Green-Lazarsfeld Condition for Toric Edge Ideals of Bipartite Graphs

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Abstract

Previously, Ohsugi and Hibi gave a combinatorial description of bipartite graphs G whose toric edge ideal I_G is generated by quadrics, showing that every cycle of G of length at least 6 must have a chord. This corresponds to the Green-Lazarsfeld condition $\mathbf{N}_1$. In this paper, we investigate the higher syzygies of I_G and give combinatorial descriptions of the Green-Lazarsfeld conditions $\mathbf{N}_p$ of toric edge ideals of bipartite graphs for all $p \geq 1$. In particular, we show that I_G is linearly presented (i.e. satisfies condition $\mathbf{N}_2$) if and only if the bipartite complement of G is a tree of diameter at most 3. We also investigate the regularity of linearly presented toric edge ideals and give criteria for polyomino ideals to satisfy the Green-Lazarsfeld conditions.

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1. Introduction

Let k be a field and let $G = (V, E)$ be a finite, simple graph. Let $k[V]$ denote the polynomial ring with variables corresponding to the vertices of G. The edge ring $k[G]$ of G is the k-subalgebra of $k[V]$ generated by the quadratic monomials corresponding to the edges of G. The toric edge ideal I_G is the presenting ideal of $k[G]$ in the polynomial ring $k[E]$ whose variables correspond to the edges of G. In particular, I_G is a homogeneous prime ideal generated by binomials. When G is a complete bipartite graph, I_G defines a Segre embedding. Such ideals are special cases of toric ideals where one finds defining equations of ideals generated by any set of monomials; the restriction to toric edge ideals corresponds to considering only subrings generated by squarefree monomials of degree two. There has been significant interest in understanding the minimal free resolutions of I_G for different classes of graphs; see e.g. [3, 15, 24, 25].

If G is a bipartite graph, then more is known about I_G. The following result is due to Ohsugi and Hibi:

Theorem 1.1 ([25, Theorem 1]). *Let G be a bipartite graph. The following are equivalent:*

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1. *Every cycle in G of length ≥ 6 has a chord.*

2. *I_G has a Gröbner basis consisting of quadratic binomials.*

3. *$k[G]$ is Koszul.*

4. *I_G is generated by quadratic binomials, corresponding to the 4-cycles of G.*

One can generalize the property of having a quadratic generating set by considering the degrees of syzygies of I_G over S. The Green-Lazarsfeld condition $\mathbf{N_p}$ describes ideals (defining normal quotient rings) generated by quadrics with linear syzygies for the first $p-1$ steps of the resolution. If G is a bipartite graph, then Theorem 1.1 says that I_G satisfies property $\mathbf{N}_1$ if and only if every cycle in G of length ≥ 6 has a chord. The main goal of this paper is to give a combinatorial description of when I_G satisfies property $\mathbf{N}_p$ for all $p \geq 0$. We first need a couple definitions to state our main result.

Let $G = (V, E)$ be a bipartite graph and let $V = X \sqcup Y$ be a partition of V so that $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$ and all edges $e \in E$ are of the form $e = \{x_i, y_j\}$ for some $1 \leq i \leq m$ and $1 \leq j \leq n$.

We define the *bipartite complement* of G as the bipartite graph $\overline{G} = (X \sqcup Y, E')$, where $E' = (X \times Y) \setminus E$, viewed as sets. A graph is *essentially a tree* if it is a tree after perhaps removing some isolated vertices; for a formal definition, see the following section.

Our main theorem is a combinatorial characterization of toric ideals of bipartite graphs which satisfy property $\mathbf{N}_p$ for arbitrary $p \geq 1$.

Theorem 1.2. *Let G be a bipartite graph with minimum vertex degree at least 2 and let k be a field.*

1. *I_G satisfies property $\mathbf{N}_1$ if and only if every cycle of length ≥ 6 has a chord.*

2. *I_G satisfies property $\mathbf{N}_2$ if and only if $\overline{G}$ is essentially a tree of diameter at most 3.*

3. *I_G satisfies property $\mathbf{N}_3$ if and only if G is a complete bipartite graph unless the characteristic of k is 3 and $G = K_{m,n}$ with $\min\{m, n\} \geq 5$.*

4. *I_G satisfies property $\mathbf{N}_p$ for some/any $p \geq 4$ if and only if $G = K_{2,n}$ for some n.*

The proof of Theorem 1.2 is given in the proofs of Theorems 5.2, 5.5, and 5.6.

The rest of this paper is organized as follows: Section 2 sets notation and basic definitions. Section 3 contains our main tools for finding obstructions to vanishing of graded Betti numbers. Section 4 gives a purely graph-theoretic result we need to connect local and global graph structure. Our main results appear in Section 5. In Section 6, we also obtain a characterization of the Green-Lazarsfeld conditions for ideals associated to convex polyominoes. This seems to correct an omission in the characterization of linearly presented polyomino ideals in [10]. Finally, in Section 7 we apply our result to a special case of a recent question of Constantinescu, Kahle, and Varbaro [6].

2. Preliminaries

Here we fix notation for the remainder of the paper. We first record the standard graph-theoretic definitions we require.

2.1. *Graph Theory*

All graphs considered in this paper are finite and simple. Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E. The graph G is bipartite if there is a partition $V = X \sqcup Y$ such that all edges in E lie in $X \times Y$; that is, all edges contain one vertex in X and one vertex in Y. For positive integers m, n, the complete bipartite graph $K_{m,n}$ has vertex set $V = X \sqcup Y$ with $|X| = m$, $|Y| = n$ and edge set $E = X \times Y$. The degree of a vertex is the number of edges incident to it. The minimum degree of a vertex in a graph G is denoted $\delta(G)$. An isolated vertex is a vertex of degree 0. A path of length t from vertex v to vertex w is a sequence of vertices $v = v_0, v_1, \ldots, v_t = w$ such that $\{v_{i-1}, v_i\} \in E$ for all $1 \leq i \leq t$. A graph is connected if for any two vertices $v, w \in V$, there is a path from v to w. A cycle of length t in G is a path of length t from v to itself. Such a cycle has a chord if $\{v_i, v_j\} \in E$ for some $1 \leq i < j \leq t$ with $j - i \geq 2$. A graph G is a tree if there is a unique path between any two distinct vertices of G, or equivalently, G is a tree if it is connected and has no cycles. Given a subset $W \subseteq V$, the induced graph G_W is the graph with vertex set W and edge set given by all edges of G both of whose vertices lie in W. The diameter of a graph is the minimum integer n such that for any pairs of vertices $v, w \in V$, there is a path of length at most n starting at v and ending at w. A perfect matching in a graph is a collection $M \subseteq E$ such that every vertex is incident to exactly one edge in M. A bridge is an edge whose deletion increases the number of connected components.

We add some new graph-theoretic definitions to the standard definitions above. For a nonnegative integer k and graph G, we define the **degree k subgraph** of G to be the largest induced subgraph G_k such that all vertices have degree at least k. Thus $G_0 = G$; G_1 is the subgraph of G with all isolated vertices removed. For a graph property P, we say that a graph is **essentially P** if G_1 satisfies property P. Thus a graph G is called **essentially a tree** if G_1 is a tree. If $G = (X \sqcup Y, E)$ is a bipartite graph, then the **bipartite complement** of G, denoted $\overline{G}$, is the graph with same vertex set $X \sqcup Y$ and with edge set $(X \times Y) \setminus E$; that is, an edge $\{x, y\}$ with $x \in X$ and $y \in Y$ is in $\overline{G}$ if and only if it is not in G.

2.2. *Toric Edge Ideals*

Let $G = (V, E)$ be a finite simple graph, with $V = \{v_1, \ldots, v_n\}$ and fix a field k. By abuse of notation, we also view the v_i as variables in the polynomial ring $k[V] = k[v_1, \ldots, v_n]$. The **edge ring** of G, denoted $k[G]$, is the k-subalgebra of $k[V]$ generated by $v_i v_j$, where $\{v_i, v_j\} \in E$. In the special case we focus on, where G is a bipartite graph, we denote by $V = X \sqcup Y$ the partition of the vertex set, where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ and all edges (viewed as ordered pairs) are contained in $X \times Y$. Denote by $S = k[e_{i,j} \,|\, \{x_i, y_j\} \in E]$ a polynomial ring with variables $e_{i,j}$ corresponding to the edges in G. The surjective map $\pi : S = k[e_{i,j} \,|\, \{x_i, y_j\} \in E] \to k[G]$ sends $e_{i,j} \mapsto x_i y_j$. The ideal $I_G = \mathrm{Ker}(\pi)$ is called the **toric edge ideal** of G. For an arbitrary graph G, it is well-known that the generators of I_G

are binomials corresponding to even closed walks of G [18, Lemma 5.9]. When G is bipartite, the ring $S/I_G \cong k[G]$ is Cohen-Macaulay [18, Corollary 5.26]. Ohsugi and Hibi [25] gave the characterization in Theorem 1.1 of bipartite graphs for which I_G is generated by quadratic binomials. It follows that all such rings are normal [18, Corollary 5.25]. It is then natural to investigate the properties of the syzygies of such ideals.

2.3. Green-Lazarsfeld Conditions

Unless otherwise noted, we regard S as a standard graded ring with $\deg(e_{i,j}) = 1$ for all i, j. Writing $S(-j)$ for the rank-one free S-module with $S(-j)_i = S_{i-j}$, we consider the minimal graded free resolution of S/I_G:

$$0 \to \bigoplus_j S(-j)^{\beta_{h,j}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}} \to S.$$

Here $\beta_{i,j}$ denotes the minimal, graded Betti numbers of S/I_G, which by the uniqueness of minimal, graded free resolutions, are invariants of S/I_G. The projective dimension is $\mathrm{pd}_S(S/I_G) = \max\{i \mid \beta_{i,j} \neq 0\} = h$ and the regularity is $\mathrm{reg}(S/I_G) = \max\{j - i \mid \beta_{i,j} \neq 0\}$. We often refer to the graded Betti numbers of I_G, noting that $\beta_{i,j}(S/I_G) = \beta_{i-1,j}(I_G)$, and therefore $\mathrm{pd}_S(S/I_G) = \mathrm{pd}_S(I_G) + 1$ and $\mathrm{reg}(S/I_G) = \mathrm{reg}(I_G) - 1$.

With the notation above, we say that I_G satisfies condition $\mathbf{N}_p$ if S/I_G is (projectively) normal and $\beta_{i,j}(I_G) = 0$ for $i < p$ and $j > i + 2$. Thus condition $\mathbf{N}_0$ means that S/I_G is normal; condition $\mathbf{N}_1$ means that in addition to $\mathbf{N}_0$, I_G is generated by quadrics; condition $\mathbf{N}_2$ means that in addition to satisfying $\mathbf{N}_1$, I_G is linearly presented; and so on. This idea was first defined by Green and Lazarsfeld [12, 13]. The $\mathbf{N}_p$ conditions and their generalizations have been well studied; see for example [9, 20].

Note that in the specific case that $G = K_{m,n}$, the ideal I_G defines to the image of the Segre embedding of $\mathbb{P}_k^{m-1} \times \mathbb{P}_k^{n-1} \hookrightarrow \mathbb{P}_k^{mn-1}$ whose resolutions in characteristic 0 are known by work of Pragacz-Weyman [29] and Lascoux [21]; see also Roberts [31]. If $\min\{m, n\} \leq 4$, Hashimoto and Kurano showed that the Betti numbers of I_G do not depend on the characteristic [17]. In particular, this includes $K_{2,n}$ whose toric edge ideal $I_{K_{2,n}}$ is resolved by the linear Eagon-Northcott resolution in all characteristics. For all m, n, the second Betti numbers $\beta_{2,i}(S/I_{K_{m,n}})$ are also independent of the characteristic [17]. However, in characteristic 3, Hashimoto [16] showed that $\beta_{3,i}(S/I_{K_{m,n}})$ does depend on the characteristic of the base field when $m, n \geq 5$. In this paper, we give a complete description of the Green-Lazarsfeld conditions for bipartite toric edge ideals. It follows from [21, 29, 31] that the precise $\mathbf{N}_p$ conditions for complete bipartite graphs in characteristic 0 are known; see [31] for a summary.

When I_G is the toric edge ideal of a bipartite graph, Ohsugi and Hibi [25, Theorem 1.1] proved that I_G is generated by quadratic binomials (i.e. satisfies condition $\mathbf{N}_1$) if and only if every cycle in G of length at least 6 has a chord. Ohsugi and Hibi [24, Theorem 4.6] also showed that I_G has a linear free resolution (i.e. satisfies condition $\mathbf{N}_p$ for all p) if and only if $G = K_{2,n}$ for some n. Thus our main theorem interpolates between these two results. In related work, Hibi, Matsuda, and Tsuchiya [19] show that the only toric edge ideals with 3-linear resolutions are hypersurfaces.

3. Obstructions to Vanishing of Graded Betti Numbers

In this section we prove that the nonvanishing of certain graded Betti numbers of the toric edge ideal of a graph G correspond in a precise way to forbidden induced subgraphs of G. A version of this result was proved previously by Ha, Kara, and O'Keefe in [15, Theorem 3.6]. Our result quantifies how large the forbidden subgraph must be relative to the index of the graded Betti number in question. Toward this end, we follow the notation in [28]. Let $G = (V, E)$ be a finite simple graph on vertex set $V = \{v_1, \dots, v_n\}$. Given a field k, the edge ring $k[G] = k[v_i v_j \mid \{v_i, v_j\} \in E]$ is a subring of the polynomial ring $k[V]$, which we view as a multigraded ring by setting $\mathrm{mdeg}(v_i) = \underline{e}_i$, where $\underline{e}_i$ denotes the ith standard basis vector of $\mathbb{Z}^n$. By setting $k[E] = k[e_{ij} \mid \{v_i, v_j\} \in E]$ to be the multigraded ring with $\mathrm{mdeg}(e_{ij}) = \mathrm{mdeg}(v_i v_j) = \underline{e}_i + \underline{e}_j$, the toric edge ideal $I_G \subset k[E]$ is also multigraded. Fix a multidegree $\underline{\alpha}$. The fiber of $\underline{\alpha}$, denoted $C_{\underline{\alpha}}$, is the set of all monomials of $k[E]$ of multidegree $\underline{\alpha}$. We let $\Gamma(\underline{\alpha})$ denote the simplicial complex associated to $\underline{\alpha}$ with vertices identified with the variables e_{ij} and whose faces are identified with the radicals of monomials in $\Gamma(\underline{\alpha})$. With this notation, we have the following result of Aramova and Herzog:

Theorem 3.1 (cf. [28, Theorem 67.5]). *For $\underline{\alpha} \in \mathbb{N}^n$ and $i \geq 0$ we have*

$$\beta_{i,\underline{\alpha}}(I_G) = \dim_k \widetilde{H}_i(\Gamma(\underline{\alpha}); k).$$

Here $\widetilde{H}_i(\Delta; k)$ denotes the reduced simplicial homology of the simplicial complex Δ with coefficients in k. Comparing the standard grading on I_G with the multigrading, we see that

$$\beta_{i,j}(I_G) = \sum_{\substack{\underline{\alpha} \\ \sum \underline{\alpha} = 2j}} \beta_{i,\underline{\alpha}}(I_G).$$

This perspective gives us a way of finding local obstructions to the vanishing of certain graded Betti numbers of I_G.

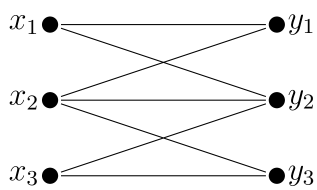
Theorem 3.2. *Let G be a graph with toric edge ideal I_G. Then $\beta_{i,j}(I_G) \neq 0$ if and only if there is an induced subgraph H of G with at most $2j$ vertices such that $\beta_{i,j}(I_H) \neq 0$.*

Proof. Let $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, v_r\}$. Suppose $\beta_{i,j}(I_G) \neq 0$. Then $\beta_{i,\underline{\alpha}}(I_G) \neq 0$ for some multidegree $\underline{\alpha}$ such that $\sum_\ell \alpha_\ell = 2j$. Let $V' = \{v_\ell \in V : \alpha_\ell \neq 0\}$. Since at most $2j$ of the α_ℓ are nonzero, we have $|V'| \leq 2j$. Let $H = (V', E')$ be the induced subgraph of G on V'. $k[E']$ is a subring of S, so it is $\mathbb{Z}^n$-graded. Let $C^G_{\underline{\alpha}}$ and $C^H_{\underline{\alpha}}$ denote the fibers of α in $k[E]$ and $k[E']$ respectively. Since $k[H]$ is a subring of $k[G]$, we know $C^H_{\underline{\alpha}} \subseteq C^G_{\underline{\alpha}}$.

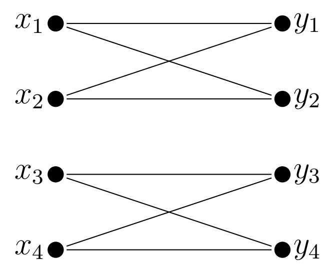
Suppose $f = e_1^{m_1} e_2^{m_2} \cdots e_r^{m_r} \in C^G_{\underline{\alpha}} \setminus C^H_{\underline{\alpha}}$. Since $f \notin k[E']$, $m_\ell > 0$ for some ℓ such that $e_\ell \notin E'$. Since H is induced, $e_\ell = \{v_{\ell_1}, v_{\ell_2}\}$ where at least one of v_{ℓ_1} and v_{ℓ_2} is not in V'. Suppose, w.l.o.g. $v_{\ell_1} \notin V'$. Since f has multidegree $\underline{\alpha}$, we have $\alpha_{\ell_1} \neq 0$, giving that $v_{\ell_1} \in V'$, a contradiction. Thus no such f exists and $C^G_{\underline{\alpha}} = C^H_{\underline{\alpha}}$. Then the associated simplicial complexes are the same and we must have $\beta_{i,\underline{\alpha}}(I_H) \neq 0$, giving $\beta_{i,j}(I_H) \neq 0$. $\square$

It follows that to characterize when a particular $\beta_{i,j}(I_G) = 0$, we could simply enumerate over all graphs H of at most $2j$ vertices for which $\beta_{i,j}(I_H) \neq 0$ and then check to see if any such H is an induced subgraph of G. While this strategy would work, it is not very efficient. In the next section adopt a more efficient strategy taking advantage of the fact that quadratic toric edge rings of bipartite graphs are Koszul. However, we do take advantage of the previous theorem by identifying a few key induced subgraphs that act as obstructions to satisfying the N_p property for various p.

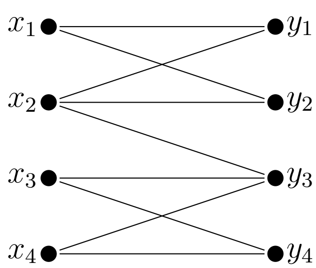
There are 8 bipartite graphs whose presence as an induced subgraph characterizes failure of I_G satisfying $\mathbf{N}_2$ for G a bipartite graph and such that every cycle of length at least 6 has a chord. These 8 forbidden graphs are those pictured in Figure 1.



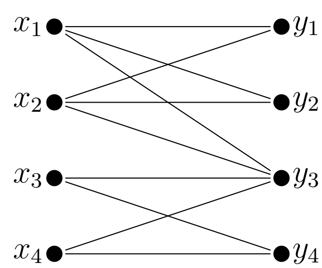
$H^{(1)}$: Two 4-cycles which share an edge.



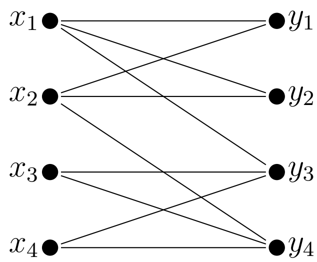
$H^{(2)}$: Two disjoint 4-cycles.



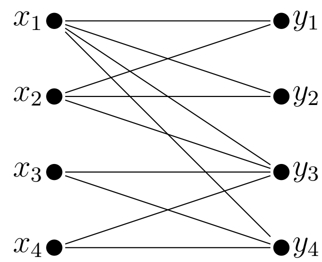
$H^{(3)}$: Two 4-cycles connected by a single edge.



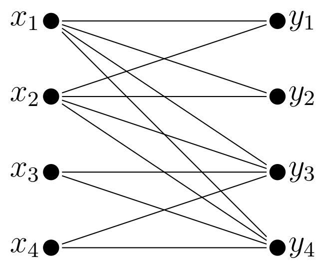
$H^{(4)}$: Two 4-cycles connected by two adjacent edges.



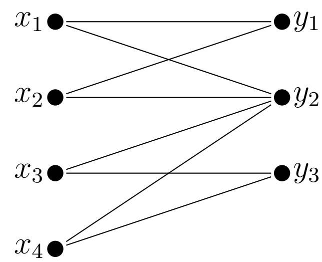
$H^{(5)}$: Two 4-cycles connected by two non-adjacent edges.



$H^{(6)}$:Two 4-cycles connected by three connected edges.



$H^{(7)}$: Two 4-cycles connected by 4 connected edges.



$H^{(8)}$: Two 4-cycles which share exactly one vertex.

Figure 1: Induced subgraphs that are obstructions to satisfying condition $\mathbf{N_2}$

Lemma 3.3. *$I_{H^{(1)}}$ and $I_{H^{(8)}}$ do not satisfy condition $\mathbf{N}_2$; specifically, $\beta_{1,4}(I_{H^{(1)}})$ and $\beta_{1,4}(I_{H^{(8)}})$ are nonzero.*

Proof. The ideals $I_{H^{(1)}}$ and $I_{H^{(8)}}$ are complete intersections generated by two quadrics. In particular $\beta_{1,4}(I_{H^{(1)}}) = 1$.

$\square$

Lemma 3.4. *For each $i = 2, \ldots, 7$, the ideal $I_{H^{(i)}}$ does not satisfy condition $\mathbf{N}_2$; specifically, $\beta_{1,4}(I_{H^{(i)}}) \neq 0$.*

Proof. Let $E^{(i)}$ be the edge set of $H^{(i)}$. Let e_{ij} denote the edge $\{x_i, y_j\}$. Let $\underline{\alpha}$ be the multidegree $(1,1,1,1,1,1,1,1)$. In each $k[E^{(i)}]$, the monomials in multidegree $\underline{\alpha}$ correspond to perfect matchings in $H^{(i)}$. Note that perfect matchings in $H^{(i)}$ for $i \geq 3$ cannot contain any of the edges e_{13}, e_{14}, e_{23}, and e_{24}, so the perfect matchings of $H^{(i)}$ are precisely the same as those of $H^{(2)}$. It follows that in each of these graphs,

$$C_{\underline{\alpha}} = \{e_{11}e_{22}e_{33}e_{44},\ e_{11}e_{22}e_{34}e_{43},\ e_{12}e_{21}e_{33}e_{44},\ e_{12}e_{21}e_{34}e_{43}\},$$

so $\Gamma(\underline{\alpha})$ has facets

$$\{\{e_{11}, e_{22}, e_{33}, e_{44}\},\ \{e_{11}, e_{22}, e_{34}, e_{43}\},\ \{e_{12}, e_{21}, e_{33}, e_{44}\},\ \{e_{12}, e_{21}, e_{34}, e_{43}\}\}.$$

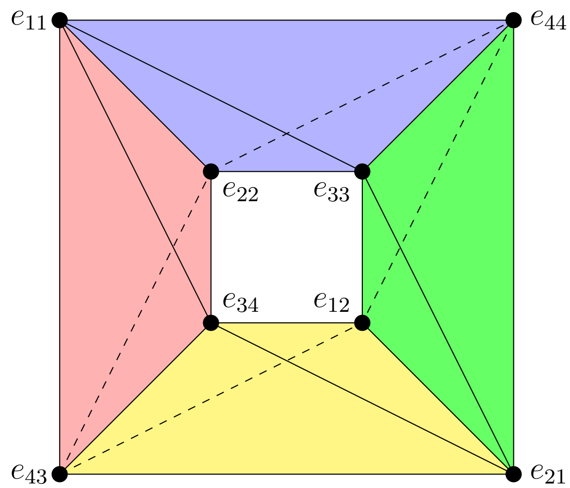


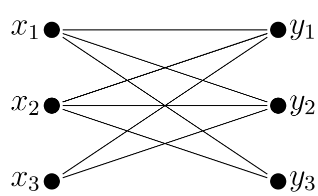
Figure 2: A visualization of $\Gamma(\underline{\alpha})$ in which each shaded tetrahedron represents a facet.

The geometric realization of the abstract simplicial complex is $\Gamma(\underline{\alpha})$ contracts to a circle. Thus $\beta_{1,\underline{\alpha}}(I_{H^{(i)}}) = \dim_k \widetilde{H}_1(\Gamma(\underline{\alpha}); k) = 1$. By Theorem 3.2, $\beta_{1,4}(I_{H^{(i)}}) \neq 0$ for $i = 2, \ldots, 7$. $\square$

Note that $H^{(2)}$ and $H^{(3)}$ are complete intersections and could similarly fit in Lemma 3.3. We also note that $H^{(5)}$ has cubic generators, in addition to the two obvious quadratic ones, because it has 6-cycles without a chord; however, including it here makes arguments later in the paper easier to state.

The following graph is the main obstruction to satisfying condition $\mathbf{N}_3$.

Lemma 3.5. *Let H be the bipartite graph pictured below.*



Then I_H is Gorenstein and has graded Betti table:

	0	1	2
2:	5	5	-
3:	-	-	1

In particular, $\beta_{2,5}(I_H) \neq 0$ and so I_H does not satisfy condition N_3.

Proof. It is easy to check that I_H is generated by the 4×4 Pfaffians of the 5×5 alternating matrix:

$$\mathbf{M} = \begin{pmatrix} 0 & e_{13} & e_{23} & e_{31} & e_{32} \\ -e_{13} & 0 & 0 & e_{11} & e_{12} \\ -e_{23} & 0 & 0 & e_{21} & e_{22} \\ -e_{31} & -e_{11} & -e_{21} & 0 & 0 \\ -e_{32} & -e_{12} & -e_{22} & 0 & 0 \end{pmatrix}.$$

By [4, Theorem 2.1], it follows that I_H is a Gorenstein, height 3 ideal. The claim follows from the symmetry of resolutions of Gorenstein ideals. □

Lemma 3.6. *The ideal $I_{K_{3,3}}$ is Gorenstein and has graded Betti table:*

	0	1	2	3
2:	9	16	9	-
3:	-	-	-	1

In particular, $\beta_{3,6}(I_{K_{3,3}}) \neq 0$ and so $I_{K_{3,3}}$ does not satisfy condition $\mathbf{N}_4$.

Proof. The ideal $I_{K_{3,3}}$ is generated by the 2×2 minors of a generic 3×3 matrix of linear forms and so is Gorenstein and has the above resolution by [14]. □

4. A Graph-theoretic Result

This section contains a purely combinatorial characterization of the types of graphs which we show in the following section define linearly presented toric edge ideals. We show that trees of diameter at most 3 can be characterized locally by the absence of certain induced subgraphs on 4 vertices. We then show that a bipartite graph with minimum vertex degree at least 2 such that every cycle of size 6 or greater has a chord and such that its bipartite complement is a tree of diameter at most 3 can also be characterized by the absence of 8 particular graphs on at most 8 vertices.

Proposition 4.1. *Let G be a graph. Then G is a tree of diameter at most 3 if and only if every induced subgraph is essentially connected and has no cycles.*

Proof. Suppose G is a tree of diameter at most 3. Since G has no cycles, clearly the same is true for any induced subgraph. Since the diameter of G is at most 3, either G has no edges or there exist two vertices v_1, v_2 such that every edge of G is incident to v_1 or v_2. Let H be an induced subgraph of G. If H contains neither v_1 nor v_2, then H contains no edges and so is essentially connected. If H contains exactly one of these vertices, say v_1 but not v_2, then H_1 consists only of edges incident to v_1 and isolated vertices, in which case H is also essentially connected. Finally if H contains both v_1 and v_2, then all edges of H are incident to v_1 or v_2 and so H is essentially connected.

The converse follows easily since G is connected and has no cycles by assumption. □

Note that there is no similar statement for trees of diameter at most 4. Indeed, consider a path graph with 4 edges and 5 vertices $v_1, \ldots, v_5$. Then the induced subgraph on vertex set $\{v_1, v_2, v_4, v_5\}$ is not essentially connected.

The following result is our main combinatorial result that allows us to take local obstructions in the form of forbidden induced subgraphs and translate them into a global statement about certain bipartite graphs.

Theorem 4.2. *Let G be a bipartite graph with $\delta(G) \geq 2$, such that each cycle of length ≥ 6 has a chord and $(\overline{G})_1$ is not a tree of diameter at most 3. Then G contains $H^{(i)}$ for some $1 \leq i \leq 8$ as an induced subgraph.*

Proof. It follows from the previous proposition that $(\overline{G})_1$ contains a 4-cycle or two nonadjacent edges. Before handling these two cases (Cases 4 and 5 below), we first prove intermediate cases.

Case 1: *G is disconnected.*

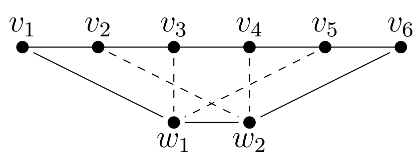
Because G has $\delta(G) \geq 2$, each connected component must have a cycle. Since each cycle of length ≥ 6 has a chord, each connected component must have a 4-cycle. Taking the induced subgraph on two four cycles from distinct connected components will then yield $H^{(2)}$.

Case 2: *G has a bridge.*

Removing the bridge results in a graph with two connected components, each of which has at most one vertex of degree 1 and all other vertices of degree at least 2. Then each connected component must have a 4-cycle, so the graph contains an induced copy of $H^{(2)}$.

Case 3: *G has a path with 5 edges as an induced subgraph.*

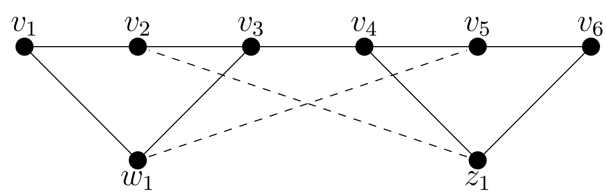
Denote the induced path by $v_1, v_2, v_3, v_4, v_5, v_6$. We emphasize that this path is induced, so there can not be any edges $\{v_i, v_j\}$ for $|i - j| \geq 2$ in the entire graph G. First we assume that there is a second path v_1, w_1, w_2, v_6 where the w_i are distinct from the v_j. These two paths form an 8-cycle.



Because G is bipartite, there are four possible chords in the 8-cycle: $\{v_2, w_2\}, \{v_3, w_1\}$, $\{v_4, w_2\}$, and $\{v_5, w_1\}$. If both of the chords $\{v_2, w_2\}$ and $\{v_5, w_1\}$ are present, then the induced subgraph on the vertices $\{v_1, v_2, v_5, v_6, w_1, w_2\}$ is $H^{(1)}$. So we may assume w.l.o.g. that $\{v_2, w_2\}$ is not present. In this case, the chord $\{v_3, w_1\}$ must be present, as must at least one more chord. If the chord $\{v_4, w_2\}$ is present, then the induced subgraph on $\{v_1, v_2, v_3, v_4, w_1, w_2\}$ is $H^{(1)}$. Otherwise, the chord $\{v_5, w_1\}$ is present, in which case the induced subgraph on $\{v_1, v_2, v_3, v_4, v_5, w_1\}$ is $H^{(1)}$.

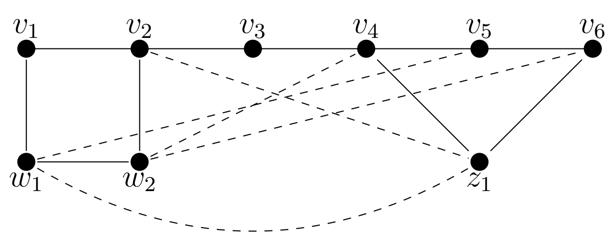
Now we may suppose there is no such w_1 and w_2. Because the edge $\{v_1, v_2\}$ is not a bridge, there is some path from v_1 to v_6 which avoids it. Consider such a path $w_1, w_2, \ldots, w_m$ of minimal length. The union of the original path with this new path must have cycle containing both of the edges $\{v_1, v_2\}$ and $\{v_1, w_1\}$. If there is an edge $\{v_1, w_i\}$ for any $i > 1$, we contradict the minimality of the new path, so any chord in this cycle must not be incident to v_1. So the edges $\{v_1, v_2\}$ and $\{v_1, w_1\}$ must be contained in the same 4-cycle created by adding chords to the large cycle. This gives two possibilities, either the fourth vertex in the cycle is v_3 or w_2. Similarly, we get a path $z_1, \ldots, z_r$ from v_6 to v_1 avoiding the edge $\{v_5, v_6\}$, which gives two possibilities for cycles v_6, z_1, z_2, v_5 or v_6, z_1, v_4, v_5. By symmetry, there are three cases to consider.

Case 3a: The two 4-cycles include v_3 and v_4.

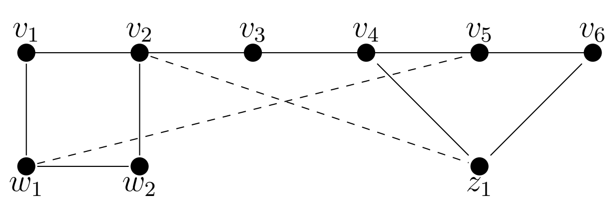


The edge $\{w_1, z_1\}$ cannot exist, as we assumed there was no disjoint path of length 3 from v_1 to v_6; so the only possible additional edges are $\{v_2, z_1\}$ and $\{w_1, v_5\}$. If both of these edges are present, the induced subgraph on $\{v_1, v_2, v_5, v_6, w_1, z_1\}$ is a 6-cycle with no chord, a contradiction, so at least one of the two edges must be missing, which w.l.o.g. we may take to be $\{v_2, z_1\}$. If the edge $\{w_1, v_5\}$ is also missing, then the induced subgraph on all 8 vertices is $H^{(3)}$. If the edge $\{w_1, v_5\}$ is present, then the induced subgraph on $\{v_1, v_2, v_3, v_4, v_5, w_1\}$ is $H^{(1)}$.

Case 3b: The two 4-cycles include w_2 and v_4.

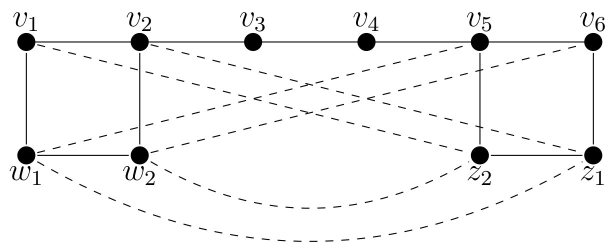


First note that the edge $\{w_1, v_3\}$ cannot be present, as then we would be in the previous case. We consider the possible edges between the two 4-cycles: $\{w_1, v_5\}, \{w_1, z_1\}, \{w_2, v_4\}$, $\{w_2, v_6\}$, and $\{v_2, z_1\}$. If the edge $\{w_2, v_4\}$ is present, the induced subgraph on vertex set $\{v_1, v_2, v_3, v_4, w_1, w_2\}$ is $H^{(1)}$, so we can assume that it is not present. Either of the edges $\{w_1, z_1\}$ and $\{w_2, v_6\}$ would give us a disjoint path of length 3, reducing to a previous case. This leaves us with the following picture:



If the edge $\{w_1, v_5\}$ were present, it would produce a 6-cycle with no chord. So the induced subgraph on the vertices $\{v_1, v_2, v_4, v_5, v_6, w_1, w_2, z_1\}$ is either $H^{(2)}$ or $H^{(3)}$, depending on whether or not the edge $\{v_2, z_1\}$ is present.

Case 3c: The two 4-cycles include w_2 and z_2.



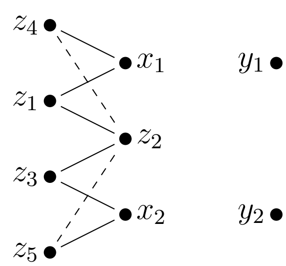
In this case, the possible edges between the two cycles are $\{v_1, z_2\}, \{w_1, v_5\}, \{w_1, z_1\}$, $\{w_2, v_6\}, \{w_2, z_2\}$, and $\{v_2, z_1\}$. If any of the edges $\{v_1, z_2\}, \{w_1, z_1\}$, or $\{w_2, v_6\}$ are present, we have a disjoint path of length 3 and are in a previous case. If either of $\{w_1, v_5\}$ or $\{v_2, z_1\}$ are present, we would have a 6-cycle with no chord. So the induced subgraph on $\{v_1, v_2, v_5, v_6, w_1, w_2, z_1, z_2\}$ is either $H^{(2)}$ or $H^{(3)}$, depending on whether $\{w_2, z_2\}$ is present.

Case 4: $(\overline{G})_1$ contains a cycle.

Let x_1, y_1, x_2, y_2 be the cycle missing from G. Since we can assume our graph is connected, there is a shortest path from x_1 to x_2. If the shortest path has length at least 6, we have an induced path of length 5 and are thus in the previous case. So the shortest path has length 2 or 4.

Case 4a: The shortest path between x_1 and x_2 is x_1, z_1, z_2, z_3, x_2.

Since $\delta(G) \geq 2$, x_1 must be adjacent to another vertex z_4 distinct from z_1 and z_3, and x_2 must be adjacent to another vertex z_5 distinct from z_1 and z_3. If $z_4 = z_5$, we have a shorter path from x_1 to x_2, so z_4 and z_5 must be distinct.



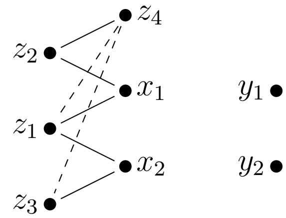
If the edges $\{x_2, z_1\}$ or $\{x_1, z_3\}$ are present, we have a shorter path, so these edges must be missing. If either of the edges $\{z_4, z_2\}$ or $\{z_5, z_2\}$ are missing, we have an induced path of length 5, putting us in the previous case. So the induced subgraph on $\{x_1, x_2, z_1, z_2, z_3, z_4, z_5\}$ is $H^{(8)}$.

Case 4b: The shortest path between x_1 and x_2 is x_1, z_1, x_2.

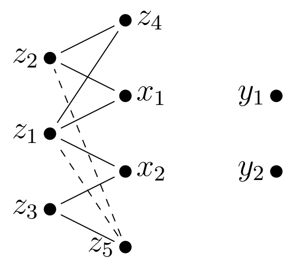
Since $\delta(G) \geq 2$, x_1 must be adjacent to another vertex z_2, and x_2 must be adjacent to another vertex z_3.

Case 4b(i): z_1 is the only common neighbor of x_1 and x_2.

If $z_2 \neq z_3$, z_2 must be adjacent to another vertex z_4. If $z_4 = x_2$, it is a common neighbor of both vertices. Otherwise, consider the induced subgraph on the vertices $\{x_1, x_2, z_1, z_2, z_3, z_4\}$. By assumption, the edges $\{z_3, x_1\}$ and $\{z_2, x_2\}$ cannot be present, so the only possibilities are $\{z_4, z_1\}$ and $\{z_4, z_3\}$.



If neither of these are present, we have an induced path of length 5, which is case 3. If both edges are present, then the induced subgraph is $H^{(1)}$. If only $\{z_4, z_3\}$ is present, we have a cycle of length 6 with no chord, a contradiction. If only $\{z_4, z_1\}$ is present, we can find a second neighbor of z_3, which we label z_5. We apply the same analysis to the induced subgraph on $\{x_1, x_2, z_1, z_2, z_3, z_5\}$ and the only case we have not already argued is if the only additional edge is $\{z_1, z_5\}$. In this case, the induced subgraph on the seven vertices $\{x_1, x_2, z_1, z_2, z_3, z_4, z_5\}$ is $H^{(8)}$.

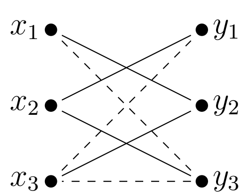


Case 4b(ii): z_2 and z_3 can be chosen to be the same.

In this case, we have a 4-cycle x_1, z_1, x_2, z_2. We then consider the shortest path from y_1 to y_2 and apply all prior case 4 analysis to this path. The only case we have not then argued is if there is also a 4-cycle y_1, w_1, y_2, w_2. In this case, the induced subgraph on the vertices $\{x_1, x_2, z_1, z_2, y_1, y_2, w_1, w_2\}$ is one of $H^{(2)}, H^{(3)}, H^{(4)}, H^{(5)}, H^{(6)}$, or $H^{(7)}$, depending on what edges are present between the z_i and w_j. (Note that $x_1, \ldots, x_4, y_1, \ldots, y_4$ in $H^{(i)}$ correspond in order to $z_1, z_2, x_1, x_2, y_1, y_2, w_1, w_2$ here.)

Case 5: $(\overline{G})_1$ has two nonadjacent edges.

In this case, G contains 4 vertices x_1, x_2, y_1, y_2 such that $\{x_1, y_2\}$ and $\{x_2, y_1\}$ are edges, while $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are non-edges. Because $\delta(G) \geq 2$, x_2 is adjacent to another vertex, y_3, and y_2 is adjacent to another vertex x_3. The edges $\{x_1, y_3\}, \{x_3, y_1\}$, and $\{x_3, y_3\}$ may or may not be present.



Case 5a: None of these three edges are present.

In this case, x_1, y_1, x_3, y_3 gives a 4-cycle in $(\overline{G})_1$ which is case 4.

Case 5b: All three edges are present.

In this case, our graph is $H^{(1)}$.

Case 5c: Exactly one of $\{x_1, y_3\}$, $\{x_3, y_1\}$, and $\{x_3, y_3\}$ are present.

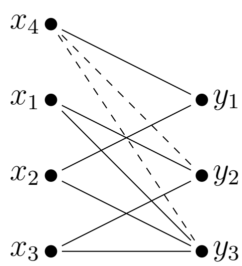
In this case, the induced subgraph on all six vertices is an induced path of length 5, which is case 3.

Case 5d: Only the edge $\{x_3, y_3\}$ is missing.

In this case, G has a 6-cycle with no chord, a contradiction.

Case 5e: Only the edge $\{x_3, y_1\}$ is missing.

In this case, y_1 must be adjacent to another vertex x_4 and the edges $\{x_4, y_2\}$ and $\{x_4, y_3\}$ may or may not be present.



Case 5e(i): Both $\{x_4, y_2\}$ and $\{x_4, y_3\}$ are present.

In this case, the induced subgraph on all seven vertices is $H^{(8)}$.

Case 5e(ii): Only $\{x_4, y_2\}$ is present.

In this case, the induced subgraph on $\{x_1, x_2, x_4, y_1, y_2, y_3\}$ is a 6-cycle with no chord, a contradiction.

Case 5e(iii): Only $\{x_4, y_3\}$ is present.

In this case, the induced subgraph on all seven vertices is $H^{(8)}$.

Case 5e(iv): Both edges are missing.

In this case, the induced subgraph on $\{x_4, y_1, x_2, y_3, x_3, y_2\}$ is a path of length 5, which is case 3.

Case 5f: Only the edge $\{x_1, y_3\}$ is missing.

This case is identical to case 5e.

□

5. Main Results

In this section we collect the proofs of our main results classifying the graphs whose toric edge ideals satisfy each of the Green-Lazarsfeld conditions $\mathbf{N}_p$. The previous sections identified certain obstructions in the form of forbidden subgraphs to a given ideal I_G satisfying conditions $\mathbf{N}_2$, $\mathbf{N}_3$, or $\mathbf{N}_4$. First we use the following result which shows we may focus our attention on the existence of minimal Koszul syzygies of toric edge ideals.

Proposition 5.1 (cf. [22, Proposition 2.8])**.** *If $R = S/I$ is a Koszul algebra, then the first syzygies of I are minimally generated by linear syzygies and Koszul syzygies.*

We note that it is also possible to show that I has only linear and Koszul syzygies when I has a quadratic Gröbner basis (as happens in our case of interest) using Schreyer's Theorem on syzygies. See [7, Theorem 3.3]. A similar observation was made in [10, Theorem 3.1]. Of course, having a quadratic Gröbner basis is sufficient but not necessary for the Koszul condition.

It would be possible to prove the following characterization of linearly presented bipartite toric edge ideals by appealing to Theorem 3.2 and enumerating all possible subgraphs with at most 8 vertices. While this strategy is useful to enumerate the obstructions to being linearly presented, it is inefficient to check all such graphs by hand. Instead, by using the previous proposition, we need only consider induced subgraphs where a potential Koszul syzygy exists and show that it is not a minimal generator of the syzygy module of I_G in every possible case. This brings us to our first main result.

Theorem 5.2. *Let G be a bipartite graph with $\delta(G) \geq 2$. Then I_G satisfies $\mathbf{N}_2$ if and only if $\overline{G}$ is essentially a tree of diameter at most 3.*

Proof. Suppose $\overline{G}$ is not essentially a tree of diameter at most 3. If there is a cycle of length ≥ 6 that has no chord, I_G is not quadratically generated by Theorem 1.1 and thus must fail condition $\mathbf{N}_1$ and also condition $\mathbf{N}_2$. So we may suppose that all chords of G of length ≥ 6 have a chord. Now by Theorem 4.2, G contains the graph $H = H^{(i)}$ for some $1 \leq i \leq 8$ as an induced subgraph. By Lemmas 3.3 and 3.4, $\beta_{2,4}(I_H) \neq 0$. By Theorem 3.2, $\beta_{2,4}(I_G) \neq 0$ and thus I_G does not satisfy property $\mathbf{N}_2$.

Now suppose that $\overline{G}$ is essentially a tree of diameter at most 3. By Theorem 1.1, I_G is generated by a Gröbner basis of quadrics corresponding to 4-cycles in G and S/I_G is Koszul. If H is any subgraph of G, then any 4-cycle in H is a 4-cycle in G, so $(I_H)_2 \subseteq I_G$. Any syzygy of I_H can be extended to a syzygy of I_G, so, by Proposition 5.1, it is sufficient to show that any Koszul syzygy is a linear combination of linear syzygies.

Koszul Syzygies correspond to pairs of distinct 4-cycles. There are five possibilities for the configuration of pairs of distinct cycles:

1. they share two edges,
2. they share an edge,
3. they share two vertices, but no edges.
4. they share a vertex but no edges,
5. or they don't intersect.

Case 1: G contains a subgraph H which has distinct 4-cycles sharing exactly two edges.

In this case, the only subgraph satisfying our assumption is $K_{2,3}$; see Figure 3. I_H is then resolved by the linear Eagon-Northcott complex and thus has no minimal quadratic syzygies.

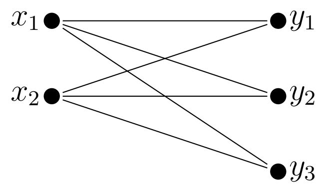
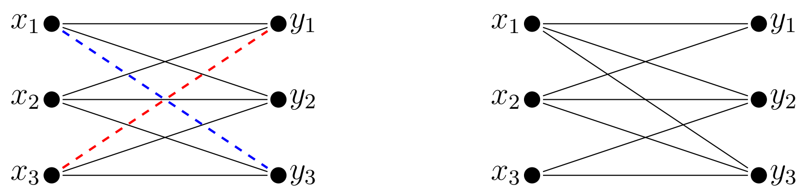


Figure 3: The graph $K_{2,3}$.

Case 2: G contains a subgraph H which has distinct 4-cycles sharing exactly one edge.

In this case, the minimal graph containing the two given 4-cycles is pictured in Figure 4(a),



(a) The minimal graph containing the two cycles.

(b) Edge $\{x_1, y_3\}$ added.

Figure 4: Possible edges in the subgraph H.

where the two 4-cycles are x_1, y_1, x_2, y_2 and x_2, y_2, x_3, y_3 and the bipartite complement consists of the two dashed lines. Since $H^{(1)}$ is not an induced subgraph of G by Theorem 4.2, at least one of the two missing edges must be present. By symmetry, we can assume that this present edge is the edge from x_1 to y_3; see Figure 4(b). Calling this graph H and labeling the edge from x_i to y_j by e_{ij}, we have

$$I_H = (e_{12}e_{21} - e_{11}e_{22}, e_{13}e_{21} - e_{11}e_{23}, e_{13}e_{22} - e_{12}e_{23}, e_{13}e_{32} - e_{12}e_{33}, e_{23}e_{32} - e_{22}e_{33}).$$

The Koszul syzygy in this case is

$$\begin{pmatrix} -e_{22}e_{33} + e_{23}e_{32} \\ 0 \\ 0 \\ 0 \\ e_{11}e_{22} - e_{12}e_{21} \end{pmatrix} = e_{32} \begin{pmatrix} e_{23} \\ -e_{22} \\ e_{21} \\ 0 \\ 0 \end{pmatrix} - e_{21} \begin{pmatrix} 0 \\ 0 \\ e_{32} \\ -e_{22} \\ e_{12} \end{pmatrix} + e_{22} \begin{pmatrix} -e_{33} \\ e_{32} \\ 0 \\ -e_{21} \\ e_{11} \end{pmatrix}.$$

The reader can verify that the terms on the right-hand side are linear syzygies.

Case 3: G contains a subgraph H which has distinct 4-cycles sharing exactly two vertices but no edges.

In this case, the only subgraph satisfying our assumptions is $K_{2,4}$,

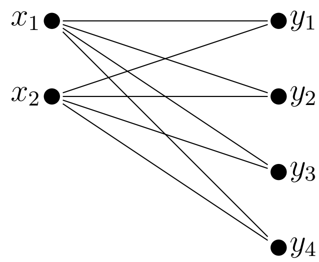
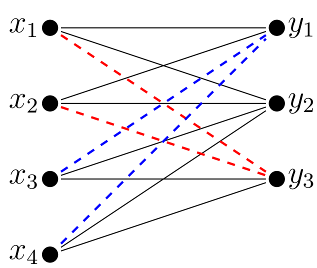


Figure 5: The graph $K_{2,4}$.

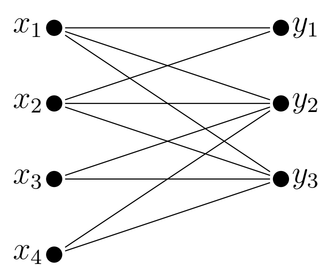
where the two cycles are x_1, y_1, x_2, y_2 and x_1, y_3, x_2, y_4. Once again this is $K_{2,4}$ and $I_{K_{2,4}}$ is resolved by a linear Eagon-Northcott resolution and so, as in Case 1, there are no minimal quadratic syzygies.

Case 4: G contains a subgraph H which has distinct 4-cycles sharing exactly one vertex and no edges.

In this case, the minimal subgraph containing the two cycles is pictured in Figure 6(a),



(a) Minimal subgraph containing the cycles.



(b) Minimal satisfactory subgraph.

Figure 6: Possible edges in the subgraph H.

where the two cycles are x_1, y_1, x_2, y_2 and x_3, y_2, x_4, y_3. To ensure that $\overline{G}$ is essentially connected, one of the two connected components of $\overline{G}$ (the dashed lines) must be present in G. By symmetry, we can assume that the edges $\{x_1, y_3\}$ and $\{x_2, y_3\}$ are present in G. The resulting graph is shown in Figure 6(b).

Calling this graph H and labeling the edge from x_i to y_j by e_{ij}, one computes that

$$\begin{aligned} I_H = (&e_{12}e_{21} - e_{11}e_{22}, e_{13}e_{21} - e_{11}e_{23}, e_{13}e_{22} - e_{12}e_{23}, e_{13}e_{32} - e_{12}e_{33}, \\ &e_{23}e_{32} - e_{22}e_{33}, e_{13}e_{42} - e_{12}e_{43}, e_{23}e_{42} - e_{22}e_{43}, e_{33}e_{42} - e_{32}e_{43}). \end{aligned}$$

The Koszul syzygy in question is then a sum of linear syzygies, as shown below.

$$\begin{pmatrix} -e_{32}e_{43}+e_{33}e_{42} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{11}e_{22}-e_{12}e_{21} \end{pmatrix} = e_{11}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{42} \\ -e_{32} \\ e_{22} \end{pmatrix} - e_{21}\begin{pmatrix} 0 \\ 0 \\ 0 \\ e_{42} \\ -e_{32} \\ 0 \\ 0 \\ e_{12} \end{pmatrix} - e_{42}\begin{pmatrix} -e_{33} \\ e_{32} \\ 0 \\ -e_{21} \\ 0 \\ e_{11} \\ 0 \\ 0 \end{pmatrix} + e_{32}\begin{pmatrix} -e_{43} \\ e_{42} \\ 0 \\ 0 \\ -e_{21} \\ 0 \\ e_{11} \\ 0 \end{pmatrix}.$$

Case 5: G contains a subgraph H which has two disjoint 4-cycles.

In this case, the minimal subgraph of the cycles is pictured in 7(a),

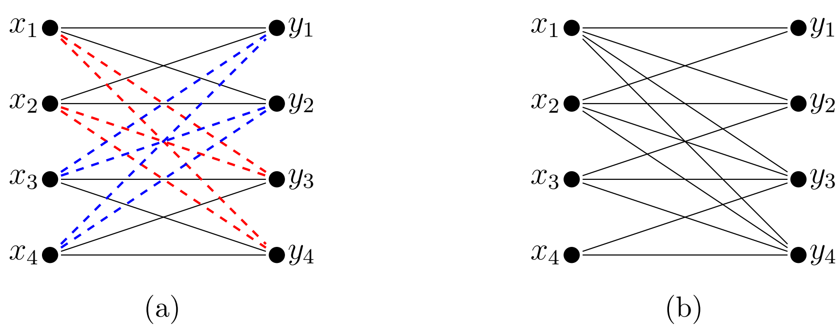


Figure 7: Possible edges in the subgraph H.

where the two cycles are x_1, y_1, x_2, y_2 and x_3, y_3, x_4, y_4. The dashed lines form the two connected components of the bipartite complement. In order to minimally satisfy our assumptions, we must add one entire connected component from the bipartite complement and at least a single edge from the other yielding the graph in Figure 7(b). If less than a full component of the bipartite complement is included, there is an induced subgraph whose bipartite complement is not essentially connected, violating Theorem 4.2. If the extra edge is not present, then the bipartite complement contains a 4-cycle, also violating Theorem 4.2.

Calling this graph in Figure 7(b) H, one computes

$$\begin{aligned} I_H = (&e_{12}e_{21}-e_{11}e_{22}, e_{13}e_{21}-e_{11}e_{23}, e_{14}e_{21}-e_{11}e_{24}, e_{13}e_{22}-e_{12}e_{23}, e_{14}e_{22}-e_{12}e_{24}, \\ &e_{14}e_{23}-e_{13}e_{24}, e_{13}e_{32}-e_{12}e_{33}, e_{14}e_{32}-e_{12}e_{34}, e_{23}e_{32}-e_{22}e_{33}, e_{24}e_{32}-e_{22}e_{34}, \\ &e_{14}e_{33}-e_{13}e_{34}, e_{24}e_{33}-e_{23}e_{34}, e_{14}e_{43}-e_{13}e_{44}, e_{24}e_{43}-e_{23}e_{44}, e_{34}e_{43}-e_{33}e_{44}). \end{aligned}$$

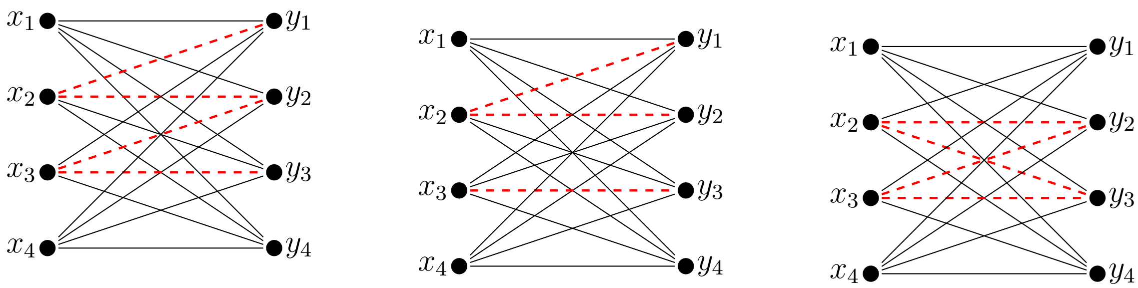
Once again, the corresponding Koszul syzygy is a sum of linear syzygies:

$$\begin{pmatrix} -e_{33}e_{44}+e_{34}e_{43} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{11}e_{22}-e_{12}e_{21} \end{pmatrix} = e_{11}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -e_{44} \\ e_{43} \\ 0 \\ -e_{32} \\ e_{22} \end{pmatrix} - e_{21}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -e_{44} \\ e_{43} \\ 0 \\ 0 \\ -e_{32} \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{12} \end{pmatrix} + e_{44}\begin{pmatrix} -e_{33} \\ e_{32} \\ 0 \\ 0 \\ 0 \\ -e_{21} \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{11} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - e_{43}\begin{pmatrix} -e_{34} \\ 0 \\ e_{32} \\ 0 \\ 0 \\ 0 \\ -e_{21} \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{11} \\ 0 \\ 0 \\ 0 \end{pmatrix} + e_{32}\begin{pmatrix} 0 \\ -e_{44} \\ e_{43} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -e_{21} \\ 0 \\ 0 \\ 0 \\ e_{11} \\ 0 \end{pmatrix}.$$

Since all possible configurations of two distinct 4-cycles produce non-minimal Koszul syzygies, we can conclude that I_G satisfies $\mathbf{N}_2$. □

Remark 5.3. The requirement that $\delta(G) \geq 2$ is not restrictive. If G has an isolated (degree 0) vertex, then removing it does not change the edge ring $k[G]$. If G has a vertex of degree 1, then removing the adjacent edge merely reduces the embedding dimension of the ring of I_G; it does not change the minimal cycles in G and thus does not affect the structure of I_G or its resolution. However, requiring $\delta(G) \geq 2$ makes our main result, which refers to $\overline{G}$, much easier to state and apply in practice.

Remark 5.4. A bipartite graph G with $\delta(G) \geq 2$ such that every cycle of length ≥ 6 has a chord can fail to satisfy Theorem 5.2 in three different ways: $\overline{G}$ could be a tree of diameter greater than 3, $\overline{G}$ could be disconnected, or $\overline{G}$ could contain a cycle. Figure 8 gives an example of each type. Dashed lines represent edges in $\overline{G}$. The corresponding Betti tables are listed below showing the corresponding toric edge ideals are not linearly presented.



	0	1	2	3	4
2:	10	16	3	-	-
3:	-	3	16	10	-
4:	-	-	-	-	1

(a) $\overline{G}$ has diameter greater than 3.

	0	1	2	3	4	5
2:	14	29	15	-	-	-
3:	-	9	41	50	21	1
4:	-	-	.	-	-	1

(b) $\overline{G}$ is disconnected.

	0	1	2	3	4
2:	11	20	6	-	-
3:	-	1	16	15	4

(c) $\overline{G}$ contains a cycle.

Figure 8: Three examples of bipartite graphs with non-linearly presented toric edge ideals and their Betti tables

Consider the complete bipartite graph $K_{5,5}$. Then $\beta_{2,5}(I_{K_{5,5}}) = 0$ if and only if the characteristic of k is not 3. In particular, in characteristic other than 3, $I_{K_{5,5}}$ has partial graded Betti table

	0	1	2	
2:	100	800	3075	$\cdots$

,

while in characteristic 3, $I_{K_{5,5}}$ has partial graded Betti table

	0	1	2	
2:	100	800	3075	$\cdots$
3:	-	-	1	$\cdots$

.

More generally, Hashimoto [16] showed that the ideal $I_t(\mathbf{M})$ of $t \times t$ minors of a generic $m \times n$ matrix has the same third Betti numbers independent of the characteristic if $t = 1$ or if $t \geq \min\{m, n\} - 2$, whereas the third Betti number is larger in characteristic 3 if $2 \leq t \leq \min\{m, n\} - 3$. The ideal $I_{K_{5,5}}$, corresponding to the 2×2 minors of a generic 5×5 matrix, is thus the minimal situation where $\beta_3(I_{K_{m,n}})$ depends on the characteristic. While this example shows that characterizing toric edge ideals of bipartite graphs satisfying $\mathbf{N}_3$ must refer to the characteristic of the coefficient field, we show below that this is the only obstruction.

Theorem 5.5. *Let G be a bipartite graph with $\delta(G) \geq 2$. The ideal I_G satisfies condition $\mathbf{N}_3$ if and only if $G = K_{m,n}$ for some m, n, unless the characteristic of k is 3 and $\min\{m, n\} \geq 5$.*

Proof. First suppose that $G = K_{5,5}$. In characteristic 0, I_G satisfies $\mathbf{N}_3$ by considering the resolution of Lascoux or Pragacz and Weyman. It follows by considering the number of boxes in the last partition of the Lascoux complex (20 in this case), the resolution is the same in characteristic $p > 20$ as it is in characteristic 0. A quick Macaulay2 [11] calculation shows that $p = 3$ is the only characteristic less than 20 which in which I_G fails to satisfy $\mathbf{N}_3$.

Now let $G = K_{m,n}$ for arbitrary m, n. If $\min\{m, n\} \leq 4$, then I_G has the same graded Betti numbers as its characteristic 0 Lascoux resolution by [17] and [1]. Thus we may assume $\min\{m, n\} \geq 5$. If the $\mathrm{char}(k) = 3$, then G has $K_{5,5}$ as an induced subgraph and so does not satisfy $\mathbf{N}_3$ as above. Thus we may assume $\mathrm{char}(k) \neq 3$. If I_G does not satisfy $\mathbf{N}_3$, then $\beta_{2,j}(I_G) \neq 0$ for some $j \geq 5$. Since S/I_G is Koszul and since I_G is linearly presented, it follows from [2, Main Theorem (2)] that $\beta_{2,j}(I_G) = 0$ for all $j \geq 6$. Thus if I_G fails to satisfy $\mathbf{N}_3$, we must have $\beta_{2,5}(I_G) \neq 0$.

Now by Theorem 3.2 there must be an induced subgraph H of G with at most 10 vertices that fails $\mathbf{N}_3$. Since H is an induced subgraph, it is also a complete bipartite graph, say $K_{m',n'}$ with $m' + n' = 10$. If $\min\{m', n'\} \leq 4$, then I_H has the same graded Betti numbers as that characteristic 0 Lascoux resolution, which satisfies $\mathbf{N}_3$ by [17] and [1]. Thus is suffices to consider the case $H = K_{5,5}$, as we have above.

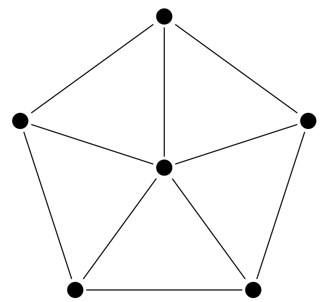
Now suppose G is not a complete bipartite graph. If I_G does not satisfy $\mathbf{N}_2$, then it doesn't satisfy $\mathbf{N}_3$, so it is enough to consider a graph G such that I_G satisfies $\mathbf{N}_2$. By Theorem 5.2, $\overline{G}$ is essentially a tree of diameter at most 3. In particular, since G is not a complete graph, $\overline{G}$ has at least one edge. Moreover, since $\overline{G}$ is a tree of diameter at most 3, there is an edge with vertices x and y such that every other edge of $\overline{G}$ is adjacent to this edge. Since $\delta(G) \geq 2$, there exist vertices x', x'', y', y'' such that $\{x, y'\}$, $\{x, y''\}$, $\{y, x'\}$, and $\{y, x''\}$ are all edges in G. Since every edge in $\overline{G}$ is adjacent to $\{x, y\}$, we must also have the edges $\{x', y'\}, \{x', y''\}, \{x'', y'\}$, and $\{x'', y''\}$ in G. So the induced subgraph on $\{x, x', x'', y, y', y''\}$ is the graph H in Lemma 3.5, which satisfies $\beta_{2,5}(I_H) \neq 0$. By Theorem 3.2, $\beta_{2,5}(I_G) \neq 0$, so I_G does not satisfy $\mathbf{N}_3$. □

The preceding work yields the following surprising characterization of toric edge ideals with linear free resolutions. That $I_{K_{2,n}}$ has a linear free resolution is well-known and follows from the Eagon-Northcott resolution. Ohsugi and Hibi [24, Theorem 4.6] showed that I_G has a linear free resolution if and only if $G = K_{2,n}$ for some n. The content of the following theorem is that the only bipartite graphs satisfying condition $\mathbf{N}_4$ are complete bipartite graphs $K_{2,n}$ for some n and that condition $\mathbf{N}_4$ is sufficient to guarantee linear free resolutions regardless of the characteristic.

Theorem 5.6. *Let G be a bipartite graph with $\delta(G) \geq 2$. The ideal I_G satisfies property $\mathbf{N}_4$ if and only if $G = K_{2,n}$ for some n. In this case, I_G has a linear free resolutions and thus satisfies $\mathbf{N}_p$ for all $p \geq 1$.*

Proof. If I_G does not satisfy $\mathbf{N}_3$, then it doesn't satisfy $\mathbf{N}_4$, so it is enough to consider a graph G such that I_G satisfies $\mathbf{N}_3$. So we can assume G is a complete bipartite graph. If $G = K_{m,n}$ with $m, n \geq 3$, then it has $C = K_{3,3}$ as an induced subgraph. By Lemma 3.6, $\beta_{3,6}(I_C) \neq 0$, so by Theorem 3.2 $\beta_{3,6}(I_G) \neq 0$. □

Keeping in mind Theorem 5.1, we see that these proofs relied on the fact that I_G was generated by a Gröbner basis of quadrics. If we attempt to directly extend these results to toric edge ideals of arbitrary (not necessarily bipartite) graphs, we lose the power of this result, even in the case where I_G is generated by quadrics. For example the graph pictured below



has toric edge ideal generated by quadrics [18, Example 5.28] but has no Gröbner basis of quadrics with respect to any monomial ordering [18, Example 1.18].

6. Linear Syzygies of Polyominoes

Ideals associated to polyominoes were introduced by Qureshi, where it was shown that the ring associated to a convex polyomino is normal and Cohen-Macaulay [30, Theorem 2.2]. Later work by Ene, Herzog, and Hibi showed that the defining ideals are generated by a quadratic Gröbner basis by viewing them as toric ideals associated to bipartite graphs [10, Proposition 2.3]. They also give a characterization of polynominoes whose associated ideals are linearly presented [10, Theorem 3.1]. However, when we translate our result on bipartite graphs whose toric edge ideals are linearly presented, we discovered a discrepancy; in particular, there are polyominoes that are not linearly presented that satisfy [10, Theorem 3.1]. The purpose of this section is to then translate our results on toric edge ideals of bipartite graphs into results on convex polynomino ideals satisfying coditions $\mathbf{N}_p$ for all p, thereby correcting the error in the above theorem. We begin with some notation.

If $a, b \in \mathbb{N}^2$ with $a \leq b$ under the natural partial order, the set $[a, b] = \{c \in \mathbb{N}^2 \mid a \leq c \leq b\}$ is called an interval. If $b = a + (1, 1)$, then $[a, b]$ is called a cell. The edges of the cell $C = [a, a+(1, 1)]$ are the sets $\{a, a+(0, 1)\}, \{a+(0, 1), a+(1, 1)\}, \{a+(1, 1), a+(1, 0)\}, \{a+(1, 0), a\}$ and the points $a, a + (0, 1), a + (1, 1), a + (1, 0)$ are the vertices of C. The vertex a is called the lower left corner of C. Let $\mathcal{P}$ be a finite collections of cells. The set of vertices $V(\mathcal{P})$ is the union of the sets of vertices of all cells in $\mathcal{P}$. If $C, D \in \mathcal{P}$, then C and D are connected if there is a sequence of cells of $\mathcal{P}$ given by $C = C_1, \ldots, C_t = D$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1 \ldots, t-1$. A collection of cells $\mathcal{P}$ is a polyomino if any two of its cells are connected. Two polyominos are isomorphic is they are mapped to each other by a finite sequence of translations, rotations, and reflections. A polyomino $\mathcal{P}$ is row convex if given any any two cells of $\mathcal{P}$ with lower left corners (i_1, j) and (i_2, j) with $i_1 < i_2$, all of the cells with lower left corners (i, j) with $i_1 < i < i_2$ are also in $\mathcal{P}$. Similarly, one defines $\mathcal{P}$ to be column convex if given any two cells of $\mathcal{P}$ with lower left corners (i, j_1) and (i, j_2) with $j_1 < j_2$, one has that all the cells with lower left corners (i, j) with $j_1 < j < j_2$ are in $\mathcal{P}$. Finally $\mathcal{P}$ is convex if it is row convex and column convex.

Now let $\mathcal{P}$ be a polyomino. We may rotate and translate $\mathcal{P}$ until $[(1,1),(m,n)]$ is the smallest interval containing $\mathcal{P}$. Fix a field k and a polynomial ring $S = k[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]$. The polyomino ideal $I_{\mathcal{P}}$ is the ideal of S generated by the 2×2 minors $x_{ij}x_{kl} - x_{il}x_{kj}$ for with $[(i,j),(k,l)] \subset V(\mathcal{P})$.

Remark 6.1. In [10, Proposition 2.3], it is shown every convex polyomino ideal is the toric edge ideal of a bipartite graph with a quadratic Gröbner basis and thus satisfies condition $\mathbf{N}_1$. One identifies the vertical line segments with one set of vertices x_i and the horizontal line segments with another set of vertices y_j; then one draws an edge between x_i and y_j if the corresponding line segments intersect. Thus every convex polynomino corresponds to a bipartite graph such that every cycle of length ≥ 6 has a chord.

Here we remark that the converse does not hold; that is, not every quadratically generated toric edge ideal is the polyomino ideal for some convex polyomino. The ideal $I_{H^{(1)}}$ which corresponds to a complete intersection of two quadratic binomials is clearly not associated to any convex polyomino, which cannot have exactly 2 minimal generators. However, the disconnected collection of cells in Figure 9 has an isomorphic ideal of inner minors to $I_{H^{(1)}}$.

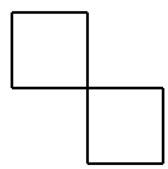


Figure 9: A collections of cells that is not a polyomino.

Proposition 6.2. *Let $G = (V, E)$ is a connected bipartite graph with $\delta(G) \geq 2$ such that every cycle with length ≥ 6 has a chord and such that $\overline{G}$ is essentially a tree of diameter at most* 3*. Then there is a convex polyomino $\mathcal{P}$ such that I_G and $I_{\mathcal{P}}$ are isomorphic.*

Proof. Let $\{x_1, \ldots, x_m\} \sqcup \{y_1, \ldots, y_n\}$ be the vertex set of G. Since $\overline{G}$ is essentially a tree of diameter at most 3, if G is not a complete bipartite graph, then $\overline{G}$ has an edge adjacent to every other edge. By relabeling, we may assume this edge is $\{x_m, y_n\}$ and every other edge is of the form $\{x_m, y_j\}$ for some $1 \leq j < n$ or $\{x_i, y_n\}$ for some $1 \leq i < m$. Again by relabeling vertices, we may assume that $\{x_i, y_n\} \in E$ if and only if $i \leq m$ and $\{x_m, y_j\} \in E$ if and only if $j \leq n'$ for some integers m' and n' with $1 \leq m' < m$ and $1 \leq n' < n$. It follows that $I_G = I_{\mathcal{P}}$ where $\mathcal{P}$ is the following polyomino:

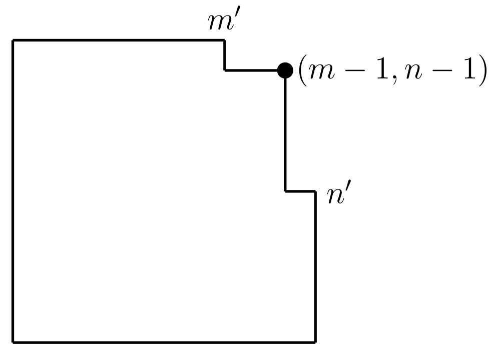


Figure 10: A linearly related polyomino.

$\square$

Using the previous dictionary between polyomino ideals and toric edge ideals of bipartite graphs, we translate our main Theorem 1.2 to characterize polyomino ideals satisfying the various Green-Lazarsfeld conditions.

Theorem 6.3. *Let $\mathcal{P}$ be a convex polyomino and let k be a field.*

1. *$I_{\mathcal{P}}$ satisfies property $\mathbf{N}_1$.*
2. *$I_{\mathcal{P}}$ satisfies property $\mathbf{N}_2$ if and only if $\mathcal{P}$ is isomorphic to a polyomino all of whose missing cells are in the first row or first column (possibly after rotating $\mathcal{P}$. See Figure 11.)*
3. *$I_{\mathcal{P}}$ satisfies property $\mathbf{N}_3$ if and only if $\mathcal{P}$ is an interval unless* $\mathrm{char}(k) = 3$ *and $\mathcal{P}$ is an interval with width and length at least 4.*
4. *$I_{\mathcal{P}}$ satisfies property $\mathbf{N}_p$ for some/any $p \geq 4$ if and only if $\mathcal{P}$ is an interval of the form $[a, (2, n) + a]$ for some $a \in \mathbb{N}^2$.*

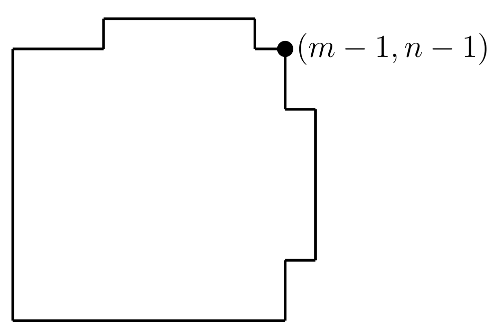


Figure 11: A general, linearly related, convex polyomino.

7. Application to a Question of Constantinescu, Kahle, and Varbaro

Our original motivation for studying linearly presented toric edge ideals comes from the following question of Constantinescu, Kahle, and Varbaro [6]:

Question 7.1 ([6, Question 1.1])**.** *Is there a family of linearly presented, quadratically generated ideals* $\{I_n \subseteq R = k[x_1, ..., x_n]\}_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} \frac{\operatorname{reg}(I_n)}{n} > 0?$$

A similar question in the Koszul setting was posed by Conca [5, Question 2.8]. Such a family of ideals would have regularity growing linearly with respect to the number of variables. In [6], Constantinescu, Kahle, and Varbaro construct a family of squarefree, quadratic monomial ideals with linear syzygies for arbitrarily many steps (i.e. satisfying property $\mathbf{N}_p$ for arbitrary p if we ignore the normality condition) and with arbitrarily large regularity, however these ideals have a very large number of variables. A result of Dao, Huneke, and Schweig [8] shows that the regularity of squarefree monomial ideals with linear syzygies is bounded logarithmically in terms of the number of variables; in particular, no such families of ideals yielding a positive answer to Question 7.1 can be monomial.

Note that if $\operatorname{depth}(R/I_n) > 0$, we can mod out by a general linear form, thereby reducing the number of variables while preserving the graded Betti numbers and regularity. It follows from the Auslander-Buchsbaum formula that the following question is equivalent to Question 7.1

Question 7.2. *Is there a family of linearly presented, quadratically generated ideals* $\{I_n \subseteq R = k[x_1, ..., x_n]\}_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} \operatorname{pd}(I_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\operatorname{reg}(I_n)}{\operatorname{pd}(I_n)} > 0?$$

The restriction that the ideals is linearly presented rules out complete intersections of n quadrics for with $\operatorname{pd}(S/I_n) = \operatorname{reg}(S/I_n) = n$. In general, both questions are still open. A corollary to our Theorem 5.2 is that no such families exist among toric edge ideals associated to bipartite graphs.

Corollary 7.3. *There are no families of graphs G_n, where G_n is bipartite and I_{G_n} satisfies property $\mathbf{N}_2$, that give a positive answer to Question 7.2. In other words, if* $\lim_{n\to\infty} \operatorname{pd}(I_{G_n}) = \infty$*, then* $\lim_{n\to\infty} \frac{\operatorname{reg}(I_{G_n})}{\operatorname{pd}(I_{G_n})} = 0$.

Proof. Fix a bipartite graph $G = (X \sqcup Y, E)$ such that I_G is linearly presented and $\delta(G) \geq 2$. Set $r = |X|$ and $s = |Y|$ and without loss of generality assume $2 \leq r \leq s$. Since I_G is Cohen-Macaulay, $\operatorname{pd}(S/I_G) = \operatorname{ht}(I_G)$. When G is a complete (r, s)-bipartite graph, it is well-known that $\operatorname{ht}(I_G) = (r-1)(s-1)$. If G is an arbitrary bipartite graph such that I_G is linearly presented, it follows from Theorem 5.2 that $\overline{G}$ is a tree of diameter at most 3. Thus there are at most $(r-3)+(s-3)+1$ edges missing from the complete bipartite graph and so

$|E| \geq rs - r - s + 5$. It follows that $\mathrm{ht}(I_G) \geq (r-1)(s-1) - r - s + 5 = (r-2)(s-1) - r + 4$. By [3, Theorem 4.9], $\mathrm{reg}(S/I_G) \leq r$. If the above limit is nonzero, we must have $\lim_{n\to\infty} r = \infty$. Since $r \leq s$, we get

$$\frac{\mathrm{reg}(I_{G_n})}{\mathrm{pd}(I_{G_n})} \leq \frac{r}{(r-2)(s-1) - r + 4} = \frac{1}{\frac{r-2}{r}(s-1) - 1 + \frac{4}{r}} \to 0$$

as $n \to \infty$. □

It is worth noting that even though quadratic toric edge ideals of bipartite graphs are generated by quadratic Gröbner bases, this does not reduce the problem of answering the above question to the monomial case. Indeed there are linearly presented, quadratic toric edge ideals whose lead term (monomial) ideals are not linearly presented. For a simple example, let G be $K_{4,3}$ with one edge removed. By Theorem 5.2, I_G is linearly presented. One checks however that $LT(I_G)$ is quadratic but not linearly presented.

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