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Linear instability of Z-pinch in plasma: Inviscid case

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The z-pinch is a classical steady state for the MHD model, where a confined plasma fluid is separated by vacuum, in the presence of a magnetic field which is generated by a prescribed current along the z direction. We develop a variational framework to study its stability in the absence of viscosity effect, and demonstrate for the first time that such a z-pinch is always unstable. Moreover, we discover a sufficient condition such that the eigenvalues can be unbounded, which leads to ill-posedness of the linearized MHD system.

Keywords: Compressible MHD system; z–pinch plasma; vacuum; linear instability.

AMS Subject Classification 2020: 35Q35, 35Q30, 76W05, 76E25

1. Introduction

1.1. *Overview*

A pinch is a cylindrical device used to contain plasma, which is typically too hot to be in contact with the device's walls. Due to this constraint, the plasma must be

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confined to the interior of the pinch with a vacuum region separating the plasma from the pinch's outer wall. In this paper, we are concerned with so-called z-pinches, in which current is flowed along the axis of the pinch (the z direction in cylindrical coordinates, and hence the name z-pinch), inducing an azimuthal magnetic field; the resulting Lorentz force acts to confine the plasma in the z-pinch's center.

The first z-pinch was, remarkably, constructed in the 18th century, but interest in these devices grew significantly in the middle of the 20th century as physicists used them in the pursuit of fusion technology. We refer to the exhaustive three-volume treatise of Cap[3–5] and the references therein for a survey of what was learned in this pursuit. One of the principal lessons was that z-pinches are subject to powerful instabilities that make long-term plasma confinement problematic. In spite of this, these devices remain of interest in the physics community due to their use as components in potential fusion devices and as sources of high energy X-rays: we refer to the survey[12] and the references therein for a discussion of the contemporary applications of z-pinches.

In order to understand the nature of the z-pinch instability, one must fix a model of the plasma. One of the simplest and most fundamental choices is to model the plasma with the inviscid, compressible, magnetohydrodynamic equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \rho(\partial_t u + (u \cdot \nabla) u) + \nabla p = (\nabla \times B) \times B \\ \partial_t B - \nabla \times (u \times B) = 0 \\ \nabla \cdot B = 0. \end{cases} \tag{1.1}$$

Here ρ is the plasma's density, u is its velocity, p is its pressure, and B is the magnetic field. Fluid viscosity and resistivity are neglected in the model, which removes typical dissipation mechanisms. The term $(\nabla \times B) \times B$ in the second equation, the balance of linear momentum, is the Lorentz force that acts on the plasma since in the MHD approximation the current is given by $\mathbb{J} = \nabla \times B$. To close the system, we posit a typical polytropic relation between p and ρ: $p = A\rho^\gamma$ for $A > 0$ an entropy constant and $\gamma > 1$ the adiabatic exponent.

With the MHD model in hand, we can look for equilibrium configurations. Suppose that the pinch is comprised of a cylinder of radius $r_w > 0$ (the subscript w is used because we think of the pinch boundary as a wall), and let us look for a steady (time independent) solution with $u = 0$ and with the plasma confined to a concentric cylinder of radius $0 < r_0 < r_w$. Within the plasma (where $0 \le r < r_0$) the equilibrium equations reduce to

$$\nabla \cdot B = 0, \quad \mathbb{J} = \nabla \times B, \quad \text{and} \quad \nabla p = \mathbb{J} \times B. \tag{1.2}$$

As we will see later, if we impose cylindrical symmetry and switch to cylindrical coordinates, then a solution to this system can be found in one of three equivalent ways: by specifying the pressure p, the current $\mathbb{J}$, or the magnetic field B, and then recovering the other two via these equations. Thus, the space of equilibria is quite large. To enforce the vacuum confinement, we force $p(r_0) = 0 = \rho(r_0)$. These equations have to be coupled to magnetic field equations in the region $r_0 < r < r_w$ as well as to boundary conditions at the vacuum boundary where $r = r_0$ and at the boundary of the pinch at $r = r_w$. We postpone a detailed discussion of these until later as well.

With an equilibrium solution in hand, its stability may be investigated via the method of normal mode analysis: linearize the PDEs around the equilibrium, assume an exponential time ansatz of the form $e^{\mu t}$, and a Fourier mode ansatz in the z and θ directions (where r, θ, z are standard cylindrical coordinates in the pinch) of the form $e^{i(m\theta+kz)}$. This results in an ODE boundary value problem in r with the equilibrium functions appearing as coefficients, the frequencies m and k appearing as parameters, and μ appearing as an eigenvalue. If solutions can be constructed with $\mu > 0$, then the equilibrium is unstable. In the literature (Chap. 11.4 in Freidberg's book,[6] Chap. 9 in Goedbloed and Poedts's book[7] and Chap. V in Schmidt's book[17]) one finds formal variational arguments suggesting the existence of unstable solutions for $m \in \{0, 1\}$, often with special assumptions on the equilibrium.

In this paper, we rigorously demonstrate that the equilibria are generically unconditionally unstable. We prove that the decay rate of the equilibrium pressure near the vacuum boundary plays a key role in the instability. Indeed, for some decay rates, μ remains bounded as a function of m and k, while for other rates μ blows up. In the latter case, we can then deduce not only linear instability but linear ill-posedness (via the same mechanism that gives ill-posedness of backward heat flow), which does not seem to have been observed before in the literature. However, as we prove in our companion paper,[1] this ill-posedness can be avoided when viscous effects are taken into account in the model, though instability persists.

The primary purpose of this paper is to place the variational arguments on a rigorous foundation for a very general class of equilibria and to employ the direct method in the calculus of variations to construct unstable solutions for the full range of frequencies k, m. Because of the vacuum boundary conditions, the Sobolev-like function spaces used in the direct method have degenerate weights of an unknown form due to the generality of the equilibrium profile. In turn, this creates a number of nontrivial difficulties in establishing the tools needed to run the direct method. Fortunately, we are able to overcome these obstacles and construct solutions variationally. The variational formulation is essential, as it then allows us to study the asymptotic behavior of the unstable growth rate μ as a function of the frequencies m and k.

1.2. *Formulation of the problem in Eulerian coordinates*

We model the pinch as the cylindrical domain $\{(x_1, x_2, z) \in \mathbb{R}^2 \times 2\pi\mathbb{T} \mid x_1^2 + x_2^2 \leq r_w^2\}$ where $2\pi\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the torus of length 2π. Here the periodicity in the z direction is not essential in our analysis and $2\pi\mathbb{T}$ could be replaced with $\mathbb{R}$. We have opted to introduce the periodicity as this is the usual choice in the physics literature since it is a simple model of a toroidal z-pinch. For each time $t \geq 0$, the z-pinch divides into two disjoint pieces, $\Omega(t) = \{\rho(t) > 0\}$ and $\Omega^v(t) = \{\rho(t) = 0\}$, with the free boundary $\Sigma_{t,pv} = \overline{\Omega(t)} \cap \overline{\Omega^v(t)}$ and the perfectly conducting wall Σ_w on the outside r_w. We may rewrite the Lorentz force to write the compressible MHD system in the plasma region as

$$\begin{cases} \partial_t \rho + (u \cdot \nabla)\rho + \rho \nabla \cdot u = 0 & \text{in } \Omega(t), \\ \rho(\partial_t u + (u \cdot \nabla)u) + \nabla \left(p + \dfrac{1}{2}|B|^2 \right) = (B \cdot \nabla)B & \text{in } \Omega(t), \\ \partial_t B - \nabla \times (u \times B) = 0 & \text{in } \Omega(t), \\ \nabla \cdot B = 0 & \text{in } \Omega(t), \end{cases} \tag{1.3}$$

where the vector-field $u = (u_1, u_2, u_3)$ denotes the Eulerian plasma velocity field, ρ denotes the density of the fluid, $B = (B_1, B_2, B_3)$ is magnetic field, and p denotes the pressure function. The above system (1.3) is called the inviscid compressible MHD equations which describe the motion of a perfectly conducting fluid interacting with a magnetic field. Here, the open, bounded subset $\Omega(t) \subset \mathbb{R}^3$ denotes the changing volume occupied by the plasma with $\rho(t) > 0$ in $\Omega(t)$. Recall that we consider the polytropic equation of state $p = A\rho^\gamma$, where A is an entropy constant and $\gamma > 1$ is the adiabatic gas exponent.

From the mass conservation equation in (1.3) and pressure satisfying γ law, one can get that

$$\partial_t p + u \cdot \nabla p + \gamma p \nabla \cdot u = 0. \tag{1.4}$$

In the vacuum domain $\Omega^v(t)$, we have the div-curl system

$$\begin{cases} \nabla \cdot \widehat{B} = 0 & \text{in } \Omega^v(t), \\ \nabla \times \widehat{B} = 0 & \text{in } \Omega^v(t) \end{cases} \tag{1.5}$$

which describes the vacuum magnetic field $\widehat{B}$. Here, we consider so-called pre-Maxwell dynamics. That is, as usual in nonrelativistic MHD, we neglect the displacement current $\frac{1}{c^2}\partial_t \widehat{E}$, where c is the speed of the light and $\widehat{E}$ is the electric field. In general, quantities with a hat $\widehat{\cdot}$ denote vacuum variables.

We assume that the plasma region $\Omega(t)$ with the fluid density $\rho(t) > 0$ is isolated from the fixed perfectly conducting wall Σ_w by a vacuum region $\Omega^v(t)$, which makes the plasma surface free to move. Hence, this model is a free boundary problem of the combined plasma–vacuum system. To solve this system, we need to prescribe appropriate boundary conditions. On the perfectly conducting wall Σ_w, the normal component of the magnetic field must vanish

$$n \cdot \widehat{B}|_{\Sigma_w} = 0, \tag{1.6}$$

where n is the outer unit normal to the boundary of Σ_w.

We prescribe the following jump conditions on the free boundary to connect the magnetic fields across the surface. These arise from the Maxwell's equations and the continuum mechanics

$$\begin{cases} n \cdot B = n \cdot \widehat{B} & \text{on } \Sigma_{t,pv}, \\ \left[\left[p + \frac{1}{2}|B|^2\right]\right] = 0 & \text{on } \Sigma_{t,pv}, \end{cases} \tag{1.7}$$

where for any quantity q, $[[q]]_{\Sigma_{t,pv}}$ denotes $\widehat{q} - q$ on the free boundary $\Sigma_{t,pv}$, and n is the outer normal to the free boundary of $\Omega(t)$.

In conclusion, denote $\mathcal{V}(\Sigma_{t,pv})$ as the normal velocity of the free surface $\Sigma_{t,pv}$, then the plasma-vacuum compressible MHD system can be written in Eulerian coordinates as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 & \text{in } \Omega(t), \\ \rho(\partial_t u + (u \cdot \nabla) u) + \nabla \left(p + \frac{1}{2}|B|^2\right) = (B \cdot \nabla) B & \text{in } \Omega(t), \\ \partial_t B - \nabla \times (u \times B) = 0, \quad \nabla \cdot B = 0 & \text{in } \Omega(t), \\ \nabla \cdot B = 0 & \text{in } \Omega(t), \\ \nabla \cdot \widehat{B} = 0, \quad \nabla \times \widehat{B} = 0 & \text{in } \Omega^v(t), \\ \mathcal{V}(\Sigma_{t,pv}) = u \cdot n & \text{on } \Sigma_{t,pv}, \\ n \cdot B = n \cdot \widehat{B} & \text{on } \Sigma_{t,pv}, \\ p + \frac{1}{2}|B|^2 - \frac{1}{2}|\widehat{B}|^2 = 0 & \text{on } \Sigma_{t,pv}, \\ n \cdot \widehat{B}|_{\Sigma_w} = 0, \quad \rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad B|_{t=0} = B_0. \end{cases} \tag{1.8}$$

1.3. *Some previous work*

The z-pinch instability in plasma for the compressible MHD system (1.8) with vacuum and free boundary is an interesting and long-time open problem since the pinch experiments of the 1960s and 1970s, see Refs. 16 and 17 and the references therein. There are many numerical simulations.[6,7] Recently, Guo–Tice[10,11] proved the linear Rayleigh–Taylor instability for inviscid and viscous compressible fluids by introducing a new variational method. Later on, using the variational framework, many authors considered the effects of magnetic field in the fluid equations. Jiang–Jiang[13] considered the magnetic inhibition theory in non-resistive incompressible MHD fluids. Jiang–Jiang[14] considered the nonlinear stability and instability in the Rayleigh–

Taylor problem of compressible MHD equations without vacuum and established the stability/instability criteria for the stratified compressible magnetic Rayleigh–Taylor problem in Lagrangian coordinates. Jiang–Jiang[15] investigated the stability and instability of the Parker problem for the three-dimensional compressible isentropic viscous magnetohydrodynamic system with zero resistivity in the presence of a modified gravitational force in a vertical strip domain in which the velocity of the fluid is non-slip on the boundary. Wang–Xin[19] proved the global well-posedness of the inviscid and resistive problem with surface tension around a non-horizontal uniform magnetic field for the incompressible MHD equations. Wang[18] got sharp nonlinear stability criterion of viscous incompressible non-resistive MHD internal waves in 3D. Gui[9] considered the Cauchy problem of the two-dimensional incompressible magnetohydrodynamics system with inhomogeneous density and electrical conductivity and has showed the global well-posedness for a generic family of the variations of the initial data and an inhomogeneous electrical conductivity. All these results do not contain vacuum. For presenting vacuum, under the Taylor sign condition of the total pressure on the free surface, Gu–Wang[8] proved the local well-posedness of the ideal incompressible MHD equations in Sobolev spaces. In this paper, we will rigorously prove the linear z-pinch instability for ideal compressible MHD system (1.8).

2. Steady State and Main Results

2.1. *Derivation of the MHD system in Lagrangian coordinates*

In this section, we mainly introduce the Lagrangian coordinates in which the free boundary becomes fixed.

First, we assume the equilibrium domains are given by

$$\begin{aligned}
\overline{\Omega} &= \{(r,\theta,z)|r<r_0, \quad \theta\in[0,2\pi], z\in 2\pi\mathbb{T}\},\\
\overline{\Omega}^v &= \{(r,\theta,z)|r_0<r<r_w, \quad \theta\in[0,2\pi], z\in 2\pi\mathbb{T}\}.
\end{aligned}$$

Here, the constant r_0 is the interface boundary and the constant r_w is the perfectly conducting wall position. This is meant to be a simplified model of the toroidal geometry employed in tokamaks.

Now we introduce the Lagrangian coordinates.

1. The flow map

Let $h(t,\mathcal{X})$ be a position of the gas particle $\mathcal{X}$ in the equilibrium domain $\overline{\Omega}$ at time t so that

$$\begin{cases} \dfrac{d}{dt}h(t,\mathcal{X}) = u(t,h(t,\mathcal{X})), & t>0, \mathcal{X}\in\overline{\Omega},\\[2ex] h|_{t=0} = \mathcal{X}+g_0(\mathcal{X}), & \mathcal{X}\in\overline{\Omega}. \end{cases} \tag{2.1}$$

Then the displacement $g(t,\mathcal{X}) = h(t,\mathcal{X}) - \mathcal{X}$ satisfies

$$\begin{cases} \dfrac{d}{dt} g(t,\mathcal{X}) = u(t,\mathcal{X} + g(t,\mathcal{X})), \\ \\ g|_{t=0} = g_0. \end{cases} \tag{2.2}$$

We define the Lagrangian quantities in the plasma as follows (where $\mathcal{X} = (x,y,z) \in \overline{\Omega}$):

$$f(t,\mathcal{X}) = \rho(t,h(t,\mathcal{X})), \quad v(t,\mathcal{X}) = u(t,h(t,\mathcal{X})), \quad q(t,\mathcal{X}) = p(t,h(t,\mathcal{X})),$$
$$b(t,\mathcal{X}) = B(t,h(t,\mathcal{X})), \quad \mathcal{A} = (Dh)^{-1}, \quad J = \det(Dh).$$

According to definitions of the flow map h and the displacement g, for $(i,j,k) \in \{1,2,3\}$ one can get the following identities:

$$\mathcal{A}_i^k \partial_k h^j = \mathcal{A}_k^j \partial_i h^k = \delta_i^j, \quad \partial_k (J\mathcal{A}_i^k) = 0, \quad \partial_i h^j = \delta_i^j + \partial_i g^j, \quad \mathcal{A}_i^j = \delta_i^j - \mathcal{A}_i^k \partial_k g^j, \tag{2.3}$$

where the Einstein notation is used and will be used in the whole paper. If the displacement g is sufficiently small in an appropriate Sobolev space, then the flow mapping h is a diffeomorphism from Ω_0 to $\Omega(t)$, which allows us to switch back and forth from Lagrangian to Eulerian coordinates.

2. Derivatives of J and $\mathcal{A}$ in Lagrangian coordinates

We write the derivatives of J and $\mathcal{A}$ in Lagrangian coordinates as follows:

$$\begin{gathered} \partial_t J = J\mathcal{A}_i^j \partial_j v^i, \quad \partial_\ell J = J\mathcal{A}_i^j \partial_j \partial_\ell g^i, \quad \partial_t \mathcal{A}_i^j = -\mathcal{A}_k^j \mathcal{A}_i^\ell \partial_\ell v^k, \\ \partial_\ell \mathcal{A}_i^j = -\mathcal{A}_k^j \mathcal{A}_i^n \partial_n \partial_\ell g^k, \quad \partial_i v^j = \partial_i h^k \mathcal{A}_k^\ell \partial_\ell v^j = \mathcal{A}_i^\ell \partial_\ell v^j + \partial_i g^k \mathcal{A}_k^\ell \partial_\ell v^j. \end{gathered} \tag{2.4}$$

3. Plasma equations in Lagrangian coordinates

Denote $(\nabla_{\mathcal{A}})_i = \mathcal{A}_i^j \partial_j$. Then we can write the plasma equations in Lagrangian coordinates as follows:

$$\begin{cases} \partial_t g = v & \text{in } \overline{\Omega}, \\ f\partial_t v + \nabla_{\mathcal{A}} \left(q + \dfrac{1}{2}|b|^2 \right) = (b \cdot \nabla_{\mathcal{A}}) b & \text{in } \overline{\Omega}, \\ \partial_t f + f \nabla_{\mathcal{A}} \cdot v = 0 & \text{in } \overline{\Omega}, \\ \partial_t b + b \nabla_{\mathcal{A}} \cdot v = (b \cdot \nabla_{\mathcal{A}}) v & \text{in } \overline{\Omega}, \\ \nabla_{\mathcal{A}} \cdot b = 0 & \text{in } \overline{\Omega}, \\ n \cdot b = n \cdot \widehat{b} & \text{on } \Sigma_{0,pv}, \\ q + \dfrac{1}{2}|b|^2 - \dfrac{1}{2}|\widehat{b}|^2 = 0 & \text{on } \Sigma_{0,pv}, \end{cases} \tag{2.5}$$

where the exterior magnetic field $\widehat{b}$ satisfies the vacuum equations (A.5) in Lagrangian coordinates which can be recalled from Appendix A.

Since $\partial_t J = J\mathcal{A}_i^j \partial_j v^i = J\,\nabla_{\mathcal{A}} \cdot v$ and $J(0) = \det(Dh_0) = \det(I + Dg_0)$, with I the identity matrix, we find from the equation of f in (2.5) that $f\,J = \rho_0(h_0)\det(I+Dg_0)$, where ρ_0 is given initial density function. Taking ρ_0 such that $\rho_0(h_0)\det(I+Dg_0) = \overline{\rho}$, we get

$$f = J^{-1}\overline{\rho}, \quad q = AJ^{-\gamma}\overline{\rho}^{\gamma}. \tag{2.6}$$

On the other hand, we multiply the magnetic field equation of (2.5) by $J\mathcal{A}^T$ to get

$$J\mathcal{A}_j^i \partial_t b^j + J\mathcal{A}_j^i b^j \mathcal{A}_k^h \partial_h v^k = J\mathcal{A}_j^i b^h \mathcal{A}_h^k \partial_k v^j,$$

which along with (2.4) implies

$$\begin{aligned}\partial_t(\mathcal{A}_j^i J b^j) &= J\mathcal{A}_j^i \partial_t b^j + \mathcal{A}_j^i b^j \partial_t J + J b^j \partial_t \mathcal{A}_j^i = J\mathcal{A}_j^i \partial_t b^j + J\mathcal{A}_j^i b^j \mathcal{A}_k^h \partial_h v^k \\ &\quad - J b^j \mathcal{A}_k^i \mathcal{A}_j^h \partial_h v^k = 0.\end{aligned}$$

Therefore, we have

$$J b^j \mathcal{A}_j^i = J(0) b_0^j \mathcal{A}_j^i(0) = \det(I + Dg_0) B_0^j(h_0) \mathcal{A}_j^i(0), \tag{2.7}$$

where B_0 is given initial magnetic field. Taking B_0 such that $\det(I + Dg_0) B_0^j(h_0)\mathcal{A}_j^i(0) = \overline{B}^i$, we obtain from (2.7) that

$$b^k = J^{-1}\overline{B}^i \partial_i h^k = J^{-1}\overline{B}^k + J^{-1}\overline{B}^i \partial_i g^k.$$

2.2. *The equilibrium for the z-pinch plasma*

In this paper, our goal is to study the linear z-pinch instability for the compressible MHD equations (1.3). Therefore, we look for the cylindrically symmetric steady solution $\overline{u} = 0$, $\overline{B} = (0, \overline{B}_\theta(r), 0)$, $\overline{p} = \overline{p}(r)$, $\widehat{\overline{B}} = (0, \widehat{\overline{B}}_\theta(r), 0)$. For notational simplicity, in the following we abuse notation to denote steady state z-pinch solutions as

$$p(r) = \overline{p}(r), \quad B_\theta(r) = \overline{B}_\theta(r), \quad \widehat{B}_\theta(r) = \widehat{\overline{B}}_\theta(r),$$

which imply that $B = \overline{B}$ and $\widehat{B} = \widehat{\overline{B}}$.

Then we can get the following lemma describing the steady solution.

Lemma 2.1. *Assume that the function $p(r)$ satisfies $p(r) \geq 0$ and $p(r) = 0$ if and only if $r = r_0$, and*

$$-\int_0^r s^2 p'(s)ds \geq 0 \quad \text{for all } 0 \leq r \leq r_0, \quad p(r) \in C^{2,1}([0, r_0]). \tag{2.8}$$

Then the cylindrically symmetric steady solution $\overline{u} = 0$, $B = B(r)$, $\mathbb{J}_z = \mathbb{J}_z(r)$, $\widehat{B} = \widehat{B}(r)$ *with a function* $p(r)$ *taking the form of*

$$\begin{cases} B_r = 0, \quad B_z = 0, \quad B_\theta(r) = \left(-\dfrac{2}{r^2}\displaystyle\int_0^r s^2 p'(s)ds\right)^{\frac{1}{2}}, \\ \mathbb{J}_z(r) = \dfrac{1}{2}\left(-\dfrac{2}{r^2}\displaystyle\int_0^r s^2 p'(s)ds\right)^{-\frac{1}{2}}\left(\dfrac{4}{r^3}\displaystyle\int_0^r s^2 p'(s)ds - 2p'(r)\right) \\ \qquad\qquad + \dfrac{1}{r}\left(-\dfrac{2}{r^2}\displaystyle\int_0^r s^2 p'(s)ds\right)^{\frac{1}{2}} \quad in\ \overline{\Omega}, \\ \widehat{B}_r = 0, \quad \widehat{B}_z = 0, \quad \widehat{B}_\theta(r) = B_\theta(r_0)\dfrac{r_0}{r} \quad in \quad \overline{\Omega}^v, \end{cases} \tag{2.9}$$

solves the equilibrium equations in plasma domain,

$$\nabla p = \mathbb{J} \times B, \quad \nabla \cdot B = 0, \quad \mathbb{J} = \nabla \times B, \tag{2.10}$$

and the system (1.5) *in the vacuum region. We can define the equilibrium density*

$$\rho(r) = \left(\frac{p(r)}{A}\right)^{1/\gamma}.$$

Moreover, we have

$$\mathbb{J}_z \in C^{1,1}([0, r_0]), \quad B_\theta \in C^{1,1}([0, r_0]). \tag{2.11}$$

Proof. In cylindrical r, θ, z-coordinates, the equilibrium equations (2.10) which are equivalent to the system

$$\nabla\left(p + \frac{1}{2}|B|^2\right) = (B \cdot \nabla)B, \quad \nabla \cdot B = 0, \quad \mathbb{J} = \nabla \times B,$$

are reduced to

$$\begin{gathered} \frac{d}{dr}\left(p(r) + \frac{1}{2}|B_\theta(r)|^2\right) = -\frac{B_\theta^2(r)}{r}, \quad \frac{1}{r}\frac{d}{dr}(rB_r(r)) = 0, \\ \frac{1}{r}\frac{d}{dr}(rB_\theta(r)) = \mathbb{J}_z(r). \end{gathered} \tag{2.12}$$

The first equation of (2.12) is equivalent to $p' = -\mathbb{J}_z B_\theta$, which together with the third equation of (2.12) implies that $-r^2 p' = rB_\theta(rB_\theta)'$. Set $C(r) = rB_\theta(r)$ to reduce this to $(C^2)' = -2r^2 p'$. Integrating and forcing $B_\theta(0)$ to be finite, which gives $C(0) = 0$, we find that $B_\theta(r) = (-\frac{2}{r^2}\int_0^r s^2 p'(s)ds)^{\frac{1}{2}}$. Thus, given the pressure p, we can compute B_θ. Then we define $\mathbb{J}_z$ by the third equation of (2.12) that

$$\begin{aligned} \mathbb{J}_z(r) = \frac{1}{2}\left(-\frac{2}{r^2}\int_0^r s^2 p'(s)ds\right)^{-\frac{1}{2}}\left(\frac{4}{r^3}\int_0^r s^2 p'(s)ds - 2p'(r)\right) \\ + \frac{1}{r}\left(-\frac{2}{r^2}\int_0^r s^2 p'(s)ds\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, solving the system (1.5) and (2.12), we get the steady solution z-pinch ($\overline{u}=0, B=B(r)=B_\theta(r)e_\theta, \mathbb{J}_z=\mathbb{J}_z(r), \widehat{B}=\widehat{B}(r)=\widehat{B}_\theta(r)e_\theta$) as follows:

$$\begin{cases} B_r=0, \quad B_z=0, \quad B_\theta(r)=\left(-\dfrac{2}{r^2}\displaystyle\int_0^r s^2p'(s)ds\right)^{\frac{1}{2}}, \\ \mathbb{J}_z(r)=\dfrac{1}{2}\left(-\dfrac{2}{r^2}\displaystyle\int_0^r s^2p'(s)ds\right)^{-\frac{1}{2}}\left(\dfrac{4}{r^3}\displaystyle\int_0^r s^2p'(s)ds-2p'(r)\right) \\ \qquad\qquad +\dfrac{1}{r}\left(-\dfrac{2}{r^2}\displaystyle\int_0^r s^2p'(s)ds\right)^{\frac{1}{2}} \quad \text{in } \overline{\Omega}, \\ \widehat{B}_r=0, \quad \widehat{B}_z=0, \quad \widehat{B}_\theta(r)=B_\theta(r_0)\dfrac{r_0}{r} \quad \text{in } \overline{\Omega}^v. \end{cases}$$

Since $p\in C^{2,1}([0,r_0])$. The equilibrium magnetic field B_θ is determined in terms of p by the equation

$$B_\theta(r)=\left(-\frac{2}{r^2}\int_0^r s^2p'(s)ds\right)^{\frac{1}{2}},$$

which by forcing the value of $B_\theta(r)$ at $r=0$ to finite $B_\theta(0)$ and the value of $\mathbb{J}_z(r)$ at $r=0$ to finite $\mathbb{J}_z(0)$, gives that $B_\theta(0):=\lim_{r\to0}B_\theta(r)=\lim_{r\to0}(-\frac{2}{r^2}\int_0^r s^2p'(s)ds)^{\frac{1}{2}}=0$. Since $p'=-\mathbb{J}_z(r)B_\theta(r)$, we have $p'(0)=0$, which gives that

$$\begin{aligned} B'_\theta(0):=\lim_{r\to0}\frac{B_\theta(r)}{r}&=\lim_{r\to0}\left(-\frac{2}{r^4}\int_0^r s^2p'(s)ds\right)^{\frac{1}{2}} \\ &=\lim_{r\to0}\left(-\frac{p'(r)}{2r}\right)^{\frac{1}{2}}=\left(-\frac{p''(0)}{2}\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, we can get $B_\theta\in C^{1,1}([0,r_0])$. By $p'(r)=-B_\theta(r)\mathbb{J}_z(r)$, we have $\mathbb{J}_z\in C^{1,1}([0,r_0])$. □

From Lemma 2.1, we know that at the plasma–vacuum interface, the steady solution $B(r)$ in cylindrical r, θ, z-coordinates satisfies naturally

$$n_0\cdot B=n_0\cdot\widehat{B}=0, \quad \text{on } \Sigma_{0,pv},$$

due to $n_0=e_r$, $B=(0,B_\theta(r),0)$ and $\widehat{B}=(0,B_\theta(r_0)\frac{r}{r_0},0)$.

Now we introduce the admissibility of the pressure p, which will be used in the following sections.

Definition 2.2. We say that p is admissible if $p(r)\geq0$ for all $r\in[0,r_0]$ and $p(r)=0$ if and only if $r=r_0$, $p'(r)\leq0$ for r near r_0, that is, $p'(s)\leq0$ for $s\in(r_0-\epsilon,r_0]$ with small constant $\epsilon>0$, and $p(r)$ satisfies (2.8) and

$$\lim_{r\to r_0}\frac{p(r)}{p'(r)}=0. \tag{2.13}$$

2.3. *Perturbed ideal MHD system and main results*

Our main results concern the stability of the linearized problem. From Appendix A, we find that this linearization is

$$\begin{cases} \partial_t g = v & \text{in } \overline{\Omega}, \\ \rho\partial_{tt} g = \nabla(g\cdot\nabla p + \gamma p \nabla\cdot g) + (\nabla\times B)\times[\nabla\times(g\times B)] & \\ \qquad\quad + \{\nabla\times[\nabla\times(g\times B)]\}\times B, & \text{in } \overline{\Omega}, \\ \nabla\cdot\widehat{Q} = 0, & \text{in } \overline{\Omega}^v, \\ \nabla\times\widehat{Q} = 0, & \text{in } \overline{\Omega}^v, \\ n\cdot\nabla\times(g\times\widehat{B}) = n\cdot\widehat{Q}, & \text{on } \Sigma_{0,pv}, \\ -\gamma p\nabla\cdot g + B\cdot Q + g\cdot\nabla\left(\frac{1}{2}|B|^2\right) = \widehat{B}\cdot\widehat{Q} + g\cdot\nabla\left(\frac{1}{2}|\widehat{B}|^2\right), & \text{on } \Sigma_{0,pv}, \\ n\cdot\widehat{Q}|_{\Sigma_w} = 0, & \\ g|_{t=0} = g_0, & \end{cases} \tag{2.14}$$

with $Q = \nabla\times(g\times B)$.

Define the energy

$$E[g,\widehat{Q}] = E^p[g] + E^s[g] + E^v[\widehat{Q}], \tag{2.15}$$

where $E^p[g]$, $E^s[g]$ and $E^v[\widehat{Q}]$ are given by

$$E^p[g] = \frac{1}{2}\int_{\overline{\Omega}}[\gamma p|\nabla\cdot g|^2 + Q^2 + (g^*\cdot\nabla p)\nabla\cdot g + (\nabla\times B)\cdot(g^*\times Q)]dx, \tag{2.16}$$

$$E^s[g] = \frac{1}{2}\int_{\Sigma_{0,pv}}|n\cdot g|^2 n\cdot\left[\left[\nabla\left(p+\frac{1}{2}|B|^2\right)\right]\right]dx, \tag{2.17}$$

$$E^v[\widehat{Q}] = \frac{1}{2}\int_{\overline{\Omega}^v}|\widehat{Q}|^2 dx.$$

The motivation for introducing E is two-fold. First, as we show in Lemma A.5, for solutions to (2.14) we have the evolution equation

$$\frac{d}{dt}\|\sqrt{\rho}g_t\|_{L^2}^2 = -\frac{d}{dt}E[g,\widehat{Q}]. \tag{2.18}$$

Second, and more important for our analysis, is that upon using the Fourier transform in the θ and z variables to decompose E into an infinite sum

$$E[g,\hat{Q}] = \sum_{m,k\in\mathbb{Z}} E_{m,k}, \tag{2.19}$$

we arrive at a mechanism for producing special normal-mode solutions to (2.14) via a variational analysis applied to $E_{m,k}$. Indeed, we seek normal mode solutions of the form

$$\begin{aligned} g(r,\theta,z,t) &= (g_{r,mk}(r), g_{\theta,mk}(r), g_{z,mk}(r))e^{\mu t+i(m\theta+kz)}, \\ \widehat{Q}(r,\theta,z) &= (i\widehat{Q}_{r,mk}(r), \widehat{Q}_{\theta,mk}(r), \widehat{Q}_{z,mk}(r))e^{\mu t+i(m\theta+kz)}, \end{aligned} \tag{2.20}$$

where (r,θ,z) are standard cylindrical coordinates, $\mu > 0$, and $m,k \in \mathbb{Z}$. As we will prove in the next section, solutions of this form correspond to constrained minimizers of $E_{m,k}$, and the value of μ can be computed in terms of m and k via this minimization.

It is convenient to introduce some new unknowns (here dropping subscripts m and k for notational simplicity) to reduce to study only real-valued functions. In the above ansatz, it is convenient to assume that g_θ and g_z are pure imaginary functions, while g_r, $\widehat{Q}_r$, $\widehat{Q}_\theta$ and $\widehat{Q}_z$ are real-valued functions. Then we define three real-valued functions

$$\xi = e_r \cdot g = g_r, \quad \eta = -ie_z \cdot g = -ig_z, \quad \zeta = ie_\theta \cdot g = ig_\theta. \tag{2.21}$$

which together with (2.20), gives that

$$\begin{aligned} Q &= \nabla \times (g \times B) = \frac{im}{r} B_\theta \xi e_r - [(B_\theta \xi)' - kB_\theta \eta] e_\theta - \frac{m}{r} \eta B_\theta e_z, \\ g \cdot \nabla p + \gamma p \nabla \cdot g &= p'\xi + \gamma p \nabla \cdot g, \quad \nabla \cdot g = \frac{1}{r}(r\xi)' - k\eta + \frac{m}{r}\zeta, \end{aligned} \tag{2.22}$$

where the factor $e^{\mu t+i(m\theta+kz)}$ is dropped for notational simplicity.

In terms of ξ, η and ζ, from the expressions in (2.22), the boundary conditions in (2.14) are transformed to

$$\widehat{Q}_r = 0, \quad \text{at } r = r_w, \tag{2.23}$$

$$m\widehat{B}_\theta \xi = r\widehat{Q}_r, \quad \text{at } r = r_0, \tag{2.24}$$

$$B_\theta^2 \xi - B_\theta^2 \xi' r + kB_\theta^2 \eta r - \widehat{B}_\theta \widehat{Q}_\theta r = 0, \quad \text{at } r = r_0. \tag{2.25}$$

We impose the boundary conditions (2.23) and (2.24) as constraint for variational problem setup and the boundary condition (2.25) follows the minimizer solution. When $m = 0$, we know that $\widehat{Q}_r = 0$ on the boundary $r = r_0$, which implies that $\widehat{Q}_r$ is separated from interior variational problem. Obviously, it holds that $\widehat{Q} = 0$. Therefore, for the case $m = 0$ and any k, the energy functional (2.15) reduces to

$$\begin{aligned} E_{0,k} = E_{0,k}(\xi,\eta) = 2\pi^2 \int_0^{r_0} &\left\{ \left[\frac{2p'}{r} + \frac{4\gamma p B_\theta^2}{r^2(\gamma p + B_\theta^2)} \right] \xi^2 \right. \\ &\left. + (\gamma p + B_\theta^2) \left[k\eta - \frac{1}{r}\left((r\xi)' - \frac{2B_\theta^2}{\gamma p + B_\theta^2}\xi \right) \right]^2 \right\} rdr. \end{aligned} \tag{2.26}$$

For the case $m \neq 0$ and any k, the solution $\widehat{Q}_r$ and ξ are related by the boundary conditions (2.24), so we cannot set $\widehat{Q}_r = 0$, therefore the energy functional takes the form of

$$\begin{aligned} E_{m,k} &= E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r) \\ &= 2\pi^2 \int_0^{r_0} \left\{ (m^2 + k^2 r^2) \left[\frac{B_\theta}{r}\eta + \frac{-kB_\theta (r\xi)' + 2kB_\theta \xi}{m^2 + k^2 r^2} \right]^2 \right. \\ &\quad \left. + \gamma p \left[\frac{1}{r}(r\xi)' - k\eta + \frac{m}{r}\zeta \right]^2 \right\} r dr + 2\pi^2 \int_0^{r_0} \frac{m^2 B_\theta^2}{r(m^2 + k^2 r^2)} (\xi - r\xi')^2 \\ &\quad + 2\pi^2 \int_0^{r_0} \left[2p' + \frac{m^2 B_\theta^2}{r} \right] \xi^2 dr \\ &\quad + 2\pi^2 \int_{r_0}^{r_w} \left[|\widehat{Q}_r|^2 + \frac{1}{m^2 + k^2 r^2} |(r\widehat{Q}_r)'|^2 \right] r dr. \end{aligned} \tag{2.27}$$

We can now state our main results.

Theorem 2.1. *Assume that the equilibrium pressure is admissible in the sense of Definition 2.2 and that (2.13) holds. For $k, m \in \mathbb{Z}$ set*

$$\lambda_{m,k} = \inf_{\mathcal{A}} E_{m,k} \tag{2.28}$$

where $\mathcal{A} = \mathcal{A}_1$ as defined in (3.24) when $m = 0$ and $\mathcal{A} = \mathcal{A}_2$ as defined in (3.50) when $m \neq 0$. Then the following hold.

(1) *$\lambda_{0,k} < 0$ for every $k \in \mathbb{Z}$. Upon setting $\mu_{0,k} = \sqrt{-\lambda_{0,k}} > 0$, this yields an unstable growing mode solution to (2.14). In particular, the equilibrium is always linearly unstable.*

(2) *If $m \neq 0$, $k \in \mathbb{Z}$, and there exists $r^* \in [0, r_0)$ such that*

$$2p'(r^*) + \frac{m^2 B_\theta^2(r^*)}{r^*} < 0, \tag{2.29}$$

then $\lambda_{m,k} < 0$ and $\mu_{m,k} = \sqrt{-\lambda_{m,k}} > 0$ is the growth rate of an unstable growing mode solution to (2.14).

(3) *If the pressure obeys the estimate $|p'| \leq C\rho$ an open neighborhood of r_0, then*

$$\sup\{-\lambda_{m,k} | m, k \in \mathbb{Z} \text{ are such that } \lambda_{m,k} < 0\} < \infty. \tag{2.30}$$

Remark 2.1. In the plasma literature, unstable modes with $m = 0$ are called sausage instabilities, and modes with $|m| = 1$ are called kink instabilities.

The first item of this theorem establishes the unconditional instability of the admissible equilibria, but the second item is conditional. In Sec. 2.4, we give some conditions for determining when this conditional is and is not satisfied. The third item is built on the assumption that $|p'(r)| \leq C\rho(r)$ for $r \in (r_0 - \epsilon, r_0)$. Since

$p = A\rho^{\gamma}$, this is a differential inequality that we can integrate to deduce that p satisfies

$$p(r) \leq C(r_0 - r)^{\gamma/(\gamma-1)} \quad \text{for } r \in (r_0 - \epsilon, r_0), \tag{2.31}$$

where $C > 0$ is some constant. Thus, the assumption in the third item bounds from below the rate at which p decays near the vacuum boundary; for instance, if $p(r) \simeq (r_0 - r)^{\beta}$ near r_0, then we must have that $\gamma/(\gamma - 1) \leq \beta$. It is then natural to examine what happens in the complementary regime. This is the content of our second main result.

Theorem 2.2. *Assume that the equilibrium pressure is admissible in the sense of Definition 2.2 and that (2.13) holds. If there exist constants $C_0, C_1 > 0$ and $1 < \beta < \gamma/(\gamma - 1)$ such that*

$$C_0(r_0 - r)^{\beta-1} \leq |p'(r)| \leq C_1(r_0 - r)^{\beta-1} \tag{2.32}$$

for r in an open neighborhood of r_0, then

$$\lim_{k\to\infty} \lambda_{0,k} = -\infty. \tag{2.33}$$

Clearly, the hypotheses on p' imply that $p(r) \simeq (r_0 - r)^{\beta}$ and $\beta < \gamma/(\gamma - 1)$, so this is indeed a sort of complementary regime compared to the third item of Theorem 2.1. The immediate consequence of Theorem 2.2 is that there exists growing normal mode solutions to (2.14) with growth rate $\mu_{0,k} \to \infty$, and as such the linearized equations are ill-posed. Thus, depending on the rate of decay of the pressure near the vacuum boundary, we either have instability with a bound on the growth-in-time rates, or else ill-posedness due to the unboundedness of the growth rates.

These theorems are proved in a number of separate results in Secs. 3 and 4. We now turn to a somewhat technical survey of how we proceed in the proofs. The key observation lies in Lemma 2.4, which leads to a growing mode in so-called sausage instability. In order to construct a growing mode, we study the variational problem for the linearized functional (2.15). It should be noted that (2.16) only $\operatorname{div} g$ is expected to be bounded, which fails to provide necessary compactness to find a minimizer or an eigenfunction.

To overcome this seemingly lack of compactness, we study carefully the variational problem in cylindrical charts. It turns out that, thanks to special symmetry of the z-pinch profile, the energy takes the form of (2.26) for $m = 0$ and any k, takes the form of (2.27) for $m \neq 0$ and any k. The only possible negative part which needs compactness in (2.26) and (2.27) are given by

$$\int_0^{r_0} 2p'\xi^2 dr.$$

We note crucially it depends only on ξ. Luckily, it is possible to control $\partial_r \xi$ so that the compactness is established away from $r = 0$ and $r = r_0$.

The compactness at $r = r_0$ and $r = 0$ is delicate, due to subtle vanishing order of the z-pinch. Near $r = r_0$, we make use of an integration by parts to derive estimate (3.13) and to gain compactness as in Lemmas 3.8 and 3.15. Near $r = 0$, however, we make use of an expansion of exact z-pinch profile in Lemma 2.3, to gain subtle but crucial higher vanishing power and positivity of lower power, more precisely, we observe

$$\begin{aligned}
&\frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi_n - r\xi_n')^2 + \left[2p' + \frac{m^2 B_\theta^2}{r}\right]\xi_n^2 \\
&\quad = \left[\frac{1}{4}\mathbb{J}_z^2(0)r + O(r^2)\right] r^2\xi_n'^2 + O(r^3)\xi_n^2, \quad \text{for } |m| = 1
\end{aligned}$$

and

$$\left[2p' + \frac{m^2 B_\theta^2}{r}\right]\xi_n^2 = \left[\left(\frac{m^2}{4} - 1\right)\mathbb{J}_z^2(0)r + O(r^2)\right]\xi_n^2, \quad \text{for } |m| \geq 2$$

to ensure compactness. See Proposition 3.19 for the details.

We establish the ill-posedness in Sec. 4. In the case $p(r) = C(r_0 - r)^\beta$ for r near r_0 and $\beta \geq 1$. If $\gamma < \frac{\beta}{\beta-1}$, we use a careful scaling analysis to construct particular test function of the form $\xi_k = w(k^\alpha[r - r_0])$, without lower bound as $k \to \infty$.

2.4. *Some more properties of the equilibria*

Here we record a few more properties of the equilibria that will be used in our subsequent analysis. From Taylor expanding and the steady state (2.9), we have the following lemma.

Lemma 2.3. *Assume the function* $\mathbb{J}_z(r)$ *satisfies* (2.11), *we have* $\mathbb{J}_z(r) = \mathbb{J}_z(0) + r\mathbb{J}_z'(0) + O(r^2)$. *Moreover, we have*

$$B_\theta(r) = \frac{1}{2}\mathbb{J}_z(0)r + \frac{1}{3}\mathbb{J}_z'(0)r^2 + O(r^3), \tag{2.34}$$

$$p'(r) = -\frac{1}{2}\mathbb{J}_z^2(0)r - \frac{5}{6}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3). \tag{2.35}$$

Proof. Since $\mathbb{J}_z(r)$ satisfies (2.11), we can expand $\mathbb{J}_z(r)$ near $r = 0$ as

$$\mathbb{J}_z(r) = \mathbb{J}_z(0) + r\mathbb{J}_z'(0) + O(r^2). \tag{2.36}$$

From the steady state (2.9), we know that

$$B_\theta(r) = \frac{1}{r}\int_0^r \mathbb{J}_z(r)rdr,$$

which together with (2.36) gives (2.34). Therefore, we have

$$B_\theta'(r) = \frac{1}{2}\mathbb{J}_z(0) + \frac{2}{3}\mathbb{J}_z'(0)r + O(r^2),$$

$$p'(r) = -B_\theta(r)B_\theta'(r) - \frac{B_\theta^2(r)}{r}$$

$$= -\left[\frac{1}{2}\mathbb{J}_z(0)r + \frac{1}{3}\mathbb{J}_z'(0)r^2 + O(r^3)\right]\left[\frac{1}{2}\mathbb{J}_z(0) + \frac{2}{3}\mathbb{J}_z'(0)r + O(r^2)\right]$$
$$- \frac{\frac{1}{4}\mathbb{J}_z^2(0)r^2 + \frac{1}{3}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^3 + O(r^4)}{r}$$
$$= -\frac{1}{2}\mathbb{J}_z^2(0)r - \frac{5}{6}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3).$$

□

Now, we give the properties of the steady solution that will be essential in proving the instability of any z-pinch equilibrium for $m = 0$.

Lemma 2.4. *There exists $r_* \in (0, r_0)$ such that*

$$p'(r_*) + \frac{2\gamma p(r_*)B_\theta^2(r_*)}{r_*(\gamma p(r_*) + B_\theta^2(r_*))} < 0. \tag{2.37}$$

Proof. First note that simple algebra reveals that (2.37) is equivalent to the existence of $r_* \in (0, r_0)$ such that

$$p'(r_*) + \frac{2\gamma}{r_*}p(r_*) - \frac{2\gamma^2 p^2(r_*)}{r_*(\gamma p(r_*) + B_\theta^2(r_*))} < 0.$$

Consider the function $q \in C^1([0, r_0])$ given by $q(r) = r^{2\gamma}p(r)$. We have that $q(0) = 0$ and $q(r_0) = r_0^{2\gamma}p(r_0) = 0$, so by the mean-value theorem there exists $r_* \in (0, r_0)$ such that

$$0 = q'(r_*) = r_*^{2\gamma}p'(r_*) + 2\gamma {r_*}^{2\gamma-1}p(r_*) = r_*^{2\gamma}\left(p'(r_*) + \frac{2\gamma p(r_*)}{r_*}\right).$$

Consequently,

$$0 = p'(r_*) + \frac{2\gamma p(r_*)}{r_*}. \tag{2.38}$$

We know that $p(r_*) > 0$, and so

$$-\frac{2\gamma^2 p^2(r_*)}{r_*(\gamma p(r_*) + B_\theta^2(r_*))} < 0. \tag{2.39}$$

Combining (2.38) and (2.39), we conclude that

$$p'(r_*) + \frac{2\gamma}{r}p(r_*) - \frac{2\gamma^2 p^2(r_*)}{r_*(\gamma p(r_*) + B_\theta^2(r_*))} = -\frac{2\gamma^2 p^2(r_*)}{r_*(\gamma p(r_*) + B_\theta^2(r_*))} < 0.$$

This concludes the proof. □

Next, we give the following property of the steady solution when $|m| \geq 2$ and $\mathbb{J}_z(r)$ is non-increasing and non-negative.

Lemma 2.5. *For $|m| \geq 2$, suppose that $\mathbb{J}_z : [0, r_0] \to [0, \infty)$ is non-increasing, then we have*

$$2p'(r) + m^2 \frac{B_\theta^2(r)}{r} \geq 0 \quad \textit{for all } r \in [0, r_0].$$

Proof. Note that

$$2p'(r) + m^2 \frac{B_\theta^2(r)}{r} = B_\theta(r) \left(-2\mathbb{J}_z(r) + m^2 \frac{B_\theta(r)}{r} \right).$$

Since $\mathbb{J}_z$ is non-increasing and non-negative, we have

$$B_\theta(r) = \frac{1}{r} \int_0^r s\mathbb{J}_z(s)ds \geq \frac{\mathbb{J}_z(r)}{r} \int_0^r sds = \frac{r\mathbb{J}_z(r)}{2}.$$

Hence,

$$-2\mathbb{J}_z(r) + m^2 \frac{B_\theta(r)}{r} \geq -2\mathbb{J}_z(r) + \frac{m^2}{2}\mathbb{J}_z(r) \geq \mathbb{J}_z(r)\frac{(m^2-4)}{2} \geq 0.$$

Since $B_\theta(r) \geq 0$ as well which can be obtained from the third equation of the ODEs (2.12), we deduce that

$$2p'(r) + m^2 \frac{B_\theta^2(r)}{r} \geq 0. \qquad \square$$

Note that Lemma 2.5 implies that the hypothesis of the second item of Theorem 2.1 fails. Now we will give an example when $|m| \geq 2$, the instability condition $2p'(r^*) + m^2 \frac{B_\theta^2(r^*)}{r^*} < 0$ holds for some $r^* \in (0, r_0)$ and $\mathbb{J}_z$ vanishing at the origin $r = 0$ in suitable order.

Lemma 2.6. *For $|m| \geq 2$, we define $\alpha = (m^2 - 2)/2 \geq 1$. Suppose that $\beta > \alpha \geq 0$ and $\mathbb{J}_z$ vanishes to order β at the origin $r = 0$ in the sense that $|\mathbb{J}_z(r)| \leq Cr^\beta$, and further suppose that $B_\theta \neq 0$ in $(0, r_0]$, i.e. B_θ has a sign. Then there exists $r^* \in (0, r_0)$ such that*

$$2p'(r^*) + m^2 \frac{B_\theta^2(r^*)}{r^*} < 0.$$

Proof. We will only prove the result assuming that $B_\theta > 0$ in $(0, r_0]$, as the other case follows similarly. For $r \in [0, r_0]$, we compute that

$$\begin{aligned}
2p'(r) + m^2 \frac{B_\theta^2(r)}{r} &= -2\frac{B_\theta^2(r)}{r} - 2B_\theta(r)B_\theta'(r) + m^2 \frac{B_\theta^2(r)}{r} \\
&= (m^2 - 2)\frac{B_\theta^2(r)}{r} - 2B_\theta(r)B_\theta'(r) \\
&= -2B_\theta(r) \left(B_\theta'(r) - \frac{\alpha}{r} B_\theta(r) \right) \\
&= -2r^\alpha B_\theta(r)(r^{-\alpha} B_\theta(r))'.
\end{aligned}$$

Now, we may estimate from the third equation of (2.12) and the assumption that

$$\frac{|B_\theta(r)|}{r^{\alpha+1}} \leq \frac{1}{r^{2+\alpha}} \int_0^r s|\mathbb{J}_z(s)|ds \leq \frac{C}{r^{2+\alpha}} \int_0^r s^{1+\beta} ds = \frac{C}{2+\beta} \frac{r^{2+\beta}}{r^{2+\alpha}} = \frac{C}{2+\beta} r^{\beta-\alpha}$$

and

$$\frac{|B_\theta'(r)|}{r^\alpha} \leq \frac{\mathbb{J}_z(r)}{r^\alpha} + \frac{1}{r^{2+\alpha}} \int_0^r s|\mathbb{J}_z(s)|ds \leq Cr^{\beta-\alpha} + \frac{C}{2+\beta} r^{\beta-\alpha}$$

in order to conclude that $[0, r_0] \ni r \mapsto B_\theta(r)/r^\alpha \in [0, \infty)$ is a continuous function. Hence if we define $q : [0, r_0] \to [0, \infty)$ via $q(r) = r^{-\alpha} B_\theta(r)$, then we find that $q \in C^1([0, r_0])$. Note that

$$q(0) = 0 \quad \text{and} \quad q(r_0) = r_0^{-\alpha} B_\theta(r_0) > 0$$

since $B_\theta(r_0) > 0$. By the mean-value theorem there exists $r^* \in (0, r_0)$ such that $q'(r^*) > 0$. Therefore, we have

$$2p'(r^*) + m^2 \frac{B_\theta^2(r^*)}{r^*} = -2r^{*\alpha} B_\theta(r^*) q'(r^*) < 0. \qquad \square$$

We remark that Lemma 2.6 implies the instability for $|m| \geq 2$ for some class of z-pinch equilibria. An interesting corollary is found if we suppose that $\mathbb{J}_z$ is non-negative and is compactly supported.

Corollary 2.7. *If $\mathbb{J}_z \geq 0$ and $\mathbb{J}_z$ is compactly supported in $(0, r_0)$, then for each $m \in \mathbb{Z} \backslash \{0\}$ there exists $r^* \in (0, r_0)$ such that*

$$2p'(r^*) + m^2 \frac{B_\theta^2(r^*)}{r^*} < 0.$$

We remark that Corollary 2.7 implies the instability for any $m \in \mathbb{Z} \backslash \{0\}$ for some class of z-pinch equilibria.

3. A Family of Variational Problems

3.1. *Growing mode ansatz and cylindrical coordinates*

In this paper, we mainly study the normal mode solution for the linearized perturbation (2.14) in cylindrical coordinates e_r, e_θ and e_z. In order to write the energy in cylindrical coordinates, we now record several computations in cylindrical coordinates. In cylindrical coordinates, under the normal mode (2.20), the result of the gradient operator becomes algebraic multipliers $\nabla = e_r \partial_r + ike_z + \frac{im}{r} e_\theta$ and we can get the following lemma.

Lemma 3.1. *We decompose $E_{m,k}$ as follows*:

$$E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r) = E^p_{m,k} + E^s_{m,k} + E^v_{m,k}, \tag{3.1}$$

where the fluid energy takes the form of

$$E^p_{m,k} = 2\pi^2 \int_0^{r_0} \left\{ \frac{m^2 B_\theta^2}{r^2(m^2+k^2r^2)} [(r\xi)']^2 + \beta_0 (r\xi)^2 \right.$$
$$+ (m^2+k^2r^2) \left[\frac{B_\theta}{r}\eta + \frac{-kB_\theta(r\xi)' + 2kB_\theta\xi}{m^2+k^2r^2} \right]^2$$
$$\left. + \gamma p \left[\frac{1}{r}(r\xi)' - k\eta + \frac{m}{r}\zeta \right]^2 \right\} rdr - 2\pi^2 \left[\frac{2m^2B_\theta^2}{m^2+k^2r^2}\xi^2 \right]_{r=r_0}, \tag{3.2}$$

with

$$\beta_0 = \frac{1}{r}\left[\frac{m^2B_\theta^2}{r^3} + \frac{2m^2B_\theta\left(\frac{B_\theta}{r}\right)'}{r(m^2+k^2r^2)} - \frac{4k^2m^2B_\theta^2}{r(m^2+k^2r^2)^2} + \frac{2k^2p'}{m^2+k^2r^2} \right],$$

the surface energy vanishes

$$E^s_{m,k} = -2\pi^2 [\widehat{B}_\theta^2 - B_\theta^2]_{r=r_0} \xi^2(r_0) = 0, \tag{3.3}$$

and when $m \neq 0$ and any k, the vacuum energy takes the form of

$$E^v_{m,k} = 2\pi^2 \int_{r_0}^{r_w} \left[|\widehat{Q}_r|^2 + \frac{1}{m^2+k^2r^2} |(r\widehat{Q}_r)'|^2 \right] rdr. \tag{3.4}$$

Proof. Recall (ξ,η,ζ) in (2.21) and (2.20). We begin with the proof of (3.2). Inserting the expressions of (2.22) into (2.16) and using $g_\theta^* = i\zeta$, we can get (3.2). In fact,

$$(g^*\cdot\nabla p)(\nabla\cdot g) = \xi p' \left(\frac{1}{r}(r\xi)' - k\eta + \frac{m}{r}\zeta \right),$$
$$\begin{aligned}(\nabla\times B)\cdot(g^*\times Q) &= \left(B_\theta' + \frac{B_\theta}{r} \right)\left[-\xi((B_\theta\xi)' - kB_\theta\eta) - \frac{im}{r}B_\theta\xi g_\theta^* \right] \\ &= \left(B_\theta' + \frac{B_\theta}{r} \right)\left[-\xi((B_\theta\xi)' - kB_\theta\eta) + \frac{m}{r}\zeta B_\theta\xi \right],\end{aligned}$$

which combining with (2.16) and (2.22) gives (3.2).

Now we turn to the proof of (3.3). From (2.17) and the equilibrium equations (2.12), we have

$$E^s_{m,k} = -2\pi^2 [\widehat{B}_\theta^2 - B_\theta^2]_{r=r_0} \xi^2(r_0),$$

which together with $p + \frac{1}{2}B_\theta^2 = \frac{1}{2}\widehat{B}_\theta^2$ and $p = 0$ on the interface boundary $r = r_0$, gives that $E^s_{m,k} = 0$.

Finally, we prove (3.4). From the vacuum equation $(2.14)_3$ and $\widehat{Q}$ in (2.20), it follows that

$$\frac{1}{r}(r\widehat{Q}_r)' + \frac{m}{r}\widehat{Q}_\theta + k\widehat{Q}_z = 0, \tag{3.5}$$

that is, $\widehat{Q}_z = -\frac{(r\widehat{Q}_r)' + m\widehat{Q}_\theta}{kr}$. Inserting the expression of $\widehat{Q}_z$ into $E^v_{m,k}[\widehat{Q}] = \frac{1}{2}\int_{\overline{\Omega}} |\widehat{Q}|^2 dx$, implies that

$$\begin{aligned} E^v_{m,k} &= 2\pi^2 \int_{r_0}^{r_w} [|\widehat{Q}_r|^2 + |\widehat{Q}_\theta|^2 + |\widehat{Q}_z|^2] r dr \\ &= 2\pi^2 \int_{r_0}^{r_w} \left[|\widehat{Q}_r|^2 + |\widehat{Q}_\theta|^2 + \frac{1}{k^2 r^2} |(r\widehat{Q}_r)' + m\widehat{Q}_\theta|^2 \right] r dr. \end{aligned} \tag{3.6}$$

From the vacuum equations $(2.14)_2$ and $\widehat{Q}$ in (2.20), we have

$$\begin{cases} \dfrac{d}{dr}\widehat{Q}_\theta + \dfrac{m}{r}\widehat{Q}_r + \dfrac{1}{r}\widehat{Q}_\theta = 0, \\ k\widehat{Q}_r + \dfrac{d}{dr}\widehat{Q}_z = 0, \\ \dfrac{m}{r}\widehat{Q}_z - k\widehat{Q}_\theta = 0. \end{cases} \tag{3.7}$$

Using the third equation in (3.7), from (3.5), we can obtain the tangential components of $\widehat{Q}$ in terms of radial component

$$\widehat{Q}_\theta = -\frac{m}{m^2 + k^2 r^2}(r\widehat{Q}_r)', \quad \widehat{Q}_z = -\frac{kr}{m^2 + k^2 r^2}(r\widehat{Q}_r)'. \tag{3.8}$$

From (3.8), we know that the first equation and the second equation in (3.7) are equivalent. From (3.6) and (3.8), the vacuum energy takes form of

$$\begin{aligned} E^v_{m,k} &= 2\pi^2 \int_{r_0}^{r_w} \left[|\widehat{Q}_r|^2 + |\widehat{Q}_\theta|^2 + \frac{1}{k^2 r^2} |(r\widehat{Q}_r)' + m\widehat{Q}_\theta|^2 \right] r dr \\ &= 2\pi^2 \int_{r_0}^{r_w} \left[|\widehat{Q}_r|^2 + \frac{1}{m^2 + k^2 r^2} |(r\widehat{Q}_r)'|^2 + \frac{m^2 + k^2 r^2}{k^2 r^2} \right. \\ &\quad \left. \times \left| \widehat{Q}_\theta + \frac{m}{m^2 + k^2 r^2}(r\widehat{Q}_r)' \right|^2 \right] r dr \\ &= 2\pi^2 \int_{r_0}^{r_w} \left[|\widehat{Q}_r|^2 + \frac{1}{m^2 + k^2 r^2} |(r\widehat{Q}_r)'|^2 \right] r dr. \end{aligned}$$

□

We will use Lemma 3.1 to prove the following equivalent energy functionals.

Proposition 3.2. *The energy functionals* (3.1) *to* (3.4) *take forms of* (2.26) *and* (2.27), *respectively.*

Proof. From Lemma 3.1 and let $m = 0$ in (3.2), we can get directly

$$E_{0,k}(\xi,\eta) = 2\pi^2 \int_0^{r_0} \left\{ \frac{2p'\xi^2}{r} + B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi) \right]^2 + \gamma p \left[\frac{1}{r}(r\xi)' - k\eta \right]^2 \right\} r dr.$$

Together with the identity

$$\begin{aligned} &\frac{4\gamma p B_\theta^2 \xi^2}{r(\gamma p + B_\theta^2)} + (\gamma p + B_\theta^2) \left[\frac{4B_\theta^4 \xi^2}{r^2(\gamma p + B_\theta^2)^2} + \left(k\eta - \frac{1}{r}(r\xi)' \right) \frac{4B_\theta^2 \xi}{r(\gamma p + B_\theta^2)} \right] r \\ &\quad = \frac{4B_\theta^2 \xi^2}{r} + 4B_\theta^2 \xi \left(k\eta - \frac{1}{r}(r\xi)' \right), \end{aligned}$$

we establish (2.26). From Lemma 3.1, it follows that

$$\begin{aligned} E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r) = 2\pi^2 \int_0^{r_0} & \left\{ \frac{m^2 B_\theta^2}{r^2(m^2+k^2r^2)} [(r\xi)']^2 + \beta_0 (r\xi)^2 \right. \\ & + (m^2+k^2r^2) \left[\frac{B_\theta}{r}\eta + \frac{-kB_\theta (r\xi)' + 2kB_\theta \xi}{m^2+k^2r^2} \right]^2 \\ & \left. + \gamma p \left[\frac{1}{r}(r\xi)' - k\eta + \frac{m}{r}\zeta \right]^2 \right\} r dr - 2\pi^2 \left[\frac{2m^2 B_\theta^2}{m^2+k^2r^2} \xi^2 \right]_{r=r_0} \\ & + 2\pi^2 \int_{r_0}^{r_w} \left[|\widehat{Q}_r|^2 + \frac{1}{m^2+k^2r^2} |(r\widehat{Q}_r)'|^2 \right] r dr, \end{aligned}$$

with

$$\beta_0 = \frac{1}{r} \left[\frac{m^2 B_\theta^2}{r^3} + \frac{2m^2 B_\theta \left(\frac{B_\theta}{r} \right)'}{r(m^2+k^2r^2)} - \frac{4k^2 m^2 B_\theta^2}{r(m^2+k^2r^2)^2} + \frac{2k^2 p'}{m^2+k^2r^2} \right].$$

By $p' = -B_\theta B_\theta' - \frac{B_\theta^2}{r}$, we have

$$\begin{aligned} &\int_0^{r_0} \left(\frac{m^2 B_\theta^2}{r^2(m^2+k^2r^2)} [(r\xi)']^2 + \beta_0 (r\xi)^2 \right) r dr - \left[\frac{2m^2 B_\theta^2}{m^2+k^2r^2} \xi^2 \right]_{r=r_0} \\ &\quad = \int_0^{r_0} \left[\frac{m^2 B_\theta^2}{r(m^2+k^2r^2)} (\xi - r\xi')^2 + 2p'\xi^2 + \frac{m^2 B_\theta^2 \xi^2}{r} \right] dr, \end{aligned}$$

which implies the energy (2.27). □

Using $g(r,\theta,z,t) = (g_r(r,t), g_\theta(r,t), g_z(r,t)) e^{i(m\theta + kz)}$, we can prove that the second equation in (2.14) is reduced to the following system.

Lemma 3.3. *Assume* $g(r,\theta,z,t) = (g_r(r,t), g_\theta(r,t), g_z(r,t))e^{i(m\theta+kz)}$ *solves the second equation in* (2.14), *then*

$$\begin{pmatrix} \frac{d}{dr}\frac{\gamma p + B_\theta^2}{r}\frac{d}{dr}r - \frac{m^2}{r^2}B_\theta^2 - r\left(\frac{B_\theta^2}{r^2}\right)' & -\frac{d}{dr}k(\gamma p + B_\theta^2) - \frac{2kB_\theta^2}{r} & \frac{d}{dr}\frac{m}{r}\gamma p \\ \frac{k(\gamma p + B_\theta^2)}{r}\frac{d}{dr}r - \frac{2kB_\theta^2}{r} & -k^2(\gamma p + B_\theta^2) - \frac{m^2}{r^2}B_\theta^2 & \frac{mk}{r}\gamma p \\ -\frac{m\gamma p}{r^2}\frac{d}{dr}r & \frac{mk}{r}\gamma p & -\frac{m^2}{r^2}\gamma p \end{pmatrix}$$
$$\times \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \rho \begin{pmatrix} \xi_{tt} \\ \eta_{tt} \\ \zeta_{tt} \end{pmatrix}. \tag{3.9}$$

Proof. Inserting the expression (2.22) into the second equation in (2.14), by $g(r,\theta,z,t) = (g_r(r,t), g_\theta(r,t), g_z(r,t))e^{i(m\theta+kz)}$ and the definitions of ξ, η and ζ in (2.21), we can easily get that the second equation in (2.14) reduces to (3.9). □

In order to study the stability to use variational methods, we use the following second-order ODE system.

Lemma 3.4. *Assume* $g(r,\theta,z,t) = (g_r(r,t), g_\theta(r,t), g_z(r,t))e^{\mu t+i(m\theta+kz)}$ *solves the second equation in* (2.14), *then*

$$\begin{pmatrix} \frac{d}{dr}\frac{\gamma p + B_\theta^2}{r}\frac{d}{dr}r - \frac{m^2}{r^2}B_\theta^2 - r\left(\frac{B_\theta^2}{r^2}\right)' & -\frac{d}{dr}k(\gamma p + B_\theta^2) - \frac{2kB_\theta^2}{r} & \frac{d}{dr}\frac{m}{r}\gamma p \\ \frac{k(\gamma p + B_\theta^2)}{r}\frac{d}{dr}r - \frac{2kB_\theta^2}{r} & -k^2(\gamma p + B_\theta^2) - \frac{m^2}{r^2}B_\theta^2 & \frac{mk}{r}\gamma p \\ -\frac{m\gamma p}{r^2}\frac{d}{dr}r & \frac{mk}{r}\gamma p & -\frac{m^2}{r^2}\gamma p \end{pmatrix}$$
$$\times \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \rho\mu^2 \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \tag{3.10}$$

Proof. Inserting the expression (2.22) into the second equation in (2.14), by $g(r,\theta,z,t) = (g_r(r,t), g_\theta(r,t), g_z(r,t))e^{\mu t+i(m\theta+kz)}$ and the definitions of ξ, η and ζ in (2.21), we can easily get that the second equation in (2.14) reduces to (3.10). □

In order to study the stability to use variational methods in vacuum domain, we use the following second-order ODE about $\widehat{Q}_r$ for $m \neq 0$ and any k.

Lemma 3.5. *The vacuum equations* (2.14)$_3$ *and* (2.14)$_4$ *can be reduced to the second-order differential equation*

$$\left[\frac{r}{m^2+k^2r^2}(r\widehat{Q}_r)'\right]' - \widehat{Q}_r = 0, \tag{3.11}$$

with the other two components $\widehat{Q}_\theta = -\frac{m}{m^2+k^2r^2}(r\widehat{Q}_r)'$ *and* $\widehat{Q}_z = -\frac{kr}{m^2+k^2r^2}(r\widehat{Q}_r)'$.

Proof. Inserting (3.8) into the first or second ODE in (3.7), we can easily deduce that the radial component satisfies (3.11). □

3.2. *Variational problem when* $m = 0$

In this section, we will introduce the definition of function space X_k, then give the variational analysis for the case $m = 0$ and any $k \in \mathbb{Z}$.

Let us first introduce the function space X_k for any k and its properties.

Definition 3.6. The weighted Sobolev space X_k is defined as the completion of $\{(\xi, \eta) \in C^\infty([0, r_0]) \times C^\infty([0, r_0]) | \xi(0) = 0\}$, with respect to the norm

$$\begin{aligned}\|(\xi,\eta)\|_{X_k}^2 = &\int_0^{r_0} \left\{ p\left|\frac{1}{r}\partial_r(r\xi(r))\right|^2 + B_\theta^2\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]^2\right\} rdr \\ &+ \int_0^{r_0} \rho(|\xi|^2 + |\eta|^2) rdr. \end{aligned} \tag{3.12}$$

Define function $g(r) = \sup_{r\le s\le r_0} \frac{p(s)}{-p'(s)}$, then we can get the following lemma.

Lemma 3.7. *Assume* s_1 *near* $r = r_0$, *then it holds that*

$$\int_{s_1}^{r_0} -p'(r)\xi^2 dr \le 2p(s_1)\xi^2(s_1) + 4g(s_1)\int_{s_1}^{r_0} p\xi'^2 dr, \tag{3.13}$$

with $g(s_1) \to 0$ *as* $s_1 \to r_0$.

Proof. From Definition 2.2/admissibility of the pressure p, we have $\frac{p}{p'} \to 0$ for $r \to r_0$, which together with the definition of function g, gives that $p(s) \le -g(s_1)p'(s)$ for $s_1 \le s \le r_0$, with $g(s_1) \to 0$ as $s_1 \to r_0$. Since $p > 0$ for all $r \in (0, r_0)$ and $p'(r) \le 0$ for r near r_0, the Hölder inequality provides

$$\begin{aligned}\int_{s_1}^{r_0} -p'(r)\xi^2 dr &= p(s_1)\xi^2(s_1) + 2\int_{s_1}^{r_0} p\xi\xi' dr \\ &\le p(s_1)\xi^2(s_1) + 2\left(\int_{s_1}^{r_0} p\xi^2 dr\right)^{\frac12}\left(\int_{s_1}^{r_0} p\xi'^2 dr\right)^{\frac12} \\ &\le p(s_1)\xi^2(s_1) + 2g^{\frac12}(s_1)\left(\int_{s_1}^{r_0} -p'\xi^2 dr\right)^{\frac12}\left(\int_{s_1}^{r_0} p\xi'^2 dr\right)^{\frac12},\end{aligned}$$

which by Cauchy inequality ensures (3.13). □

From the Definition 3.6, we can show the following compactness results.

Lemma 3.8. *Assume s_1 is near $r = r_0$ and p is admissible. Let π_1 denote the projection operator onto the first factor. Then $\pi_1 : X_k \to Z$ is a bounded, linear, compact map, with the norm*

$$\|\xi\|_Z^2 = \int_0^{s_1} \xi^2 dr + \int_{s_1}^{r_0} -p'\xi^2 dr, \tag{3.14}$$

and we denote it by

$$X_k \subset\subset Z. \tag{3.15}$$

Proof. For any $(\xi, \eta) \in X_k$, we have for $r \in (0, \frac{r_0}{2})$ that

$$\begin{aligned}
|r\xi(r)p(r)| &= \left|\int_0^r \partial_s(s\xi(s)p(s))ds\right| \leq \left|\int_0^r \partial_s(s\xi(s))p(s)ds\right| + \left|\int_0^r s\xi(s)p'(s)ds\right| \\
&\leq \left(\int_0^r p(s)\left|\frac{1}{s}\partial_s(s\xi(s))\right|^2 sds\right)^{\frac{1}{2}} \left(\int_0^r p(s)sds\right)^{\frac{1}{2}} + \left|\int_0^r \mathbb{J}_z B_\theta \xi(s)sds\right| \\
&\leq \frac{r}{\sqrt{2}}\|p\|_{L^\infty(0,r)} \left(\int_0^r p(s)\left|\frac{1}{s}\partial_s(s\xi(s))\right|^2 sds\right)^{\frac{1}{2}} \\
&\quad + \frac{r}{\sqrt{2}}\|\mathbb{J}_z\|_{L^\infty(0,r)}\|B_\theta\|_{L^\infty(0,r)} \left(\int_0^r \xi^2(s)sds\right)^{\frac{1}{2}},
\end{aligned}$$

which gives that

$$\begin{aligned}
|\xi(r)p(r)| &\leq \frac{1}{\sqrt{2}}\|p\|_{L^\infty(0,r)} \left(\int_0^r p(s)\left|\frac{1}{s}\partial_s(s\xi(s))\right|^2 sds\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{\sqrt{2}}\|\mathbb{J}_z\|_{L^\infty(0,r)}\|B_\theta\|_{L^\infty(0,r)} \left(\int_0^r \xi^2(s)sds\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

Here $\|p\|_{L^\infty}$, $\|\mathbb{J}_z\|_{L^\infty}$ and $\|B_\theta\|_{L^\infty}$ are bounded from Lemma 2.1 and (2.11).

Assume that $\|(\xi_n, \eta_n)\|_{X_k} \leq C$, for $n \in \mathbb{N}$. Fix any $\kappa > 0$. We claim that there exists a subsequence $\{\xi_{n_i}\}$ so that

$$\sup_{i,j} \|\xi_{n_i} - \xi_{n_j}\|_Z \leq \kappa. \tag{3.17}$$

To prove the claim, let s_0 with $0 < s_0 < s_1 < r_0$ and s_0 be chosen small enough, so that

$$3s_0C^2\left(\|p\|_{L^\infty}^2 + \frac{\|\mathbb{J}_z\|_{L^\infty}^2\|B_\theta\|_{L^\infty}^2}{\inf_{0<r\leq s_0}\rho}\right)\frac{1}{\inf_{0<r\leq s_0}p^2} \leq \kappa. \tag{3.18}$$

From Definition 2.2/admissibility of p, we have $\frac{p}{p'} \to 0$ for $r \to r_0$, which together with the definition of g, gives that $g(s_1) \to 0$ as $s_1 \to r_0$. Choose s_1 close enough to r_0, such that

$$g(s_1)C \le \frac{\kappa}{6}, \quad \frac{Cp(s_1)}{3(s_1-s_0)} \le \frac{1}{6}. \tag{3.19}$$

Since the subinterval (s_0, s_1) avoids the singularity of $\frac{1}{r}$ and degenerate of p at the boundary $r = r_0$, the function ξ_n is uniformly bounded in $H^1(s_0, s_1)$. By the compact embedding $H^1(s_0, s_1) \subset\subset C^0(s_0, s_1)$, one can extract a subsequence $\{\xi_{n_i}\}$ that converges in $L^\infty(s_0, s_1)$. So for i, j large enough, it holds that

$$\sup_{i,j} \|\xi_{n_i} - \xi_{n_j}\|^2_{L^\infty(s_0,s_1)} \le \frac{\kappa}{3(s_1-s_0)}.$$

Since $p'(r) \le 0$ near $r = r_0$, by Lemma 3.7 and (3.19), we deduce for i and j large enough that

$$\begin{aligned}\int_{s_1}^{r_0} -p'(r)(\xi_{n_i}-\xi_{n_j})^2 dr &\le 2p(s_1)(\xi_{n_i}-\xi_{n_j})^2(s_1) + 4g(s_1)\int_{s_1}^{r_0} p(\xi'_{n_i}-\xi'_{n_j})^2 dr \\ &\le \frac{Cp(s_1)\kappa}{3(s_1-s_0)} + g(s_1)C \le \frac{\kappa}{3},\end{aligned}$$

where we have used the facts $\|(\xi_n, \eta_n)\|_{X_k} \le C$ and

$$\begin{aligned}\int_{s_1}^{r_0} p{\xi'_n}^2 dr &\lesssim \int_{s_1}^{r_0} p\left|\frac{1}{r}\partial_r(r\xi_n(r))\right|^2 rdr + \int_{s_1}^{r_0} p\xi_n^2 rdr \\ &\lesssim \int_{s_1}^{r_0} p\left|\frac{1}{r}\partial_r(r\xi_n(r))\right|^2 rdr + \|\rho\|^{\gamma-1}_{L^\infty(s_1,r_0)}\int_{s_1}^{r_0} \rho\xi_n^2 rdr \\ &\lesssim \int_{0}^{r_0} p\left|\frac{1}{r}\partial_r(r\xi_n(r))\right|^2 rdr + \|\rho\|^{\gamma-1}_{L^\infty(s_1,r_0)}\int_{0}^{r_0} \rho\xi_n^2 rdr.\end{aligned}$$

Then along the above subsequence one can get from (3.16) and (3.18) that

$$\begin{aligned}\|\xi_{n_i}-\xi_{n_j}\|_Z^2 &= \int_0^{s_1} |\xi_{n_i}-\xi_{n_j}|^2 dr + \int_{s_1}^{r_0} -p'|\xi_{n_i}-\xi_{n_j}|^2 dr \\ &= \left(\int_0^{s_0} + \int_{s_0}^{s_1}\right)|\xi_{n_i}-\xi_{n_j}|^2 dr + \int_{s_1}^{r_0} -p'|\xi_{n_i}-\xi_{n_j}|^2 dr \\ &\le s_0 C^2\left(\|p\|^2_{L^\infty} + \frac{\|\mathbb{J}_z\|^2_{L^\infty}\|B_\theta\|^2_{L^\infty}}{\inf_{0<r\le s_0}\rho}\right)\frac{1}{\inf_{0<r\le s_0} p^2} \\ &\quad + (s_1-s_0)\sup_{i,j}\|\xi_{n_i}-\xi_{n_j}\|^2_{L^\infty(s_0,s_1)} \\ &\quad + \frac{Cp(s_1)\kappa}{3(s_1-s_0)} + g(s_1)C \le \kappa,\end{aligned} \tag{3.20}$$

which implies the claim (3.17) and the compactness result (3.15). $\square$

Now, we consider the case $m=0$ and any $k\in\mathbb{Z}$. In order to understand μ, we consider the following energy

$$E_{0,k}(\xi,\eta)=2\pi^2\int_0^{r_0}\left\{\frac{2p'\xi^2}{r}+B_\theta^2\left[k\eta-\frac{1}{r}((r\xi)'-2\xi)\right]^2\right.$$
$$\left.+\gamma p\left[\frac{1}{r}(r\xi)'-k\eta\right]^2\right\}rdr, \tag{3.21}$$

which is equivalent to (2.26) from Proposition 3.2. We denote

$$\mathcal{J}(\xi,\eta)=2\pi^2\int_0^{r_0}\rho(|\xi|^2+|\eta|^2)rdr.$$

From the Definition 3.6 and Lemma 3.7, it follows that $E_{0,k}$ and $\mathcal{J}$ are both well-defined on the space X_k.

Lemma 3.9. *$E_{0,k}(\xi,\eta)$ and $\mathcal{J}(\xi,\eta)$ are both well-defined on the space X_k.*

Proof. From (2.35), we have for every $s_1<r_0$ that

$$\left|2\pi^2\int_0^{s_1}2p'\xi^2dr\right|=\left|4\pi^2\int_0^{s_1}\left[-\frac{1}{2}\mathbb{J}_z^2(0)r-\frac{5}{6}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2+O(r^3)\right]\xi^2dr\right|$$
$$\leq C\mathcal{J}(\xi,\eta). \tag{3.22}$$

Using Lemma 3.7 for s_1 near r_0, we get

$$\left|\int_{s_1}^{r_0}2p'(r)\xi^2dr\right|=\int_{s_1}^{r_0}-2p'\xi^2dr\leq 4p(s_1)\xi^2(s_1)+8g(s_1)\int_{s_1}^{r_0}p\xi'^2dr$$
$$\leq C\mathcal{J}(\xi,\eta)+C\|(\xi,\eta)\|_{X_k}^2, \tag{3.23}$$

where we have used the facts

$$\xi^2(s_1)\leq\|\xi\|_{H^1(s_0,s_1)}^2\leq C\mathcal{J}(\xi,\eta)+C\|(\xi,\eta)\|_{X_k}^2\quad\text{for } 0<s_0<s_1<r_0,$$

and

$$\int_{s_1}^{r_0}p\xi'^2dr\lesssim\int_{s_1}^{r_0}p\left|\frac{1}{r}\partial_r(r\xi(r))\right|^2rdr+\int_{s_1}^{r_0}p\xi^2rdr$$
$$\lesssim\int_0^{r_0}p\left|\frac{1}{r}\partial_r(r\xi(r))\right|^2rdr+\|\rho\|_{L^\infty(s_1,r_0)}^{\gamma-1}\int_0^{r_0}\rho\xi^2rdr$$
$$\leq C\|(\xi,\eta)\|_{X_k}^2.$$

Hence, we get

$$\left|\int_0^{r_0}2p'\xi^2dr\right|\leq C\mathcal{J}(\xi,\eta)+C\|(\xi,\eta)\|_{X_k}^2,$$

which implies that

$$|E_{0,k}(\xi,\eta)| \le C\mathcal{J}(\xi,\eta) + C\|(\xi,\eta)\|_{X_k}^2 + C\int_0^{r_0}\left\{B_\theta^2\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]^2\right\} rdr$$
$$+ C\int_0^{r_0} p\left|\frac{1}{r}(r\xi)'\right|^2 rdr + C\|\rho\|_{L^\infty}^{\gamma-1}\int_0^{r_0}\rho|\eta|^2 rdr \le C\|(\xi,\eta)\|_{X_k}^2.$$

Therefore, $E_{0,k}(\xi,\eta)$ and $\mathcal{J}(\xi,\eta)$ are well-defined on the space X_k. □

Now we define

$$\lambda = \inf_{(\xi,\eta)\in X_k}\frac{E_{0,k}(\xi,\eta)}{\mathcal{J}(\xi,\eta)}.$$

Consider the set

$$\mathcal{A}_1 = \{(\xi,\eta)\in X_k | \mathcal{J}(\xi,\eta) = 1\}. \tag{3.24}$$

We want to show that the infimum of $E_{0,k}(\xi,\eta)$ over the set $\mathcal{A}_1$ is achieved and is negative and that the minimizer solves (3.10) with $m = 0$ and any $k \in \mathbb{Z}$ and the corresponding boundary conditions.

First, we prove that the energy $E_{0,k}$ has a lower bound on the set $\mathcal{A}_1$.

Lemma 3.10. *The energy $E_{0,k}(\xi,\eta)$ has a lower bound on the set $\mathcal{A}_1$.*

Proof. We can directly get from the energy (3.21) for $0 < s_0 < s_1 < r_0$ that

$$\begin{aligned} E_{0,k}(\xi,\eta) &\ge 2\pi^2\int_0^{r_0}\gamma p\left[\frac{1}{r}(r\xi)' - k\eta\right]^2 rdr + 2\pi^2\int_0^{r_0} 2p'\xi^2 dr \\ &\ge 2\pi^2\int_{s_0}^{r_0}\gamma p\left[\frac{1}{r}(r\xi)' - k\eta\right]^2 rdr \\ &\quad + 2\pi^2\left(\int_0^{s_1} + \int_{s_1}^{r_0}\right) 2p'\xi^2 dr, \quad \forall\,(\xi,\eta,\zeta)\in\mathcal{A}_1. \end{aligned}$$

Recall (3.22), for every $s_1 < r_0$, we have $|2\pi^2\int_0^{s_1} 2p'\xi^2 dr| \le C\mathcal{J}(\xi,\eta)$. Hence, we get

$$E_{0,k}(\xi,\eta) \ge 2\pi^2\int_{s_0}^{r_0}\gamma p\left|\frac{1}{r}(r\xi)'\right|^2 rdr + 2\pi^2\int_{s_1}^{r_0} 2p'\xi^2 dr - C\mathcal{J}(\xi,\eta).$$

The key is to control $\int_{s_1}^{r_0} 2p'\xi^2 dr$. Since in the interval (s_1, r_0), using Lemma 3.7, we know that

$$\begin{aligned} \left|\int_{s_1}^{r_0} 2p'(r)\xi^2 dr\right| &= \int_{s_1}^{r_0} -2p'\xi^2 dr \le 4p(s_1)\xi^2(s_1) + 8g(s_1)\int_{s_1}^{r_0} p\xi'^2 dr \\ &\le C(\sigma)\mathcal{J}(\xi,\eta)p(s_1) + (Cp(s_1)\sigma + Cg(s_1))\int_{s_0}^{r_0} p\xi'^2 dr, \end{aligned} \tag{3.25}$$

where we have used the facts for $0 < s_0 < s_1 < r_0$ and σ small enough

$$\xi^2(s_1) \le \|\xi\|^2_{L^\infty} \le \|\xi\|_{L^2(s_0,s_1)}\|\xi\|_{H^1(s_0,s_1)} \le C(\sigma)\|\xi\|^2_{L^2(s_0,s_1)} + \sigma\|\xi\|^2_{H^1(s_0,s_1)}$$
$$\le C(\sigma)\mathcal{J}(\xi,\eta) + C\sigma\int_{s_0}^{s_1} p\xi'^2 dr \tag{3.26}$$

and

$$\int_{s_0}^{s_1} p\xi'^2 dr \le \int_{s_0}^{r_0} p\xi'^2 dr, \quad \int_{s_1}^{r_0} p\xi'^2 dr \le \int_{s_0}^{r_0} p\xi'^2 dr.$$

Choosing s_1 close enough to r_0 and σ small enough such that $Cp(s_1)\sigma + Cg(s_1) \le \frac{\gamma}{2}$, yields that

$$E_{0,k}(\xi,\eta) \ge \pi^2\int_{s_0}^{r_0} \gamma p\left|\frac{1}{r}(r\xi)'\right|^2 rdr - 3C\mathcal{J}(\xi,\eta) \ge -3C\mathcal{J}(\xi,\eta) = -3C,$$

which implies that the energy $E_{0,k}(\xi,\eta)$ has a lower bound on the set $\mathcal{A}_1$. □

Using the fact that $\mathcal{J}(\xi,\eta) = 1$ and $E_{0,k}$ has a lower bound on the set $\mathcal{A}_1$, we can choose a minimizing sequence such that along the minimizing sequence, we have $M \le E_{0,k}(\xi_n,\eta_n) < M+1$, and for the minimizing sequence, we can show coercivity estimate

$$\|(\xi_n,\eta_n)\|^2_{X_k} \le \mathcal{J} + C(M+1) + C\int_0^{r_0} (2p'\xi_n^2 + \gamma pk^2\eta_n^2 r)dr \le C. \tag{3.27}$$

We now show that the infimum of $E_{0,k}$ over the set $\mathcal{A}_1$ is negative.

Proposition 3.11. *It holds that* $\lambda = -\mu^2 = \inf E_{0,k} < 0$.

Proof. Since both $E_{0,k}$ and $\mathcal{J}$ are homogeneous degree 2, it suffices to show that

$$\inf_{(\xi,\eta)\in X_k} \frac{E_{0,k}(\xi,\eta)}{\mathcal{J}(\xi,\eta)} < 0.$$

But since $\mathcal{J}$ is positive definite, one may reduce to constructing any $(\xi,\eta) \in X_k$ (see (3.12)) such that $E_{0,k}(\xi,\eta) < 0$. Using Lemma 2.4, we can choose a smooth function $\xi^* \in C_c^\infty(0,r_0)$ such that

$$2\pi^2\int_0^{r_0}\left[\frac{2p'}{r} + \frac{4\gamma pB_\theta^2}{r^2(\gamma p + B_\theta^2)}\right]{\xi^*}^2 rdr < 0.$$

Then, we can assume that $k\eta^* = \frac{1}{r}((r\xi^*)' - \frac{2B_\theta^2}{\gamma p + B_\theta^2}\xi^*)$, such that the second term of $E_{0,k}(\xi^*,\eta^*)$ in (2.26) vanishes. Here, ξ^* and η^* are smooth functions and belong to the space X_k.

Therefore, the energy (2.26) becomes

$$\widetilde{E}(\xi^*) = 2\pi^2\int_0^{r_0}\left[\frac{2p'}{r} + \frac{4\gamma pB_\theta^2}{r^2(\gamma p + B_\theta^2)}\right]{\xi^*}^2 rdr < 0,$$

which implies the result. □

With Proposition 3.11 in hand, we apply the direct methods to deduce the existence of a minimizer of $E_{0,k}$ on the set $\mathcal{A}_1$.

Proposition 3.12. *$E_{0,k}$ achieves its infimum on the set $\mathcal{A}_1$.*

Proof. First note that $E_{0,k}$ is bounded below on the set $\mathcal{A}_1$. Let $(\xi_n,\eta_n)\in\mathcal{A}_1$ be a minimizing sequence. Then, we know that (ξ_n,η_n) is bounded in X_k (see (3.12)), so up to the extraction of a subsequence $\psi_n=|B_\theta|[k\eta_n-\frac{1}{r}((r\xi_n)'-2\xi_n)]r^{\frac12}\rightharpoonup\psi=|B_\theta|[k\eta-\frac{1}{r}((r\xi)'-2\xi)]r^{\frac12}$ weakly in L^2, and $\xi_n\to\xi$ strongly in Z from the compact embedding in Lemma 3.8. By weak lower semi-continuity, since $\psi_n\rightharpoonup\psi$ in the space $L^2(0,r_0)$, we have

$$\int_0^{r_0}B_\theta^2\left[k\eta-\frac{1}{r}((r\xi)'-2\xi)\right]^2rdr\le\liminf_{n\to\infty}\int_0^{r_0}B_\theta^2\left[k\eta_n-\frac{1}{r}((r\xi_n)'-2\xi_n)\right]^2rdr.$$

Because of the quadratic structure of all the terms in the integrals defining $E_{0,k}$, similarly by weak lower semi-continuity and $\xi_n\to\xi$ strongly in Z, we get that

$$E_{0,k}(\xi,\eta)\le\liminf_{n\to\infty}E_{0,k}(\xi_n,\eta_n)=\inf_{\mathcal{A}_1}E_{0,k}.$$

All that remains is to show that $(\xi,\eta)\in\mathcal{A}_1$.

Again by lower semi-continuity, we know that $\mathcal{J}(\xi,\eta)\le1$. Suppose by way of contradiction that $\mathcal{J}(\xi,\eta)<1$. By the homogeneity of $\mathcal{J}$, we may find $\alpha>1$ so that $\mathcal{J}(\alpha\xi,\alpha\eta)=1$, i.e. we may scale up (ξ,η) so that $(\alpha\xi,\alpha\eta)\in\mathcal{A}_1$. By Proposition 3.11, we know that $\inf E_{0,k}<0$, and from this we deduce that

$$E_{0,k}(\alpha\xi,\alpha\eta)=\alpha^2E_{0,k}(\xi,\eta)=\alpha^2\inf E_{0,k}<\inf E_{0,k},$$

which is a contradiction since $(\alpha\xi,\alpha\eta)\in\mathcal{A}_1$. Hence $\mathcal{J}(\xi,\eta)=1$, so that we show that $(\xi,\eta)\in\mathcal{A}_1$. □

We now prove that the minimizer constructed in the previous result satisfies Euler–Lagrange equations equivalent to (3.10) with $m=0$ and any $k\neq0$.

Proposition 3.13. *Let $(\xi,\eta)\in\mathcal{A}_1$ be the minimizers of $E_{0,k}$ constructed in Proposition 3.12. Then (ξ,η) are smooth when restricted to $(0,r_0)$ and satisfy*

$$\begin{pmatrix}\dfrac{d}{dr}\dfrac{\gamma p+B_\theta^2}{r}\dfrac{d}{dr}r-r\left(\dfrac{B_\theta^2}{r^2}\right)' & -\dfrac{d}{dr}k(\gamma p+B_\theta^2)-\dfrac{2kB_\theta^2}{r}\\ \dfrac{k(\gamma p+B_\theta^2)}{r}\dfrac{d}{dr}r-\dfrac{2kB_\theta^2}{r} & -k^2(\gamma p+B_\theta^2)\end{pmatrix}$$
$$\times\begin{pmatrix}\xi\\ \eta\end{pmatrix}=-\rho\lambda\begin{pmatrix}\xi\\ \eta\end{pmatrix},\tag{3.28}$$

along with the interface boundary condition

$$B_\theta^2[k\eta r-\xi'r+\xi]|_{r=r_0}=0.\tag{3.29}$$

Proof. Since we want to use the structure of the energy and properties of functional space, we first change the spectral formula (3.28) into the following equations by a simple computation

$$\begin{cases} -\dfrac{d}{dr}\left\{\gamma p\left[k\eta - \dfrac{1}{r}(r\xi)'\right]\right\} - \dfrac{d}{dr}\left\{B_\theta^2\left[k\eta - \dfrac{1}{r}((r\xi)' - 2\xi)\right]\right\} \\ -\dfrac{2B_\theta^2}{r}\left[k\eta - \dfrac{1}{r}((r\xi)' - 2\xi)\right] - \dfrac{2p'\xi}{r} = -\rho\lambda\xi, \\ -k(\gamma p + B_\theta^2)\left[k\eta - \dfrac{1}{r}(r\xi)' + \dfrac{2B_\theta^2}{r(\gamma p + B_\theta^2)}\xi\right] = -\rho\lambda\eta. \end{cases} \tag{3.30}$$

Next, we prove that the minimization ξ and η satisfy Eqs. (3.30) in weak sense on $(0, r_0)$.

Fix $(\xi_0, \eta_0) \in X_k$ (see (3.12)). Define

$$j(t, \tau(t)) = \mathcal{J}(\xi + t\xi_0 + \tau(t)\xi, \eta + t\eta_0 + \tau(t)\eta)$$

and note that $j(0,0) = 1$. Moreover, j is smooth,

$$\frac{\partial j}{\partial t}(0,0) = 2\pi^2 \int_0^{r_0} 2\rho(\xi_0\xi + \eta_0\eta) r dr,$$

$$\frac{\partial j}{\partial \tau}(0,0) = 2\pi^2 \int_0^{r_0} 2\rho(\xi^2 + \eta^2) r dr = 2.$$

So, by the inverse function theorem, we can solve for $\tau = \tau(t)$ in a neighborhood of 0 as a C^1 function of t so that $\tau(0) = 0$ and $j(t, \tau(t)) = 1$. We may differentiate the last equation to find

$$\frac{\partial j}{\partial t}(0,0) + \frac{\partial j}{\partial \tau}(0,0)\tau'(0) = 0,$$

hence that

$$\tau'(0) = -\frac{1}{2}\frac{\partial j}{\partial t}(0,0) = -2\pi^2 \int_0^{r_0} \rho(\xi_0\xi + \eta_0\eta) r dr.$$

Since (ξ, η) are minimizers over $\mathcal{A}_1$, we may make variations with respect to (ξ_0, η_0) to find that

$$0 = \left.\frac{d}{dt}\right|_{t=0} E_{0,k}(\xi + t\xi_0 + \tau(t)\xi, \eta + t\eta_0 + \tau(t)\eta),$$

which implies that

$$\begin{aligned} 0 = 4\pi^2 &\int_0^{r_0} 2p'\xi(\xi_0 + \tau'(0)\xi) dr + 4\pi^2 \int_0^{r_0} B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right] \\ &\times \left\{k\eta_0 + \tau'(0)k\eta - \frac{1}{r}[(r(\xi_0 + \tau'(0)\xi))' - 2(\xi_0 + \tau'(0)\xi)]\right\} r dr \\ &+ 4\pi^2 \int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi)' - k\eta\right] \left[\frac{1}{r}(r(\xi_0 + \tau'(0)\xi))' - k(\eta_0 + \tau'(0)\eta)\right] r dr \end{aligned}$$

$$= 4\pi^2 \int_0^{r_0} 2p'\xi\xi_0 dr + 4\pi^2 \int_0^{r_0} B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]\left[k\eta_0 - \frac{1}{r}((r\xi_0)' - 2\xi_0)\right] r dr$$

$$+ 4\pi^2 \int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi)' - k\eta\right] \left[\frac{1}{r}(r\xi_0)' - k\eta_0\right] r dr$$

$$+ 4\tau'(0)\pi^2 \int_0^{r_0} \left\{2p'\xi^2 + B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]^2 r + \gamma p \left[\frac{1}{r}(r\xi)' - k\eta\right]^2 r\right\} dr.$$

Hence, we have

$$\begin{aligned} -\tau'(0)\lambda &= 2\pi^2 \int_0^{r_0} \rho\lambda(\xi_0\xi + \eta_0\eta) r dr \\ &= 2\pi^2 \int_0^{r_0} 2p'\xi\xi_0 dr + 2\pi^2 \int_0^{r_0} B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right] \\ &\quad \times \left[-\frac{1}{r}((r\xi_0)' - 2\xi_0)\right] r dr + 2\pi^2 \int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi)' - k\eta\right] (r\xi_0)' dr \\ &\quad + 2\pi^2 \int_0^{r_0} k B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right] \eta_0 r dr \\ &\quad + 2\pi^2 \int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi)' - k\eta\right] (-k\eta_0) r dr. \end{aligned}$$

Since ξ_0 and η_0 are independent, we deduce that

$$\begin{aligned} &\int_0^{r_0} 2p'\xi\xi_0 dr - \int_0^{r_0} B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right] [(r\xi_0)' - 2\xi_0] dr \\ &\qquad + \int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi)' - k\eta\right] (r\xi_0)' dr = \int_0^{r_0} \rho\lambda\xi_0\xi r dr, \\ &\int_0^{r_0} k B_\theta^2 \left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right] \eta_0 r dr + \int_0^{r_0} \gamma p k \left[k\eta - \frac{1}{r}(r\xi)'\right] \eta_0 r dr \\ &\quad = \int_0^{r_0} \rho\lambda\eta_0\eta r dr. \end{aligned} \tag{3.31}$$

Therefore, ξ and η satisfy (3.30) in weak sense on $(0, r_0)$. Now we prove that the interface boundary condition (3.29) is satisfied. From the first equation of (3.30), we get

$$\begin{aligned} &-\frac{d}{dr}\left\{(\gamma p + B_\theta^2)\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]\right\} + \frac{d}{dr}\left(\gamma p \frac{2\xi}{r}\right) \\ &\qquad - \frac{2B_\theta^2}{r}\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right] - \frac{2p'\xi}{r} = -\rho\lambda\xi, \end{aligned} \tag{3.32}$$

that is

$$-\frac{d}{dr}\left\{(\gamma p + B_\theta^2)\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]\right\} + \frac{2\gamma p\xi'}{r} + \frac{2\gamma p'\xi}{r} - \frac{2\gamma p\xi}{r^2}$$
$$- \frac{2B_\theta^2}{r}\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right] - \frac{2p'\xi}{r} = -\rho\lambda\xi, \tag{3.33}$$

which together with $(\xi, \eta) \in X_k$ (see (3.12)), $\xi \in Z$ (see (3.14)) and the claim $\sqrt{p}\xi' \in L^2(\frac{r_0}{2}, r_0)$ gives that

$$\frac{d}{dr}\left\{(\gamma p + B_\theta^2)\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]\right\} \in L^2\left(\frac{r_0}{2}, r_0\right). \tag{3.34}$$

Now we prove the claim $\sqrt{p}\xi' \in L^2(\frac{r_0}{2}, r_0)$. In fact, from $(\xi, \eta) \in X_k$, it follows that

$$\|(\xi,\eta)\|^2_{X_k(\frac{r_0}{2}, r_0)} = \int_{\frac{r_0}{2}}^{r_0} \left\{ p\left|\frac{1}{r}\partial_r(r\xi(r))\right|^2 + B_\theta^2\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right]^2\right\} rdr$$
$$+ \int_{\frac{r_0}{2}}^{r_0} \rho(|\xi|^2 + |\eta|^2) rdr \leq C. \tag{3.35}$$

Together with $\frac{1}{r}\partial_r(r\xi(r)) = \frac{\xi}{r} + \xi'$, we deduce that $\sqrt{p}\xi' \in L^2(\frac{r_0}{2}, r_0)$. So $(\gamma p + B_\theta^2)[k\eta - \frac{1}{r}((r\xi)' - 2\xi)]$ is well-defined at the endpoint $r = r_0$. Make variations with respect to $\xi_0 \in C_c^\infty((0, r_0])$. Integrating the terms in (3.31) by parts and using that ξ solves the first equation of (3.30) on $(0, r_0)$, we obtain

$$-B_\theta^2\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right](r\xi_0)|_{r=r_0} + \gamma p\left[\frac{1}{r}(r\xi)' - k\eta\right](r\xi_0)|_{r=r_0} = 0, \tag{3.36}$$

which implies by $p = 0$ on the boundary $r = r_0$ that

$$-B_\theta^2\left[k\eta - \frac{1}{r}((r\xi)' - 2\xi)\right](r\xi_0)\Big|_{r=r_0} = 0. \tag{3.37}$$

Since ξ_0 may be chosen arbitrarily, we get the interface boundary condition

$$B_\theta^2[k\eta r - \xi' r + \xi]|_{r=r_0} = 0. \tag{3.38}$$

This completes the proof. □

3.3. *Variational problem when $m \neq 0$*

In this section, we prove the Kink ($|m| = 1$) instability, in fact, we give the analysis for any $m \neq 0$ and any $k \in \mathbb{Z}$. Let us first introduce the definition of the space $Y_{m,k}$.

Definition 3.14. The weighted Sobolev space $Y_{m,k}$ is defined as the completion of $\{(\xi, \eta, \zeta) \in C^\infty([0, r_0]) \times C^\infty([0, r_0]) \times C^\infty([0, r_0]) | \xi(0) = 0)\}$, with respect to the norm

$$\begin{aligned}\|(\xi,\eta,\zeta)\|_{Y_{m,k}}^2 &= \int_0^{r_0} p\left[\frac{1}{r}(r\xi)' + \frac{m\zeta}{r}\right]^2 rdr + \int_0^{r_0} \frac{B_\theta^2}{r}\xi^2 dr \\ &\quad + \int_0^{r_0} B_\theta^2\left[\frac{\eta}{r} - \frac{k}{m^2+k^2r^2}((r\xi)'-2\xi)\right]^2 rdr \\ &\quad + \int_0^{r_0} \frac{B_\theta^2}{r}(\xi - r\xi')^2 dr + \int_0^{r_0} \rho(|\xi|^2+|\eta|^2+|\zeta|^2) rdr. \end{aligned} \tag{3.39}$$

From Definition 3.14, we can get the following compactness result.

Lemma 3.15. *Assume s_1 is near $r = r_0$ and p is admissible. Let Π_1 denote the projection operator onto the first factor. Then $\Pi_1 : Y_{m,k} \to V$ is a bounded, linear, compact map, with the norm*

$$\|\xi\|_V^2 = \int_0^{s_1} B_\theta^2 \xi^2 dr + \int_{s_1}^{r_0} -p'\xi^2 dr, \tag{3.40}$$

and we denote it by

$$Y_{m,k} \subset\subset V. \tag{3.41}$$

Proof. For any $(\xi,\eta,\zeta) \in Y_{m,k}$, we have for $r \in (0, \frac{r_0}{2})$ that

$$\begin{aligned}|r\xi(r)B_\theta(r)| &= \left|\int_0^r \partial_s(s\xi(s)B_\theta(s))ds\right| \\ &\le \left|\int_0^r s\xi'(s)B_\theta(s)ds\right| + \left|\int_0^r s\xi(s)B_\theta'(s)ds\right| + \left|\int_0^r \xi(s)B_\theta(s)ds\right| \\ &\le \left(\int_0^r B_\theta^2(s)(\xi'(s))^2 sds\right)^{\frac12}\left(\int_0^r sds\right)^{\frac12} + \left(\int_0^r (B_\theta'(s))^2 sds\right)^{\frac12} \\ &\quad \times \left(\int_0^r \xi^2(s)sds\right)^{\frac12} + \left(\int_0^r \frac{B_\theta^2(s)}{s}\xi^2(s)ds\right)^{\frac12}\left(\int_0^r sds\right)^{\frac12} \\ &\le \frac{r}{\sqrt2}\left(\int_0^r B_\theta^2(s)(\xi'(s))^2 sds\right)^{\frac12} + \frac{r}{\sqrt2}\|B_\theta'\|_{L^\infty}\left(\int_0^r \xi^2(s)sds\right)^{\frac12} \\ &\quad + \frac{r}{\sqrt2}\left(\int_0^r \frac{B_\theta^2(s)}{s}\xi^2(s)ds\right)^{\frac12},\end{aligned}$$

which gives that

$$\begin{aligned}|\xi(r)B_\theta(r)| &\le \frac{1}{\sqrt2}\left(\int_0^r B_\theta^2(s)(\xi'(s))^2 sds\right)^{\frac12} + \frac{1}{\sqrt2}\|B_\theta'\|_{L^\infty}\left(\int_0^r \xi^2(s)sds\right)^{\frac12} \\ &\quad + \frac{1}{\sqrt2}\left(\int_0^r \frac{B_\theta^2(s)}{s}\xi^2(s)ds\right)^{\frac12}.\end{aligned} \tag{3.42}$$

Assume that $\|(\xi_n,\eta_n,\zeta_n)\|_{Y_{m,k}} \le C$, for $n\in\mathbb{N}$, then we have

$$\begin{aligned}
&\|(\xi_n(r),\eta_n(r),\zeta_n(r))\|^2_{Y(0,\frac{r_0}{2})}\\
&\quad= \int_0^{\frac{r_0}{2}} p\left[\frac{1}{r}(r\xi_n)' + \frac{m\zeta_n}{r}\right]^2 rdr + \int_0^{\frac{r_0}{2}} \frac{B_\theta^2}{r}\xi_n^2 dr\\
&\qquad+ \int_0^{\frac{r_0}{2}} B_\theta^2\left[\frac{\eta_n}{r} - \frac{k}{m^2+k^2r^2}((r\xi_n)' - 2\xi_n)\right]^2 rdr\\
&\qquad+ \int_0^{\frac{r_0}{2}} \frac{B_\theta^2}{r}(\xi_n - r\xi_n')^2 dr + \int_0^{\frac{r_0}{2}} \rho(|\xi_n|^2+|\eta_n|^2+|\zeta_n|^2) rdr \le C^2,
\end{aligned}$$

which implies that for $r\in(0,\frac{r_0}{2})$,

$$\begin{cases}
\displaystyle\int_0^r B_\theta^2(s)(\xi_n'(s))^2 sds \le C^2,\\
\displaystyle\int_0^r \xi_n^2(s) sds \le \frac{1}{\min_{(0,r)}\rho} C^2,\\
\displaystyle\int_0^r \frac{B_\theta^2(s)}{s}\xi_n^2(s) ds \le C^2.
\end{cases} \tag{3.43}$$

Fix any $\kappa>0$. We claim that there exists a subsequence $\{\xi_{n_i}\}$ so that

$$\sup_{i,j}\|\xi_{n_i} - \xi_{n_j}\|_V \le \kappa. \tag{3.44}$$

To prove the claim, from (3.42) and (3.43), let $s_0\in(0,\frac{r_0}{2})$ be chosen small enough so that

$$3s_0\left(2C^2 + \frac{1}{\min_{(0,s_0)}\rho}\|B_\theta'\|^2_{L^\infty} C^2\right) \le \kappa. \tag{3.45}$$

From Definition 2.2/admissibility of p, we have $\frac{p}{p'}\to 0$ for $r\to r_0$, which together with the definition of g, gives that $g(s_1)\to 0$ as $s_1\to r_0$. Choose s_1 close enough to r_0, such that

$$g(s_1)C \le \frac{\kappa}{6}, \quad \frac{Cp(s_1)}{3(s_1-s_0)\|B_\theta\|^2_{L^\infty(0,r_0)}} \le \frac{1}{6}. \tag{3.46}$$

Since the subinterval (s_0,s_1) avoids the singularity of $\frac{1}{r}$ and the degenerate of pressure p on the boundary $r=r_0$, the function ξ_n is uniformly bounded in $H^1(s_0,s_1)$. By the compact embedding $H^1(s_0,s_1)\subset\subset C^0(s_0,s_1)$, one can extract a subsequence $\{\xi_{n_i}\}$ that converges in $L^\infty(s_0,s_1)$. So for i,j large enough, it holds that

$$\sup_{i,j}\|\xi_{n_i}-\xi_{n_j}\|^2_{L^\infty(s_0,s_1)} \le \frac{\kappa}{3(s_1-s_0)\|B_\theta\|^2_{L^\infty(0,r_0)}}.$$

Since $p'(r) \le 0$ near $r = r_0$, by Lemma 3.7 and (3.46), we deduce for i and j large enough

$$
\begin{aligned}
&\int_{s_1}^{r_0} -p'(r)(\xi_{n_i} - \xi_{n_j})^2 dr \\
&\quad \le 2p(s_1)(\xi_{n_i} - \xi_{n_j})^2(s_1) + 4g(s_1) \int_{s_1}^{r_0} p(\xi'_{n_i} - \xi'_{n_j})^2 dr \\
&\quad \le \frac{Cp(s_1)\kappa}{3(s_1 - s_0)\|B_\theta\|^2_{L^\infty(0,r_0)}} + g(s_1)C \le \frac{\kappa}{3}.
\end{aligned} \tag{3.47}
$$

Then along the above subsequence we can get from (3.45) that

$$
\begin{aligned}
\|\xi_{n_i} - \xi_{n_j}\|_V^2 &= \int_0^{s_1} B_\theta^2 |\xi_{n_i} - \xi_{n_j}|^2 dr + \int_{s_1}^{r_0} -p' |\xi_{n_i} - \xi_{n_j}|^2 dr \\
&= \left(\int_0^{s_0} + \int_{s_0}^{s_1} \right) B_\theta^2 |\xi_{n_i} - \xi_{n_j}|^2 dr + \int_{s_1}^{r_0} -p' |\xi_{n_i} - \xi_{n_j}|^2 dr \\
&\le s_0 \left(2C^2 + \frac{1}{\min_{(0,s_0)} \rho} \|B'_\theta\|^2_{L^\infty} C^2 \right) \\
&\quad + (s_1 - s_0) \sup_{i,j} \|\xi_{n_i} - \xi_{n_j}\|^2_{L^\infty(s_0,r_0)} \|B_\theta\|^2_{L^\infty(0,r_0)} \\
&\quad + \frac{Cp(s_1)\kappa}{3(s_1 - s_0)\|B_\theta\|^2_{L^\infty(0,r_0)}} + g(s_1)C \le \kappa,
\end{aligned}
$$

which implies the claim (3.44) and the compactness result (3.41). □

Now, consider the case any $m \neq 0$ and any $k \in \mathbb{Z}$. We need to consider the energy (2.27) in Proposition 3.2 and

$$
\mathcal{J}(\xi, \eta, \zeta) = 2\pi^2 \int_0^{r_0} \rho(|\xi|^2 + |\eta|^2 + |\zeta|^2) r dr. \tag{3.48}
$$

Then from the Definition 3.14, we can get the following lemma.

Lemma 3.16. *$E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r)$ defined in (2.27) and $\mathcal{J}(\xi, \eta, \zeta)$ are both well-defined on the space $Y_{m,k} \times H^1(r_0, r_w)$.*

Proof. From (2.35) and Lemma 3.7, similar to the proof of Lemma 3.9, we can get

$$
\int_0^{r_0} 2p'\xi^2 dr \le C\mathcal{J}(\xi, \eta, \zeta) + C\|(\xi, \eta, \zeta)\|^2_{Y_{m,k}},
$$

which implies that

$$
\begin{aligned}
|E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r)| &\le C\mathcal{J}(\xi, \eta, \zeta) + C\|(\xi, \eta, \zeta)\|^2_{Y_{m,k}} \\
&\quad + C \int_0^{r_0} B_\theta^2 \left[\frac{\eta}{r} - \frac{k}{m^2 + k^2 r^2} ((r\xi)' - 2\xi) \right]^2 r dr
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^{r_0} p\left[\frac{1}{r}(r\xi)'+\frac{m\zeta}{r}\right]^2 rdr + C\int_0^{r_0}\frac{B_\theta^2}{r}\xi^2 dr\\
&+C\int_0^{r_0}\frac{B_\theta^2}{r}(\xi-r\xi')^2 dr + C\|\rho\|_{L^\infty}^{\gamma-1}\int_0^{r_0}\rho|\eta|^2 rdr\\
&+C\int_{r_0}^{r_w}[|\widehat{Q}_r|^2+|\widehat{Q}_r'|^2]rdr\\
\leq{}& C\|(\xi,\eta,\zeta)\|_{Y_{m,k}}^2+\|\widehat{Q}_r\|_{H^1}^2.
\end{aligned}
$$

Therefore, $E_{m,k}(\xi,\eta,\zeta)$ and $\mathcal{J}(\xi,\eta,\zeta)$ are well-defined on the space $Y_{m,k}\times H^1(r_0,r_w)$. □

Now, we define

$$
\lambda=\inf_{((\xi,\eta,\zeta),\widehat{Q}_r)\in Y_{m,k}\times H^1(r_0,r_w)}\frac{E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)}{\mathcal{J}(\xi,\eta,\zeta)}. \tag{3.49}
$$

Consider the set

$$
\mathcal{A}_2=\left\{\begin{matrix}((\xi,\eta,\zeta),\widehat{Q}_r)\in Y_{m,k}\times H^1(r_0,r_w)|\mathcal{J}(\xi,\eta,\zeta)=1,\\ m\widehat{B}_\theta\xi=r\widehat{Q}_r \text{ at } r=r_0 \text{ and } \widehat{Q}_r=0 \text{ at } r=r_w\end{matrix}\right\}, \tag{3.50}
$$

where the functions ξ, η and ζ are restricted to $(0,r_0)$, and the function $\widehat{Q}_r$ is restricted to (r_0,r_w). We want to show that the infimum of $E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)$ over the set $\mathcal{A}_2$ is achieved and is negative and that the minimizer solves (3.10) and (3.11) with the corresponding boundary conditions. First, we study the lower bound of the energy $E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)$ on the set $\mathcal{A}_2$.

Lemma 3.17. *The energy $E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)$ has a lower bound on the set $\mathcal{A}_2$.*

Proof. We can directly get from the energy (2.27) that for $0<s_0<s_1<r_0$,

$$
\begin{aligned}
E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)&\geq 2\pi^2\int_0^{r_0}\gamma p\left[\frac{1}{r}(r\xi)'-k\eta+\frac{m}{r}\zeta\right]^2 rdr+2\pi^2\int_0^{r_0}2p'\xi^2dr\\
&\geq 2\pi^2\int_{s_0}^{r_0}\gamma p\left[\frac{1}{r}(r\xi)'-k\eta+\frac{m}{r}\zeta\right]^2 rdr\\
&\quad+2\pi^2\left(\int_0^{s_1}+\int_{s_1}^{r_0}\right)2p'\xi^2dr,\quad\forall\,(\xi,\eta,\zeta)\in\mathcal{A}_2.
\end{aligned}
$$

Recalling (3.22), for every $s_1<r_0$, we have $|2\pi^2\int_0^{s_1}2p'\xi^2dr|\leq C\mathcal{J}$. Hence, we get

$$
E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)\geq 2\pi^2\int_{s_0}^{r_0}\gamma p\left|\frac{1}{r}(r\xi)'\right|^2 rdr+2\pi^2\int_{s_1}^{r_0}2p'\xi^2dr-C\mathcal{J}.
$$

The key is to control $\int_{s_1}^{r_0} 2p'\xi^2 dr$. Since in the interval (s_1, r_0), using Lemma 3.7, similarly as (3.25), we know that

$$\left|\int_{s_1}^{r_0} 2p'(r)\xi^2 dr\right| = \int_{s_1}^{r_0} -2p'\xi^2 dr \leq Cp(s_1)\xi^2(s_1) + Cg(s_1)\int_{s_1}^{r_0} p\xi'^2 dr$$
$$\leq C(\sigma)\mathcal{J}p(s_1) + (Cp(s_1)\sigma + Cg(s_1))\int_{s_0}^{r_0} p\xi'^2 dr. \qquad (3.51)$$

Choosing s_1 close enough to r_0 and σ small enough such that $Cp(s_1)\sigma + Cg(s_1) \leq \frac{\gamma}{2}$, yields that

$$E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r) \geq \pi^2 \int_{s_0}^{r_0} \gamma p \left|\frac{1}{r}(r\xi)'\right|^2 r dr - 3C\mathcal{J} \geq -3C\mathcal{J} = -3C,$$

which implies that the energy $E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)$ has a lower bound on the set $\mathcal{A}_2$.

□

Now we prove the coercivity estimate. Using the fact that $\mathcal{J}(\xi,\eta,\zeta) = 1$ and $E_{m,k}$ has a lower bound on the set $\mathcal{A}_2$, we can choose a minimizing sequence such that along the minimizing sequence, we know that $M \leq E_{m,k}(\xi_n,\eta_n,\zeta_n,\widehat{Q}_{rn}) < M+1$, and therefore we have coercivity estimate:

$$\|(\xi_n,\eta_n,\zeta_n)\|^2_{Y_{m,k}} \leq \mathcal{J} + C(M+1) + C\int_0^{r_0} (2p'\xi_n^2 + \gamma p k^2 \eta_n^2 r) dr \leq C. \qquad (3.52)$$

Next, we prove that the infimum of $E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)$ over the set $\mathcal{A}_2$ is negative.

Proposition 3.18. *If there exists $r^* \in (0, r_0)$ such that $2p'(r^*) + \frac{m^2 B_\theta^2(r^*)}{r^*} < 0$, then it holds that $\lambda = \inf E_{m,k} < 0$.*

Proof. Since both $E_{m,k}$ and $\mathcal{J}$ are homogeneous degree 2, it suffices to show that

$$\inf_{((\xi,\eta,\zeta),\widehat{Q}_r))\in Y_{m,k}\times H^1(r_0,r_w)} \frac{E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r)}{\mathcal{J}(\xi,\eta,\zeta)} < 0.$$

But since $\mathcal{J}$ is positive definite, one may reduce to constructing any $((\xi,\eta,\zeta),\widehat{Q}_r)) \in Y_{m,k} \times H^1(r_0,r_w)$ such that

$$E_{m,k}(\xi,\eta,\zeta,\widehat{Q}_r) < 0.$$

If there exists $r^* \in (0, r_0)$ such that $2p' + \frac{m^2 B_\theta^2}{r} < 0$, then we can choose a smooth function $\xi^* \in C_c^\infty(0, r_0)$ such that

$$2\pi^2 \int_0^{r_0} \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi^{*2} dr < 0.$$

Then, we can assume that $\eta^* = \frac{rk(r\xi^*)'-2rk\xi^*}{m^2+k^2r^2}$ and $\zeta^* = \frac{r}{m}(k\eta^* - \frac{1}{r}(r\xi^*)')$, such that the first and second terms of $E_{m,k}(\xi^*, \eta^*, \zeta^*, \widehat{Q}_r^*)$ in (2.27) vanish, that is

$$2\pi^2 \int_0^{r_0} (m^2 + k^2r^2) \left[\frac{B_\theta}{r}\eta^* + \frac{-kB_\theta(r\xi^*)' + 2kB_\theta\xi^*}{m^2 + k^2r^2}\right]^2 rdr = 0,$$

$$2\pi^2 \int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi^*)' - k\eta^* + \frac{m\zeta^*}{r}\right]^2 rdr = 0.$$

Here, ξ^*, η^* and ζ^* are smooth functions and belong to the space $Y_{m,k}$.

Minimizing energy $E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r)$ with respect to η^* and ζ^*, so we consider the limit $k \to \infty$, then we can get that

$$\begin{aligned}\widetilde{E}(\xi^*, \widehat{Q}_r^*) &= \lim_{k\to\infty} \left(2\pi^2 \int_0^{r_0} \frac{m^2B_\theta^2}{r(m^2 + k^2r^2)}(\xi^* - r\xi^{*\prime})^2 + 2\pi^2 \int_0^{r_0} \left[2p' + \frac{m^2B_\theta^2}{r}\right]\right. \\ &\qquad \left. \times\, {\xi^*}^2 dr - 2\pi^2 \left[\frac{r}{m^2 + k^2r^2}(r\widehat{Q}_r^*)'r\widehat{Q}_r^*\right]_{r=r_0}\right) \\ &= 2\pi^2 \int_0^{r_0} \left[2p' + \frac{m^2B_\theta^2}{r}\right] {\xi^*}^2 dr < 0,\end{aligned}$$

which implies the result. □

Using Proposition 3.18 we can achieve the minimizer of the energy $E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r)$.

Proposition 3.19. *If* $\lambda = \inf E_{m,k} < 0$, *then* $E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r)$ *achieves its infimum on the set* $\mathcal{A}_2$.

Proof. First from Lemma 3.17, we have that $E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r)$ is bounded below on the set $\mathcal{A}_2$. Assume that $((\xi_n, \eta_n, \zeta_n), \widehat{Q}_{rn}) \in \mathcal{A}_2$ be a minimizing sequence. Then (ξ_n, η_n, ζ_n) is bounded in $Y_{m,k}$ (see (3.39)), and $\widehat{Q}_{rn}$ is bounded in $H^1(r_0, r_w)$, so up to the extraction of a subsequence $\psi_n = \sqrt{m^2 + k^2r^2}|B_\theta|[\frac{\eta_n}{r} - \frac{k}{m^2+k^2r^2}((r\xi_n)' - 2\xi_n)]r^{\frac{1}{2}} \rightharpoonup \psi = \sqrt{m^2 + k^2r^2}|B_\theta|[\frac{\eta}{r} - \frac{k}{m^2+k^2r^2}((r\xi)' - 2\xi)]r^{\frac{1}{2}}$ weakly in L^2, and $(\xi_n, \widehat{Q}_{rn}) \to (\xi, \widehat{Q}_r)$ strongly in $V \times L^2(r_0, r_w)$ from the compact embedding in Lemma 3.15 and the compact embedding $H^1(r_0, r_w) \subset\subset L^2(r_0, r_w)$. By weak lower semi-continuity, since $\psi_n \rightharpoonup \psi$ in the space $L^2(0, r_0)$, we have

$$\begin{aligned}&\int_0^{r_0} (m^2 + k^2r^2)B_\theta^2 \left[\frac{\eta}{r} - \frac{k}{m^2 + k^2r^2}((r\xi)' - 2\xi)\right]^2 rdr \\ &\qquad \le \liminf_{n\to\infty} \int_0^{r_0} (m^2 + k^2r^2)B_\theta^2 \left[\frac{\eta_n}{r} - \frac{k}{m^2 + k^2r^2}((r\xi_n)' - 2\xi_n)\right]^2 rdr.\end{aligned}$$

Because of the quadratic structure of all the terms in the integrals defining $E_{m,k}$, similarly by weak lower semi-continuity, we have

$$\int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi)' - k\eta + \frac{m}{r}\zeta\right]^2 rdr \le \liminf_{n\to\infty} \int_0^{r_0} \gamma p \left[\frac{1}{r}(r\xi_n)' - k\eta_n + \frac{m}{r}\zeta_n\right]^2 rdr.$$

Now let us deal with $\int_0^{r_0} \frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi_n - r\xi_n')^2 + \int_0^{r_0}[2p' + \frac{m^2B_\theta^2}{r}]\xi_n^2 dr$ by the following two cases.

Case I. When $|m| = 1$, we can write for every $s_0 > 0$,

$$\begin{aligned}
&\int_0^{r_0} \frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi_n - r\xi_n')^2 dr + \int_0^{r_0}\left[2p' + \frac{m^2B_\theta^2}{r}\right]\xi_n^2 dr \\
&\quad = \left(\int_0^{s_0} + \int_{s_0}^{r_0}\right) \frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi_n - r\xi_n')^2 dr \\
&\qquad + \left(\int_0^{s_0} + \int_{s_0}^{r_0}\right)\left[2p' + \frac{m^2B_\theta^2}{r}\right]\xi_n^2 dr.
\end{aligned} \tag{3.53}$$

Applying (2.34) and (2.35) in Lemma 2.3, for fixed k we have

$$\begin{aligned}
&\int_0^{s_0} \frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi_n - r\xi_n')^2 dr + \int_0^{s_0}\left[2p' + \frac{m^2B_\theta^2}{r}\right]\xi_n^2 dr \\
&\quad = \int_0^{s_0}\left[\frac{1}{4}\mathbb{J}_z^2(0)r + \frac{1}{3}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3)\right](\xi_n - r\xi_n')^2 dr \\
&\qquad + \int_0^{s_0}\left[-\frac{3}{4}\mathbb{J}_z^2(0)r - \frac{4}{3}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3)\right]\xi_n^2 dr \\
&\quad = \int_0^{s_0}\left[\frac{1}{4}\mathbb{J}_z^2(0)r + \frac{1}{3}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3)\right](r^2\xi_n'^2 - 2r\xi_n\xi_n') dr \\
&\qquad + \int_0^{s_0}\left[-\frac{1}{2}\mathbb{J}_z^2(0)r - \mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3)\right]\xi_n^2 dr \\
&\quad = \int_0^{s_0}\left[\frac{1}{4}\mathbb{J}_z^2(0)r + \frac{1}{3}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3)\right]r^2\xi_n'^2 dr \\
&\qquad + \int_0^{s_0}\left[\frac{1}{2}\mathbb{J}_z^2(0)r + \mathbb{J}_z'(0)\mathbb{J}_z(0)r^2\right]\xi_n^2 dr \\
&\qquad + \int_0^{s_0}\left[-\frac{1}{2}\mathbb{J}_z^2(0)r - \mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + O(r^3)\right]\xi_n^2 dr \\
&\quad = \int_0^{s_0}\left[\frac{1}{4}\mathbb{J}_z^2(0)r + O(r^2)\right]r^2\xi_n'^2 dr + \int_0^{s_0} O(r^3)\xi_n^2 dr.
\end{aligned}$$

On the other hand, it follows from $B_\theta = \frac{1}{r}\int_0^r s\mathbb{J}_z(s)ds$ that

$$\sup_{0\le r\le r_0}\frac{|B_\theta|}{r} \le \frac{\|\mathbb{J}_z\|_{L^\infty}}{2},$$

which gives that $B_\theta(0) = 0$. Assume $(\xi_n, \eta_n, \zeta_n) \in Y_{m,k}$ and $\mathcal{J}(\xi_n, \eta_n, \zeta_n) = 1$, by (3.43) and $B_\theta(0) = 0$, we choose s_0 small enough, such that $Cs_0 + Cs_0^2 \le \kappa$ with

$\kappa = \frac{1}{l}$, $l \in \mathbb{N}$, then we have for n large enough such that

$$\begin{aligned}
&\int_0^{s_0} O(r^2) r^2 |\xi_n' - \xi'|^2 dr + \int_0^{s_0} O(r^3)|\xi_n - \xi|^2 dr \\
&\quad \leq C s_0 \int_0^{s_0} r^3 |\xi_n' - \xi'|^2 dr + C s_0^2 \int_0^{s_0} r|\xi_n - \xi|^2 dr \\
&\quad \leq C s_0 + C s_0^2 \leq \kappa,
\end{aligned} \tag{3.54}$$

which together with weak lower semi-continuity, gives that

$$\begin{aligned}
&\int_0^{s_0} \left[\frac{1}{4}\mathbb{J}_z^2(0) r + O(r^2)\right] r^2 \xi'^2 dr + \int_0^{s_0} O(r^3)\xi^2 dr \\
&\quad \leq \liminf_{n\to\infty} \left\{ \int_0^{s_0} \left[\frac{1}{4}\mathbb{J}_z^2(0) r + O(r^2)\right] r^2 \xi_n'^2 dr + \int_0^{s_0} O(r^3)\xi_n^2 dr \right\}.
\end{aligned} \tag{3.55}$$

Hence, we have

$$\begin{aligned}
&\int_0^{s_0} \frac{m^2 B_\theta^2}{r(m^2 + k^2 r^2)} (\xi - r\xi')^2 dr + \int_0^{s_0} \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi^2 dr \\
&\quad = \int_0^{s_0} \left[\frac{1}{4}\mathbb{J}_z^2(0) r + O(r^2)\right] r^2 \xi'^2 dr + \int_0^{s_0} O(r^3)\xi^2 dr \\
&\quad \leq \liminf_{n\to\infty} \left\{ \int_0^{s_0} \left[\frac{1}{4}\mathbb{J}_z^2(0) r + O(r^2)\right] r^2 \xi_n'^2 dr + \int_0^{s_0} O(r^3)\xi_n^2 dr \right\} \\
&\quad = \liminf_{n\to\infty} \left\{ \int_0^{s_0} \frac{m^2 B_\theta^2}{r(m^2 + k^2 r^2)} (\xi_n - r\xi_n')^2 dr + \int_0^{s_0} \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi_n^2 dr \right\}.
\end{aligned} \tag{3.56}$$

On the other hand, by weak lower semi-continuity, we can show that

$$\begin{aligned}
&\int_{s_0}^{r_0} \frac{m^2 B_\theta^2}{r(m^2 + k^2 r^2)} (\xi - r\xi')^2 dr + \int_{s_0}^{r_0} \frac{m^2 B_\theta^2}{r} \xi^2 dr \\
&\quad \leq \liminf_{n\to\infty} \int_{s_0}^{r_0} \frac{m^2 B_\theta^2}{r(m^2 + k^2 r^2)} (\xi_n - r\xi_n')^2 dr + \liminf_{n\to\infty} \int_{s_0}^{r_0} \frac{m^2 B_\theta^2}{r} \xi_n^2 dr.
\end{aligned} \tag{3.57}$$

From Lemma 3.15, for s_1 near $r = r_0$, it holds that as $n \to \infty$,

$$\begin{aligned}
\int_{s_0}^{s_1} p' |\xi_n - \xi|^2 dr &\leq \left(\int_{s_0}^{s_1} B_\theta^2 |\xi_n - \xi|^2 dr\right)^{\frac{1}{2}} \left(\int_{s_0}^{s_1} \mathbb{J}_z^2 |\xi_n - \xi|^2 dr\right)^{\frac{1}{2}} \\
&\leq \left(\int_{s_0}^{s_1} B_\theta^2 |\xi_n - \xi|^2 dr\right)^{\frac{1}{2}} \left(\int_{s_0}^{s_1} |\xi_n - \xi|^2 dr\right)^{\frac{1}{2}} \|\mathbb{J}_z\|_{L^\infty} \\
&\leq C \left(\int_{s_0}^{s_1} B_\theta^2 |\xi_n - \xi|^2 dr\right)^{\frac{1}{2}} \to 0,
\end{aligned}$$

and

$$\int_{s_1}^{r_0} -p'|\xi_n - \xi|^2 dr \to 0,$$

where we have used ξ_n is uniformly bounded in $H^1(s_0, s_1)$ and $\mathbb{J}_z \in L^\infty([0, r_0])$.
Case II. When $|m| \geq 2$, the positive term $\int_0^{r_0} \frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi_n - r\xi_n')^2$ is dealt by weak lower semi-continuity, which implies that

$$\int_0^{r_0} \frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi - r\xi')^2 \leq \liminf_{n\to\infty} \int_0^{r_0} \frac{m^2 B_\theta^2}{r(m^2+k^2r^2)}(\xi_n - r\xi_n')^2.$$

For the term $\int_0^{r_0}[2p' + \frac{m^2 B_\theta^2}{r}]\xi_n^2 dr$, we have for every $s_0 > 0$,

$$\int_0^{r_0} \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi_n^2 dr = \left(\int_0^{s_0} + \int_{s_0}^{r_0}\right) \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi_n^2 dr. \tag{3.58}$$

Applying (2.34) and (2.35) in Lemma 2.3, we deduce

$$\begin{aligned}
&\int_0^{s_0} \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi_n^2 dr \\
&= \int_0^{s_0} \left[-\mathbb{J}_z^2(0)r - \frac{5}{3}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2 + m^2\left(\frac{1}{4}\mathbb{J}_z^2(0)r + \frac{1}{3}\mathbb{J}_z'(0)\mathbb{J}_z(0)r^2\right) + O(r^3)\right] \\
&\times \xi_n^2 dr = \int_0^{s_0} \left[\left(\frac{m^2}{4} - 1\right)\mathbb{J}_z^2(0)r + O(r^2)\right] \xi_n^2 dr.
\end{aligned} \tag{3.59}$$

Assume $\mathcal{J}(\xi_n, \eta_n, \zeta_n) = 1$, we choose s_0 small enough, such that $Cs_0 \leq \kappa$, with $\kappa = \frac{1}{l}$, $l \in \mathbb{N}$, then we have for n large enough such that

$$\int_0^{s_0} O(r^2)|\xi_n - \xi|^2 dr \leq Cs_0 \int_0^{s_0} r|\xi_n - \xi|^2 dr \leq Cs_0 \leq \kappa, \tag{3.60}$$

which together with weak lower semi-continuity, gives that

$$\begin{aligned}
\int_0^{s_0} \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi^2 dr &= \int_0^{s_0} \left[\left(\frac{m^2}{4} - 1\right)\mathbb{J}_z^2(0)r + O(r^2)\right] \xi^2 dr \\
&\leq \liminf_{n\to\infty} \int_0^{s_0} \left[\left(\frac{m^2}{4} - 1\right)\mathbb{J}_z^2(0)r + O(r^2)\right] \xi_n^2 dr \\
&= \liminf_{n\to\infty} \int_0^{s_0} \left[2p' + \frac{m^2 B_\theta^2}{r}\right] \xi_n^2 dr.
\end{aligned} \tag{3.61}$$

On the other hand, by weak lower semi-continuity, we can show that

$$\int_{s_0}^{r_0} \frac{m^2 B_\theta^2}{r} \xi^2 dr \leq \liminf_{n\to\infty} \int_{s_0}^{r_0} \frac{m^2 B_\theta^2}{r} \xi_n^2 dr. \tag{3.62}$$

From Lemma 3.15, for s_1 near $r = r_0$, similarly it holds that as $n \to \infty$,

$$\begin{aligned}
&\int_{s_0}^{s_1} p'|\xi_n - \xi|^2 dr \leq C\left(\int_{s_0}^{s_1} B_\theta^2 |\xi_n - \xi|^2 dr\right)^{\frac{1}{2}} \to 0, \\
&\int_{s_1}^{r_0} -p'|\xi_n - \xi|^2 dr \to 0.
\end{aligned}$$

Therefore, we get that for any fixed k and $m \neq 0$,

$$E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r) \leq \liminf_{n\to\infty} E_{m,k}(\xi_n, \eta_n, \zeta_n, \widehat{Q}_{rn}) = \inf_{\mathcal{A}_2} E_{m,k}.$$

All that remains is to show that $((\xi, \eta, \zeta), \widehat{Q}_r) \in \mathcal{A}_2$.

Again by lower semi-continuity, we know that $\mathcal{J}(\xi, \eta, \zeta) \leq 1$. Suppose by way of contradiction that $\mathcal{J}(\xi, \eta, \zeta) < 1$. By the homogeneity of $\mathcal{J}$, we may find $\alpha > 1$ so that $\mathcal{J}(\alpha\xi, \alpha\eta, \alpha\zeta) = 1$, i.e. we may scale up $((\xi, \eta, \zeta), \widehat{Q}_r)$ so that $((\alpha\xi, \alpha\eta, \alpha\zeta), \alpha\widehat{Q}_r) \in \mathcal{A}_2$. By Proposition 3.18, we know that $\inf E_{m,k} < 0$, and from this we deduce that

$$E_{m,k}(\alpha\xi, \alpha\eta, \alpha\zeta, \alpha\widehat{Q}_r) = \alpha^2 E_{m,k}(\xi, \eta, \zeta, \widehat{Q}_r) = \alpha^2 \inf E_{m,k} < \inf E_{m,k},$$

which is a contradiction since $((\alpha\xi, \alpha\eta, \alpha\zeta), \alpha\widehat{Q}_r) \in \mathcal{A}_2$. Hence $\mathcal{J}(\xi, \eta, \zeta) = 1$, so that $((\xi, \eta, \zeta), \widehat{Q}_r) \in \mathcal{A}_2$. □

We now prove that the minimizer constructed in the previous result satisfies Euler–Lagrange equations equivalent to (3.10) and (3.11) with suitable boundary conditions.

Proposition 3.20. *Let $((\xi, \eta, \zeta), \widehat{Q}_r) \in \mathcal{A}_2$ be the minimizers of $E_{m,k}$ constructed in Proposition 3.19. Then (ξ, η, ζ) are smooth in $(0, r_0)$ and satisfy*

$$\begin{pmatrix} \dfrac{d}{dr}\dfrac{\gamma p + B_\theta^2}{r}\dfrac{d}{dr}r - \dfrac{m^2}{r^2}B_\theta^2 - r\left(\dfrac{B_\theta^2}{r^2}\right)' & -\dfrac{d}{dr}k(\gamma p + B_\theta^2) - \dfrac{2kB_\theta^2}{r} & \dfrac{d}{dr}\dfrac{m}{r}\gamma p \\ \dfrac{k(\gamma p + B_\theta^2)}{r}\dfrac{d}{dr}r - \dfrac{2kB_\theta^2}{r} & -k^2(\gamma p + B_\theta^2) - \dfrac{m^2}{r^2}B_\theta^2 & \dfrac{mk}{r}\gamma p \\ -\dfrac{m\gamma p}{r^2}\dfrac{d}{dr}r & \dfrac{mk}{r}\gamma p & -\dfrac{m^2}{r^2}\gamma p \end{pmatrix}$$
$$\times \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = -\rho\lambda \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \tag{3.63}$$

the solution $\widehat{Q}_r$ is smooth on (r_0, r_w) and satisfies

$$\left[\frac{r}{m^2 + k^2r^2}(r\widehat{Q}_r)'\right]' - \widehat{Q}_r = 0, \tag{3.64}$$

along with the interface boundary condition

$$B_\theta^2\xi - B_\theta^2\xi' r + kB_\theta^2\eta r = \widehat{B}_\theta\widehat{Q}_\theta r, \quad \textit{at } r = r_0, \tag{3.65}$$

where the other two components of $\widehat{Q}$ are denoted by $\widehat{Q}_\theta = -\frac{m}{m^2+k^2r^2}(r\widehat{Q}_r)'$ and $\widehat{Q}_z = -\frac{kr}{m^2+k^2r^2}(r\widehat{Q}_r)'$.

Proof. Fix $((\xi_0,\eta_0,\zeta_0),q_r)\in Y_{m,k}\times H^1(r_0,r_w)$ and assume they satisfy $m\widehat{B}_\theta\xi_0 = rq_r$ on the boundary $r=r_0$ and $q_r=0$ on the boundary $r=r_w$. Define

$$j(t,\tau)=\mathcal{J}(\xi+t\xi_0+\tau\xi,\eta+t\eta_0+\tau\eta,\zeta+t\zeta_0+\tau\zeta).$$

Note that $j(0,0)=1$. Moreover, j is smooth

$$\frac{\partial j}{\partial t}(0,0)=2\pi^2\int_0^{r_0}2\rho(\xi_0\xi+\eta_0\eta+\zeta_0\zeta)rdr,$$

$$\frac{\partial j}{\partial \tau}(0,0)=2\pi^2\int_0^{r_0}2\rho(\xi^2+\eta^2+\zeta^2)rdr=2.$$

So, by the inverse function theorem, we can solve for $\tau=\tau(t)$ in a neighborhood of 0 as a C^1 function of t so that $\tau(0)=0$ and $j(t,\tau(t))=1$. We may differentiate the last equation to find

$$\frac{\partial j}{\partial t}(0,0)+\frac{\partial j}{\partial \tau}(0,0)\tau'(0)=0,$$

which gives that

$$\tau'(0)=-\frac{1}{2}\frac{\partial j}{\partial t}(0,0)=-2\pi^2\int_0^{r_0}\rho(\xi_0\xi+\eta_0\eta+\zeta_0\zeta)rdr.$$

Since $((\xi,\eta,\zeta),\widehat{Q}_r)$ are minimizers over the set $\mathcal{A}_2$, we may make variations with respect to $((\xi_0,\eta_0,\zeta_0),q_r)$ to find that

$$0=\left.\frac{d}{dt}\right|_{t=0}E(\xi+t\xi_0+\tau(t)\xi,\eta+t\eta_0+\tau(t)\eta,\zeta+t\zeta_0+\tau(t)\zeta,\widehat{Q}_r+tq_r+\tau\widehat{Q}_r),$$

which implies that

$$\begin{aligned}
0={}&4\pi^2\int_0^{r_0}\left\{\frac{m^2B_\theta^2}{r^2(m^2+k^2r^2)}(r\xi)'(r\xi_0)'+\beta_0(r\xi)\cdot(r\xi_0)+\frac{(m^2+k^2r^2)B_\theta^2}{r^2}\eta\eta_0\right.\\
&-\frac{kB_\theta^2(r\xi_0)'\eta}{r}+\frac{2kB_\theta^2}{r}\xi_0\eta+\frac{k^2B_\theta^2}{m^2+k^2r^2}(r\xi)'(r\xi_0)'-\frac{2k^2B_\theta^2}{m^2+k^2r^2}\xi(r\xi_0)'\\
&-\frac{kB_\theta^2}{r}(r\xi)'\eta_0+\frac{2kB_\theta^2}{r}\xi\eta_0-\frac{2k^2B_\theta^2}{m^2+k^2r^2}\xi_0(r\xi)'+\frac{4k^2B_\theta^2}{m^2+k^2r^2}\xi\xi_0\\
&+\gamma p\left[\frac{1}{r^2}(r\xi)'(r\xi_0)'+\frac{-k(r\xi)'\eta_0+\frac{m}{r}(r\xi)'\zeta_0}{r}+\frac{-k(r\xi_0)'\eta+\frac{m}{r}(r\xi_0)'\zeta}{r}\right.\\
&\left.\left.+\left(-k\eta+\frac{m}{r}\zeta\right)\left(-k\eta_0+\frac{m}{r}\zeta_0\right)\right]\right\}rdr-4\pi^2\left[\frac{2m^2B_\theta^2}{m^2+k^2r^2}\xi\xi_0\right]_{r=r_0}\\
&+4\pi^2\tau'(0)\int_0^{r_0}\left\{\frac{m^2B_\theta^2}{r^2(m^2+k^2r^2)}|(r\xi)'|^2+\beta_0(r\xi)^2+(m^2+k^2r^2)\left[\frac{B_\theta}{r}\eta\right.\right.\\
&\left.\left.+\frac{-kB_\theta(r\xi)'+2kB_\theta\xi}{m^2+k^2r^2}\right]^2+\gamma p\left[\frac{1}{r}(r\xi)'-k\eta+\frac{m}{r}\zeta\right]^2\right\}rdr
\end{aligned}$$

$$
\begin{aligned}
&-4\pi^2\tau'(0)\left[\frac{2m^2B_\theta^2}{m^2+k^2r^2}\xi^2\right]_{r=r_0}\\
&+4\pi^2\int_{r_0}^{r_w}\left[\widehat{Q}_rq_r+\frac{1}{m^2+k^2r^2}(r\widehat{Q}_r)'(rq_r)'\right]rdr\\
&+4\pi^2\tau'(0)\int_{r_0}^{r_w}\left[|\widehat{Q}_r|^2+\frac{1}{m^2+k^2r^2}|(r\widehat{Q}_r)'|^2\right]rdr,
\end{aligned}
$$

that is

$$
\begin{aligned}
0 = 4\pi^2\int_0^{r_0}&\left\{\frac{\gamma p+B_\theta^2}{r}(r\xi)'(r\xi_0)'+r\beta_0(r\xi)\cdot(r\xi_0)-kB_\theta^2(r\xi_0)'\eta+\frac{2kB_\theta^2}{r}(r\xi_0)\cdot\eta\right.\\
&-\frac{2k^2B_\theta^2}{m^2+k^2r^2}r\xi(r\xi_0)'-\frac{2k^2B_\theta^2}{m^2+k^2r^2}r\xi_0(r\xi)'+\frac{4k^2B_\theta^2}{m^2+k^2r^2}\xi(r\xi_0)-k\gamma p(r\xi_0)'\eta\\
&+\frac{m}{r}\gamma p(r\xi_0)'\zeta+\frac{(m^2+k^2r^2)B_\theta^2}{r^2}\eta\cdot(r\eta_0)-\frac{kB_\theta^2}{r}(r\xi)'(r\eta_0)+\frac{2kB_\theta^2}{r}\xi\cdot(r\eta_0)\\
&-\frac{k\gamma p}{r}(r\xi)'\cdot(r\eta_0)+k^2\gamma p\eta\cdot(r\eta_0)-\frac{mk}{r}\gamma p\zeta\cdot(r\eta_0)+\frac{m\gamma p}{r^2}(r\xi)'(r\zeta_0)\\
&\left.-\frac{mk}{r}\gamma p\eta(r\zeta_0)+\frac{m^2}{r^2}\gamma p\zeta\cdot(r\zeta_0)\right\}dr-4\pi^2\left[\frac{2m^2B_\theta^2}{m^2+k^2r^2}\xi\xi_0\right]_{r=r_0}\\
&+4\pi^2\int_{r_0}^{r_w}\left[\widehat{Q}_rq_r+\frac{1}{m^2+k^2r^2}(r\widehat{Q}_r)'(rq_r)'\right]rdr+2\tau'(0)\lambda.
\end{aligned}
$$

Therefore, we can prove that

$$
\begin{aligned}
-\tau'(0)\lambda &= 2\pi^2\int_0^{r_0}\rho\lambda(\xi_0\xi+\eta_0\eta+\zeta\zeta_0)rdr\\
&=2\pi^2\int_0^{r_0}\left\{\frac{\gamma p+B_\theta^2}{r}(r\xi)'(r\xi_0)'+r\beta_0(r\xi)\cdot(r\xi_0)-kB_\theta^2(r\xi_0)'\eta\right.\\
&\quad+\frac{2kB_\theta^2}{r}(r\xi_0)\cdot\eta-\frac{2k^2B_\theta^2}{m^2+k^2r^2}r\xi(r\xi_0)'-\frac{2k^2B_\theta^2}{m^2+k^2r^2}r\xi_0(r\xi)'\\
&\quad+\frac{4k^2B_\theta^2}{m^2+k^2r^2}\xi(r\xi_0)-k\gamma p(r\xi_0)'\eta+\frac{m}{r}\gamma p(r\xi_0)'\zeta+\left(\frac{m^2}{r^2}+k^2\right)B_\theta^2\eta\\
&\quad\cdot(r\eta_0)-\frac{kB_\theta^2}{r}(r\xi)'(r\eta_0)+\frac{2kB_\theta^2}{r}\xi\cdot(r\eta_0)-\frac{k\gamma p}{r}(r\xi)'\cdot(r\eta_0)+k^2\gamma p\eta\\
&\quad\cdot(r\eta_0)-\frac{mk}{r}\gamma p\zeta\cdot(r\eta_0)+\frac{m\gamma p}{r^2}(r\xi)'(r\zeta_0)-\frac{mk}{r}\gamma p\eta(r\zeta_0)+\frac{m^2}{r^2}\gamma p\zeta\\
&\quad\left.\cdot(r\zeta_0)\right\}dr-2\pi^2\left[\frac{2m^2B_\theta^2}{m^2+k^2r^2}\xi\xi_0\right]_{r=r_0}\\
&\quad+2\pi^2\int_{r_0}^{r_w}\left[\widehat{Q}_rq_r+\frac{1}{m^2+k^2r^2}(r\widehat{Q}_r)'(rq_r)'\right]rdr. \qquad (3.66)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\beta_0 &= \frac{1}{r}\left[\frac{m^2B_\theta^2}{r^3} + \frac{2m^2B_\theta\left(\frac{B_\theta}{r}\right)'}{r(m^2+k^2r^2)} - \frac{4k^2m^2B_\theta^2}{r(m^2+k^2r^2)^2} + \frac{2k^2p'}{m^2+k^2r^2}\right],\\
p' &= -\frac{1}{r}(rB_\theta)'B_\theta,
\end{aligned}
$$

we can get that

$$
\begin{aligned}
&\int_0^{r_0}\left\{r\beta_0(r\xi)\cdot(r\xi_0) - \frac{2k^2B_\theta^2}{m^2+k^2r^2}r\xi(r\xi_0)' - \frac{2k^2B_\theta^2}{m^2+k^2r^2}r\xi_0(r\xi)'\right.\\
&\qquad\left. + \frac{4k^2B_\theta^2}{m^2+k^2r^2}\xi(r\xi_0)\right\}dr\\
&\quad = \int_0^{r_0}\left\{\frac{m^2}{r^2}B_\theta^2\xi\cdot(r\xi_0) + \left(\frac{B_\theta^2}{r^2}\right)'(r\xi)\cdot(r\xi_0)\right\}dr - \left[\frac{2k^2r^2B_\theta^2}{m^2+k^2r^2}\xi\xi_0\right]_{r=r_0}.
\end{aligned}
$$

Since ξ_0, η_0, ζ_0 and q_r are independent, and using

$$
-\left[\frac{2k^2r^2B_\theta^2}{m^2+k^2r^2}\xi\xi_0\right]_{r=r_0} - \left[\frac{2m^2B_\theta^2}{m^2+k^2r^2}\xi\xi_0\right]_{r=r_0} = -2[B_\theta^2\xi\xi_0]_{r=r_0},
$$

inserting the above identity into (3.66), one has the triplet of equations

$$
\begin{aligned}
&-\int_0^{r_0}(\gamma p + B_\theta^2)k\eta(r\xi_0)'dr + \int_0^{r_0}2B_\theta^2k\eta\xi_0dr + \int_0^{r_0}\left(\frac{B_\theta^2}{r^2}\right)'(r\xi)\cdot(r\xi_0)dr\\
&\qquad + \int_0^{r_0}\frac{\gamma p + B_\theta^2}{r}(r\xi)'(r\xi_0)'dr + \int_0^{r_0}\frac{m^2}{r^2}B_\theta^2\xi\cdot(r\xi_0)dr + \int_0^{r_0}\frac{m}{r}\gamma p(r\xi_0)'\zeta dr\\
&\qquad - 2[B_\theta^2\xi\xi_0]_{r=r_0} + \int_{r_0}^{r_w}\left[\widehat{Q}_rq_r + \frac{1}{m^2+k^2r^2}(r\widehat{Q}_r)'(rq_r)'\right]rdr = \int_0^{r_0}\rho\lambda\xi_0\xi rdr,\\
&\int_0^{r_0}\left(\frac{m^2}{r^2}+k^2\right)B_\theta^2\eta\cdot(r\eta_0)dr - \int_0^{r_0}\frac{k(\gamma p + B_\theta^2)}{r}(r\xi)'(r\eta_0)dr + \int_0^{r_0}\frac{2kB_\theta^2}{r}\xi\\
&\qquad \cdot(r\eta_0)dr + \int_0^{r_0}k^2\gamma p\eta\cdot(r\eta_0)dr - \int_0^{r_0}\frac{mk}{r}\gamma p\zeta\cdot(r\eta_0)dr = \int_0^{r_0}\rho\lambda\eta_0\eta rdr,\\
&\int_0^{r_0}\frac{m\gamma p}{r^2}(r\xi)'(r\zeta_0)dr - \int_0^{r_0}\frac{mk}{r}\gamma p\eta(r\zeta_0)dr + \int_0^{r_0}\frac{m^2}{r^2}\gamma p\zeta\cdot(r\zeta_0)dr\\
&\qquad = \int_0^{r_0}\rho\lambda\zeta_0\zeta rdr.
\end{aligned}
\tag{3.67}
$$

By making variation with ξ_0 compactly supported in $(0, r_0)$, and make variations q_r compactly supported in (r_0, r_w), one gets that (ξ, η, ζ) satisfy (3.63) in a weak sense in $(0, r_0)$ and $\widehat{Q}_r$ satisfies (3.64) in a weak sense in (r_0, r_w).

Now we show that the interface boundary condition (3.65) is satisfied. From the first equation (3.63), we know that

$$\frac{d}{dr}\left[\gamma p\left(\frac{1}{r}(r\xi)'-k\eta+\frac{m}{r}\zeta\right)\right]+\frac{d}{dr}\left[B_\theta^2\left(\frac{1}{r}(r\xi)'-k\eta\right)\right]-\frac{2B_\theta B_\theta'\xi}{r}$$
$$-\frac{B_\theta^2\xi'}{r}+\frac{B_\theta^2\xi}{r^2}+\frac{B_\theta^2}{r}\left(\frac{1}{r}(r\xi)'-2k\eta-\frac{m^2}{r}\xi\right)=-\rho\lambda\xi,$$

which together with $(\xi,\eta,\zeta)\in Y_{m,k}$ (see (3.39)) and $\xi\in V$ (see (3.40)), gives that

$$\frac{d}{dr}\left[\gamma p\left(\frac{1}{r}(r\xi)'-k\eta+\frac{m}{r}\zeta\right)+B_\theta^2\left(\frac{1}{r}(r\xi)'-k\eta\right)\right]\in L^2\left(\frac{r_0}{2},r_0\right).$$

After a similar argument, we have

$$\frac{d}{dr}\left[\frac{r}{m^2+k^2r^2}(r\widehat{Q}_r)'\right]\in L^2(r_0,r_w),$$

so $\gamma p(\frac{1}{r}(r\xi)'-k\eta+\frac{m}{r}\zeta)+B_\theta^2(\frac{1}{r}(r\xi)'-k\eta)$ and $\frac{r}{m^2+k^2r^2}(r\widehat{Q}_r)'$ are well-defined at the endpoint $r=r_0$. Make variations with respect to $\xi_0\in C_c^\infty(0,r_0]$, $q_r\in C_c^\infty[r_0,r_w)$. Integrating the terms in (3.67) with derivatives of ξ_0 and q_r by parts, using ξ solves the first equation of (3.63) on $(0,r_0)$ and $\widehat{Q}_r$ solves (3.64) on (r_0,r_w), we get that

$$[(\gamma p+B_\theta^2)(r\xi)'\xi_0-k(\gamma p+B_\theta^2)\eta\xi_0 r-2B_\theta^2\xi\xi_0+m\gamma p\zeta\xi_0]_{r=r_0}$$
$$-\left[\frac{r}{m^2+k^2r^2}(r\widehat{Q}_r)'rq_r\right]_{r=r_0}=0.$$

Since ξ_0 and q_r may be chosen arbitrarily, and q_r satisfies $m\widehat{B}_\theta\xi_0=rq_r$ on the boundary $r=r_0$, using $p=0$ on the boundary $r=r_0$ and $\widehat{Q}_\theta=-\frac{m}{m^2+k^2r^2}(r\widehat{Q}_r)'$, we deduce the interface boundary condition

$$[B_\theta^2\xi-B_\theta^2\xi'r+kB_\theta^2\eta r-\widehat{B}_\theta\widehat{Q}_\theta r]|_{r=r_0}=0. \qquad \square$$

4. Analysis About the Growing Mode as a Function of *m* and *k*

In this section, we first prove the growing mode is bounded for any $(m,k)\in\mathbb{Z}\times\mathbb{Z}$, if the pressure satisfies $|p'|\le C\rho$ for r near r_0, and then show that the growing mode has no lower bound under suitable condition of the pressure p. First, we prove the growing mode is bounded for any m and k, under the condition $|p'|\le C\rho$ for r near r_0.

Proposition 4.1. *If $|p'|\le C\rho$ for r near r_0, then the growing mode is bounded for any m and k.*

Proof. We can directly get from the energy $E_{0,k}$ and $E_{m,k}$ in (2.26) and (2.27) that for any m and k

$$E_{m,k}\ge 2\pi^2\int_0^{r_0}2p'\xi^2dr=2\pi^2\int_0^{s_1}2p'\xi^2dr+2\pi^2\int_{s_1}^{r_0}2p'\xi^2dr.$$

From $p' = -B_\theta B_\theta' - \frac{B_\theta^2}{r} = -B_\theta \mathbb{J}_z$, it follows that for s_1 near r_0

$$\left|\int_0^{s_1} p'\xi^2 dr\right| = \left|\int_0^{s_1} \frac{p'}{r}\xi^2 r dr\right| = \left|-\int_0^{s_1} \frac{B_\theta \mathbb{J}_z}{r}\xi^2 r dr\right| \leq \sup_{0\leq r\leq r_0}\left|\frac{B_\theta}{r}\right| \sup_{0\leq r\leq r_0}|\mathbb{J}_z|\mathcal{J}$$

and

$$\left|\int_{s_1}^{r_0} p'\xi^2 dr\right| = \left|\int_{s_1}^{r_0} \frac{p'}{\rho}\rho\xi^2 dr\right| \leq \sup_{0\leq r\leq r_0}\left|\frac{p'}{\rho}\right|\mathcal{J}.$$

On the other hand, it follows from $B_\theta = \frac{1}{r}\int_0^r s\mathbb{J}_z(s)ds$ that

$$\sup_{0\leq r\leq r_0}\frac{|B_\theta|}{r} \leq \frac{\|\mathbb{J}_z\|_{L^\infty}}{2}.$$

Therefore, we get from $|p'| \leq C\rho$ for r near r_0 that

$$\left|\int_0^{r_0} 2p'\xi^2 dr\right| \leq C(\mathcal{J}),$$

which ensures that the energy $E_{0,k}$ and $E_{m,k}$ have a uniform lower bound. Therefore, the growing mode is bounded. □

Now we introduce the following examples which ensures the condition in Proposition 4.1.

Example 4.1. (I) Assume $p(r) = C(r_0 - r)^\beta$ for r near r_0 and $\beta \geq 1$. If $\gamma \geq \frac{\beta}{\beta-1}$, then $|p'| \leq C\rho$ for r near r_0 and the growing mode is bounded for any m and k.

(II) Assume $p = C\exp\{-(r_0 - r)^{-\beta}\}$ for r near r_0 and $\beta > 0$. If $\gamma > 1$, then $|p'| \leq C\rho$ for r near r_0 and the growing mode is bounded for any m and k.

Proof. (I) Since $p = C(r_0 - r)^\beta$ for r near r_0, we deduce that $p'(r) \sim -(r_0 - r)^{\beta-1}$ for r near r_0. By $p = A\rho^\gamma$, we have $\rho \sim (r_0 - r)^{\frac{\beta}{\gamma}}$ for r near r_0. Hence, if $\gamma \geq \frac{\beta}{\beta-1}$, then $|p'| \leq C\rho$ for r near r_0. By Proposition 4.1, we get that the growing mode is bounded for any m and k.

(II) Since $p = C\exp\{-(r_0 - r)^{-\beta}\}$ for r near r_0, we get that for r near r_0,

$$p'(r) \sim -\frac{\beta}{(r_0 - r)^{\beta+1}}\exp\{-(r_0 - r)^{-\beta}\}.$$

By $p = A\rho^\gamma$, we have $\rho \sim \exp\{-\frac{1}{\gamma}(r_0 - r)^{-\beta}\}$. Hence, if $\gamma > 1$, then $|p'| \leq C\rho$ for r near r_0. By Proposition 4.1, similarly we get that the growing mode is bounded for any m and k. □

Finally, we prove that the growing mode has no lower bound under suitable condition of the pressure.

Proposition 4.2. *Assume $p(r) = C(r_0 - r)^\beta$ for r near r_0 and $\beta \geq 1$. If $\gamma < \frac{\beta}{\beta-1}$, then $\frac{p'}{\rho} \to -\infty$ as $r \to r_0$ and the growing mode has no lower bound.*

Proof. Since $p(r) = C|r-r_0|^\beta$ for r near r_0, from (I) in Example 4.1, we know that $p' \sim -|r-r_0|^{\beta-1}$ and $\rho \sim |r-r_0|^{\frac{\beta}{\gamma}}$ for r near r_0. Hence, if $\gamma < \frac{\beta}{\beta-1}$, then $\frac{p'}{\rho} \to -\infty$ as $r \to r_0$.

Now we prove the ill-posedness by the above facts. We can choose w as any smooth function with compact support near 0 and define a sequence of test functions $\xi_k = w(k^\alpha[r-r_0])$, $\eta_k = \frac{1}{kr}((r\xi_k)' - \frac{2B_\theta^2}{\gamma p + B_\theta^2}\xi_k)$, such that

$$2\pi^2 \int_0^{r_0} k^2(\gamma p + B_\theta^2)\left[\eta_k - \frac{1}{kr}\left((r\xi_k)' - \frac{2B_\theta^2}{\gamma p + B_\theta^2}\xi_k\right)\right]^2 rdr = 0.$$

It follows that

$$\partial_r \xi_k = k^\alpha w'(k^\alpha[r-r_0])$$

and

$$\eta_k \sim k^{\alpha-1}w'(k^\alpha[r-r_0]) + k^{-1}w(k^\alpha[r-r_0]).$$

Therefore, we get for $0 < \alpha < 1$ that

$$\begin{aligned}
\mathcal{J}_{0,k} &= \int_0^{r_0} \rho(\xi_k^2 + \eta_k^2) rdr \\
&\sim \int_0^{r_0} |r-r_0|^{\frac{\beta}{\gamma}}(\xi_k^2 + \eta_k^2) rdr \\
&\sim \int_0^{r_0} |r-r_0|^{\frac{\beta}{\gamma}} |w(k^\alpha[r-r_0])|^2 dr \\
&\quad + \int_0^{r_0} |r-r_0|^{\frac{\beta}{\gamma}} k^{2\alpha-2}|w'(k^\alpha[r-r_0])|^2 dr \\
&\quad + \int_0^{r_0} |r-r_0|^{\frac{\beta}{\gamma}} k^{-2}|w'(k^\alpha[r-r_0])|^2 dr \quad (\text{let } z = k^\alpha[r-r_0]) \\
&\sim k^{-\alpha-\frac{\alpha\beta}{\gamma}} \int_0^{k^\alpha r_0} z^{\frac{\beta}{\gamma}} |w(z)|^2 dz + k^{\alpha-2-\frac{\alpha\beta}{\gamma}} \int_0^{k^\alpha r_0} z^{\frac{\beta}{\gamma}} |w'(z)|^2 dz \\
&\quad + k^{-2-\alpha-\frac{\alpha\beta}{\gamma}} \int_0^{k^\alpha r_0} z^{\frac{\beta}{\gamma}} |w'(z)|^2 dz \\
&\sim k^{-\alpha-\frac{\alpha\beta}{\gamma}}.
\end{aligned} \tag{4.1}$$

Since $p(r) \sim |r-r_0|^\beta$ and $p' \sim -|r-r_0|^{\beta-1}$ for r near r_0, we obtain

$$\begin{aligned}
E_{0,k} = E_{0,k}(\xi_k, \eta_k) = 2\pi^2 \int_0^{r_0} \Bigg\{ &\left[\frac{2p'}{r} + \frac{4\gamma p B_\theta^2}{r^2(\gamma p + B_\theta^2)}\right]\xi_k^2 \\
&+ k^2(\gamma p + B_\theta^2)\left[\eta_k - \frac{1}{kr}\left((r\xi_k)' - \frac{2B_\theta^2}{\gamma p + B_\theta^2}\xi_k\right)\right]^2 \Bigg\} rdr
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi^2 \int_0^{r_0} \left[\frac{2p'}{r} + \frac{4\gamma p B_\theta^2}{r^2(\gamma p + B_\theta^2)}\right] \xi_k^2 r dr \\
&\sim -\int_0^{r_0} |r - r_0|^{\beta-1} w^2(k^\alpha [r - r_0]) dr \\
&\quad + \int_0^{r_0} |r - r_0|^{\beta} w^2(k^\alpha [r - r_0]) dr \quad (\text{let } z = k^\alpha [r - r_0]) \\
&\sim -k^{-\alpha-\alpha(\beta-1)} \int_0^{k^\alpha r_0} z^{\beta-1} w^2(z) dz \\
&\quad + k^{-\alpha-\alpha\beta} \int_0^{k^\alpha r_0} z^\beta w^2(z) dz \sim -k^{-\alpha-\alpha(\beta-1)}.
\end{aligned} \tag{4.2}
$$

Choosing $0 < \alpha < 1$, if $\gamma < \frac{\beta}{\beta-1}$, then we get as $k \to \infty$ that

$$
\lambda_k = \min \frac{E_{0,k}}{\mathcal{J}_{0,k}} \sim \frac{-k^{-\alpha-\alpha(\beta-1)}}{k^{-\alpha-\frac{\alpha\beta}{\gamma}}} = -k^{\frac{\alpha\beta}{\gamma}-\alpha(\beta-1)} \to -\infty. \qquad \square
$$

Appendix A. Perturbed MHD System in Lagrangian Coordinates

A.1. *Harmonic extension of the free surface*

According to Sec. 2.1, we know that the equation of the free surface $S_{t,pv}$ may be read as follows:

$$
h(t, \mathcal{X}) = \mathcal{X} + g(t, \mathcal{X}), \quad (\mathcal{X} \in \Sigma_{0,pv} = \{x = r_0\}),
$$

that is

$$
\begin{cases}
h^r(t, r_0, \theta, z) = r_0 + g^r(t, r_0, \theta, z), \\
h^\theta(t, r_0, \theta, z) = \theta + g^\theta(t, r_0, \theta, z), \\
h^z(t, r_0, \theta, z) = z + g^z(t, r_0, \theta, z).
\end{cases}
$$

We consider the fixed equilibrium vacuum domain

$$
\Omega_0^v = \{(r, \theta, z) \in C(0; r_0, r_w) \times [0, 2\pi] \times 2\pi\mathbb{T} | r_0 < r < r_w, \theta \in [0, 2\pi], z \in 2\pi\mathbb{T}\} \tag{A.1}
$$

for which we will write the coordinates as $\widehat{\mathcal{X}} \in \Omega_0^v$. We will think of $\Sigma_{0,pv} = \{(r, \theta, z) | r = r_0, \theta \in [0, 2\pi], z \in 2\pi\mathbb{T}\}$ as the plasma–vacuum interface of Ω_0^v, and we will write $\Sigma_w = \{(r, \theta, z) | r = r_w, \theta \in [0, 2\pi], z \in 2\pi\mathbb{T}\}$ for the outer perfectly conducting wall.

We continue to view $g(t, r_0, \theta, z) = g^r(t, r_0, \theta, z)\, e_r + g^\theta(t, r_0, \theta, z)\, e_\theta + g^z(t, r_0, \theta, z)\, e_z$ as a vector field on $\mathbb{R}^+ \times \Sigma_{0,pv}$. We then define a vector field in cylindrical coordinates $\Psi(t, r, \theta, z) = \Psi^r(t, r, \theta, z)\, e_r + \Psi^\theta(t, r, \theta, z)\, e_\theta + \Psi^z(t, r, \theta, z)\, e_z$ as the displacement in vacuum

$$
\Psi(t, r, \theta, z) = \mathcal{H}_v g = \text{generalized harmonic extension of } g \text{ into } \Omega_0^v, \tag{A.2}
$$

where $\mathcal{H}_v g$ solves the following Laplacian equations:

$$\begin{cases} \Delta\,\Psi = 0, & \text{in } \Omega_0^v, \\ \Psi|_{\mathbb{R}^+\times\Sigma_{0,pv}} = g, \quad \Psi|_{\mathbb{R}^+\times\Sigma_w} = 0, \end{cases} \tag{A.3}$$

that is

$$\begin{cases} \left(\widetilde{\Delta} - \dfrac{1}{r^2}\right)\Psi^r - \dfrac{2}{r^2}\partial_\theta \Psi^\theta = 0, & \text{in } \Omega_0^v, \\ \Psi^r|_{\mathbb{R}^+\times\Sigma_{0,pv}} = g^r, \quad \Psi^r|_{\mathbb{R}^+\times\Sigma_w} = 0, \end{cases}$$

$$\begin{cases} \left(\widetilde{\Delta} - \dfrac{1}{r^2}\right)\Psi^\theta + \dfrac{2}{r^2}\partial_\theta \Psi^r = 0, & \text{in } \Omega_0^v, \\ \Psi^\theta|_{\mathbb{R}^+\times\Sigma_{0,pv}} = g^\theta, \quad \Psi^\theta|_{\mathbb{R}^+\times\Sigma_w} = 0, \end{cases}$$

$$\begin{cases} \widetilde{\Delta}\,\Psi^z = 0, & \text{in } \Omega_0^v, \\ \Psi^z|_{\mathbb{R}^+\times\Sigma_{0,pv}} = g^z, \quad \Psi^z|_{\mathbb{R}^+\times\Sigma_w} = 0, \end{cases}$$

with $\widetilde{\Delta} = \partial_r^2 + \frac{\partial_r}{r} + \partial_z^2 + \frac{1}{r^2}\partial_\theta^2$.

The generalized harmonic extension $\Phi := \mathrm{Id} + \Psi$ in vacuum of the flow map h allows us to flatten the coordinate domain via the mapping

$$\Omega_0^v \ni (r_s, \theta_s, z_s) \mapsto \Phi(t, r_s, \theta_s, z_s) = (x, y, z) \in \Omega^v(t).$$

Remark A.1. Note that

$$\Phi(t, \Sigma_{0,pv}) = \Sigma_{t,pv}, \quad \Phi(t,\cdot)|_{\Sigma_w} = Id|_{\Sigma_w},$$

that is, Φ maps $\Sigma_{0,pv}$ to the free surface and keeps the outer perfectly conducting wall fixed.

A.2. *Vacuum equations in Lagrangian coordinates*

According to the extended co-moving frame $\widehat{\mathcal{X}}(t) = \Phi(t, r, \theta, z)$, we may introduce the "virtual velocity" field $\widehat{u}(t, \Phi) = \frac{d}{dt}\Phi(t, r, \theta, z)$ reduced by the virtual particle in vacuum (which satisfies $|\frac{\widehat{u}}{c}| = o(1)$ when we consider the non-relativistic MHD, here c is the light speed).

We define Lagrangian quantities in vacuum as follows:

$$\widehat{b}(t, \widehat{\mathcal{X}}) = \widehat{B}(t, \Phi(t, \widehat{\mathcal{X}})), \quad \widehat{v}(t, \widehat{\mathcal{X}}) = \widehat{u}(t, \Phi(t, \widehat{\mathcal{X}})), \quad \widehat{\mathcal{A}} = (D\Phi)^{-1}, \quad \widehat{J} = \det(D\Phi).$$

Similar to (2.3) and (2.4), thanks to definitions of the mapping η and the displacement Ψ in vacuum, we may also get the following identities:

$$\begin{aligned} &\widehat{\mathcal{A}}_i^k \partial_k \Phi^j = \widehat{\mathcal{A}}_k^j \partial_i \Phi^k = \delta_i^j, \quad \partial_k(\widehat{J}\widehat{\mathcal{A}}_i^k) = 0, \quad \partial_i \Phi^j = \delta_i^j + \partial_i \Psi^j, \\ &\qquad \widehat{\mathcal{A}}_i^j = \delta_i^j - \widehat{\mathcal{A}}_i^k \partial_k \Psi^j, \\ &\quad \partial_\ell \widehat{\mathcal{A}}_i^j = -\widehat{\mathcal{A}}_k^j \widehat{\mathcal{A}}_i^h \partial_h \partial_\ell \Psi^k, \quad \partial_i \widehat{v}^j = \partial_i \Phi^k \widehat{\mathcal{A}}_k^h \partial_h \widehat{v}^j = \widehat{\mathcal{A}}_i^h \partial_h \widehat{v}^j + \partial_i \Psi^k \widehat{\mathcal{A}}_k^h \partial_h \widehat{v}^j, \\ &\qquad \partial_t \widehat{J} = \widehat{J}\widehat{\mathcal{A}}_i^j \partial_j \widehat{v}^i, \quad \partial_\ell \widehat{J} = \widehat{J}\widehat{\mathcal{A}}_i^j \partial_j \partial_\ell \Psi^i, \quad \partial_t \widehat{\mathcal{A}}_i^j = -\widehat{\mathcal{A}}_k^j \widehat{\mathcal{A}}_i^\ell \partial_\ell \widehat{v}^k. \end{aligned} \tag{A.4}$$

If the displacement Ψ is sufficiently small in an appropriate Sobolev space, then the flow mapping Φ is a diffeomorphism from Ω_0^v to $\Omega^v(t)$, which allows us to switch back and forth from Lagrangian to Eulerian coordinates.

Denote $(\nabla_{\widehat{\mathcal{A}}})_i = \widehat{\mathcal{A}}_i^j \partial_j$, then we may write the vacuum equations in Lagrangian coordinates as follows:

$$\nabla_{\widehat{\mathcal{A}}} \cdot \widehat{b} = 0, \quad \nabla_{\widehat{\mathcal{A}}} \times \widehat{b} = 0 \quad \text{in } \Omega_0^v. \tag{A.5}$$

A.3. *Decompositions of Lagrangian quantities around the equilibrium*

A.3.1. *Decompositions of J and b*

We may compute the Jacobian of the Lagrangian transformation as follows:

$$\begin{aligned} J = \det(D(h)) &= \left(1 + \frac{\partial_\theta g^\theta}{r} + \frac{1}{r} g^r\right)((1+\partial_z g^z)(1+\partial_r g^r) - \partial_z g^r \partial_r g^z) \\ &\quad + \left(\frac{\partial_\theta g^r}{r} - \frac{g^\theta}{r}\right)\partial_z g^\theta \partial_r g^z + \frac{\partial_\theta g^z}{r}\partial_r g^\theta \partial_z g^r - \left(\frac{\partial_\theta g^r}{r} - \frac{g^\theta}{r}\right)\partial_r g^\theta(1+\partial_z g^z) \\ &\quad - \frac{\partial_\theta g^z}{r}\partial_z g^\theta (1+\partial_r g^r) = 1 + J_1 = 1 + \nabla\cdot g + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &:= \nabla\cdot g + J_2, \quad \nabla\cdot g = \partial_r g^r + \frac{1}{r}g^r + \partial_z g^z + \frac{\partial_\theta g^\theta}{r}, \\ J_2 &:= \partial_r g^r \partial_z g^z + \left(\frac{\partial_\theta g^\theta}{r} + \frac{1}{r}g^r\right)(\partial_z g^z + \partial_r g^r + \partial_r g^r \partial_z g^z - \partial_r g^z \partial_z g^r) - \partial_z g^r \partial_r g^z \\ &\quad + \left(\frac{\partial_\theta g^r}{r} - \frac{g^\theta}{r}\right)\partial_z g^\theta \partial_r g^z + \frac{\partial_\theta g^z}{r}\partial_r g^\theta \partial_z g^r - \left(\frac{\partial_\theta g^r}{r} - \frac{g^\theta}{r}\right)\partial_r g^\theta(1+\partial_z g^z) \\ &\quad - \frac{\partial_\theta g^z}{r}\partial_z g^\theta(1+\partial_r g^r). \end{aligned}$$

Denote

$$\begin{aligned} b_1 &= b_0\cdot\nabla g - b_0 \nabla\cdot g, \\ b_2 &= -J^{-1}((J_2 + \nabla\cdot g)b_1 + J_2 b_0), \end{aligned} \tag{A.6}$$

then we can split b into three parts

$$b = b_0 + b_1 + b_2. \tag{A.7}$$

A.3.2. *Decompositions of the pressure q*

We first write

$$\begin{aligned} q = A\,\rho_0^\gamma\, J^{-\gamma} &= p_0\, J^{-\gamma} = p_0 + p_0\, J^{-\gamma}(1 - J^\gamma) \\ &= p_0 + p_0\, J^{-\gamma}(1 - (1+\nabla\cdot g + J_2)^\gamma), \end{aligned}$$

which implies $q = p_0(1 - \gamma\,\nabla\cdot g + Q_2)$, with $Q_2 = (1+\nabla\cdot g + J_2)^{-\gamma} - (1-\gamma\nabla\cdot g)$.

Since we expect that $J-1=\nabla\cdot g+J_2$ is small, we obtain from the Taylor expansion that

$$\begin{aligned}Q_2 &= -\gamma J_2+\frac{1}{2}\gamma(\gamma+1)(\nabla\cdot g+J_2)^2\\ &\quad -\frac{1}{2}\gamma(\gamma+1)(\gamma+2)(\nabla\cdot g+J_2)^3\int_0^1(1-\tau)^2(1+\tau(\nabla\cdot g+J_2))^{-\gamma-3}\,d\tau.\end{aligned}$$

Therefore, we split q into three parts $q=p_0+q_1+q_2$, with $q_1=-\gamma p_0\nabla\cdot g$ and $q_2=p_0Q_2$.

A.3.3. *Decompositions of the normal vector $\mathcal{N}$ on the free surface*

Let $n_0=e_r$, $\mathcal{N}_i=J\mathcal{A}_i^j n_{0,j}$, then we have

$$\mathcal{N}=n_0+n_1+n_2, \tag{A.8}$$

with

$$\begin{aligned}n_1 &= \left(\partial_z g^z+\frac{\partial_\theta g^\theta}{r}+\frac{g^r}{r}\right)e_r+\left(\frac{g^\theta}{r}-\frac{\partial_\theta g^r}{r}\right)e_\theta-\partial_z g^r e_z,\\ n_2 &= \left[\left(\frac{\partial_\theta g^\theta}{r}+\frac{g^r}{r}\right)\partial_z g^z-\frac{\partial_z g^\theta\partial_\theta g^z}{r}\right]e_r+\left[\left(\frac{g^\theta}{r}-\frac{\partial_\theta g^r}{r}\right)\partial_z g^z+\frac{\partial_\theta g^z\partial_z g^r}{r}\right]e_\theta\\ &\quad+\left[\left(\frac{\partial_\theta g^r}{r}-\frac{g^\theta}{r}\right)\partial_z g^\theta-\left(\frac{g^r}{r}+\frac{\partial_\theta g^\theta}{r}\right)\partial_z g^r\right]e_z.\end{aligned}$$

A.3.4. *Decompositions of Lagrangian quantities around the equilibrium in vacuum*

From Lemma 2.1, we know that the equilibrium vacuum magnetic field $\widehat{B}=\widehat{B}_\theta(r)e_\theta=B_\theta(r_0)\frac{r_0}{r}e_\theta$. So in vacuum, we will use (A.2) and (A.3) to split $\widehat{b}$ into three parts in Lagrangian coordinates as

$$\widehat{b}=\widehat{b}_0+\widehat{b}_1+\widehat{b}_2, \tag{A.9}$$

where

$$\begin{aligned}&\widehat{b}_0=\overline{B}_\theta(r_0)\frac{r_0}{r}e_\theta,\\ &\widehat{b}_1\ (\text{first order about }\Psi),\\ &\widehat{b}_2=O\,(\text{nonlinear term about }\Psi).\end{aligned}$$

From the vacuum equations in Lagrangian coordinates (A.5), we can get the linearized vacuum equations in a perturbation around steady solution as follows:

$$\begin{cases} \nabla\cdot\widehat{b}_1+\nabla_{\widehat{\mathcal{A}_1}}\cdot\widehat{b}_0=0, \\ \nabla\times\widehat{b}_1+\nabla_{\widehat{\mathcal{A}_1}}\times\widehat{b}_0=0, \\ n\cdot\widehat{b}_1=0, & \text{on } r=r_w, \\ n_0\cdot(b_1-\widehat{b}_1)=n_1\cdot(\widehat{b}_0-b_0) & \text{on } r=r_0, \end{cases}$$

with

$$n_0=e_r,\quad b_1=b_0\cdot\nabla g-b_0\nabla\cdot g,$$

$$n_1=\left(\partial_z\Psi^z+\frac{\partial_\theta\Psi^\theta}{r}+\frac{\Psi^r}{r}\right)e_r+\left(\frac{\Psi^\theta}{r}-\frac{\partial_\theta\Psi^r}{r}\right)e_\theta-\partial_z\Psi^r e_z,$$

$$\widehat{\mathcal{A}}_1=\begin{pmatrix} -\partial_r\Psi^r & -\partial_r\Psi^\theta & -\partial_r\Psi^z \\ -\frac{\partial_\theta\Psi^r}{r}+\frac{\Psi^\theta}{r} & -\frac{\partial_\theta\Psi^\theta}{r}-\frac{\Psi^r}{r} & -\frac{\partial_\theta\Psi^z}{r} \\ -\partial_z\Psi^r & -\partial_z\Psi^\theta & -\partial_z\Psi^z \end{pmatrix}.$$

From the steady solution $\widehat{b}_0=\widehat{B}_\theta(r)e_\theta=B_\theta(r_0)\frac{r_0}{r}e_\theta$ in vacuum domain, it follows that

$$\nabla\widehat{b}_0=\begin{pmatrix} 0 & -\frac{\widehat{B}_\theta}{r} & 0 \\ \partial_r\widehat{B}_\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which gives that

$$\nabla_{\widehat{\mathcal{A}}_1}\cdot\widehat{b}_0=\mathrm{Tr}(\widehat{\mathcal{A}}_1^T\nabla\widehat{b}_0)=\frac{\widehat{B}_\theta}{r}\partial_r\Psi^\theta-\partial_r\widehat{B}_\theta\left(\frac{1}{r}\partial_\theta\Psi^r-\frac{\Psi^\theta}{r}\right)=-\nabla\cdot(\Psi\cdot\nabla\widehat{b}_0).$$

On the other hand, we have

$$\nabla_{\widehat{\mathcal{A}}_1}\times\widehat{b}_0=\epsilon_{ijk}\widehat{\mathcal{A}}_{1jl}\partial_l\widehat{b}_0^k=\epsilon_{ijk}(\widehat{\mathcal{A}}_{1jl}(\nabla\widehat{b}_0)_{kl})=\epsilon_{ijk}(\widehat{\mathcal{A}}_1(\nabla\widehat{b}_0)^T)_{jk}$$

$$=\epsilon_{ijk}\begin{pmatrix} \partial_r\Psi^\theta\frac{\widehat{B}_\theta}{r} & -\partial_r\Psi^r\partial_r\widehat{B}_\theta & 0 \\ \left(\frac{\partial_\theta\Psi^\theta}{r}+\frac{\Psi^r}{r}\right)\frac{\widehat{B}_\theta}{r} & \left(-\frac{\partial_\theta\Psi^r}{r}+\frac{\Psi^\theta}{r}\right)\partial_r\widehat{B}_\theta & 0 \\ \partial_z\Psi^\theta\frac{\widehat{B}_\theta}{r} & -\partial_z\Psi^r\partial_r\widehat{B}_\theta & 0 \end{pmatrix}_{jk}$$

$$=e_r\partial_z\Psi^r\partial_r\widehat{B}_\theta+e_\theta\partial_z\Psi^\theta\frac{\widehat{B}_\theta}{r}$$

$$+ e_z \left[- \left(\frac{\partial_\theta \Psi^\theta}{r} + \frac{\Psi^r}{r} \right) \frac{\widehat{B}_\theta}{r} - \partial_r \Psi^r \partial_r \widehat{B}_\theta \right],$$

$$\begin{aligned} -\nabla \times (\Psi \cdot \nabla \widehat{b}_0) &= - \left(e_r \partial_r + \frac{e_\theta}{r} \partial_\theta + e_z \partial_z \right) \times \left(e_\theta \Psi^r \partial_r \widehat{B}_\theta - e_r \frac{\Psi^\theta \widehat{B}_\theta}{r} \right) \\ &= e_r \partial_z \Psi^r \partial_r \widehat{B}_\theta + e_\theta \frac{\partial_z \Psi^\theta \widehat{B}_\theta}{r} - e_z \frac{\partial_\theta \Psi^\theta \widehat{B}_\theta}{r} \\ &\quad - e_z \partial_r \Psi^r \partial_r \widehat{B}_\theta - e_z \frac{\Psi^r \widehat{B}_\theta}{r^2}. \end{aligned}$$

So we can show that $\nabla_{\widehat{\mathcal{A}}_1} \times \widehat{b}_0 = -\nabla \times (\Psi \cdot \nabla \widehat{b}_0)$. Therefore, from $b_1 = b_0 \cdot \nabla g - b_0 \nabla \cdot g$ in (A.6), it follows that

$$\begin{cases} \nabla \cdot (\widehat{b}_1 - \Psi \cdot \nabla \widehat{b}_0) = 0, \\ \nabla \times (\widehat{b}_1 - \Psi \cdot \nabla \widehat{b}_0) = 0, \\ n \cdot \widehat{b}_1 = 0, & \text{on } r = r_w, \\ n_0 \cdot (b_0 \cdot \nabla g - b_0 \nabla \cdot g - \widehat{b}_1) = n_1 \cdot (\widehat{b}_0 - b_0), & \text{on } r = r_0, \end{cases}$$

with $n_1 = (\partial_z \Psi^z + \frac{\partial_\theta \Psi^\theta}{r} + \frac{\Psi^r}{r}) e_r + (\frac{\Psi^\theta}{r} - \frac{\partial_\theta \Psi^r}{r}) e_\theta - \partial_z \Psi^r e_z$. Denoting $\widehat{Q} = \widehat{b}_1 - \Psi \cdot \nabla \widehat{b}_0$, using the fact that $\widehat{b}_0 = b_0$ on the boundary $r = r_0$, from (A.2) and (A.3), we can show that on the boundary $r = r_0$

$$\begin{aligned} n_0 \cdot (b_0 \cdot \nabla g - b_0 \nabla \cdot g - \widehat{b}_1) &= n_0 \cdot (b_0 \cdot \nabla g - b_0 \nabla \cdot g - g \cdot \nabla \widehat{b}_0 - \widehat{Q}) \\ &= n_0 \cdot (b_0 \cdot \nabla g - b_0 \nabla \cdot g - g \cdot \nabla b_0 - \widehat{Q}) = n_0 \cdot (\nabla \times (g \times b_0) - \widehat{Q}) = 0. \end{aligned}$$

Therefore, in vacuum domain, we obtain

$$\begin{cases} \nabla \cdot \widehat{Q} = 0, \\ \nabla \times \widehat{Q} = 0, \\ n \cdot \widehat{Q} = 0, & \text{on } r = r_w, \\ n_0 \cdot \nabla \times (g \times \widehat{b}_0) = n_0 \cdot \widehat{Q} & \text{on } r = r_0. \end{cases} \tag{A.10}$$

A.4. *Perturbed MHD system in plasma*

Thanks to the decomposition of b and q again, we have

$$\begin{aligned} q + \frac{1}{2}|b|^2 &= p_0 + q_1 + q_2 + \frac{1}{2}|b_0 + b_1 + b_2|^2 \\ &= p_0 + \frac{1}{2}|b_0|^2 - \gamma p_0 \nabla \cdot g + b_0 \cdot b_1 + R_{1,p} \end{aligned} \tag{A.11}$$

with

$$R_{1,p} = b_0 \cdot b_2 + q_2 + \frac{1}{2}|b_1 + b_2|^2. \tag{A.12}$$

While for $b\cdot\nabla_{\mathcal{A}}b$, it shows that $J\,b\cdot\nabla_{\mathcal{A}}b=b_0\cdot\nabla b$, where we have used the equality (2.7) $Jb^j\mathcal{A}^i_j=b^i_0$.

Since $b=J^{-1}(b_0+b_0\cdot\nabla g)$, we decompose $J\,b\cdot\nabla_{\mathcal{A}}b$ as

$$J\,b\cdot\nabla_{\mathcal{A}}b=b_0\cdot\nabla b_0+b_0\cdot\nabla b_1+b_0\cdot\nabla b_2=b_0\cdot\nabla b_0+b_0\cdot\nabla b_1+JR_{1,b}. \tag{A.13}$$

Using (A.11), we may deduce that

$$\left(\nabla_{\mathcal{A}}\left(q+\frac{1}{2}|b|^2\right)\right)^k=\mathcal{A}^j_k\partial_j\left(p_0+\frac{1}{2}|b_0|^2-\gamma p_0\nabla\cdot g+b_0\cdot b_1\right)+\mathcal{A}^j_k\partial_j\,R_{1,p}. \tag{A.14}$$

Combining (A.13) with (A.14), we obtain

$$\begin{aligned}J\,\nabla_{\mathcal{A}}\left(q+\frac{1}{2}|b|^2\right)-J\,b\cdot\nabla_{\mathcal{A}}b&=J\,\nabla_{\mathcal{A}}\left(p_0+\frac{1}{2}|b_0|^2-\gamma p_0\nabla\cdot g+b_0\cdot b_1\right)\\&\quad-b_0\cdot\nabla b_0-b_0\cdot\nabla b_1+J\,\nabla_{\mathcal{A}}\,R_{1,p}-J\,R_{1,b}.\end{aligned} \tag{A.15}$$

Substituting (2.6) and (A.15) into the momentum equations of (2.5) results in

$$\begin{aligned}&\rho_0\partial_t v+J\nabla_{\mathcal{A}}\left(p_0+\frac{1}{2}|b_0|^2-\gamma p_0\nabla\cdot g+b_0\cdot b_1\right)-b_0\cdot\nabla b_0-b_0\cdot\nabla b_1\\&\quad=J\,R_{1,b}-J\,\nabla_{\mathcal{A}}\,R_{1,p}.\end{aligned}$$

Let us now deal with the jump conditions on $\Sigma_{0,pv}$ in (2.5). In fact, thanks to (A.11), we know that

$$\left(q+\frac{1}{2}|b|^2-\frac{1}{2}|\widehat{b}|^2\right)\Big|_{\Sigma_{0,pv}}=\left(p_0+\frac{1}{2}|b_0|^2-\gamma p_0\nabla\cdot g+b_0\cdot b_1+R_{1,p}-\frac{1}{2}|\widehat{b}|^2\right)\Big|_{\Sigma_{0,pv}}.$$

From the decomposition of $\widehat{b}$ in (A.9), it follows that

$$\begin{aligned}\frac{1}{2}|\widehat{b}|^2&=\frac{1}{2}|\widehat{b}_0|^2+\widehat{b}_0\cdot\widehat{b}_1+\frac{1}{2}|\widehat{b}_1|^2+\frac{1}{2}|\widehat{b}_2|^2+\widehat{b}_0\cdot\widehat{b}_2+\widehat{b}_1\cdot\widehat{b}_2\\&=\frac{1}{2}|\widehat{b}_0|^2+\widehat{b}_0\cdot\widehat{b}_1+\widehat{R}_{1,p},\end{aligned} \tag{A.16}$$

which along with (2.9), (A.2), (A.3) and (A.6), yields that

$$\begin{aligned}\left(q+\frac{1}{2}|b|^2-\frac{1}{2}|\widehat{b}|^2\right)\Big|_{\Sigma_{0,pv}}&=\left(-\gamma p_0\nabla\cdot g+b_0\cdot b_1+R_{1,p}-\widehat{b}_0\cdot\widehat{b}_1-\widehat{R}_{1,p}\right)\Big|_{\Sigma_{0,pv}}\\&=\left(-\gamma p_0\nabla\cdot g+b_0\cdot Q+g\cdot\nabla\left(\frac{1}{2}|b_0|^2\right)-\widehat{b}_0\cdot\widehat{Q}\right.\\&\quad\left.-g\cdot\nabla\left(\frac{1}{2}|\widehat{b}_0|^2\right)+R_{1,p}-\widehat{R}_{1,p}\right)\Big|_{\Sigma_{0,pv}},\end{aligned} \tag{A.17}$$

with $Q=\nabla\times(g\times b_0)$, $\widehat{Q}=\widehat{b}_1-\Psi\cdot\nabla\widehat{b}_0$.

In conclusion, we rephrase the MHD system (2.5) in a perturbation formulation around the steady solution (see special steady solution for a z-pinch in (2.9)) as follows:

$$
\begin{cases}
\partial_t g = v & \text{in } \Omega_0,\\
\rho_0 \partial_t v + J\nabla_{\mathcal{A}}\left(p_0 + \dfrac{1}{2}|b_0|^2 - \gamma p_0 \nabla\cdot g + b_0\cdot b_1\right) & \\
\quad - b_0\cdot\nabla b_0 - b_0\cdot\nabla b_1 & \\
= J\, R_{1,b} - J\,\nabla_{\mathcal{A}}\, R_{1,p} & \text{in } \Omega_0,\\
\nabla_{\widehat{\mathcal{A}}}\cdot\widehat{b} = 0, \quad \nabla_{\widehat{\mathcal{A}}}\times\widehat{b} = 0 & \text{in } \Omega_0^v,\\
n\cdot b|_{\Sigma_{0,pv}} = n\cdot\widehat{b}|_{\Sigma_{0,pv}} = 0, \quad n\cdot\widehat{b}|_{\Sigma_w} = 0, & \\
-\gamma p_0\nabla\cdot g + b_0\cdot Q + g\cdot\nabla\left(\dfrac{1}{2}|b_0|^2\right) - \widehat{b_0}\cdot(\widehat{b_1} - g\cdot\nabla\widehat{b_0}) & \\
\qquad - g\cdot\nabla\left(\dfrac{1}{2}|\widehat{b}_0|^2\right) + R_{1,p} - \widehat{R}_{1,p} = 0 & \text{on } \Sigma_{0,pv},\\
g|_{t=0} = g_0, v|_{t=0} = v_0, &
\end{cases}
\tag{A.18}
$$

with $Q = \nabla\times(g\times b_0)$, b_1 defined in (A.6), $R_{1,p}$ defined in (A.12), $JR_{1,b}$ defined in (A.13) and $\widehat{R}_{1,p}$ defined in (A.16). Let the initial data as the steady solution, from the force $\nabla(p + \frac{1}{2}|B|^2) = B\cdot\nabla B$ of the z-pinch $(p, B, \widehat{B})$, then the linearized MHD system in a perturbation formulation around the steady solution takes the following form:

$$
\begin{cases}
\partial_t g = v & \text{in } \overline{\Omega},\\
\rho\partial_{tt} g + \nabla(-\gamma p\nabla\cdot g + B\cdot b_1) - B\cdot\nabla b_1 + \nabla_{\mathcal{A}_1}\left(p + \dfrac{1}{2}|B|^2\right) & \\
\qquad + (\nabla\cdot g)\nabla\left(p + \dfrac{1}{2}|B|^2\right) = 0 & \text{in } \overline{\Omega},\\
\nabla\cdot\widehat{b}_1 + \nabla_{\widehat{\mathcal{A}_1}}\cdot\widehat{B} = 0, & \text{in } \overline{\Omega}^v,\\
\nabla\times\widehat{b}_1 + \nabla_{\widehat{\mathcal{A}_1}}\times\widehat{B} = 0, & \text{in } \overline{\Omega}^v,\\
n_0\cdot(b_1 - \widehat{b}_1) = n_1\cdot(\widehat{B} - B) & \text{on } \Sigma_{0,pv},\\
-\gamma p\nabla\cdot g + B\cdot Q + g\cdot\nabla\left(\dfrac{1}{2}|B|^2\right) & \\
\qquad = \widehat{B}\cdot(\widehat{b}_1 - g\cdot\nabla\widehat{B}) + g\cdot\nabla\left(\dfrac{1}{2}|\widehat{B}|^2\right), & \text{on } \Sigma_{0,pv},\\
n\cdot\widehat{b}_1|_{\Sigma_w} = 0,\ g|_{t=0} = g_0, &
\end{cases}
\tag{A.19}
$$

with $Q = \nabla \times (g \times B)$, $b_1 = B \cdot \nabla g - B\nabla \cdot g$ and $\mathcal{A}_1$ is the first order of $\mathcal{A}$, that is, in cylindrical coordinates

$$\mathcal{A}_1 = \begin{pmatrix} -\partial_r g^r & -\partial_r g^\theta & -\partial_r g^z \\ -\dfrac{\partial_\theta g^r}{r} + \dfrac{g^\theta}{r} & -\dfrac{\partial_\theta g^\theta}{r} - \dfrac{g^r}{r} & -\dfrac{\partial_\theta g^z}{r} \\ -\partial_z g^r & -\partial_z g^\theta & -\partial_z g^z \end{pmatrix}, \tag{A.20}$$

$$n_1 = \left(\partial_z \Psi^z + \frac{\partial_\theta \Psi^\theta}{r} + \frac{\Psi^r}{r}\right) e_r + \left(\frac{\Psi^\theta}{r} - \frac{\partial_\theta \Psi^r}{r}\right) e_\theta - \partial_z \Psi^r e_z,$$

$$\widehat{\mathcal{A}}_1 = \begin{pmatrix} -\partial_r \Psi^r & -\partial_r \Psi^\theta & -\partial_r \Psi^z \\ -\dfrac{\partial_\theta \Psi^r}{r} + \dfrac{\Psi^\theta}{r} & -\dfrac{\partial_\theta \Psi^\theta}{r} - \dfrac{\Psi^r}{r} & -\dfrac{\partial_\theta \Psi^z}{r} \\ -\partial_z \Psi^r & -\partial_z \Psi^\theta & -\partial_z \Psi^z \end{pmatrix}. \tag{A.21}$$

Denote the new function $\widehat{Q} = \widehat{b}_1 - \Psi \cdot \nabla \widehat{b}_0 = \widehat{b}_1 - \Psi \cdot \nabla \widehat{B}$, after a computation, we can get the above system which is equivalent to the following equations:

$$\begin{cases} \partial_t g = v & \text{in } \overline{\Omega}, \\ \rho \partial_{tt} g = \nabla(g \cdot \nabla p + \gamma p \nabla \cdot g) + (\nabla \times B) \times [\nabla \times (g \times B)] & \\ \qquad + \{\nabla \times [\nabla \times (g \times B)]\} \times B, & \text{in } \overline{\Omega}, \\ \nabla \cdot \widehat{Q} = 0, & \text{in } \overline{\Omega}^v, \\ \nabla \times \widehat{Q} = 0, & \text{in } \overline{\Omega}^v, \\ n \cdot \nabla \times (g \times \widehat{B}) = n \cdot \widehat{Q}, & \text{on } \Sigma_{0,pv}, \\ -\gamma p \nabla \cdot g + B \cdot Q + g \cdot \nabla \left(\dfrac{1}{2}|B|^2\right) = \widehat{B} \cdot \widehat{Q} + g \cdot \nabla \left(\dfrac{1}{2}|\widehat{B}|^2\right), & \text{on } \Sigma_{0,pv}, \\ n \cdot \widehat{Q}|_{\Sigma_w} = 0,\ g|_{t=0} = g_0, & \end{cases} \tag{A.22}$$

with $Q = \nabla \times (g \times B)$.

From the divergence free condition about the magnetic field in (2.5), we know that $\nabla_{\mathcal{A}} \cdot b = 0$ holds in Lagrangian coordinates. We now prove that the linear perturbation of $\nabla_{\mathcal{A}} \cdot b = 0$ holds automatically.

Remark A.2. Assume the steady solution $b_0 = B_\theta(r) e_\theta$, then $\nabla_{\mathcal{A}_1} \cdot b_0 + \nabla \cdot b_1 = 0$ is the linear perturbation of $\nabla_{\mathcal{A}} \cdot b = 0$ and this linear perturbation holds for any function g, where b_1 is defined in (A.6) and $\mathcal{A}_1$ is defined in (A.20).

Proof. From the decomposition of $\mathcal{A}$ and b in Lagrangian coordinates, that is, $\mathcal{A} = I + \mathcal{A}_1 + O(\text{nonlinear matrix about } g)$ and (A.7), it follows that the corresponding linear perturbation is $\nabla_{\mathcal{A}_1} \cdot b_0 + \nabla \cdot b_1 = 0$. From the steady solution $b_0 = B_\theta(r)e_\theta$, it follows that

$$\nabla b_0 = \begin{pmatrix} 0 & -\dfrac{B_\theta}{r} & 0 \\ \partial_r B_\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which gives that

$$\nabla_{\mathcal{A}_1} \cdot b_0 = \mathrm{Tr}(\mathcal{A}_1^T \nabla b_0) = \frac{B_\theta}{r}\partial_r g^\theta - \partial_r B_\theta \left(\frac{1}{r}\partial_\theta g^r - \frac{g^\theta}{r}\right) = -\nabla \cdot (g \cdot \nabla b_0).$$

Therefore, $\nabla_{\mathcal{A}_1} \cdot b_0 + \nabla \cdot b_1 = \nabla \cdot (b_0 \cdot \nabla g - b_0 \nabla \cdot g - g \cdot \nabla b_0)$. On the other hand, we have $\nabla \cdot b_0 = 0$, which implies the identity $b_0 \cdot \nabla g - b_0 \nabla \cdot g - g \cdot \nabla b_0 = \nabla \times (g \times b_0)$. Since from $b_0 = B_\theta(r)e_\theta$, we can show that

$$\begin{aligned}
&\nabla \cdot [\nabla \times (g \times b_0)] \\
&\quad = \left(e_r\partial_r + \frac{e_\theta}{r}\partial_\theta + e_z\partial_z\right) \cdot \left[\left(e_r\partial_r + \frac{e_\theta}{r}\partial_\theta + e_z\partial_z\right) \times (g^r B_\theta e_z - g^z B_\theta e_r)\right] \\
&\quad = \left(e_r\partial_r + \frac{e_\theta}{r}\partial_\theta + e_z\partial_z\right) \cdot \left[e_r \frac{B_\theta \partial_\theta g^r}{r} - e_\theta \left(\partial_r(g^r B_\theta) + B_\theta \partial_z g^z\right) + e_z \frac{B_\theta \partial_\theta g^z}{r}\right] \\
&\quad = \partial_r \left(\frac{B_\theta \partial_\theta g^r}{r}\right) + \frac{B_\theta \partial_\theta g^r}{r^2} - \frac{1}{r}\left(\partial_r(\partial_\theta g^r B_\theta) + B_\theta \partial_z \partial_\theta g^z\right) + \frac{B_\theta \partial_\theta \partial_z g^z}{r} = 0.
\end{aligned}$$

Hence, $\nabla_{\mathcal{A}_1} \cdot b_0 + \nabla \cdot b_1 = 0$ holds automatically, which implies the result. □

In order to see the property of the force operator

$$\begin{aligned}
F(g) = \nabla(g \cdot \nabla p + \gamma p \nabla \cdot g) + (\nabla \times B) \times [\nabla \times (g \times B)] \\
+ \{\nabla \times [\nabla \times (g \times B)]\} \times B,
\end{aligned}$$

we consider two displacement vector fields g and h defined over the plasma volume V, their associated magnetic field perturbations

$$Q = \nabla \times (g \times B), \quad R = \nabla \times (h \times B),$$

and the vacuum perturbations $\widehat{Q}$ and $\widehat{R}$ defined over the vacuum volume $\widehat{V}$ are their extensions, that is, to "extend" the function g into the vacuum by means of the magnetic field variable $\widehat{Q}$, and likewise to "extend" h by means of $\widehat{R}$. Then by Chap. 6 in Ref. 7, we have the following lemma.

Lemma A.3. *Assume $g \in H^2$ is a solution of (A.22), then we get a meaning expression for the potential energy of interface plasma by identifying g, h, $\widehat{Q}$ and $\widehat{R}$,* *in the quadratic form (Appendix A)*

$$\int_{\overline{\Omega}} h\cdot F(g)dx = -\int_{\overline{\Omega}}\left[\gamma p\nabla\cdot g\nabla\cdot h + Q\cdot R + \frac{1}{2}\nabla p\cdot(g\nabla\cdot h + h\nabla\cdot g)\right.$$
$$\left. + \frac{1}{2}\nabla\times B\cdot(g\times R + h\times Q)\right] dx - \int_{\overline{\Omega}^v}\widehat{Q}\cdot\widehat{R}\,dx$$
$$- \int_{\Sigma_{0,pv}} n\cdot g\, n\cdot h\, n\cdot\left[\left[\nabla\left(p + \frac{1}{2}|B|^2\right)\right]\right] dx,$$

which is symmetric in the variables g and h, and their extensions $\widehat{Q}$ and $\widehat{R}$.

Proof. The proof can be recalled from Chap. 6 of Ref. 7, for completeness, we give it as follows. By the equilibrium equation $\nabla p = (\nabla\times B)\times B$, we have

$$\begin{aligned}\nabla(g\cdot\nabla p) &= (\nabla g)\cdot\nabla p + g\cdot\nabla\nabla p = (\nabla p\times\nabla)\times g + \nabla p\nabla\cdot g + g\cdot\nabla\nabla p\\ &= (((\nabla\times B)\times B)\times\nabla)\times g + \nabla p\nabla\cdot g + g\cdot\nabla\nabla p\\ &= (B(\nabla\times B)\cdot\nabla - (\nabla\times B)B\cdot\nabla)\times g + \nabla p\nabla\cdot g + g\cdot\nabla\nabla p\\ &= B\times((\nabla\times B)\cdot\nabla g) - (\nabla\times B)\times(B\cdot\nabla g) + \nabla p\nabla\cdot g + g\cdot\nabla\nabla p,\end{aligned}$$

which together with

$$\begin{aligned}(\nabla\times B)\times[\nabla\times(g\times B)] &= (\nabla\times B)\times(B\cdot\nabla g - B\nabla\cdot g - g\cdot\nabla B)\\ &= (\nabla\times B)\times(B\cdot\nabla g) - (\nabla\times B)\times B\nabla\cdot g\\ &\quad - g\cdot\nabla((\nabla\times B)\times B) - B\times(g\cdot\nabla(\nabla\times B)),\end{aligned}$$

implies that

$$\begin{aligned}&\nabla(g\cdot\nabla p) + (\nabla\times B)\times[\nabla\times(g\times B)]\\ &\quad = B\times((\nabla\times B)\cdot\nabla g) - B\times(g\cdot\nabla(\nabla\times B))\\ &\quad = -B\times(\nabla\times(\nabla\times B\times g)) - (\nabla\times B)\times B\nabla\cdot g\\ &\quad = -B\times(\nabla\times(\nabla\times B\times g)) - \nabla p\nabla\cdot g.\end{aligned} \tag{A.23}$$

Exploiting the inner products and by the expression (A.23), we can rewrite $h\cdot F(g)$ as

$$\begin{aligned}h\cdot F(g) = h\cdot\nabla(\gamma p\nabla\cdot g) - h\cdot B\times\{\nabla\times[\nabla\times(g\times B)]\\ + \nabla\times(\nabla\times B\times g)\} - h\cdot\nabla p\nabla\cdot g.\end{aligned} \tag{A.24}$$

The first term in (A.24) gives the following expression

$$h\cdot\nabla(\gamma p\nabla\cdot g) = -\gamma p\nabla\cdot g\nabla\cdot h + \nabla\cdot(h\gamma p\nabla\cdot g). \tag{A.25}$$

From the definitions of Q and R, the second term in (A.24) can be rewritten as

$$\begin{aligned}
-h\cdot B\times\{\nabla\times[\nabla\times(g\times B)]\} &= -\{\nabla\times[\nabla\times(g\times B)]\}\cdot(h\times B)\\
&= -\nabla\times(g\times B)\cdot\nabla\times(h\times B)+\nabla\cdot[(h\times B)\times\nabla\times(g\times B)]\\
&= -Q\cdot R+\nabla\cdot[(h\times B)\times Q] = -Q\cdot R+\nabla\cdot[Bh\cdot Q-hB\cdot Q]. \qquad \text{(A.26)}
\end{aligned}$$

Applying the definitions of R and the equilibrium equation $\nabla p=(\nabla\times B)\times B=\nabla\times B\times B$, we can rewrite the third and fourth terms in (A.24) as

$$\begin{aligned}
&-h\cdot B\times[\nabla\times(\nabla\times B\times g)]-(h\cdot\nabla p)\nabla\cdot g\\
&\quad= -\nabla\times B\times g\cdot R+\nabla\cdot[(h\times B)\times(\nabla\times B\times g)]-(h\cdot\nabla p)\nabla\cdot g\\
&\quad= g\cdot(\nabla\times B)\times R+\nabla\cdot[(\nabla\times B)B\cdot(g\times h)+gh\cdot(\nabla\times B\times B)]\\
&\qquad -(h\cdot\nabla p)\nabla\cdot g\\
&\quad= g\cdot[\nabla(h\cdot\nabla p)+(\nabla\times B)\times R]+\nabla\cdot[(\nabla\times B)B\cdot(g\times h)],
\end{aligned}$$

which together with (A.23) can be symmetrized as

$$\begin{aligned}
&h\cdot\{\nabla(g\cdot\nabla p)+(\nabla\times B)\times[\nabla\times(g\times B)]\} = h\cdot[\nabla(g\cdot\nabla p)+(\nabla\times B)\times Q]\\
&= \frac{1}{2}h\cdot[\nabla(g\cdot\nabla p)+(\nabla\times B)\times Q]+\frac{1}{2}g\cdot[\nabla(h\cdot\nabla p)+(\nabla\times B)\times R]\\
&\quad+\frac{1}{2}\nabla\cdot[(\nabla\times B)B\cdot(g\times h)]\\
&= \frac{1}{2}\nabla\cdot[\nabla p\cdot(gh+hg)]-\frac{1}{2}\nabla p\cdot(g\nabla\cdot h+h\nabla\cdot g)-\frac{1}{2}\nabla\\
&\quad\times B\cdot(g\times R+h\times Q)+\frac{1}{2}\nabla\cdot[(\nabla\times B)B\cdot(g\times h)]-\frac{1}{2}\nabla\\
&\quad\cdot[(\nabla\times B\times B-\nabla p)\cdot(gh-hg)]\\
&= -\frac{1}{2}\nabla p\cdot(g\nabla\cdot h+h\nabla\cdot g)-\frac{1}{2}\nabla\times B\cdot(g\times R+h\times Q)\\
&\quad+\nabla\cdot\left[h(g\cdot\nabla p)+\frac{1}{2}(\nabla\times B)B\cdot(g\times h)\right]\\
&\quad-\frac{1}{2}\nabla\cdot[(\nabla\times B\times B)\cdot(gh-hg)]. \qquad \text{(A.27)}
\end{aligned}$$

Adding up (A.25), (A.26) and (A.27) shows that

$$\begin{aligned}
h\cdot F(g) = -\gamma p\nabla\cdot g\nabla\cdot h-Q\cdot R-\frac{1}{2}\nabla p\cdot(g\nabla\cdot h+h\nabla\cdot g)\\
-\frac{1}{2}\nabla\times B\cdot(g\times R+h\times Q)\\
+\nabla\cdot[h(g\cdot\nabla p)]-\nabla\cdot(hB\cdot Q)+\nabla\cdot(h\gamma p\nabla\cdot g)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\nabla\cdot[(\nabla\times B)B\cdot(g\times h)]+\nabla\cdot(Bh\cdot Q)\\
&-\frac{1}{2}\nabla\cdot[(\nabla\times B\times B)\cdot(gh-hg)].
\end{aligned}
\tag{A.28}
$$

Integrating (A.28) gives that

$$
\begin{aligned}
\int_{\overline{\Omega}} h\cdot F(g)dx = -\int_{\overline{\Omega}}\Big[&\gamma p\nabla\cdot g\nabla\cdot h+Q\cdot R+\frac{1}{2}\nabla p\cdot(g\nabla\cdot h+h\nabla\cdot g)\\
&+\frac{1}{2}\nabla\times B\cdot(g\times R+h\times Q)\Big]\,dx\\
&+\int_{\Sigma_{0,pv}} n\cdot h(g\cdot\nabla p-B\cdot Q+\gamma p\nabla\cdot g)dx.
\end{aligned}
\tag{A.29}
$$

There are no contributions from the eighth, ninth and tenth terms of (A.28) to the surface integral, since $n\cdot B=0$ and $n\cdot\nabla\times B=0$ on the plasma surface, whereas $\nabla\times B\times B$ is parallel to n. From the second interface condition of (A.22), the surface integral takes the form of

$$
\begin{aligned}
&\int_{\Sigma_{0,pv}} n\cdot h(g\cdot\nabla p-B\cdot Q+\gamma p\nabla\cdot g)dx\\
&\quad=-\int_{\Sigma_{0,pv}} n\cdot hg\cdot\left[\left[\nabla\left(p+\frac{1}{2}|B|^2\right)\right]\right]dS-\int_{\Sigma_{0,pv}} n\cdot h\widehat{B}\cdot\widehat{Q}dx\\
&\quad=-\int_{\Sigma_{0,pv}} n\cdot g\,n\cdot h\,n\cdot\left[\left[\nabla\left(p+\frac{1}{2}|B|^2\right)\right]\right]dS-\int_{\Sigma_{0,pv}} n\cdot h\widehat{B}\cdot\widehat{Q}dx.
\end{aligned}
\tag{A.30}
$$

Here, we have used the facts the equilibrium jump condition $[[p+\frac{1}{2}|B|^2]]=0$, which implies that the tangential derivative of the jump vanishes as well $\mathbf{t}\cdot[[\nabla(p+\frac{1}{2}|B|^2)]]=0$, where $\mathbf{t}$ is an arbitrary unit vector tangential to the surface.

Next, let us transform the last term in (A.30).

For some of the derivations here, it is useful to exploit the alternative representation of test function $\widehat{R}$ in vacuum in terms of the vector potential $\widehat{R}=\nabla\times\widehat{C}$, and using the first interface condition $(\text{A.22})_5$ in terms of the vector potential $\widehat{C}$, that is, $n\cdot h\widehat{B}=-n\times\widehat{C}$, one has

$$
\begin{aligned}
-\int_{\Sigma_{0,pv}} n\cdot h\widehat{B}\cdot\widehat{Q}dx&=\int_{\Sigma_{0,pv}} n\times\widehat{C}\cdot\widehat{Q}dx=-\int_{\Sigma_{0,pv}}\widehat{Q}\times\widehat{C}\cdot ndx\\
&=\int_{\overline{\Omega}^v}\nabla\cdot[\widehat{Q}\times\widehat{C}]dx=\int_{\overline{\Omega}^v}[\widehat{C}\cdot\nabla\times\widehat{Q}-\widehat{Q}\cdot\nabla\times\widehat{C}]dx\\
&=-\int_{\overline{\Omega}^v}\widehat{Q}\cdot\nabla\times\widehat{C}dx=-\int_{\overline{\Omega}^v}\widehat{Q}\cdot\widehat{R}dx.
\end{aligned}
\tag{A.31}
$$

Now we prove $\widehat{R} = \nabla \times \widehat{C}$. First, extend function $\widehat{R}$ to the domain $\overline{\Omega} \cup \overline{\Omega}^v$ as follows:

$$A = \begin{cases} R & \text{in } \overline{\Omega}, \\ \widehat{R} & \text{in } \overline{\Omega}^v. \end{cases}$$

Note that $\mathrm{div} R = 0$ in $\overline{\Omega}$ and $\mathrm{div}\widehat{R} = 0$ in $\overline{\Omega}^v$. So for test function ψ in $\overline{\Omega} \cup \overline{\Omega}^v$, we have

$$\begin{aligned}
\int_{\overline{\Omega}\cup\overline{\Omega}^v} A \cdot \nabla\psi dx &= \int_{\overline{\Omega}^v} \widehat{R} \cdot \nabla\psi dx + \int_{\overline{\Omega}} R \cdot \nabla\psi dx \\
&= n \cdot \widehat{R}\psi|_{\partial\overline{\Omega}^v} + n \cdot R\psi|_{\partial\overline{\Omega}} - \int_{\overline{\Omega}^v} \mathrm{div}\widehat{R} \cdot \psi dx - \int_{\overline{\Omega}} \mathrm{div} R \cdot \psi dx \\
&= n \cdot \widehat{R}\psi|_{r_w} + n \cdot (R - \widehat{R})\psi|_{r_s} - \int_{\overline{\Omega}^v} \mathrm{div}\widehat{R} \cdot \psi dx - \int_{\overline{\Omega}} \mathrm{div} R \cdot \psi dx \\
&= -\int_{\overline{\Omega}\cup\overline{\Omega}^v} \mathrm{div} A \cdot \psi dx = 0,
\end{aligned}$$

where we have used the boundary conditions $n \cdot \widehat{R}\psi|_{r_w} = 0$ and $n \cdot (R - \widehat{R})\psi|_{r_s} = 0$, with r_w the solid boundary and r_s the interface. Hence, we get $\mathrm{div} A = 0$ in the domain $\overline{\Omega} \cup \overline{\Omega}^v$ in the sense of distributions, which together with that the domain $\overline{\Omega} \cup \overline{\Omega}^v$ is simply connected and the weak Poincáre lemma, see Theorem IV 4.11 in Ref. 2, gives that $A = \nabla \times C$. When restricted to the vacuum domain Ω^v, we obtain $A = \widehat{R} = \nabla \times \widehat{C}$, and denote $\widehat{C} = C|_{\overline{\Omega}^v}$. Combining (A.29) with (A.30) and (A.31) yields (Appendix A), which concludes the proof. □

Remark A.4. Even though it is natural to expect the existence of such H^2 solutions, their construction is beyond the focus of this paper, which will be left for the future.

From Lemma A.3, we can get the following energy identity.

Lemma A.5. *Assume g is a H^2 solution to the system (A.22) with the corresponding jump and boundary conditions, then we can get*

$$\begin{aligned}
&\int_{\overline{\Omega}} [|Q|^2 + \gamma p|\nabla \cdot g|^2] dV + \int_{\overline{\Omega}} [(\nabla \times B) \cdot (g^* \times Q) + \nabla \cdot g(g^* \cdot \nabla p)] dx \\
&\quad + \int_{\overline{\Omega}^v} |\widehat{Q}|^2 dx + \|\sqrt{\rho} g_t\|_{L^2}^2 \\
&= \int_{\overline{\Omega}} [|Q_0|^2 + \gamma p|\nabla \cdot g_0|^2] dx + \int_{\overline{\Omega}} [(\nabla \times B) \cdot (g_0^* \times Q_0) \\
&\quad + \nabla \cdot g_0(g_0^* \cdot \nabla p)] dx + \int_{\overline{\Omega}^v} |\widehat{Q}_0|^2 dx + \|\sqrt{\rho} g_{0t}\|_{L^2}^2, \qquad (A.32)
\end{aligned}$$

with $Q = \nabla \times (g \times B)$.

Proof. Multiplying (A.22)$_2$ by g_t^*, similarly as the proof of Lemma A.3, we can show

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}g_t\|_{L^2}^2 = -\frac{1}{2}\frac{d}{dt}\int_{\overline{\Omega}^v}|\widehat{Q}|^2dx - \frac{1}{2}\frac{d}{dt}\int_{\overline{\Omega}}[|Q|^2+\gamma p|\nabla\cdot g|^2]dx$$
$$-\frac{1}{2}\frac{d}{dt}\int_{\overline{\Omega}}[(\nabla\times B)\cdot(g^*\times Q)+\nabla\cdot g(g^*\cdot\nabla p)]dx, \tag{A.33}$$

with $Q=\nabla\times(g\times B)$. Integrating (A.33) about time, we have (A.32). □

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References

1. D. Bian, Y. Guo and I. Tice, Linear instability of Z-pinch in plasma: Viscous case, *Math. Models Methods Appl. Sci.* **30** (2020) 2827–2908.
2. F. Boyer and P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models* (Springer, 2013).
3. F. Cap, *Handbook on Plasma Instabilities*, Vol. 1 (Academic Press, 1976).
4. F. Cap, *Handbook on Plasma Instabilities*, Vol. 2 (Academic Press, 1978).
5. F. Cap, *Handbook on Plasma Instabilities*, Vol. 3 (Academic Press, 1982).
6. J. Freidberg, *Ideal Magnetohydrodynamics* (Cambridge Univ. Press, 2014).
7. J. P. Goedbloed and S. Poedts, *Principle of Magnetohydrodynamics, with Applications to Laboratory and Astrophysical Plasma* (Cambridge Univ. Press, 2004).
8. X. Gu and Y. Wang, On the construction of solutions to the free-surface incompressible ideal magnetohydrodynamic equations, *J. Math. Pures Appl.* **128** (2019) 1–41.
9. G. Gui, Global well-posedness of the two-dimensional incompressible magnetohydrodynamics system with variable density and electrical conductivity, *J. Funct. Anal.* **267** (2014) 1488–1539.
10. Y. Guo and I. Tice, Compressible, inviscid Rayleigh–Taylor instability, *Indiana Univ. Math. J.* **60** (2011) 677–711.
11. Y. Guo and I. Tice, Linear Rayleigh–Taylor instability for viscous, compressible fluids, *SIAM J. Math. Anal.* **42** (2010) 1688–1720.
12. M. Haines, S. Lebedev, J. Chittenden, F. Beg, S. Bland and A. Dangor, The past, present, and future of Z pinches, *Phys. Plasmas* **7** (2000) 1672–1680.
13. F. Jiang and S. Jiang, On magnetic inhibition theory in non-resistive magnetohydrodynamic fluids, *Arch. Rational Mech. Anal.* **233** (2019) 749–798.
14. F. Jiang and S. Jiang, Nonlinear stability and instability in the Rayleigh–Taylor problem of stratified compressible MHD fluids, *Calc. Var. Partial Differential Equations* **58** (2019) 29.
15. F. Jiang and S. Jiang, On the dynamical stability and instability of Parker problem, *Physica D* **391** (2019) 17–51.
16. A. B. Mikhailovskii, *Theory of Plasma Instabilities* (Consultants Bureau, 1974). Translated from the Russian. An exhaustive description of plasma instabilities: Vol. I. Instabilities of a Homogeneous Plasma; Vol. II. Instabilities in an Inhomogeneous Plasma.

17. G. Schmidt, *Physics of High Temperature Plasma* (Academic press, 1979).
18. Y. Wang, Sharp Nonlinear stability criterion of viscous non-resistive MHD internal waves in 3D, *Arch. Rational Mech. Anal.* **231** (2019) 1675–1743.
19. Y. Wang and Z. Xin, Incompressible inviscid resistive MHD surface waves in 2D, prepint (2018), arXiv: 1801.04694 [math. AP].