial L(q_0)$. The proof of Lemma 3.4 is complete. □

As a consequence of Lemma 3.4, one has the following.

propH **Corollary 3.5.** *Suppose that H satisfies* (1.4) *and Theorem 1.1(i). Then* $0 \in \partial H(0)$*,* $H(p) > 0$ *and* $0 \notin \partial H(p)$ *for all* $p \in \mathbb{R}^n \setminus \{0\}$.

Proof. First we see that $H(p) = 0$ if and only if $p = 0$. Indeed, if $H(p) = 0$ for some $p \neq 0$, by convexity of H and $H(0) = 0$, we know that $H = 0$ in $[0, p]$, which is a contradiction. Thus $H(p) > 0$ and $0 \notin \partial H(p)$ whenever $p \neq 0$. Moreover, by Lemma 3.4 $0 \in \partial H(p)$ if and only if $H(p) = 0$, and hence if and only if $p = 0$. □

Lemma 3.6 gives some geoemtric/analytic property when H is a constant in some line-segment.

LEMconst **Lemma 3.6.** *Suppose that H satisfies* (1.4)*. For any $a, b \in \mathbb{R}^n$ with $a \neq b$, the following are equivalent:*

(i) *H is a constant in the line-segment* $[a, b]$*;*
(ii) $b - a \perp \partial H(\frac{1}{2}a + \frac{1}{2}b)$ *and*

Hcon (3.2)
$$\partial H(\frac{1}{2}a + \frac{1}{2}b) = \partial H(\lambda a + (1-\lambda)b) \subset \partial H(a) \cap \partial H(b) \quad \forall \lambda \in (0,1).$$

(iii) *there exists a $q \in \mathbb{R}^n$ such that $b - a \perp q$ and $a, b \in \partial L(q)$ (or $[a, b] \subset \partial L(q)$).*
(iv) *there exists a $q \in \mathbb{R}^n$ such that $H(a) = H(b)$ and $a, b \in \partial L(q)$ (or $[a, b] \subset \partial L(q)$).*

Proof. (*i*) ⇒ (*ii*) Assume that H is a constant in the line-segment $[a, b]$. For any $\lambda \in (0, 1)$ and $q_\lambda \in \partial H(\lambda a + (1-\lambda)b)$, we have

$$0 = H(a) - H(\lambda a + (1-\lambda)b) \geq (1-\lambda)q_\lambda \cdot (a - b) \text{ and } 0 = H(b) - H(\lambda a + (1-\lambda)b) \geq \lambda q_\lambda \cdot (b - a),$$

which implies that $q_\lambda \perp (b - a)$. Thus for any $\mu \in [0, 1]$, we have

$$\begin{aligned} H(p) - H(\mu a + (1-\mu)b) &= H(p) - H(\lambda a + (1-\lambda)b) \\ &\geq q_\lambda \cdot [p - (\lambda a + (1-\lambda)b)] \\ &= q_\lambda \cdot [p - (\mu a + (1-\mu)b)] + (\mu - \lambda)q_\lambda \cdot (a - b) \\ &= q_\lambda \cdot [p - (\mu a + (1-\mu)b)] \quad \forall p \in \mathbb{R}^n, \end{aligned}$$

that is, $q_\lambda \in \partial H(\mu a + (1-\mu)b)$. In particular, this gives (3.2).

(*ii*) ⇒ (*iii*) Let $q \in \partial H(\frac{a+b}{2})$. By (ii) and Lemma 3.4 (i), $[a, b] \in \partial L(q)$ and $a - b \perp q$.

(*iii*) ⇒ (*iv*) Let q be as in (iii). By (iii), $a, b \in \partial L(q)$ and hence $H(b) - H(a) \geq q \cdot (b - a) = 0$ and $H(a) - H(b) \geq q \cdot (a - b) = 0$, which gives (iv). Note that by Lemma 3.4 (ii), $a, b \in \partial L(q)$ implies $[a, b] \subset \partial L(q)$.

(*iv*) ⇒ (*i*) Let q be as in (iv). By (iv) and Lemma 3.4 (ii), $[a, b] \subset \partial L(q)$. By Lemma 3.4 (i), for any $\lambda \in [0, 1]$,

$$H(\lambda a + (1-\lambda)b) - H(a) = q \cdot [(\lambda a + (1-\lambda)b) - a] = \lambda q \cdot a + (1-\lambda)q \cdot b = \lambda H(a) + (1-\lambda)H(b) = H(a),$$

which implies that H is a constant in $[a, b]$. □

Now we are able to prove Theorem 3.1, Theorem 3.2 and Theorem 3.3 as below.

Proof of Theorem 3.1. Proof of (i). (i-a)⇒(i-b). Assume that H is strictly convex. If for some $q \in \mathbb{R}^n$, $\partial L(q)$ contains two distinct points p_1, p_2, then Lemma 3.4 (ii) implies that H is linear in $[p_1, p_2]$, which contradicts with the strictly convexity of H. Thus for any $q \in \mathbb{R}^n$, $\partial L(q)$ must consists of a single point.

(i-b)⇒(i-c). Next, assume that for any $q \in \mathbb{R}^n$, $\partial L(q)$ must consists of a single point. We show that $L \in C^1(\mathbb{R}^n)$. By Lemma 3.4 (iv), it suffices to show that L is differentiable everywhere. We prove by contradiction. Assume L is not differentiable at $q_0 \in \mathbb{R}^n$. Write $\partial L(q_0) = \{p_0\}$. Then there exists $\epsilon_0 > 0$ and a sequence $\{q_i\}$ which converges to q_0 such that $|L(q_i) - L(q_0) - p_0 \cdot (q_i - q_0)| \geq \epsilon_0 |q - q_0|$. Write $\partial L(q_i) = \{p_i\}$. Then

$$p_0 \cdot (q_i - q_0) \leq L(q_i) - L(q_0) \leq p_i \cdot (q_i - q_0) = p_0 \cdot (q_i - q_0) + (p_i - p_0) \cdot (q_i - q_0)$$

and hence

$$|L(q_i) - L(q_0) - p_0 \cdot (q_i - q_0)| \leq |p_i - p_0||q_i - q_0|.$$

But by Lemma 3.4 we have $p_i \to p_0$ as $i \to \infty$, which is a contradiction.

(i-c)⇒(i-a). assume that $L \in C^1(\mathbb{R}^n)$ and hence $\partial L(q) = \{DL(q)\}$ for any $q \in \mathbb{R}^n$. We show that H is strictly convex. Otherwise,

$$H(\lambda p_1 + (1-\lambda)p_2) = \lambda H(p_1) + (1-\lambda)H(p_2) \text{ for some } \lambda \in (0,1) \text{ and } p_1 \neq p_2.$$

Let $q \in \partial H(\lambda p_1 + (1-\lambda)p_2)$,. By Lemma 3.4 (iii), $q \in \partial H(p_1) \cap \partial H(p_2)$, and hence by Lemma 3.4, $p_1, p_2 \in \partial L(q) = \{DL(q)\}$, which is a contradiction.

Proof of (ii). (ii-b)⇒(ii-a). If H is a constant in a line-segment, say $[a,b]$, by Lemma 3.6 (iv), there exists $q \in \mathbb{R}^n$ such that $[a,b] \subset \partial L(q)$; but note that H is not strictly monotone in $[a,b]$.

(ii-a)⇒(ii-b). assume that H is not a constant in any line-segment. For any $q \in \mathbb{R}^n$, assume that $\partial L(q)$ contains more than one point, say p_1, p_2 with $p_1 \neq p_2$. We only need to show that $\partial L(q)$ is contained lie in the line determined by p_1 and p_2. Indeed, Lemma 3.4 $\partial L(q)$ is a bounded convex set; but bounded convex set contained in some line must be a line-segment.

Let $p_0 \in \partial L(q)$ with $p_0 \neq p_1, p_2$. Then $H(p_i) \neq H(p_j)$ for $i, j = 0, 1, 2$ and $i \neq j$. Indeed, if $H(p_i) = H(p_j)$ for some $i, j = 0, 1, 2$ and $i \neq j$, by $(iv) \Rightarrow (i)$ in Lemma 3.6, H is a constant in $[p_i, p_j]$. This is a contradiction.

Reorder p_0, p_1, p_2 as $\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2$ so that $H(\widetilde{p}_0) < H(\widetilde{p}_1) < H(\widetilde{p}_2)$. Then there exists $\widetilde{\lambda} \in (0,1)$ such that

$$H\left(\widetilde{\lambda}\widetilde{p}_0 + \left(1 - \widetilde{\lambda}\right)\widetilde{p}_2\right) = H(\widetilde{p}_1).$$

Since H is not a constant in any line-segment, by $(iv) \Rightarrow (i)$ in Lemma 3.6 again, we know that

$$\widetilde{p}_1 = \widetilde{\lambda}\widetilde{p}_0 + (1-\widetilde{\lambda})\widetilde{p}_2,$$

which implies that $\widetilde{p}_1, \widetilde{p}_2$ and $\widetilde{p}_3$ lies in the same line, that is, p_0 must lie in the line determined by p_1 and p_2 as desired.

Proof of (iii). (iii-b)⇒(iii-a). If H is a constant in a 2-simplex Δ, which is the convex hull of p_1, p_2, p_3, let $q \in \partial H(\frac{1}{3}[p_1 + p_2 + p_3])$. For any line segment $I \subset \Delta$ with $\frac{1}{3}[p_1 + p_2 + p_3] \in I$, since H is a constant in I, by Lemma 3.6 (ii), $I \subset \partial L(q)$. This implies that $\Delta \subset \partial L(q)$. Obviously, $\partial L(q)$ is not given by a) or b).

Now we show that $\partial L(q)$ is not given c). If $\partial L(q)$ is given by c), and then $\partial L(q)$ is bounded by $\{\gamma_i\}_{i=0}^3$ in order. Since H is strictly increasing in γ_1 and strictly decreasing in γ_3, for any $m := \min_{p \in \partial L(q)} H(p) < k < \max_{p \in \partial L(q)} H(p)) =: M$, we can only find one point $a_k \in \gamma_1$ and one point $b_k \in \gamma_1$ such that $H(a_k) = H(b_k) = k$. We also write $[a_m, b_m] = \gamma_0$ and $[a_M, b_M] = \gamma_2$. Thus H is constant k in $[a_k, b_k]$. Since the unions of $[a_k, b_k]$ is exactly $\partial L(q)$, we know that $H^{-1}(k) \cap \partial L(q) = [a_k, b_k]$. This is a contradiction.

(iii-a)⇒(iii-b). Conversely, assume that H is not a constant in any 2-simplex. First we see that $\partial L(q)$ must be contained in some 2-dimensional hyperplane P; otherwise, we can find 4 distinct points p_0, p_1, p_2, p_3 which are not contained in any 2-dimensional hyperplane. After reordering, we may assume that $H(p_0) \leq H(p_1) \leq H(p_2) \leq H(p_3)$. Since H is not constant in any 2-simplex, it can only happen that $H(p_0) \leq H(p_1) < H(p_2) \leq H(p_3)$, and $H(p_0) < H(p_1) \leq H(p_2) < H(p_3)$. In the first we can find 3 points $p_0' \in [p_0, p_2]$, $p_1' \in [p_0, p_3]$ and $p_2' \in [p_1, p_2]$ such that

$$H(p_0') = H(p_1') = H(p_2') = H\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right),$$

Note that p_0', p_1', p_2' are not contained in the same line, and hence its convex hull is a 2-simplex denoted by Δ'. Moreover, for any $\lambda_i' > 0$ and $\sum_{i=0}^2 \lambda_i = 1$ we have $\sum_{i=0}^2 \lambda_i p_i' \in \partial L(q)$ and hence

$$H(\sum_{i=0}^{2} \lambda_i p_i') = (\sum_{i=0}^{2} \lambda_i p_i') \cdot q - L(q) = \sum_{i=0}^{2} \lambda_i [p_i' \cdot q - L(q)] = \sum_{i=0}^{2} \lambda_i H(p_i') = H\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right)$$

That is, H is a constant in Δ', which is a contradiction. In the second case, we can find 3 points $p_0' \in [p_0, p_1]$, $p_1' \in [p_0, p_2]$ and $p_2' \in [p_0, p_3]$ such that

$$H(p_0') = H(p_1') = H(p_2') = H\left(\frac{1}{2}p_0 + \frac{1}{2}p_1\right),$$

Similarly, H is a constant in the convex hull of $p_0'p_1'p_2'$; this is a contradiction.

Next, assume that $\partial L(q)$ is not a single point or a line-segment. Then $\partial L(q)$ is a closed bounded domain in P. Thus $\partial L(q)$ is bounded by a simple closed curve γ. Note that H reaches its minimum and maximum in ∂L only at boundary. Denote by γ_0 (resp. γ_2) the part on which H reaches its minimum (resp. maximum) in ∂L. Since H is not a constant in any 2-simplex, γ_0 and γ_2 must be a single point or a line-segment. The other two components of $\gamma \setminus (\gamma_0 \cup \gamma_2)$ are denoted by γ_1 and γ_3. Up to some reorientation we may assume that the ending point of γ_i is the starting point of γ_{i+1} for $i = 0, 1, 2, 3$ (where $\gamma_4 = \gamma_0$). Now we show that H is strict increasing along γ_1; otherwise, there exists two distinct points $p_0, p_1 \in \gamma_1$ so that $H(p_0) = H(p_1)$. Observe that there exists $p_2 \in \gamma_3$ with $H(p_2) = H(p_1)$. Note that H is a constant in the convex hull of $\{p_0, p_1, p_2\}$, which is a 2-simplex, and hence it is a contradiction. Similarly, we know that H is strict decreasing along γ_3. The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.2. $(ii) \Rightarrow (i)$. Assume that (i) fails, that is H is a constant in some line-segment, say $[a, b]$. Let $p = \frac{1}{2}(a + b)$, $q \in \partial H(p)$ and $v = \frac{b-a}{2}$. Then $H(p) = H(p + v)$ and, by Lemma 3.6, $q \perp v$. Thus (ii) fails.

$(i) \Rightarrow (ii)$. Suppose that (ii) is not correct. There exists $\epsilon_0 > 0$ so that for any $i \in \mathbb{N}$ we can find vectors $v_i \in \overline{B(0, 1/\epsilon_0)}$ with $|v_i| \geq \epsilon_0$ satisfying $H(p_i + v_i) - H(p_i) \leq 1/i$ and $|\measuredangle(q_i, v_i) - \pi/2| \leq 1/i$ for some $p_i \in \overline{B(0, 1/\epsilon_0)}$ $q_i \in \partial H\left(p_i'\right)$ and $p_i' \in \overline{B(p_i, 1/i)}$. Note that $p_i, p_i' \to p_0$, $v_i \to v_0$ and $q_i \to q_0 \in \partial H(p_0)$ as $i \to \infty$ (up to some subsequence). Moreover,

$$H(p_0 + v_0) - H(p_0) \leq 0, \quad \angle(q_0, v_0) = \pi/2$$

By the convexity of H and $H(p_0 + v_0) - H(p_0) \leq 0$, we have

$$H(p_0 + tv_0) \leq (1 - t)H(p_0) + tH(p_0 + v_0) \leq H(p_0) \quad \forall t \in [0, 1].$$

Moreover, by $q_0 \in \partial H(p_0)$ and $\angle(q_0, v_0) = \pi/2$, one also has

$$H(p_0 + tv_0) - H(p_0) \geq q_0 \cdot tv_0 = 0, \quad \forall t \in [0, 1].$$

Thus $H(p_0 + tv_0) = H(p_0)$ for all $t \in [0, 1]$, which leads to a contradiction. The proof of Theorem 3.2 is complete. □

Proof of Theorem 3.3. $(ii) \Rightarrow (i)$. Assume that (i) fails, that is H is a constant in some line-segment, say $[a, b]$. Let $p = a$, $e = b$ and $q \in \partial H(\frac{1}{2}[a + b])$. Then $H(p) = H(p + v)$ and, by Lemma 3.6, $q \perp (b - a)$. Thus $\phi_R(|b - a|) = 0$, that is, (ii) fails.

$(i) \Rightarrow (ii)$. Suppose that (ii) is not correct. There exists a $\eta_0 > 0$ such that $\delta_R(\eta_0) = 0$, that is, we can find p_i and e_i with $|p_i - e_i| \geq \eta_0$ and $H(e_i) = H(p_i) \leq R$, and $q_i \in \partial H(p_i)$ such that

$$(p_i - e_i) \cdot \frac{q_i}{|q|} \leq 1/i.$$

Letting $p_i \to p$, $e_i \to e$ and $q_i \to q \in \partial H(p)$ when $i \to \infty$, we get $H(p) = H(e) \leq R$, $|p - e| \geq \eta_0$ and

$$(p - e) \cdot \frac{q}{|q|} \leq 0.$$

Since $q \in \partial H(p)$ implies

$$(e - p) \cdot q \leq H(e) - H(p) = 0,$$

we obtain

$$(e - p) \cdot q = 0.$$

Therefore, by $q \in \partial H(p)$ again,

$$H(e) + L(q) = H(p) + L(q) = p \cdot q = e \cdot q,$$

which together with Lemma 3.4(i) yields that $e, p \in \partial L(q)$ and hence $[p, e] \in \partial L(q)$. By $H(p) = H(e)$ and Lemma 3.4 (ii), we know that H is constant in $[p, e]$, which is a contradiction. □

Finally, we also use the following Lemma 3.7 and its corollary 3.8.

la-mu **Lemma 3.7.** *Let H be as* (1.4) *and satisfy Theorem 1.1(i).*

(i) *If $q \in \partial H(p)$, then $C_{H(p)}(q) = \langle p, q\rangle$.*
(ii) *If $C_k(z) = \langle p_0, z\rangle$ with $k > 0$ and $|z| > 0$, then $tz \in \partial H(p_0)$ for some $t > 0$.*

Proof. (i) If $q \in \partial H(p)$, then

$$\langle p, q\rangle = H(p) + L(q) \geq H(p) + \sup_{p' \in \mathbb{R}^n} [\langle p', q\rangle - H(p')] \geq \sup_{H(p') \leq H(p)} \langle p', q\rangle = C_{H(p)}(q),$$

which together with $\langle p, q\rangle \leq C_{H(p)}(q)$ gives $C_{H(p)}(q) = \langle p, q\rangle$.

(ii) By Theorem 3.1, $\partial L(tz)$ is either a single point or a line-segment for each $t \geq 0$. We may write $I_t = [a_t, b_t] = H(\partial L(tz))$ for each $t \geq 0$. It suffices to show that $\cup_{t\geq 0} I_t = [0, \infty)$. Indeed, if this correct, by Lemma 3.5 and $H(p_0) \neq 0$ we know that $H(p_0) \in I_{t_0}$ for some $t_0 > 0$. Thus $H(p_0) = H(p_{t_0 z})$ with $p_{t_0 z} \in \partial L(t_0 z)$, then by (i),

$$C_{H(p_{t_0 z})}(t_0 z) = p_{t_0 z} \cdot t_0 z = H(p_{t_0 z}) + L(t_0 z) = H(p_0) + L(t_0 z).$$

Since

$$p_0 \cdot t_0 z = C_{H(p_0)}(t_0 z) = \frac{1}{t} C_{H(p_{t_0 z})}(t_0 z),$$

we obtain $p_0 \cdot t_0 z = H(p_0) + L(t_0 z)$, which together with Lemma 3.4 gives $tz \in \partial H(p_0)$.

For any $k > 0$, note that there exist $t_1 < t_2$ such that $b_s < k < a_t$ for all $t \geq t_2$ and $s < t_1$. Indeed, for any $p_t \in \partial L(tz)$ with $t > 0$, by Lemma 3.4 (i) we have $H(p_t) + L(tz) = p_t \cdot tz$ and hence $|p_t| \geq \frac{1}{t} L(tz) \geq M(tz)|z| \to \infty$ as $t \to \infty$. Moreover, by Lemma 3.4 (iv), $C = \sup_{t\leq 1} \sup_{p \in \partial H(tz)} |p| < \infty$ and hence $H(p_t) \leq Ct|z| \to 0$ as $t \to 0$. Let t_z be the supremum of $t > 0$ so that $b_t < k$. Then $a_{t_z} \leq k \leq b_{t_z}$. Indeed, for any $\epsilon > 0$, we have $b_{t_z+\epsilon} \geq k$. Let $p_{t_z+\epsilon} \in \partial L((t_z + \epsilon)z)$ such that $H(p_{t_z+\epsilon}) = b_{t_z+\epsilon}$. Since Lemma (3.4) (iv) implies $p_{t_z+\epsilon} \to p \in \partial L(t_z z)$ as $\epsilon \to 0$ (up to some subsequence), we have $H(p) \geq k$ and hence $b_{t_z} \geq k$. Moreover, by $a_{t_z-\epsilon} \leq b_{t_z-\epsilon}$ and a similar argument, we $a_{t_z} \leq k$. Thus $k \in I_{t_z}$ as desired. The proof of Lemma 3.7 is complete. □

xew4 **Corollary 3.8.** *Let H be as (1.4) and satisfy Theorem 1.1(i). For any $R \geq 1$, there exists a constant C_R such that for any $\delta \in (0, 1)$ and $0 \leq k \leq R$, one has*

xew5 (3.3)
$$e \cdot x + \delta|x| \leq \mathscr{C}^H_k(x) + \delta|x| \leq \mathscr{C}^H_{k+C_R\delta}(x), \quad \forall x \in \mathbb{R}^n, e \in H^{-1}(k)$$

Proof. If $x = 0$, (3.3) holds obviously. By Lemma 3.7, $x = q_x/|q_x|$ for some $q_x \in \partial H(p_x)$ with $H(p_x) = k$. Thus

$$e \cdot x + \delta \leq C_k(x) + \delta = p_x \cdot x + \delta = (p_x + \delta \frac{x}{|x|}) \cdot x.$$

By the convexity of H we know that

$$H(p_x + \delta \frac{x}{|x|}) \leq H(p_x) + q \cdot \delta \frac{x}{|x|} \quad \forall q \in \partial H(p_x + \delta \frac{x}{|x|}).$$

Letting

$$C_R := \sup\{|q| : q \in \partial H(p + v), H(p) \leq R, |v| \leq 1\},$$

we get (3.3) as desired. □

4. Proofs of Lemma 1.3 and (*i*) ⇔ (*ii*) in Theorem 1.1

By using Lemmas 3.4 and 3.1, we first establish the following weaker version of Lemma 1.3.

LEMC5 **Lemma 4.1.** *Let H be as in (1.4) with $n = 2$ and satisfy (i). If $u \in C^{0,1}(\mathbb{R}^n)$ satisfies (1.5) for some $0 < k < \infty$ and also*

(4.1)
$$u(e) = k + L(e) \text{ and } u(se) = su(e) \quad \forall s \in \mathbb{R}$$

for some vector $e \in \mathbb{R}^n$, then there exists a vector $p_0 \in \partial L(e)$ such that

$$u(x) = p_0 \cdot x \quad \forall x \in \mathbb{R}^n, \text{ and } H(p_0) \equiv k.$$

Proof. Note that by (4.1), $u(0) = 0$. Since $S_t^+ u((t-s)e) \le k$ for all $t > 0$ and $s \in \mathbb{R}$, we have

$$\frac{u(x+se) - u((s-t)e)}{t} - L\left(\frac{x + se - (s-t)e}{t}\right) \le k \quad \forall x \in \mathbb{R}^n.$$

Thus, by (4.1) we obtain

$$-\frac{u(x+se)}{t} + \frac{s}{t}u(e) \ge u(e) - k - L\left(e + \frac{x}{t}\right) = L(e) - L\left(e + \frac{x}{t}\right) \quad \forall x \in \mathbb{R}^n, t > 0, s \in \mathbb{R}.$$

By the convexity of L,

$$u(x+se) - su(e) \le p_{t,x} \cdot x \quad \forall x \in \mathbb{R}^n, t > 0, s \in \mathbb{R}.$$

where $p_{t,x} \in \partial L(e + x/t)$. By Lemma 3.4 (iv), $p_{t,x}$ converges to some $p_x \in \partial L(e)$ as $t \to \infty$ (up to some subsequence). Therefore,

F3 (4.2) $$u(x+se) - su(e) \le p_x \cdot x \quad \forall x \in \mathbb{R}^n, s \in \mathbb{R}.$$

Similarly, applying $-S_t^- u(-(t-s)e) \le k$ for $t > 0$ and $s \in \mathbb{R}$ we also have

F4 (4.3) $$u(x+se) - su(e) \ge \hat{p}_x \cdot x \quad \forall x \in \mathbb{R}^n, s \in \mathbb{R}$$

for some $\hat{p}_x \in \partial L(e)$.

If $\partial L(e)$ contains only one point say p_0, then $p_x = p_0 = \hat{p}_x$ for all $x \in \mathbb{R}^n$. Thus by (4.2) and (4.3), one has $u(x) = p_0 \cdot x$ as desired.

Below, assume that $\partial L(e)$ contains more than one point. By Theorem 3.1, $\partial L(e)$ must be a line-segment contained in some line ℓ. Therefore, $p_{te}, \hat{p}_{te} \in \ell$ for all $t \in \mathbb{R}$. Applying (4.2) and (4.3) with $x = te$ and $s = 0$ we have

$$\hat{p}_{te} \cdot te \le u(te) \le p_{te} \cdot te \quad \forall t \in \mathbb{R}.$$

By (4.1), one has

$$\hat{p}_{te} \cdot e = u(e) = p_{te} \cdot e \quad \forall t \in \mathbb{R}.$$

This together with (4.1) and Lemma 3.4 (i) further gives

$$k = L(e) - u(e) = L(e) - p_{te} \cdot e = H(p_{te}) \quad \forall t \in \mathbb{R},$$

and similarly, $k = H(\hat{p}_{te})$ for all $t \in \mathbb{R}$. Since H is not a constant in any line-segment, we conclude that $p_{te}, \hat{p}_{se}$ must coincide for all $t, s \in \mathbb{R}$, and is denoted by p_0. Obviously, $H(p_0) = k > 0$ and $H(p_0) + L(e) = p_0 \cdot e$. By Lemma 3.4 again, $p_0 \in \partial L(e)$.

To see $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$, we note that $e \cdot (p_0 - p) \ne 0$ for any $p \in \partial L(e) \setminus p_0$ and hence for any $p \in \ell$, that is,

F5 (4.4) $$\mathbb{R}^n = \bigcup_{z \perp \ell - p_0} (z + \mathbb{R}e).$$

Otherwise, by Theorem 3.1, one has

$$H(p_0) = p_0 \cdot e - L(e) = p_1 \cdot e - L(e) = H(p_1)$$

for some $p_1 \in \partial L(e) \setminus p_0$. For any $\lambda \in (0,1)$, by Lemma 3.4 (ii) we have

$$H(\lambda p_0 + (1-\lambda)p_1) = (\lambda p_0 + (1-\lambda)p_1) \cdot e - L(e) = \lambda[p_0 \cdot e - L(e)] + (1-\lambda)[p_1 \cdot e - L(e)] = H(p_0),$$

that is, H is a constant in the line-segment $[p_0, p_1]$, which is a contradiction with the assumption (i).

For any $z \perp \ell - p_0$, since $p_z, \hat{p}_z \in \partial L(e) \subset \ell$ we have $\hat{p}_z \cdot z = p_0 \cdot z = p_z \cdot z$. By (4.2) and (4.3) one further gets

$$p_0 \cdot z = \hat{p}_z \cdot z \le u(z+se) - u(se) \le p_z \cdot z = p_0 \cdot z \quad \forall z \perp \ell - p_0, s \in \mathbb{R}$$

which implies

$$u(z+se) = u(se) + p_0 \cdot z = sp_0 \cdot e + p_0 \cdot z = p_0 \cdot (z+se) \quad \forall z \perp \ell - p_0, s \in \mathbb{R}.$$

By (4.4), one has $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$ as desired.

□

By using Lemmas 3.4, 3.1 and 4.1, we are ready to prove Lemma 1.3.

Proof of Lemma 1.3. Assume that $u(0)=0$ without loss of generality. If $k=0$, then by $\pm S_t^{\pm}u(0)=k$ for all $t>0$ we have

$$\pm\frac{u(x)}{t}-L\left(\frac{x}{t}\right)\le 0\quad\forall x\in\mathbb{R}^n.$$

By the sup-linear growth of L one gets

$$\pm u(x)\le\liminf_{t\to\infty}tL\left(\frac{x}{t}\right)=0\quad\forall x\in\mathbb{R}^n,$$

that is, $u\equiv 0$ in $\mathbb{R}^n$.

Assume that $k>0$ below. First we have the following claim.

Claim I. There exists $y^{\pm}\in\mathbb{R}^n$ such that

E6
$$u(y^+)-L(y^+)=k\quad\text{and}\quad u(sy^+)=su(y^+)>0\quad\forall s>0,\tag{4.5}$$

E7
$$-u(y^-)-L(-y^-)=k\quad\text{and}\quad u(sy^-)=su(y^-)<0\quad\forall s>0,\tag{4.6}$$

E8
$$L(\lambda y^+-(1-\lambda)y^-)=\lambda L(y^+)+(1-\lambda)L(-y^-)\quad\forall\lambda\in(0,1)\tag{4.7}$$

and

E10
$$u(\lambda ty^++(1-\lambda)sy^-)=\lambda tu(y^+)+(1-\lambda)su(y^-)\quad\forall\lambda\in(0,1),\ t,s>0.\tag{4.8}$$

To find $y^{\pm}$ in Claim I, since $S_t^+u(0)=k=-S_t^-u(0)$ for all $t>0$, by Lemma 2.6 there exist $y_t^{\pm}\in\overline{B(0,Rt)}$ such that

E1
$$\frac{u(y_t^+)}{t}-L\left(\frac{y_t^+}{t}\right)=k=-\frac{u(y_t^-)}{t}-L\left(\frac{-y_t^-}{t}\right)\quad\forall t>0.\tag{4.9}$$

This gives

E2
$$\frac{u(y_t^+)}{2t}-\frac{u(y_t^-)}{2t}=k+\frac{1}{2}L\left(\frac{y_t^+}{t}\right)+\frac{1}{2}L\left(\frac{-y_t^-}{t}\right)\quad\forall t>0.\tag{4.10}$$

Since $y_t^{\pm}/t$ are bounded, as $t\to\infty$ (up to some subsequence) $y_t^{\pm}/t$ must converge to some points $y^{\pm}\in\mathbb{R}^n$ as desired.

To see (4.7) in Claim I, by $S_{2t}^+u(y_t^-)\le k$ for all $t>0$, one has

$$\frac{u(y_t^+)-u(y_t^-)}{2t}-L\left(\frac{y_t^+-y_t^-}{2t}\right)\le k\quad\forall t>0.$$

Thus, by (4.10) one gets

$$\frac{1}{2}L\left(\frac{y_t^+}{t}\right)+\frac{1}{2}L\left(\frac{-y_t^-}{t}\right)\le L\left(\frac{y_t^+-y_t^-}{2t}\right)\quad\forall t>0,$$

which together with the convexity of L yields that

$$\frac{1}{2}L\left(\frac{y_t^+}{t}\right)+\frac{1}{2}L\left(\frac{-y_t^-}{t}\right)=L\left(\frac{y_t^+-y_t^-}{2t}\right)\quad\forall t>0.$$

By the convexity of L again, we know that L must be linear in $[-y_t^-/t,y_t^+/t]$, that is,

E3
$$L\left(\lambda\frac{y_t^+}{t}-(1-\lambda)\frac{y_t^-}{t}\right)=\lambda L\left(\frac{y_t^+}{t}\right)+(1-\lambda)L\left(\frac{-y_t^-}{t}\right)\quad\forall t>0,\lambda\in(0,1).\tag{4.11}$$

Letting $t\to\infty$ and by $y_t^{\pm}/t\to y^{\pm}$ one see that (4.7) follows from (4.11).

To see (4.5) and (4.6), by $S^+_{(\theta_2-\theta_1)t}u(\theta_1y_t^+)\le k$ for $0\le\theta_1<\theta_2\le 1$ and $t>0$, we have

$$\frac{u(\theta_2y_t^+)-u(\theta_1y_t^+)}{(\theta_2-\theta_1)t}-L\left(\frac{y_t^+}{t}\right)\le k$$

that is,

$$u(\theta_2 y_t^+) - u(\theta_1 y_t^+) \le \left[k + L\left(\frac{y_t^+}{t}\right)\right](\theta_2 t - \theta_1 t).$$

By this and (4.9), one gets

$$u(\theta_2 y_t^+) - u(\theta_1 y_t^+) = \left[k + L\left(\frac{y_t^+}{t}\right)\right](\theta_2 t - \theta_1 t) \quad \forall t > 0, 0 \le \theta_1 < \theta_2 \le 1.$$

In particular, for all $0 \le s \le t$, letting $\theta_1 = 0$ and $\theta_2 = s/t$ in above identity and by (4.9) we have

E4 (4.12) $$\frac{u(sy_t^+/t)}{s} - L\left(\frac{y_t^+}{t}\right) = k \quad \text{and} \quad u\left(\frac{sy_t^+}{t}\right) = su\left(\frac{y_t^+}{t}\right).$$

Similarly, for all $0 \le s \le t$, we have

E5 (4.13) $$-\frac{u(sy_t^-/t)}{s} - L\left(-\frac{y_t^-}{t}\right) = k \quad \text{and} \quad u\left(\frac{sy_t^-}{t}\right) = su\left(\frac{y_t^-}{t}\right).$$

Letting $t \to \infty$ and by $y_t^\pm/t \to y^\pm$ one see that (4.5) follows from (4.12), (4.6) from (4.13).

To see the (4.8) in Claim I, since $S_{t\lambda}^+ u((1-\lambda)sy^-) \le k$ for all $t, s > 0$ and $\lambda \in (0,1)$, we have

$$\frac{u(\lambda t y^+ + (1-\lambda)sy^-) - u((1-\lambda)sy^-)}{t\lambda} - L(y^+) \le k,$$

which together with (4.5) and (4.6) yields

E9 (4.14) $$u(\lambda t y^+ + (1-\lambda)sy^-) \le \lambda t u(y^+) + (1-\lambda)su(y^-) \quad \forall t, s > 0.$$

Similarly, by $-S_{(1-\lambda)s}^- u(\lambda t y^+) \le k$ for $t, s > 0$, we have

$$-\frac{u(\lambda t y^+ + (1-\lambda)sy^-) - u(\lambda t y^+)}{(1-\lambda)s} - L(-y^-) \le k,$$

which together with (4.5) and (4.6) yields again

$$u(\lambda t y^+ + (1-\lambda)sy^-) \ge \lambda t u(y^+) + (1-\lambda)su(y^-) \quad \forall t, s > 0.$$

From this and (4.14), one deduce that (4.8) as desired. The Claim I is then proved.

Observe that by (4.5), (4.6) and (4.8) in Claim I, there exists a $\lambda_0 \in (0,1)$ such that

E11 (4.15) $$u(\lambda_0 sy^+ + (1-\lambda_0)sy^-) = s[\lambda_0 u(y^+) + (1-\lambda_0)u(y^-)] = 0 \quad \forall s > 0.$$

This leads us to consider two cases: $\lambda_0 y^+ + (1-\lambda_0)y^- = 0$ and $\lambda_0 y^+ + (1-\lambda_0)y^- \ne 0$.

Case 1. $\lambda_0 y^+ + (1-\lambda_0)y^- = 0$.

In this case, $-y^- = s_0 y^+$ with $s_0 = \lambda_0/(1-\lambda_0)$. By (4.15), one has $u(y^-) = -u(y^+)/s_0$ and hence

$$u(-y^+) = u(y^-/s_0) = (1-\lambda_0)u(y^-)/\lambda_0 = -u(y^+).$$

This together with (4.5) gives

E12 (4.16) $$u(y^+) - L(y^+) = k \quad \text{and} \quad u(sy^+) = su(y^+) \quad \forall s \in \mathbb{R}.$$

Note that (4.16) says that u satisfies the condition (1.5) with $e = y^+$ in Lemma 4.1. Therefore, applying Lemma 4.1, we can find a vector $p_0 \in \partial L(y^+)$ such that $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$ and $H(p_0) = k$ as desired.

Case 2. $\lambda_0 y^+ + (1-\lambda_0)y^- \ne 0$.

For any $s > 0$, let $x_s = \lambda_0 sy^+ + (1-\lambda_0)sy^-$, and define a function $v_s(z) = u(x_s + z)$ for all $z \in \mathbb{R}^n$. By (4.15), we have $v_s(0) = 0$. By $u \in C^{0,1}(\mathbb{R}^n)$ and the Arzela-Ascolli theorem, we know that v_s converges to some function $v \in C^{0,1}(\mathbb{R}^n)$ locally uniformly as $s \to \infty$ (up to some subsequence). Letting $y^0 = (y^+ - y^-)/2$, we have the following claim:

Claim II. There exists a $p_0 \in \partial L(y^0)$ such that

E-3 (4.17) $$v(x) = p_0 \cdot x \text{ for all } x \in \mathbb{R}^n, \text{ and } H(p_0) = k.$$

Note that Claim II follows from Lemma 4.1 provided that v satisfies (1.5) with the same k here and also satisfies (4.1) with $e = y^0$. To see this, observe that, by (4.8) and (4.15), one has

E15 (4.18)
$$u\left(x_s + \delta y^0\right) = \left(\lambda_0 s + \frac{\delta}{2}\right) u\left(y^+\right) + \left[(1-\lambda_0)\, s - \frac{\delta}{2}\right] u\left(y^-\right)$$
$$= \lambda_0 s u\left(y^+\right) + (1-\lambda_0)\, s u\left(y^-\right) + \frac{\delta}{2}\left[u\left(y^+\right) - u\left(y^-\right)\right]$$
$$= \frac{\delta}{2}\left[u\left(y^+\right) - u\left(y^-\right)\right] \quad \forall s > 0, -2\lambda_0 s < \delta < 2\,(1-\lambda_0)\, s.$$

Thus

E16 (4.19)
$$v\left(\delta y^0\right) = \lim_{s\to\infty} u\left(x_s + \delta y^0\right) = \frac{\delta}{2}\left[u\left(y^+\right) - u\left(y^-\right)\right] = \delta v(y^0) \quad \forall \delta \in \mathbb{R}.$$

In particular, by (4.7)

E17 (4.20)
$$v(y^0) = \frac{1}{2}\left[u\left(y^+\right) - u\left(y^-\right)\right] = k + \frac{1}{2}\left[L\left(y^+\right) + L\left(-y^-\right)\right] = k + L\left(\frac{1}{2}y^+ - \frac{1}{2}y^-\right) = k + L(y^0).$$

By this, one has

E18 (4.21)
$$\pm S_t^{\pm} v\,(0) \geq \pm\frac{v\,(\pm t y_0)}{t} - L(y^0) = v(y^0) - L(y^0) = k \quad \forall t > 0.$$

One the other hand,

E19 (4.22)
$$\pm S_t^{\pm} v\,(x) = \sup_{y\in\mathbb{R}^n}\left[\pm\frac{v\,(\pm y + x) - v\,(x)}{t} - L\left(\frac{y}{t}\right)\right]$$
$$= \sup_{y\in\mathbb{R}^n}\lim_{s\to\infty}\left[\pm\frac{u\,(\pm y + x + x_s) - u\,(x + x_s)}{t} - L\left(\frac{y}{t}\right)\right]$$
$$\leq \limsup_{s\to\infty} \pm S_t^{\pm} u\,(x + x_s)$$
$$\leq k \quad \forall t > 0, x \in \mathbb{R}^n.$$

Combining (4.19), (4.20), (4.21) and (4.22), we see that v satisfies (1.5) and (4.1) as required by Lemma 4.1. This prove Claim II.

We also have the following claim.

Claim III. There exists vectors $p_x^{\pm} \in \partial L(\pm y^{\pm})$ such that

E22 (4.23)
$$u\,(x - s y^-) + s u\,(y^-) \leq p_x^- \cdot x \quad \forall x \in \mathbb{R}^n, s \in \mathbb{R}$$

and

E23 (4.24)
$$u\,(x - s y^+) + s u\,(y^+) \geq p_x^+ \cdot x \quad \forall x \in \mathbb{R}^n, s \in \mathbb{R}.$$

To see Claim III, since $S_t^+ u\,((t-s)\,y^-) \leq k$ for all $t > 0$ and $s < t$, one has
$$\frac{u\,(x - s y^-) - u\,((t-s)\,y^-)}{t} - L\left(\frac{x - s y^- - (t-s)\,y^-}{t}\right) \leq k,$$
which together with (4.6) and $t - s > 0$ implies that
$$\frac{u\,(x - s y^-) + s u\,(y^-)}{t} \leq k + u\,(y^-) + L\left(\frac{x - t y^-}{t}\right) = -L\,(y^-) + L\left(-y^- + \frac{x}{t}\right)$$
and hence
$$-[u\,(x - s y^-) + s u\,(y^-)] \geq \frac{1}{t}\left[L\,(y^-) - L\left(-y^- + \frac{x}{t}\right)\right] \geq -p_{t,x}^- \cdot x$$
for any $p_{t,x}^- \in \partial L(-y^- + \frac{x}{t})$. By Lemma 3.4 (iii), $p_{t,x}^-$ converges to some $p_x^- \in \partial L\,(-y^-)$ as $t \to \infty$ (up to some subsequence). Thus (4.23) follows. Similarly, using $-S_t^- u\,((t-s)\,y^+) \leq k$ for all $t > 0$ and $s < t$, one can prove (4.24). This prove Claim III.

In the reminder, we prove $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$, which together with $H(p_0) = k$ given in (4.17) yields Lemma 1.3. We consider the following 4 subcases as below. Note that, by Lemma 3.4 (iv) and (4.7), $p_0 \in \partial L(y^0)$ implies that $p_0 \in \partial L(\pm y^\pm)$ and hence, by Lemma 3.4 (i), one has

E20 (4.25) $$\pm p_0 \cdot y^\pm = H(p_0) + L(\pm y^\pm) = k + L(\pm y^\pm) = \pm u(y^\pm),$$

which will be used below.

Subcase 2.1. $\partial L(y^+) \cup \partial L(-y^-)$ contains the unique point p_0. Then $p_x^\pm = p_0$. By (4.23) and (4.24) with $s = 0$, we have

$$p_0 \cdot x = p_x^+ \cdot x \le u(x) \le p_x^- \cdot x = p_0 \cdot x \quad \forall x \in \mathbb{R}^n$$

and hence $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$ as desired.

Subcase 2.2. $\partial L(y^+)$ contains more than one point ; $\partial L(-y^-)$ constants only one point p_0. Then $p_x^- = p_0$, and hence by (4.25) with $s = 1$, we know that

$$u(x) \le p_x^- \cdot x = p_0 \cdot x \quad \forall x \in \mathbb{R}^n.$$

On the other hand, by Theorem 3.1, $\partial L(y^+)$ is a line-segment contained in the same line, say ℓ_{y^+}. Observe that y^+ is not perpendicular to $\ell - p_0$, and hence $\mathbb{R}^n = \sup_{z \perp \ell_{y^+} - p_0}(z + \mathbb{R}y^+)$. Otherwise, $y^+ \cdot (p - p_0) = 0$ for any $p \in \partial L(y^+) \setminus \{p_0\}$. By Lemma 3.4 (i) this gives

$$H(p) + L(y^+) = p \cdot y^+ = p_0 \cdot y^+ = H(p_0) + L(y^+)$$

and hence $H(p_0) = H(p)$. This contradicts with our assumption that H is not a constant in any line-segment.

For any $z \perp \ell_{y^+} - p_0$, we have $p_x^+ \cdot z = p_0 \cdot z$. Thus by (4.24), one has

$$u(z - sy^+) + su(y^+) \ge p_x^+ \cdot z = p_0 \cdot z \quad \forall x \in \mathbb{R}^n, s \in \mathbb{R}.$$

This and (4.25) give

$$u(z - sy^+) \ge p_0 \cdot (z - sy^+) \quad \forall x \in \mathbb{R}^n, s \in \mathbb{R}.$$

For any $x \in \mathbb{R}^n$, by $\mathbb{R}^n = \sup_{z \perp \ell_{y^+} - p_0}(z + \mathbb{R}y^+)$, we can find s such that $x + sy^+ \perp \ell_{y^+} - p_0$. Thus

$$u(x) = u(x + sy^+ - sy^+) \ge p_0 \cdot (x + sy^+ - sy^+) = p_0 \cdot x \quad \forall x \in \mathbb{R}^n.$$

We conclude that $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$.

Subcase 2.3. $\partial L(-y^-)$ contains more than one point; $\partial L(y^+)$ constants only one point p_0. Similarly to the Subcase 2.2, we have $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$.

Subcase 2.4. $\partial L(y^+)$ contains more than one point; $\partial L(-y^-)$ contains more than one point. By Theorem 3.1, $\partial L(y^+)$ is a line-segment contained in the line, say ℓ_{y^+}; $\partial L(-y^-)$ is a line-segment contained in the line, say ℓ_{y^-}. By an argument similar to the Subcase 2.2, we know that $y^\pm$ is not perpendicular to $\ell_{y^\pm} - p_0$. Hence

$$\mathbb{R}^n = \bigcup_{z \perp \ell_{y^+} - p_0} (z + \mathbb{R}y^+) = \bigcup_{z \perp \ell_{y^-} - p_0} (z + \mathbb{R}y^-).$$

For any $z \perp \ell_{y^-} - p_0$, we have $p_0 \cdot z = p_x^- \cdot z$. Thus by (4.23),

$$u(z - sy^-) + su(y^-) \le p_0 \cdot z \quad \forall s \in \mathbb{R}.$$

which together with (4.25) gives

$$u(z - sy^-) \le p_0 \cdot (z - sy^-) \quad \forall s \in \mathbb{R}.$$

By $\mathbb{R}^n = \sup_{z \perp \ell_{y^-} - p_0}(z + \mathbb{R}y^-)$, for any $x \in \mathbb{R}^n$, we can find s such that $x + sy^- \perp \ell_{y^-} - p_0$. Hence, we conclude that $u(x) \le p_0 \cdot x$ for all $x \in \mathbb{R}^n$.

Similarly, by (4.24), (4.25) and $\mathbb{R}^n = \sup_{z \perp \ell_{y^+} - p_0}(z + \mathbb{R}y^+)$ we have $u(x) \ge p_0 \cdot x$ for all $x \in \mathbb{R}^n$. Thus $u(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^n$ as desired. The proof of Lemma 1.3 is complete. □

We also need the following to prove $(ii) \Rightarrow (i)$ in Theorem 1.1.

LEM3.4 **Lemma 4.2.** *Let H be as* (1.4)*. If $H^{-1}(k) = [a, b]$ with $a \neq b$ for some $k \geq 0$, then the following function u_f given in* (1.3) *is an absolute minimizer with $\|H(Du_f)\|_{L^\infty(\mathbb{R}^n)} = k$ whenever $f \in C^{0,1}(\mathbb{R})$ and $\|f'\|_{L^\infty(\mathbb{R})} \leq 1$.*

Proof. Note that $u_f \in C^{0,1}(\mathbb{R}^n)$ and

$$Du_f(x) = \frac{b+a}{2} + \frac{b-a}{2} f'\left(\frac{b-a}{2} \cdot x\right) \quad \text{for almost all } x \in \mathbb{R}^n.$$

Since $\|f'\|_{L^\infty(\mathbb{R})} \leq 1$ implies that $|f'\left(\frac{b-a}{2} \cdot x\right)| \leq 1$ for almost all $x \in \mathbb{R}^n$, we know that $Du_f(x) \in [a, b]$ and hence $H(Du_f(x)) = k$ for all such $x \in \mathbb{R}^n$. Thus $\|H(Du)\|_{L^\infty(V)} = k$.

To see $u_f \in AM_H(\mathbb{R}^n)$, by definition, it suffices to show that for some domain $V \Subset \mathbb{R}^n$ and some function $v \in C^{0,1}\left(\overline{V}\right)$ with $v = u_f$ on ∂V, we have $\|H(Dv)\|_{L^\infty(V)} := s \geq \|H(Du)\|_{L^\infty(V)} = k$.

If $k = 0$, we always have $\|H(Dv)\|_{L^\infty(V)} \geq \|H(Du)\|_{L^\infty(V)}$. Assume that $k > 0$ below. Let $q_0 \in \partial H((a+b)/2)$. Then $q_0 \neq 0$; otherwise,

$$H(0) - H\left(\frac{a+b}{2}\right) \geq -q_0 \cdot \frac{a+b}{2} = 0,$$

which implies that $k = H((a+b)/2) = 0$. By Lemma 3.6, we have $q_0 \perp (b-a)$. Let $x_0 \in V$ and the line $\ell_0 = x_0 + \mathbb{R}q_0$. Denote by γ_0 the component (an open interval) containing x_0 of $\ell_0 \cap V$, and $t_0 > 0$ the length of γ_0. Write $\gamma_0 = (x_0, y_0)$ with $x_0, y_0 \in \partial V$ and $y_0 = x_0 + t_0 q_0/|q_0|$.

Since $q_0 \perp (b-a)$ implies

$$\frac{b-a}{2} \cdot y_0 = \frac{b-a}{2} \cdot x_0 + \frac{t_0}{|q_0|} \frac{b-a}{2} \cdot q_0 = \frac{b-a}{2} \cdot x_0,$$

we get

$$u(y_0) - u(x_0) = \frac{b+a}{2} \cdot (y_0 - x_0) + f\left(\frac{b-a}{2} \cdot y_0\right) - f\left(\frac{b-a}{2} \cdot x_0\right) = \frac{t_0}{|q_0|} \frac{b+a}{2} \cdot q_0.$$

By $q_0 \in \partial H((a+b)/2)$ and Lemma 3.4, this yields that

E25 (4.26)
$$u(y_0) - u(x_0) = \frac{t_0}{|q_0|}\left[H\left(\frac{b+a}{2}\right) + L(q_0)\right] = \frac{t_0}{|q_0|}[k + L(q_0)].$$

On the other hand, by Lemma 2.2, one has

$$v(y) - v(x) \leq C_s^H(y - x) \text{ whenever } [x, y] \subset V.$$

Up to some approximation to x_0, y_0, this gives that

$$v(y_0) - v(x_0) \leq C_s^H(y_0 - x_0) = \frac{t_0}{|q_0|} \sup_{H(p)<s} p \cdot q_0 \leq \frac{t_0}{|q_0|}[s + L(q_0)].$$

By this, $u(y_0) - u(x_0) = v(y_0) - v(x_0)$ and (4.26), we have

$$\frac{t_0}{|q_0|}[s + L(q_0)] \geq \frac{t_0}{|q_0|}[k + L(q_0)]$$

which gives $s \geq k$, that is, $\|H(Dv)\|_{L^\infty(V)} \geq \|H(Du)\|_{L^\infty(V)}$ as desired. This completes the proof of Lemma 4.2. □

With the aid of Lemmas 1.3 and 4.2, we are ready to prove $(i) \Leftrightarrow (ii)$ in Theorem 1.1.

Proof of $(i) \Leftrightarrow (ii)$ *in Theorem 1.1.* As clarified in the introduction, we only need to prove Theorem 1.1 under the assumption (1.4).

Proof of $(i) \Rightarrow (ii)$. Assume that H is convex, satisfies (1.4) and H is not a constant in any line-segment. Let $u \in AM_H(\Omega)$ and $x^0 \in \Omega$. Up to some translation, we assume that $x^0 = 0 \in \Omega$. We also assume $u(0) = 0$. Let $U = B(0, \delta_0) \subset \Omega$ and $K := \|u\|_{C^{0,1}(U)} < \infty$. Note that $u \in AM_H(U)$ is bounded. For any $0 < r < 1$, $u_r(x) = \frac{1}{r}u(rx)$ for $x \in \frac{1}{r}U$. Then $u_r \in AM_H\left(\frac{1}{r}U\right)$. Let $B(0, \delta_0) \Subset U$. Since $u \in C^{0,1}(U)$, we know that $u_r \in C^{0,1}(B(0, \delta_0/r))$ and $\|u_r\|_{C^{0,1}(B(0,\delta_0/r))} = K$. Up to considering some subsequence we may assume that

$u_r \to v \in C^{0,1}(\mathbb{R}^2)$ locally uniformly in $\mathbb{R}^2$, and $\|v\|_{C^{0,1}(\mathbb{R}^2)} \le \|u\|_{C^{0,1}(B(0,\delta_0))} = K$. By (2.1), it suffice to show that v is a linear function and $H(Dv) = Su(0)$. This will follow if v fulfils all the assumptions in Lemma 1.3 with $k = Su(0)$.

To this end, let R_K be as in Lemma 2.6. For any $x \in \mathbb{R}^2$, by Lemma 2.6 we write

$$\begin{aligned} S_t^+ v(x) &= \sup_{|x-y|\le R_K t} \frac{1}{t}\left[v(y) - v(x) - tL\left(\frac{y-x}{t}\right)\right] \\ &= \sup_{|x-y|\le R_K t} \lim_{r\to 0} \frac{1}{t}\left[\frac{u(ry) - u(rx)}{r} - tL\left(\frac{y-x}{t}\right)\right] \\ &\le \liminf_{r\to 0} \sup_{|rx-y|\le R_K tr} \frac{1}{tr}\left[u(y) - u(rx) - trL\left(\frac{y-rx}{tr}\right)\right] \\ &\le \liminf_{r\to 0} S_{tr}^+ u(rx). \end{aligned}$$

By Lemma 2.5, we have

E26 (4.27) $$S_t^+ v(x) \le \liminf_{r\to 0} S_{tr}^+ u(rx) \le \liminf_{\delta\to 0} \liminf_{r\to 0} S_\delta^+ u(rx) = \liminf_{\delta\to 0} S_\delta^+ u(0) = Su(0) \quad \forall x \in \mathbb{R}^n.$$

Similarly, we have

E27 (4.28) $$-S_t^- v(x) \le -S^- u(0) = Su(0) \quad \forall x \in \mathbb{R}^n.$$

One the other hand, for any $0 < r < \delta_0/Rt$, by Lemma 4.2 let $z_r \in \overline{B(0, R_K tr)}$ such that

$$\frac{u(z_r)}{tr} - trL\left(\frac{z_r}{tr}\right) = S_{tr}^+ u(0).$$

Then $z_r/r \in \overline{B(0, Rt)}$. For any $\epsilon > 0$, there exists $r_{\epsilon,t} > 0$ such that for any $y \in \overline{B(0, Rt)}$, we have $v(y) \ge u_r(y) - \epsilon$ for all $r \in (0, r_{\epsilon,t})$. For r sufficiently small, we have

$$S_t^+ v(0) \ge \frac{1}{t}\left[v(z_r/r) - tL\left(\frac{z_r}{tr}\right)\right] \ge \frac{1}{tr}\left[u(z_r) - trL\left(\frac{y}{t}\right)\right] - \frac{\epsilon}{t} = S_{tr}^+ u(0) - \frac{\epsilon}{t} \ge Su(0) - \frac{\epsilon}{t}.$$

By the arbitrariness of $\epsilon > 0$, we have $S_t^+ v(0) \ge Su(0)$. Similarly we have $-S_t^- v(0) \ge Su(0)$. Combining these, (4.27) and(4.28), we see that the assumptions of Lemma 1.3 is fulfilled with $k = Su(0)$. This proves $(i) \Rightarrow (ii)$.

Proofs of $(ii) \Rightarrow (i)$. Assume that $H^{-1}(k) = [a, b]$ with $a \ne b$ for some $k \ge 0$. It suffices to show that (ii) fail.

To this end, let $f(t) = |t|$ for $t \in \mathbb{R}$, and u_f be as in (1.3). By Lemma 4.2, $u_f \in AM_H(\mathbb{R}^n)$ and $\|H(Du_f)\|_{L^\infty(\mathbb{R}^n)} = k$. To see the failure of (ii), we only need to show that u_f does not have the linear approximation property, that is, for any vector $e \in \mathbb{R}^n$ and $r \in (0, 1)$, one always has

$$\liminf_{r\to 0} \sup_{x\in B(0,1)} \left|\frac{u_f(rx) - u_f(0)}{r} - e \cdot x\right| > c$$

for some constant $c > 0$ depending only on a, b, e. Write

$$\liminf_{r\to 0} \sup_{x\in B(0,1)} \left|\frac{u_f(rx) - u_f(0)}{r} - e \cdot x\right| = \sup_{x\in B(0,1)} \left|\left|\frac{b-a}{8} \cdot x\right| - \left(e - \frac{b+a}{2}\right) \cdot x\right|.$$

If $e - \frac{b+a}{2} \ne 0$, letting $x = \frac{1}{2} \frac{2e-(b+a)}{|2e-(b+a)|}$, we have

$$\liminf_{r\to 0} \sup_{x\in B(0,1)} \left|\frac{u_f(rx)}{r} - e \cdot x\right| \ge \frac{1}{2}|e - \frac{b+a}{2}|.$$

If $e - \frac{b+a}{2} = 0$, letting $x = \frac{1}{2}\frac{b-a}{|b-a|}$, we obtain

$$\liminf_{r\to 0} \sup_{x\in B(0,1)} \left| \frac{u_f(rx)}{r} - e\cdot x \right| \geq \frac{|b-a|}{16}$$

as desired. This gives $(ii) \Rightarrow (i)$. The proof of $(i) \Leftrightarrow (ii)$ in Theorem 1.1 is complete. □

5. Proof of Proposition 1.4

Suppose that vectors $e_{0,6}, e_6 \in \mathbb{R}^2$ and $u \in AM_H(B(0,6))$ satisfy (1.6),(1.7) and (1.8). We want to show that for any $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then $|e_{0,6} - e_6| \leq C\epsilon$.

Since u is not linear in $B(0,r)$, there are a line segment $[z_1, z_2] \subset B(0,r)$, a linear function $l(x) = a_0 \cdot x + b_0$, $z_3 \in [z_1, z_2]$ with $a_0 = \frac{u(z_2)-u(z_1)}{|z_2-z_1|}$ such that either

xcase1 (5.1) $$u \geq l \quad \text{on } [z_1,z_2], \quad u(z_1) > l(z_1), \quad u(z_3) = l(z_3), \quad u(z_2) > l(z_2),$$

or

xcase2 (5.2) $$u \leq l \quad \text{on } [z_1,z_2], \quad u(z_1) < l(z_1), \quad u(z_3) = l(z_3), \quad u(z_2) < l(z_2).$$

Without loss of generality, we assume that (5.1) holds.

xcon-com **Lemma 5.1.** *There exists $e \in \mathbb{R}^2$, with $H(e) = S^+(u)(z_3)$, such that z_1 and z_2 belong two distinct connected components of the set $\{y \in \mathbb{R}^2 : u(y) > u(z_3) + e\cdot(y - z_3)\} \cap B(0,6)$.*

Proof of Lemma 5.1. By Theorem 1.1 (ii), for an sequence $\{s_k\}_{k\in\mathbb{N}}$ which converges to 0, up to considering subsequence there exists $e \in \mathbb{R}^2$ such that $H(e) = S^+u(z_3)$ and

xeq4.1 (5.3) $$\lim_{s_k\to 0} \sup_{B(z_3,s_k)} \frac{|u(y) - u(z_3) - e\cdot(y - z_3)|}{s_k} = 0.$$

Since u is not linear in $B(0,r)$, for any $z_k \in [z_1,z_2] \cap \partial B(z_3, s_k)$, by (5.3) we have

$$(a_0 - e)\cdot\left(\frac{z_k - z_3}{s_k}\right) = \frac{l(z_k) - l(z_3) - e\cdot(z_k - z_3)}{s_k} \leq \frac{u(z_k) - u(z_3) - e\cdot(z_k - z_3)}{s_k} \to 0,$$

as $k\to\infty$. Therefore we have $(a_0 - e)\cdot(z - z_3) = 0$ for any $z \in [z_1, z_2]$, that is

$$u(z_i) - u(z_3) > l(z_i) - l(z_3) = a_0\cdot(z_i - z_3) = e\cdot(z_i - z_3), \quad i = 1,2,$$

and hence $z_1, z_2 \in \{y \in \mathbb{R}^2 : u(y) > u(z_3) + e\cdot(y - z_3)\}$. Now we assume that z_1, z_2 belong to the same connected component of $\{y \in \mathbb{R}^2 : u(y) > u(z_3) + e\cdot(y - z_3)\} \cap B(0,6)$. Then there exists a ploygonal line $\Gamma \subset \{y \in \mathbb{R}^2 : u(y) > u(z_3) + e\cdot(y - z_3)\} \cap B(0,6)$ joining z_1 to z_2. Let $S = \Gamma \cup [z_1, z_2]$ be the closed curve and $U \subset B(0,6)$ be the open set such that $S = \partial U$. Without loss of the generality, we may assume that there exists a $\beta > 0$ such that

$$B^+(z_3,\beta) := B(z_3,\beta) \cap \{y \in \mathbb{R}^2 : 0 < \angle(y - z_3, z_2 - z_1) < \pi\} \subset U.$$

It is clear that there exists an δ_0 such that $u(y) - u(z_3) - e\cdot(y - z_3) \geq \delta_0$ for any $y \in \Gamma$. Therefore there are a small $\epsilon > 0$ and a unit vector $v \in \mathbb{R}^2$, with $\angle(v, z_2 - z_1) = \frac{\pi}{2}$ and $e + \epsilon v \neq 0$, such that

$$u(y) \geq u(z_3) + (e + \epsilon v)\cdot(y - z_3), \quad \forall y \in S.$$

Note that $\phi(y) = u(z_3) + (e + \epsilon v)\cdot(y - z_3)$ is linear and $D\phi = e + \epsilon v \neq 0$ so that Lemma 2.1 implies that

$$u(y) \geq u(z_3) + (e + \epsilon v)\cdot(y - z_3), \quad y \in U.$$

On the other hand, we have

$$\lim_{s_k\to 0} \max_{y\in B^+(z_3,\beta)\cap B(z_3,\beta)} \frac{|u(y) - u(z_3) - e\cdot(y - z_3)|}{s_k} \geq \epsilon > 0,$$

this is contradicts (5.3). The proof of Lemma 5.1 is complete. □

The proof (1.9) is reduced to proving

xeq4.2
$$|e - e_6|^2 \le \epsilon/2 \tag{5.4}$$

and

xxeq4.2
$$|e - e_{0,6}|^2 \le \epsilon/2. \tag{5.5}$$

Proof of (5.4). Note that $e_6 - e \in \overline{B(0, 2R)}$. For any $\epsilon > 0$ the function $\psi_{2R}(\epsilon/2)$ be as in Theorem 3.2. By Lemma 5.2 and Lemma 5.3 as below, for $0 < \delta < \delta(H, R, \epsilon)$ as there, the assumptions of Theorem 3.2 with $p = e$ and $v = e_6 - e$ are fulfilled, and hence $|e_6 - e| \le \epsilon/2$ as desired.

xlength **Lemma 5.2.** *For any $\epsilon > 0$, there exists a $\delta = \delta(H, R, \epsilon) > 0$ such that*

xeq4.3
$$H(e_6) - H(e) \le \psi_{2R}(\epsilon/2). \tag{5.6}$$

xangle **Lemma 5.3.** *For any $\epsilon > 0$, there is $\delta = \delta(H, R, \epsilon) > 0$ such that*

xeq4.28
$$\left|\angle(q, e_6 - e) - \frac{\pi}{2}\right| \le \psi_{2R}(\epsilon/2) \quad \text{for some } q \in \partial H(e') \text{ and some } e' \in \overline{B(e, \psi_{2R}(\epsilon/2))} \text{ with } H(e) = H(e'). \tag{5.7}$$

Below we prove Lemma 5.2 and Lemma 5.3. Write $\eta = \frac{1}{2}\psi_{2R}(\epsilon/2)$ and $f = e_6 - e$ for simple. We may assume that $|f| \ge \eta/2$ without loss of generality. Write

$$\mathscr{S} := \{y \in \mathbb{R}^2 : |f \cdot (y - z_3)| \le 2\delta\}, \mathscr{S}_- := \{y \in \mathbb{R}^2 : f \cdot (y - z_3) < -2\delta\} \text{ and } \mathscr{S}_+ := \{y \in \mathbb{R}^2 : f \cdot (y - z_3) > 2\delta\}.$$

The width of $\mathscr{S}$ is $2\delta/|f|$, and hence, at most $\frac{2\delta}{\epsilon}$. Moreover, (1.7) implies that

$$|u(y) - u(z_3) - e_6 \cdot (y - z_3)| \le 2\delta, \quad \forall y \in B(0, 6).$$

Since

$$\mathscr{S}_- \cap B(0, 6) \subset \{y \in \mathbb{R}^2 : u(y) < u(z_3) + e \cdot (y - z_3)\}$$

and

$$\mathscr{S}_+ \cap B(0, 6) \subset \{y \in \mathbb{R}^2 : u(y) > u(z_3) + e \cdot (y - z_3)\},$$

by Lemma 5.1 above there is a connected component of $\{y \in \mathbb{R}^2 : u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B(0, 6)$, called U, that contains either z_1 or z_2, and hence intersects $B(0, 1)$, and is contained in the strip $\mathscr{S}$.

Observe that $U \not\subset B(0, 6)$; otherwise, thanks to $u(y) = u(z_3) + e \cdot (y - z_3)$ on $y \in \partial U$, by Lemma 2.1, $u = u(z_3) + e \cdot (y - z_3)$ in U, which contradicts with the definition of U. Therefore there exists a polygonal line Γ inside U starting in $B(0, 1)$ and and exiting $B(0, 6)$. Now we find $z_4 \in B(0, 6)$, with $|z_4 - z_3| = 3$ and $z_4 - z_3 \perp f$, such that

(A1) $\sup_{B(z_4, 2)} |u(y) - u(z_3) - e_6 \cdot (y - z_3)| \le 2\delta$ and $1 \le H(e_6) \le 2$.

(A2) $\{y \in \mathbb{R}^2 : u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B(0, 6)$ has a connected component $U \subset \mathscr{S}$ that contains a polygonal line Γ connecting the two arcs $\mathscr{S} \cap \partial B(z_4, 2)$.

xlengthU **Lemma 5.4.** *If $\delta \in (0, 2/C)$ for some constant $C(H) \ge 1$, then*

xeq4.6
$$S u(x) \le H(e) + C\delta \quad \forall x \in U \cap B(z_4, 1). \tag{5.8}$$

Proof of Lemma 5.4. For any $x_0 \in B(z_4, 1) \cap U$, we have $B(x_0, 1) \subset B(z_4, 2)$ and $u(z_3) + e \cdot (x_0 - z_3) < u(x_0)$. Note that

xeq4.7
$$u(y) = u(z_3) + e \cdot (y - x_0) \le u(x_0) + \mathscr{C}^H_{H(e)}(y - x_0), \quad \forall y \in \partial U \cap B(x_0, 1), \tag{5.9}$$

and

$$u(y) \le u(z_3) + e \cdot (y - x_0) + 4\delta \le u(x_0) + e \cdot (y - x_0) + 4\delta, \quad \forall y \in U \cap \partial B(x_0, 1), \tag{5.10}$$

By (3.3), (5.9) and (5.10) we have

xeq4.9
$$u(y) \le u(x_0) + \mathscr{C}^H_{H(e)+C\delta}(y - x_0), \quad \forall y \in \partial(U \cap B(x_0, 1)). \tag{5.11}$$

Since $u \in CC_H(B(0,6))$, we have

$$u(y) \leq u(x_0) + \mathscr{C}^H_{H(e)+C\delta}(y - x_0), \quad \text{in } U \cap B(x_0, 1).$$

This, combined with the Proposition 2.2 in [28], implies (5.8). □

We are able to prove Lemma 5.2 as below.

Proof of Lemma 5.2. If $H(e) > H(e_6) - \eta$, then (5.6) follows. Assume that $H(e) \leq H(e_6) - \eta$ now. Thanks to $1 \leq H(e_6) \leq 2$, by (i) and Lemma 3.5, we know that $e_6 \neq 0$ and $0 \notin \partial H(e_6)$. The convexity of H implies

xeq4.4

$$q \cdot f = q \cdot (e_6 - e) \geq H(e_6) - H(e) \geq \eta, \qquad \forall q \in \partial H(e_6). \tag{5.12}$$

We always let $\delta > 0$ is small so that $50|q|\delta \leq \frac{1}{2}H(e_6)$ for all $q \in \partial H(e_6)$ below.

Claim 5.2 (a) For any $q \in \partial H(e_6)$, there exists

$$w_q \in L := \left\{ z_4 + s\frac{q}{|q|} : -\frac{1}{3} \leq s \leq -\frac{1}{4} \right\} \subset B(z_4, 1)$$

such that

xeq4.10

$$Su(y_q) \geq H(e_6) - 50|q|\delta. \tag{5.13}$$

Proof of Claim 5.2 (a). Let $q \in \partial H(e_6)$ and $c := \sup_{x\in L} Su(x)$. To find y_q it suffices to show that

$$H(e_6) - 48|q|\delta \leq c.$$

This is further reduced this to proving

xeq4.14

$$e_6 \cdot q - 48|q|\delta \leq \mathscr{C}^H_c(q). \tag{5.14}$$

Indeed, since

$$\mathscr{C}^H_c(q) = \sup_{H(p)\leq c} p \cdot q \leq \sup_{H(p)\leq c} [H(p) + L(q)] = c + L(q),$$

by $e_6 \cdot q = H(e_6) + L(q)$ as in Lemma 3.4, one has $H(e_6) - 48|q|\delta \leq c$.

To see (5.14), thanks to $L \subset B(z_4, 2)$, (A1) implies

xeq4.11

$$u\left(z_4 - \frac{1}{4}\frac{q}{|q|}\right) - u\left(z_4 - \frac{1}{3}\frac{q}{|q|}\right) \geq \frac{1}{12} e_6 \cdot \frac{q}{|q|} - 4\delta. \tag{5.15}$$

On the other hand, by the semi-continuity of Su, for any $\eta > 0$ there exists an open neighborhood $V_\eta(L)$ such that $\sup_{x\in V_\eta(L)} Su(x) \leq c + \eta$. and hence $\sup_{x\in V_\eta(L)} H(Du(x)) \leq c + \eta$. By Lemma 2.2, We have

$$u\left(z_4 - \frac{1}{3}\frac{q}{|q|}\right) - u\left(z_4 - \frac{1}{4}\frac{q}{|q|}\right) \leq \mathscr{C}^H_{c+\eta}(\frac{q}{12|q|}).$$

Letting $\eta \to 0$, we conclude (5.14) from this and (5.15).

□

Fix a $q^0 \in \partial H(e_6)$ such that

EQ4.x1

$$f \cdot \frac{q^0}{|q^0|} = \min \left\{ f \cdot \frac{q}{|q|} : q \in \partial H(e_6) \right\} \tag{5.16}$$

Let $y_0 := w_{q^0} = z_4 + s_0 \frac{q^0}{|q^0|}$ for some $s_0 \in [-1/3, -1/4]$ as determined in Claim 5.2(a). Since $f \cdot q^0 \geq H(e_6) - H(e) \geq \eta$, we have

$$f \cdot (y^0 - z_3) = f \cdot (y^0 - z_4) = s_0 f \cdot \frac{q^0}{|q^0|} < -\frac{1}{4}\frac{\eta}{|q^0|}.$$

If $\delta < \eta/|q^0|2$, then $f \cdot (y^0 - z_3) < -2\delta$, that is, $y^0 \in \mathscr{S}_-$.

For any $0 < \delta < \eta/8$, let

$$t = t(\delta) := \mathrm{dist}(\Gamma, \partial U \cap B(z_4, 2)) \le \frac{2\delta}{|f|}.$$

Applying Lemma 2.8, we get a discrete gradient flow $\{y_i\}_{i=1}^m$ for some $m = m(\delta)$ satisfying

xeq4.18

$$y_i = y_i(\delta) \in B(z_4, 2),\ |y_i - y_{i-1}| = t,\ u(y_i) = u(y_{i-1}) + \mathscr{C}^H_{S_t^+(u)(y_{i-1})}(y_i - y_{i-1}), \quad 1 \le i \le m, \tag{5.17}$$

but $y_{m+1} \notin B(z_4, 2)$. Note that $y_{m+1} \notin B(z_4, 2)$ implies that

xeq4.19

$$\mathrm{dist}(y_m, \partial B(z_4, 2)) \le t. \tag{5.18}$$

We prove the existence of such $m \ge 1$ by contradiction. Assume (5.17) holds for all $i \ge 1$. By Claim 5.2(a),

xeq4.20

$$S^+(u)(y_i) \ge S^+(u)(y_0) \ge H(e_6) - 50|q^0|\delta, \quad 0 \le i \le m. \tag{5.19}$$

Thus, for any $j \ge 1$

xeq4.20

$$\begin{aligned} u(y_j) - u(y_0) &= \sum_{i=1}^m (u(y_i) - u(y_{i-1})) \ge \sum_{i=1}^j \mathscr{C}^H_{S^+(u)(y_{i-1})}(y_i - y_{i-1}) \\ &\ge \sum_{i=1}^j \mathscr{C}^H_{S^+(u)(y_0)}(y_i - y_{i-1}) \ge \sum_{i=1}^j \mathscr{C}^H_{H(e_6)-50|q^0|\delta}(y_i - y_{i-1}), \end{aligned} \tag{5.20}$$

By $50|q^0|\delta \le \frac{1}{2}H(e_6)$, then there exists $C > 0$ depending only on H such that

$$\mathscr{C}^H_{H(e_6)-50|q^0|\delta}(x) \ge C|x|, \quad \forall |x| = 1,$$

and hence,

$$u(y_j) - u(y_0) \ge \sum_{i=1}^j \mathscr{C}^H_{H(e_6)-50|q^0|\delta}(y_i - y_{i-1}) \ge Cjt,$$

Noteing $|u(y_j) - u(y_0)| \le \|Du\|_{L^\infty(B(0,6))}|y_j - y_0|$, one has $|y_j - y_0| \to Cjt \to \infty$ as $j \to \infty$, which is a contradiction.

Claim 5.2 (b) There exists $\delta(\epsilon) > 0$ such that for any $0 < \delta < \delta(\epsilon)$, we can find $1 \le j_\delta \le m$ such that $y_m \in \mathscr{S}_+$ and $y_{j_\delta} \in B(z_4, 1) \cap U$.

Proof of Claim 5.2(b). We first show that $y_m \in \mathscr{S}_+$ if $\delta > 0$ is sufficiently small. Note that

xeq4.23

$$\mathscr{C}^H_{H(e_6)-50|q^0|\delta}(y_m - y_0) \le e_6 \cdot (y_m - y_0) + 4\delta. \tag{5.21}$$

Indeed, noting that (5.20) also holds with $j = m$, and applying the triangle inequality for $\mathscr{C}^H_k$, we have

xeq4.21

$$u(y_m) - u(y_0) \ge \sum_{i=1}^m \mathscr{C}^H_{H(e_6)-50|q^0|\delta}(y_i - y_{i-1}) \ge \mathscr{C}^H_{H(e_6)-50|q^0|\delta}\left(\sum_{i=1}^m (y_i - y_{i-1})\right) = \mathscr{C}^H_{H(e_6)-50|q^0|\delta}(y_m - y_0). \tag{5.22}$$

On the other hand, by (A1) and $y_m, y_0 \in B_2(z_4)$ one has

xeq4.22

$$u(y_m) - u(y_0) \le e_6 \cdot (y_m - y_0) + 4\delta. \tag{5.23}$$

From this and (5.22) one gets (5.21).

Thanks to $\mathrm{dist}(y_m, \partial B(z_4, 2)) \le t$ as in (5.18), there exists $y \in \partial B(z_4, 2)$ such that $t \ge |y_m - y|$. For $0 < \delta < \eta/6$, by $|f| \ge \eta/2$ and $t < 2\delta/|f| \le 2/3$, one gets

$$|y_m - z_4| \ge |y - z_4| - |y - y_m| \ge 2 - t \ge \frac{4}{3}.$$

Since $y_0 = w_{q^0} \in \{z_4 + t\frac{q^0}{|q^0|} : t \in [-\frac{1}{3}, -\frac{1}{4}]\}$, it is clear that

$$|y_m - y_0| \ge |y_m - z_4| - |z_4 - y_0| \ge \frac{4}{3} - \frac{1}{3} = 1.$$

Writing $e_m = \frac{y_m - y_0}{|y_m - y_0|}$, by (5.21), one has

xeq4.24 (5.24)
$$\mathscr{C}^H_{H(e_6)-50|q|\delta}(e_m) \le e_6 \cdot e_m + 4\delta.$$

Note that e_m converges to some unit vector $\hat{e}$ as $\delta \to 0$ (up to some subsequence), and hence $\mathscr{C}^H_{H(e_6)}(\hat{e}) \le e_6 \cdot \hat{e}$. By definition, $\mathscr{C}^H_{H(e_6)}(\hat{e}) = e_6 \cdot \hat{e}$. By Lemma 3.7, we have $\hat{e} = \hat{q}/|\hat{q}|$ for some $\hat{q} \in \partial H(e_6)$. Recall that $\hat{q}$ and q^0 do not coincide necessarily.

Next we show that if $\delta > 0$ is sufficiently small, then $y_m \in \{y \in \mathbb{R}^2 : f \cdot (y - z_4) \ge 2\delta\}$. Indeed, by $y_0 = z_4 + s_0 \frac{q^0}{|q^0|}$ for some $s_0 \in [-1/3, -1/4]$ we have

$$f \cdot (y_m - z_4) = f \cdot (y_m - y_0) + f \cdot (y_0 - z_4) = |y_m - y_0| f \cdot \hat{e} + s_0 f \cdot \frac{q^0}{|q^0|} + |y_m - y_0| f \cdot (e_m - \hat{e}).$$

Since $1 \le |y_m - y^0| \le 4$ and $f \cdot \hat{e} \ge f \cdot \frac{q^0}{|q^0|} \ge \eta/|q^0|$, we get

$$f \cdot (y_m - z_4) \ge \frac{2}{3} f \cdot \frac{q^0}{|q^0|} - 4|f||e_m - \hat{e}| \ge \frac{2\eta}{3|q^0|} - 2\eta|e_m - \hat{e}|.$$

Since $e_m \to \hat{e}$ as $\delta \to 0$, we have $f \cdot (y_m - z_4) > 2\delta$ if δ is sufficiently small. Since $f \cdot (y_m - z_3) = f \cdot (y_m - z_4) > 2\delta$ and hence $y_m \in \mathscr{S}_+$ by definition.

Now we find $1 \le j_\delta \le m$ such that $y_{j_\delta} \in U \cap B(z_4, 1)$ when δ is sufficiently small.

When δ is sufficiently small, by $y^0 \in \mathscr{S}_-$, $y_m \in \mathscr{S}_+$ and the choice of the step size t, there exists $1 \le j_\delta \le m$ such that $y_{j_\delta} \in U \cap B(z_4, 2)$. It remains to show that $|y_{j_\delta} - z_4| \le 1$. If $|y_{j_\delta} - z_4| > 1$, we have

$$|y_{j_\delta} - y_0| \ge |y_{j_\delta} - z_4| - |y_0 - z_4| \ge 1 - \frac{1}{3} = \frac{2}{3}.$$

By an argument similar to above, we also have $f \cdot (y_{j_\delta} - z_4) > 2\delta$, that is, $y_{j\delta} \in \mathscr{S}_+$ which is a contradiction. Thus we obtain $|y_{j_\delta} - z_4| \le 1$. The Claim 5.2 (b) is proven. □

Combing (5.8), (5.13),(5.20) and Claim 5.2 (b), we obtain

$$H(e_6) - 50|q|\delta \le S^+(H, u, y_{j_\delta}) \le H(e) + C\delta,$$

this implies that (5.6), where $\delta = \delta(\eta, H)$ is chosen to be sufficiently small. The proof of Lemma 5.2 is complete. □

The proof of Lemma 5.3 is as below.

Proof of the Lemma 5.3. If there exist $e' \in B(e, \eta)$ and $q_1, q_2 \in \partial H(e')$ with $H(e) = H(e')$ and $\angle(q_1, f) \le \pi/2 \le \angle(q_2, f)$, then we can find $\lambda \in [0, 1]$ such that $\angle(\lambda q_1 + (1 - \lambda) q_2, f) = \pi/2$. Then $\lambda q_1 + (1 - \lambda) q_2 \in \partial H(e')$ is the desired result. Below we may assume

x1 (5.25)
$$\angle(q, f) \in [0, \tfrac{\pi}{2}) \text{ for all } q \in \partial H(e') \text{ and all } e' \in B(e, \eta) \text{ with } H(e) = H(e').$$

The case $\angle(q, f) \in (\pi/2, \pi]$ for all $q \in \partial H(e')$ and all $e' \in B(e, \eta)$ with $H(e) = H(e')$ is similar.

Note that there exists $e' \in \overline{B(e, \eta)}$ with $H(e) = H(e')$ and $q_{e'} \in \partial H(e')$ such that

$$\alpha := \angle(q_{e'}, f) = \max_{e' \in \overline{B}(e, \eta)} \max_{q \in \partial H(e)} \angle(q, f) < \frac{\pi}{2}.$$

Assume $\alpha < \frac{\pi}{2} - \eta$ without loss of generality.

Let $x_\delta = z_4 - \frac{2q_{e'}}{q_{e'} \cdot f}\delta$ be the intersection of $L := \{z_4 + s q_{e'} : t \in \mathbb{R}\}$ and $\{y \in \mathbb{R}^2 : (y - z_4) \cdot f = -2\delta\}$. Observe that

$$|x_\delta - z_4| = \frac{2\delta}{|f| \cos \angle(q, f)} \le \frac{2\delta}{\eta \sin \eta} \le 1,$$

provided that $\delta > 0$ is chosen to be sufficiently small. This implies $B(x_\delta, 1) \subset (z_4, 2)$. By (A1), we have

xeq4.29 (5.26)
$$u(y) - u(z_3) - e \cdot (y - z_3) \le u(y) - u(z_3) - e_6 \cdot (y - z_3) + f \cdot (y - z_3) \le 4\delta, \quad \forall y \in U \cap B(x_\delta, 1),$$

and

xeq4.30 (5.27)
$$u(y) = u(z_3) + e \cdot (y - z_3), \quad \forall y \in \partial U \cap B(x_\delta, 1)$$

Thus

xeq4.32 (5.28)
$$u(y) = u(z_3) + e \cdot (x_\delta - z_3) + e \cdot (x - x_\delta) \le u(z_3) + e \cdot (x_\delta - z_3) + \mathscr{C}^H_{H(e)}(x - x_\delta), \quad \forall x \in \partial U \cap B(x_\delta, 1).$$

For any $x \in U \cap \partial B(x_\delta, 1)$, we have

xeq4.33 (5.29)
$$u(x) \le u(z_3) + e \cdot (x_\delta - z_3) + e \cdot (x - x_\delta) + 4\delta.$$

Let $p_x \in H^{-1}(H(e))$ such that $t_x(x - x_\delta) \in \partial H(p_x)$ for some $t_x > 0$; see Lemma 3.7 for the existence of t_x. Since $\angle(x - x_\delta, f) > \frac{\pi}{2} - C\delta/|f|$ for some constant $C \ge 2$, we know that if $\delta < 2R\eta/C$, then $\angle(x - x_\delta, f) > \frac{\pi}{2} - \eta$ and hence, by the assumption (5.25), $|p_x - e| \ge \eta$. If $\delta < \frac{1}{4}\phi_{2R}(\eta)$ as needed in Theorem 3.3 we further have

$$(p_x - e) \cdot (x - x_\delta) \ge 4\delta.$$

Since $C_{H(e)}(x - x_\delta) = p_x \cdot (x - x_\delta)$, we obtain,

$$C_{H(e)}(x - x_\delta) = p_x \cdot (x - x_\delta) \ge e \cdot (x - x_\delta) + 4\delta.$$

By this and (5.29), one has

xeq4.34 (5.30)
$$u(x) \le u(z_3) + e \cdot (x_\delta - z_3) + e \cdot (x - x_\delta) + 4\delta \le u(z_3) + e \cdot (x_\delta - z_3) + C_{H(e)}(x - x_\delta).$$

It follows from (5.28), (5.29) and (5.30) that for sufficiently small $\delta > 0$, we have

eq4.37 (5.31)
$$u(y) \le u(z_3) + e \cdot (x_\delta - z_3) + \mathscr{C}^H_{H(e)}(y - x_\delta), \quad \forall y \in \partial B(U \cap B(x_\delta, 1)).$$

Applying comparison property with cones, we have

eq4.38 (5.32)
$$u(y) \le u(z_3) + e \cdot (x_\delta - z_3) + \mathscr{C}^H_{H(e)}(y - x_\delta), \quad \text{in } U \cap B(x_\delta, 1).$$

By (A2), we have $\{x_\delta + tq_e : t \ge 0\} \cap (U \cap B(x_\delta, 1)) \ne \emptyset$ for any $q_e \in \partial H(e)$. For any $y_0 = x_\delta + t_0 q_e \in U \cap B(x_\delta, 1)$, by (5.32) and Lemma 3.7 one has

$$\begin{aligned} u(y_0) &\le u(z_3) + e \cdot (x_\delta - z_3) + \mathscr{C}^H_{H(e)}(y_0 - x_\delta) \\ &\le u(z_3) + e \cdot (x_\delta - z_3) + t_0 \mathscr{C}^H_{H(e)}(q_e) \\ &= u(z_3) + e \cdot (x_\delta - z_3) + t_0 e \cdot q_e \\ &= u(z_3) + e \cdot (y_0 - z_3), \end{aligned}$$

which contradicts with $y_0 \in U$, that $u(y_0) > u(z_3) + e \cdot (y_0 - z_3)$. This completes the proof of Lemma 5.3. □

Proof of (5.5). To see (5.5), define $v_r(x) = \frac{u(rx)}{r}$ for $x \in B(0, 6)$. Then we have

$$\sup_{x \in B(0,6)} |v_r(x) - e_{0,6} \cdot x| \le \delta.$$

Moreover, Lemma 5.1 also implies that $\{y \in \mathbb{R}^2 : u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B(0, 6r)$ has two connected components that intersect $B(0, r)$. This implies that for $z_3^r = \frac{z_3}{r}$, $\{y \in \mathbb{R}^2 : v_r(y) > v_r(z_3^r) + e \cdot (y - z_3^r)\} \cap B(0, 6)$ has two connected components that intersect $B(0, 1)$. Therefore similarly to the proof (5.4) above, we could also derive (5.5) provided that

ew2 (5.33)
$$\frac{1}{2} \le H(e_{0,6}) \le 4.$$

Below we verify (5.33). By Lemma 3.8 and (1.7), one has

xew13 (5.34)
$$u(x) \le e_6 \cdot x + \delta \le \mathscr{C}^H_{H(e_6)}(x) + \delta \le \mathscr{C}^H_{H(e_6) + C\delta}(x), \quad \forall x \in \partial B(0, 3).$$

Thus for $\delta \in (0, 2/C)$, we have

xew14 (5.35)
$$H(e_{0,6}) = S u(0) \le H(e_6) + C\delta \le 4.$$

On the other hand, we have $\psi_{2R}(\epsilon) \le 1/4$ for $0 < \epsilon < \epsilon_0$. Lemma 5.2 implies that for $\delta > 0$ sufficiently small, we have

ew3
$$H(e) \ge H(e_6) - \psi_{2R}(\epsilon) \ge 1 - \psi_{2R}(\epsilon) \ge \frac{3}{4}. \tag{5.36}$$

Since $e_{0,6} \in \mathscr{D}(u)(0; 6r; \delta)$, we have

eew
$$\begin{aligned} u(x) &\le u(z_3) + e_{0,6} \cdot (x - z_3) + 2\delta r \\ &\le u(z_3) + \mathscr{C}^H_{H(e_{0,6})}(x - z_3) + 2\delta r \le u(z_3) + \mathscr{C}^H_{H(e_{0,6})+C\delta}(x - z_3) \end{aligned} \tag{5.37}$$

for any $x \in \partial B(z_3, 2r) \subset B(0, 6r)$, and hence by comparison property with cones, for any $x \in B(z_3, 2r)$. By the definition of $S^+_{2r}(u)(z_3)$ and (5.37), if $C\delta < 1/4$ we have

ew10
$$S^+_{2r}(u)(z_3) \le H(e_{0,6}) + C\delta \le H(e_{0,6}) + \frac{1}{4}. \tag{5.38}$$

By Lemma 2.5 and Lemma 2.1, we have

$$H(e) = S^+(u)(z_3) \le S^+_{2r}(u)(z_3),$$

which together with (5.38) yields $H(e_{0,6}) \ge 1/2$ as desired. The proof of (5.5) is complete.

6. Proofs of Theorem 1.5 and (*i*) $\Leftrightarrow$ (*iii*) or (*iv*) in Theorem 1.1

In the following Corollary 6.1, the condition (1.6) needed in Proposition 1.4 is reduced to $\|H(Du)\|_{L^\infty(B(0,6))} \le R$, that is, u is may be linear in some neighborhood of 0.

xcor
Corollary 6.1. *Suppose that H satisfies* (1.4) *with $n = 2$ and (i). Let $R \ge 1$. For any $\epsilon > 0$, there exists $\delta = \delta(R, \epsilon) > 0$ such that for any e_6 and $e_{0,6}$, if we can find a $u \in AM_H(B(0,6))$ satisfying $\|H(Du)\|_{L^\infty(B(0,6))} \le R$,* (1.7) *and* (1.8)*, then we have*

xxlin3
$$|e_6 - e_{0,6}| \le \epsilon. \tag{6.1}$$

Proof. Let $u \in AM_H(B(0,6))$ satisfy $\|H(Du)\|_{L^\infty(B(0,6))} \le R$, (1.7) and (1.8). If u is not linear in any neighborhood of 0, this follows from Proposition 1.4. Below, we assume that $u \equiv e \cdot x$ near 0 for some vector $e \in \mathbb{R}^2$. Let $r_0 \in (0, 6)$ be the largest $r \in (0, 6)$ so that $u(x) = u(0) + e \cdot x$ in $B(0, r)$. Obviously, $u(x) = u(0) + e \cdot x$ in $B(0, r_0)$. By (1.8) one has

$$\sup_{B(0,6r)} |e \cdot y - e_{0,6} \cdot y| \le \delta r$$

that is, $\|e - e_{0,6}\| \le \delta/6$.

If $r_0 > 1/2$, by (1.7) we have

$$\sup_{B(0,6)} |e \cdot x - e_6 \cdot x| \le \delta,$$

that is, $\|e - e_6\| \le \delta/6$. Therefore, $\|e_{0,6} - e_6\| \le \delta/3 \le \epsilon$ whenever $\delta < \epsilon$.

If $r_0 < 1/2$, by the choice of r_0 we can find $x_0 \in \partial B(0, r_0)$, $u \ne e \cdot x$ in some neighborhood. Moreover, we see that u is not linear in any neighborhood of x^0. Indeed, if $u(x) = u(x^0) + p \cdot (x - x^0)$ in $B(x^0, s)$ for some p and $s > 0$, then $(e - p) \cdot (x - x^0) = 0$ in $B(0, r_0) \cap B(x^0, s)$. If $e - p \perp x^0$, then we can find $z_j \in B(0, r_0) \cap B(z, x^0)$ such that $(z_j - x^0)/|z_j - x^0|$ converges to $(e - p)/|e - p|$ as $j \to \infty$, and hence $e - p = 0$, that is, $e = p$. If $e - p \not\perp x^0$, then we can find $w \in B(0, r_0) \cap B(x^0, s)$ such that either $(w - x^0)/|w - x^0|$ or $(x^0 - w)/|w - x^0|$ equals to $(e - p)/|e - p|$, and hence $e = p$. But this is impossible by the choice of x^0.

Next we have $H(e) = Su(x^0)$ and $e \in \mathscr{D}(u)(x^0; r; \delta)$ for any $\delta > 0$ and sufficiently small $r > 0$. Indeed, by Theorem 1.1 (ii), for any sequence $\mathbf{r} = \{r_j\}$ which converges to 0, up to considering its subsequence, there exists a vector $e_{x_0,\mathbf{r}}$ such that $H(e_{x_0,\mathbf{r}}) = Su(x^0)$ and

$$\lim_{j\to\infty} \sup_{y \in B(0,1)} \frac{|u(y + x_0) - u(x_0) - e_{x_0,\mathbf{r}} \cdot y|}{r_j} = 0.$$

Since $u(y) = e \cdot y \in \overline{B(0, r_0)}$, we have

$$\lim_{r_j \to 0} \sup_{y \in B(x_0, r_j) \cap B(0, r_0)} \frac{|(e - e_{x_0,\mathbf{r}}) \cdot (y - x_0)|}{r_j} = 0.$$

If $e - e_{x_0,\mathbf{r}} \perp x_0$, then we can find vectors $p_j \in \partial B(x_0, r_j) \cap B(0, r_0))$ so that $p_j/|p_j| \to \frac{e - e_{x_0,\mathbf{r}}}{|e - e_{x_0,\mathbf{r}}|}$. Thus $|e - e_{x_0,\mathbf{r}}| = 0$, that is, $e = e_{x_0,\mathbf{r}}$. If $e - e_{x_0,\mathbf{r}} \not\perp x_0$, then we can find vectors $p \in \partial B(x_0, r_j) \cap B(0, r_0))$ so that either $w/|w|$ or $-w/|w|$ is exactly $\frac{e - e_{x_0,\mathbf{r}}}{|e - e_{x_0,\mathbf{r}}|}$, and hence $e = e_{x_0,\mathbf{r}}$. Thus $H(e) = Su(x^0)$ and, for any $\delta > 0$, $e \in \mathscr{D}(u)(x^0; r; \delta)$ for sufficiently small $r > 0$.

We also note that (1.7) implies

$$|u(y) - u(x_0) - e_6 \cdot (y - x_0)|_{L^\infty(B(x_0,5))} \le 2\delta, \quad 1 \le H(e_6) \le 2.$$

That is, $e_6 \in \mathscr{D}(u)(x^0; 5/6; 2\delta)$.

Let $v(x) = \frac{6}{5}[u(\frac{5}{6}x + x_0) - u(x_0)]$ for $x \in B(0, 6)$. We see that $v \in AM_H(B(0, 6))$, $\|Dv\|_{L^\infty(B(0,6))} \le 2R$, $e_6 \in \mathscr{D}(v)(0; 1; 2\delta)$ with $1 \le H(e_6) \le 2$, and $e \in \mathscr{D}(v)(0; r; 2\delta)$ for some sufficiently small $r > 0$, $Sv(0) = H(e)$. By Proposition 1.4, if $\delta < \delta(H, 2R, 2\delta)$ we know that $|e_6 - e| \le \epsilon$. Thus $|e_6 - e_{0,6}| \le \epsilon$ as desired. □

Using Corollary 6.1 and Theorem 1.1 (ii) we are able to prove Theorem 1.5.

Proof of Theorem 1.5. Step 1. We prove that u is differentiable everywhere and $H(Du) = Su$ everywhere. Let $x^0 \in \Omega$. Up to considering $u(\cdot + x_0) - u(x^0)$, we may assume that $x^0 = 0$ and $u(0) = 0$. Up to considering $\frac{1}{r}u(rx)$ we may assume that $B(0, 2) \subset \Omega$.

To see the differentiability of u at 0, it suffices to find a vector e such that for any sequence $\{r_j\}_{j\in\mathbb{N}}$ which converges to 0 we have

c21 (6.2)
$$\lim_{j\to\infty} \sup_{y \in B(0,1)} \left| \frac{1}{r_j} u(r_j y) - e \cdot y \right| = 0,$$

and $H(e) = Su(0)$. Thus $Du(0) = e$. Indeed, if u is not differentiable at 0, then there exists a sequence s_j such that $\lim_{j\to\infty} \left| \frac{1}{s_j} u(s_j y) - e \cdot y \right| > 0$.

If $Su(0) = 0$, by Theorem 1.1 (ii) we know that (6.2) holds with $e = 0$.

If $Su(0) > 0$, up to considering $\widetilde{H} = \frac{1}{Su(0)} H$, we may assume that $Su(0) = 1$. By Theorem 1.1 (ii), for any sequence $\mathbf{r} = \{r_j\}_{j\in\mathbb{N}}$ which converges to 0, up to considering its subsequence we can find a vector $e_{\mathbf{r}}$ such that

c21x (6.3)
$$\lim_{j\to\infty} \sup_{y \in B(0,1)} \left| \frac{1}{r_j} u(r_j y) - e_{\mathbf{r}} \cdot y \right| = 0,$$

and $H(e_{\mathbf{r}}) = Su(0)$. Let $\mathbf{s} = \{s_k\}_{k\in\mathbb{N}}$ be any another sequence with vector $e_{\mathbf{s}}$ satisfying

c21xx (6.4)
$$\lim_{k\to\infty} \sup_{y \in B(0,1)} \left| \frac{1}{s_k} u(s_k y) - e_{\mathbf{s}} \cdot y \right| = 0$$

and $H(e_{\mathbf{s}}) = Su(0)$. To get (6.2), we only need to prove $e_{\mathbf{r}} = e_{\mathbf{s}}$.

For any j, set $v_j(y) = \frac{6}{r_j} u(r_j y/6)$ for all $y \in B(0, 6)$. Note that, for all $j \in \mathbb{N}$, we have $Sv_j(0) = Su(0)$ and

$$\|H(Dv_j)\|_{L^\infty(B(0,6))} = \|H(Du)\|_{L^\infty(B(0,r_j))} \le \|H(Du)\|_{L^\infty(B(0,1))} =: R < \infty.$$

For any $\delta > 0$, by (6.3) and (6.4), there exists a j_δ such that $e_{\mathbf{r}} \in \mathscr{D}(v_j)(0; 1; \delta)$ and $e_{\mathbf{s}} \in \mathscr{D}(v_j)(0; s_k/r_j; \delta)$ for all $j \ge j_\delta$ and large k with $s_k << r_j$. For any $\epsilon > 0$, applying Corollary 6.1 with $\delta < \delta(H, R, \epsilon)$, we obtain $|e_{\mathbf{s}} - e_{\mathbf{r}}| \le C\epsilon$. By the arbitrariness of ϵ, we have $e_{\mathbf{s}} = e_{\mathbf{r}}$ as desired.

Step 2. We prove (1.10) and hence the continuity of Du.

Suppose that the conclusion (1.10) in Theorem 1.5 is not correct. Then there exist $k_0 > 0$, $\epsilon_0 > 0$ so that we can find a sequence $\{s_j > 0\}$ and a family $u_j \subset AM_H(B(x_j, 2r_j))$ satisfying $s_j/r_j \to 0$,

c24 (6.5)
$$\|H(Du_j)\|_{L^\infty(B(x_j,2r_j))} \le k_0, \quad |Du_j(x_j) - Du_j(y_j)| \ge \epsilon_0 \text{ for some } y_j \in B(x_j, s_j).$$

Up to considering $\frac{1}{r_j}u_j(r_jx+x_j)-u_j(x_j)$ for $x\in B(0,1)$ we may assume that $x_j=0$, $u_j(0)=0$ $r_j=1$ and $s_j\to 0$. Since $\|H(Du_j)\|_{L^\infty(B(0,2))}\le k_0$ implies that $\|Du_j\|_{L^\infty(B(0,2))}\le C(H,k)$, we know that $u_j\to u_\infty$ locally uniformly in $B(0,2)$ as $j\to\infty$ (up to some subsequence). Note that $u_\infty\in AM_H(B(0,2))$, $u_\infty(0)=0$. By Step 1, u_∞ is differentiable in $B(0,2)$, and hence for any $\delta>0$, there exists $r_0>0$ such that we have

c25 (6.6)
$$\sup_{x\in B(0,r_0)}|u_\infty(x)-Du_\infty(0)\cdot x|\le\frac{\delta r_0}{2},\quad |Du_j(y_j)-Du_j(0)|\ge\epsilon_0.$$

Therefore there exists a sufficiently large $j_\delta>0$ such that

c26 (6.7)
$$\sup_{x\in B(0,r_0)}|u_j(x)-Du_\infty(0)\cdot x|\le\delta r_0,\quad \forall j\ge j_\delta.$$

This implies that for $j\ge j_\delta$, we have

c27 (6.8)
$$|u_j(x)-u_j(y)-Du_\infty(0)\cdot(y-x)|\le 2\delta r_0,\quad \forall x,y\in B(0,r_0).$$

In particular, for $j\ge j_\delta$, we have

c28 (6.9)
$$\sup_{x\in B(0,r_0/2)}|u_j(x+y_j)-u_j(y_j)-Du_\infty(0)\cdot x|\le 2\delta r_0.$$

If $Du_\infty(0)=0$, by (6.6) we have

$$u_j(x)\le\delta r_0=\delta|x|\le\mathscr{C}^H_{C\delta}(x)\quad\forall x\in\partial B(0,r_0)$$

and hence by Corollary 3.8, $u_j(x)\le\mathscr{C}^H_{C\delta}(x)\quad\forall x\in B(0,r_0)$. This implies that $H(Du_j(0))=Su_j(0)\le C\delta$. Similarly, by (6.9), one has $H(Du_j(y_j))=Su_j(y_j)\le C\delta$. If $\delta<<\epsilon$, this leads to a contraction with $|H(Du_j(y_j))-H(Du_j(0))|\ge\epsilon_0$.

If $Du_\infty(0)\ne 0$, up to considering $\widetilde{H}=\frac{1}{H(Du_\infty(0))}H$ we may assume that $H(Du_\infty(0))=1$. It suffices to show that for any $\epsilon>0$,

$$|Du_j(y_j)-Du_\infty(0)|\le C\epsilon,\quad |Du_j(0)-Du_\infty(0)|\le C\epsilon \text{ for sufficiently large } j.$$

If this is correct, letting $\epsilon<\epsilon_0/4C$ we get contradiction with $|H(Du_j(y_j))-H(Du_j(0))|\ge\epsilon_0$.

Below we only verify $|Du_j(y_j)-Du_\infty(0)|\le C\epsilon$ for large j; the proof of $|Du_j(0)-Du_\infty(0)|\le C\epsilon$ for large j is similar and easier. Let $v_j(x)=\frac{12}{r_0}u(r_0x/12+y_j)$. We have $\|H(Dv_j)\|_{L^\infty(B(0,6))}\le k_0$. Moreover, (6.9) implies $Du_\infty(0)\in\mathscr{D}(v_j)(0;1;2\delta)$ for $j\ge j_0$. Since v_j is differentiable at 0, we know that $Du(y_j)=Dv_j(0)\in\mathscr{D}(v_j)(0;r;2\delta)$ for sufficiently small $r>0$. For any $\epsilon>0$, applying Corollary 6.1 with $\delta<\delta(H,k_0,\epsilon)$ we know that $|Du_j(y_j)-Du_\infty(0)|\le C\epsilon$. Thus we complete proof of the Theorem 1.5. □

Withe the aid of Theorem 1.5 we prove $(i)\Leftrightarrow(iii)$ and $(i)\Leftrightarrow(iv)$ in Theorem 1.1 as below.

Proof of $(i)\Leftrightarrow(iii)$ and $(i)\Leftrightarrow(iv)$ in Theorem 1.1. As clarified in the introduction, we only need to prove Theorem 1.1 under the assumption (1.4).

Proof of $(i)\Rightarrow(iii)$. This follows from Theorem 1.5 directly.

Proof of $(i)\Rightarrow(iv)$. Let $u\in AM_H(\mathbb{R}^n)$ and assume that there exists some constant $K\ge 1$ such that $|u(x)|\le K(1+|x|)$ for all $x\in\mathbb{R}^n$. By Theorem 1.5, $u\in C^1(\mathbb{R}^n)$. Moreover we have $\|H(Du)\|_{L^\infty(\mathbb{R}^n)}<\infty$. Indeed, since Lemma 2.5 implies $H(Du(x))=Su(x)\le S_tu(x)$ for any $t>0$ and $x\in\mathbb{R}^n$, it suffices to show $S^+_tu(x)<C(K)$ whenever t is sufficiently large for any given $x\in\mathbb{R}^n$. By $|u(y)|\le K(1+|y|)$ for all $y\in\mathbb{R}^n$, if $t\ge(1+|x|)$, we have

$$S^+_tu(x)=\sup_{y\in\mathbb{R}^n}\left[\frac{u(y)-u(x)}{t}-L\left(\frac{y-x}{t}\right)\right]\le 2K+\sup_{y\in\mathbb{R}^n}\left[\frac{K|y-x|}{t}-L\left(\frac{y-x}{t}\right)\right]=2K+\sup_{z\in\mathbb{R}^n}[K|z|-L(z)].$$

... a 2.6,

$$S^+_tu(x)\le 2K+\sup_{|z|\le R_K}[K|z|-L(z)]\le 2K+KR_K$$

as desired.

Write $k = \|H(Du)\|_{L^\infty(\mathbb{R}^2)}$. For any $R > 0$ let $u_R(x) = u(Rx)/R$ for all $x \in \mathbb{R}^n$. Then $u_R \in AM_H(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$ and $\|H(Du_R)\|_{L^\infty(\mathbb{R}^2)} = k$. By Theorem 1.5 again,

$$|Du(x) - Du(0)| = \limsup_{R\to\infty} \left| Du_R\left(\frac{x}{R}\right) - Du(0)\right| \le \limsup_{R\to\infty} \rho_k\left(\frac{|x|}{R}\right) = 0 \quad \forall x \in \mathbb{R}^n.$$

This implies that $Du \equiv Du(0)$ and hence $u(x) = u(0) + Du(0)\cdot x$ for all $x \in \mathbb{R}^n$.

Proofs of (iii) *or* $(vi) \Rightarrow (i)$. Assume that $H^{-1}(k) = [a, b]$ with $a \neq b$ for some $k \geq 0$. It suffices to show that (iii) and (iv) fail.

Let f and u_f be as in the proof of $(ii) \Rightarrow (i)$. Obviously, $u_f \notin C^1(\mathbb{R}^n)$ and hence (iii) fails. This gives $(iii) \Rightarrow (i)$. Note also that $\|H(Du_f)\|_{L^\infty(\mathbb{R}^n)} = k$ implies that $\|Du_f\|_{L^\infty(\mathbb{R}^n)} < \infty$ and hence u_f enjoys a linear growth. But, obviously, u_f is not a linear function and hence (iv) fails. This gives $(iv) \Rightarrow (i)$. The proof of Theorem 1.1 is complete. □

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Department of Mathematics, Beihang University, Beijing 100191, P.R. China
E-mail address: SY1609131@buaa.edu.cn

Department of Mathematics, Purdue University, 150 N. University Street West Lafayette, IN 47907-2067, US
E-mail address: wang2482@wang2482@purdue.edu

Department of Mathematics, Beihang University, Beijing 100191, P.R. China
E-mail address: yuanzhou@buaa.edu.cn