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A CLASSIFICATION OF FINITE SIMPLE AMENABLE $\mathcal{Z}$-STABLE C^*-ALGEBRAS, I: C^*-ALGEBRAS WITH GENERALIZED TRACIAL RANK ONE

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ABSTRACT. A class of C^*-algebras, to be called those of generalized tracial rank one, is introduced. A second class of unital simple separable amenable C^*-algebras, those whose tensor products with UHF-algebras of infinite type are in the first class, to be referred to as those of rational generalized tracial rank one, is proved to exhaust all possible values of the Elliott invariant for unital finite simple separable amenable $\mathcal{Z}$-stable C^*-algebras. A number of results toward the classification of the second class are presented including an isomorphism theorem for a special sub-class of the first class, leading to the general classification of all unital simple C^*-algebras with rational generalized tracial rank one in Part II.

RÉSUMÉ. . Dans cet article et le prochain, on donne une classification complète, au moyen de l'invariant d'Elliott, d'une sous-classe de la classe des C*-algèbres simples, moyennables, séparables, à élément unité, absorbant l'algèbre de Jiang-Su, et satisfaisant au UCT, qui épuise l'ensemble des valeurs possibles de l'invariant pour cette class. La partie I réalise une grande partie de ce projet, et la partie II l'achève.

1. **Introduction** The concept of a C^*-algebra exists harmoniously in many areas of mathematics. The abstract definition of a C^*-algebra axiomatized the norm closed self-adjoint subalgebras of $B(H)$, the algebra of all bounded linear operators on a Hilbert space H. Thus, C^*-algebras are operator algebras. The study of C^*-algebras may also be viewed as the study of a non-commutative analogue of topology. This is because every unital commutative C^*-algebra is isomorphic to $C(X)$, the algebra of continuous functions on a compact Hausdorff space X (by means of the Gelfand transform). If we take our space X and equip it with a group action via homeomorphisms, we enter the realm of topological dynamical systems, where remarkable progress has been made by considering the transformation C^*-algebra $C(X) \rtimes G$ arising via the crossed product construction. Analogously, we can consider the study of general crossed products as the study of non-commutative topological dynamical systems. The most special case is the group C^*-algebra of G, which is fundamental in the study of abstract

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harmonic analysis. C^*-algebra theory is also basic to the non-commutative geometry of Connes. There are deep and extremely important interactions between the theory of C^*-algebras and the theory of the very special concrete C^*-algebras (weak operator closed) called von Neumann algebras. One might continue in this vein. Therefore, naturally, it would be of great interest to classify C^*-algebras. There has already been noteworthy progress in this direction.

Early classification theorems start with the work of Glimm in the late 1950s who classified infinite tensor products of matrix algebras, which he called uniformly hyperfinite algebras (UHF-algebras) by supernatural numbers. Dixmier, a few years later, classified non-unital inductive limits of UHF-algebras, noting that these were more complex. Bratteli, in 1972, generalized the results of Glimm and Dixmier to arbitrary inductive limits of finite-dimensional C^*-algebras (AF-algebras), using an analogue of the combinatorial data of Glimm which is now called a Bratteli diagram. (For instance, Pascal's triangle is a Bratteli diagram.) Bratteli's classification of AF-algebras was reformulated in a striking way by Elliott in 1976. He showed in effect that Bratteli's equivalence relation on Bratteli diagrams, which corresponded to isomorphism of the AF-algebras they described, was in fact the same as isomorphism of certain ordered groups arising in a natural way from these diagrams: in a sense, just their inductive limits. These ordered groups turned out to be just the K-groups of the algebras (generated by their Murray-von Neumann semigroups).

By 1989, Elliott had begun his classification program by classifying AT-algebras of real rank zero by scaled ordered K-theory. Notably, this used an approximate version of an (exact) intertwining argument used by both Bratteli and Elliott in the AF case. Since then there has been rapid progress in the program to classify separable amenable C^*-algebras now known as the Elliott program. Elliott and Gong ([32]) and Elliott, Gong, and Li ([33]) (together with a deep reduction theorem by Gong ([43])) classified simple AH-algebras with no dimension growth by means of the Elliott invariant (see Definition 2.4 below).

The Elliott intertwining argument provided a framework for further classification proofs and focused attention on the invariants of and the maps between certain building block algebras (for AF-algebras these building blocks would be finite dimensional algebras and for AH-algebras these building blocks would be certain homogeneous C^*-algebras). In particular, one wants to know when maps between invariants are induced by maps between building blocks (sometimes referred to as an existence theorem) and to know when maps between the building blocks are approximately unitarily equivalent (often called an uniqueness theorem). To classify C^*-algebras without assuming some particular inductive limit structure, one would like to establish abstract existence theorems and uniqueness theorems. These efforts became the engine for these rapid developments ([64], [60], [69] and [21] for example). Both existence theorems and uniqueness theorems used KL-theory ([105]) and the total K-theory ($\underline{K}(A)$) developed by Dadarlat and Loring ([20]). These existence and uniqueness theorems provide not only the technical tools for the classification program but also the foundation for understanding the morphisms in the category of C^*-algebras.

The rapid developments mentioned above include the Kirchberg-Phillips classification ([54], [55], and [97]) of purely infinite simple separable amenable C^*-algebras which satisfy the UCT, by means of their K-theory. There is also the classification of unital simple amenable C^*-algebras in the UCT class which have tracial rank zero or one ([60], [67], and [71]).

On the other hand, it had been suggested in [22] and [4] that unital simple AH-algebras without a dimension growth condition might behave differently. It was Villadsen ([116] and [117]) who showed that unital simple AH-algebras may have perforated K_0-groups and may have stable rank equal to any non-zero natural number. Rørdam exhibited an amenable separable simple C^*-algebra which is finite but not stably finite ([106]). It was shown by Toms ([115]) that there are unital simple AH-algebras of stable rank one with the same Elliott invariant that are not isomorphic. Before that, Jiang and Su ([53]) constructed a unital simple ASH-algebra $\mathcal{Z}$ of stable rank one which has the same Elliott invariant as that of $\mathbb{C}$. In particular, $\mathcal{Z}$ has no non-trivial projections.

If A is a simple separable amenable C^*-algebra with weakly unperforated $K_0(A)$ which belongs to a (reasonable) classifiable class, then one would expect that A must be isomorphic to $A \otimes \mathcal{Z}$, since these two algebras have the same Elliott invariant (Theorem 1 of [44]). If A is isomorphic to $A \otimes \mathcal{Z}$, then A is called $\mathcal{Z}$-stable. The existence of non-elementary simple C^*-algebras which are not $\mathcal{Z}$-stable was first proved by Gong, Jiang and Su (see [44]). Toms's counterexample is in particular not $\mathcal{Z}$-stable. Thus, $\mathcal{Z}$-stability should be added to the hypotheses if one uses the conventional Elliott invariant. (The class of simple AH-algebras of [33] are known to be $\mathcal{Z}$-stable. In fact, all unital separable simple amenable C^*-algebras with finite tracial rank are $\mathcal{Z}$-stable; see Corollary 8.4 of [81].)

The next development in this direction came from a new approach due to Winter, who made use of the assumption of $\mathcal{Z}$-stability in a remarkably innovative way ([119]). His idea was to view $A \otimes \mathcal{Z}$ as an inductive limit of algebras of paths in $A \otimes Q$ with endpoints in $A \otimes M_{\mathbf{p}}$ and $A \otimes M_{\mathbf{q}}$ (where Q is the UHF-algebra with $K_0(Q) = \mathbb{Q}$, $\mathbf{p}$ and $\mathbf{q}$ are coprime supernatural numbers, and $M_{\mathbf{p}}$ and $M_{\mathbf{q}}$ their associated UHF algebras). Suppose that the endpoint algebras are classifiable. Winter showed that if somehow there is a continuous path of isomorphisms from one endpoint to the other, then the algebra A itself is also classifiable.

Winter's procedure provided a new framework to carry out classification. However, to actually execute the continuation from endpoint to endpoint alluded to above, one needs new types of uniqueness and existence theorems. In other words, just like before, in Elliott's intertwining argument, the new procedure ultimately, but not surprisingly, depends on certain existence and uniqueness theorems. However, this time we need the uniqueness and existence theorems with respect to (one-parameter) asymptotic unitary equivalence of the maps involved rather than just (sequential) approximate unitary equivalence. This is significantly more demanding. For example, in the case of the existence theorem, we need to construct a map which lifts a prescribed KK-element rather than just a KL-element. This was once thought to be out of reach for general stably finite algebras since the KK-functor does not preserve inductive limits (as the

KL-functor does; see [20]). It was an unexpected usage of the Basic Homotopy Lemma that made this possible. Moreover, the existence theorem also needs to respect a prescribed rotation related map. The existence theorems are very different from those developed in the early study of the subject. Inevitably, the uniqueness theorem also becomes more complicated (again the Basic Homotopy Lemma plays the key role).

Once we overcame these new hurdles arising in following the Winter approach, we were able to classify the class of all unital separable simple $\mathcal{Z}$-stable C^*-algebras A which satisfy the UCT and whose tensor products with all UHF-algebras of infinite type are of tracial rank zero, by means of the Elliott invariant, in [83]. We were then able in [77] to extend this result to the class $\mathcal{A}$ of all unital simple separable amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT and have tracial rank one (not just zero) after tensoring with (just) some infinite dimensional UHF-algebra. The class $\mathcal{A}$, and already for that matter just the subclass mentioned above, the C^*-algebras rationally of tracial rank zero, carry classification substantially beyond earlier results. For instance, the Jiang-Su algebra $\mathcal{Z}$ is unital projectionless, but $\mathcal{Z} \otimes U \cong U$ for any infinite dimensional UHF-algebra U. In fact, the class $\mathcal{A}$ exhausts all Elliott invariants with simple weakly unperforated rationally Riesz groups as ordered K_0-group and pairing with the trace simplex taking extreme traces to extreme K_0-states ([84]).

The class $\mathcal{A}$ not only contains all unital simple separable amenable C^*-algebras with tracial rank one in the UCT class, and the Jiang-Su algebra, but also contains many other simple C^*-algebras. In fact it unifies the previously classified classes such as the simple limits of dimension drop interval algebras and dimension drop circle algebras which, like the Jiang-Su algebra, are not AH-algebras ([76]). However, the restriction on the pairing between traces and K_0 prevents the class $\mathcal{A}$ from including inductive limits of "point–line" algebras, which we called Elliott-Thomsen building blocks. This brings us to the main goal of our work.

The goal of the present article, and its sequel (Part II), is to give a classification of a new class of unital simple separable amenable C^*-algebras satisfying the UCT, by means of the Elliott invariant. This class is significant because it exhausts all possible values of the Elliott invariant for all unital, simple, finite, $\mathcal{Z}$-stable, separable amenable C^*-algebras. (It strictly contains the class $\mathcal{A}$ mentioned above.)

First, we shall introduce a class of unital simple separable C^*-algebras which we shall refer to as the C^*-algebras of *generalized* tracial rank one. The definition is in the same spirit as that of tracial rank one, but, instead of using only matrix algebras of continuous functions on a one-dimensional finite CW complex, it uses the point–line algebras. These were first introduced into the Elliott program by Elliott and Thomsen ([38]), in connection with determining the range of the Elliott invariant. Some time later, these C^*-algebras were also called one dimensional non-commutative CW complexes (NCCW). This class of unital simple separable C^*-algebras (of generalized tracial rank one) will be denoted by $\mathcal{B}_1$. If we insist that the point–line algebras used in the definition have trivial K_1, then the resulted sub-class will be denoted by $\mathcal{B}_0$. Amenable C^*-algebras in the class $\mathcal{B}_1$ are proved here to be Jiang-Su stable.

Denote by $\mathcal{N}_1$ the family of unital simple separable amenable C^*-algebras A which satisfy the UCT such that $A \otimes Q \in \mathcal{B}_1$, and by $\mathcal{N}_0$ the subclass of those C^*-algebras A such that $A \otimes Q \in \mathcal{B}_0$, where Q is the UHF-algebra with $K_0(Q) = \mathbb{Q}$.

As earlier for C^*-algebras of tracial rank one, we shall expand the new class of unital simple separable C^*-algebras of generalized tracial rank one (the class $\mathcal{B}_1$) to include those $\mathcal{Z}$-stable C^*-algebras such that the tensor product with some infinite-dimensional UHF algebra belongs to this class ($\mathcal{B}_1$). We shall show, in the present Part I of this work, that this expanded new class, the class of $\mathcal{Z}$-stable unital simple separable amenable C^*-algebras rationally of generalized tracial rank one, exhausts the Elliott invariant for finite $\mathcal{Z}$-stable unital simple separable C^*-algebras. We shall also prove, in the present Part I of this work, that, if A and B are amenable C^*-algebras in $\mathcal{B}_0$ satisfying the UCT, then $A \otimes U \cong B \otimes U$ for some UHF-algebra U of infinite type if, and only if, $A \otimes U$ and $B \otimes U$ have isomorphic Elliott invariants.

In other words, we classify a certain sub-class of C^*-algebras of generalized tracial rank one. In Part II, we shall classify the class of all C^*-algebras rationally of generalized tracial rank one.

The present part of the paper, Part I, is organized as follows. Section 2 serves as preliminaries and establishes some conventions. In Section 3, we study the class of unital Elliott-Thomsen building blocks, denoted by $\mathcal{C}$ (see [26] and [31]). Elliott-Thomsen building blocks are also called point–line algebras, or one dimensional non-commutative CW complexes (NCCW complexes, studied in [27] and [28]). Sections 4 and 5 discuss the uniqueness theorem for maps from C^*-algebras in $\mathcal{C}$ to finite dimensional C^*-algebras. Section 8 presents a uniqueness theorem for maps from a C^*-algebra in $\mathcal{C}$ to another C^*-algebra in $\mathcal{C}$. This is done by using a homotopy lemma established in Section 6 and existence theorems established in Section 7 to bridge the uniqueness theorems of Sections 4 and 5 with those of Section 8. In Section 9, the classes $\mathcal{B}_1$ (as above) and $\mathcal{B}_0$ (as above) are introduced. Properties of C^*-algebras in the class $\mathcal{B}_1$ are discussed in Sections 9, 10, and 11. These C^*-algebras, unital separable simple C^*-algebras of generalized tracial rank (at most) one, can also be characterized as being tracially approximable by (arbitrary) subhomogeneous C^*-algebras with one-dimensional spectrum. For example, we show, in Section 9, that $\mathcal{B}_1$ and $\mathcal{B}_0$ are not the same (unlike the previous case, in which unital simple separable C^*-algebras of tracial rank one are TAI), and in Section 10, that amenable C^*-algebras in $\mathcal{B}_1$ are $\mathcal{Z}$-stable. Section 12 is dedicated to the main uniqueness theorem used in the isomorphism theorem of Section 21.

Sections 13 and 14 are devoted to the range theorem, one of the main results: given any possible Elliott invariant sextuple for a $\mathcal{Z}$-stable unital simple separable amenable C^*-algebra, namely, any countable weakly unperforated simple order-unit abelian group, paired with an arbitrary metrizable Choquet simplex mapping onto the state space of the order-unit group, together with an arbitrary countable abelian group, there is a C^*-algebra in the class $\mathcal{N}_1$ realizing this sextuple as its Elliott invariant. (The construction of this model, an inductive limit of subhomogeneous C^*-algebras, is similar to that in [31], but additional work is needed to show that the inductive limits in question belong to the class $\mathcal{N}_1$.) This is shown in Section 13; Section 14 gives a similar construction, for a restricted class of invariants, and yielding a correspondingly restricted class of (inductive limit) algebras. For reasons that will become clear later, this second, restricted, model construction is very important.

Sections 15 to 19 could all be described as different stages in the development of the existence theorem, to be used in the later sections as well as final isomorphism theorem (in Part II). These deal with the issue of existence for maps from C^*-algebras in $\mathcal{C}$ to finite dimensional C^*-algebras and then to C^*-algebras in $\mathcal{C}$ that match prescribed K_0-maps and tracial information. The ordered K_0-structure and combined simplex information of these C^*-algebras become complicated. We also need to consider maps from homogeneous C^*-algebras to C^*-algebras in $\mathcal{C}$. The mixture with higher dimensional noncommutative CW complexes does not ease the difficulties. However, in Section 18, we show that, at least under certain restrictions, any given compatible triple which consists of a strictly positive KL-element, a map on the tracial state space, and a homomorphism on a quotient of the unitary group, it is possible to construct a homomorphism from a separable amenable C^*-algebra A satisfying the UCT of the form $B \otimes U$ for some $B \in \mathcal{B}_0$ and some UHF-algebra U of infinite type to another separable C^*-algebra C of the form $D \otimes V$, where $D \in \mathcal{B}_0$ and V is a UHF-algebra of infinite type, which matches the triple. Variations of this are also discussed. In Section 19, we show that $\mathcal{N}_1 = \mathcal{N}_0$, even though $\mathcal{B}_1 \neq \mathcal{B}_0$, as indicated in Section 9. In Section 20, we continue to study the existence theorem. In Section 21, we show that any unital simple C^*-algebra, which satisfies the UCT, in the class $\mathcal{N}_1$ absorbing tensorially a UHF algebra of infinite type is isomorphic to an inductive limit C^*-algebra as constructed in Section 14, and any two such C^*-algebras are isomorphic if they have the same Elliott invariant. This isomorphism theorem is special but is also the foundation of our main isomorphism theorem in Part II of the paper.

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2. Notation and Preliminaries

This section includes a list of notations and definitions most of which are standard. We list them here for the reader's convenience. However, we also recommend skipping this section until some of these notions appear.

Definition 2.1. Let A be a unital C^*-algebra. Denote by $A_{s.a.}$ the self-adjoint part of A and A_+ the set of all positive elements of A. Denote by $U(A)$ the unitary group of A, and denote by $U_0(A)$ the normal subgroup of $U(A)$ consisting of those unitaries which are in the connected component of $U(A)$ containing 1_A. Denote by $DU(A)$ the commutator subgroup of $U_0(A)$ and $CU(A)$ the closure of $DU(A)$ in $U(A)$.

Definition 2.2. Let A be a unital C^*-algebra and let $T(A)$ denote the simplex of tracial states of A, a compact subset of A^*, the dual of A, with the weak* topology (see also [109] and II.4.4 of [5]). Let $\tau \in T(A)$. We say that τ is faithful if $\tau(a) > 0$ for all $a \in A_+ \setminus \{0\}$. Denote by $T_f(A)$ the set of all faithful tracial states.

For each integer $n \geq 1$ and $a \in M_n(A)$, write $\tau(a) = (\tau \otimes \mathrm{Tr})(a)$, where Tr is the (non-normalized) standard trace on M_n.

Let S be a compact convex set. Denote by $\mathrm{Aff}(S)$ the space of all real continuous affine functions on S and denote by $\mathrm{LAff}_b(S))$ the set of all bounded lower semi-continuous real affine functions on S. Denote by $\mathrm{Aff}(S)_+$ the set of those non-negative valued functions in $\mathrm{Aff}(S)$ and $\mathrm{Aff}(S)^+ = \mathrm{Aff}(S)_+ \setminus \{0\}$. Also define $\mathrm{Aff}(S)^{++} = \{f \in \mathrm{Aff}(A)^+ : f(s) > 0,\ s \in S\}$. Define $\mathrm{LAff}_b(S)_+$ to be the set of those non-negative valued functions in $\mathrm{LAff}_b(S)$, and $\mathrm{LAff}_b(S)^{++}$ to be the set $\{f \in \mathrm{LAff}_b(S)^+ : f(s) > 0,\ s \in S\}$.

Suppose that $T(A) \neq \varnothing$. There is a linear map $r_{\mathrm{aff}} : A_{s.a.} \to \mathrm{Aff}(T(A))$ defined by

$$r_{\mathrm{aff}}(a)(\tau) = \hat{a}(\tau) = \tau(a) \text{ for all } \tau \in T(A)$$

and for all $a \in A_{s.a.}$.

Let A_0 denote the closure of the set of all self–adjoint elements of A of the form $\sum x_i x_i^* - \sum x_i^* x_i$. Then by Theorem 2.9 of [17] (also see the proof of Lemma 3.1 of [112] for further explanation), we know that $A_0 = \ker(r_{\mathrm{aff}})$. As in [17], denote by A^q the quotient space $A_{s.a.}/A_0$. It is a real Banach space. Denote by A^q_+ the image of A_+ in A/A_0. Denote by $q : A_{s.a.} \to A^q$ the quotient map. It follows from Proposition 2.7 of [17] that $T(A)$ is precisely the set of those real bounded linear functionals f on A^q such that $f(x) \geq 0$ for all $x \in A^q_+$ and $f(q(1_A)) = 1$. Moreover, the topology on $T(A)$ is the weak* topology of $T(A)$ as a subset of the dual space of A^q. For a given element $g \in \mathrm{Aff}(T(A))$, by Proposition 2.8 of [17], g can be uniquely extended to a bounded linear functional on $(A^q)^*$, the dual space of A^q. Since g is continuous on $T(A)$, g is weak*-continuous. Therefore this gives an element $\Gamma(g)$ in A^q (this also shows that the map r_{aff} is surjective). Let $\tau \in T(A)$. Then we view τ as an element in $(A^q)^*$ as above. Then $\tau(\Gamma(g)) = g(\tau)$ for all $\tau \in T(A)$. It follows that Γ is a linear map from $\mathrm{Aff}(T(A))$ to A^q. It is clear that Γ is injective. To see it is surjective, let $x \in A^q$. Then, viewing $T(A)$ as a subset of the dual of A^q, $\hat{x}(\tau) = \tau(x)$ (for all $\tau \in T(A)$) defines an element of $\mathrm{Aff}(T(A))$. It is clear that $\Gamma(\hat{x}) = x$. That the map Γ is an isometry now follows from the equation (3.1) of the proof of Lemma 3.1 of [112] (see also Theorem 2.9 of [17]). (Hence Γ is also positive.)

Suppose that A and B are two unital separable C^*-algebras such that $\mathrm{Aff}(T(A)) \cong \mathrm{Aff}(T(B))$, i.e., there is an isometric order isomorphism γ from the real Banach space $\mathrm{Aff}(T(A))$ onto the real Banach space $\mathrm{Aff}(T(B))$ which preserves the constant function 1. Then there is an isometric isomorphism from A^q onto B^q which maps A^q_+ into B^q_+ and $q(1_A)$ to $q(1_B)$, and the inverse maps B^q_+ into A^q_+ and maps $q(1_B)$ to $q(1_A)$. Since we have identified $T(A)$ and $T(B)$ with the subset of $(A^q)^*$ which preserves the order and has value 1 on $q(1_A)$ and the subset of $(B^q)^*$ which preserves the order and has value 1 on $q(1_B)$, respectively, the map γ induces an affine homeomorphism from $T(B)$ onto $T(A)$.

DEFINITION 2.3. Let A be a unital stably finite C^*-algebra with $T(A) \neq \varnothing$. Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the order preserving homomorphism defined by $\rho_A([p])(\tau) = \tau(p)$ for any projection $p \in M_n(A)$, $n = 1, 2, ...$ (see the convention above).

A map $s : K_0(A) \to \mathbb{R}$ is said to be a state if s is an order preserving homomorphism such that $s([1_A]) = 1$. The set of states on $K_0(A)$ is denoted by $S_{[1_A]}(K_0(A))$.

Denote by $r_A : T(A) \to S_{[1_A]}(K_0(A))$ the map defined by $r_A(\tau)([p]) = \tau(p)$ for all projections $p \in M_n(A)$ (for all integers n) and for all $\tau \in T(A)$.

DEFINITION 2.4. Let A be a unital simple C^*-algebra. The Elliott invariant of A, denoted by $\mathrm{Ell}(A)$, is the sextuple

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A).$$

Suppose that B is another unital simple C^*-algebra. We say that $\Gamma : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is a homomorphism if there are an order preserving homomorphism $\kappa_0 : K_0(A) \to K_0(B)$ such that $\kappa_0([1_A]) = [1_B]$, a homomorphism $\kappa_1 : K_1(A) \to K_1(B)$, and a continuous affine map $\kappa_T : T(B) \to T(A)$ such that

$$r_A(\kappa_T(t))(x) = r_B(t)(\kappa_0(x)) \text{ for all } x \in K_0(A) \text{ and} \tag{e 2.1}$$

for all $t \in T(B)$, and we write $\Gamma = (\kappa_0, \kappa_1, \kappa_T)$.

We write $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ if there is Γ as above such that κ_0 is an order isomorphism such that $\kappa_0([1_A]) = [1_B]$, κ_1 is an isomorphism, and κ_T is an affine homeomorphism. If, in addition, A is separable and satisfies the UCT, then there exists an element $\alpha \in KL(A, B)$ such that $\alpha|_{K_i(A)} = \kappa_i$, $i = 0, 1$ (recall that, by [105], in this case, $KL(A, B) = KK(A, B)/Pext$, where $Pext$ is the subgroup corresponding to the pure extensions of $K_*(A)$ by $K_*(B)$). If B is also separable and satisfies the UCT, then there also exists $\alpha^{-1} \in KL(B, A)$ with $\alpha^{-1} \times \alpha = [\mathrm{id}_A]$ and $\alpha \times \alpha^{-1} = [\mathrm{id}_B]$ (see 23.10.1 of [2]).

Any continuous affine map $\kappa_T : T(B) \to T(A)$ induces a map $\kappa_T^\sharp : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ defined by $\kappa_T^\sharp(l)(\tau) = l(\kappa(\tau))$ for all $\tau \in T(B)$ and $l \in \mathrm{Aff}(T(A))$. Furthermore, κ_T is compatible with κ_0 in the sense of (e 2.1) if and only if $\kappa_T^\sharp$ and κ_0 are compatible in the following sense:

$$\rho_B(\kappa_0(x)) = \kappa_T^\sharp(\rho_A(x)) \text{ for all } x \in K_0(A).$$

Note that any unital homomorphism $\varphi : A \to B$ induces maps $\varphi_{*,0} : K_0(A) \to K_0(B)$ and $\varphi^\# : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$, which are compatible.

DEFINITION 2.5. Let X be a compact metric space, let $x \in X$ be a point, and let $r > 0$. Denote by $B(x, r)$ the open unit ball $\{y \in X : \mathrm{dist}(x, y) < r\}$.

Let $\varepsilon > 0$. Define $f_\varepsilon \in C_0((0, \infty))$ to be the function with $f_\varepsilon(t) = 0$ if $t \in [0, \varepsilon/2]$, $f_\varepsilon(t) = 1$ if $t \in [\varepsilon, \infty)$, and $f_\varepsilon(t) = (2 - \varepsilon)/\varepsilon$ if $t \in [\varepsilon/2, \varepsilon]$. Note that $0 \leq f \leq 1$ and $f_\varepsilon f_{\varepsilon/2} = f_\varepsilon$.

Denote by t_+ the continuous function $g(t) := (1/2)(t + |t|)$ for all $t \in \mathbb{R}$. Then, if A is a C^*-algebra and $a \in A_+$, the element $a_+ := g(a) = (1/2)(a + |a|)$ is the positive part of a.

DEFINITION 2.6. Let A be a C^*-algebra. Let $a, b \in M_n(A)_+$. Following Cuntz ([16]), we write $a \lesssim b$ if there exists a sequence $\{x_n\}$ in $M_n(A)$ such that $\lim_{n\to\infty} x_n^* b x_n = a$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \sim b$. The relation "$\sim$" is an equivalence relation. Denote by $W(A)$ the Cuntz semigroup, consisting of the equivalence classes of positive elements in $\bigcup_{m=1}^\infty M_m(A)$ with orthogonal addition (i.e., $[a + b] = [a \oplus b]$).

If $p, q \in M_n(A)$ are projections, then $p \lesssim q$ if and only if p is Murray-von Neumann equivalent to a subprojection of q. In particular, when A is stably finite, $p \sim q$ if and only if p and q are Murray-von Neumann equivalent.

Recall (see II. 1.1 of [5]) that a (normalized) 2-quasi-trace of a C^*-algebra A is a function $\tau : A \to \mathbb{C}$ satisfying

(1) $\tau(1) = 1$,
(2) $0 \leq \tau(xx^*) = \tau(x^*x)$, $x \in A$,
(3) $\tau(a + ib) = \tau(a) + i\tau(b)$, $a, b \in A_{s.a.}$,
(4) τ is linear on abelian C^*-subalgebra of A, and
(5) τ extends to a function from $M_2(A)$ to $\mathbb{C}$ satisfying the conditions above.

Denote by $QT(A)$ the set of normalized 2-quasi-traces on A. It follows from II 4.1 of [5] that every 2-quasi-trace τ extends to a quasi-trace on $M_n(A)$ (for all $n \geq 1$). For $a \in A_+$ and $\tau \in QT(A)$, define

$$d_\tau(a) = \lim_{\varepsilon \to 0} \tau(f_\varepsilon(a)).$$

Suppose that $QT(A) \neq \varnothing$. We say A has strict comparison for positive elements if, for any $a, b \in M_n(A)$ (for all integers $n \geq 1$), $d_\tau(a) < d_\tau(b)$ for $\tau \in QT(M_n(A))$ implies $a \lesssim b$.

DEFINITION 2.7. Let A be a C^*-algebra. Denote by $A^{\mathbf{1}}$ the unit ball of A, and by $A_+^{q,\mathbf{1}}$ the image of the intersection of $A_+ \cap A^{\mathbf{1}}$ in A_+^q. (Recall that $A_+^q = r_{\mathrm{aff}}(A_+)$; see Definition 2.2.)

DEFINITION 2.8. Let A be a unital C^*-algebra and let $u \in U(A)$. We write $\mathrm{Ad}\, u$ for the automorphism $a \mapsto u^*au$ for all $a \in A$. Suppose $B \subset A$ is a unital C^*-subalgebra. Denote by $\overline{\mathrm{Inn}}(B, A)$ the set of all those monomorphisms $\varphi : B \to A$ such that there exists a sequence of unitaries $\{u_n\} \subset A$ with $\varphi(b) = \lim_{n\to\infty} u_n^* b u_n$ for all $b \in B$.

DEFINITION 2.9. Denote by $\mathcal{N}$ the class of separable amenable C^*-algebras which satisfy the Universal Coefficient Theorem (UCT).

Denote by $\mathcal{Z}$ the Jiang-Su algebra ([53]). Note that $\mathcal{Z}$ has a unique trace state and $K_i(\mathcal{Z}) = K_i(\mathbb{C})$ $(i = 0, 1)$. A C^*-algebra A is said to be $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$.

DEFINITION 2.10. Let A be a unital C^*-algebra . Recall that, following Dădărlat and Loring ([20]), one defines

$$\underline{K}(A) = \bigoplus_{i=0,1} K_i(A) \oplus \bigoplus_{i=0,1} \bigoplus_{k\geq 2} K_i(A, \mathbb{Z}/k\mathbb{Z}). \tag{e 2.2}$$

There is a commutative C^*-algebra C_k such that one may identify $K_i(A \otimes C_k)$ with $K_i(A, \mathbb{Z}/k\mathbb{Z})$. Let A be a unital separable amenable C^*-algebra , and let B be a σ-unital C^*-algebra. Following Rørdam ([105]), $KL(A, B)$ is the quotient of $KK(A, B)$ by those elements represented by limits of trivial extensions (see [69]). In the case that A satisfies the UCT, Rørdam defines $KL(A, B) = KK(A, B)/Pext$, where $Pext$ is the subgroup corresponding to the pure extensions of $K_*(A)$ by $K_*(B)$. In [20], Dădărlat and Loring proved that (if A satisfies the UCT)

$$KL(A, B) = \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \tag{e 2.3}$$

(see page 362 of [20] for the definition of the group of module homomorphisms $\mathrm{Hom}_\Lambda(-, -)$).

Now suppose that A is stably finite. Denote by $KK(A,B)^{++}$ the set of those elements $\kappa \in KK(A,B)$ such that $\kappa(K_0(A)_+) \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$. (Warning: the notation here may be different from other papers.) In the absence of the UCT, we denote by $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$ the set of those elements $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ such that $\kappa(K_0(A) \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$. Suppose further that both A and B are unital. Denote by $KK_e(A,B)^{++}$ the subset of those $\kappa \in KK(A,B)^{++}$ such that $\kappa([1_A]) = [1_B]$. Denote by $KL_e(A,B)^{++}$ the image of $KK_e(A,B)^{++}$ in $KL(A,B)$.

DEFINITION 2.11. Let A and B be C^*-algebras and $\varphi : A \to B$ be a linear map. We will sometimes, without notice, continue to use φ for the induced map $\varphi \otimes \mathrm{id}_{M_n} : A \otimes M_n \to B \otimes M_n$. Also, $\varphi \otimes 1_{M_n} : A \to B \otimes M_n$ is used for the amplification which maps a to $\varphi(a) \otimes 1_{M_n}$, the diagonal element with $\varphi(a)$ repeated n times. Throughout the paper, if φ is a homomorphism, we will use $\varphi_{*i} : K_i(A) \to K_i(B)$, $i = 0, 1$, for the induced homomorphism. We will use $[\varphi]$ for the element of $KL(A,B)$ (or $KK(A,B)$ if there is no confusion) which is induced by φ. Suppose that A and B are unital and $\varphi(1_A) = 1_B$. Then φ induces an affine map $\varphi_T : T(B) \to T(A)$ defined by $\varphi_T(\tau)(a) = \tau(\varphi(a))$ for all $\tau \in T(B)$ and $a \in A_{s.a.}$. Denote by $\varphi^\sharp : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ the affine continuous map defined by $\varphi^\sharp(f)(\tau) = f(\varphi_T(\tau))$ for all $f \in \mathrm{Aff}(T(A))$ and $\tau \in T(B)$.

DEFINITION 2.12. Let A be a unital separable amenable C^*-algebra and let $x \in A$. Suppose that $\|xx^* - 1\| < 1$ and $\|x^*x - 1\| < 1$. Then $x|x|^{-1}$ is a unitary. Let us use $\langle x \rangle$ to denote $x|x|^{-1}$.

Let $\mathcal{F} \subset A$ be a finite subset and $\varepsilon > 0$ be a positive number. We may assume that $1_A \in \mathcal{F}$. We say a map $L : A \to B$ is $\mathcal{F}$-ε-multiplicative if

$$\|L(xy) - L(x)L(y)\| < \varepsilon \text{ for all } x, y \in \mathcal{F}.$$

Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset. Let us first assume that $\mathcal{P} \subset K_0(A) \oplus K_1(A)$ Assume also that $\{p_i, p_i' : 1 \le i \le m_0\} \subset M_N(A)$ is a finite subset of projections and $\{u_j : 1 \le j \le m_1\} \subset M_N(A)$ is a finite subset of unitaries such that $\{[p_i] - [p_i'], [u_j] : 1 \le i \le m_0, 1 \le j \le m_1\} = \mathcal{P}$. Then there is $\varepsilon > 0$ and a finite subset $\mathcal{F}$ of A satisfying the following condition: for any unital C^*-algebra B and any unital $\mathcal{F}$-ε-multiplicative completely positive linear map $L : A \to B$, the map L induces a homomorphism $[L]$ defined on $G(\mathcal{P})$, where $G(\mathcal{P})$ is the subgroup generated by $\mathcal{P}$, to $\underline{K}(B)$ such that there are projections $q_i, q_i' \in M_N(B)$ with $[q_i] = [L]([p_i])$, $[q_i'] = [L]([p_i'])$ in $K_0(B)$ $(1 \le i \le m_0)$ and unitaries $u_j \in M_N(B)$ $(1 \le j \le m_1)$ with $[v_j] = [L]([u_j])$ such that

$$\|L(p_i) - q_i\| < 1/2,\ \|L(p_i') - q_i'\| < 1/2\ (1 \le i \le m_0) \text{ and} \tag{e 2.4}$$

$$\|\langle L(u_j)\rangle - v_j\| < 1/2\ (1 \le j \le m_1). \tag{e 2.5}$$

In general, $\mathcal{P} \cap K_i(A, \mathbb{Z}/k\mathbb{Z}) \neq \varnothing$. Then the above also applies to $\mathcal{P} \cap K_i(A, \mathbb{Z}/k\mathbb{Z})$ with a necessary modification, by replacing L, by $L \otimes \mathrm{id}_{C_k}$, where C_k is the commutative C^*-algebra referred to in 2.10. Suppose that the triple $(\varepsilon, \mathcal{F}, \mathcal{P})$ also has the following property: if L' is another such map with the property that $\|L(a) - L'(a)\| < 2\varepsilon$ for all $a \in \mathcal{F}$, then $[L]|_{\mathcal{P}} = [L']|_{\mathcal{P}}$. Then such a triple $(\varepsilon, \mathcal{F}, \mathcal{P})$ may be called a KL-triple for A (see, for example, 1.2 of [58] and 3.3 of [21]). Note that these considerations, in particular, imply that, if $u_j \in U_0(A)$, then $[L]([u_j]) \in U_0(B)$.

Suppose that A is unital and L is a contractive completely positive linear map which is not unital. We may always assume that $1_A \in \mathcal{F}$. When $\varepsilon < 1/4$, let $p = \chi(L(1_A))$, where $0 \le \chi(t) \le 1$ is a function in $C([0,1])$ which is zero on $[0, 1/3]$ and 1 on $[1/2, 1]$. For small ε, $pL(1_A)p$ is close to p and so invertible in pBp. Let $b \in pBp$ be the inverse of $pL(1_A)p$ in B. Then $\|b^{1/2} - p\| < 4\varepsilon$. Define $L' : A \to pBp$ by $L'(a) = b^{1/2}pL(a)pb^{1/2}$ for $a \in A$. Note that, for any $\eta > 0$, $\|L - L'\| < \eta$ if ε is sufficiently small. The convention of this article, as usual, will be that we may always assume that $L(1_A)$ is a projection when we mention an $\mathcal{F}$-ε-multiplicative map L.

Then, if u is a unitary, with sufficiently large $\mathcal{G}$ and small ε, as above, then $\langle L(u) \rangle$ is a unitary in pBp. By $[L]([u])$, in $K_1(B)$, we then mean $[\langle L(u)\rangle + (1-p)]$.

Suppose that $K_i(A)$ is finitely generated. Then, by Proposition 2.4 of [74], for some large $\mathcal{P}$, if $(\varepsilon, \mathcal{F}, \mathcal{P})$ (with sufficiently small ε, and sufficiently large $\mathcal{F}$) is a KL-triple for A, then $[L]$ defines an element in $KL(A, B) = KK(A, B)$. In this case, we say $(\varepsilon, \mathcal{F})$ is a KK-pair.

LEMMA 2.13 (Lemma 2.8 of [74]). *Let A be a unital separable amenable C^*-algebra. Let $\varepsilon > 0$, let $\mathcal{F}_0 \subset A$ be a finite subset and let $\mathcal{F} \subset A \otimes C(\mathbb{T})$ be a finite subset. There exist a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ satisfying the following condition: For any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\varphi : A \to B$ (for some unital C^*-algebra B) and any unitary $u \in B$ such that*

$$\|\varphi(g)u - u\varphi(g)\| < \delta \text{ for all } g \in \mathcal{G}, \tag{e 2.6}$$

there exists a unital $\mathcal{F}$-ε-multiplicative completely positive linear map $L : A \otimes C(\mathbb{T}) \to B$ such that

$$\|\varphi(f) - L(f \otimes 1)\| < \varepsilon \text{ and } \|L(1 \otimes z) - u\| < \varepsilon \tag{e 2.7}$$

for all $f \in \mathcal{F}_0$, where $z \in C(\mathbb{T})$ is the identity function on the unit circle.

PROOF. This is well known. The proof follows the same lines as that of 2.1 of [85]. We sketch the proof here. Claim: Suppose that there exists a sequence of unital completely positive linear maps $\varphi_n : A \to B_n$ for some sequence of unital C^*-algebras $\{B_n\}$ and a sequence of unitaries $u_n \in B_n$ such that $\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0$ for all $a, b \in A$ and $\lim_{n\to\infty} \|\varphi_n(a)u_n - u_n\varphi_n(a) = 0$ for all $a \in A$. Then there exists a sequence of unital completely positive linear maps $L_n : A \otimes C(\mathbb{T}) \to B_n$ such that $\lim_{n\to\infty} \|L_n(a \otimes 1_{C(\mathbb{T})}) - \varphi_n(a)\| =$

0 and $\lim_{n\to\infty}\|L_n(1\otimes g)-g(u_n)\|=0$ for all $g\in C(\mathbb{T})$. As in the proof of 2.1 of [85], the lemma follows from the claim.

Now we prove the claim with exactly the same argument as that of 2.1 of [85]. Consider the C^*-algebras $C=\prod_{n=1}^{\infty}B_n$ and $C_0=\bigoplus_{n=1}^{\infty}B_n$. Let $\pi: C\to C/C_0$ be the quotient map. Define $\Phi_A: A\to C$ by $\Phi_A(a)=\{\varphi(a)\}$ for all $a\in A$ and $\Phi_T: C(\mathbb{T})\to C$ by $\Phi_T(g)=\{g(u_n)\}$ for all $g\in C(\mathbb{T})$. Then $\pi\circ\Phi_A$ and $\pi\circ\Phi_T$ are unital homomorphisms such that $\pi\circ\Phi_A(a)$ and $\pi\circ\Phi_T(g)$ commute for each $a\in A$ and $g\in C(\mathbb{T})$. Thus, there is a unital homomorphism $\Psi: A\otimes C(\mathbb{T})\to C/C_0$ such that $\Psi(a\otimes g)=\Phi_A(a)\Phi_T(g)$ for all $a\in A$ and $g\in C(\mathbb{T})$. Since $A\otimes C(\mathbb{T})$ is amenable, by a result of Choi and Effros, just as in the proof of 2.1 of [85], we obtain a unital completely positive linear map $L: A\otimes C(\mathbb{T})\to C$ such that $\pi\circ L=\Psi$. Write $L(a)=\{L_n(a)\}$, where each $L_n: A\to B_n$ is a unital completely positive linear map. Then

$$\lim_{n\to\infty}\|L_n(a\otimes 1_{C(\mathbb{T})})-\varphi_n(a)\|=0 \text{ for all } a\in A \text{ and} \tag{e 2.8}$$

$$\lim_{n\to\infty}\|L_n(1\otimes g)-g(u_n)\|=0 \text{ for all } g\in C(\mathbb{T}), \tag{e 2.9}$$

as desired. □

DEFINITION 2.14. Let A be a unital C^*-algebra. Consider the tensor product $A\otimes C(\mathbb{T})$. By the Künneth Formula (note that $K_*(C(\mathbb{T}))$ is finitely generated), the tensor product induces two canonical injective homomorphisms

$$\beta^{(0)}: K_0(A)\to K_1(A\otimes C(\mathbb{T})) \quad\text{and}\quad \beta^{(1)}: K_1(A)\to K_0(A\otimes C(\mathbb{T})). \tag{e 2.10}$$

In this way (with further application of the Künneth Formula), one may write

$$K_i(A\otimes C(\mathbb{T}))=K_i(A)\oplus\beta^{(i-1)}(K_{i-1}(A)),\ \ i=0,1. \tag{e 2.11}$$

For each $i\geq 2$, one also obtains the following injective homomorphisms

$$\beta_k^{(i)}: K_i(A,\mathbb{Z}/k\mathbb{Z})\to K_{i-1}(A\otimes C(\mathbb{T}),\mathbb{Z}/k\mathbb{Z}),\ \ i=0,1. \tag{e 2.12}$$

Moreover, one may write

$$K_i(A\otimes C(\mathbb{T}),\mathbb{Z}/k\mathbb{Z})=K_i(A,\mathbb{Z}/k\mathbb{Z})\oplus\beta_k^{(i-1)}(K_{i-1}(A,\mathbb{Z}/k\mathbb{Z})),\ \ i=0,1. \tag{e 2.13}$$

If $x\in\underline{K}(A)$, let us write $\boldsymbol{\beta}(x)$ for $\beta^{(i)}(x)$ if $x\in K_i(A)$ and for $\beta_k^{(i)}(x)$ if $x\in K_i(A,\mathbb{Z}/k\mathbb{Z})$. So we have an injective homomorphism

$$\boldsymbol{\beta}: \underline{K}(A)\to\underline{K}(A\otimes C(\mathbb{T})), \tag{e 2.14}$$

and

$$\underline{K}(A\otimes C(\mathbb{T}))=\underline{K}(A)\oplus\boldsymbol{\beta}(\underline{K}(A)). \tag{e 2.15}$$

Let $h : A \otimes C(\mathbb{T}) \to B$ be a unital homomorphism. Then h induces a homomorphism $h_{*i,k} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z})$, $k = 0, 2, 3, ...$ and $i = 0, 1$. Suppose that $\varphi : A \to B$ is a unital homomorphism and $v \in U(B)$ is a unitary such that $\varphi(a)v = v\varphi(a)$ for all $a \in A$. Then φ and v determine a unital homomorphism $h : A \otimes C(\mathbb{T}) \to B$ by $h(a \otimes z) = \varphi(a)v$ for all $a \in A$, where $z \in C(\mathbb{T})$ is the identity function on the unit circle $\mathbb{T}$, and every unital homomorphism $A \otimes C(\mathbb{T}) \to B$ arises in this way. We use $\mathrm{Bott}(\varphi,\, v) : \underline{K}(A) \to \underline{K}(B)$ to denote the collection of all homomorphisms $h_{*i-1,k} \circ \beta_k^{(i)}$, where $h : A \otimes C(\mathbb{T}) \to B$ is the homomorphism determined by (φ, v), and we write

$$\mathrm{Bott}(\varphi,\, v) = 0 \tag{e 2.16}$$

if $h_{*i-1,k} \circ \beta_k^{(i)} = 0$ for all k and i. In particular, since A is unital, (e 2.16) implies that $[v] = 0$ in $K_1(B)$. We also use $\mathrm{bott}_i(\varphi,\, v)$ for $h_{*i-1} \circ \beta^{(i)}$, $i = 0, 1$.

Suppose that A is a unital separable amenable C^*-algebra. Let $\mathcal{Q} \subset \underline{K}(A \otimes C(\mathbb{T}))$ be a finite subset, let $\mathcal{F}_0 \subset A$ be a finite subset, and let $\mathcal{F}_1 \subset A \otimes C(\mathbb{T})$ also be a finite subset. Suppose that $(\varepsilon, \mathcal{F}_0, \mathcal{Q})$ is a KL-triple. Then, by 2.13 there exist a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ satisfying the following condition: For any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\varphi : A \to B$ (where B is a unital C^*-algebra) and any unitary $v \in B$ such that

$$\|[\varphi(g),\, v]\| < \delta \ \text{ for all } \ g \in \mathcal{G}, \tag{e 2.17}$$

there exists a unital $\mathcal{F}_1$-ε-multiplicative completely positive linear map $L : A \otimes C(\mathbb{T}) \to B$ such that

$$\|L(f \otimes 1) - \varphi(f)\| < \varepsilon \ \text{ for all } \ f \in \mathcal{F}_0 \ \text{ and } \ \|L(1 \otimes z) - v\| < \varepsilon, \tag{e 2.18}$$

where $z \in C(\mathbb{T})$ is the standard unitary generator of $C(\mathbb{T})$. In particular, $[L]|_{\mathcal{Q}}$ is well defined (see 2.12). Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset. There are $\delta_{\mathcal{P}} > 0$ and a finite subset $\mathcal{F}_{\mathcal{P}}$ satisfying the following condition: if $\varphi : A \to B$ is a unital $\mathcal{F}_{\mathcal{P}}$-$\delta_{\mathcal{P}}$-multiplicative completely positive linear map and (e 2.17) holds for $\delta_{\mathcal{P}}$ (in place of δ) and $\mathcal{F}_{\mathcal{P}}$ (in place of $\mathcal{G}$), then there exists a unital completely positive linear map $L : A \otimes C(\mathbb{T}) \to B$ which satisfies (e 2.18) such that $[L]|_{\boldsymbol{\beta}(\mathcal{P})}$ is well defined, and $[L']|_{\boldsymbol{\beta}(\mathcal{P})} = [L]|_{\boldsymbol{\beta}(\mathcal{P})}$ if L' also satisfies (e 2.18) (for the same φ and v) (see 2.12). In this case, we will write

$$\mathrm{Bott}(\varphi,\, v)|_{\mathcal{P}}(x) = [L]|_{\boldsymbol{\beta}(\mathcal{P})}(x) \tag{e 2.19}$$

for all $x \in \mathcal{P}$. In particular, when $[L]|_{\boldsymbol{\beta}(\mathcal{P})} = 0$, we will write

$$\mathrm{Bott}(\varphi,\, v)|_{\mathcal{P}} = 0. \tag{e 2.20}$$

When $K_*(A)$ is finitely generated, $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ is determined by a finitely generated subgroup of $\underline{K}(A)$ (see [20]). Let $\mathcal{P}$ be a finite subset which generates this subgroup. Then, in this case, instead of (e 2.20), we may write

$$\mathrm{Bott}(\varphi,\, v) = 0. \tag{e 2.21}$$

In general, if $\mathcal{P} \subset K_0(A)$, we will write

$$\mathrm{bott}_0(\varphi,\, v)|_{\mathcal{P}} = \mathrm{Bott}(\varphi,\, v)|_{\mathcal{P}}, \tag{e 2.22}$$

and if $\mathcal{P} \subset K_1(A)$, we will write

$$\mathrm{bott}_1(\varphi,\, v)|_{\mathcal{P}} = \mathrm{Bott}(\varphi,\, v)|_{\mathcal{P}}. \tag{e 2.23}$$

DEFINITION 2.15. Let A be a unital C^*-algebra. Each element $u \in U_0(A)$ can be written as $u = e^{ih_1}e^{ih_2}\cdots e^{ih_k}$ for $h_1, h_2, ..., h_k \in A_{s.a.}$. We write $\mathrm{cer}(u) \le k$ if $u = e^{ih_1}e^{ih_2}\cdots e^{ih_k}$ for selfadjoint elements $h_1, h_2, ..., h_k$. We write $\mathrm{cer}(u) = k$ if $\mathrm{cer}(u) \le k$ and u is *not* a norm limit of unitaries $\{u_n\}$ with $\mathrm{cer}(u_n) \le k-1$. We write $\mathrm{cer}(u) = k + \varepsilon$ if $\mathrm{cer}(u) \not\le k$ and there exists a sequence of unitaries $\{u_n\} \subset A$ such that $u_n \in U_0(A)$ with $\mathrm{cer}(u_n) \le k$.

Let $u = u(t) \in C([0,1], U(A))$ be a unitary. Let $\mathcal{Q} = \{0 = t_0 < t_1 < \cdots t_m = b\}$ be a partition of $[0,1]$. Define $L_{\mathcal{Q}}((u(t))_{0\le t\le 1}) = \sum_{i=1}^m \|u(t_i) - u(t_{i-1})\|$, and

$$\mathrm{length}(u(t))_{0\le t\le 1} = \sup\{L_{\mathcal{P}}((u(t))_{0\le t\le 1} : \mathcal{P}\}, \tag{e 2.24}$$

where the supremum is taken among all possible partitions $\mathcal{Q}$. Define

$$\mathrm{cel}(u) = \inf\big\{\text{length of } (u(t))_{0\le t\le 1} \mid\ u(t) \in C([0,1], U_0(A)),\ u(0)) = u,\ u(1) = 1\big\}.$$

Obviously, if $u = e^{ih_1}e^{ih_2}\cdots e^{ih_k}$, then $\mathrm{cel}(u) \le \|h_1\| + \|h_2\| + \cdots + \|h_k\|$. In fact (see [101]),

$$\mathrm{cel}(u) = \inf\{\sum_{j=1}^n \|h_j\| : u = \prod_{i=1}^n e^{ih_j},\ h_j \in A_{s.a.}\}.$$

DEFINITION 2.16. Suppose that A is a unital C^*-algebra with $T(A) \ne \varnothing$. Recall that $CU(A)$ is the closure of the commutator subgroup of $U_0(A)$. Let $u \in U(A)$. We shall use $\bar{u}$ to denote the image in $U(A)/CU(A)$. It was proved in [112] that there is a splitting short exact sequence

$$0 \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to \bigcup_{n=1}^\infty U(M_n(A))/\bigcup_{n=1}^\infty CU(M_n(A)) \stackrel{\kappa_1^A}{\to} K_1(A) \to 0. \tag{e 2.25}$$

In what follows, we will use $U(M_\infty(A))$ for $\bigcup_{n=1}^\infty U(M_n(A))$, $U_0(M_\infty(A))$ for $\bigcup_{n=1}^\infty U_0(M_n(A))$, and $CU(M_\infty(A))$ for $\bigcup_{n=1}^\infty CU(M_n(A))$. Let J_c (or J_c^A) be a fixed splitting map. Then one may write

$$U(M_\infty(A))/CU(M_\infty(A)) = \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \oplus J_c(K_1(A)). \tag{e 2.26}$$

As in [46], denote by $P_n(A)$ the subgroup of $K_0(A)$ generated by the projections in $M_n(A)$. Denote by $\rho_A^n(K_0(A))$ the subgroup $\rho_A(P_n(A))$ of $\rho_A(K_0(A))$.

In particular, $\rho_A^1(K_0(A))$ is the subgroup of $\rho_A(K_0(A))$ generated by the images of the projections in A under the map ρ_A. Let $\beta : \pi_1(U(M_\infty(A))) \to K_0(A)$ denote the inverse of the Bott periodicity isomorphism. Then it is well known that the image of $\pi_1(U(M_n(A)))$, under β, contains $P_n(A)$. (Namely, for any $[p] \in K_0(A)$ represented by a projection $p \in M_n(A)$, we have $\beta([u]) = [p]$, for $u \in C(\mathbb{T}, U_n(A))$ defined by $u(z) = z \cdot p + (1_n - p)$, for all $z \in \mathbb{T}$.) By Theorem 3.2 of [112] (see pertinent notation at the beginning of §3 of [112]), if $\rho_A^n(K_0(A)) = \rho_A(K_0(A))$, then

$$U_0(M_n(A))/CU(M_n(A)) \cong \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \cong U_0(M_\infty(A))/CU(M_\infty(A))$$

(cf. [46]).

In general, let $\widetilde{r}_{\text{aff}}^k : A_{s.a.} \to U_0(M_k(A))/CU(M_k(A))$ be defined by the composition of $r_{\text{aff}} : A_{s.a.} \to \operatorname{Aff}(T(A))$ and the quotient map $\mathfrak{q}_k : \operatorname{Aff}(T(A)) \to \operatorname{Aff}(T(A))/\overline{\rho_A(\pi_1(U_0(M_k(A))))} \cong U_0(M_k(A))/CU(M_k(A))$, where the last isomorphism is given by Theorem 3.2 of [112]. Denote by

$$\widetilde{r}_{\text{aff}} : A_{s.a.} \to U_0(M_\infty(A))/CU(M_\infty(A))$$

the composition of r_{aff} and the quotient map

$$\mathfrak{q} : \operatorname{Aff}(T(A)) \to \operatorname{Aff}(T(A))/\overline{\rho_A(K_0(A))}.$$

Suppose that $\operatorname{Aff}(T(A))/\overline{\rho_A(K_0(A))} \cong U_0(M_k(A))/CU(M_k(A))$. Then the map $\widetilde{r}_{\text{aff}} = \widetilde{r}_{\text{aff}}^k$ defined above can be defined concretely as

$$\text{(e 2.27)}\ \widetilde{r}_{\text{aff}}(h) = [\operatorname{diag}(\exp(2\pi i h), 1_{k-1})] \in U_0(M_k(A))/CU(M_k(A))$$

for any $h \in A_{s.a}$, (see Cor 2.12 of [46]). The map $\widetilde{r}_{\text{aff}}$ is surjective, since r_{aff} is surjective.

If A has stable rank k, then $K_1(A) = U(M_k(A))/U_0(M_k(A))$. Note that

$$\operatorname{csr}(C(\mathbb{T}, A)) \leq \operatorname{tsr}(A) + 1 = k + 1.$$

It follows from Theorem 3.10 of [46] that

$$\text{(e 2.28)} \qquad U_0(M_\infty(A))/CU(M_\infty(A)) = U_0(M_k(A))/CU(M_k(A)),$$

whence it follows that this holds with U in place of U_0. Then, combining these facts, one has the split short exact sequence

$$\begin{aligned} 0 \to \operatorname{Aff}(T(A))/\overline{\rho_A(K_0(A))} &\to U(M_k(A))/CU(M_k(A)) \\ \text{(e 2.29)} \qquad &\xrightarrow{\kappa_1^A} U(M_k(A))/U_0(M_k(A)) \to 0, \end{aligned}$$

and one may write

$$U(M_k(A))/CU(M_k(A)) = \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \oplus J_c(K_1(A)) \tag{e 2.30}$$

$$= \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \oplus J_c(U(M_k(A))/U_0(M_k(A))). \tag{e 2.31}$$

Note that $\kappa_1^A \circ J_c = \mathrm{id}_{K_1(A)}$. For each continuous and piecewise smooth path and $\{u(t) : t \in [0,1]\} \subset U(M_k(A))$, define

$$D_A(\{u(t)\})(\tau) = \frac{1}{2\pi i}\int_0^1 \tau(\frac{du(t)}{dt}u^*(t))dt, \quad \tau \in T(A).$$

For each $\{u(t)\}$, the map $D_A(\{u\})$ is a real continuous affine function on $T(A)$. Let

$$\overline{D}_A : U_0(M_k(A))/CU(M_k(A)) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$$

denote the de la Harpe and Skandalis determinant ([18]) given by

$$\overline{D}_A(\bar{u}) = D_A(\{u\}) + \overline{\rho_A(K_0(A))}, \quad u \in U_0(M_k(A)),$$

where $\{u(t) : t \in [0,1]\} \subset M_k(A)$ is a continuous and piecewise smooth path of unitaries with $u(0) = 1$ and $u(1) = u$. It is known that the de la Harpe and Skandalis determinant is independent of the choice of representative for $\bar{u}$ and the choice of path $\{u(t)\}$. Define

$$\|\overline{D}_A(\bar{u})\| = \inf\{\|D_A(\{v\})\| : v(0) = 1,\ \ v(1) = v \ \text{ and } \ \bar{v} = \bar{u}\}, \tag{e 2.32}$$

where $\|D_A(\{v\})\| = \sup_{\tau \in T(A)} \|D_A(\{v\})(\tau)\|$.

Suppose that $u, v \in U(M_k(A))$. Define

$$\mathrm{dist}(\overline{u}, \overline{v}) = \inf\{\|uv^* - c\| : c \in CU(M_k(A))\}. \tag{e 2.33}$$

It is a metric. Note that $\mathrm{dist}(\overline{uv^{-1}}, \overline{1}) = \mathrm{dist}(\overline{u}, \overline{v}) \le \mathrm{dist}(\overline{u}, \overline{1}) + \mathrm{dist}(\overline{v}, \overline{1}) = \mathrm{dist}(\overline{u}, \overline{1}) + \mathrm{dist}(\overline{v^{-1}}, \overline{1})$. Define

$$\mathrm{d}(\bar{u}, \bar{v}) = \begin{cases} 2, & \text{if } uv^* \notin U_0(M_k(A)), \text{ or } \|\overline{D}_A(\overline{uv^*})\| \ge 1/2, \\ \|e^{2\pi i \|\overline{D}_A(\overline{uv^*})\|} - 1\|, & \text{otherwise.} \end{cases} \tag{e 2.34}$$

This is also a metric (see the lines preceding Theorem 6.4 of [113]).

Note that, if $u, v \in U_0(M_k(A))$, then $\mathrm{d}(\overline{uv^*}, \overline{1_k}) = \mathrm{dist}(\overline{u}, \overline{v})$. Now suppose that A has the property that $\overline{\rho_A^k(K_0(A))} \supset \rho_A(K_0(A))$. This means $\overline{\rho_A^1(P_k(A))} \supset \rho_A(P_k(A))$, where $P_k(A)$ is the subgroup of $K_0(A)$ which is generated by the elements in $K_0(A)$ represented by projections in $M_k(A)$. By 3.6 of [46],

$$U_0(M_k(A))/CU(M_k(A)) = U_0(M_m(A))/CU(M_m(A))$$

for all $m \ge k$. It follows from the proof of Theorem 3.1 of [89] that for any unitaries $u, v \in U_0(M_k(A))$, $\mathrm{d}(\overline{u}, \overline{v}) = \mathrm{dist}(\overline{u}, \overline{v}) = \inf\{\|uv^* - w\| : w \in CU(M_k(A))\|\}$. On the other hand, if $\|\overline{D}_A(\overline{uv^*})\| = \delta < 1/2$, then

$$\inf\{\|uv^* - w\| : w \in CU(M_k(A))\|\} < 2\pi\delta. \tag{e 2.35}$$

See Proposition 3.23 below for further discussion.

DEFINITION 2.17. Let A be a unital separable amenable C^*-algebra and B be another unital C^*-algebra. If $\varphi : A \to B$ is a unital homomorphism, then φ induces a continuous homomorphism $\varphi^{\ddagger} : U(M_m(A))/CU(M_m(A)) \to U(M_m(B))/CU(M_m(B))$ which maps $U_0(M_m(A))/CU(M_m(A))$ to $U_0(M_m(B))/CU(M_m(B)))$ for each m. Moreover, $\kappa_1^B \circ \varphi^{\ddagger} = \varphi_{*1} \circ \kappa_1^A$.

For any finite subset $\mathcal{U} \subset U(A)$, there exists $1 > \delta > 0$ and a finite subset $\mathcal{G} \subset A$ with the following property: If $L : A \to B$ is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, then $\overline{\langle L(u) \rangle}$ is a well-defined element in $U(B)/CU(B)$ for all $u \in \mathcal{U}$. Recall that we have assumed that $L(1_A) = p$ is a projection in B. Here $\langle L(u) \rangle$ is originally defined as a unitary in pBp. But we will also use $\langle L(u) \rangle$ for $\langle L(u) \rangle + (1-p)$, whenever it is convenient, and $\overline{\langle L(u) \rangle}$ is defined to be $\overline{\langle L(u) \rangle + (1-p)}$.

Let $G(\mathcal{U})$ denote the subgroup generated by $\mathcal{U}$ and let $1/4\pi > \varepsilon > 0$. Denote by $\overline{G(\mathcal{U})}$ the image of $G(\mathcal{U})$ in $U(A)/CU(A)$. Let $\mathcal{V} \subset G(\mathcal{U})$ be another finite subset. By the Appendix of [81], there is a homomorphism $L^{\ddagger} : \overline{G(\mathcal{U})} \to U(B)/CU(B)$ such that $\mathrm{dist}(L^{\ddagger}(\overline{u}), \overline{\langle L(u) \rangle}) < \varepsilon$ for all $u \in \mathcal{V}$, if $\mathcal{G}$ is sufficiently large and δ is sufficiently small. In particular, $\mathrm{dist}(\overline{\langle L(u) \rangle}, CU(B))) < \varepsilon$ if $u \in CU(A) \cap \mathcal{V}$. Suppose that $u, v \in U(A)$ and $\mathrm{dist}(\overline{u}, \overline{v}) < \varepsilon$. Then $\mathrm{dist}(\overline{uv^*}, CU(A)) < \varepsilon$. With sufficiently large $\mathcal{G}$ and small δ, we may assume that $\mathrm{dist}(\overline{\langle L(uv^*) \rangle}, \overline{1}_B) < \varepsilon$. It follows that we may also assume that $\mathrm{dist}(L^{\ddagger}(\overline{u}), L^{\ddagger}(\overline{v}))$ $= \mathrm{dist}(L^{\ddagger}(\overline{uv^{-1}}), \overline{1_B}) < 2\varepsilon$.

Note also that we may assume $L^{\ddagger}((G(\mathcal{U}) \cap U_0(A))/CU(A)) \subset U_0(B)/CU(B)$. To see this, let $\mathcal{U}_0$ be a finite subset of $U_0(A) \cap G(\mathcal{U})$ which generates $U_0(A) \cap G(\mathcal{U})$. (With sufficiently small δ and large $\mathcal{G}$—see 2.12), we may assume $\langle L(u) \rangle \in U_0(B)$ for each $u \in \mathcal{U}_0$, and $[L]$ is well defined on the subgroup generated by the image of $\mathcal{U}$ in $K_1(A)$. Let $z \in U(B)$ be such that $\overline{z} = L^{\ddagger}(\overline{u})$. Then there exists $\zeta \in CU(B)$ such that $\|z(\langle L(u) \rangle)^{-1} - \zeta\| < \varepsilon < 1/2$. It follows that there exists $y \in U_0(B)$ such that $z(\langle L(u) \rangle)^{-1} = y\zeta$. Since $\langle L(u) \rangle \in U_0(B)$, $\overline{z} \in U_0(B)/CU(B)$. This proves the assertion above. It follows that $\kappa_1^B \circ L^{\ddagger}(\overline{u}) = [L] \circ \kappa_1^A([u])$ for all $u \in G(\mathcal{U})$, where $\kappa_1^C : \bigcup_{n=1}^{\infty} U(M_n(C))/\bigcup_{n=1}^{\infty} CU(M_n(C)) \to K_1(C)$ is the quotient map for a unital C^*-algebra C (see 2.16).

In what follows, whenever we write $L^{\ddagger}$ (associated with $\mathcal{U}$ and ε), $\mathcal{U}$ is specified, and $1/2 > \varepsilon > 0$ is given, we mean that δ is small enough and $\mathcal{G}$ is large enough that we may choose $L^{\ddagger}|_{\overline{G(\mathcal{U})}}$ as above to be a homomorphism such that, if $u, v \in \mathcal{U}$, then

$$\mathrm{dist}(L^{\ddagger}(\overline{u}), \overline{\langle L(u) \rangle}) < \varepsilon/2 \text{ and} \tag{e 2.36}$$

$$\mathrm{dist}(L^{\ddagger}(\overline{u}), L^{\ddagger}(\overline{v})) < \varepsilon, \text{ if } \mathrm{dist}(\overline{u}, \overline{v}) < \varepsilon/2, \tag{e 2.37}$$

and $\kappa_1^B \circ L^{\ddagger}(\overline{u}) = [\langle L(u) \rangle] = [L] \circ \kappa_1^B(\overline{u})$ for all $u \in G(\mathcal{U})$. The latter equation implies that $L^{\ddagger}(\overline{u})^{-1}(\overline{\langle L(u) \rangle}) \in U_0(B)/CU(B)$. Note that such a choice is not unique. However, if $L^{\dagger}$ is another choice which satisfies the requirements above, then $\mathrm{dist}(L^{\dagger}(\overline{u}), L^{\ddagger}(\overline{u})) < \varepsilon$ and $\kappa_1^B \circ L^{\dagger}(\overline{u}) = \kappa_1^B \circ L^{\ddagger}(\overline{u})$ for all $u \in \mathcal{U}$. Moreover, for an integer $k \geq 1$, we will also use $L^{\ddagger}$ for the map on a subgroup of $U(M_k(A))/CU(M_k(A))$ induced by $L \otimes \mathrm{id}_{M_k}$. Recall also, if $e \in B$ is a projection, and $w \in eBe$ is a unitary, then, we also write $\bar{w}$ for $\overline{w + (1-e)}$ in $U(B)/CU(B)$. Finally, if $e_0, e_1 \in A$ are two mutually orthogonal projections and $\varphi_i : C \to e_iAe_i$ $(i = 0, 1)$ are two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps, then we shall write $(\varphi_0 \oplus \varphi_1)^{\ddagger}(\overline{u}) := \varphi_0^{\ddagger}(\overline{u})\varphi_1^{\ddagger}(\overline{u})$ for $\overline{u} \in G(\overline{\mathcal{U}})$.

LEMMA 2.18. *Let A be a unital C^*-algebra and let $\mathcal{U} \in U_0(M_k(A))$ be a finite subset. There exists a finite subset $\mathcal{H} \in A_+$ with $\widetilde{r^k_{\rm aff}}(\mathcal{H}) \supset \overline{\mathcal{U}} := \{\bar{u} : u \in \mathcal{U}\} \subset U_0(M_k(A))/CU(M_k(A))$ with the following property: for any $\varepsilon > 0$, there are a finite subset $\mathcal{G} \subset A$, $\delta > 0$, and $\eta > 0$ such that, for any two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps $\varphi, \psi : A \to B$ (for any unital C^*-algebra B) with the property $|\tau(\varphi(h) - \psi(h))| < \eta$ for all $h \in \mathcal{H}$ and $\tau \in T(B)$, we have, in $U_0(M_k(B))/CU(M_k(B))$, $\mathrm{dist}(\varphi^{\ddagger}(\bar{u}), \psi^{\ddagger}(\bar{u})) < \varepsilon$ for all $u \in \mathcal{U}$.*

PROOF. Assume that $\varepsilon < 1/4$. For the finite set $\mathcal{U}$, since $\widetilde{r^k_{\rm aff}}$ is surjective, there is a finite set $\mathcal{H} \subset A_{s.a.}$ such that $\widetilde{r^k_{\rm aff}}(\mathcal{H}) \supset \overline{\mathcal{U}}$. We may assume that $h \in A_+$ for all $h \in \mathcal{H}$; otherwise replace h by $h + 2m\pi 1_A$, for a large enough positive integer m, which has the same image in $\mathrm{Aff}(T(A))/\overline{\rho_A(\pi_1(U_0(M_k(A))))} \cong U_0(M_k(A))/CU(M_k(A))$. For each $u \in \mathcal{U}$, there are an $h \in \mathcal{H}$ and finitely many unitaries $u_i, v_i \in U_0(M_k(A))$ such that $\|u^*\mathrm{diag}(\exp(2\pi i h), 1_{k-1}) - \prod_j u_jv_ju_j^*v_j^*\| < \varepsilon/8$. Choose a finite set $\mathcal{G} \subset A$ which contains all the elements h, 1_A, and all entries of $u, u^*, u_j, u_j^*, v_j, v_j^*$ (as matrices in $M_k(A)$) for $u \in \mathcal{U}$. Let $\eta = \varepsilon/4\pi$. If $\delta > 0$ is small enough, then for any unital $\mathcal{G}$-δ-multiplicative completely positive map $\varphi : A \to B$, $\varphi^{\ddagger}|_{\overline{\mathcal{U}}}$ can be defined so that

$$\mathrm{dist}(\varphi^{\ddagger}(\overline{u}), \langle\varphi(u)\rangle) < \varepsilon/32,$$

$$\|\varphi(u^*\mathrm{diag}(\exp(2\pi i h), 1_{k-1})) - \langle\varphi(u)\rangle^*\mathrm{diag}(\exp(2\pi i\varphi(h)), 1_{k-1})\| < \varepsilon/32, \quad \text{and}$$

$$\|\varphi(\prod_j u_jv_ju_j^*v_j^*) - \prod_j \langle\varphi(u_j)\rangle\langle\varphi(v_j)\rangle\langle\varphi(u_j)\rangle^*\langle\varphi(v_j)\rangle^*\| < \varepsilon/32.$$

Consequently, in $U_0(M_k(B))/CU(M_k(B))$,

$$\mathrm{dist}(\varphi^{\ddagger}(\bar{u}), \overline{\mathrm{diag}(\exp(2\pi i\varphi(h)), 1_{k-1})}) \quad < \quad 3 \cdot (\varepsilon/32) + \varepsilon/8 < \varepsilon/4.$$

If $\psi : A \to B$ is another unital $\mathcal{G}$-δ-multiplicative completely positive map such that
$|\tau(\varphi(h) - \psi(h))| < \eta$ for all $h \in \mathcal{H}$ and $\tau \in T(B)$, then by (e 2.35),

$$\mathrm{dist}(\overline{\mathrm{diag}(\exp(2\pi i\varphi(h)), 1_{k-1})}, \overline{\mathrm{diag}(\exp(2\pi i\psi(h)), 1_{k-1})}) < 2\pi\eta = \varepsilon/2.$$

Hence $\mathrm{dist}(\varphi^{\ddagger}(\bar{u}), \psi^{\ddagger}(\bar{u})) < \varepsilon$ for all $u \in \mathcal{U}$.

□

The following lemma is known and has appeared implicitly in some of the proofs earlier. We present it here for convenience.

LEMMA 2.19. *For any* $0 < \delta < 1/2^8$, *any pair of projections* p *and* q, *and any element* x *in a* C^**-algebra* C, *if* $\|p-x^*x\| < \delta$ *and* $\|q-xx^*\| < \delta$, *then there exists* $w \in C$ *such that* $w^*w = p$, $ww^* = q$, *and* $\|w-x\| < (4/3)(1+\delta)(\delta)^{1/4} + 4(\delta)^{1/2}$.

For any positive number $\varepsilon < 1/4$, *and positive integer* $K \in \mathbb{N}$, *there is a positive number* $\delta < \varepsilon$ *such that the following statements are true:*

(a) *For any unital* C^**-algebra* C *and any* C^**-algebra* $C_1 \subset C$ *with* $1_{C_1} = p$, *if* $e \in C$ *is a projection with* $\|ep - pe\| < \delta$ *and* $pep \in_\delta C_1$, *then there exist projections* $q \in C_1 \subset pCp$ *and* $q_0 \in (1-p)C(1-p)$ *such that*

$$\|q - pep\| < \varepsilon \quad \text{and} \quad \|q_0 - (1-p)e(1-p)\| < \varepsilon.$$

(b) *Under the assumptions of (a), if unitaries* $v \in C$, $u \in eCe$, *and* $w = u \oplus (1-e)$ *are such that* $\|vp-pv\| < \delta$, $\|up-pu\| < \delta$, $\|wp-pw\| < \delta$, *and* $pvp \in_\delta C_1$, $pup \in_\delta C_1$, *then there exist unitaries* $v' \in C_1 \subset C$, $v'' \in (1-p)C(1-p)$, *and* $z \in qC_1q$ *(*q *as in (a)) such that*

$$\|v'-pvp\| < \varepsilon,\ \|v''-(1-p)v(1-p)\| < \varepsilon,\ \|z-quq\| < \varepsilon,\ \text{and}\ \|z\oplus(p-q)-pwp\| < \varepsilon.$$

(c) *Under the assumptions of (a), if mutually orthogonal projections* $e_1, e_2, ..., e_K \in C$ *and partial isometries* $s_1, s_2, ..., s_K \in C$ *satisfy* $s_i^*s_i = e$ *and* $s_is_i^* = e_i$ *for all* $i = 1, 2, ..., K$, *where* $1 - e = \sum_{i=1}^K e_i$, *and also satisfy*

$$\|pe_i - e_ip\| < \delta, \quad pe_ip \in_\delta C_1, \quad \|ps_i - s_ip\| < \delta, \quad \text{and} \quad ps_ip \in_\delta C_1,$$

for all $i = 1, 2, ..., K$, *then there are mutually orthogonal projections* $q, e_1', e_2', ..., e_K' \in C_1 \subset pCp$, $q_0, e_1'', e_2'', ..., e_K'' \in (1-p)C(1-p)$, *and partial isometries* $s_1', s_2', ..., s_K' \in C_1 \subset pCp$, $s_1'', s_2'', ..., s_k'' \in (1-p)C(1-p)$, *such that* $\|q - pep\| < \varepsilon$,

$$\|(p-q)-\sum_{i=1}^K e_i'\| < \varepsilon, \quad \|(s_i')^*s_i'-q\| < \varepsilon, \qquad s_i'(s_i')^* = e_i', \quad \text{for all } i = 1, 2, ..., K,$$

and

$$\|((1-p)-q_0)-\sum_{i=1}^K e_i''\| < \varepsilon,\ \|(s_i'')^*s_i''-q_0\| < \varepsilon,\ s_i''(s'')_i^* = e_i'' \text{ for all } i = 1, 2, ..., K.$$

Consequently, $[p-q] = K[q] \in K_0(C_1)$ *and* $[(1-p)-q_0] = K[q_0] \in K_0(C)$.

(d) *Let* $1/2 > \varepsilon > 0$. *Under the assumptions of (a), if* $\{e_1, e_2, ..., e_m\}$ *and* $\{e_1', e_2', ..., e_m'\}$ *are two sets of projections in* C *and* $\{v_1, v_2, ..., v_m\}$ *is a set of partial isometries such that* $v_j^*v_j = e_j$ *and* $v_jv_j^* = e_j'$, *and if* $u_1, u_2, ..., u_m$, $u_1', u_2', ..., u_m' \in C$ *are unitaries such that* $u_j = z_ju_j'$, *with* $z_j = \prod_{k=1}^{n_j} \exp(ih_{k,j})$, *where* $h_{k,j} \in C_{s.a.}$, $j = 1, 2, ..., m$, *then there exist* $\varepsilon' > 0$ *and a finite subset* $\mathcal{F}' \subset C$ *with the following properties:*

If $\|px - xp\| < \varepsilon'$ *and* $\mathrm{dist}(pxp, C_1) < 2\varepsilon'$ *for all* $x \in \mathcal{F}'$, *then there are projections* $e_{j,0}, e_{j,0}' \in (1-p)C(1-p)$ *and* $v_{j,0} \in (1-p)A(1-p)$, *and there are* *projections* $e_{j,1}, e'_{j,1} \in C_1$ *and* $v_{j,1} \in C_1$, *such that* $\|e_{j,0} - (1-p)e_j(1-p)\| < \varepsilon$, $\|e_{j,1} - (1-p)e'_j(1-p)\| < \varepsilon$, $\|e_{j,1} - pe_jp\| < \varepsilon$, $\|e'_{j,1} - pe'_jp\| < \varepsilon$, $v^*_{j,0}v_{j,0} = e_{j,0}$ *and* $v_{j,0}v^*_{j,0} = e'_{j,0}$, *and* $v^*_{j,1}v_{j,1} = e_{j,1}$ *and* $v_{j,1}v^*_{j,1} = e'_{j,1}$, *and there are unitaries* $u_{j,0}, u'_{j,0} \in (1-p)C(1-p)$, *and* $h_{k,j,0} \in (1-p)A(1-p)_{s.a.}$, *and unitaries* $u_{j,1}, u'_{j,1} \in C_1$, *and* $h_{k,j,1} \in (C_1)_{s.a.}$, $k = 1, 2, ..., n_j + 1$, *such that* $\|u_{j,0} - (1-p)u_j(1-p)]\| < \varepsilon$, $\|u'_{j,0} - (1-p)u'_j(1-p)\| < \varepsilon$, $\|u_{j,1} - pu_jp\| < \varepsilon$, $\|u'_{j,1} - pu'_jp\| < \varepsilon$, *and* $u_{j,0} = z_{j,0}u'_{j,0}$ *and* $u_{j,1} = z_{j,1}u'_{j,1}$, *where* $z_{j,0} = \prod_{k=1}^{n_j+1} \exp(ih_{k,j,0}) \in (1-p)A(1-p)$ *and* $z_{j,1} = \prod_{k=1}^{n_j+1} \exp(ih_{k,j,1}) \in C_1$, *with* $\|h_{n_j+1,k,0}\|, \|h_{n_j+1,k,1}\| \le \varepsilon$ *and* $\|h_{j,k,0}\|, \|h_{k,j,1}\| \le \|h_{k,j}\|$, $1 \le k \le n_j$, $j = 1, 2, ..., m$.

PROOF. Variations of the first part of the statement are known. In fact, $\||x| - p\| < \sqrt{\delta}$ (see Lemma 2.3 of [14]). Write $x = u|x|$ as the polar decomposition of x in C^{**}. Then $x = u|x| \approx_{\sqrt{\delta}} up = up^2 \approx_{\sqrt{\delta}} u|x|p = xp$. Consider the polar decomposition $x^* = v(xx^*)^{1/2}$ in C^{**}. Then, similarly, $\|qx - x\| < 2\sqrt{\delta}$. Put $y = qxp$ and $\eta := (1+\delta)\sqrt{\delta} + \delta < 1/16$. Then $\|y - x\| < 4\sqrt{\delta}$ and $\|p - y^*y\| < \|x^*x\|\sqrt{\delta} + \delta \le (1+\delta)\sqrt{\delta} + \delta = \eta$. Also, $\|q - yy^*\| < \eta$. Thus, y^*y is invertible in pCp and yy^* is invertible in qCq. One computes that $\|p - |y|^{-1}\| < \frac{\sqrt{\eta}}{1-\sqrt{\eta}} < (4/3)\sqrt{\eta}$. Set $w = y|y|^{-1}$. Then $\|w - y\| \le \|y\|\|p - |y|^{-1}\| \le (1+\delta)(4/3)\sqrt{\eta}$. It follows that $\|w - x\| < (1+\delta)(4/3)\sqrt{\eta} + 4\sqrt{\delta}$. As in Lemma 2.5.3 of [63], one checks that $w^*w = p$ and $ww^* = q$.

For the second part of the statement, one notes that it is straightforward to prove part (a) and part (b) by standard perturbation arguments (see 2.5 of [63]).

For part (c), let $e = e_0$. Let $\{e_{i,j} : 1 \le i, j \le K+1\}$ be a system of matrix units for M_{K+1}. There is a unital homomorphism $\varphi : M_{K+1} \to C$ defined by $\varphi(e_{i,i}) = e_{i-1}$, $i = 1, 2, ..., K+1$, and $\varphi(e_{1,j}) = s_{j-1}$, $j = 2, 3, ..., K$. Define $\varphi_1 : M_{K+1} \to pCp$ by $\varphi_1(a) = p\varphi(a)p$, and define $\varphi_2 : M_{K+1} \to (1-p)C(1-p)$ by $\varphi_2(a) = (1-p)\varphi(a)(1-p)$ for all $a \in M_{K+1}$, respectively. Fix $\eta > 0$. Then, by semiprojectivity of M_{K+1}, if δ is sufficiently small, there is a unital homomorphism $\psi_1 : M_{K+1} \to pCp$ and a unital homomorphism $\psi_2 : M_{K+1} \to (1-p)C(1-p)$ such that (note that the unital ball of M_{K+1} is compact)

$$\max\{\|\varphi_1(a) - \psi_1(a)\| : \|a\| \le 1\} < \eta \qquad \text{(e 2.38)}$$
$$\text{and } \max\{\|\varphi_2(a) - \psi_2(a)\| : \|a\| = 1\} < \eta.$$

Since $pep, pe_ip, ps_ip \in_\delta C_1$, $i = 1, 2, ..., K$, for sufficiently small δ we may assume that there are $a_{i,j} \in C_1$ such that

$$\|\psi(e_{i,j}) - a_{i,j}\| < 2\eta, \ \ 1 \le i, j \le K+1. \qquad \text{(e 2.39)}$$

Then, by semiprojectivity of M_{K+1} again (see, for example, 2.5.9 of [63]), with sufficiently small η (in other words, with sufficiently small δ), there is a unital homomorphism $\psi_3 : M_{K+1} \to C_1$ such that

$$\sup\{\|\psi_3(a) - \psi_1(a)\| : \|a\| \le 1\} < \varepsilon. \qquad \text{(e 2.40)}$$

Now let $q = \psi_3(e_{1,1})$, $e_i' = \psi_3(e_{i+1,i+1})$, $s_i' = \psi_3(e_{1,i+1})$, $q_0 = \varphi_2(e_{1,1})$, $e_i'' = \varphi_2(e_{i+1}, e_{i+1})$, and $s_i'' = \varphi_2(e_{1,i+1})$, $i = 1, 2, ..., K$. One then verifies that part (c) of the lemma follows.

For part (d), the statement for projections follows a standard perturbation argument as above. In fact, it is a direct consequence of (a) and the first part of the statement of the lemma.

To see the second part of (d), let $1/2 > \varepsilon > 0$ be fixed, and let u_j, u_j', and $h_{k,j}$ be given. With sufficiently small $\varepsilon' > 0$ and sufficiently large finite subset $\mathcal{F}'$, we have

$$\|(1-p)u_j(1-p) - z_{j,0}'(1-p)u_j'(1-p)\| < \varepsilon/16 \quad \text{and} \quad \|pu_jp - z_{j,1}'pu_j'p\| < \varepsilon/16, \tag{e 2.41}$$

with the unitaries $z_{j,0}' = \prod_{k=1}^{n_j} \exp(i(1-p)h_{k,j}(1-p)) \in (1-p)C(1-p)$ and $z_{j,1}' = \prod_{k=1}^{n_j} \exp(ih_{k,j,1}) \in C_1$, where $h_{k,j,1} \in (C_1)_{s.a.}$ with $\|h_{k,j,1} - ph_{k,j}p\| < \varepsilon'$ and $\|h_{k,j,1}\| \leq \|h_{k,j}\|$, $j = 1, 2, ..., m$. Moreover, there are unitaries $u_{j,0}, u_{j,0}' \in (1-p)C(1-p)$ and $u_{j,1}, u_{j,1}' \in C_1$ such that

$$\|(1-p)u_j(1-p) - u_{j,0}\| < \varepsilon/16, \ \|(1-p)u_j'(1-p) - u_{j,0}'\| < \varepsilon/16, \tag{e 2.42}$$

$$\|pu_jp - u_{j,1}\| < \varepsilon/16, \ \text{and} \ \|pu_j'p - u_{j,1}'\| < \varepsilon/16, \tag{e 2.43}$$

$j = 1, 2, ..., m$. It follows that

$$\|u_{j,0} - z_{j,0}'u_{j,0}'\| < \varepsilon/4 \ \text{and} \ \|u_{j,1} - z_{j,1}'u_{j,1}'\| < \varepsilon/4. \tag{e 2.44}$$

Then, there are $h_{n_j+1,j,0} \in (1-p)A(1-p)_{s.a.}$ and $h_{n_j+1,j,1} \in (C_1)_{s.a.}$ such that $\|h_{n_j+1,j,i}\| \leq 2\arcsin(\varepsilon/4)$, $i = 0, 1$, and $u_{j,0} = z_{j,0}u_{j,0}'$ and $u_{j,1} = z_{j,1}u_{j,1}'$, where $z_{j,0} = \exp(ih_{n_j+1,j,0})z_{j,0}$ and $z_{j,1} = \exp(ih_{n_j+1,j,1})z_{j,1}'$, $j = 1, 2, ..., m$. This proves the second part of (d). □

DEFINITION 2.20. Let C and B be unital C^*-algebras and let $\varphi, \psi : C \to B$ be two monomorphisms. Consider the mapping torus

$$M_{\varphi,\psi} = \{(f, c) : C([0,1], B) \oplus C : f(0) = \varphi(c) \ \text{and} \ f(1) = \psi(c)\}. \tag{e 2.45}$$

Denote by $\pi_t : M_{\varphi,\psi} \to B$ the point evaluation at $t \in [0,1]$. One has the short exact sequence

$$0 \to SB \xrightarrow{\imath} M_{\varphi,\psi} \xrightarrow{\pi_e} C \to 0,$$

where $\imath : SB \to M_{\varphi,\psi}$ is the embedding and π_e is the quotient map from $M_{\varphi,\psi}$ to C. Denote by $\pi_0, \pi_1 : M_{\varphi,\psi} \to C$ the point evaluations at 0 and 1, respectively. Since both φ and ψ are injective, one may identify π_e with the point evaluation at 0 for convenience.

Suppose that $[\varphi] = [\psi]$ in $KL(C, B)$. Then $M_{\varphi,\psi}$ corresponds to the zero element of $KL(C, B)$. In particular, the corresponding extensions

$$0 \to K_i(B) \xrightarrow{\imath_*} K_i(M_{\varphi,\psi}) \xrightarrow{\pi_e} K_i(C) \to 0 \quad (i = 0, 1)$$

are pure (see Lemma 4.3 of [69]).

DEFINITION 2.21. Suppose that $T(B) \neq \varnothing$. Let $u \in M_l(M_{\varphi,\psi})$ (for some integer $l \geq 1$) be a unitary which is a piecewise smooth continuous function on $[0,1]$. Recall from 2.16

$$D_B(\{u(t)\})(\tau) = \frac{1}{2\pi i}\int_0^1 \tau(\frac{du(t)}{dt}u^*(t))dt \text{ for all } \tau \in T(B).$$

(see 2.2 for the extension of τ to $M_l(B)$). Suppose that $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(B)$. Then there exists a homomorphism

$$R_{\varphi,\psi} : K_1(M_{\varphi,\psi}) \to \mathrm{Aff}(T(B)),$$

defined by $R_{\varphi,\psi}([u])(\tau) = D_B(\{u(t)\})(\tau)$ as above, which is independent of the choice of the piecewise smooth path u in $[u]$. We have the following commutative diagram:

$$\begin{tikzcd} K_0(B) \arrow[rr, "\imath_*"] \arrow[dr, "\rho_B"'] & & K_1(M_{\varphi,\psi}) \arrow[dl, "R_{\varphi,\psi}"] \\ & \mathrm{Aff}(T(B)) & \end{tikzcd} .$$

Suppose, in addition, that $[\varphi] = [\psi]$ in $KK(C,B)$. Then the following exact sequence splits:

$$0 \to \underline{K}(SB) \to \underline{K}(M_{\varphi,\psi}) \underset{\theta}{\overset{[\pi_e]}{\rightleftharpoons}} \underline{K}(C) \to 0. \qquad \text{(e 2.46)}$$

We may assume that $[\pi_0] \circ [\theta] = [\varphi]$ and $[\pi_1] \circ [\theta] = [\psi]$. In particular, one may write $K_1(M_{\varphi,\psi}) = K_0(B) \oplus K_1(C)$. Then we obtain a homomorphism

$$R_{\varphi,\psi} \circ \theta|_{K_1(C)} : K_1(C) \to \mathrm{Aff}(T(B)).$$

We shall say "the rotation map vanishes" if there exists a splitting map θ, as above, such that $R_{\varphi,\psi} \circ \theta|_{K_1(C)} = 0$.

Denote by $\mathcal{R}_0$ the set of those elements $\lambda \in \mathrm{Hom}(K_1(C), \mathrm{Aff}(T(B)))$ for which there is a homomorphism $h : K_1(C) \to K_0(B)$ such that $\lambda = \rho_B \circ h$. It is a subgroup of $\mathrm{Hom}(K_1(C), \mathrm{Aff}(T(B)))$. If $[\varphi] = [\psi]$ in $KK(C,B)$ and $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(B)$, one has a well-defined element $\overline{R_{\varphi,\psi}} \in \mathrm{Hom}(K_1(C), \mathrm{Aff}(T(B)))/\mathcal{R}_0$ (which is independent of the choice of θ).

Under the assumptions that $[\varphi] = [\psi]$ in $KK(C,B)$, $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(B)$, and C satisfies the UCT, there exists a homomorphism $\theta_1' : K_1(C) \to K_1(M_{\varphi,\psi})$ such that $(\pi_e)_{*1} \circ \theta_1' = \mathrm{id}_{K_1(C)}$ and $R_{\varphi,\psi} \circ \theta_1' \in \mathcal{R}_0$ if, and only if, there is $\Theta \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(M_{\varphi,\psi}))$ such that

$$[\pi_e] \circ \Theta = [\mathrm{id}_C] \text{ in } KK(C,B) \text{ and } R_{\varphi,\psi} \circ \Theta|_{K_1(C)} = 0.$$

(See the proof of 4.5 of [83].) In other words, $\overline{R_{\varphi,\psi}} = 0$ if, and only if, there is Θ as described above such that $R_{\varphi,\psi} \circ \Theta|_{K_1(C)} = 0$. When $\overline{R_{\varphi,\psi}} = 0$, one has that $\theta(K_1(C)) \subset \ker R_{\varphi,\psi}$ for some θ such that (e 2.46) holds. In this case θ also gives the following decomposition:

$$\ker R_{\varphi,\psi} = \ker \rho_B \oplus K_1(C).$$

DEFINITION 2.22. Let C be a C^*-algebra, let $a, b \in C$ be two elements, and let $\varepsilon > 0$. We write $a \approx_\varepsilon b$ if $\|a - b\| < \varepsilon$. Suppose that A is another C^*-algebra, $L_1, L_2 : C \to A$ are two maps, and $\mathcal{F} \subset C$ is a subset (usually finite). We write

$$L_1 \approx_\varepsilon L_2 \text{ on } \mathcal{F}, \tag{e 2.47}$$

if $\|L_1(c) - L_2(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.

DEFINITION 2.23. Let A and B be C^*-algebras, and assume that B is unital. Let $\mathcal{H} \subset A_+ \setminus \{0\}$ be a finite subset, and let functions $T : A_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ and $N : A_+ \setminus \{0\} \to \mathbb{N}$ be given. A map $L : A \to B$ is said to be $T \times N$-$\mathcal{H}$-full if for any $h \in \mathcal{H}$, there are $b_1, b_2, ..., b_{N(h)} \in B$ such that $\|b_i\| \le T(h)$ and

$$\sum_{i=1}^{N(h)} b_i^* L(h) b_i = 1_B.$$

We say L is $T \times N$-full, if it is $T \times N$-$\mathcal{H}$-full for every finite $\mathcal{H} \subset A_+ \setminus \{0\}$.

PROPOSITION 2.24. *Let A, T, N be as in Definition 2.23. Let $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset A^{\mathbf{1}}$ be two increasing sequences of finite subsets such that $\bigcup_{n=1}^\infty \mathcal{H}_n$ is dense in the unit ball $A_+^{\mathbf{1}}$ of A_+ and $\bigcup_{n=1}^\infty \mathcal{G}_n$ is dense in the unit ball $A^{\mathbf{1}}$ of A. Suppose also that, if $h \in \mathcal{H}_n$ then $f_{1/2}(h) \in \mathcal{H}_{n+1} \cup \{0\}$. Let $\{\delta_n\}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} \delta_n = 0$, let $\{B_n\}$ be a sequence of unital C^*-algebras, and let $\{\sigma_n : A \to B_n\}$ be a sequence of $\mathcal{G}_n$-δ_n-multiplicative and $T \times N$-$\mathcal{H}_n$-full unital completely positive maps. Then $\{\sigma_n\}$ induces a full homomorphism $\Sigma : A \to Q(B)$, where $B = \prod_{n=1}^\infty B_n$ and $B^0 = \bigoplus_{n=1}^\infty B_n$, and $Q(B) = B/B^0$.*

PROOF. Denote by $S : A \to \prod_{n=1}^\infty B_n$ the map defined by $S(a) = \{\sigma_n(a)\}$ for all $a \in A$. Note that $\Sigma = \pi \circ S$, where $\pi : B \to Q(B)$ is the quotient map. By the properties of $\mathcal{G}_n$ and δ_n, the induced map $\Sigma : A \to Q(B)$ is a homomorphism. To prove Σ is full, we need to prove that for any $h \in A_+ \setminus \{0\}$, $\Sigma(h)$ is full in $Q(B)$. For any $h \in \mathcal{H}_n$, and for any $m \ge n$, there are $b_1^m, b_2^m, ..., b_{N(h)}^m \in B_m$ such that $\|b_i^m\| \le T(h)$ and

$$\sum_{i=1}^{N(h)} (b_i^m)^* \sigma_m(h) b_i^m = 1_{B_m}.$$

Set $b_i = (\underbrace{0, \cdots, 0}_{n-1}, b_i^n, b_i^{n+1}, \cdots)$. Since $\|b_i^n\| \le T(h)$ for all n, $b_i \in B$, $i = 1, 2, ..., N(h)$. Then $\sum_{i=1}^{N(h)} \pi(b_i)^* \Sigma(h) \pi(b_i) = 1_{Q(B)}$. That is, $\Sigma(h)$ is full for any $h \in \mathcal{H}_n$. Now let $a \in A_+$ with $\|a\| = 1$. Then there exists $h \in \mathcal{H}_n$ for some n such that $\|a - h^2\| < 1/16$. It follows from Proposition 2.2 and part (a) of Lemma 2.3 of [104] that there exists $r \in A$ such that $r^* a r = f_{1/2}(h)$. Since

$\|a\| = 1$, $\|h\| > 15/16$. Thus, $f_{1/2}(h) \neq 0$. It follows that $f_{1/2}(h) \in \mathcal{H}_{n+1}$. Since $\Sigma(f_{1/2}(h))$ is full in $Q(B)$, in other words, $\Sigma(r)^*\Sigma(a)\Sigma(r)$ is full, one concludes that $\Sigma(a)$ is full. Thus, $\Sigma(a)$ is full in $Q(B)$ for any $a \in A_+ \setminus \{0\}$. □

2.25. Let A be a unital separable C^*-algebra and U an infinite dimensional UHF-algebra. Write $U = \overline{\bigcup_{n=1}^{\infty} B_n}$, where B_n is a full matrix algebra and $B_{n+1} = B_n \otimes M_{r_n}$ for some integer $r_n \to \infty$ and where B_n is identified with $B_n \otimes 1_{r_n}$ as a C^*-subalgebra of B_{n+1}.

Let $p \in A \otimes U$ be a projection, $r \in (0,1)$, and $\varepsilon > 0$. Choose $n_0 \geq 1$ such that $r_{n_0} > 1/2\varepsilon$. There is a projection $q \in A \otimes B_n$, for some $n \geq n_0$, such that $\|q - p\| < 1/2$. Then there is $k \in \mathbb{N}$ such that $|r - k/r_n| < \varepsilon$ and a projection $e = q \otimes q'$ with $q' \in M_{r_n}$ and $t(q') = k/r_n$, where tr is the unique tracial state of M_{r_n}. Moreover, if $\tau \in T(A)$ then $\tau(e) = \mathrm{tr}(q')\tau(p)$. In other words, $|\tau(e) - r\tau(p)| < \varepsilon$ for all $\tau \in T(A)$. We will use this fact later.

2.26. Let A be a unital C^*-algebra and let $e_1, e_2, ..., e_n$ be mutually orthogonal and mutually equivalent projections in A and set $p = \sum_{i=1}^{n} e_i$. Let $v_i \in pAp$ be partial isometries such that $v_i v_i^* = e_1$ and $v_i^* v_i = e_i$, $i = 1, 2, ..., n$. Then one may identify pAp with $M_n(e_1Ae_1)$. Let C be another C^*-algebra and let $\varphi : C \to e_1Ae_1$ be a map. Define $\varphi_i : C \to e_iAe_i$ by $\varphi_i(c) = v_i^*\varphi(c)v_i$ for all $c \in C$. In this paper, we shall often write

$$\sum_{i=1}^{n} \varphi_i(c) = \mathrm{diag}(\overbrace{\varphi(c), \varphi(c), ..., \varphi(c)}^{n}) = \varphi(c) \otimes 1_n \quad \text{for all } c \in C. \tag{e 2.48}$$

If a, $b \in A$ and $ab = ba = 0$, we will write $a + b = a \oplus b$.

3. The Elliott-Thomsen Building Blocks

To generalize the class of C^*-algebras of tracial rank at most one, it would be natural to consider all subhomogeneous C^*-algebras with one dimensional spectrum which, in particular, include circle algebras as well as dimension drop interval algebras. We begin, however, with the following special class:

DEFINITION 3.1 (See [38] and [31]). Let F_1 and F_2 be two finite dimensional C^*-algebras. Suppose that there are two unital homomorphisms $\varphi_0, \varphi_1 : F_1 \to F_2$. Consider the mapping torus M_{φ_0,φ_1} (see 2.20):

$$\begin{aligned} A &= A(F_1, F_2, \varphi_0, \varphi_1) \\ &= \{(f, g) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_0(g) \text{ and } f(1) = \varphi_1(g)\}. \end{aligned}$$

These C^*-algebras were introduced into the Elliott program by Elliott and Thomsen ([38]), and in [31], Elliott used this class of C^*-algebras and some other building blocks with 2-dimensional spectra to realize any weakly unperforated simple ordered group with order unit as the K_0-group of a simple ASH C^*-algebra. Denote by $\mathcal{C}$ the class of all unital C^*-algebras of the form $A = A(F_1, F_2, \varphi_0, \varphi_1)$ (which includes all finite dimensional C^*-algebras). These C^*-algebras will be called Elliott-Thomsen building blocks.

A unital C^*-algebra $C \in \mathcal{C}$ is said to be *minimal* if it is not the direct sum of two non-zero of C^*-algebras. If $A \in \mathcal{C}$ is minimal and is not finite dimensional, then $\ker\varphi_0 \cap \ker\varphi_1 = \{0\}$. In general, if $A \in \mathcal{C}$, and $\ker\varphi_0 \cap \ker\varphi_1 \neq \{0\}$, then $A = A_1 \oplus (\ker\varphi_0 \cap \ker\varphi_1)$, where $A_1 = A(F_1', F_2, \varphi_0', \varphi_1')$, $F_1 = F_1' \oplus (\ker\varphi_0 \cap \ker\varphi_1)$ and $\varphi_i' = \varphi_i|_{F_1'}$ for $i = 1, 2$. (Note that $A_1 = A(F_1', F_2, \varphi_0', \varphi_1')$ satisfies the condition $\ker\varphi_0' \cap \ker\varphi_1' = \{0\}$, and that $\ker\varphi_0 \cap \ker\varphi_1$ is a finite dimensional C^*-algebra.)

Let $\lambda : A \to C([0,1], F_2)$ be defined by $\lambda((f, a)) = f$. Note that if A is infinite dimensional and minimal, then λ is injective, since $\ker\varphi_0 \cap \ker\varphi_1 = \{0\}$. As in Definition of 2.20, for $t \in (0,1)$, define $\pi_t : A \to F_2$ by $\pi_t((f,g)) = f(t)$ for all $(f,g) \in A$. For $t = 0$, define $\pi_0 : A \to \varphi_0(F_1) \subset F_2$ by $\pi_0((f,g)) = \varphi_0(g)$ for all $(f,g) \in A$. For $t = 1$, define $\pi_1 : A \to \varphi_1(F_1) \subset F_2$ by $\pi_1((f,g)) = \varphi_1(g))$ for all $(f,g) \in A$. In what follows, we will call π_t a point evaluation of A at t. There is a canonical map $\pi_e : A \to F_1$ defined by $\pi_e(f,g) = g$ for all pair $(f,g) \in A$. It is a surjective map. *The notation π_e will be used for this map throughout this paper.*

If $A \in \mathcal{C}$, then A is the pull-back corresponding to the diagram

(e 3.1)
$$\begin{tikzcd} A \arrow[r, dashed, "\lambda"] \arrow[d, dashed, "\pi_e"] & C([0,1], F_2) \arrow[d, "(\pi_0,\pi_1)"] \\ F_1 \arrow[r, "(\varphi_0,\varphi_1)"] & F_2 \oplus F_2 . \end{tikzcd}$$

Conversely, every such pull-back is an algebra in $\mathcal{C}$. Infinite dimensional C^*-algebras in $\mathcal{C}$ are also called *one-dimensional non-commutative finite CW complexes* (NCCW) (see [27] and [28]).

We would like to mention that the C^*-algebras $C([0,1], F_2)$ and $C(\mathbb{T}, F_2)$ are in $\mathcal{C}$. Suppose that F_1, F_2, φ_0 are as mentioned above, and fix a point $t_0 \in [0,1]$. Define

$$B = \{(f,g) \in C([0,1], F_2) \oplus F_1 : f(t_0) = \varphi_0(g)\}.$$

Then one easily verifies that B is also in $\mathcal{C}$.

Denote by $\mathcal{C}_0$ the sub-class of those C^*-algebras A in $\mathcal{C}$ such that $K_1(A) = \{0\}$.

C^*-algebras in $\mathcal{C}$ are finitely generated (Lemma 2.4.3 of [27]) and semiprojective (Theorem 6.22 of [27]). We will use these important features later without further mention.

LEMMA 3.2. *Let $f \in C([0,1], M_k)$ and let $a_0, a_1 \in M_k$ be invertible elements with*

$$\|a_0 - f(0)\| < \varepsilon \text{ and } \|a_1 - f(1)\| < \varepsilon.$$

Then there exists an invertible element $g \in C([0,1], M_k)$ *such that* $g(0) = a_0,\ g(1) = a_1$*, and*

$$\|f(t) - g(t)\| < \varepsilon \qquad \textit{for all } t \in [0,1].$$

PROOF. Let $S \subset M_k$ denote the set consisting of all singular matrices. Then M_k is a $2k^2$-dimensional differential manifold (diffeomorphic to $\mathbb{R}^{2k^2}$), and S is a finite union of closed submanifolds of codimension at least two. Since each continuous map between two differential manifolds (perhaps with boundary) can be approximated arbitrarily well by smooth maps, we can find $f_1 \in C^\infty([0,1], M_k)$ with $f_1(0) = a_0$ and $f_1(1) = a_1$ and $\|f_1(t) - f(t)\| < \varepsilon' < \varepsilon$. Apply the relative version of the transversality theorem—the corollary on page 73 of [49] and its proof (see pages 70 and 68 of [49]), for example, with $Z = S$, $Y = M_k$, $X = [0,1]$ and with $\partial X = \{0,1\}$—to obtain $g \in C^\infty([0,1], M_k)$ with $g(0) = f_1(0)$, $g(1) = f_1(1)$, $g(t) \notin S$, and $\|g(t) - f_1(t)\| < \varepsilon - \varepsilon'$. Hence, $\|f(t) - g(t)\| < \varepsilon$, $t \in [0,1]$. □

PROPOSITION 3.3. *If* $A \in \mathcal{C}$*, then* A *has stable rank one.*

PROOF. Let (f, a) be in A with $f \in C([0,1], F_2)$ and $a \in F_1$ with $f(0) = \varphi_0(a)$ and $f(1) = \varphi_1(a)$. For any $\varepsilon > 0$, since F_1 is a finite dimensional C^*-algebra, there is an invertible element $b \in F_1$ such that $\|b - a\| < \varepsilon$. Since φ_0 and φ_1 are unital, $\varphi_0(b)$ and $\varphi_1(b)$ are invertible. Also,

$$\|\varphi_0(b) - f(0)\| < \varepsilon \ \text{ and } \ \|\varphi_1(b) - f(1)\| < \varepsilon.$$

By Lemma 3.2 (applied to each direct summand of F_2), there exists an invertible element $g \in C([0,1], F_2)$ such that $g(0) = \varphi_0(b)$, $g(1) = \varphi_1(b)$, and

$$\|g - f\| < \varepsilon.$$

This is what was desired. □

3.4. Let $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$, let $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(k)}$ and let $\varphi_0,\ \varphi_1 :\ F_1 \to F_2$ be unital homomorphisms, where $R(j)$ and $r(i)$ are positive integers. Then φ_0 and φ_1 induce homomorphisms

$$\varphi_{0*}, \varphi_{1*} :\ K_0(F_1) = \mathbb{Z}^l \longrightarrow K_0(F_2) = \mathbb{Z}^k$$

represented by matrices $(a_{ij})_{k \times l}$ and $(b_{ij})_{k \times l}$, respectively, where

$$r(i) = \sum_{j=1}^{l} a_{ij} R(j) = \sum_{j=1}^{l} b_{ij} R(j)$$

for $i = 1, 2, ..., k$. Note that a_{ij} and b_{ij} are non-negative integers. The matrices $(a_{ij})_{k \times l}$ and $(b_{ij})_{k \times l}$ will be called the multiplicities of φ_0 and φ_1, respectively.

PROPOSITION 3.5 (see also 2.1 of [114]). *Let F_1 and F_2 be as in 3.4, and let $A = A(F_1, F_2, \varphi_0, \varphi_1)$, where $\varphi_1, \varphi_2 : F_1 \to F_2$ are two unital homomorphisms. Then $K_1(A) = \mathbb{Z}^k/\mathrm{Im}(\varphi_{0*0} - \varphi_{1*0})$ and*

$$K_0(A) \cong \left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_l \end{pmatrix} \in \mathbb{Z}^l \ , \quad \varphi_{0*}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_l \end{pmatrix} = \varphi_{1*}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_l \end{pmatrix} \right\}, \tag{e 3.2}$$

with positive cone $K_0(A) \cap \mathbb{Z}^l_+$, and scale $\begin{pmatrix} R(1) \\ R(2) \\ \vdots \\ R(l) \end{pmatrix} \in \mathbb{Z}^l$*, where*

$$\mathbb{Z}^l_+ = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_l \end{pmatrix} \ ; \quad v_i \geq 0 \right\} \subset \mathbb{Z}^l \ .$$

Moreover, the map $\pi_e : A \to F_1$ induces the natural order embedding $(\pi_e)_{*0} : K_0(A) \to K_0(F_1) = \mathbb{Z}^l$; in particular, $\ker\rho_A = \{0\}$ (see Definition 2.3 for ρ_A).

Furthermore, if $K_1(A) = \{0\}$, then $\mathbb{Z}^l/K_0(A) \cong K_1(C_0((0,1), F_2))$ is torsion free, and in this case, $[\pi_e]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})}$ is injective for all $k \geq 2$ and $i = 0, 1$.

PROOF. Most of these statements are known. We sketch the proof here. Consider the short exact sequence

$$0 \longrightarrow \mathrm{C}_0((0,1), F_2) \longrightarrow A \xrightarrow{\pi_e} F_1 \longrightarrow 0.$$

We obtain the exact sequence

$$0 \longrightarrow K_0(A) \xrightarrow{\pi_{e*}} K_0(F_1) \longrightarrow K_0(F_2) \longrightarrow K_1(A) \longrightarrow 0, \tag{e 3.3}$$

where the map $K_0(F_1) \to K_0(F_2)$ is given by $\varphi_{0*0} - \varphi_{1*0}$. In particular, $(\pi_e)_{*0}$ is injective. If $x \in K_0(A)$ is such that $(\pi_e)_{*0}(x) = [p]$, where $p \in \pi_e(M_m(A))$ is a projection, then, by (e 3.3), $((\varphi_0)_{*0} - (\varphi_1)_{*0})([p]) = 0$, or $(\varphi_0)_{*0}([p]) = (\varphi_1)_{*0}([p])$. Therefore, $\varphi_0(p)$ and $\varphi_1(p)$ have the same rank. It follows that there is a projection $q \in M_m(A)$ such that $\pi_e(q) = p$. Thus, $(\pi_e)_{*0}(x - [q]) = 0$. Since $(\pi_e)_{*0}$ is injective, $x = [q] \in K_0(A)_+$. This implies that $(\pi_e)_{*0}$ is an order embedding. This also implies that $\ker\rho_A = \{0\}$. The descriptions of $K_i(A)$ ($i = 0, 1$) also follow. The quotient group $K_0(F_1)/(\pi_e)_{*0}(K_0(A))$ is always torsion free, since $K_0(F_1)/(\pi_e)_{*0}(K_0(A))$ is a subgroup of $K_0(F_2)$. It happens that $K_0(F_1)/(\pi_e)_{*0}(K_0(A)) = K_0(F_2)$ holds if and only if $K_1(A) = \{0\}$.

In the case that $K_1(A) = \{0\}$, one also computes that $K_0(A, \mathbb{Z}/k\mathbb{Z})$ may be identified with $K_0(A)/kK_0(A)$ and $K_1(A, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ for all $k \geq 2$.

To see that $[\pi_e]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})}$ is injective for $k \geq 2$, let $\bar{x} \in K_0(A, \mathbb{Z}/k\mathbb{Z}) = K_0(A)/kK_0(A)$ and let $x \in K_0(A)$ be such that its image in $K_0(A)/kK_0(A)$ is $\bar{x}$. If $[\pi_e](\bar{x}) = 0$, then $(\pi_e)_{*0}(x) \in kK_0(F_1)$. Let $y \in K_0(F_1)$ be such that $ky = (\pi_e)_{*0}(x)$. Then $k\bar{y} = \overline{(\pi_e)_{*0}(x)} = 0$ in $K_0(F_1)/(\pi_e)_{*0}(K_0(A))$. This implies that $K_0(F_2)$ has torsion, a contradiction. Therefore $[\pi_e]|_{K_0(A/\mathbb{Z}/k\mathbb{Z})}$ is injective for $k \geq 2$. Then the rest of the proposition also follows from (e 3.3). ☐

PROPOSITION 3.6. *For fixed finite dimensional C^*-algebras F_1, F_2, the C^*-algebra $A = A(F_1, F_2, \varphi_0, \varphi_1)$ is completely determined (up to isomorphism) by the maps $\varphi_{0*}, \varphi_{1*} : \mathbb{Z}^l \longrightarrow \mathbb{Z}^k$.*

PROOF. Let $B = A(F_1, F_2, \varphi_0', \varphi_1')$ with $\varphi_{0*}' = \varphi_{0*}, \varphi_{1*}' = \varphi_{1*}$. It is well known that there exist two unitaries $u_0, u_1 \in F_2$ such that

$$u_0\varphi_0(a)u_0^* = \varphi_0'(a), \quad a \in F_1, \text{ and } u_1\varphi_1(a)u_1^* = \varphi_1'(a), \quad a \in F_1.$$

Since $U(F_2)$ is path connected, there is a unitary path $u: [0,1] \to U(F_2)$ with $u(0) = u_0$ and $u(1) = u_1$. Define $\varphi: A \to B$ by

$$\varphi(f, a) = (g, c),$$

where $g(t) = u(t)f(t)u(t)^*$. Then a straightforward calculation shows that the map φ is a *-isomorphism. ☐

3.7. Let $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$ and $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(k)}$ and let $A = (F_1, F_2, \varphi_0, \varphi_1)$. Denote by m the greatest common divisor of $\big\{R(1), R(2), \cdots, R(l)\big\}$. Then each $r(i)$ is also a multiple of m. Let $\widetilde{F}_1 = M_{\frac{R(1)}{m}} \oplus M_{\frac{R(2)}{m}} \oplus \cdots \oplus M_{\frac{R(l)}{m}}$ and $\widetilde{F}_2 = M_{\frac{r(1)}{m}} \oplus M_{\frac{r(2)}{m}} \oplus \cdots \oplus M_{\frac{r(k)}{m}}$. Let $\widetilde{\varphi}_0, \widetilde{\varphi}_1: \widetilde{F}_1 \to \widetilde{F}_2$ be maps such that the K_0-maps

$$\widetilde{\varphi}_{0*}, \widetilde{\varphi}_{1*}: K_0(\widetilde{F}_1) = \mathbb{Z}^l \longrightarrow \mathbb{Z}^k$$

satisfy $\widetilde{\varphi}_{0*} = (a_{ij})_{k\times l}$ and $\widetilde{\varphi}_{1*} = (b_{ij})_{k\times l}$. That is, $\widetilde{\varphi}_{0*}$ and $\widetilde{\varphi}_{1*}$ are the same as φ_{0*} and φ_{1*}. By 3.6,

$$A(F_1, F_2, \varphi_0, \varphi_1) \cong M_m(A(\tilde{F}_1, \widetilde{F}_2, \widetilde{\varphi}_0, \widetilde{\varphi}_1)).$$

3.8. It is well known that the extreme points of $T(A)$ are in canonical one-to-one correspondence with the irreducible representations of A, which are given by

$$\coprod_{j=1}^{k}(0,1)_j \cup \{\rho_1, \rho_2, ..., \rho_l\} = \mathrm{Irr}(A),$$

where each $(0,1)_j$ is the open interval $(0,1)$. (We use the subscript j to indicate the j-th copy.)

It follows from Lemma 2.3 of [114] that the affine function space $\mathrm{Aff}(\mathrm{T}(A))$ can be identified with the subset of

$$\bigoplus_{j=1}^{k} C([0,1]_j,\mathbb{R}) \oplus \underbrace{(\mathbb{R}\oplus\mathbb{R}\oplus\cdots\oplus\mathbb{R})}_{l \quad copies}$$

consisting of the elements $(f_1, f_2, ..., f_k, g_1, g_2, ..., g_l)$ satisfying the conditions

$$f_i(0) = \frac{1}{R(i)}\sum_{j=1}^{l} a_{ij}g_j \cdot r_j \qquad \text{and} \qquad f_i(1) = \frac{1}{R(i)}\sum_{j=1}^{l} b_{ij}g_j \cdot r_j,$$

where $(a_{ij})_{k\times l} = \varphi_{0*}$ and $(b_{ij})_{k\times l} = \varphi_{1*}$ as in 3.4.

Denote by $\tau_{t,j} \in T(A)$ the tracial state defined by $\tau_{t,j}((g,a)) = \mathrm{tr}_j(\pi_j(g(t)))$ for all $(g,a) \in A$, where $t \in (0,1)$, tr_j is the tracial state of $M_{r(j)}$, and $\pi_j : F_2 \to M_{r(j)}$ is the quotient map, $j = 1,2,...,k$. Denote by $\tau_{0,j} \in T(A)$ the tracial state defined by $\tau_{0,j}((g,a)) = \mathrm{tr}_j \circ \pi_j(g(0))$ and $\tau_{1,j}((g,a)) = \mathrm{tr}_j \circ \pi_j(g(1))$ for all $(g,a) \in A$. If $t \to 0$ in $[0,1]$, then $\tau_{t,j} \circ \pi_j(g(t)) \to \tau_{0,j} \circ \pi_j(g(0))$ for all $(g,a) \in A$, $j = 1,2,...,k$, It follows that $\tau_{t,j} \to \tau_{0,j}$ in (the weak* topology of) $T(A)$. For exactly same reason $\tau_{t,j} \to \tau_{1,j}$ in $T(A)$ if $t \to 1$ in $[0,1]$. Let $f = (f_1, f_2, ..., f_k, g_1, g_2, ..., g_l) \in \mathrm{Aff}(T(A))$. Then $f(\tau_{0,j}) = f_j(0)$ and $f(\tau_{1,j}) = f_j(1)$, $j = 1,2,...,k$. Denote by $\tau_{e,i} \in T(A)$ the tracial state defined by $\tau_{e,i}((g,a)) = \mathrm{tr}_{e,i}(\psi_{e,i}(a))$ for all $(g,a) \in A$, where $\mathrm{tr}_{e,i}$ is the tracial state on $M_{R(i)}$ and $\psi_{e,i} : F_1 \to M_{R(i)}$ is the quotient map, $i = 1,2,...,l$.

If $h, h' \in (F_1)_+$ and $\tau(h) = \tau(h')$ for all $\tau \in T(F_1)$, then, $\mathrm{Tr}_j \circ \pi \circ \varphi_0$ and $\mathrm{Tr}_j \circ \pi_j \circ \varphi_1$ are traces of F_1. It follows that

$$\mathrm{Tr}_j(\pi_j(\varphi_0(h))) = \mathrm{Tr}_j(\pi_j(\varphi_0(h')) \text{ and } \mathrm{Tr}_j(\pi_j(\varphi_1(h))) = \mathrm{Tr}_j(\pi_j(\varphi_1(h')),$$

where $\pi_j : F_2 \to M_{r(j)}$ is the quotient map and Tr_j is the standard trace on $M_{r(j)}$, $j = 1,2,...,k$. Furthermore, if $h_1, h_2, .., h_m, , h'_1, h'_2, ..., h'_{m'} \in (F_1)_+$, and $\sum_{n=1}^{m}\tau(h_n) = \sum_{s=1}^{m'}\tau(h'_s)$ for all $\tau \in T(F_1)$, then

$$\text{(e 3.4)} \qquad \begin{aligned} \mathrm{Tr}_j(\pi_j(\sum_{n=1}^{m}\varphi_0(h_n))) &= \mathrm{Tr}_j(\pi_j \circ \varphi_0(\sum_{n=1}^{m}(h_n))) \\ &= \mathrm{Tr}_j(\pi_j \circ \varphi_0(\sum_{s=1}^{m'}(h'_s)) \\ &= \mathrm{Tr}_j(\pi_j(\sum_{s=1}^{m'}\varphi_0(h'_s))), \end{aligned}$$

$j = 1, 2, ..., k$. For exactly the same reason, we have

$$\mathrm{Tr}_j(\pi_j(\sum_{n=1}^{m}\varphi_1(h_n))) = \mathrm{Tr}_j(\pi_j(\sum_{s=1}^{m'}\varphi_1(h'_s))),\ j = 1, 2, ..., k. \tag{e 3.5}$$

Let $f \in \mathrm{LAff}_b(T(A))_+$ (see 2.2) and let $f_n \in \mathrm{Aff}(T(A))_+$ be such that $f_n \nearrow f$. Then

$$\begin{aligned} f(\tau_{0,i}) &= \lim_{n\to\infty} f_n(\tau_{0,i}) = \lim_{n\to\infty}\frac{1}{R(i)}\sum_{j=1}^{l} a_{ij} f_n(\tau_{e,j})\cdot r(j) \\ &= \frac{1}{R(i)}\sum_{j=1}^{l} a_{ij} f(\tau_{e,j})\cdot r(j) \end{aligned} \tag{e 3.6}$$

$$\text{and}\ \ f(\tau_{1,i}) = \frac{1}{R(i)}\sum_{j=1}^{l} b_{ij} f(\tau_{e,j})\cdot r(j),\ i = 1, 2, ..., k. \tag{e 3.7}$$

PROPOSITION 3.9. *Let A be as in Proposition 3.5 and let $u = (f, a)$ be a unitary in A, where $a = (a_1, a_2, ..., a_l) = e^{ih} \in \bigoplus_{j=1}^{l} M_{R(j)} = F_1$ and where $h = (h_1, h_2, ..., h_l) \in (F_1)_{s.a.}$.*

(1) *Then $uv^* \in U_0(A)$ (hence $[u] = [v]$ in $K_1(A)$), where $v = (fe^{-ig}, 1_{F_1})$ and where*

$$g(t) = \begin{cases} (1-2t)\varphi_0(h), & 0 \le t \le \frac{1}{2}, \\ (2t-1)\varphi_1(h), & \frac{1}{2} < t \le 1. \end{cases}$$

(2) *If $\det(u)(\psi) = 1$ for all irreducible representations ψ of A, then $\det(v)(\psi) = 1$ for all ψ.*

(3) *Write $fe^{-ig} = (v_1, v_2, ..., v_k)$, where $v_i \in U(C_0((0,1), F_2)^\sim)$. Then, $[u] = [(s_1, s_2, ..., s_k)] \in \mathbb{Z}^k/((\varphi_0)_{*0} - (\varphi_1)_{(0)}(\mathbb{Z}^k)$, where s_j is the winding number of the map*

$$[0,1] \ni t \mapsto \det(v_j(t)) \in \mathbb{T} \subset \mathbb{C}\,(j = 1, 2, ..., k),$$

(4) *If $\det(u)(\psi) = 1$ for all irreducible representations ψ of A, then $u \in U_0(A)$.*

PROOF. Let $u = (f, a) \in U(A)$. Then $a \in U(F_1)$. Therefore, $a = e^{ih}$ for some $h = (h_1, h_2, ..., h_l) \in \bigoplus_{j=1}^{l} M_{R(j)} = F_1$. Define

$$g(t) = \begin{cases} (1-2t)\varphi_0(h), & 0 \le t \le \frac{1}{2}, \\ (2t-1)\varphi_1(h), & \frac{1}{2} < t \le 1. \end{cases} \tag{e 3.8}$$

Then $(g, h) \in A$. Let $v = (f_1, 1_{F_1})$, where $f_1(t) = f(t)e^{-ig(t)}$. Let $U(s) = (f(t)e^{-ig(t)s}, e^{ih(1-s)})$ for $s \in [0,1]$. Then $\{U(s) : s \in [0,1]\}$ is a continuous path of unitaries in A with $U(0) = u$ and $U(1) = (f_1, 1_{F_1})$. Therefore $uv^* \in U_0(A)$. In particular, $[u] = [v] \in K_1(A)$. This proves (1).

To prove (2), suppose that $\det(u(\psi)) = 1$ for all irreducible representations ψ of A. Then $\det(e^{ih}) = 1$. So we may choose h above so that $\mathrm{tr}_{e,j}(h_j) = 0$, where $\mathrm{tr}_{e,j}$ is the normalized tracial state on $M_{R(j)}$, $j = 1, 2, ..., l$. Write $g(t) = (g_1(t), g_2(t), ..., g_k(t)) \in C([0, 1], F_2)$. Let $\pi_m : F_2 \to M_{r(m)}$ be the quotient map. Then $\pi_m \circ \varphi_i : F_1 \to M_{r(m)}$ is a homomorphism. Write $\varphi_i(h) = (g_1(i), g_2(i), ..., g_k(i))$, $i = 0, 1$, where $g_m(i) \in M_{r(m)}$, $m = 1, 2, ..., k$. Then $\mathrm{tr}_m(g_m(i)) = \mathrm{tr}_m(\varphi_i(h))$, $i = 0, 1$, where tr_m is the normalized trace on $M_{r(m)}$, $m = 1, 2, ..., k$. Since φ_i is unital, $\mathrm{tr}_m \circ \varphi_i$ is a trace on F_1, $i = 0, 1$ and $m = 1, 2, ..., k$. Thus $\mathrm{tr}_m(g_m(i)) = 0$, $i = 0, 1$ and $m = 1, 2, ..., k$. It follows that (see (e 3.8)) $\mathrm{tr}_m(g(t)) = 0$ for all $t \in [0, 1]$ and $1 \le m \le k$. In other words, $\det(e^{-g(t)}) = 1$ for all $t \in [0, 1]$, as well as $\det(e^{ih}) = 1$. This implies $\det(v(\psi)) = 1$ for all irreducible representations ψ of A.

For (3), we may write $fe^{-ig} = (v_1, v_2, ..., v_k)$ with $v_m(0) = v_m(1) = 1_{M_{r(m)}}$, where $v_m \in C([0, 1], M_{r(m)})$, $m = 1, 2, ..., k$. Thus, we may view v as an element of $C_0((0, 1), F_2)^\sim$, the unitalization of the ideal $C_0((0, 1), F_2)$ of A, and write $v = (v_1, v_2, ..., v_k) \in U(C_0((0, 1), F_2)^\sim)$ (as $v = (fe^{-ig}, 1_{F_1})$). Put $w' = (\det(v_1(t)), \det(v_2(t)), ..., \det(v_k(t))) \in U(C([0, 1])$ and $w = \mathrm{diag}(d_1, d_2, ..., d_k) \in U(C_0([0, 1], F_2)^\sim)$, where $d_m = \mathrm{diag}(1_{r(m)-1}, \det(v_m(t))) \in U(C_0((0, 1), M_{r(m)})^\sim)$, $m = 1, 2, ..., k$. Then $w \in U(C_0([0, 1], F_2)^\sim)$. Note $\det(vw^*)(t) = 1$ for all $t \in (0, 1)$. Then in $C_0([0, 1], F_2)^\sim$, $vw^* \in U_0(C_0((0, 1), F_2)^\sim) \subset U_0(A)$.

Let $(s_1, s_2, ..., s_k) \in \mathbb{Z}^k = K_1(C_0((0, 1), F_2))$, where s_j is the winding number of the map

$$[0, 1] \ni t \mapsto \det(v_j(t)) \in \mathbb{T} \subset \mathbb{C}.$$

In $K_1(C_0(0, 1), F_2)$, $[v] = (s_1, s_2, ..., s_k)$. Note that (as $uv^* \in U_0(A)$, $[u]$ is the image of $[v]$ under the map from $K_0(F_2) = K_1(C_0((0, 1), F_2))$ onto $K_1(A)$ given by (e 3.3). Thus

$$[u] = (s_1, s_2, ..., s_k) \in \mathbb{Z}^k/(\varphi_{1*} - \varphi_{0*})(\mathbb{Z}^l) \ .$$

Finally (4) follows from (3) and the facts that A has stable rank one (by 3.3) and stable rank one C^*-algebras are K_1-injective by [100]. □

LEMMA 3.10. *Let $A = A(F_1, F_2, \varphi_0, \varphi_1)$ be as in Definition 3.1. A unitary $u \in U(A)$ is in $CU(A)$ (see the definition 2.16) if, and only if, for each irreducible representation ψ of A, one has $\det(\psi(u)) = 1$.*

PROOF. One direction is obvious. It remains to show that the condition is also sufficient. From Proposition 3.9 above, if $u \in U(A)$ with $\det(\psi(u)) = 1$ for all irreducible representations ψ, then $u \in U_0(A)$.

Write $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$ and $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(k)}$.

Then, since $u \in U_0(A)$, we may write $u = \prod_{n=1}^m \exp(i2\pi h_n)$ for some $h_n \in A_{s.a.}$, $n = 1, 2, ..., m$. We may write $h_n = (h_{nI}, h_{nq}) \in A$, where $h_{nI} \in C([0, 1], F_2)_{s.a.}$ and $h_{nq} \in (F_1)_{s.a.}$ with $\varphi_i(h_{nq}) = h_{nI}(i)$, $i = 0, 1$. Write $h_{nq} =$

$(h_{nq,1}, h_{nq,2}, ..., h_{nq,l})$, where $h_{nq,j} \in (M_{R(j)})_{s.a.}$, $j = 1, 2, ..., l$. Let $\pi_j^{F_1} : F_1 \to M_{R(j)}$ denote the quotient map and set $\pi_{e,j} = \pi_j^{F_1} \circ \pi_e$, $j = 1, 2, ..., l$.

For any irreducible representation ψ of A, as $\det(\psi(u)) = 1$, $\sum_{n=1}^m \mathrm{Tr}_\psi(\psi(h_n)) \in \mathbb{Z}$, where Tr_ψ is the standard (unnormalized) trace on $\psi(A) \cong M_{n(\psi)}$ for some integer $n(\psi)$. By replacing each h_n with $h_n + J$ for a large enough integer J, one may assume that $h_n \in A_+$ and $\sum_{n=1}^m \mathrm{Tr}_\psi(\psi(h_n))$ is positive. Put $H_j(t) = \mathrm{Tr}_j(\sum_{n=1}^m \pi_j(h_{nI}(t)))$ and $H(t) = (H_1(t), H_2(t), ..., H_k(t))$ for $t \in [0, 1]$, where Tr_j is the standard trace on $M_{r(j)}$ and $\pi_j : F_2 \to M_{r(j)}$ is the projection map, for $j = 1, 2, ..., k$. Note that $H_j \in C([0, 1])$ and $H_j(t) \in \mathbb{Z}$ for all $t \in [0, 1]$, $j = 1, 2, ..., k$. It follows that

$$H_j(t) = H_j(0) = H_j(1) \text{ for all } t \in [0, 1], \ \ j = 1, 2, ..., k. \tag{e 3.9}$$

Put $b(\psi) = \sum_{n=1}^m \mathrm{Tr}_\psi(\psi(h_n))$ for $\psi = \pi_{e,j}$, $j = 1, 2, ..., l$.

Since $\sum_{n=1}^m \mathrm{Tr}_\psi(\psi(h_n)) \in \mathbb{Z}$ and is positive, there is a projection $p \in M_N(F_1)$ such that $\mathrm{Tr}'_{\pi_{e,j}}((\mathrm{id}_{M_N} \otimes \pi_j^{F_1})(p)) = b(\pi_{e,j})$ for some $N \geq 1$, where $\mathrm{Tr}'_{\pi_{e,j}} = Tr'_N \otimes \mathrm{Tr}_{\pi_{e,j}}$, and where Tr'_N is the standard trace on M_N. Write $p = (p_1, p_2, ..., p_l)$, where $p_s \in M_N(M_{R(s)})$ is a projection, $s = 1, 2, ..., l$. Then

$$\mathrm{rank}(p_s) = \sum_{n=1}^m \mathrm{Tr}_{e,s}(h_{nq,s}),$$

where $h_{nq} = (h_{nq,1}, h_{nq,2}, ..., h_{nq,l}) \in F_1$, where $h_{nq,s} \in (M_{r(s)})_+$, and where $\mathrm{Tr}_{e,s}$ is the standard trace on $M_{r(s)}$, $s = 1, 2, ..., l$. It follows that from (e 3.4) that

$$\mathrm{Tr}'_j(\pi'_j \circ (\mathrm{id}_{M_N} \otimes \varphi_0)(p)) = \sum_{n=1}^m \mathrm{Tr}_j(\pi_j \circ \varphi_0(h_{nq})) = H_j(0) \text{ and} \tag{e 3.10}$$

$$\mathrm{Tr}'_j(\pi'_j \circ (\mathrm{id}_{M_N} \otimes \varphi_1)(p)) = \sum_{n=1}^m \mathrm{Tr}_j(\pi_j \circ \varphi_1(h_{nq})) = H_j(1), \tag{e 3.11}$$

$j = 1, 2, ..., k$, where Tr'_j is the standard trace on $M_N(M_{r(j)})$ and $\pi'_j : M_N(F_2) \to M_N(M_{r(j)})$ is the projection map. Then, by (e 3.9),

$$\mathrm{Tr}'_j(\pi'_j \circ (\mathrm{id}_{M_N} \otimes \varphi_0)(p)) = H_j(0) = H_j(1) = \mathrm{Tr}'_j(\pi'_j(\mathrm{id}_{M_N} \otimes \varphi_1)(p)),$$

$j = 1, 2, ..., k$. It follows that $\pi'_j(\mathrm{id}_{M_N} \otimes \varphi_0)(p))$ and $\pi'_j(\mathrm{id}_{M_N} \otimes \varphi_1)(p))$ have the same rank. Therefore, there is a projection $P_j \in M_N(C([0, 1], M_{r(j)}))$ such that $P_j(0) = \pi'_j \circ (\mathrm{id}_{M_n} \otimes \varphi_0)(p)$ and $P_j(1) = \pi'_j \circ (\mathrm{id}_{M_n} \otimes \varphi_1)(p)$. Choose $P \in M_N(C([0, 1], F_2))$ such that $\pi'_j(P) = P_j$ and put $e = (P, p)$. Then $e \in M_N(A)$. Note that since $P_j(t)$ has the same rank as $P_j(0)$ (and $P_j(1)$), $\mathrm{Tr}'_j(P_j(t)) = H_j(t) = H_j(0)$ for all $t \in [0, 1]$, $j = 1, 2, ..., k$. Consider the continuous path $u(t) = \prod_{n=1}^m \exp(i2\pi h_n t)$ for $t \in [0, 1]$. Then $u(0) = 1$, $u(1) = u$, and

$$\tau(\frac{du(t)}{dt} u^*(t)) = i2\pi \sum_{n=1}^m \tau(h_n) \text{ for all } \tau \in T(A).$$

But (see Lemma 2.6 of [61]), for all $a \in A$ and $\tau \in T(A)$,

$$\tau(a) = \sum_{s=1}^{l} \alpha_s \mathrm{tr}_{e,s}(\pi_{e,s}(a)) + \sum_{j=1}^{k} \int_{(0,1)} \mathrm{tr}_j(\pi_j(a(t)))d\mu_j(t),$$

where μ_j is a Borel measure on $(0,1)$, $\mathrm{tr}_{e,s}$ is the tracial state on $M_{R(s)}$ and tr_j is the tracial state on $M_{r(j)}$, and $\alpha_i \geq 0$ and $\sum_{s=1}^{l} \alpha_s + \sum_{j=1}^{k} \|\mu_j\| = 1$. Note that

$$\begin{aligned}
\mathrm{tr}_{e,s}(\pi_{e,s}(\sum_{n=1}^{m} h_n)) &= (1/R(s))(\mathrm{Tr}_{\pi_{e,s}}(\pi_{e,s}(\sum_{n=1}^{m} h_n))) \qquad \text{(e 3.12)}\\
&= (1/R(s))b(\pi_{e,s})\\
&= (1/R(s))(\mathrm{Tr}'_{\pi_{e,s}}((\mathrm{id}_{M_N} \otimes \pi_s^{F_1})(p)). \qquad \text{(e 3.13)}
\end{aligned}$$

Thus (recall that we write $\tau(q) = (\tau \otimes Tr_N)(q)$ for $q \in M_N(A)$ as in 2.2)

$$\begin{aligned}
&\sum_{n=1}^{m} \tau(h_n) = \tau(\sum_{n=1}^{m} h_n)\\
&= \sum_{s=1}^{l} \alpha_s \mathrm{tr}_{e,s}(\pi_{e,s}(\sum_{n=1}^{m} h_n)) + \sum_{j=1}^{k} \int_{(0,1)} \mathrm{tr}_j(\pi_j(\sum_{n=1}^{m} h_{nI}(t)))d\mu_j(t)\\
&= \sum_{s=1}^{l} \alpha_s \mathrm{tr}_{e,s}((\mathrm{id}_{M_N} \otimes \pi_s^{F_1})(p)) + \sum_{j=1}^{k} \int_{(0,1)} (1/r(j))H_j(t)d\mu_j(t) \qquad \text{(e 3.14)}\\
&= \sum_{s=1}^{l} \alpha_s \mathrm{tr}_{e,s}((\mathrm{id}_{M_N} \otimes \pi_{e,s})(e)) + \sum_{j=1}^{k} \int_{(0,1)} (1/r(j))\mathrm{Tr}'_j(P_j(t))d\mu_j(t) \qquad \text{(e 3.15)}\\
&= \sum_{s=1}^{l} \alpha_s \mathrm{tr}_{e,s}((\mathrm{id}_{M_N} \otimes \pi_{e,s})(e))\\
&\quad + \sum_{j=1}^{k} \int_{(0,1)} \mathrm{tr}_j((\mathrm{id}_{M_N} \otimes \pi_j)(P(t)))d\mu_j(t) = \tau(e). \qquad \text{(e 3.16)}
\end{aligned}$$

It follows that

$$\frac{1}{2\pi i} \int_0^1 \tau(\frac{du(t)}{dt} u(t)^*)dt = \tau(e) \text{ for all } \tau \in T(A).$$

In other words, $D_A(\{u(t)\}) \in \rho_A(K_0(A))$ (see 2.16). It follows from Lemma 3.1 of [112] and the fact that A has stable rank one that $u \in CU(A)$ (see also Corollary 3.11 of [46]). □

The following statement is known (see [30], [13], and [96]).

LEMMA 3.11. *Let u be a unitary in $C([0,1],M_n)$. Then, for any $\varepsilon>0$, there exist continuous functions $h_j\in C([0,1])_{s.a.}$, $j=1,2,...,n$, such that*

$$\|u-u_1\|<\varepsilon,$$

where $u_1=\exp(2i\pi H)$, $H=\sum_{j=1}^n h_jp_j$ where $\{p_1,p_2,...,p_n\}$ is a set of mutually orthogonal rank one projections in $C([0,1],M_n)$, and $\exp(2i\pi h_j(t))\neq \exp(2i\pi h_k(t))$ if $j\neq k$ for all $t\in(0,1)$, and $u_1(0)=u(0)$ and $u_1(1)=u(1)$.

Furthermore, if $\det(u(t))=1$ for all $t\in[0,1]$, then u_1 can be chosen so that $\det(u_1(t))=1$ for all $t\in[0,1]$.

PROOF. The statement follows from Lemma 1.5 of [96] with $m=2$ and $d=1$, whose proof was inspired by the proof of Theorem 4 of [13] for self adjoint elements (rather than unitaries).

□

LEMMA 3.12. *Let $A=A(F_1,F_2,\varphi_0,\varphi_1)$ be as in Definition 3.1. For any unitary $(f,a)\in U(A)$, $\varepsilon>0$, there is a unitary $(g,a)\in U(A)$ such that $\|g-f\|<\varepsilon$ and, for each block $M_{r(j)}\subset F_2=\bigoplus_{j=1}^k M_{r(j)}$, there are real valued functions $h_1^j,h_2^j,...,h_{r(j)}^j:[0,1]\to\mathbb{R}$ such that $g^j=\sum_{i=1}^{r(j)}h_i^jp_i$ and $\{p_1,p_2,...,p_n\}$ is a set of mutually orthogonal rank one projections in $C([0,1],M_{r(j)})$ and $\exp(2i\pi h_s^j(t))\neq\exp(2i\pi h_{s'}^j(t))$ if $s\neq s'$ for all $t\in(0,1)$. Moreover, if $(f,a)\in CU(A)$, one can choose $(g,a)\in CU(A)$.*

PROOF. By Lemma 3.11, for each unitary $f^j\in C([0,1],M_{r(j)})$, one can approximate f^j by g_1^j to within ε such that $g^j(0)=f^j(0)$, $g^j(1)=f^j(1)$, and for each t in the open interval $(0,1)$, $g^j(t)$ has distinct eigenvalues. If $(f,a)\in U(A)$, then $(g,a)\in U(A)$, too. On combining Lemma 3.11 with Lemma 3.10, the last statement also follows. □

REMARK 3.13. In Lemma 3.12, one may ensure that $h_1^j(0),h_2^j(0),...,h_{r(j)}^j(0)\in[0,1)$, and that, for some $\delta\in(0,1)$, for all $t\in(0,\delta)$,

$$\max\{h_i^j(t);\ 1\le i\le r(j)\}-\min\{h_i^j(t);\ 1\le i\le r(j)\}<1, \tag{e 3.17}$$

and $h_{i_1}^j(t)\neq h_{i_2}^j(t)$ for $i_1\neq i_2$. From the choice of g^j, we know that for any $t\in(0,1)$, $e^{2\pi ih_{i_1}^j(t)}\neq e^{2\pi ih_{i_2}^j(t)}$. That is, $h_{i_1}^j(t)-h_{i_2}^j(t)\notin\mathbb{Z}$. This implies that (e 3.17) in fact holds for all $t\in(0,1)$. Hence, also,

$$\max\{h_i^j(1);\ 1\le i\le r(j)\}-\min\{h_i^j(1);\ 1\le i\le r(j)\}\le 1. \tag{e 3.18}$$

LEMMA 3.14. *Let $A=A(F_1,F_2,\varphi_0,\varphi_1)$ be as in Definition 3.1. For any $u\in CU(A)$, one has $\mathrm{cer}(u)\le 2+\varepsilon$ and $\mathrm{cel}(u)\le 4\pi$. Moreover, there exists a continuous path of unitaries $\{u(t):t\in[0,1]\}\subset CU(A)$ with length at most 4π such that $u(0)=1_A$ and $u(1)=u$.*

PROOF. Case (i): the case that $u = (f, a)$ with $a = 1_{F_1}$. By Lemma 3.12, up to approximation to within an arbitrarily small pre-specified number $\varepsilon > 0$, u is unitarily equivalent to $v = (g, a) \in CU(A)$ with $g = (g_1, g_2, ..., g_k) \in C([0,1], F_2)$, where, for each $t \in (0,1)$, the unitary

$$g_j(t) = \mathrm{diag}\big(e^{2\pi i h_1^j(t)}, e^{2\pi i h_2^j(t)}, ..., e^{2\pi i h_{r(j)}^j(t)}\big)$$

has distinct eigenvalues. (Note that since $f(0) = f(1) = 1 = g(0) = g(1)$, the unitary intertwining the approximation of u and v can be chosen to be 1 at $t = 0, 1$, and therefore, the unitary is in $A = A(F_1, F_2, \varphi_0, \varphi_1)$.) Since $v \in CU(A)$, $\det(\psi(v)) = 1$ for every irreducible representation ψ of A (see Lemma 3.10). Let $N \geq 1$ be an integer such that every irreducible representation ψ of A has rank no more than N. Then, we may assume that $\|u - v\| < \varepsilon < 1/4N\pi$. So $\|uv^* - 1_A\| < \varepsilon$. Write $uv^* = e^{ih}$ for some $h \in A_{s.a.}$ with $\|h\| < 2\arcsin(\varepsilon/2) < 1/2N$. Note $uv^* \in CU(A)$. It follows that $\det(\psi(uv^*)) = 1$ for every irreducible representation ψ of A (see Lemma 3.10). Then, $\mathrm{Tr}_\psi(\psi(h)) \in 2\pi\mathbb{Z}$ for every irreducible representation ψ of A, where Tr_ψ is the standard trace on $\psi(A) = M_{n(\psi)}$ (for some integer $n(\psi) \leq N$). Since $\|h\| < 1/2N$, $|\mathrm{Tr}_\psi(\psi(h))| < 1/2$. It follows that $\mathrm{Tr}_\psi(\psi(h)) = 0$ for every irreducible representation ψ of A. Define $w(t) = e^{iht}v$ $(t \in [0,1])$. Since $\mathrm{Tr}_\psi(\psi(ht)) = 0$ for every irreducible representation ψ of A, $\det(\psi(w(t))) = 1$ for each irreducible representation ψ of A. It follows from Lemma 3.10 that $w(t) \in CU(A)$ for all $t \in [0,1]$. Note that $w(0) = v$ and $w(1) = u$. Moreover, the length of $\{w(t) : t \in [0,1]\}$ is no more than $2\arcsin(\varepsilon/2)$. Therefore, without loss of generality, we may assume that $u = v = (g, a)$ with g as above.

Furthermore, one may assume that

$$h_1^j(0) = h_2^j(0) = \cdots = h_{r(j)}^j(0) = 0.$$

Since $\det(g_j(t)) = 1$ for all $t \in [0,1]$, one has $h_1^j(t) + h_2^j(t) + \cdots + h_{r(j)}^j(t) \in \mathbb{Z}$.

By the continuity of each $h_s^j(t)$, It follows that

$$\sum_{s=1}^{r(j)} h_s^j(t) = 0. \tag{e 3.19}$$

Furthermore, as $h_s^j(1) \in \mathbb{Z}$ (since $g_j(1) = 1_{F_2}$), we know that $h_s^j(1) = 0$ for all $s \in \{1, 2, ..., r(j)\}$. Otherwise, $\min_s\{h_s^j(1)\} \leq -1$ and $\max_s\{h_s^j(1)\} \geq 1$ which implies that

$$\max_s\{h_s^j(1)\} - \min_s\{h_s^j(1)\} \geq 2,$$

and this contradicts Remark 3.13. That is, one has proved that $h = \big((h^1, h^2, \cdots, h^k), 0\big)$, where $h^j(t) = \mathrm{diag}\big(h_1^j(t), h_2^j(t), ..., h_{r(j)}^j(t)\big)$ is an element of $A = A(F_1, F_2, \varphi_0, \varphi_1)$ with $h(0) = h(1) = 0$. As $g = e^{2\pi i h}$, we have $\mathrm{cer}(u) \leq 1 + \varepsilon$. We also have $\mathrm{tr}(h(t)) = 0$ for all $t \in [0,1]$.

It follows from (e 3.19) above and $\max_s\{h_s^j(t)\} - \min_s\{h_s^j(t)\} \le 1$ (see (e 3.18) in Remark 3.13) that

$$h_s^j(t) \subset (-1,1), \quad t \in [0,1],\ s = 1,2,...,r(j)\ .$$

Hence $\|2\pi h\| \le 2\pi$, which implies $\mathrm{cel}(u) \le 2\pi$. Moreover, let $u(s) = \exp(is2\pi h)$. Then $u(0) = 1_A$ and $u(1) = u$. Since $\mathrm{tr}(s2\pi h(t)) = 2s\pi \cdot \mathrm{tr}(h(t)) = 0$ for all $t \in [0,1]$, by Lemma 3.10, one has $u(s) \in CU(A)$ for all $s \in [0,1]$.

Case (ii): The general case. In this case $a = (a^1, a^2, ..., a^l)$ with $\det(a^j) = 1$ for $a^j \in F_1^j$. So $a^j = \exp(2\pi i h^j)$ for $h^j \in F_1^j$ with $\mathrm{tr}(h^j) = 0$ and $\|h^j\| < 1$. Define $H \in A(F_1, F_2, \varphi_0, \varphi_1)$ by

$$H(t) = \begin{cases} \varphi_0(h^1, h^2, ..., h^l) \cdot (1-2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ \\ \varphi_1(h^1, h^2, ..., h^l) \cdot (2t-1), & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

Note $H(\frac{1}{2}) = 0$, $H(0) = \varphi_0(h^1, h^2, ..., h^l)$, and $H(1) = \varphi_1(h^1, h^2, ..., h^l)$, and therefore $H \in A(F_1, F_2, \varphi_0, \varphi_1)$. Moreover, $\mathrm{tr}(H(t)) = 0$ for all t. Then $u' = u \cdot \exp(-2\pi i H) \in A(F_1, F_2, \varphi_0, \varphi_1)$ with $u'(0) = u'(1) = 1_{F_2}$. By Case (i), $\mathrm{cer}(u') \le 1+\varepsilon$ and $\mathrm{cel}(u') \le 2\pi$, and so $\mathrm{cer}(u) \le 2+\varepsilon$ and $\mathrm{cel}(u) \le 2\pi + 2\pi\|H\| \le 4\pi$. Furthermore, we note that $\exp(-2\pi s H) \in CU(A)$ for all s as in Case (i). $\square$

THEOREM 3.15. *Let G be a subgroup of the ordered group $\mathbb{Z}^l$ (with the usual positive cone $\mathbb{Z}_+^l$). Then the semigroup $G_+ = G \cap \mathbb{Z}_+^l$ is finitely generated. In particular, one has the following special case:*

Let $A = A(F_1, F_2, \varphi_1, \varphi_2)$ be in $\mathcal{C}$. Then $K_0(A)_+$ is finitely generated (by its minimal elements); in other words, there are an integer $m \ge 1$ and finitely many minimal projections of $M_m(A)$ such that these minimal projections generate the positive cone $K_0(A)_+$.

PROOF. Recall that an element $e \in G_+ \setminus \{0\}$ is minimal, if $x \in G_+ \setminus \{0\}$ and $x \le e$, then $x = e$. We first show that $G_+ \setminus \{0\}$ has only finitely many minimal elements.

Suppose otherwise that $\{q_n\}$ is an infinite set of minimal elements of $G_+ \setminus \{0\}$. Write $q_n = (m(1,n), m(2,n), ..., m(l,n)) \in \mathbb{Z}_+^l$, where $m(i,n)$ are non-negative integers, $i = 1,2,...,l$, and $n = 1,2,....$ If there is an integer $M \ge 1$ such that $m(i,n) \le M$ for all i and n, then $\{q_n\}$ is a finite set. So we may assume that $\{m(i,n)\}$ is unbounded for some $1 \le i \le l$. There is a subsequence of $\{n_k\}$ such that $\lim_{k\to\infty} m(i,n_k) = \infty$. To simplify the notation, without loss of generality, we may assume that $\lim_{n\to\infty} m(i,n) = \infty$. We may assume that, for some j, the set $\{m(j,n)\}$ is bounded. Otherwise, by passing to a subsequence, we may assume that $\lim_{n\to\infty} m(i,n) = \infty$ for all $i \in \{1,2,...,l\}$. Therefore $\lim_{n\to\infty} m(i,n) - m(i,1) = \infty$. It follows that, for some $n \ge 1$, $m(i,n) > m(i,1)$ for all $i \in \{1,2,...,l\}$. Therefore $q_n \ge q_1$, which contradicts the fact that q_n is minimal. By passing to a subsequence, we may write $\{1,2,...,l\} = N \sqcup B$ in such a way that $\lim_{n\to\infty} m(i,n) = \infty$ if $i \in N$ and $\{m(i,n)\}$ is bounded if $i \in B$.

Therefore $\{m(j,n)\}$ has only finitely many different values for fixed $j \in B$. Thus, by passing to a subsequence again, we may assume that $m(j,n) = m(j,1)$ for fixed $j \in B$. Therefore, for some $n > 1$, $m(i,n) > m(i,1)$ for all n if $i \in N$ and $m(j,n) = m(j,1)$ for all n if $j \in B$. It follows that $q_n \geq q_1$. This is impossible since q_n is minimal. This shows that G_+ has only finitely many minimal elements.

To show that G_+ is generated by these minimal elements, fix an element $q \in G_+ \setminus \{0\}$. If q is not minimal, consider the set of all elements of $G_+ \setminus \{0\}$ which are strictly smaller than q. This set is finite. Choose one which is minimal among them, say p_1. Then p_1 is a minimal element of $G_+ \setminus \{0\}$, as otherwise there is one smaller than p_1. Since q is not minimal, $q \neq p_1$. Consider $q - p_1 \in G_+ \setminus \{0\}$. If $q - p_1$ is minimal, then $q = p_1 + (q - p_1)$. Otherwise, we repeat the same argument to obtain a minimal element $p_2 \leq q - p_1$. If $q - p_1 - p_2$ is minimal, then we have the decomposition $q = p_1 + p_2 + (q - p_1 - p_2)$. Otherwise we repeat the argument again. This process is finite. Therefore q is a finite sum of minimal elements of $G_+ \setminus \{0\}$. ☐

THEOREM 3.16. *The exponential rank of* $A = A(F_1, F_2, \varphi_0, \varphi_1)$ *is at most* $3+\varepsilon$.

PROOF. For each unitary $u \in U_0(A)$, as in Proposition 3.9, $u = (f, a) \in A$ where $a = e^{ih}$ for some $h \in (F_2)_{s.a.}$. Therefore there is $x \in A_{s.a.}$ such that $\pi_e(x) = h$ and hence $\pi_e(e^{ix}) = e^{ih} = a$. Therefore one may write $u = ve^{ix}$ for some $x \in A_{s.a.}$, where $v = (g, 1_{F_1})$ with $g(0) = g(1) = 1_{F_2} \in F_2$. So we only need to prove the exponential rank of v is at most $2 + \varepsilon$. Consider v as an element in $C_0((0,1), F_2)^{\sim}$ which defines an element $(s_1, s_2, ..., s_k) \in \mathbb{Z}^k = K_1(C_0((0,1), F_2))$. Since $[v] = 0$ in $K_1(A)$, there are $(m_1, m_2, ..., m_l) \in \mathbb{Z}^l$ such that

$$(s_1, s_2, ..., s_k) = ((\varphi_1)_{*0} - (\varphi_0)_{*0})((m_1, m_2, ..., m_l)).$$

Note that

$$\begin{aligned}(\varphi_0)_{*0}((R(1), R(2), ..., R(l))) &= (\varphi_1)_{*0}((R(1), R(2), ..., R(l))) \\ &= (r(1), r(2), ..., r(k)) = [1_{F_2}] \in K_0(F_2).\end{aligned}$$

Increasing $(m_1, m_2, ..., m_l)$ by adding a positive multiple of $\big(R(1), R(2), ..., R(l)\big)$, we can assume $m_j \geq 0$ for all $j \in \{1, 2, ..., l\}$. Let $a = (m_1 P_1, m_2 P_2, ..., m_l P_l)$, where

$$P_j = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \in M_{R(j)} \subset F_1.$$

Let h be defined by

$$h(t) = \begin{cases} \varphi_0(a)(1-2t), & 0 \leq t \leq \frac{1}{2}, \\ \varphi_1(a)(2t-1), & \frac{1}{2} < t \leq 1. \end{cases}$$

Then (h, a) defines a self-adjoint element in A. One also has $e^{2\pi i h} \in C_0((0,1), F_2)^{\sim}$, since $e^{2\pi i h(0)} = e^{2\pi i h(1)} = 1_{F_2}$. Furthermore, $e^{2\pi i h}$ defines

$$(\varphi_{1*} - \varphi_{0*})((m_1, m_2, ..., m_l)) \in \mathbb{Z}^k$$

as an element in $K_1(C_0((0,1), F_2)) = \mathbb{Z}^k$. Let $w = ve^{-2\pi i h}$. Then w satisfies $w(0) = w(1) = 1_{F_2}$ and $w \in C_0(\widetilde{(0,1)}, F_2)$ defines the element

$$(0, 0, ..., 0) \in K_1(C_0((0,1), F_2)).$$

Up to an approximation to within a sufficiently small ε, by Lemma 3.12, one may assume that $w = (w_1, w_2, ..., w_k)$ with, for all $j = 1, 2, ..., k$,

(1) $w_j(t) = \mathrm{diag}\big(e^{2\pi i h_1^j(t)}, e^{2\pi i h_2^j(t)}, ..., e^{2\pi i h_{r(j)}^j(t)}\big)$,
(2) the numbers $e^{2\pi i h_1^j(t)}, e^{2\pi i h_2^j(t)}, ..., e^{2\pi i h_{r(j)}^j(t)}$ are distinct for all $t \in (0,1)$, and
(3) $h_1^j(0) = h_2^j(0) = \cdots = h_{r(j)}{}^j(0) = 0$.

Since $w_j(1) = 1_{F_2}$, one has that $h_i^j(1) \in \mathbb{Z}$.

On the other hand, the unitary w defines

$$h_1^j(1) + h_2^j(1) + \cdots + h_{r(j)}^j(1) \in \mathbb{Z} \cong K_1(C_0((0,1), F_2^j))$$

which is zero since $h_1^j(0) = h_2^j(0) = \cdots = h_{r(j)}{}^j(0) = 0$. From (e 3.18), one has $h_{i_1}^j(1) - h_{i_2}^j(1) \le 1$ for $i_1 \ne i_2$. This implies

$$h_1^j(1) = h_2^j(1) = \cdots = h_{r(j)}^j(1) = 0.$$

Hence, $h = \big((h^1, h^2, ..., h^k), 0\big)$ defines a selfadjoint element of A and $w = e^{2\pi i h}$. $\square$

3.17. Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$, where $F_1 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(l)}$ and $F_2 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(k)}$. By 3.4, we may write $(\varphi_0)_{*0} = (a_{ij})_{k\times l}$ and $(\varphi_1)_{*0} = (b_{ij})_{k\times l}$. Denote by $\pi_i : F_2 \to M_{r(i)}$ the projection map. Let us calculate the Cuntz semigroup of A.

For any $h \in (A \otimes \mathcal{K})_+$, consider the map $D_h : \mathrm{Irr}(A) \to \mathbb{Z}_+ \cup \{\infty\}$, for any $\pi \in \mathrm{Irr}(A)$

$$D_h(\pi) = \lim_{n\to\infty} \mathrm{Tr}((\pi \otimes \mathrm{id}_{\mathcal{K}})(h^{\frac{1}{n}}))\ ,$$

where Tr is the standard unnormalized trace. Then we write

$$D_h = (D_h^1, D_h^2, ..., D_h^k, D_h(\rho_1), D_h(\rho_2), ..., D_h(\rho_l)),$$

where $D_h^i = D_h|_{(0,1)_i}$ satisfy the following conditions:

(1) D_h^j is lower semi-continuous on each $(0,1)_j$,
(2) $\liminf_{t\to 0} D_h^i(t) \ge \sum_j a_{ij} D_h(\rho_j)= \mathrm{rank}(\pi_i(\varphi_0(h)))$ and
$\liminf_{t\to 1} D_h^i(t) \ge \sum_j b_{ij} D_h(\rho_l)= \mathrm{rank}(\pi_i(\varphi_1(h)))$.

It is straightforward to verify that the image of the map $h \in (A\otimes \mathcal{K})_+ \to D_h$ is the subset of $\mathrm{Map}(\mathrm{Irr}(A), \mathbb{Z}_+\cup\{\infty\})$ consisting elements satisfying the above two conditions.

Note that $D_h(\pi) = \mathrm{rank}((\pi\otimes \mathrm{id}_{\mathcal{K}})(h))$ for each $\pi \in \mathrm{Irr}(A)$ and $h \in (A\otimes\mathcal{K})_+$.

The following result is well known to experts (for example, see [15]). We also keep the notation above in the following statement.

THEOREM 3.18. *Let $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ and let $n \ge 1$ be an integer.*
(a) *The following statements are equivalent:*

(1) *$h \in M_n(A)_+$ is Cuntz equivalent to a projection;*
(2) *0 is an isolated point in the spectrum of h;*
(3) *D_h^i is continuous on each $(0,1)_i$, $\lim_{t\to 0} D_h^i(t) = \sum_j a_{ij} D_h(\rho_j)$, and $\lim_{t\to 1} D_h^i(t) = \sum_j b_{ij} D_h(\rho_j)$.*

(b) *For $h_1, h_2 \in M_n(A)_+$, $h_1 \lesssim h_2$ if and only if $D_{h_1}(\pi) \le D_{h_2}(\pi)$ for each $\pi \in \mathrm{Irr}(A)$. In particular, A has strict comparison for positive elements.*

PROOF. For part (b), without loss of generality, by Subsection 3.7, we may assume that $h_1, h_2 \in A$. Obviously, $h_1 \lesssim h_2$ implies $D_{h_1}(\varphi) \le D_{h_2}(\varphi)$ for each $\varphi \in \mathrm{Irr}(A)$. Conversely, assume that $h_1 = (f, a)$ and $h_2 = (g, b)$ satisfy $D_{h_1}(\varphi) \le D_{h_2}(\varphi)$ for each $\varphi \in \mathrm{Irr}(A)$.

First, there are strictly positive functions $s_1, s_2 \in C_0((0,1])$ such that $s_1(a)$ and $s_2(b)$ are projections in F_1. Note that $s_i(h_i)$ are Cuntz equivalent to h_i $(i = 1,2)$. By replacing h_i by $s_i(h_i)$, without loss of generality we may assume that a and b are projections.

Let $\pi_i : F_2 \to M_{R_i}$ be the projection map. Fix $1/4 > \varepsilon > 0$. There exists $\delta_1 > 0$ such that

$$\|f(t) - \varphi_0(a)\| < \varepsilon/64 \qquad \text{for all } t \in [0, 2\delta_1]$$

and

$$\|f(t) - \varphi_1(a)\| < \varepsilon/16 \qquad \text{for all } t \in [1-2\delta_1, 1].$$

It follows that $f_{\varepsilon/8}(f(t))$ is a projection in $[0, 2\delta_1]$ and $[1-2\delta_1, 1]$. Put $h_0 = f_{\varepsilon/8}(h_1)$. Note that $h_0 = (f_{\varepsilon/8}(f), a)$. Then $D^i_{h_0}(\pi)$ is constant in $(0, 2\delta_1] \subset (0,1)_i$ and $[1-2\delta_1, 1] \subset (0,1)_i$, $i = 1, 2, ..., k$. Choose $\delta_2 > 0$ such that

$$\begin{aligned} &D^i_{h_2}(t) \ge \mathrm{rank}(\pi_i(\varphi_0(b))) \ \text{ for all } \ t \in (0, 2\delta_2]_i \ \text{ and} \\ &D_{h_2}(t) \ge \mathrm{rank}(\pi_i(\varphi_1(b))) \ \text{ for all } \ t \in [2\delta_2, 1)_i, \end{aligned} \tag{e 3.20}$$

$i = 1, 2, ..., k$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Since $a \lesssim b$ in F_1, there is a unitary $u_e \in F_1$ such that $u_e^* a u_e = q \le b$, where $q \le b$ is a subprojection. Let $u_0 = \varphi_0(u_e)$

and $u_1 = \varphi_1(u_e) \in F_2$. Then one can find a unitary $u \in A$ such that $u(0) = u_0$ and $u(1) = u_1$. Without loss of generality, replacing h_0 by u^*h_0u, we may assume that $a = q\leq b$.

Let $\beta : [0,1] \to [0,1]$ be a continuous function which is 1 on the boundary and 0 on $[\delta, 1-\delta]$. Let $h_3 = (f_2, b-a)$ with $f_2(t) = \beta(t)\varphi_0(b-a)$ for $t \in [0,\delta]$, $f_2(t) = \beta(t)\varphi_1(b-a)$ for $t \in [1-\delta, \delta]$, and $f_2(t) = 0$ for $t \in [\delta, 1-\delta]$. Define $h_1' = h_0 + h_3$. Note that h_1' has the form (f', b) for some $f' \in C([0,1], F_2)$.

Then, for any $\pi \in (\delta, 1-\delta) \subset (0,1)_i$,

$$h_0 \leq h_1',\ \ D^i_{h_1'}(\pi) = D^i_{h_0}(\pi),$$

for any $\pi \in (0,\delta) \subset (0,1)_i$,

$$\begin{aligned}
\mathrm{rank}(h_1'(\pi)) &\leq \mathrm{rank}(h_0(t)) + \mathrm{rank}(h_3(t))\\
&\leq \mathrm{rank}(\pi_i(\varphi_0(a))) + \mathrm{rank}(\pi_i(\varphi_0(b-a)))\\
&= \mathrm{rank}(\pi_i(\varphi_0(b))) \leq D^i_{h_2}(\pi),
\end{aligned}$$

and for any $\pi \in (1-\delta, 1) \subset (0,1)_i$,

$$\begin{aligned}
\mathrm{rank}(h_1'(\pi)) &\leq \mathrm{rank}(h_0(t)) + \mathrm{rank}(h_3(t))\\
&\leq \mathrm{rank}(\pi_i(\varphi_1(a))) + \mathrm{rank}(\pi_i(\varphi_1(b-a)))\\
&= \mathrm{rank}(\pi_i(\varphi_1(b))) \leq D^i_{h_2}(\pi).
\end{aligned}$$

It follows that

$$D_{h_1'}(\pi) \leq D_{h_2}(\pi) \text{ for all } \pi \in \mathrm{Irr}(A). \tag{e 3.21}$$

It follows from (e 3.21) and Theorem 1.1 of [102] that $h_1' \lesssim h_2$ in $C([0,1], F_2)$. Since $C([0,1], F_2)$ has stable rank one, there is a unitary $w \in C([0,1], F_2)$ such that $w^*h_1'w = h_4 \in \overline{h_2C([0,1], F_2)h_2}$. Since $h_1' = (f', b)$, $h_2 = (g, b)$, and b is a projection, $h_4(0) = \varphi_0(b)$ and $h_4 = \varphi_1(b)$. In other words, $h_4 \in A$. In particular, $h_4 \in \overline{h_2Ah_2}$ and $h_4 \lesssim h_2$. Note that $w^*\varphi_i(b)w = \varphi_i(b)$, $i = 0, 1$. By using a continuous path of unitaries which commutes with $\varphi_i(b)$ $(i = 0, 1)$ and connects to the identity, it is easy to find a sequence of unitaries $u_n \in C([0,1], F_2)$ with $u_n(0) = u_n(1) = 1_{F_2}$ such that

$$\lim_{n\to\infty} u_n^*h_1'u_n = h_4. \tag{e 3.22}$$

Notice that $u_n \in A$. We also have that $u_nh_4u_n^* \to h_1'$. Thus, $h_1' \lesssim h_4$. It follows that

$$f_\varepsilon(h_1) \lesssim h_0 \lesssim h_1' \lesssim h_4 \lesssim h_2 \tag{e 3.23}$$

for all $\varepsilon > 0$. This implies that $h_1 \lesssim h_2$ and part (b) follows.

To prove part (a), we note that (1) and (2) are obviously equivalent and both imply (3). That (3) implies (1) follows from the computation of $K_0(A)$ in 3.5 and part (b).

□

LEMMA 3.19. *Let $C \in \mathcal{C}$, and let $p \in C$ be a projection. Then $pCp \in \mathcal{C}$. Moreover, if p is full and $C \in \mathcal{C}_0$, then $pCp \in \mathcal{C}_0$.*

PROOF. We may assume that C is finite dimensional. Write $C = C(F_1, F_2, \varphi_0, \varphi_1)$. Denote by $p_e = \pi_e(p)$, where $\pi_e : C \to F_1$ is the map defined in 3.1.

For each $t \in [0,1]$, write $\pi_t(p) = p(t)$ and $\widetilde{p} \in C([0,1], F_2)$ such that $\pi_t(\widetilde{p}) = p(t)$ for all $t \in [0,1]$. Then $\varphi_0(p_e) = p(0)$, $\varphi_1(p_e) = p(1)$, and

$$pCp = \{(f,g) \in C : f(t) \in p(t)F_2p(t), \text{ and } g \in p_eF_1p_e\}. \tag{e 3.24}$$

Put $p_0 = p(0)$. There is a unitary $W \in C([0,1], F_2)$ such that $(W^*\widetilde{p}W)(t) = p_0$ for all $t \in [0,1]$. Define $\Phi : \widetilde{p}C([0,1], F_2)\widetilde{p} \to C([0,1], p_0F_2p_0)$ by $\Phi(f) = W^*fW$ for all $f \in \widetilde{p}C([0,1], F_2)\widetilde{p}$. Put $F_1' = p_1F_1p_1$ and $F_2' = p_0F_2p_0$. Define $\psi_0 = \mathrm{Ad}\, W(0) \circ \varphi_0|_{F_1'}$ and $\psi_1 = \mathrm{Ad}\, W(1) \circ \varphi_1|_{F_1'}$. Put

$$C_1 = \{(f,g) \in C([0,1], F_2') \oplus F_1' : f(0) = \psi_0(g) \text{ and } f(1) = \psi_1(g)\},$$

and note that $C_1 \in \mathcal{C}$. Define $\Psi : pCp \to C_1$ by

$$\Psi((f,g)) = (\Phi(f), g) \text{ for all } f \in \widetilde{p}C([0,1], F_2)\widetilde{p} \text{ and } g \in F_1'. \tag{e 3.25}$$

It is readily verified that Ψ is an isomorphism.

If p is full and $C \in \mathcal{C}_0$, then, by Brown's theorem ([9]), the hereditary C^*-subalgebra pCp is stably isomorphic to C, and hence $K_1(pCp) = K_1(C) = \{0\}$; that is, $pCp \in \mathcal{C}_0$. ☐

The classes $\mathcal{C}$ and $\mathcal{C}_0$ are not closed under passing to quotients. However, we have the following approximation rersult:

LEMMA 3.20. *Any quotient of a C^*-algebra in $\mathcal{C}$ (or in $\mathcal{C}_0$) can be locally approximated by C^*-algebras in $\mathcal{C}$ (or in $\mathcal{C}_0$). More precisely, let $A \in \mathcal{C}$ (or $A \in \mathcal{C}_0$), let B be a quotient of A, let $\mathcal{F} \subset B$ be a finite set, and let $\varepsilon > 0$. There exists a unital C^*-subalgebra $B_0 \subset B$ with $B_0 \in \mathcal{C}$ (or $B_0 \in \mathcal{C}_0$) such that*

$$\mathrm{dist}(x, B_0) < \varepsilon \text{ for all } x \in \mathcal{F}.$$

PROOF. Let $A \in \mathcal{C}$. We may consider only those A's which are minimal and are not finite dimensional. Let I be an ideal of A. Write $A = A(E, F, \varphi_0, \varphi_1)$, where $E = E_1 \oplus \cdots \oplus E_l$, $E_i \cong M_{k_i}$, and where $F = F_1 \oplus \cdots \oplus F_s$ with $F_j \cong M_{m_j}$. Let $J = \{f \in C([0,1], F) : f(0) = f(1) = 0\} \subset A$. As before, we may write $[0,1]_j$ for the the spectrum of the j-th summand of $C([0,1], F_j)$, whenever it is convenient. Put $\varphi_{i,j} = \pi_j \circ \varphi_i : E \to F_j$, where $\pi_j : F \to F_j$ is the quotient map, $i = 0, 1$. Then A/I may be written (with a re-indexing) as

$$\begin{aligned}\{(f,a) : a \in \widetilde{E}, f \in \bigoplus_{1 \le j \le s'} C(\tilde{I}_j, F_j), f(0_j) &= \widetilde{\varphi}_{0,j}(a), \text{ if } 0_j \in \tilde{I}_j, f(1_j) \\ &= \widetilde{\varphi}_{1,j}(a), \text{if } 1_j \in \tilde{I}_j\}, \quad (*)\end{aligned}$$

where $s' \le s$, $l' \le l$, $\widetilde{E} = \bigoplus_{i=1}^{l'} E_i$ and $\widetilde{\varphi}_{i,j} = \varphi_{i,j}|_{\widetilde{E}}$ and $\tilde{I}_j \subset [0,1]_j$ is a compact subset.

It follows from [42] that, for any j, there is a sequence of spaces $X_{n,j}$ which are finite disjoint unions of closed intervals (including points) such that $\tilde{I}_j$ is the inverse limit of a sequence $(X_{n,j}, s_{n,n+1,j})$ and each map $s_{n,n+1,j} : X_{n+1,j} \to X_{n,j}$ is surjective and continuous. Moreover, $X_{n,j}$ can be identified with a disjoint union of closed subintervals of $[0,1]$. Let $s_{n,j} : \tilde{I}_j \to X_{n,j}$ be the surjective continuous map induced by the inverse limit system.

We may assume that $\tilde{I}_j = [0,1]_j$, $j = 1, 2, ..., t' \le s'$, and $\tilde{I}_j = \tilde{I}_j^- \sqcup \tilde{I}_j^+$, where $\tilde{I}_j^- \subset [0, t_j^-]$ and $\tilde{I}_j^+ \subset [t_j^+, 1]$ are compact subsets for some $0 \le t_j^- < t_j^+ \le 1$, $t' < j \le s'$. Without loss of generality (by applying [42] to $\tilde{I}_j^-$ and to $\tilde{I}_j^+$), we may assume that, for $j > t'$, each of the disjoint closed interval in $X_{n,j}$ contains at most one of $s_{n,j}(0_j)$ and $s_{n,j}(1_j)$. Let $s_n^* : \bigoplus_{j=1}^{s'} C(X_{n,j}, F_j) \to \bigoplus_{j=1}^{s'} C(\tilde{I}_j, F_j)$ be the map induced by $s_{n,j}$. Put $C_j = C(X_{n,j}, F_j)$. We may write $C(\tilde{I}_j, F_j) = \overline{\bigcup_{n=1}^{\infty} s_{n,j}^*(C(X_{n,j}, F_j))}$. Then, for all sufficiently large n, for each $f \in \mathcal{F}$, there is $g \in \bigoplus_{j=1}^{s'} C_j$ such that $g(s_{n,j}(0_j)) = f(0_j)$, if $0_j \in \tilde{I}_j$ and $g(s_{n,j}(1_j)) = f(1_j)$, if $1_j \in \tilde{I}_j$, and

$$\|f|_{\tilde{I}_j} - s_{n,j}^*(g|_{X_{n,j}})\| < \varepsilon/4. \tag{e 3.26}$$

Note that C_j is a unital C^*-subalgebra of $C(\tilde{I}_j, F_j)$, $j = 1, 2, ..., s$. Define

$$\begin{aligned} C &= \{(f,a) : f \in \bigoplus_{j=1}^{s'} C_j,\, a \in \widetilde{E}, f(s_{n,j}(0_j)) = \widetilde{\varphi}_{0,j}(a), \text{ if } 0_j \in \tilde{I}_j \\ &\qquad \text{and } f(s_{n,j}(1_j)) = \widetilde{\varphi}_{1,j}(a), \text{if } 1_j \in \tilde{I}_j\}. \end{aligned}$$

Then g in (e 3.26) is in C and $\mathcal{F} \subset_{\varepsilon} C$. Since each C_j is a C^*-subalgebra of $C(\tilde{I}_j, F_j)$, one sees that C is a C^*-subalgebra of A/I. Note, $X_{n,j}$ is either equal to $[0,1]_j$, or, $s_{n,j}(0_j)$ and $s_{n,j}(1_j)$ are in different disjoint intervals of $X_{n,j}$. It is then easy to check that $C \in \mathcal{C}$ (see 3.1). This proves the lemma in the case that $A \in \mathcal{C}$.

Now suppose that $A \in \mathcal{C}_0$. We will show that $K_1(A/I) = \{0\}$. Consider the following six-term exact sequence:

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d] \\
K_1(A/I) \arrow[u, "\delta_1"] & K_1(A) \arrow[l] & K_1(I). \arrow[l]
\end{tikzcd}$$

By Lemma 3.3, A and A/I have stable rank one. It follows from Proposition 4 of [86] that $\delta_1 = 0$. Since $K_1(A) = \{0\}$, it follows that $K_1(A/I) = \{0\}$.

Denote by $\Pi : A \to A/I$ the quotient map. Set $\widetilde{F} = \bigoplus_{i=1}^{s'} F_j$ and $\widetilde{J} = \Pi(J) = \Pi(C_0((0,1),F))$. Note, since $K_1(C_0(\tilde{I}_j^- \setminus \{s_{n,j}(0_j)\}, F_j)) = \{0\}$ and $K_1(C_0(\tilde{I}_j^+ \setminus \{s_{n,j}(1_j)\}, F_j)) = \{0\}$ (for $t' < j \leq s'$), one obtains $K_1(J) = K_1(\bigoplus_{i=1}^{t'} C_0((0,1)_j, F_j)) \cong K_0(\bigoplus_{i=1}^{t'} F_j)$. Consider the short exact sequence $0 \to J \to A/I \to \widetilde{E} \to 0$. Note that $K_1(\widetilde{E}) = 0$. Thus we have

$$0 \to K_0(\widetilde{J}) \to K_0(A/I) \to K_0(\widetilde{E}) \to K_1(\widetilde{J}) \to K_1(A/I) \to K_1(\widetilde{E}) = 0.$$

The fact that $K_1(A/I) = \{0\}$ implies that the map $\oplus_{j=1}^{t'}((\widetilde{\varphi}_{0,j})_{*0} - (\widetilde{\varphi}_{1,j})_{*1})$ from $K_0(\widetilde{E})$ to $K_1(\widetilde{J}) \cong K_0(\bigoplus_{j=1}^{t'} F_j)$ is surjective.

Now let $J_0 = \widetilde{J} \cap C$. Then $K_1(J_0) = K_1(\widetilde{J}) \cong K_0(\bigoplus_j^{t'} F_j)$. The short exact sequence $0 \to J_0 \to C \to \tilde{E} \to 0$ gives the exact sequence

$$0 \to K_0(J_0) \to K_0(C) \to K_0(\widetilde{E}) \to K_1(J_0) \to K_1(C) \to K_1(\widetilde{E}) = 0.$$

The map from $K_0(\widetilde{E})$ to $K_1(J_0)$ in the diagram above is the same as $\oplus_{j=1}^{t'}((\widetilde{\varphi}_{0,j})_{*0} - (\widetilde{\varphi}_{1,j})_{*1})$ which is surjective. It follows that $K_1(C) = 0$. The lemma follows. $\square$

We would like to return briefly to the beginning of this section by stating the following proposition which will not be used.

PROPOSITION 3.21 (Theorem 2.15 of [37]). *Let A be a unital C^*-algebra which is a subhomogeneous C^*-algebra with one dimensional spectrum. Then, for any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there exists a unital C^*-subalgebra B of A in the class $\mathcal{C}$ such that*

$$\operatorname{dist}(x, B) < \varepsilon \textit{ for all } x \in \mathcal{F}.$$

PROOF. We use the fact that A is an inductive limit of C^*-algebras in $\mathcal{C}$ (Theorem B and Definition 1.3 of [37]). Therefore, there is a C^*-algebra $C \in \mathcal{C}$ and a unital homomorphism $\varphi : C \to A$ such that

$$\operatorname{dist}(x, \varphi(C)) < \varepsilon/2 \text{ for all } x \in \mathcal{F}.$$

Then we apply Lemma 3.20. $\square$

In [88], the following type of unital C^*-algebras (dimension drop circle algebra) is studied:

$$A = \{f \in C(\mathbb{T}, M_n) : f(x_i) \in M_{d_i},\ i = 1, 2, ..., N\}, \tag{e 3.27}$$

where $\{x_1, x_2, ..., x_N\}$ are distinct points in $\mathbb{T}$, and $d_i \in \mathbb{N}$ such that $m_i d_i = n$ for some integer $m_i > 1$.

The following fact will be used later in the paper.

PROPOSITION 3.22. *Every dimension drop circle algebra is in $\mathcal{C}$.*

PROOF. Let A be as in (e 3.27). We identify $\mathbb{T}$ with the unit circle of the plane. Without loss of generality, we may assume that $x_i \neq 1 \in \mathbb{C}$, $i = 1, 2, ..., N$. Then,

$$A \cong \{f \in C([0,1], M_n) : f(0) = f(1), f(x_i) \in M_{d_i}, \, i = 1, 2, ..., N\}.$$

We write $\bigoplus_{j=0}^{N} C([0,1], M_n) = \bigoplus_{j=0}^{N} C(I_j, M_n)$, where $I_j = [0,1]$ and we identify the end points of I_j as 0_j and 1_j, $j = 0, 1, ..., N$. We may further write

$$\begin{aligned} A \cong \{f \in \bigoplus_{j=1}^{N+1} C(I_j, M_n) : f(0_0) = f(1_N), \; f(1_j) \; &= \; f(0_{j+1}) \in M_{d_j}, \\ & \qquad j = 1, 2, ..., N\}. \end{aligned} \tag{e 3.28}$$

Let $h_i : M_{d_i} \to M_n$ be defined by $h_i(a) = a \otimes 1_{M_{m_i}}$ for all $a \in M_{d_i}$, $i = 1, 2, ..., N$. Consider $F_1 = M_n \oplus (\bigoplus_{j=1}^{N} M_{d_i})$ and $F_2 = \bigoplus_{i=0}^{N+1} S_i$, where $S_i \cong M_n$. Let $\pi_0 : F_1 \to M_n$ and $\pi_i : F_1 \to M_{d_i}$, $i = 1, 2, ..., N$, denote the quotient maps. Also let $\pi^i : F_2 \to S_i$ denote the quotient map, $i = 0, 1, ..., N$. Define $\varphi_0 : F_1 \to F_2$ by $(\pi^0 \circ \varphi_0)|_{M_n} = \mathrm{id}_{M_n}$, $(\pi^i \circ \varphi_0)|_{M_n} = 0$, if $i \neq 0$, $(\pi^i \circ \varphi_0)|_{M_{d_i}} = h_i$, and $\pi^j \circ \varphi_0|_{M_{d_i}} = 0$, $j \neq i$, $i = 1, 2, ..., N$, and define $\varphi_1 : F_1 \to F_2$ by $(\pi^N \circ \varphi_1)|_{M_n} = \mathrm{id}_{M_n}$, where we identify S_N with M_n, $(\pi^j \circ \varphi_1)|_{M_n} = 0$, $j \neq N$, and $(\pi^{i-1} \circ \varphi_1)|_{M_{d_i}} = h_i$ and $(\varphi^j \circ \varphi_1)|_{M_{d_i}} = 0$, if $j \neq i - 1$, $i = 1, 2, ..., N$. One checks that both φ_0 and φ_1 are unital. Define

$$B = \{(f, a) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_0(a) \text{ and } f(1) = \varphi_1(a)\}.$$

Let $(f, a) \in B$, where $f = (f_0, f_1, ..., f_N)$, $f_i \in C([0,1], M_n)$, $i = 0, 1, ..., N$, and $a = (a_0, a_1, ..., a_N)$, $a_0 \in M_n$, $a_i \in M_{d_i}$, $i = 1, 2, ..., N$. Then $f_0(0) = \pi^0(f(0)) = a_0$ and $f_N(1) = \pi^N(f(1)) = \pi^N(\varphi_1(a)) = a_0$. So $f_0(0) = f_N(1)$, $f_0(1) = \pi^0(f(1)) = \pi^0 \circ \varphi_1(a) = h_1(a_1)$, and $f_1(0) = \pi^1(f(0)) = \pi^1 \circ \varphi_0(a) = h_1(a_1)$. So $f_0(1) = f_1(0) = h_1(a_1)$; for $j = 1, 2, ..., N$, $f_j(1) = \pi^j(f(1)) = \pi^j \circ \varphi_1(a) = h_{j+1}(a_j)$ and $f_{j+1}(0) = \pi^{j+1} \circ \varphi_0(a) = h_{j+1}(a_j)$. So $f_j(1) = f_{j+1}(0) = h_{j+1}(a_j)$, $j = 1, 2, ..., N$. Note that we may write $C([0,1], F_2) = \bigoplus_{i=0}^{N} C(I_j, M_n)$. Then,

$$B \cong \{(f, a) \in (\bigoplus_{i=0}^{N} C(I_j, M_n)) : f(0_0) = f(1_N), \; f(1_j) = f(0_{j+1}), \; j = 1, 2, ..., N\}.$$

Thus, $A \cong B$. Since $B \in \mathcal{C}$, $A \in \mathcal{C}$. ☐

PROPOSITION 3.23. *Let A be a unital C^*-algebra with $T(A) \neq \varnothing$. Suppose that $\overline{\rho_A^k(K_0(A))} \supset \rho_A(K_0(A))$ (see Definition* 2.16 *for the definition of $\rho_A^k(K_0(A))$). Then, for any pair $u, v \in U(M_k(A))$ with $uv^* \in U_0(M_k(A))$,*

$$\mathrm{d}(\overline{u}, \overline{v}) = \mathrm{dist}(\overline{u}, \overline{v}) \tag{e 3.29}$$

(see 2.16*). Moreover, if A also has stable rank one, then* (e 3.29) *holds for any $u, v \in U(A)$. In particular, this is true if $A \in \mathcal{C}$.*

PROOF. If $u, v \in U(M_k(A))$ and $uv^* \in U_0(M_k(A))$, then by 2.16,

$$\mathrm{d}(\overline{u}, \overline{v}) = \mathrm{d}(\overline{uv^*}, \overline{1}) = \mathrm{dist}(\overline{uv^*}, \overline{1}) = \mathrm{dist}(\overline{u}, \overline{v}). \tag{e 3.30}$$

Suppose A also has stable rank one. Let $u, v \in U(A)$ with $uv^* \in U_0(A)$. Let $\mathrm{d}(\overline{u}, \overline{v}) = \delta < 2$. Let $u_1 = \mathrm{diag}(u, 1_{k-1})$ and $v_1 = \mathrm{diag}(v, 1_{k-1})$. By Corollary 2.11 of [46], for example, there is $a \in A_{s.a.}$ such that $\tau(a) = D_A(\zeta)(\tau)$ for all $\tau \in T(A)$ for some piecewise smooth path ζ in $U_0(M_k(A))$ with $\zeta(0) = u_1 v_1^*$ and $\zeta(1) = 1_k$, and $u_1 v_1^* = \mathrm{diag}(\exp(i2\pi a), 1_{k-1})z$ for some $z \in CU(M_k(A))$. We may assume that $\omega(a) = \sup\{|\tau(a)| : \tau \in T(A)\} < 1/2$ as $\delta < 2$. Since A has stable rank one, by Corollary 3.11 of [46], $uv^* \exp(-i2\pi a) \in CU(A)$. This implies that

$$\begin{aligned} \mathrm{dist}(\overline{u}, \overline{v}) = \mathrm{dist}(\overline{uv^*}, \overline{1}) &= \mathrm{dist}(\overline{\exp(-i2\pi a)}, \overline{1}) \\ &= \mathrm{dist}(\overline{\exp(i2\pi a)}, \overline{1}). \end{aligned} \tag{e 3.31}$$

By the first paragraph of the proof of Lemma 3.1 of [113], for any $\varepsilon > 0$, there is $h \in A_0$ such that $\omega(a) \le \|a - h\| \le \omega(a) + \varepsilon < 1/2$. Let $b = a - h$. Then, since $\exp(ih) \in CU(A)$ (see Lemma 3.1 of [112]), $\mathrm{dist}(\overline{\exp(i2\pi a)}, \overline{1}) = \mathrm{dist}(\overline{\exp(i2\pi b)}, \overline{1}) \le \|\exp(i2\pi(\omega(a) + \varepsilon)) - 1\|$. Since this holds for any $\varepsilon > 0$, we conclude that

$$\mathrm{dist}(\overline{u}, \overline{v}) = \mathrm{dist}(\overline{\exp(i2\pi a)}, \overline{1}) \le \|\exp(i2\pi\omega(a)) - 1\|. \tag{e 3.32}$$

This holds for any choice of ζ as above, and therefore,

$$\mathrm{dist}(\overline{u}, \overline{v}) \le \|\exp(i2\pi\|\overline{D}_A(uv^*)\|) - 1\| = \mathrm{d}(\overline{u}, \overline{v}). \tag{e 3.33}$$

On the other hand, by definition, $\mathrm{d}(\overline{u}, \overline{v}) = \mathrm{d}(\overline{\exp(i2\pi a)}, \overline{1}) = \delta$. If $\mathrm{dist}((\overline{u}, \overline{v}) < \delta$, then there is $z_1 \in CU(A)$ such that $\|uv^* z_1 - 1\| < \delta < 2$. There exists $b_1 \in A_{s.a.}$ such that $\|b_1\| < 2\arcsin(\delta/2)$ and $uv^* z_1 = \exp(i2\pi b_1)$. It follows that $\tau(b) \le 2\arcsin(\delta/2)$ for all $\tau \in T(A)$. This implies that $\mathrm{d}(\overline{\exp(i2\pi b_1)}, \overline{1}) < \delta$. Note that $[z_1] = 0$ in $K_1(A)$. Since A has stable rank one, $z_1 \in U_0(A)$. Therefore, by Lemma 3.1 of [112], $\overline{D}_A(\overline{z_1}) = 0$. It follows that

$$\begin{aligned} \overline{D}_A(\overline{\exp(i2\pi b_1)}) = \overline{D}(\overline{uv^* z_1}) &= \overline{D}_A(\overline{uv^*}) + \overline{D}_A(\overline{z_1}) \\ &= \overline{D}_A(\overline{uv^*}). \end{aligned} \tag{e 3.34}$$

Hence $\mathrm{d}(\overline{uv^*}, \overline{1}) = \|\exp(i2\pi\|\overline{D}_A(\overline{\exp(i2\pi b_1)})\|) - 1\| \le \|\exp(i2\pi\|b_1\|) - 1\| < \delta$. This contradicts the fact that $\mathrm{d}(\overline{u}, \overline{v}) = \delta$. Therefore,

$$\mathrm{d}(\overline{u}, \overline{v}) = \mathrm{dist}(\overline{u}, \overline{v}). \tag{e 3.35}$$

Now, $uv^* \notin U_0(A)$, then, by Definition (see 2.16), $\mathrm{d}(\overline{u}, \overline{v}) = 2$. Suppose that A has stable rank one. Since each element of $CU(A)$ gives the zero element of

$K_1(A)$, $CU(A) \subset U_0(A)$. Suppose that $\|uv^* - z_2\| < 2$ for some $z_2 \in CU(A)$. Then $\|uv^*z_2^* - 1\| < 2$. It follows that $uv^*z_2^* \in U_0(A)$. Therefore $uv^* \in U_0(A)$, a contradiction. This shows that $\mathrm{dist}(\overline{u}, \overline{v}) = \mathrm{dist}(\overline{uv^*}, 1) = 2$.

To see the last part of the statement, let $A \in \mathcal{C}$. We note that, by Proposition 3.3, A has stable rank one. Moreover, by Theorem 3.15, there exists $k \geq 1$ such that $M_k(A)$ contains a set of projections whose images in $K_0(A)$ generate $K_0(A)$. It follows that $\overline{\rho_A^k(K_0(A))} = \overline{\rho_A(K_0(A))}$. Thus, the lemma applies in this case. □

4. Maps to Finite Dimensional C^*-algebras

Lemma 4.1. *Let $z_1, z_2, ..., z_n$ be positive integers which may not be distinct. Set $T = n \cdot \max\{z_iz_j : 1 \leq i, j \leq n\}$. Then, for any two nonnegative integer linear combinations $a = \sum_{i=1}^n a_i \cdot z_i$ and $b = \sum_{i=1}^n b_i \cdot z_i$, there are two integer combinations $a' = \sum_{i=1}^n a_i' \cdot z_i$ and $b' = \sum_{i=1}^n b_i' \cdot z_i$ with $a' = b'$, $0 \leq a_i' \leq a_i$, $0 \leq b_i' \leq b_i$, and $\min\{a - a', b - b'\} \leq T$.*

Consequently, if $\delta > 0$ and $|a-b| < \delta$, we also have $\max\{a-a', b-b'\} < \delta+T$.

Proof. To prove the first part, note first that if $\min\{a, b\} \leq T$, we may choose $a' = b' = 0$. Therefore it is enough to prove that if $\min\{a, b\} > T$, then there are non-zero $0 < a' = \sum_{i=1}^n a_i' \cdot z_i = b' = \sum_{i=1}^n b_i' \cdot z_i$ with $0 \leq a_i' \leq a_i$, $0 \leq b_i' \leq b_i$, and $\min\{a - a', b - b'\} \leq T$.

Suppose that a, $b > T$. Then there is i' such that $a_{i'}z_{i'} > \max_{i,j} z_iz_j$, and there is j' such that $b_{j'}z_{j'} \geq \max_{i,j} z_iz_j$. Thus, $a_{i'}z_{i'} \geq z_{i'}z_{j'}$ and $b_{j'}z_{j'} \geq z_{i'}z_{j'}$. It follows that $a_{i'} \geq z_{j'}$ and $b_{j'} \geq z_{i'}$. Then choose $a^{(1)'} = a_{i'}^{(1)}z_{i'}$ and $b^{(1)'} = b_{j'}^{(1)}z_{j'}$ with $a_{i'}^{(1)} = z_{j'}$ and $b_{j'}^{(1)} = z_{i'}$. Then $a^{(1)'} = b^{(1)'} \geq 1$ and $a_{i'}^{(1)} \leq a_{i'}$ and $b_{j'}^{(1)} \leq b_{j'}$. Moreover, $\min\{a-a^{(1)'}, b-b^{(1)'}\} \leq \min\{a, b\}-1$. If $\min\{a-a^{(1)'}, b-b^{(1)'}\} \leq T$, then we are done, by choosing $a_i^{(1)} = 0$ if $i \neq i'$ and $b_j^{(1)} = 0$, if $j \neq j'$. In particular, the lemma holds if $T-\min\{a, b\} \leq 1$. If $T-\min\{a-a^{(1)'}, b-b^{(1)'}\} > 0$, we repeat this on $a - a^{(1)'}$ and $b - b^{(1)'}$ and obtain $a^{(2)} \leq a - a^{(1)}$ and $b^{(2)} \leq b - b^{(1)}$ such that $a^{(2)} = b^{(2)} \geq 1$. Put $a^{(2)'} = a^{(1)'} + a^{(2)}$ and $b^{(2)'} = b^{(1)'} + b^{(2)}$. Note we also have $a^{(2)'} = \sum_i a_i^{(2)}z_i$ and $b^{(2)'} = \sum_i b_i^{(2)}z_i$ with $0 \leq a_i^{(2)} \leq a_i$ and $0 \leq b_i^{(2)} \leq b_i$ for all i. Moreover, $a^{(2)'} = b^{(2)'} \geq a^{(1)'} + 1 \geq 2$. It follows that $\min\{a - a^{(2)'}, b - b^{(2)'}\} \leq \min\{a, b\} - 2$. If $\min\{a - a^{(2)'}, b - b^{(2)'}\} \leq T$, then we are done. In particular, the lemma holds for $T - \min\{a, b\} \leq 2$. If $T-\min\{a-a^{(2)'}, b-b^{(2)'}\} > 0$, we continue. An inductive argument establishes the first part of the lemma.

To see the second part, assume that $a - a' \leq T$. Then $b - b' < |a - b| + T$. □

Theorem 4.2 (2.10 of [82]; see also Theorem 4.6 of [70] and 2.15 of [56]). *Let $A = PC(X, F)P$, where X is a compact metric space, F is a finite dimensional C^*-algebra, and $P \in C(X, F)$ is a projection, and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ be an order preserving map.*

For any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A$, there exist a finite set $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite set $\mathcal{H}_2 \subset A_{s.a.}$, and $\delta > 0$ with the *following property: If* $\varphi_1, \varphi_2 : A \to M_n$ *(for some integer* $n \geq 1$*) are two unital homomorphisms such that*

$$[\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}}, \ \ \mathrm{tr} \circ \varphi_1(h) \geq \Delta(\hat{h}) \ \ \textit{for all } h \in \mathcal{H}_1, \ \ \textit{and}$$
$$|\mathrm{tr} \circ \varphi_1(g) - \mathrm{tr} \circ \varphi_2(g)| < \delta \ \ \textit{for all } g \in \mathcal{H}_2,$$

where tr *is the tracial state of* M_n*, then there exists a unitary* $u \in M_n$ *such that*

$$\|\mathrm{Ad}\, u \circ \varphi_1(f) - \varphi_2(f)\| < \varepsilon \ \ \textit{for all } f \in \mathcal{F}. \tag{e 4.1}$$

PROOF. We first prove this for the case $A = C(X)$. This actually follows (as we shall show) from Theorem 2.10 of [82]. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F}$. Let $\eta > 0$ be as in Theorem 2.10 of [82].

Let $x_1, x_2, ..., x_K \in X$ be a finite subset such that $X \subset \bigcup_{i=1}^K B(x_i, \eta/8)$. Set $\mathcal{H}_1 = \{f_i : 1 \leq i \leq K\}$, where $f_i \in C(X)$, $0 \leq f_i \leq 1$ and the support of f_i lies in $B(x_i, \eta/4)$ and $f_i(x) = 1$ if $x \in B(x_i, \eta/8)$, $i = 1, 2, ..., K$. Choose $\sigma = \min\{\Delta(\hat{f}_i) : 1 \leq i \leq K\} > 0$. Suppose that $\mathrm{tr}(\varphi_1(f_i)) \geq \Delta(\hat{f}_i)$, $i = 1, 2, ..., K$. Let O_r be an open ball of X with radius at least $r \geq \eta$ and with center $x \in X$. Then $x \in B(x_i, \eta/8)$ for some i. It follows that $B(x_i, \eta/4) \subset O_r$. Therefore,

$$\mu_{\mathrm{tr} \circ \varphi_1}(O_r) \geq \mathrm{tr} \circ \varphi_1(f_i) \geq \sigma. \tag{e 4.2}$$

Now let $\gamma > 0$, $\mathcal{P} \subset \underline{K}(A)$, and $\mathcal{H}_2 \subset A_{s.a.}$ (in place of $\mathcal{H}$) be as given by 2.10 of [82] for the above ε, $\mathcal{F}$, η and σ (note we do not need to mention $\mathcal{G}$ and δ since φ_1 and φ_2 are homomorphisms).

Choose $\delta = \gamma$, $\mathcal{P}$, and $\mathcal{H}_2$ as above. Then Theorem 2.10 of [82] applies. This proves that the theorem holds for $A = C(X)$. The case that $A = M_m(C(X))$ follows easily. By considering each summand separately, it is also easy to see that the theorem also holds for $A = C(X, F)$.

Now let $A = PC(X, F)P$. Again, by considering each summand separately, we may reduce the general case to the case that $A = PC(X, M_r)P$ for some integer $r \geq 1$.

Note that $\{\mathrm{rank}(P(x)) : x \in X\}$ is a finite set of positive integers. Therefore the set $Y = \{x \in X : \mathrm{rank}(P(x)) > 0\}$ is compact and open. Then we may write $A = PC(Y, M_r)P$. Thus, without loss of generality, we may assume that $P(x) > 0$ for all $x \in X$. With this assumption, we may assume that P is a full projection in $C(X, M_r)$. Then, by [9], $A \otimes \mathcal{K} \cong C(X, F) \otimes \mathcal{K}$. It follows that there is an integer $m \geq r$ and a full projection $e \in M_m(A)$ such that $eM_m(A)e \cong C(X, M_r)$. In particular, e has rank r everywhere. Moreover, there exist an integer $m_1 \geq 1$, a full projection e_A in $M_{m_1}(eM_m(A)e)$, and a unitary $u \in M_{m_1m}(A)$ such that $u^*e_Au = 1_A$ (we identify 1_A with $1_A \otimes e_{11}$ in $M_{m_1m}(A)$). In particular, we may write $u^*e_A(M_{m_1}(eM_m(A)e))e_Au = A$ (where we identify A with $A \otimes e_{11}$).

Put $B = M_{m_1m}(A)$. Since $e \in M_m(A)$, $e \in B$. Moreover, $eBe = eM_m(A)e$ and $M_{m_1}(eBe) \subset M_{m_1}(eM_m(A)e) \subset M_{m_1m}(A) = B$. Put $e_1 = \mathrm{diag}(\overbrace{e, e, ..., e}^{m_1})$.

Then we may identify $M_{m_1}(eBe)$ with $e_1M_{m_1m}(A)e_1$. Set $B_1 = M_{m_1}(eBe) = e_1M_{m_1m}(A)e_1 \subset B$. Note $B_1 \cong M_{m_1}(C(X, M_r))$. Put $B_2 = e_AM_{m_1}(eM_m(A)e)e_A = e_AB_1e_A$. So $e_A \leq e_1$ and $B_2 = e_ABe_A$. Note e_A is full in B_1. Define $\Delta_1 : B_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ by $\Delta_1(\widehat{(a_{i,j})}) = (1/m_1m)\sum_{i=1}^{m_1m} \Delta(\hat{a_{ii}})$ for $0 \leq (a_{i,j}) \leq 1$ in $B_1^{\mathbf{1}} \setminus \{0\}$.

Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. Without loss of generality, we may assume that $\|a\| \leq 1$ for all $a \in \mathcal{F}$. Put $\mathcal{F}_1 = \{ufu^* : f \in \mathcal{F}\} \cup \{e_A\}$. Let $\varepsilon_1 = \min\{\varepsilon/15, 1/3 \cdot 64\}$. Let $\mathcal{P}_1 \subset \underline{K}(B_1)$ (in place of $\mathcal{P}$), $\mathcal{H}_1' \subset B_1 \setminus \{0\}$ (in place of $\mathcal{H}_1$), and let $\mathcal{H}_2' \subset (B_1)_{s.a.}$ (in place of $\mathcal{H}_2$), and $\delta' > 0$ (in place of δ) denote the finite subsets and constant provided by the theorem for the case $A = C(X, M_r)$ for Δ_1, ε_1, and $\mathcal{F}_1$ (and for B_1 instead of A). Without loss of generality, we may assume that $[e_A]$ and $[e_1] \in \mathcal{P}_1$.

Notice that $B_1 \subset M_{m_1m}(A)$. Let $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ be a finite subset such that a finite subset of the set $\{(b_{i,j})_{m_1m\times m_1m} \in B_1 \setminus \{0\} : b_{ii} \in \mathcal{H}\}$ contains $\mathcal{H}_1$. We may view $\mathcal{P}_1$ as a finite subset of $\underline{K}(A)$ since $A \otimes \mathcal{K} \cong B_1 \otimes \mathcal{K}$. Let $\mathcal{H}_2$ be a finite subset of A such that $\{(a_{ij}) : a_{ii} \in \mathcal{H}_2''\} \supset \mathcal{H}_2'$. Note if $(a_{i,j}) \in M_{m_1m}(A)_{s.a.}$, then $a_{ii} \in A_{s.a.}$ for all i. Choose $\delta = \delta'$.

Now suppose that $\varphi_1, \varphi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms which satisfies the assumption for the above $\mathcal{H}_1, \mathcal{H}_2$, $\mathcal{P}_1$ (in place of $\mathcal{P}$) and δ.

Set $\varphi_i^\sim = (\varphi_i \otimes \mathrm{id}_{m_1m})$. Define $\tilde{\varphi}_i = \varphi_i^\sim|_{B_1} : B_1 \to M_{m_1mn}$, $i = 1, 2$. Note that

$$\varphi_i(a) = \varphi_i^\sim(u^*)\tilde{\varphi}_i(uau^*)\varphi_i^\sim(u) \text{ for all } a \in A,\ i = 1, 2. \tag{e 4.3}$$

Then,

$$[\tilde{\varphi}_1]|_{\mathcal{P}_1} = [\tilde{\varphi}_2]|_{\mathcal{P}_1}. \tag{e 4.4}$$

In particular, $[\tilde{\varphi}_1(e_1)] = [\tilde{\varphi}_2(e_1)]$. Therefore, replacing $\tilde{\varphi}_1$ by $\mathrm{Ad}\, w \circ \tilde{\varphi}_1$ with a unitary w in M_{m_1mn}, without loss of generality, we may assume $\tilde{\varphi}_1(e_1) = \tilde{\varphi}_2(e_1)$. Put $F = \tilde{\varphi}_1(e_1)M_{m_1mn}\tilde{\varphi}_1(e_1) \cong M_{m_1rn}$.

Then, for any $h = (h_{i,j}) \in \mathcal{H}_1'$,

$$t(\tilde{\varphi}_1(h)) = (1/m_1m)\sum_{i=1}^{m_1m} \tau(\varphi_1(h_{i,i})) \geq \Delta_1(\hat{h}), \tag{e 4.5}$$

where τ is the tracial state of M_n and t is the tracial state of $M_{m_1m}(M_n)$. Denote by $\bar{t}$ the tracial state of F. Then, by (e 4.5),

$$\bar{t}(\tilde{\varphi}_1(h) \geq \Delta_1(\hat{h}) \text{ for all } h \in \mathcal{H}_1'. \tag{e 4.6}$$

If $h = (h_{i,j}) \in \mathcal{H}_2'$, then

$$|t(\tilde{\varphi}_1(h)) - t(\tilde{\varphi}_2(h))| = (1/m_1m)\sum_{i=1}^{m_1m} |\tau(\varphi_1(h_{i,i}) - \tau(\varphi_2(h_{ii}))| \tag{e 4.7}$$

$$\leq (1/m_1m)\sum_{i=1}^{m_1m} \delta = \delta. \tag{e 4.8}$$

It follows that

$$|\bar{t}(\tilde{\varphi}_1(h)) - \bar{t}(\tilde{\varphi}_2(h))| < \delta \ \text{ for all } \ h \in \mathcal{H}_2'. \tag{e 4.9}$$

Since (as shown above) the theorem holds for $M_{m_1}(C(X, M_r)) \cong B_1$, by (e 4.4), (e 4.6), and (e 4.9), we conclude that there is a unitary $w_1 \in F$ such that

$$\|\mathrm{Ad}\, w_1 \circ \tilde{\varphi}_1(a) - \tilde{\varphi}_2(a)\| < \varepsilon_1 \ \text{ for all } \ a \in \mathcal{F}_1. \tag{e 4.10}$$

Note that $uau^* \in \mathcal{F}_1$ if $a \in \mathcal{F}$. Set $w_2 = w_1^* \varphi_1^{\sim}(u^*) w_1 \varphi_2^{\sim}(u)$. Then, by (e 4.3) and (e 4.10),

$$\begin{align*}
\mathrm{Ad}\, w_1 w_2 \circ \varphi_1(a) &= w_2^* w_1^* \varphi_1^{\sim}(u)^* \tilde{\varphi}_1(uau^*) \varphi_1^{\sim}(u) w_1 w_2^* \\
&= w_2^* w_1^* \varphi_1^{\sim}(u)^* w_1 w_1^* \tilde{\varphi}_1(uau^*) w_1 w_1^* \varphi_1^{\sim}(u) w_1 w_2 \tag{e 4.11}\\
&\approx_{2\varepsilon_1} \varphi_2^{\sim}(u^*)(\mathrm{Ad}\, w_1 \circ \tilde{\varphi}_1(uau^*)) \varphi_2^{\sim}(u) \tag{e 4.12}\\
&\approx_{\varepsilon_1} \tilde{\varphi}_2(u^*) \tilde{\varphi}_2(uau^*) \tilde{\varphi}_2(u) = \varphi_2(a). \tag{e 4.13}
\end{align*}$$

Denote by e_0 the identity of M_n. We also view e_0 as an element of $M_{m_1 mn}$. Note that $\varphi_1(1_A) = \varphi_2(1_A) = e_0$. The above estimate implies that $\|e_0 w_2^* w_1^* e_0 w_1 w_2 e_0 - e_0\| < 3\varepsilon_1$. Thus there exists a unitary $w_3 \in e_0 M_{m_1 mn} e_0 = M_n$ such that, as an element of $M_{m_1 mn}$, $\|w_3 - e_0 w_1 w_2 e_0\| < 6\varepsilon_1$. Thus we have, for all $a \in A$,

$$\|\mathrm{Ad}\, w_2 \circ \varphi_1(a) - \varphi_2(a)\| < (6 + 3 + 6)\varepsilon_1 < \varepsilon \ \text{ for all } \ a \in \mathcal{F}. \tag{e 4.14}$$

□

Recall that $K_0(M_n) = \mathbb{Z}$ and $K_0(M_n) = \{0\}$. So, if X is connected, then, for any two unital homomorphisms $\varphi_1, \varphi_2 : C(X) \to M_n$, $[\varphi_1] = [\varphi_2]$ in $KL(C(X), M_n)$.

We state the following version of 2.10 of [82] for convenience.

THEOREM 4.3 (see 2.10 of [82], Theorem 4.6 of [70] and 2.15 of [56]). *Let X be a connected compact metric space, and let $C = C(X)$. Let $\mathcal{F} \subset C$ be a finite set, and let $\epsilon > 0$ be a constant. There is a finite set $\mathcal{H}_1 \subset C_+ \backslash \{0\}$ such that, for any $\sigma_1 > 0$, there are a finite subset $\mathcal{H}_2 \subset C$ and $\sigma_2 > 0$ such that for any unital homomorphisms $\varphi, \psi : C \to M_n$ for a matrix algebra M_n, satisfying*

(1) *$\varphi(h) > \sigma_1$ and $\psi(h) > \sigma_1$ for any $h \in \mathcal{H}_1$, and*
(2) *$|\mathrm{tr} \circ \varphi(h) - \mathrm{tr} \circ \psi(h)| < \sigma_2$ for any $h \in \mathcal{H}_2$,*

there is a unitary $u \in M_n$ such that

$$\|\varphi(f) - u^* \psi(f) u\| < \epsilon \quad \text{for any } f \in \mathcal{F}.$$

REMARK 4.4. Let X be a compact metric space and let $A = C(X)$. Then $C(X) = \lim_{n \to \infty} C(Y_n)$, where each Y_n is a finite CW complex. It follows that $C(X, M_r) = \lim_{n \to \infty} C(Y_n, M_r)$ for any integer $r \geq 1$. Let $P \in C(X, M_r)$ be a projection. Consider $A = PC(X, M_r)P$. Note, since the rank function is continuous and has integer values, that $\{x \in X : P(x) > 0\}$ is a clopen subset of X. Thus, without loss of generality, we may assume that $P(x) > 0$ for all $x \in X$. In particular, P is a full projection in $C(X, M_r)$. We also have $A = \lim_{n\to\infty} P_nC(Y_n, M_r)P_n$, where $P_n \in C(Y_n, M_r)$ is a projection. Let $h_n : P_nC(Y_n, M_r)P \to A$ denote the homomorphism induced by the inductive limit. Without loss of generality, we may assume that $P_n(y) > 0$ for all $y \in Y_n$, $n = 1, 2, ...$

Suppose that $\varphi_1, \varphi_2 : A \to M_n$ are two homomorphisms. Then $(\varphi_i)_{*1} = 0$, $i = 1, 2$. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset and denote by G be the subgroup generated by $\mathcal{P}$. There exists an integer $n \geq 1$ such that $G \subset [h_n](\underline{K}(C(Y_n)))$. Define $\bar{G} = [h_n](\underline{K}(C(Y_n)))$. Suppose that $\{p_1, p_2, ..., p_k\}$ in $C(Y_n)$ are mutually orthogonal projections which correspond to k different path connected components $Y_1, Y_2, ..., Y_k$ of Y with $\bigsqcup_{i=1}^k Y_i = Y$. Fix $\xi_i \in Y_i$, Let $C_i = C_0(Y_i \setminus \{\xi_i\})$, $i = 1, 2, ..., k$. Since Y_i is path connected, by considering the point evaluation at ξ_i, it is easy to see that, for any homomorphism $\varphi : PC(Y_n, M_r)P \to M_n$, $[\varphi]|_{\underline{K}(C_i)} = 0$. Let $P_{n,i} = P_n|_{Y_i}$, $i = 1, 2, ..., k$. We may assume that $h_n(P_{n,i})(x)$ has two rank values $r_i \geq 1$ or zero. Suppose that $\tau \circ \varphi_1(P_{n,i}) = \tau \circ \varphi_2(P_{n,i})$, $i = 1, 2, ..., k$. Then $[\varphi_1]([p_i]) = [\varphi_2]([p_i])$, $i = 1, 2, ..., k$. It follows that $[\varphi_1 \circ h] = [\varphi_2 \circ h]$ in $KL(C(Y), M_n)$. One then computes that

$$[\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}}. \tag{e 4.15}$$

We will use this fact in the next proof.

LEMMA 4.5. *Let X be a compact metric space, let F be a finite dimensional C^*-algebra and let $A = PC(X, F)P$, where $P \in C(X, F)$ is a projection. Let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ be an order preserving map.*

For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$, and $\delta > 0$ satisfying the following condition: If $\varphi_1, \varphi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$\tau \circ \varphi_1(h) \geq \Delta(\hat{h}) \textit{ for all } h \in \mathcal{H}_1, \textit{ and}$$
$$|\tau \circ \varphi_1(g) - \tau \circ \varphi_2(g)| < \delta \textit{ for all } g \in \mathcal{H}_2,$$

then, there exist a projection $p \in M_n$, a unital homomorphism $H : A \to pM_np$, unital homomorphisms $h_1, h_2 : A \to (1 - p)M_n(1 - p)$, and a unitary $u \in M_n$ such that

$$\|\mathrm{Ad}\, u \circ \varphi_1(f) - (h_1(f) + H(f))\| < \varepsilon,$$
$$\|\varphi_2(f) - (h_2(f) + H(f)\| < \varepsilon \textit{ for all } f \in \mathcal{F},$$
$$\text{and } \tau(1 - p) < \sigma,$$

where τ is the tracial state of M_n.

PROOF. Note that we may rewrite $A = PC(X, M_r)P$ for some possibly large r.

Let $\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite set, $\mathcal{H}_1' \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite set, $\mathcal{H}_2' \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be a finite set and $\delta_1 > 0$ (in place of δ) be required by 4.2 for $\varepsilon/2$ (in place of ε), $\mathcal{F}$ and Δ_1.

Without loss of generality, we may assume that $1_A \in \mathcal{F}$, $1_A \in \mathcal{H}_1' \subset \mathcal{H}_2'$ and $\mathcal{H}_2' \subset A_+^{\mathbf{1}} \setminus \{0\}$. So, in what follows, $\mathcal{H}_2' \subset A_+^{\mathbf{1}} \setminus \{0\}$. Put

$$\sigma_0 = \min\{\Delta_1(\hat{g}) : g \in \mathcal{H}_2'\}. \tag{e 4.16}$$

Let G be the subgroup generated by $\mathcal{P}$ and let $\bar{G}$ be as defined in Remark 4.4. We keep the notation of 4.4. Set $Q_i = h_n(P_{n,i})$, $i = 1, 2, ..., k$. Denote by $h_{n*} : X \to Y_n$ the continuous map induced by h_n. Let $\mathcal{P}_0 = \{[Q_1], [Q_2], ..., [Q_k]\}$ and let $r_i \geq 1$ denote the rank of $Q_i(x)$ when $Q_k(x) \neq 0$ (see 4.4).

Let $\mathcal{H}_1 = \mathcal{H}_1' \cup \{Q_i : 1 \leq i \leq k\}$ and $\mathcal{H}_2 = \mathcal{H}_2' \cup \mathcal{H}_1$. Set $\sigma_1 = \min\{\Delta(\hat{g}) : g \in \mathcal{H}_2\}$. Let $r = \max\{r_1, r_2, ..., r_k\}$. Choose $\delta = \min\{\sigma_0 {\cdot} \sigma/4kr, \sigma_0 {\cdot} \delta_1/4kr, \sigma_1/16kr\}$.

Suppose now that $\varphi_1, \varphi_2 : A \to M_n$ are two unital homomorphisms described in the lemma for the above $\mathcal{H}_1$, $\mathcal{H}_2$ and Δ.

If $x \in X$, denote by $\pi_x : A \to M_{r(x)}$ the point evaluation at x. We may write $\varphi_j(f) = \sum_{s=1}^{k}(\sum_{i=1}^{n(s,j)} \psi_{s,i,j}(\pi_{x_{s,i,j}}(f)))$ for all $f \in P(C(X, M_r)P$ $(j = 1, 2)$, where $h_{n*}(x_{s,i,j}) \in Y_{n,i}$, $\psi_{s,i,j} : M_{r_i} \to M_n$ is a homomorphism such that $\psi_{s,i,j}(1_{M_{r_i}})$ has rank r_i. Note that $x_{s,i,j}$ may be repeated, and $\varphi_j(Q_s) = \sum_{i=1}^{n(s,j)} \psi_{s,i,j}(\pi_{x_{s,i,j}}(Q_s))$, $s = 1, 2, ..., k$, $j = 1, 2$. Note also that $\psi_{s,i,j}(\pi_{x_{s,i,j}}(Q_s))$ has rank r_s, and the rank of $\varphi_j(Q_s)$ is $n(s, j)r_s$, $1 \leq s \leq k$, $j = 1, 2$,

We have, $i = 1, 2, ..., k$,

$$(\frac{r_i}{n})|n(i,1) - n(i,2)| \quad = \quad |\tau \circ \varphi_1(Q_i) - \tau \circ \varphi_2(Q_i)| < \delta, \tag{e 4.17}$$

where τ is the tracial state on M_n. Therefore, by comparing the ranks of $\varphi_j(Q_i)$, $1 \leq i \leq k$ $(j = 1, 2)$, one finds a projection $P_{0,j} \in M_n$ such that

$$\tau(P_{0,j}) < k\delta < \sigma_0 \cdot \sigma, \;\; j = 1, 2, \tag{e 4.18}$$

and $\mathrm{rank}(P_{0,1}) = \mathrm{rank}(P_{0,2})$, and unital homomorphisms $\varphi_{1,0} : A \to P_{0,1}M_nP_{0,1}$, $\varphi_{2,0} : A \to P_{0,2}M_nP_{0,2}$, $\varphi_{1,1} : A \to (1 - P_{0,1})M_n(1 - P_{0,1})$ and $\varphi_{1,2} : A \to (1 - P_{0,2})M_n(1 - P_{0,2})$ such that

$$\varphi_1 = \varphi_{1,0} \oplus \varphi_{1,1}, \;\; \varphi_2 = \varphi_{2,0} \oplus \varphi_{2,1}, \tag{e 4.19}$$

$$\tau \circ \varphi_{1,1}(Q_i) = \tau \circ \varphi_{1,2}(Q_i), \;\; i = 1, 2, ..., k. \tag{e 4.20}$$

Replacing φ_1 by $\mathrm{Ad}\, v \circ \varphi_1$, simplifying the notation, without loss of generality, we may assume that $P_{0,1} = P_{0,2}$. It follows from (e 4.20) that (see 4.4)

$$[\varphi_{1,1}]|_{\mathcal{P}} = [\varphi_{2,1}]|_{\mathcal{P}}. \tag{e 4.21}$$

By (e 4.18) and the choice of σ_0, we also have

$$\tau\circ\varphi_{1,1}(g)\ge\Delta_1(\hat{g})\ \text{ for all }\ g\in\mathcal{H}_1'\ \text{ and} \tag{e 4.22}$$

$$|\tau\circ\varphi_{1,1}(g)-\tau\circ\varphi_{1,2}(g)|<\sigma_0\cdot\delta_1\ \text{ for all }\ g\in\mathcal{H}_2'. \tag{e 4.23}$$

Therefore,

$$t\circ\varphi_{1,1}(g)\ge\Delta_1(\hat{g})\ \text{ for all }\ g\in\mathcal{H}_1'\ \text{ and} \tag{e 4.24}$$

$$|t\circ\varphi_{1,1}(g)-t\circ\varphi_{1,2}(g)|<\delta_1\ \text{ for all }\ g\in\mathcal{H}_2', \tag{e 4.25}$$

where t is the tracial state on $(1-P_{1,0})M_n(1-P_{1,0})$. By applying 4.2, there exists a unitary $v_1\in(1-P_{1,0})M_n(1-P_{1,0})$ such that

$$\|\mathrm{Ad}\,v_1\circ\varphi_{1,1}(f)-\varphi_{2,1}(f)\|<\varepsilon/16\ \text{ for all }\ f\in\mathcal{F}. \tag{e 4.26}$$

Put $H=\varphi_{2,1}$ and $p=P_{1,0}$. The lemma follows.

□

COROLLARY 4.6. *Let X be a compact metric space, let F be a finite dimensional C^*-algebra, and let $A=PC(X,F)P$, where $P\in C(X,F)$ is a projection. Let $\Delta: A_+^{q,\mathbf{1}}\setminus\{0\}\to(0,1)$ be an order preserving map. Let $1>\alpha>1/2$.*

For any $\varepsilon>0$, any finite subset $\mathcal{F}\subset A$, any finite subset $\mathcal{H}_0\subset A_+^{\mathbf{1}}\setminus\{0\}$, and any integer $K\ge 1$, there are an integer $N\ge 1$, a finite subset $\mathcal{H}_1\subset A_+^{\mathbf{1}}\setminus\{0\}$, a finite subset $\mathcal{H}_2\subset A_{s.a.}$, and $\delta>0$ satisfying the following condition: If $\varphi_1,\varphi_2: A\to M_n$ (for any integer $n\ge N$) are two unital homomorphisms such that

$$\tau\circ\varphi_1(h)\ge\Delta(\hat{h})\ \textit{for all}\ h\in\mathcal{H}_1,\ \textit{and}$$
$$|\tau\circ\varphi_1(g)-\tau\circ\varphi_2(g)|<\delta\ \textit{for all}\ g\in\mathcal{H}_2,$$

then there exist mutually orthogonal non-zero projections $e_0,e_1,e_2,...,e_K\in M_n$ such that $e_1,e_2,...,e_K$ are equivalent, $e_0\lesssim e_1$, and $e_0+\sum_{i=1}^K e_i=1_{M_n}$, and there are unital homomorphisms $h_1,h_2: A\to e_0M_ne_0$, $\psi: A\to e_1M_ne_1$ and a unitary $u\in M_n$ such that

$$\|\mathrm{Ad}\,u\circ\varphi_1(f)-(h_1(f)+\mathrm{diag}(\overbrace{\psi(f),\psi(f),...,\psi(f)}^{K}))\|<\varepsilon,$$
$$\|\varphi_2(f)-(h_2(f)+\mathrm{diag}(\overbrace{\psi(f),\psi(f),...,\psi(f)}^{K}))\|<\varepsilon\ \textit{for all}\ f\in\mathcal{F},$$
$$\text{and}\ \tau\circ\psi(g)\ge\alpha\frac{\Delta(\hat{g})}{K}\ \textit{for all}\ g\in\mathcal{H}_0,$$

where τ is the tracial state of M_n.

PROOF. By applying 4.5, it is easy to see that it suffices to prove the following statement:

Let X, F, P A and α be as in the corollary.

Let $\varepsilon > 0$, let $\mathcal{F} \subset A$ be a finite subset, let $\mathcal{H}_0 \subset A_+^{\mathbf{1}} \setminus \{0\}$ and let $K \geq 1$. There are an integer $N \geq 1$ and a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ with the following property:

Suppose that $H : A \to M_n$ (for some $n \geq N$) is a unital homomorphism such that

$$\tau \circ H(g) \geq \Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_0. \tag{e 4.27}$$

Then there are mutually orthogonal projections $e_0, e_1, e_2, ..., e_{2K} \in M_n$, a unital homomorphism $\varphi : A \to e_0 M_n e_0$, and a unital homomorphism $\psi : A \to e_1 M_n e_1$ such that, for all $f \in \mathcal{F}$,

$$\|\mathrm{Ad}\, U \circ H(f) - (\varphi(f) \oplus \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{2K}))\| < \varepsilon$$
$$\text{and } \tau \circ \psi(g) \geq \alpha \Delta(\hat{g})/2K \text{ for all } g \in \mathcal{H}_0$$

for some unitary $U \in M_n$. (Note $2K$ is used since we will have $h_i(1_A) \oplus e_0 \lesssim e_1 + e_2$ $(i = 1, 2.)$

We now prove the above statement. We may rewrite $A = P(X, M_r)P$ for some large integer r. Note that there are finitely many values of the rank of $P(x)$. Thus, we may write $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$, where each $A_i = P_i C(X_i, M_r) P_i$, X_i is a clopen subset of X and $P_i(x)$ has constant rank for $x \in X_i$, $i = 1, 2, ..., m$. Considering each summand separately, without loss of generality, we may assume that $A = PC(X, M_r)P$ and P has constant rank, say r_0.

Put

$$\sigma_0 = ((1 - \alpha)/4) \min\{\Delta(\hat{g}) : g \in \mathcal{H}_0\} > 0. \tag{e 4.28}$$

Let $\varepsilon_1 = \min\{\varepsilon/16, \sigma_0, 1/2\}$ and let $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{H}_0$. Choose $d_0 > 0$ such that

$$|f(x) - f(x')| < \varepsilon_1 \text{ for all } f \in \mathcal{F}_1, \tag{e 4.29}$$

provided that $x, x' \in X$ and $\mathrm{dist}(x, x') < d_0$.

Choose $\xi_1, \xi_2, ..., \xi_m \in X$ such that $\bigcup_{j=1}^m B(\xi_j, d_0/2) \supset X$, where $B(\xi, r) = \{x \in X : \mathrm{dist}(x, \xi) < r\}$. There is $d_1 > 0$ such that $d_1 < d_0/2$ and

$$B(\xi_j, d_1) \cap B(\xi_i, d_1) = \varnothing, \tag{e 4.30}$$

if $i \neq j$. There is, for each j, a function $h_j \in C(X)$ with $0 \leq h_j \leq 1$, $h_j(x) = 1$ if $x \in B(\xi_j, d_1/2)$ and $h_j(x) = 0$ if $x \notin B(\xi_j, d_1)$. Define $\mathcal{H}_1 = \mathcal{H}_0 \cup \{h_j : 1 \leq j \leq m\}$ and put

$$\sigma_1 = \min\{\Delta(\hat{g}) : g \in \mathcal{H}_1\}. \tag{e 4.31}$$

Choose an integer $N_0 \geq 1$ such that $1/N_0 < \sigma_1 \cdot (1-\alpha)/4$ and $N = 4r_0 m(N_0+1)^2(2K+1)^2$.

Now let $H : PC(X, M_r)P \to M_n$ be a unital homomorphism with $n \geq N$ satisfying the assumption (e 4.27). Let $Y_1 = \overline{B(\xi_1, d_0/2)} \setminus \bigcup_{i=2}^m B(\xi_i, d_1)$, $Y_2 = \overline{B(\xi_2, d_0/2)} \setminus (Y_1 \cup \bigcup_{i=3}^m B(\xi_i, d_1)$, $Y_j = \overline{B(\xi_j, d_0/2)} \setminus (\bigcup_{i=1}^{j-1} Y_i \cup \bigcup_{i=j+1}^m B(\xi_i, d_1))$, $j = 1, 2, ..., m$. Note that $Y_j \cap Y_i = \varnothing$ if $i \neq j$ and $B(\xi_j, d_1) \subset Y_j$. We write that

$$\begin{aligned} H(f) = \sum_{i=1}^{k_0} \psi_i(\pi_{x_i}(f)) \quad &= \quad \sum_{j=1}^m \sum_{x_i \in Y_j} \psi_i(\pi_{x_i}(f)) \\ \text{(e 4.32)} \qquad\qquad & \qquad \text{for all } f \in PC(X, M_r)P, \end{aligned}$$

where $\pi_{x_i} : A \to M_{r_0}$ is a point evaluation and $\psi_i : M_{r_0} \to M_n$ is a homomorphism such that $\psi_i : M_{r_0} \to p_i M_n p_i$, is a unital homomorphism with multiplicity 1, and where $\{p_1, p_2, ..., p_{k_0}\}$ is a set of mutually orthogonal rank r_0 projections in M_n, $n = k_0 r_0$, and $x_1, x_2, ..., x_{k_0}$ are in X (some of the x_i could be repeated). Let R_j be the cardinality of $\{x_i : x_i \in Y_j\}$, counting multiplicities. Then, by (e 4.27), for $j = 1, 2, ..., m$,

$$\begin{aligned} R_j \geq N\tau \circ H(h_j) \geq N\Delta(\hat{h_j}) \quad &\geq \quad mr_0(N_0+1)^2(2K+1)^2\sigma_1 \\ \text{(e 4.33)} \qquad\qquad\qquad &\geq \quad mr_0(N_0+1)(2K+1)^2. \end{aligned}$$

Write $R_j = S_j 2K + r_j$, where $S_j \geq N_0 2Kmr_0$ and $0 \leq r_j < 2K$, $j = 1, 2, ..., m$. Choose $x_{j,1}, x_{j,2}, ..., x_{j,r_j}$ in $\{x_i : x_i \in Y_j\}$ and set $Z_j = \{x_{j,1}, x_{j,2}, ..., x_{j,r_j}\}$, $j = 1, 2, ..., m$, counting multiplicities.

Then, we may write

$$\text{(e 4.34)} \quad H(f) = \sum_{j=1}^m \Big(\sum_{x_i \in Y_j \setminus Z_j} \psi_i(\pi_{x_{i,j}}(f)) \Big) + \sum_{j=1}^m \Big(\sum_{i=1}^{r_j} \psi_i(\pi_{x_{i,j}}(f)) \Big)$$

for $f \in C(X)$. Note that the cardinality of $\{x_i : x_i \in Y_j \setminus Z_j\}$ (counting multiplicities) is $2KS_j$, $j = 1, 2, ..., m$. We write, counting multiplicities, $\{x_i : x_i \in Y_j \setminus Z_j\} = \bigsqcup_{k=1}^{2K} \Omega_{k,j}$, where each $\Omega_{k,j}$ has exactly S_j points in $\{x_i : x_i \in Y_j \setminus Z_j\}$, counting multiplicities. Put $e_k = \sum_{x_i \in \Omega_{k,j}} p_i$, $k = 1, 2, ..., 2K$. Then each e_k has rank $(\sum_{j=1}^m S_j) r_0$, $k = 1, 2, ..., 2K$. Define

$$\begin{aligned} \text{(e 4.35)}\ \varphi(f) \quad &= \quad \sum_{j=1}^m \Big(\sum_{i=1}^{r_j} \psi_i(\pi_{x_{i,j}}(f)) \Big) \text{ and} \\ \text{(e 4.36)}\ \Psi(f) \quad &= \quad \sum_{j=1}^m \Big(\sum_{x_i \in Y_j \setminus Z_j} \psi_i(\pi_{\xi_j}(f)) \Big) = \sum_{j=1}^m \Big(\sum_{k=1}^{2K} \Big(\sum_{x_i \in \Omega_{k,j}} \psi_i(\pi_{\xi_j}(f)) \Big) \\ \text{(e 4.37)} \qquad\quad &= \quad \sum_{k=1}^{2K} \Big(\sum_{j=1}^m \Big(\sum_{x_i \in \Omega_{k,j}} \psi_i(\pi_{\xi_j}(f)) \Big) = \sum_{k=1}^{2K} \Big(\sum_{j=1}^m \Psi_{k,j}(\pi_{\xi_j}(f)) \Big), \end{aligned}$$

where $\Psi_{k,j}$ is a direct sum of S_j homomorphisms ψ_i with $x_i \in \Omega_{k,j}$. We estimate that

$$\|H(f) - (\varphi(f) \oplus \Psi(f))\| < \varepsilon_1 \text{ for all } f \in \mathcal{F}_1. \tag{e 4.38}$$

Note that each ψ_i is unitarily equivalent to ψ_1. So each $\Psi_{k,j}$ is unitarily equivalent to S_j copies of ψ_1. It follows that, for each k, $\sum_{j=1}^m \Psi_{k,j} \circ \psi_{\xi_j}$ is unitarily equivalent to $\psi := \sum_{j=1}^m \Psi_{1,j} \circ \pi_{\xi_j}$. Thus there is a unitary $U_1 \in M_n$ such that

$$\mathrm{Ad}\, U_1 \circ \Psi(f) = \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{2K}) \text{ for all } f \in A. \tag{e 4.39}$$

Put $e_0 = \sum_{j=1}^m (\sum_{x_i \in Z_j}) p_i$. Then e_0 has rank $\sum_{j=1}^m r_j r_0$. Moreover, φ is a unital homomorphism from A into $e_0 M_n e_0$. Note that

$$\begin{aligned} &\mathrm{rank}(e_0) = \sum_{j=1}^m r_j r_0 < m r_0 2K \text{ and} \\ &S_j \geq N_0 2K m r_0 > m r_0 2K, \;\; j = 1, 2, ..., 2K. \end{aligned} \tag{e 4.40}$$

It follows that $e_0 \lesssim e_1$ and e_i is equivalent to e_1. Thus, by (e 4.38), for some unitary $U \in M_n$,

$$\begin{aligned} \|\mathrm{Ad}\, U \circ H(f) - (\varphi(f) &\oplus \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{2K}))\| < \varepsilon_1 \\ &\text{for all } f \in \mathcal{F}_1. \end{aligned} \tag{e 4.41}$$

We also compute that, for all $g \in \mathcal{H}_0$,

$$\tau \circ \psi(g) \geq (1/2K)(\Delta(\hat{g}) - \varepsilon_1 - \frac{m r_0 2K}{N_0 2K m r_0}) \geq \alpha \frac{\Delta(\hat{g})}{2K}. \tag{e 4.42}$$

□

REMARK 4.7. If we also assume that X has infinitely many points, then Lemma 4.6 holds without mentioning the integer N. This can be seen by taking larger $\mathcal{H}_1$ which contains at least N mutually orthogonal non-zero elements. This will force the integer n to be larger than N.

DEFINITION 4.8. Denote by $\bar{\mathcal{D}}_0$ the class of all C^*-algebras with the form $PC(X,F)P$, where X is a compact metric space, F is a finite dimensional C^*-algebra, and $P \in C(X,F)$ is a projection. For $k \geq 1$, denote by $\bar{\mathcal{D}}_k$ the class of all C^*-algebras with the form:

$$A = \{(f,a) \in PC(X,F)P \oplus B : f|_Z = \Gamma(a)\},$$

where X is a compact metric space, F is a finite dimensional C^*-algebra, $P \in C(X,F)$ is a projection, $Z \subset X$ is a non-empty proper closed subset of X, $B \in \bar{\mathcal{D}}_m$ for some $0 \le m < k$, and $\Gamma : B \to P|_Z C(Z,F)P|_Z$ is a unital homomorphism, where we assume that there is $d_{X,Z} > 0$ such that, for any $0 < d \le d_{X,Z}$, there exists $s_*^d : \overline{X^d} \to Z$ such that, for all $x \in Z$ and $f \in C(X,F)$,

$$s_*^d(x) = x \text{ and } \lim_{d\to 0} \|f|_Z \circ s_*^d - f|_{\overline{X^d}}\| = 0, \tag{e 4.43}$$

where $X^d = \{x \in X : \mathrm{dist}(x,Z) < d\}$. We also assume that, for any $0 < d < d_{X,Z}/2$ and for any $d > \delta > 0$, there is a homeomorphism $r : X \setminus X^{d-\delta} \to X \setminus X^d$ such that

$$\mathrm{dist}(r(x),x) < \delta \text{ for all } x \in X \setminus X^{d-\delta}. \tag{e 4.44}$$

In what follows, as in 3.1, we will use $\lambda : A \to PC(X,F)P$ for the homomorphism defined by $\lambda((f,b)) = f$ for all $(f,b) \in A$.

Note that, $\bar{D}_{k-1} \subset \bar{D}_k$. Suppose that $A, B \in \bar{D}_k$. Then, from the definition above, it is routine to check that $A \oplus B \in \bar{D}_k$.

Let A_m be a unital C^*-algebra in $\bar{\mathcal{D}}_m$. For $0 \le k < m$, let $A_k \in \bar{\mathcal{D}}_k$ such that $A_{k+1} = \{(f,a) \in P_{k+1}C(X_{k+1},F_{k+1})P_{k+1} \oplus A_k : f|_{Z_{k+1}} = \Gamma_{k+1}(a)\}$, where F_{k+1} is a finite dimensional C^*-algebra, $P_{k+1} \in C(X_{k+1},F_{k+1})$ is a projection, $\Gamma_{k+1} : A_k \to P_{k+1}C(Z_{k+1},F_{k+1})P_{k+1}$ is a unital homomorphism, $k = 0,1,2,...,m-1$. Denote by $\partial_{k+1} : P_{k+1}C(X_{k+1},F_{k+1})P_{k+1} \to Q_{k+1}C(Z_{k+1},F_{k+1})Q_{k+1}$ the map defined by $f \mapsto f|_{Z_{k+1}}$ $(Q_{k+1} = P_{k+1}|_{Z_{k+1}})$. We use $\pi_e^{(k+1)} : A_{k+1} \to A_k$ for the quotient map and $\lambda_{k+1} : A_{k+1} \to P_{k+1}C(X_{k+1},F_{k+1})P_{k+1}$ for the map $(f,a) \mapsto f$.

For each k, one has the following commutative diagram:

$$\begin{tikzcd}
A_k \arrow[r, dashed, "\lambda_k"] \arrow[d, dashed, "\pi_e^{(k)}"] & P_kC(X_k,F_k)P_k \arrow[d, "\partial_k"] \\
A_{k-1} \arrow[r, "\Gamma_k"] & Q_kC(Z_k,F_k)Q_k.
\end{tikzcd} \tag{e 4.45}$$

In general, suppose that $A = A_m \in \bar{\mathcal{D}}_m$ is constructed as in the following sequence

$$\begin{array}{rcl}
A_0 & \in & \overline{\mathcal{D}}_0, \\
A_1 & = & P_1C(X_1,F_1)P_1 \oplus_{Q_1C(Z_1,F_1)Q_1} A_0, \\
A_2 & = & P_2C(X_2,F_2)P_2 \oplus_{Q_2C(Z_2,F_2)Q_2} A_1, \cdots, \\
A_m & = & P_mC(X_m,F_m)P_m \oplus_{Q_mC(Z_m,F_m)Q_m} A_{m-1},
\end{array}$$

where $Q_i = P_i|_{Z_i}$, $i = 1, 2, ..., m$. With $\pi_e^{(k+1)}$ and λ_k above we can define the quotient map $\Pi_k : A = A_m \to A_k$ and the homomorphism $\Lambda_k : A = A_m \to P_k C(X_k, F_k) P_k$ as follows:

$$\Pi_k = \pi_e^{(k+1)} \circ \pi_e^{(k+2)} \circ \cdots \circ \pi_e^{(m-1)} \circ \pi_e^{(m)} \qquad \text{and} \qquad \Lambda_k = \lambda_k \circ \Pi_k.$$

Combining all Λ_k we get the inclusion homomorphism

$$\Lambda : A \to \bigoplus_{k=0}^{m} P_k C(X_k, F_k) P_k$$

with X_0 being the single point set. In particular, A is a subhomogeneous C^*-algebra. For each $k \geq 1$, we may write

$$\begin{aligned} & P_k(C(X_k, F_k))P_k \\ = \ & P_{k,1}C(X_k, M_{s(k,1)})P_{k,1} \oplus P_{k,2}C(X_k, M_{s(k,2)})P_{k,2} \\ & \oplus \cdots \oplus P_{k,t_k}C(X_k, M_{s(k,t_k)})P_{t_k}, \end{aligned}$$

where $P_{k,j} \in C(X_k, M_{s(k,j)})$ is a projection of rank $r(k,j)$ at each $x \in X_k$. For each $x \in X_k$ and $j \leq t_k$ and $f \in A$, denote by $\pi_{(x,j)}(f) \in M_{r(k,j)}$ the evaluation of the j^{th} component of $\Lambda_k(f)$ at the point x. Then for each pair (x, j), $\pi_{(x,j)}$ is a finite dimensional representation of A, and, furthermore if we assume $x \in X_k \backslash Z_k$ (and $P_{k,j}(x) \neq 0$) then $\pi_{(x,j)}$ is an irreducible representation.

In the definition of $\bar{\mathcal{D}}_m$ above, if, in addition, X_k is path connected, Z_k has finitely many path connected components, and $X_k \setminus Z_k$ is path connected, then we will use $\mathcal{D}_m$ for the resulting class of C^*-algebras. Note that $\mathcal{D}_m \subset \bar{\mathcal{D}}_m$ and $\mathcal{C} \subset \mathcal{D}_1$.

Note that if X_k is a simplicial complex, $Z_k \subset X_k$ is sub-complex, $k = 1, 2, ..., m$, then any iterated pull-back

$$\begin{aligned} & (P_m C(X_m, F_m) P_m \oplus_{Q_k C(Z_k, F_k) Q_k} P_{k-1} C(X_{k-1}, F_{k-1}) P_k) \\ & \quad \oplus \cdots \oplus_{Q_1 C(Z_2, F_1) Q_1} P_0 C(X_0, F_0) P_0 \end{aligned}$$

is in $\mathcal{D}_m$.

REMARK 4.9. Let $A \in \bar{\mathcal{D}}_k$ (or $A \in \mathcal{D}_k$). It is easy to check that $C(\mathbb{T}) \otimes A \in \bar{\mathcal{D}}_{k+1}$ (or $C(\mathbb{T}) \otimes A \in \mathcal{D}_{k+1}$). First, if F_0 is a finite dimensional C^* algebra, then $C(\mathbb{T}) \otimes F_0 \in \bar{\mathcal{D}}_1$ by putting $F_1 = F_0$ and $X_1 = \mathbb{T}$ with $Z_1 = \{1\} \subset \mathbb{T}$ and $\Gamma_1 : F_0 \to C(Z_1, F_1) \cong F_0$ to be the identity map. And if a pair of spaces (X_k, Z_k) satisfies the conditions described for the pair (X, Z) in the definition above, in particular, the existence of the retraction s_*^d and homeomorphism r as in (e 4.43) and (e 4.44), then the pair $(X_k \times \mathbb{T}, Z_k \times \mathbb{T})$ also satisfies the same conditions.

LEMMA 4.10. *Let X be a measurable space with infinitely many points with a specified set $\mathcal{M}$ of k probability measures. Suppose $Y_1, Y_2, ..., Y_m$ are disjoint measurable sets with $m \geq kN$, where $N \geq 1$ is an integer. Then, for some j, $\mu(Y_j) < 1/N$ for all $\mu \in \mathcal{M}$.*

PROOF. Write $\mathcal{M} = \{\mu_1, \mu_2, ..., \mu_m\}$. Since $\sum_{i=1}^m \mu_1(Y_i) \leq 1$, there are at least $(k-1)N$ many Y_i's such that $\mu_1(Y_i) < 1/N$. We may assume that $\mu_1(Y_i) < 1/N$, $i = 1, 2, ..., (k-1)N$. Then among $\{Y_1, Y_2, ..., Y_{(k-1)N}\}$, there are at least $(k-2)N$ Y_i's such that $\mu_2(Y_i) < 1/N$. By induction, one finds at least one Y_i with the property that $\mu_j(Y_i) < 1/N$ for all j.

□

LEMMA 4.11. *Let $A \in \bar{\mathcal{D}}_k$ be a unital C^*-algebra. Let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $1 > \alpha > 1/2$.*

Let $\varepsilon > 0$, $\mathcal{F} \subset A$ be a finite subset, $\mathcal{H}_0 \subset A_+^{\mathbf{1}} \setminus \{0\}$ be a finite subset, and $K \geq 1$ be an integer. There exist an integer $N \geq 1$, $\delta > 0$, a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, and a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ satisfying the following condition:

If $\varphi_1, \varphi_2 : A \to M_n$ (for some integer $n \geq N$) are two unital homomorphisms such that

$$\text{(e 4.46)} \qquad \mathrm{tr} \circ \varphi_1(g) \geq \Delta(\hat{g}) \ \ \textit{for all } g \in \mathcal{H}_1 \ \textit{and}$$

$$\text{(e 4.47)} \qquad |\mathrm{tr} \circ \varphi_1(g) - \mathrm{tr} \circ \varphi_2(g)| < \delta \ \ \textit{for all } g \in \mathcal{H}_2,$$

where tr *is the tracial state on M_n, then there exist mutually orthogonal projections*
$e_0, e_1, e_2, ..., e_K \in M_n$ such that $e_1, e_2, ..., e_K$ are mutually equivalent, $e_0 \lesssim e_1$, and $e_0 + \sum_{i=1}^K e_i = 1_{M_n}$, unital homomorphisms $h_1, h_2 : A \to e_0 M_n e_0$, a unital homomorphism $\psi : A \to e_1 M_n e_1$, and a unitary $u \in M_n$ such that

$$\text{(e 4.48)} \qquad \|\mathrm{Ad}\, u \circ \varphi_1(f) - (h_1(f) \oplus \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K}))\| < \varepsilon,$$

$$\text{(e 4.49)} \qquad \|\varphi_2(f) - (h_2(f) \oplus \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K}))\| < \varepsilon \ \ \textit{for all } f \in \mathcal{F},$$

$$\text{(e 4.50)} \qquad \textit{and } \mathrm{tr} \circ \psi(g) \geq \alpha \frac{\Delta(\hat{g})}{K} \ \ \textit{for all } g \in \mathcal{H}_0,$$

where tr *is the tracial state on M_n.*

PROOF. We will use the induction on the integer $k \geq 0$. The case $k = 0$ follows from Corollary 4.6. Assume that the conclusion of the lemma holds for integers $0 \leq k \leq m$.

We assume that $A \in \bar{\mathcal{D}}_{m+1}$. We will retain the notation for A as an algebra in $\bar{\mathcal{D}}_{m+1}$ in the later part of Definition 4.8. Put $X_{m+1} = X$, $Z_{m+1} = Z$, $Y = X \setminus Z$, $X^0 = Z = X \setminus Y$, and $I = PC_0(Y, F)P \subset A$. We will write

$$\text{(e 4.51)} \qquad A = \{(f, b) \in PC(X, F)P \oplus B : f|_{X^0} = \Gamma(b)\},$$

where $B \in \bar{\mathcal{D}}_m$ is a unital C^*-algebra and will be identified with A/I. We also keep the notation $\lambda : A \to PC(X,F)P$ in the pull-back of 4.8. We will write $f|_S$ for $\lambda(f)|_S$ for $f \in A$ and $S \subset X$ in the proof when there is no confusion. Let $d_{X,Z} > 0$ be as given in 4.8. Denote by $\pi_I : A \to A/I$ the quotient map. We may write

$$PC(X,F)P = \bigoplus_{j=1}^{k_2} P_j C(X_j, M_{s(k,j)}) P_j, \tag{e 4.52}$$

where $P_j \in C(X, M_{s(k,j)})$ is a projection of rank $r(j)$ at each $x \in X_j$. Let us say that the dimensions of the irreducible representations of A/I are $l_1, l_2, ..., l_{k_1}$. Set

$$T \;=\; (k_1 k_2 \cdot \max_{i,j}\{z_i z_j : z_i, z_j \in \{l_1, l_2, ..., l_{k_1}, r(1), r(2), ..., r(k_2)\}\} \tag{e 4.53}$$

$$\text{and } \delta_{00} = \min\{\Delta(\hat{g})/2 : g \in \mathcal{H}_0\} > 0. \tag{e 4.54}$$

Let $\beta = \sqrt{1-(1-\alpha)/8} = \sqrt{(7+\alpha)/8}$. Note that $1 > \beta^2 > \alpha$. Fix $N_{00} \geq 4$ such that

$$1/N_{00} < \frac{(1-\beta)\delta_{00}}{64}. \tag{e 4.55}$$

Fix $\varepsilon > 0$, a finite subset $\mathcal{F} \subset A$, and a finite subset $\mathcal{H}_0 \subset A_+^{\mathbf{1}} \setminus \{0\}$, and let $K > 0$ be an integer. Let $K_0 = N_{00}K$. We may assume that $1_A \in \mathcal{H}_0 \subset \mathcal{F}$. Without loss of generality, we may also assume that $\mathcal{F} \subset A_{s.a.}$ and $\|f\| \leq 1$ for all $f \in \mathcal{F}$. Write $I = \{f \in PC(X,F)P : f|_{X^0} = 0\}$. There is $d > 0$ such that

$$\|\pi_{x,j}(f) - \pi_{x',j}(f)\| < \min\{\varepsilon, \delta_{00}\}/256KN_{00} \text{ for all } f \in \mathcal{F}, \tag{e 4.56}$$

provided that $\mathrm{dist}(x, x') < d$ for any pair $x, x' \in X$ (here we identify $\pi_{x,j}(f)$ with $\pi_{x,j}(\Lambda(f))$ —see Definition 4.8). Put $\varepsilon_0 = \min\{\varepsilon, \delta_{00}\}/16KN_{00}$.

We also assume that, for any $x \in X^d = \{x \in X : \mathrm{dist}(x, X^0) < d\}$, choosing a smaller d if necessary,

$$\|\pi_{x,j} \circ s^d \circ (\lambda(f)|_Z) - \pi_{x,j}(f)\| < \varepsilon_0/16 \text{ for all } f \in \mathcal{F}, \tag{e 4.57}$$

where $s^d : QC(Z,F)Q \to P|_{X^d} C(\overline{X^d}, F) P|_{X^d}$ is induced by $s_*^d : \overline{X^d} \to Z$ (see 4.8). Note that s^d also induces a map (still denoted by s^d)

$$s^d : B \to P|_{\overline{X^d}} C(\overline{X^d}, F) P|_{\overline{X^d}} \oplus_{QC(Z,F)Q} B, \tag{e 4.58}$$

where $Q = P|_Z$, by $s^d(a) = (s^d(\Gamma(a)), a)$ for all $a \in B$. To simplify notation, let us assume that $d < d_{X,Z}/2$.

For any $b > 0$, as in 4.8, we will continue to use X^b for $\{x \in X : \mathrm{dist}(x, X^0) < b\}$.

Let $Y_{0,d/2} = X \setminus X^{d/2}$. Note $Y_{0,d/2}$ is closed. Put $C_{I,0} = PC(Y_{0,d/2}, F)P$. Let $\mathcal{F}_{I,0} = \{f|_{Y_{0,d/2}} : f \in \mathcal{F}\}$ and set $\mathcal{H}_{0,I,0} = \{h|_{Y_{0,d/2}} : h \in \mathcal{H}_0\}$. Let $f_{0,0} \in C_0(Y)_+$ be such that $0 \le f_{0,0} \le 1$, $f_{0,0}(x) = 1$ if $x \in X \setminus X^d$, $f_{0,0}(x) = 0$ if $x \notin Y_{0,d/2}$, and $f_{0,0}(x) > 0$ if $\mathrm{dist}(x, X^0) > d/2$.

Let $\Delta_{I,0} : (C_{I,0})_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be defined by

$$\Delta_{I,0}(\hat{g}) = \beta\Delta(\widehat{g'}) \ \text{ for all } \ g \in (C_{I,0})_+^{\mathbf{1}} \setminus \{0\}, \tag{e 4.59}$$

where $g' = (f_{0,0} \cdot P) \cdot g$ is viewed as an element in $I_+^{\mathbf{1}}$. Note that if $g \in C_{I,0}$ and $g \ne 0$, then $(f_{0,0} \cdot P) \cdot g \ne 0$. So $\Delta_{I,0} : (C_{I,0})_+^{q,\mathbf{1}} \to (0,1)$ is an order preserving map. Let $N^I \ge 1$ be an integer (in place of N) as provided by 4.6 for $C_{I,0}$ (in place of A), $\Delta_{I,0}$ (in place of Δ), $\varepsilon_0/16$ (in place of ε), $\mathcal{F}_{I,0}$ (in place of $\mathcal{F}$), $\mathcal{H}_{0,I,0}$ (in place of $\mathcal{H}_0$), and $2K_0$ (in place of K).

Let $\mathcal{H}_{1,I,0} \subset (C_{I,0})_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\mathcal{H}_{2,I,0} \subset (C_{I,0})_{s.a.}$ (in place of $\mathcal{H}_2$), and $\delta_1 > 0$ (in place of δ) be the finite subsets and constant provided by 4.6 for $\varepsilon_0/16$ (in place of ε), $\mathcal{F}_{I,0}$ (in place of $\mathcal{F}$), $2K_0$ (in place of K), $\mathcal{H}_{0,I,0}$ associated with $C_{I,0}$ (in place of A), $\Delta_{I,0}$ (in place of Δ), and β (in place of α). Without loss of generality, we may assume that $\|g\| \le 1$ for all $g \in \mathcal{H}_{2,I,0}$. We may assume that $1_{C_{I,0}} \in \mathcal{H}_{1,I,0}$ and $1_{C_{I,0}} \in \mathcal{H}_{2,I,0}$.

Let $\mathcal{F}_\pi = \pi_I(\mathcal{F})$. Let $g_0' \in C(X)_+$ with $0 \le g_0' \le 1$ such that $g_0'(x) = 0$ if $\mathrm{dist}(x, X^0) < d/256$ and $g_0'(x) = 1$ if $\mathrm{dist}(x, Y_{0,d/2}) \le d/16$. Define $g_0 = 1_A - g_0' \cdot P = ((1 - g_0')P, 1_B)$. Since $g_0' \cdot P \in I$, we view g_0 as an element of A. Hence, for $g \in A/I = B$, $g_0 \cdot s^d(g) = ((1 - g_0')P \cdot s^d(\Gamma(g)), g) \in A$ (see (e 4.58)). Define

$$\Delta_\pi(\hat{g}) = \beta\Delta(\widehat{g_0 \cdot s^d(g)}) \ \text{ for all } \ g \in (A/I)_+^{\mathbf{1}}. \tag{e 4.60}$$

We will later use the fact that $g_0(x) = 0$ if $\mathrm{dist}(x, Y_{0,d/2}) \le d/16$.

Note if g is non-zero, so is $s^d(g)$. Since $g_0|_{X^0} = 1$, we have $g_0 \cdot s^d(g) \ne 0$. It follows that $\Delta_\pi : (A/I)_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ is an order preserving map.

Put $\mathcal{H}_{0,\pi} = \pi_I(\mathcal{H}_0)$. Let $N^\pi \ge 1$ be the integer associated with $A/I\,(= B)$, Δ_π, $\varepsilon_0/16$, $\mathcal{F}_\pi$ and $\mathcal{H}_{0,\pi}$ (as required by the inductive assumption that the lemma holds for integer m).

Let $\mathcal{H}_{1,\pi} \subset (A/I)_+^{q,\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\mathcal{H}_{2,\pi} \subset A/I_{s.a.}$ (in place of $\mathcal{H}_2$), and $\delta_2 > 0$ (in place of δ) denote the finite subsets and constant provided by the inductive assumption that the lemma holds for the case that $k = m$ for $\varepsilon_0/16$ (in place of ε), $\mathcal{F}_\pi$ (in place of $\mathcal{F}$), $\mathcal{H}_{0,\pi}$ (in place of $\mathcal{H}_0$), $2K_0$ associated with A/I (in place of A), Δ_π (in place of Δ), and β (in place of α). Without loss of generality, we may assume that $\|h\| \le 1$ for all $h \in \mathcal{H}_{2,\pi}$.

Set $\delta_{000} = \min\{\delta_1, \delta_2, \varepsilon_0\}$. There is an integer $N_0 \ge 256$ such that

$$\begin{aligned} 1/N_0 < \Delta(\widehat{f_{0,0} \cdot P}) \cdot {\delta_{000}}^2 \quad &\cdot \quad \min\{\Delta_{I,0}(\hat{g}) : g \in \mathcal{H}_{1,I,0}\} \\ &\cdot \quad \min\{\Delta_\pi(\hat{g}) : g \in \mathcal{H}_{1,\pi}\}/64K_0N_{00}. \end{aligned} \tag{e 4.61}$$

Choose $0 < d_0 < d$ such that, if $\mathrm{dist}(x, x') < d_0$, then, for all $g \in \mathcal{H}_{1,I,0}$.

$$\|g(x) - g(x')\| < 1/16N_0^2 \ \text{ and } \ \|f_{00}Pg(x) - f_{00}Pg(x')\| < 1/16N_0^2 \tag{e 4.62}$$

Define Y_k to be the closure of $\{y \in Y : \mathrm{dist}(y, Y_{0,d/2}) < kd_0/64N_0^2\}$, $k = 1, 2, ..., 4N_0^2$.

Let $\mathcal{F}_{I,k} = \{f|_{Y_k} : f \in \mathcal{F}\}$ and let $\mathcal{H}_{0,I,k} = \{h|_{Y_k} : h \in \mathcal{H}_0\}$. Put $C_{I,k} = P|_{Y_k}C(Y_k, F)P|_{Y_k}$. Let $f_{0,k} \in C_0(Y)_+$ be such that $0 \le f_{0,k} \le 1$, $f_{0,k}(x) = 1$ if $x \in Y_{k-1}$, $f_{0,k}(x) = 0$ if $x \notin Y_k$ and $f_{0,k}(x) > 0$ if $\mathrm{dist}(x, Y_{0,d/2}) < kd_0/64N_0^2$, $k = 1, 2, ..., 4N_0^2$.

Let $r_k : Y_k \to Y_{0,d/2}$ be a homeomorphism such that

$$\text{(e 4.63)} \quad \mathrm{dist}(r_k(x), x) < d_0/16 \text{ for all } x \in Y_k, \ \ k = 1, 2, ..., 4N_0^2$$

(see 4.8). Let $\mathcal{F}'_{I,k} = \{f \circ r_k : f \in \mathcal{F}_{I,0}\}$ and $\mathcal{H}'_{0,I,k} = \{g \circ r_k : g \in \mathcal{H}_{0,I,0}\}$, $k = 1, 2, ..., 4N_0^2$.

Any unital homomorphism $\Phi : C_{I,k} \to D$ (for a unital C^*-algebra D) induces a unital homomorphism $\Psi : C_{I,0} \to D$ defined by $\Psi(f) = \Phi(f \circ r_k)$ for all $f \in C_{I,0}$. Note that $f \mapsto f \circ r_k$ is an isomorphism from $C_{I,0}$ onto $C_{I,k}$. Therefore, applying Corollary 4.6 $4N_0$ times, for $\varepsilon_0/16$ (in place of ε), $\mathcal{F}'_{I,k}$ (in place of $\mathcal{F}$), $2K_0$ (in place of K), and $\mathcal{H}'_{0,I,k}$ (in place of $\mathcal{H}_0$) associated with $C_{I,k}$ (in place of A), $\Delta_{I,0}$ (in place of Δ), and β (in place of α), we obtain $\mathcal{H}_{1,I,k}$ (in place of $\mathcal{H}_1$), which we may suppose equal to $\mathcal{H}_{1,I,0} \circ r_k$, $\mathcal{H}_{2,I,k}$ (in place of $\mathcal{H}_2$), which we may suppose equal to $\mathcal{H}_{2,I,0} \circ r_k$, and δ_1 (in place of δ). (Note that $\Delta_{I,0}$ is the same as above.)

We also note that

$$\text{(e 4.64)} \ \|f - f|_{Y_{0,d/2}} \circ r_k\| < \min\{\varepsilon, \delta_{00}\}/64K_0N_{00} \text{ for all } f \in \mathcal{F}_{I,k}.$$

Define

$$\text{(e 4.65)} \quad \sigma_0 = \min\{\min_{0 \le k \le 4N_0^2}\{\min\{\Delta_{I,0}(\widehat{g \circ r_k}) : g \in \mathcal{H}_{1,I,k}\}\}, \\ \min\{\Delta_\pi(\hat{g}) : g \in \mathcal{H}_{1,\pi}\}\}.$$

Then $\Delta(\widehat{f_{00} \cdot P}) \ge \sigma_0$. Choose an integer $N \ge (N^\pi + N^I)$ such that

$$\text{(e 4.66)} \quad T/N < \sigma_0 \cdot \min\{\delta_1/64, \delta_2/64, \varepsilon_0/64K_0\}/N_{00}(N^l + N^\pi).$$

Put

$$\text{(e 4.67)} \quad \mathcal{H}_1 = \bigcup_{k=0}^{4N_0^2}\{f_{0,k} \cdot P \circ g \circ r_k : g \in \mathcal{H}_{1,I,0}\} \\ \cup\{(g_0 \cdot P \cdot s^d(g), g) : g \in \mathcal{H}_{1,\pi}\}.$$

With the convention that $r_0 : Y_{0,d/2} \to Y_{0,d/2}$ is the identity map, put

$$\text{(e 4.68)} \quad \mathcal{H}'_2 = \bigcup_{k=0}^{4N_0^2}\{f_{0,k} \cdot P \cdot g \circ r_k : g \in \mathcal{H}_{2,I,0}\} \\ \cup\{f_{0,k} \cdot P : 0 \le k \le 4N_0^2\}.$$

Put $g_{0.k} = 1_A - f_{0,k} \cdot P$. Note, since $f_{0,k} \cdot P \in I$, $g_{0,k}= ((1-f_{0,k})P, 1_B) \in A$. Define

$$\mathcal{H}_2'' = \bigcup_{k=1}^{4N_0^2} \{(g_{0,k} \cdot s^d(\Gamma(g)), g) \in A : g \in \mathcal{H}_{2,\pi}\} \cup \mathcal{F}. \tag{e 4.69}$$

Put $\mathcal{H}_2 = \mathcal{H}_2' \cup \mathcal{H}_2''$. Let

$$\delta = \frac{\sigma_0 \cdot \min\{\delta_1/64, \delta_2/64, \varepsilon_0/64K\}}{4K_0 N_0 N_{00}}. \tag{e 4.70}$$

Now let $\varphi_1, \varphi_2 : A \to M_n$ (for some integer $n \geq N$) be two unital homomorphisms satisfying the assumptions for the above $\mathcal{H}_1$, $\mathcal{H}_2$, and δ.

Consider the two finite Borel measures on Y defined by

$$\int_Y f \mu_i = \mathrm{tr} \circ \varphi_i(f \cdot P) \ \text{ for all } \ f \in C_0(Y), \ \ i = 1, 2. \tag{e 4.71}$$

Note that $\{Y_k \setminus Y_{k-1} : k = 1, 2, ..., 4N_0^2\}$ is a family of $4N_0^2$ disjoint Borel sets. By 4.10, there exists k such that

$$\mu_i(Y_k \setminus Y_{k-1}) < 1/N_0, , \ \ i = 1, 2. \tag{e 4.72}$$

We fix this k. We may write

$$\varphi_1 = \Sigma_\pi^1 \oplus \Sigma_b^1 \oplus \Sigma_s^1 \oplus \Sigma_I^1 \ \text{ and } \ \varphi_2 = \Sigma_\pi^2 \oplus \Sigma_b^2 \oplus \Sigma_s^2 \oplus \Sigma_I^2, \tag{e 4.73}$$

where Σ_I^1 and Σ_I^2 are finite direct sums of terms of the form $\pi_{x,j}$ for $x \in Y_{k-1}$, Σ_s^1 and Σ_s^2 are finite direct sums of terms of the form $\pi_{x,j}$ for $x \in Y_k \setminus Y_{k-1}$, Σ_b^1 and Σ_b^2 are finite direct sums of terms of the form $\pi_{x,j}$ for $x \in Y \setminus Y_k$, and Σ_π^1 and Σ_π^2 are finite direct sums of terms of the form $\bar{\pi}_{x,i}$ given by irreducible representations of A/I (note that these $\pi_{x,j}$ or $\bar{\pi}_{x,i}$ can be repeated).

Define $\psi_I^{1,0}, \psi_I^{2,0} : C_{I,k} \to M_n$ by

$$\psi_I^{i,0}(f) = \Sigma_I^i(f) \ \text{ for all } \ f \in C_{I,k}, \ \ i = 1, 2. \tag{e 4.74}$$

By (e 4.72), the choice of $\mathcal{H}_2$, (e 4.47), and (e 4.61), we estimate that

$$\begin{aligned}
&|\mathrm{tr} \circ \psi_I^{1,0}(1_{C_{I,k}}) - \mathrm{tr} \circ \psi_I^{2,0}(1_{C_{I,k}})| && \text{(e 4.75)}\\
\leq\ & |\mathrm{tr} \circ \Sigma_I^1(f_{0,k} \cdot P) - \mathrm{tr} \circ \varphi_1(f_{0,k} \cdot P)| + |\mathrm{tr} \circ \varphi_1(f_{0,k} \cdot P) \\
&-\mathrm{tr} \circ \varphi_2(f_{0,k} \cdot P)| && \text{(e 4.76)}\\
&+|\mathrm{tr} \circ \varphi_2(f_{0,k} \cdot P) - \Sigma_I^2(f_{0,k} \cdot P)| < 1/N_0 + \delta + 1/N_0 && \text{(e 4.77)}\\
\leq\ & \delta + \Delta(\widehat{f_{00} \cdot P})\delta_1 \min\{\Delta_{I,0}(\hat{g}) : g \in \mathcal{H}_{1,I,0}\}/32K_0 N_{00}. && \text{(e 4.78)}
\end{aligned}$$

Note that $n\mathrm{tr}\circ\psi_I^{1,0}(1_{C_{I,k}}) = \sum a_i z_i$ and $n\mathrm{tr}\circ\psi_I^{2,0}(1_{C_{I,k}}) = \sum_j b_j z_j$ (integer combinations), where $z_i \in \{r(1), r(2), ..., r(k_2)\} \subset \{l_1, l_2, ..., l_{k_1}, r(1), r(2), ..., r(k_2)\}$ (see (e 4.53)). It follows from Lemma 4.1 that there are two equivalent projections $p_{1,0}$, $p_{2,0} \in M_n$ such that $p_{i,0}$ commutes with $\psi_I^{i,0}(f)$ for all $f \in C_{I,k}$ and $p_{i,0}\psi_I^{i,0}(1_{C_{I,k}}) = p_{i,0}$, $i = 1, 2$, and so (by (e 4.77)), for $i = 1, 2$,

$$\begin{array}{lrcl}
 & 0 & \le & \mathrm{tr}\circ\psi_I^{i,0}(1_{C_{I,k}}) - \mathrm{tr}(p_{i,0}) \\
\text{(e 4.79)} & & < & (1/N_0 + \delta + 1/N_0) + T/n \\
 & & < & \delta + \Delta(\widehat{f_{00}\cdot P})\delta_1 \min\{\Delta_{I,0}(\hat{g}) : g \in \mathcal{H}_{1,I,0}\}/32K_0N_{00} \\
\text{(e 4.80)} & & & +T/n (< 1/2).
\end{array}$$

Since $Y_{0,d/2} \subset Y_{k-1}$, supp($f_{00}$)= $Y_{0,d/2} \subset Y_{k-1}$. Therefore (by (e 4.66)),

$$\begin{array}{lrcl}
 & \mathrm{tr}\circ\psi_I^{1,0}(1_{C_{I,k}}) & \ge & \mathrm{tr}\circ\psi_I^{1,0}(f_{00}\cdot P) \\
\text{(e 4.81)} & & \ge & \Delta(\widehat{f_{00}\cdot P}) \ge \sigma_0 > 4N^I/n.
\end{array}$$

This, in particular, shows that $\psi_I^{1,0}(1_{C_{I,k}})$ has rank at least $4N^I$. Then, by (e 4.80), $p_{1,0}$ has rank at least N^l. Moreover, by (e 4.81), (e 4.80), (e 4.61), (e 4.65), (e 4.70), and (e 4.66),

$$\begin{array}{lrcl}
 & \mathrm{tr}(p_{2,0}) & > & \Delta(\widehat{f_{00}\cdot P}) \\
\text{(e 4.82)} & & & -(\delta + \Delta(\widehat{f_{00}\cdot P})\delta_1 \min\{\Delta_{I,0}(\hat{g}) : g \in \mathcal{H}_{1,I,0}\}/32K_0N_{00} + T/n) \\
 & & \ge & 31\Delta(\widehat{f_{00}\cdot P})/32 - (\delta + T/N) \\
\text{(e 4.83)} & & \ge & \max\{31\sigma_0/32, 64/(N_0\delta_1)\} - (\delta + T/N) \\
\text{(e 4.84)} & & \ge & \max\{\sigma_0, 64/(N_0\delta_1)\}/2.
\end{array}$$

Put $q_{i,0} = \psi_I^{i,0}(1_{C_{I,k}}) - p_{i,0}$, $i = 1, 2$. There is a unitary $U_1 \in M_n$ such that $U_1^* p_{1,0} U_1 = p_{2,0}$. Define $\psi_I^1 : C_{I,k} \to p_{2,0}M_n p_{2,0}$ by $\psi_I^1(f) = U_1^* p_{1,0}\psi_I^{1,0}(f)U_1$ for all $f \in C_{I,k}$ and define $\psi_I^2 : C_{I,k} \to p_{2,0}M_n p_{2,0}$ by $\psi_I^2(f) = p_{2,0}\psi_I^{2,0}(f)$ for all $f \in C_{I,k}$. We compute (using (e 4.62), (e 4.79), (e 4.70), (e 4.61), (e 4.66), (e 4.46), and (e 4.55)) that

$$\begin{array}{lcl}
 & & \mathrm{tr}\circ\psi_I^1(g\circ r_k) \\
\text{(e 4.85)} & \ge & \mathrm{tr}\circ\psi_I^{1,0}((f_{0,0}Pg)\circ r_k) - \mathrm{tr}(q_{1,0}) \\
\text{(e 4.86)} & > & \mathrm{tr}\circ\psi_I^{1,0}(f_{0,0}Pg) - 1/16N_0^2 - (\delta + 2/N_0 + T/N) \\
\text{(e 4.87)} & > & \mathrm{tr}\circ\psi_I^{1,0}(f_{0,0}Pg) - 5\min\{\Delta_{I,0}(\hat{g}) : g \in \mathcal{H}_{1,I,0}\}/64N_{00} \\
\text{(e 4.88)} & > & (1 - (1-\beta)/64)\mathrm{tr}\circ\psi_I^{1,0}(f_{0,0}Pg) \\
\text{(e 4.89)} & > & \beta\Delta(\widehat{f_{0,0}Pg}) = \Delta_{I,0}(g)
\end{array}$$

for all $g \in \mathcal{H}_{1,I,0}$. Therefore,

$$t \circ \psi_I^1(g) \ge \Delta_{I,0}(g) \ \text{ for all } \ g \in \mathcal{H}_{1,I,k}, \tag{e 4.90}$$

where t is the tracial state on $p_{2,0}M_np_{2,0}$. We also estimate that (using (e 4.72), (e 4.47), (e 4.84), (e 4.79), (e 4.70), and (e 4.66)),

$$\begin{aligned}
& |t \circ \psi_I^1(g) - t \circ \psi_I^2(g)| \\
= \ & (1/\mathrm{tr}(p_{1,0}))|\mathrm{tr} \circ \psi_I^1(g) - \mathrm{tr} \circ \psi_I^2(g)| && \text{(e 4.91)} \\
\le \ & (1/\mathrm{tr}(p_{1,0}))|\mathrm{tr} \circ \psi_I^1(g) - \mathrm{tr} \circ \psi_I^{1,0}(g)| && \text{(e 4.92)} \\
+ \ & (1/\mathrm{tr}(p_{1,0}))|\mathrm{tr} \circ \psi_I^{1,0}(g) - \mathrm{tr}(\varphi_1(f_{0,k} \cdot 1_A \cdot g))| && \text{(e 4.93)} \\
+ \ & (1/\mathrm{tr}(p_{2,0}))|\mathrm{tr}(\varphi_1(f_{0,k} \cdot 1_A \cdot g)) - \mathrm{tr}(\varphi_2(f_{0,k} \cdot 1_A \cdot g))| && \text{(e 4.94)} \\
+ \ & (1/\mathrm{tr}(p_{2,0}))|\mathrm{tr}(\varphi_2(f_{0,k} \cdot 1_A \cdot g) - \mathrm{tr} \circ \psi_I^{2,0}(g)| && \text{(e 4.95)} \\
+ \ & |\mathrm{tr} \circ \psi_I^{2,0}(g) - \mathrm{tr} \circ \psi_I^2(g)| && \text{(e 4.96)} \\
< \ & (1/\mathrm{tr}(p_{2,0}))(\mathrm{tr}(q_{1,0}) + 1/N_0 + \delta + 1/N_0 + \mathrm{tr}(q_{2,0})) && \text{(e 4.97)} \\
< \ & (N_0\delta_1/32)((4/N_0 + 2\delta + 2T/n) < \delta_1 && \text{(e 4.98)}
\end{aligned}$$

for all $g \in \mathcal{H}_{2,I,k}$. Recall that $p_{2,0}$ has rank at least N^I. It follows (by (e 4.90) and (e 4.98), by Corollary 4.6, and by the choice of $\mathcal{H}_{1,I,k}$, $\mathcal{H}_{2,I,k}$, and δ_1) that there are mutually orthogonal projections $e_0^I, e_1^I, e_2^I, ..., e_{2K_0}^l \in p_{2,0}M_np_{2,0}$ such that $e_0^I + \sum_{i=1}^{2K_0} e_i^I = p_{2,0}$, $e_0^I \lesssim e_1^I$, and all e_j^I are equivalent to e_1^I, two unital homomorphisms $\psi_{1,I,0}, \psi_{2,I,0} : C_{I,k} \cong C_{I,0} \to e_0^I M_n e_0^I$, a unital homomorphism $\psi_I : C_{I,k} \to e_1^I M_n e_1^I$, and a unitary $u_1 \in p_{2,0}M_np_{2,0}$ such that

$$\|\mathrm{Ad}\, u_1 \circ \psi_I^1(f) - (\psi_{1,I,0}(f) \oplus \mathrm{diag}(\overbrace{\psi_I(f), \psi_I(f), ..., \psi_I(f)}^{2K_0}))\| < \varepsilon_0/16 \tag{e 4.99}$$

$$\text{and} \quad \|\psi_I^2(f) - (\psi_{2,I,0}(f) \oplus \mathrm{diag}(\overbrace{\psi_I(f), \psi_I(f), ..., \psi_I(f)}^{2K_0}))\| < \varepsilon_0/16 \tag{e 4.100}$$

for all $f \in \mathcal{F}'_{I,k}$. By (e 4.64), it follows that, for all $f \in \mathcal{F}_{I,k}$,

$$\|\mathrm{Ad}\, u_1 \circ \psi_I^1(f) - (\psi_{1,I,0}(f) \oplus \mathrm{diag}(\overbrace{\psi_I(f), \psi_I(f), ..., \psi_I(f)}^{2K_0}))\| < \varepsilon_0/8 \tag{e 4.101}$$

$$\text{and} \quad \|\psi_I^2(f) - (\psi_{2,I,0}(f) \oplus \mathrm{diag}(\overbrace{\psi_I(f), \psi_I(f), ..., \psi_I(f)}^{2K_0}))\| < \varepsilon_0/8. \tag{e 4.102}$$

For each $x \in X \setminus Y_k$ such that $\pi_{x,j}$ appeares in Σ_b^1, or Σ_b^2, by (e 4.57),

$$\|\pi_{x,j}(f) - \pi_{x,j} \circ s^d \circ \pi_I(f))\| < \varepsilon_0/16 \text{ for all } f \in \mathcal{F}. \tag{e 4.103}$$

Define $\Sigma_{\pi,b,i} := \Sigma_b^i \circ s^d : A/I \to M_n$, $i = 1, 2$.

Define $\Phi_1 : A/I \to (1 - p_{2,0})M_n(1 - p_{2,0})$ by

$$\Phi_1(f) = \mathrm{Ad}\, U_1 \circ (\Sigma_\pi^1 \oplus \Sigma_{\pi,b,1})(f) \text{ for all } f \in A/I. \tag{e 4.104}$$

Define $\Phi_2 : A/I \to (1 - p_{2,0})M_n(1 - p_{2,0})$ by

$$\Phi_2(f) = (\Sigma_\pi^1 \oplus \Sigma_{\pi,b,2})(f) \text{ for all } f \in A/I. \tag{e 4.105}$$

Note that

$$\Phi_1(1_{A/I}) = \Sigma_\pi^1(g_{0,k}) \oplus \Sigma_b^1(g_{0,k}) \text{ and } \Phi_2(1_{A/I}) = \Sigma_\pi^1(g_{0,k}) \oplus \Sigma_b^2(g_{0,k}). \tag{e 4.106}$$

We compute that

$$\begin{aligned}
&|\mathrm{tr} \circ \Phi_1(1_{A/I}) - \mathrm{tr} \circ \Phi_2(1_{A/I})| \\
\text{(e 4.107)} \quad &\leq |\mathrm{tr} \circ \Phi_1(1_{A/I}) - \mathrm{tr} \circ \varphi_1(g_{0,k})| \\
&\quad + |\mathrm{tr} \circ \varphi_1(g_{0,k}) - \mathrm{tr} \circ \varphi_2(g_{0,k})| \\
\text{(e 4.108)} \quad &\quad + |\mathrm{tr} \circ \varphi_2(g_{0,k}) - \mathrm{tr} \circ \Phi_2(g_{0,k})| \\
\text{(e 4.109)} \quad &< 1/N_0 + \delta + 1/N_0 \\
\text{(e 4.110)} \quad &< \delta + \Delta(\widehat{f_{00} \cdot P})\delta_2 \min\{\Delta_\pi(\hat{g}) : g \in \mathcal{H}_{1,\pi}\}/32N_{00}.
\end{aligned}$$

It follows from Lemma 4.1 that there are two mutually equivalent projections $p_{1,1}$ and $p_{2,1} \in (1 - p_{2,0})M_n(1 - p_{2,0})$ such that $p_{i,1}$ commutes with $\Phi_i(f)$ for all $f \in A/I$ and $p_{i,1}\Phi_i(1_{A/I}) = p_{i,1}$. $i = 1, 2$, and, for $i = 1, 2$,

$$\begin{aligned}
\text{(e 4.111)} \quad & 0 \leq \mathrm{tr} \circ \Phi_i(1_{A/I}) - \mathrm{tr}(p_{i,1}) < \delta + 2/N_0 + T/n \\
\text{(e 4.112)} \quad & < \delta + \Delta(\widehat{f_{00}})\delta_2 \min\{\Delta_\pi(\hat{g}) : g \in \mathcal{H}_{1,\pi}\}/(32N_{00}) + T/n(< 1/2).
\end{aligned}$$

Since $g_0(x) = 0$, if $\mathrm{dist}(x, Y_{0,d/2}) \leq d/16$, we have, by (e 4.61),

$$\mathrm{tr} \circ \Phi_1(1_{A/I}) > \Delta(\widehat{g_0 \cdot s^d(1)}) > \Delta_\pi(\hat{1}) \geq \max\{\sigma_0, 64KN_{00}/(N_0\delta_2)\},$$

and (by (e 4.112)) $\mathrm{tr}(p_{2,1}) \geq \max\{\sigma_0, 64K_0N_{00}/(N_0\delta_2)\}/2$. Since $\sigma_0 > 4N^\pi/N \geq 4N^\pi/n$ (see (e 4.66)), $\Phi_i(1_{A/I})$ has rank at least $4N^\pi$. Then, by (e 4.112), it follows that $p_{i,1}$ has rank at least N^π, $i = 1, 2$.

Put $q_{i,1}=\Phi_i(1_{A/I})-p_{i,1}$, $i=1,2$. There is a unitary $U_2\in(1-p_{2,0})M_n(1-p_{2,0})$ such that $U_2^*p_{1,1}U_2=p_{2,1}$. Define $\Phi_\pi^1: A/I\to p_{2,1}M_np_{2,1}$ by $\Phi_\pi^1(f)=U_2^*p_{1,1}\Phi_1(f)U_2$ for all $f\in A/I$ and define $\Phi_\pi^2: A/I\to p_{2,1}M_np_{2,1}$ by $\Phi_\pi^2(f)=p_{2,1}\Phi_2(f)$ for all $f\in A/I$.

Since $g_0(x)=0$, if $\mathrm{dist}(x,Y_{0,d/2})\le d/16$, and $\{g_0s^d(g): g\in\mathcal{H}_{1,\pi}\}\subset\mathcal{H}_1$, we compute (using (e 4.46), (e 4.111), (e 4.70), (e 4.66), (e 4.55), and (e 4.60)) that

$$\mathrm{tr}\circ\Phi_\pi^1(g) \ge \varphi_1(g_0s^d(g))-\mathrm{tr}(q_{1,1})>\Delta(\widehat{g_0s^d(g)})-\sigma_0/N_{00} \tag{e 4.113}$$

$$> \beta\Delta(\widehat{g_0s^d(g)})=\Delta_\pi(g) \tag{e 4.114}$$

for all $g\in\mathcal{H}_{1,\pi}$. Therefore, for the tracial state t_1 of $p_{2,1}M_np_{2,1}$,

$$t_1\circ\Phi_\pi^1(g)\ge\Delta_\pi(g)\ \text{ for all }\ g\in\mathcal{H}_{1,\pi}. \tag{e 4.115}$$

We also estimate (in a way similar to the estimate of (e 4.98)) that, for all $g\in\mathcal{H}_{2,\pi}$,

$$|t_1\circ\Phi_\pi^1(g)-t_1\circ\Phi_\pi^2(g)|$$

$$= (1/\mathrm{tr}(p_{2,1}))|\mathrm{tr}\circ\Phi_\pi^1(g)-\mathrm{tr}\circ\Phi_\pi^2(g)| \tag{e 4.116}$$

$$\le (1/\mathrm{tr}(p_{2,1}))|\mathrm{tr}\circ\Phi_\pi^1(g)-\mathrm{tr}\circ\Phi_1(g)| \tag{e 4.117}$$

$$+ (1/\mathrm{tr}(p_{2,1}))|\mathrm{tr}\circ\Phi_1(g)-\mathrm{tr}\circ\varphi_1(g_{0,k}s^d(g))| \tag{e 4.118}$$

$$+(1/\mathrm{tr}(p_{2,1}))|\mathrm{tr}\circ\varphi_1(g_{0,k}s^d(g))-\mathrm{tr}\circ\varphi_2(g_{0,k}s^d(g))| \tag{e 4.119}$$

$$+ (1/\mathrm{tr}(p_{2,1}))|\mathrm{tr}\circ\varphi_2(g_{0,k}s^d(g))-\mathrm{tr}\circ\Phi_2(g)| \tag{e 4.120}$$

$$+ (1/\mathrm{tr}(p_{2,1}))|\mathrm{tr}\circ\Phi_2(g)-\mathrm{tr}\circ\Phi_\pi^2(g)| \tag{e 4.121}$$

$$< (1/\mathrm{tr}(p_{2,1}))(1/N_0+\delta+1/N_0)<\delta_2. \tag{e 4.122}$$

Recall that $p_{2,1}$ has rank at least N^π. It follows from the induction assumption that the theorem holds for A/I (and from (e 4.115) and (e 4.122)) that there are mutually orthogonal projections $e_0^\pi,e_1^\pi,e_2^\pi,...,e_{2K_0}^\pi\in p_{2,1}M_np_{2,1}$ such that $e_0^\pi\lesssim e_1^\pi$ and all e_j^π are equivalent to e_1^π, two unital homomorphisms $\psi_{1,\pi,0},\psi_{2,\pi,0}: A/I\to e_0^\pi M_ne_0^\pi$, a unital homomorphism $\psi_\pi: A/I\to e_1^\pi M_ne_1^\pi$, and a unitary $u_2\in p_{2,1}M_np_{2,1}$, such that

$$\|\mathrm{Ad}\,u_2\circ\Phi_\pi^1(f)-(\psi_{1,\pi,0}(f)\oplus\mathrm{diag}(\overbrace{\psi_\pi(f),\psi_\pi(f),...,\psi_\pi(f)}^{2K_0}))\|<\varepsilon_0/16 \tag{e 4.123}$$

$$\text{and}\quad \|\Phi_\pi^2(f)-(\psi_{2,\pi,0}(f)\oplus\mathrm{diag}(\overbrace{\psi_\pi(f),\psi_\pi(f),...,\psi_\pi(f)}^{2K_0}))\|<\varepsilon_0/16 \tag{e 4.124}$$

for all $f \in \mathcal{F}_\pi$. Let $\psi_\pi^1 : A \to p_{2,1}M_np_{2,1}$ be defined by $\psi_\pi^1(f) = \mathrm{Ad}\, u_2 \circ \mathrm{Ad}\, U_2(p_{2,1}(\Sigma_\pi^1 \oplus \Sigma_b^1)(f))$ and define $\psi_\pi^2 : A \to p_{2,1}M_np_{2,1}$ by $\psi_\pi^2(f) = p_{2,1}(\Sigma_\pi^2 \oplus \Sigma_b^2)(f)$ for all $f \in A$. Then, by (e 4.103),

$$\|\psi_\pi^i(f) - (\psi_{i,\pi,0} \circ \pi_I(f) \oplus \mathrm{diag}(\overbrace{\psi_\pi(\pi_I(f)), \psi_\pi(\pi_I(f)), ..., \psi_\pi(\pi_I(f))}^{2K_0}))\| < \varepsilon_0/8 \tag{e 4.125}$$

for all $f \in \mathcal{F}$, $i = 1, 2$.

Put $e_i = \sum_{j=1}^{2N_{00}} e^I_{2(i-1)N_{00}+j} \oplus \sum_{j=1}^{2N_{00}} e^\pi_{2(i-1)N_{00}+j}$, $i = 1, 2, ..., K$. Denote by $\bar{\psi}_I$ and $\bar{\psi}_\pi$ the direct sums of $2N_{00}$ copies of ψ_I and ψ^π, respectively. Define $\psi : A \to e_1M_ne_1$ by

$$\psi(f) = \mathrm{diag}(\bar{\psi}_I(f|_{Y_k}), \bar{\psi}_\pi(\pi_I(f)))$$

for all $f \in A$. By (e 4.79), (e 4.111), and (e 4.72), for $i = 1, 2$,

$$\mathrm{tr}(q_{i,0}) + \mathrm{tr}(q_{i,1}) + \mathrm{tr}(\Sigma_s^i(1_A)) < 1/64K_0 + 1/64K_0 + 1/N_0 < 1/16K. \tag{e 4.126}$$

We have, for $f \in A$,

$$\varphi_2(f) = \psi_\pi^2(f) \oplus q_{2,1}(\Sigma_\pi^2 + \Sigma_b^2)(f) \oplus \Sigma_s^2(f) \oplus \psi_I^2(f|_{Y_k}) \oplus q_{2,0}\psi_I^{2,0}(f|_{Y_k}). \tag{e 4.127}$$

Put $e_0 = e_0^I \oplus e_0^\pi + q_{2,1} + \Sigma_s^2(1_A) + q_{2,0}$. Then

$$\mathrm{tr}(e_0) < \mathrm{tr}(e_0^I) + \mathrm{tr}(e_0^\pi) + 1/16K \le \mathrm{tr}(e_1). \tag{e 4.128}$$

In other words, $e_0 \lesssim e_1$. Moreover e_1 is equivalent to each e_i, $i = 1, 2, ..., K$. Define $h_2 : A \to e_0M_ne_0$ by, for each $f \in A$,

$$\begin{aligned} h_2(f) \quad = \quad & \psi_{2,I,0}(f|_{Y_k}) \oplus \psi_{2,\pi,0}(\pi_I(f)) \oplus q_{2,1}(\Sigma_\pi^2 + \Sigma_b^2)(f) \\ & \oplus \Sigma_s^2(f) \oplus q_{2,0}\psi_I^{2,0}(f|_{Y_k}). \end{aligned} \tag{e 4.129}$$

It follows from (e 4.102), (e 4.125), and (e 4.129) that

$$\|\varphi_2(f) - (h_2(f) \oplus \mathrm{diag}(\overbrace{\bar{\psi}(f), \bar{\psi}(f), ..., \bar{\psi}(f)}^{K})\| < \varepsilon_0/8 \text{ for all } f \in \mathcal{F}. \tag{e 4.130}$$

Similarly, there exist a unitary $U \in M_n$ and a unital homomorphism $h_1 : A \to e_0 M_n e_0$ such that

$$\|\mathrm{Ad}\, U \circ \varphi_1(f) - (h_1(f) \oplus \mathrm{diag}(\overbrace{\bar{\psi}(f), \bar{\psi}(f), ..., \bar{\psi}(f)}^{K}))\| < \varepsilon_0/8 \ \text{ for all } \ f \in \mathcal{F}.$$
(e 4.131)

Since we assume that $\mathcal{H}_0 \subset \mathcal{F}$, by (e 4.46), the choice of ε_0, and (e 4.55), we also have that

$$\begin{aligned}
\mathrm{tr} \circ \bar{\psi}(g) &\geq (1/K)(\mathrm{tr} \circ \varphi_1(g) - \varepsilon_0/8 - \mathrm{tr}(h_1(g))) \quad &\text{(e 4.132)}\\
&> (1/K)(\Delta(\hat{g}) - \delta_{00}/16KN_{00} - 1/KN_{00})\\
&> \alpha \frac{\Delta(\hat{g})}{K} \ \text{ for all } \ g \in \mathcal{H}_0. &\text{(e 4.133)}
\end{aligned}$$

Thus the conclusion of the theorem holds for m. ☐

REMARK 4.12. If we assume that A is infinite dimensional, then Lemma 4.11 still holds without the assumption about the integer N. This could be easily seen by taking a larger $\mathcal{H}_1$ which contains at least N mutually orthogonal non-zero elements as we remarked in 4.7.

COROLLARY 4.13. *Let $A_0 \in \overline{\mathcal{D}}_s$ be a unital C^*-algebra, let $\varepsilon > 0$, and let $\mathcal{F} \subset A_0$ be a finite subset. Let $\Delta : (A_0)_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map.*

Suppose that $\mathcal{H}_1 \subset (A_0)_+^{\mathbf{1}} \setminus \{0\}$ is a finite subset, $\sigma > 0$ is a positive number and $n \geq 1$ is an integer. There exists a finite subset $\mathcal{H}_2 \subset (A_0)_+^{\mathbf{1}} \setminus \{0\}$ satisfying the following condition: Suppose that $\varphi : A = A_0 \otimes C(\mathbb{T}) \to M_k$ (for some integer $k \geq 1$) is a unital homomorphism and

$$tr \circ \varphi(h \otimes 1) \geq \Delta(\hat{h}) \ \text{ for all } \ h \in \mathcal{H}_2. \quad \text{(e 4.134)}$$

Then there exist mutually orthogonal projections $e_0, e_1, e_2, ..., e_n \in M_k$ such that $e_1, e_2, ..., e_n$ are equivalent and $\sum_{i=0}^n e_i = 1$, and there exist unital homomorphisms $\psi_0 : A = A_0 \otimes C(\mathbb{T}) \to e_0 M_k e_0$ and $\psi : A = A_0 \otimes C(\mathbb{T}) \to e_1 M_k e_1$ such that

$$\begin{aligned}
&\|\varphi(f) - \mathrm{diag}(\psi_0(f), \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{n})\| < \varepsilon &\text{(e 4.135)}\\
&\text{and } \mathrm{tr}(e_0) < \sigma &\text{(e 4.136)}
\end{aligned}$$

for all $f \in \mathcal{F}$, where tr is the tracial state on M_k. Moreover, ψ can be chosen such that

$$tr(\psi(g \otimes 1)) \geq \frac{\Delta(\hat{g})}{2n} \ \text{ for all } \ g \in \mathcal{H}_1. \quad \text{(e 4.137)}$$

PROOF. The statement follows directly from Lemma 4.11 with $\varphi_1 = \varphi_2 = \varphi$, $K = n$, and $\alpha = \frac{1}{2}$. ☐

The following is known and is taken from Theorem 3.9 of [69].

THEOREM 4.14. *Let A be a unital separable amenable C^*-algebra which satisfies the UCT and let B be a unital C^*-algebra. Suppose that h_1, $h_2 : A \to B$ are homomorphisms such that*

$$[h_1] = [h_2] \text{ in } KL(A, B).$$

Suppose that $h_0 : A \to B$ is a unital full monomorphism. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist an integer $n \geq 1$ and a partial isometry $W \in M_{n+1}(B)$ such that

$$\|W^*\mathrm{diag}(h_1(a), h_0(a), ..., h_0(a))W - \mathrm{diag}(h_2(a), h_0(a), ..., h_0(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$, and $W^*pW = q$, where

$$p = \mathrm{diag}(h_1(1_A), h_0(1_A), ..., h_0(1_A)) \text{ and } q = \mathrm{diag}(h_2(1_A), h_0(1_A), ..., h_0(1_A)).$$

In particular, if $h_1(1_A) = h_2(1_A)$, we may choose $W \in U(pM_{n+1}(B)p)$.

PROOF. This is a slight variation of Theorem 3.9 of [69]. If h_1 and h_2 are both unital, then it is exactly the same as Theorem 3.9 of [69]. So suppose that h_1 is not unital. Let $A' = \mathbb{C} \oplus A$. Choose $p_0 = 1_B - h_1(1_A)$ and $p_1 = \mathrm{diag}(p_0, 1_B)$. Put $B' = p_1 M_2(B) p_1$. Define $h_1' : A' \to B'$ by $h_1'(\lambda \oplus a) = \lambda \cdot \mathrm{diag}(p_0, p_0) \oplus h_1(a)$ for all $\lambda \in \mathbb{C}$ and $a \in A$, and define $h_2' : A' \to B'$ by $h_2'(\lambda \oplus a) = \lambda \cdot \mathrm{diag}(p_0, 1_B - h_2(1_A)) \oplus h_2(a)$ for all $\lambda \in \mathbb{C}$ and $a \in A$. Then h_1' and h_2' are unital and $[h_1'] = [h_2']$ in $KL(A', B')$. Define $h_0' : A' \to B'$ by $h_0'(\lambda \oplus a) = \lambda \cdot p_0 \oplus h_0(a)$ for all $\lambda \in \mathbb{C}$ and $a \in A$. Note that h_0' is full in B'. So, Theorem 3.9 of [69] applies. It follows that there are an integer $n \geq 1$ and a unitary $W' \in M_{n+1}(B')$ such that

$$\begin{aligned}\|(W')^*\mathrm{diag}(h_1'(a), h_0'(a), ..., h_0'(a))W' &- \mathrm{diag}(h_2'(a), h_0'(a), ..., h_0'(a))\| \\ &< \min\{1/2, \varepsilon/2\}\end{aligned}$$

for all $a \in \mathcal{F} \cup \{1_A\}$. In particular,

$$\|(W')^*pW' - q\| < \min\{1/2, \varepsilon/2\}. \tag{e 4.138}$$

There is a unitary $W_1 \in M_{n+1}(B')$ such that

$$\|W_1 - 1_{M_{n+1}(B')}\| < \varepsilon/2 \text{ and } W_1^*(W')^*pW'W_1 = q. \tag{e 4.139}$$

Put $W = pW'W_1q$. Then

$$\|W^*\mathrm{diag}(h_1(a), h_0(a), ..., h_0(a))W - \mathrm{diag}(h_2(a), h_0(a), ..., h_0(a))\| < \varepsilon \tag{e 4.140}$$

for all $a \in \mathcal{F}$, as desired. (The last statement is clear.) ☐

LEMMA 4.15. (cf. 5.3 of [64], Theorem 3.1 of [45], [19], 5.9 of [69], and Theorem 7.1 of [75]) *Let A be a unital separable amenable C^*-algebra which satisfies the UCT and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$, and an integer $K \geq 1$ satisfying the following condition: For any two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps $\varphi_1, \varphi_2 : A \to M_n$ (for some integer n) and any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\psi : A \to M_m$ with $m \geq n$ such that*

$$\text{(e 4.141)} \qquad \mathrm{tr} \circ \psi(g) \geq \Delta(\hat{g}) \ \ \textit{for all} \ \ g \in \mathcal{H} \ \ \textit{and} \ \ [\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}},$$

where tr *is the tracial state of M_m, there exists a unitary $U \in M_{Km+n}$ such that*

$$\text{(e 4.142)} \quad \|\mathrm{Ad}\, U \circ (\varphi_1 \oplus \Psi)(f) - (\varphi_2 \oplus \Psi)(f)\| < \varepsilon \ \ \textit{for all} \ \ f \in A,$$

where

$$\Psi(f) = \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K}) \ \ \textit{for all} \ \ f \in A.$$

PROOF. This follows, as we shall show, from Theorem 4.14.

Fix Δ as given. Suppose that the conclusion is false. Then there exist $\varepsilon_0 > 0$ and a finite subset $\mathcal{F}_0 \subset A$, an increasing sequence of finite subsets $\{\mathcal{P}_n\}$ of $\underline{K}(A)$ with union $\underline{K}(A)$, an increasing sequence of finite subsets $\{\mathcal{H}_n\} \subset A_+^{\mathbf{1}} \setminus \{0\}$ with union dense in $A_+^{\mathbf{1}}$ and such that if $a \in \mathcal{H}_n$ and $f_{1/2}(a) \neq 0$, then $f_{1/2}(a) \in \mathcal{H}_{n+1}$, three increasing sequences of integers $\{R(n)\}$, $\{r(n)\}$, and $\{s(n)\}$ (with $s(n) \geq r(n)$), two sequences of unital completely positive linear maps $\varphi_{1,n}, \varphi_{2,n},: A \to M_{r(n)}$ with the properties that

$$\text{(e 4.143)} \qquad [\varphi_{1,n}]|_{\mathcal{P}_n} = [\varphi_{2,n}]|_{\mathcal{P}_n}$$

and, for all $a, b \in A$,

$$\text{(e 4.144)} \qquad \lim_{n\to\infty} \|\varphi_{i,n}(ab) - \varphi_{i,n}(a)\varphi_{i,n}(b)\| = 0, \quad i = 1, 2,$$

and a sequence of unital completely positive linear maps $\psi_n : A \to M_{s(n)}$ with the properties that

$$\text{(e 4.145)} \qquad \mathrm{tr}_n \circ \psi_n(g) \geq \Delta(\hat{g}) \ \text{ for all } \ g \in \mathcal{H}_n \ \text{ and}$$

$$\text{(e 4.146)} \qquad \lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = 0 \ \text{ for all } \ a, b \in A$$

such that

$$\inf\{\sup\{\|\mathrm{Ad}\, U_n \circ (\varphi_{1,n}(f) \oplus \tilde{\psi}_n^{R(n)}(f)) - (\varphi_{2,n}(f) \quad \oplus \quad \tilde{\psi}_n^{R(n)}(f)\| : f \in \mathcal{F}\}\}$$
$$\text{(e 4.147)} \qquad \geq \varepsilon_0,$$

where tr_n is the normalized trace on $M_{s(n)}$, $\tilde{\psi}^{(R(n))}(f) = \mathrm{diag}(\overbrace{\psi_n(f), \psi_n(f), ..., \psi_n(f)}^{R(n)})$ for all $f \in A$, and the infimum is taken among all unitaries $U_n \in M_{r(n)+R(n)s(n)}$.

Note that, by (e 4.145), since $\{\mathcal{H}_n\}$ is increasing, for any $g \in \mathcal{H}_n \subset A_+^{\mathbf{1}}$, we compute that

$$\mathrm{tr}_m(p_m) \geq \Delta(\hat{g})/2, \tag{e 4.148}$$

where p_m is the spectral projection of $\psi_m(g)$ corresponding to the subset $\{\lambda > \Delta(\hat{g})/2\}$ for all $m \geq n$. It follows that (for all sufficiently large m) there are elements $x_{g,i,m} \in M_{s(m)}$ with $\|x_{g,i,m}\| \leq 1/\Delta(\hat{g})$, $i = 1, 2, ..., N(g)$, such that

$$\sum_{i=1}^{N(g)} x_{g,i,m}^* \psi_m(g) x_{g,i,m} = 1_{s(m)}, \tag{e 4.149}$$

where $1 \leq N(g) \leq 1/\Delta(\hat{g}) + 1$. Define $X_{g,i} = \{x_{g,i,m}\}$, $i = 1, 2, ..., N(g)$. Then $X_{g,i} \in \prod_{n=1}^{\infty} M_{r(n)}$. Let $Q(\{M_{r(n)}\}) = \prod_{n=1}^{\infty} M_{r(n)} / \bigoplus_{n=1}^{\infty} M_{r(n)}$, $Q(\{M_{s(n)}\}) = \prod_{n=1}^{\infty} M_{s(n)} / \bigoplus_{n=1}^{\infty} M_{s(n)}$, and let $\Pi_1 : \prod_{n=1}^{\infty} M_{r(n)} \to \prod_{n=1}^{\infty} M_{r(n)} / \bigoplus_{n=1}^{\infty} M_{r(n)}$, $\Pi_2 : \prod_{n=1}^{\infty} M_{s(n)} \to \prod_{n=1}^{\infty} M_{s(n)} / \bigoplus_{n=1}^{\infty} M_{s(n)}$ be the quotient maps. Denote by $\Phi_i : A \to Q(\{M_{r(n)}\})$ the homomorphisms $\Pi_1 \circ \{\varphi_{i,n}\}$ and denote by $\bar{\psi} : A \to Q(\{M_{s(n)}\})$ the homomorphism $\Pi_2 \circ \{\psi_n\}$. For each $g \in \bigcup_{n=1}^{\infty} \mathcal{H}_n$,

$$\sum_{i=1}^{N(g)} \Pi_2(X_{i,g})^* \bar{\psi}(g) \Pi_2(X_{i,g}) = 1_{Q(\{M_{s(n)}\})}. \tag{e 4.150}$$

Note that if $g \in \bigcup_{n=1} \mathcal{H}_n$ and $f_{1/2}(g) \neq 0$, then $g_{1/2}(g) \in \bigcup_{n=1} \mathcal{H}_n$, and $\bigcup_{n=1} \mathcal{H}_n$ is dense in $A_+^{\mathbf{1}}$. This implies that $\bar{\psi}$ is full (see the proof of Proposition 2.24). Note that both $\prod_{n=1}^{\infty} M_{r(n)}$ and $Q(\{M_{r(n)}\})$ have stable rank one and real rank zero. One then computes (see Corollary 2.1 of [45]) that

$$[\Phi_1] = [\Phi_2] \ \text{ in } \ KL(A, Q(\{M_{r(n)}\}). \tag{e 4.151}$$

By applying Theorem 4.14 (Theorem 3.9 of [69]), one obtains an integer $K \geq 1$ and a unitary $U \in PM_{K+1}(Q(\{M_{s(n)}\})P$, where $P = \mathrm{diag}(1_{Q(\{M_{r(n)}\}}, 1_{M_K(Q(\{M_{s(n)}\}))})$, such that

$$\begin{aligned} \|\mathrm{Ad}\, U \circ (\Phi_1(f) \oplus \mathrm{diag}(\overbrace{\bar{\psi}(f), ..., \bar{\psi}(f)}^{K}) \\ -(\Phi_2(f) \oplus \mathrm{diag}(\overbrace{\bar{\psi}(f), ..., \bar{\psi}(f)}^{K}))\| < \varepsilon_0/2 \end{aligned} \tag{e 4.152}$$

for all $f \in \mathcal{F}_0$. It follows that there are unitaries

$$\{U_n\} \in \prod_{n=1}^{\infty} M_{r(n)+Ks(n)}$$

such that, for all large n,

$$\|\mathrm{Ad}\, U_n \circ \varphi_{1,n}(f) \oplus \mathrm{diag}(\overbrace{\psi_n(f), ..., \psi_n(f)}^{K}) \\ -\varphi_{2,n}(f) \oplus \mathrm{diag}(\overbrace{\psi_n(f), ..., \psi_n(f)}^{K})\| < \varepsilon_0/2 \tag{e 4.153}$$

for all $f \in \mathcal{F}_0$. This is in direct contradiction with (e 4.147) when we choose n with $R(n) \geq K$. □

REMARK 4.16. The preceding lemma holds in a much more general setting and variations of it have appeared. We state this version here for our immediate purpose (see 12.3 and part (1) of 12.4 for more comments).

It should be noted that, in the following statement the integer L and the map Ψ depend not only on ε, $\mathcal{F}$, and $\mathcal{G}$, but also on B, as well as φ_1 and φ_2.

LEMMA 4.17. *Let C be a unital amenable separable residually finite dimensional C^*-algebra which satisfies the UCT. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist a finite subset $\mathcal{G} \subset C$, $\delta > 0$, and a finite subset $\mathcal{P} \subset \underline{K}(C)$ satisfying the following condition: For any unital $\mathcal{G}$-δ-multiplicative completely positive linear maps $\varphi_1, \varphi_2 : C \to A$ (for any unital C^*-algebra A) such that*

$$[\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}}, \tag{e 4.154}$$

there exist an integer $L \geq 1$, a unital homomorphism $\Psi : C \to M_L \subset M_L(A)$, and a unitary $U \in U(M_{L+1}(A))$ such that, for all $f \in \mathcal{F}$,

$$\|\mathrm{Ad}\, U \circ \mathrm{diag}(\varphi_1(f), \Psi(f)) - \mathrm{diag}(\varphi_2(f), \Psi(f)\| < \varepsilon. \tag{e 4.155}$$

PROOF. The proof is almost the same as that of Theorem 9.2 of [73]. Suppose that the conclusion is false. We then obtain a positive number $\varepsilon_0 > 0$, a finite subset $\mathcal{F}_0 \subset C$, a sequence of finite subsets $\mathcal{P}_n \subset \underline{K}(C)$ with $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ and $\bigcup_n \mathcal{P}_{n+1} = \underline{K}(C)$, a sequence of unital C^*-algebras $\{A_n\}$, and sequences of unital completely positive linear maps $\{L_n^{(1)}\}$ and $\{L_n^{(2)}\}$, from C to A_n, such that

$$\lim_{n\to\infty} \|L_n^{(i)}(ab) - L_n^{(i)}(a)L_n^{(i)}(b)\| = 0 \text{ for all } a, b \in C, \tag{e 4.156}$$

$$[L_n^{(1)}]|_{\mathcal{P}_n} = [L_n^{(2)}]|_{\mathcal{P}_n}, \text{ and} \tag{e 4.157}$$

$$\inf\{\sup\{\|u_n^*\mathrm{diag}(L_n^{(1)}(a), \Psi_n(a))u_n - \mathrm{diag}(L_n^{(2)}(a), \Psi_n(a)\| : a \in \mathcal{F}_0\}\} \geq \varepsilon_0, \tag{e 4.158}$$

where the infimum is taken among all integers $k > 1$, all possible unital homomorphisms $\Psi_n : C \to M_k$, and all possible unitaries $u_n \in M_{k+1}(A_n)$. We may assume that $1_C \in \mathcal{F}$. Define $B_n = A_n \otimes \mathcal{K}$, $B = \prod_{n=1}^{\infty} B_n$, and $Q_1 = B/\bigoplus_{n=1}^{\infty} B_n$. Let $\pi : B \to Q_1$ be the quotient map. Define $\varphi_j : C \to B$ by $\varphi_j(a) = \{L_n^{(j)}(a)\}$ and let $\bar{\varphi_j} = \pi \circ \varphi_j$, $j = 0, 1$. Note that $\bar{\varphi_j} : C \to Q_1$ is a homomorphism. As in the proof of 9.2 of [73], we have

$$[\bar{\varphi}_1] = [\bar{\varphi}_2] \text{ in } KL(C, Q_1).$$

Fix an irreducible representation $\varphi_0' : C \to M_r$. Denote by p_n the unit of the unitization $\tilde{B}_n$ of B_n, $n = 1, 2, ...$ Define a homomorphism $\varphi_0^{(n)} : C \to M_r(\tilde{B}_n) = M_r \otimes \tilde{B}_n$ by $\varphi_0^{(n)}(c) = \varphi_0'(c) \otimes 1_{\tilde{B}_n}$ for all $c \in C$. Put

$$e_A = \{1_{A_n}\}, \; P = \{1_{M_r(\tilde{B}_n)}\} + e_A.$$

Put also $Q_2 = \pi(P)M_{r+1}(\tilde{Q}_1)\pi(P)$ and define $\bar{\varphi}_j' = \bar{\varphi}_j \oplus \pi \circ \{\varphi_0^{(n)}\}$, $j = 1, 2$. Then

$$[\bar{\varphi}_1'] = [\bar{\varphi}_2'] \text{ in } KL(C, Q_2). \tag{e 4.159}$$

The reason for adding $\pi \circ \{\varphi_0^{(n)}\}$ is that, now, $\bar{\varphi}_1'$ and $\bar{\varphi}_2'$ are unital. It follows from Theorem 4.3 of [19] that there exist an integer $K > 0$, a unitary $u \in M_{1+K}(Q_2)$, and a unital homomorphism $\psi : C \to M_K \subset M_K(Q_2)$ (M_K idenitified with the natural unital subalgebra of $M_K(Q_2)$) such that

$$\operatorname{Ad} u \circ \operatorname{diag}(\bar{\varphi}_1', \psi) \approx_{\varepsilon_0/4} \operatorname{diag}(\bar{\varphi}_2', \psi) \text{ on } \mathcal{F}_0.$$

There exists a unitary $V = \{V_n\} \in M_{1+K}(PM_{r+1}(\tilde{B})P)$ such that $\pi(V) = u$. It follows (on identifying M_K with $M_K \otimes 1_{Q_2}$ as above) that for all sufficiently large n,

$$\operatorname{Ad} V_n \circ \operatorname{diag}(L_1^{(n)} \oplus \varphi_n^{(n)}, \psi) \approx_{\varepsilon_0/3} \operatorname{diag}(L_2^{(n)} \oplus \varphi_n^{(n)}, \psi) \text{ on } \mathcal{F}_0.$$

For each integer $k \geq 1$, write $e_{n,k,0} = \operatorname{diag}(\overbrace{1_{A_n}, 1_{A_n}, ..., 1_{A_n}}^{k}) \in A_n \otimes \mathcal{K} = B_n$,

$$\begin{aligned} e_{n,k}' &= \operatorname{diag}(1_{A_n}, \overbrace{e_{n,k,0}, e_{n,k,0}, ..., e_{n,k,0}}^{r}) \in PM_{1+r}(B_n)P, \text{ and} \\ e_{n,k}'' &= \operatorname{diag}(\overbrace{e_{n,k}', e_{n,k}', ..., e_{n,k}'}^{K}) \in M_K(PM_{1+r}(B_n)P). \end{aligned}$$

It should be noted that $e_{n,k}''$ commutes with ψ and $e_{n,k}'$ commutes with $\varphi_n^{(0)}$. Put $e_{n,k} = e_{n,k}' \oplus e_{n,k}''$ in $M_{1+K}(PM_{1+r}(B_n)P)$. Then $\{e_{n,k}\}$ is an approximate identity for $M_{1+K}(PM_{1+r}(B_n)P)$. Note that $V_n \in M_{1+K}(PM_{r+1}(\tilde{B})P)$. It is easy to check that

$$\lim_{k\to\infty} \|[V_n, e_{n,k}]\| = 0. \tag{e 4.160}$$

It follows that there exists a unitary $U_{n,k} \in e_{n,k}M_{1+K}(PM_{1+r}(B_n)P)e_{n,k}$ for each n and k such that

$$\lim_{k\to\infty} \|e_{n,k}V_n e_{n,k} - U_{n,k}\| = 0. \tag{e 4.161}$$

For each k, there is $N(k) = rk + K(rk+1)$ such that

$$\begin{aligned} &M_{N(k)}(A_n)\\ &\quad = ((e'_{n,k} - 1_{A_n}) \oplus e''_{n,k})M_{1+K}(PM_{r+1}(B_n)P)((e'_{n,k} - 1_{A_n}) \oplus e''_{n,k}). \end{aligned}$$

Moreover, $e_{n,k}M_{1+K}(PM_{1+r}(B_n)P)e_{n,k} = M_{N(k)+1}(A_n)$. Define $\Psi_n(c) = (e'_{n,k} - 1_{A_n})\varphi_0^{(n)}(c)(e'_{n,k} - 1_{A_n}) \oplus e''_{n,k}\psi(c)e_{n,k}$ for $c \in C$. Then, for large k and large n,

$$\operatorname{Ad} U_n \circ \operatorname{diag}(L_1^{(n)}, \Psi_n) \approx_{\varepsilon_0/2} \operatorname{diag}(L_2^{(n)}, \Psi_n) \text{ on } \mathcal{F}_0. \tag{e 4.162}$$

This is in contradiction with (e 4.157). □

THEOREM 4.18. *Let $A \in \bar{\mathcal{D}}_s$ and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map.*

For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, and a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ satisfying the following condition:

If $\varphi_1, \varphi_2 : A \to M_n$ are unital homomorphisms such that

$$[\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}}, \tag{e 4.163}$$

$$\operatorname{tr} \circ \varphi_1(g) \geq \Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_1, \text{ and} \tag{e 4.164}$$

$$|\operatorname{tr} \circ \varphi_1(h) - \operatorname{tr} \circ \varphi_2(h)| < \delta \text{ for all } h \in \mathcal{H}_2, \tag{e 4.165}$$

where $\operatorname{tr} \in T(M_n)$, *then there exists a unitary $u \in M_n$ such that*

$$\|\operatorname{Ad} u \circ \varphi_1(f) - \varphi_2(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}. \tag{e 4.166}$$

PROOF. If A has finite dimension, the lemma is known. So, in what follows, we will assume that A is infinite dimensional.

Define $\Delta_0 : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ by $\Delta_0 = (3/4)\Delta$. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. Let $\mathcal{P} \subset \underline{K}(A)$, $\mathcal{H}_0 \subset A_+^{q,\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}$), and $K \geq 1$ be the finite subsets and integer provided by Lemma 4.15 for $\varepsilon/2$ (in place of ε), $\mathcal{F}$, and Δ_0.

Choose $\varepsilon_0 > 0$ and a finite subset $\mathcal{G} \subset A$ such that $\varepsilon_0 < \varepsilon$ and

$$[\Phi_1']|_{\mathcal{P}} = [\Phi_2']|_{\mathcal{P}} \tag{e 4.167}$$

for any pair of unital homomorphisms from A satisfying

$$\|\Phi_1'(g) - \Phi_2'(g)\| < \varepsilon_0 \text{ for all } g \in \mathcal{G}. \tag{e 4.168}$$

We may assume that $\mathcal{F} \subset \mathcal{G}$ and $\varepsilon_0 < \varepsilon/2$.

Let $\alpha = 3/4$. Let $N \geq 1$, $\delta_1 > 0$ (in place of δ), $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, and $\mathcal{H}_2 \subset A_{s.a.}$ be the constant and finite subsets provided by 4.11 for $\varepsilon_0/2$ (in place ε), $\mathcal{G}$ (in place of $\mathcal{F}$), $\mathcal{H}_0$, K, and Δ_0 (in place of Δ). Choosing a larger $\mathcal{H}_1$, since A has infinite dimension, we may assume that $\mathcal{H}_1$ contains at least N mutually orthogonal non-zero positive elements.

Now suppose that φ_1, φ_2 are unital homomorphisms satisfying the assumptions for the $\mathcal{P}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ above. The assumption (e 4.164) implies that $n \geq N$. Applying 4.11, we obtain a unitary $u_1 \in M_n$, mutually orthogonal non-zero projections $e_0, e_1, e_2, ..., e_K \in M_n$ with $\sum_{i=0}^K e_i = 1_{M_n}$, $e_0 \lesssim e_1$, e_1 equivalent to e_i, $i = 1, 2, ..., K$, unital homomorphisms $\Phi_1, \Phi_2 : A \to e_0 M_n e_0$, and a unital homomorphism $\psi : A \to e_1 M_n e_1$ such that

$$\|\mathrm{Ad}\, u_1 \circ \varphi_1(f) - (\Phi_1(f) \oplus \Psi(f))\| < \varepsilon_0/2 \text{ for all } f \in \mathcal{G}, \tag{e 4.169}$$

$$\|\varphi_2(f) - (\Phi_2(f) \oplus \Psi(f))\| < \varepsilon_0/2 \text{ for all } f \in \mathcal{G}, \text{ and} \tag{e 4.170}$$

$$\mathrm{tr} \circ \psi(g) \geq (3/4)\Delta(\hat{g})/K \text{ for all } g \in \mathcal{H}_0, \tag{e 4.171}$$

where $\Psi(a) = \mathrm{diag}(\overbrace{\psi(a), \psi(a), ..., \psi(a)}^{K})$ for all $a \in A$ and tr is the tracial state on M_n.

Since $[\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}}$, by the choice of ε_0 and $\mathcal{G}$, we compute that

$$[\Phi_1]|_{\mathcal{P}} = [\Phi_2]|_{\mathcal{P}}. \tag{e 4.172}$$

Moreover,

$$t \circ \psi(g) \geq (3/4)\Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_0, \tag{e 4.173}$$

where t is the tracial state of $e_1 M_n e_1$. By 4.15, there is a unitary $u_2 \in M_n$ such that, for all $f \in \mathcal{F}$,

$$\|\mathrm{Ad}\, u_2 \circ (\Phi_1 \oplus \Psi)(f) - (\Phi_1 \oplus \Psi)(f))\| < \varepsilon/2. \tag{e 4.174}$$

Put $U = u_2 u_1$. Then, by (e 4.169), (e 4.170), and (e 4.174),

$$\|\mathrm{Ad}\, U \circ \varphi_1(f) - \varphi_2(f)\| < \varepsilon_0/2 + \varepsilon/2 + \varepsilon_0/2 < \varepsilon \text{ for all } f \in \mathcal{F}.$$

□

LEMMA 4.19. *Let $A \in \bar{\mathcal{D}}_s$ be a unital C^*-algebra and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\mathcal{P}_0 \subset K_0(A)$ be a finite subset. Then there exist an integer $N(\mathcal{P}_0) \geq 1$ and a finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$ satisfying the following condition: For any unital homomorphism $\varphi : A \to M_k$ (for some $k \geq 1$) and any unital homomorphism $\psi : A \to M_R$ for some integer $R \geq N(\mathcal{P}_0)k$ such that (with* tr *the tracial state of M_R)*

$$\mathrm{tr} \circ \psi(g) \geq \Delta(\hat{g}) \ \textit{for all}\ g \in \mathcal{H}, \tag{e 4.175}$$

there exists a unital homomorphism $h_0 : A \to M_{R-k}$ such that

$$[\varphi \oplus h_0]|_{\mathcal{P}_0} = [\psi]|_{\mathcal{P}_0}. \tag{e 4.176}$$

PROOF. Denote by G_0 the subgroup of $K_0(A)$ generated by $\mathcal{P}_0$. We may assume, without loss of generality, that $\mathcal{P}_0 = \{[p_1], [p_2], ..., [p_{m_1}]\} \cup \{z_1, z_2, ..., z_{m_2}\}$, where $p_1, p_2, ..., p_{m_1} \in M_l(A)$ are projections (for some integer $l \geq 1$) and $z_j \in \ker \rho_A$, $j = 1, 2, ..., m_2$.

We prove the lemma by induction. Assume first that $A = PC(X, F)P$, where X is a compact metric space. This, of course, includes the case that X is a single point. There is $d > 0$ such that

$$\|\pi_{x,j} \circ p_i - \pi_{x',j} \circ p_i\| < 1/2, \ \ i = 1, 2, ..., m_1, \tag{e 4.177}$$

provided that $\mathrm{dist}(x, x') < d$, where $\pi_{x,j}$ is identified with $\pi_{x,j} \otimes \mathrm{id}_{M_l}$. Since X is compact, we may assume that $\{x_1, x_2, ..., x_{m_3}\}$ is a $d/2$-dense set. Write $P_{x_i} F P_{x_i} = M_{r(i,1)} \oplus M_{r(i,2)} \oplus \cdots \oplus M_{r(i,k(x_i))}$, $i = 1, 2, ..., m_3$.

There are $h_{i,j} \in C(X)$ with $0 \leq h_{i,j} \leq 1$, $h_{i,j}(x_i) = 1_{M_{r(i,j)}}$, and $h_{i,j} h_{i',j'} = 0$ if $(i, j) \neq (i', j')$. Moreover we may assume that $h_{i,j}(x) = 0$ if $\mathrm{dist}(x, x_i) \geq d$.

Put $g_{i,j} = h_{i,j} \cdot P \in A$, $j = 1, 2, ..., k(x_i), i = 1, 2, ..., m_3$. Let

$$\sigma_0 = \min\{\Delta(\hat{h_{i,j}}) : 1 \leq j \leq k(x_i),\ 1 \leq i \leq m_3\} \tag{e 4.178}$$

and choose an integer $N(\mathcal{P}_0) \geq 2/\sigma_0$. Put $\mathcal{H} = \{h_{i,j} : 1 \leq j \leq k(x_i),\ 1 \leq i \leq m_3\}$.

Now suppose that maps $\varphi : A \to M_k$ and $\psi : A \to M_R$ are given with $R \geq N(\mathcal{P}_0)k$ and

$$\mathrm{tr} \circ \psi(g) \geq \Delta(\hat{g}) \ \text{for all}\ g \in \mathcal{H}. \tag{e 4.179}$$

Write $\varphi = \bigoplus_{i,j}^{m_3} \Pi_{y_i,j}$, where $\Pi_{y_i,j(i)}$ is $T_{i,j}$ copies of $\pi_{y_i,j}$. Note that $k - T_{i,j} > 0$ for all i, j. Since $R \geq N(\mathcal{P}_0)k$, (e 4.179) implies that ψ is the direct sum of at least

$$\Delta(\widehat{h_{j,i}}) \cdot (2k/\sigma_0) > 2k \tag{e 4.180}$$

copies of $\pi_{x,j}$ with $\mathrm{dist}(x, x_i) < d$, $i = 1, 2, ..., m_3$. Rewrite ψ as $\Sigma_1 \oplus \Sigma_2$, where Σ_1 contains exactly $T_{i,j}$ copies of $\pi_{x,j}$ with $\mathrm{dist}(x, x_i) < d$ for each i and j. Then

$$\mathrm{rank}\, \Sigma_1(p_i) = \mathrm{rank}\varphi(p_i), \ \ i = 1, 2, ..., m_1. \tag{e 4.181}$$

Put $h_0 = \Sigma_2$. Note for any unital homomorphism $h : A \to M_n$, $[h(z)] = 0$ for all $z \in \ker\rho_A$. So $[\varphi \oplus h_0]|_{\mathcal{P}_0} = [\psi]|_{\mathcal{P}_0}$. This proves the case that $A = PC(X, F)P$ as above, in particular, the case that $A \in \bar{\mathcal{D}}_0$.

Now assume the conclusion of the lemma holds for any C^*-algebra $A \in \bar{\mathcal{D}}_m$.

Let A be a C^*-algebra in $\bar{\mathcal{D}}_{m+1}$. We assume that $A \subset PC(X, F)P \oplus B$ is a unital C^*-subalgebra and $I = \{f \in PC(X, F)P : f|_{X^0} = 0\}$, where $X^0 = X \setminus Y$ and Y is an open subset of X and $B \in \mathcal{D}'_m$ and $A/I \cong B$. We assume that, if $\mathrm{dist}(x, x') < 2d$, then

$$\|\pi_{x,j}(p_i) - \pi_{x',j}(p_i)\| < 1/2 \ \text{ and } \ \|\pi_{x,j} \circ s \circ \pi_I(p_i) - \pi_{x,j}(p_i)\| < 1/2,$$

where $s : A/I \to A^d = \{(f|_{\overline{X^d}}, b) : (f, b) \in A\}$ is the injective homomorphism given by 4.8, and $X^d = \{x \in X : \mathrm{dist}(x, X^0) < d\}$. We also assume that $2d < d_{X,X^0}$. Define $\Delta_\pi : (A/I)_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ by

$$\Delta_\pi(\hat{g}) = \Delta(\widehat{g_0 \cdot P \cdot s(g)}) \ \text{ for all } \ g \in (A/I)_+^{\mathbf{1}} \setminus \{0\}, \tag{e 4.182}$$

where $g_0 \in C(X^d)_+$ with $0 \le g \le 1$, $g_0(x) = 1$ if $x \in X^0$, $g_0(x) > 0$ if $\mathrm{dist}(x, X^0) < d/2$, and $g_0(x) = 0$ if $\mathrm{dist}(x, X^0) \ge d/2\}$. Note that $g_0 \cdot Ps(g) > 0$ if $g \in (A/I)_+ \setminus \{0\}$. Therefore, Δ_π is indeed an order preserving map from $(A/I)_+^{q,\mathbf{1}} \setminus \{0\}$ into $(0, 1)$.

Note that $A/I \in \mathcal{D}_m$ (see the later part of Definition 4.8). By the inductive assumption, there is an integer $N_\pi(\mathcal{P}_0) \ge 1$, a finite subset $\mathcal{H}_\pi \subset (A/I)_+^{\mathbf{1}} \setminus \{0\}$ satisfying the following condition: if $\varphi' : A/I \to M_{k'}$ is a unital homomorphism and $\psi' : A/I \to M_{R'}$ is a unital homomorphism for some $R' > N_\pi(\mathcal{P}_0)k'$ such that

$$t \circ \psi'(\hat{g}) \ge \Delta_\pi(\hat{g}) \ \text{ for all } \ g \in \mathcal{H}_\pi,$$

where t is the tracial state of $N_{R'}$, then there exists a unital homomorphism $h_\pi : A/I \to M_{R'-k'}$ such that

$$(\varphi' \oplus h_\pi)_{*0}|_{\bar{\mathcal{P}}_0} = (\psi_\pi)_{*0}|_{\bar{\mathcal{P}}_0},$$

where $\bar{\mathcal{P}}_0 = \{(\pi_I)_{*0}(p) : p \in \mathcal{P}_0\}$.

For $\delta > 0$, define $Y^\delta = X \setminus X^\delta$. Let $r : Y^{d/2} \to Y^d$ be a homeomorphism such that $\mathrm{dist}(r(x), x) < d$ for all $x \in Y^{d/2}$ (see Definition 4.8). Set $C = \{f|_{Y^d} : f \in I\}$. Let $\pi_C : A \to C$ be defined by $\pi_C(a) = \lambda(a)|_{Y^d}$ for all $a \in A$ (see 4.8 for $\lambda : A \to PC(X, F)P$). Define $\Delta_I : C_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ by

$$\Delta_I(\hat{g}) = \Delta(\widehat{(f_0 Pg)} \circ r) \ \text{ for all } \ g \in C_+^{q,\mathbf{1}} \setminus \{0\}, \tag{e 4.183}$$

where $f_0 \in C_0(Y)_+$ with $0 \le f_0 \le 1$, $f_0(x) = 1$ if $x \in Y^d$, $f_0(x) = 0$ if $\mathrm{dist}(x, X^0) \le d/2$ and $f_0(x) > 0$ if $\mathrm{dist}(x, X^0) > d/2\}$.

Note that $C = P|_{Y^d} C(Y^d, F) P_{Y^d}$. By what has been proved, there are an integer $N_I(\mathcal{P}_0) \ge 1$ and a finite subset $\mathcal{H}_I \subset C_+^{\mathbf{1}} \setminus \{0\}$ satisfying the following condition: if $\varphi'' : C \to M_{k''}$ is a unital homomorphism and $\psi'' : C \to M_{R''}$ (for some $R'' \ge N_I(\mathcal{P}_0)k''$) is another unital homomorphism such that (t is the tracial state on $M_{R''}$)

$$t \circ \psi''(\hat{g}) \ge \Delta_I(\hat{g}) \ \text{ for all } \ g \in \mathcal{H}_I,$$

then there exists a unital homomorphism $h'' : C \to M_{R''-k''}$ such that

$$(\varphi'' \oplus h'')_{*0}|_{[\pi_C](\mathcal{P}_0)} = (\psi''|_{*0})|_{[\pi_C](\mathcal{P}_0)}. \tag{e 4.184}$$

Put

$$\sigma = \min\{\min\{\Delta_\pi(\hat{g}) : g \in \mathcal{H}_\pi\}, \min\{\Delta_I(\hat{g}) : g \in \mathcal{H}_\pi\}\} > 0. \tag{e 4.185}$$

Let $N = (N_\pi(\mathcal{P}_0) + N_I(\mathcal{P}_0))/\sigma$ and let

$$\mathcal{H} = \{g_0 \circ s(g) : g \in \mathcal{H}_\pi\} \cup \{f_0 \cdot g \circ r\}. \tag{e 4.186}$$

Now suppose that $\varphi : A \to M_k$ and $\psi : A \to M_R$ satisfy the assumptions for $N = N(\mathcal{P}_0)$ and $\mathcal{H}$ as above (and $R > Nk$). We may write $\varphi = \Sigma_{\varphi,\pi} \oplus \Sigma_{\varphi,I}$, where $\Sigma_{\varphi,\pi}$ is the (finite) direct sum of irreducible representations of A which factor through A/I and $\Sigma_{\varphi,I}$ is the (finite) direct sum of irreducible representations of I. We may also write

$$\psi = \Sigma_{\psi,\pi} \oplus \Sigma_{\psi,b} \oplus \Sigma_{\psi,I'}, \tag{e 4.187}$$

where $\Sigma_{\psi,\pi}$ is the direct sum of irreducible representations of A which factor through A/I, $\Sigma_{\psi,b}$ is the (finite) direct sum of irreducible representations which factor through point evaluations at $x \in Y$ with $\mathrm{dist}(x, X^0) < d/2$ and $\Sigma_{\psi,I'}$ is the direct sum of irreducible representations which factor through point evaluations at $x \in Y$ with $\mathrm{dist}(x, X^0) \ge d/2$.

Put $q_\pi = (\Sigma_{\psi,\pi} \oplus \Sigma_{\psi,b})(1_A)$ and $k' = \mathrm{rank}\Sigma_{\varphi,\pi}(1_A)$. Define $\psi_\pi : A/I \to M_{\mathrm{rank}(q_\pi)}$ by $\psi_\pi(a) = (\Sigma_{\psi,\pi} \oplus \Sigma_{\psi,b}) \circ s(a)$ for all $a \in A/I$. Then, by (e 4.175) and the choice of $\mathcal{H}$,

$$t_1 \circ \psi_\pi(g) \ge \mathrm{tr} \circ \psi(g_0 P \cdot s(g)) \ge \Delta(g_0\widehat{P \cdot s(g)}) = \Delta_\pi(\hat{g}) \ \text{ for all } \ g \in \mathcal{H}_\pi,$$

where t_1 is the tracial state on $M_{\mathrm{rank}(q_\pi)}$. Note that

$$\mathrm{tr} \circ \psi(g_0 P) \ge \Delta(\widehat{g_0 P}) = \Delta_\pi(\widehat{1_{A/I}}), \tag{e 4.188}$$

Therefore,

$$\mathrm{rank}(q_\pi) \ge R\Delta_\pi(1_{A/I}) \ge N_\pi(\mathcal{P}_0)k'. \tag{e 4.189}$$

By the inductive assumption, there is a unital homomorphism $h_\pi : A/I \to M_{\mathrm{rank}q_\pi - k'}$ such that

$$(\Sigma_{\varphi,\pi} \oplus h_\pi)_{*0}|_{\bar{\mathcal{P}}_0} = (\psi_\pi)_{*0}|_{\bar{\mathcal{P}}_0}. \tag{e 4.190}$$

Put $q_I = \Sigma_{\psi,I'}(1_A)$ and $k'' = \mathrm{rank}(\Sigma_{\varphi,I}(1_A))$. Define $\psi_I : C \to M_{\mathrm{rank}q_I}$ by

$$\psi_I(a) = \Sigma_{\psi,I'}(a \circ r) \ \text{ for all } \ a \in C. \tag{e 4.191}$$

Then

$$\begin{aligned} t_2 \circ \psi_I(g) &= \mathrm{tr} \circ \Sigma_{\psi,I'}(g \circ r) \ge \psi(f_0 P g \circ r) \ge \Delta(\widehat{(f_0Pg)} \circ r) \\ &= \Delta_I(\hat{g}) \ \text{ for all } \ g \in \mathcal{H}_I, \end{aligned} \tag{e 4.192}$$

where t_2 is the tracial state on $M_{\mathrm{rank}(q_I)}$. Note that

$$\mathrm{tr} \circ \psi(f_0 P) \ge \Delta(\widehat{f_0P}) = \Delta_I(\hat{1_A}). \tag{e 4.193}$$

Therefore,

$$\mathrm{rank}(q_I) \ge R\Delta_I(1_A) \ge N_I(\mathcal{P}_0)k''. \tag{e 4.194}$$

There are $0 < d_1 < d < d_{X,X^0}$ such that all irreducible representations appearing in $\Sigma_{\varphi,I}$ factor through point evaluations at x with $\mathrm{dist}(x, X^d) \ge d_1$. Choose a homomorphism $r' : Y^{d_1} \to Y^d$ as in 4.8. Define $\varphi_I : C \to M_{k''}$ by $\varphi_I(f) = \Sigma_{\varphi,I}(f \circ r')$.

By what has been proved, the choice of $N_I(\mathcal{P}_0)$, and by (e 4.194), there is a unital homomorphism $h_I : C \to M_{\mathrm{rank}(q_I) - k''}$ such that

$$(\Sigma_{\varphi,I} \oplus h_I)_{*0}|_{[\pi_C](\mathcal{P}_0)} = (\psi_I)_{*0}|_{[\pi_C](\mathcal{P}_0)}. \tag{e 4.195}$$

Define $h : A \to M_{R-k}$ by $h(a) = h_\pi(\pi_I(a)) \oplus h_I(a|_{Y^d}) \oplus \Sigma_{\psi,b}(a)$ for all $a \in A$. Then, for each i, by (e 4.190) and (e 4.195),

$$\begin{aligned} &\mathrm{rank}\varphi(p_i) + \mathrm{rank}h(p_i) \\ &= \mathrm{rank}(\Sigma_{\varphi,\pi}(p_i)) + \mathrm{rank}(\Sigma_{\varphi,I}(p_i)) && \text{(e 4.196)} \\ &\quad +\mathrm{rank}(\Sigma_{\psi,b}(p_i)) + \mathrm{rank}h_\pi(p_i) + \mathrm{rank}h_I(p_i) && \text{(e 4.197)} \\ &= \mathrm{rank}\psi_\pi(p_i) + \mathrm{rank}(\Sigma_{\psi,b}(p_i)) + \mathrm{rank}\psi_I(p_i) && \text{(e 4.198)} \\ &= \mathrm{rank}\psi(p_i), \ \ i = 1, 2, ..., m_1. && \text{(e 4.199)} \end{aligned}$$

Since $(\varphi)_{*0}(z_j) = h_{*0}(z_j) = \psi_{*0}(z_j) = 0$, $j = 1, 2, ..., m_2$, we conclude that

$$(\varphi \oplus h)_{*0}|_{\mathcal{P}_0} = \psi_{*0}|_{\mathcal{P}_0}. \tag{e 4.200}$$

This completes the induction process. ☐

LEMMA 4.20. *Let $A \in \bar{\mathcal{D}}_s$ be a unital C^*-algebra and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist a finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$ and an integer $L \geq 1$ satisfying the following condition: For any unital homomorphism $\varphi : A \to M_k$ and any unital homomorphism $\psi : A \to M_R$ for some $R \geq Lk$ such that*

$$\mathrm{tr} \circ \psi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H} \ \ (\mathrm{tr} \in T(M_R)), \tag{e 4.201}$$

there exist a unital homomorphism $\varphi_0 : A \to M_{R-k}$ and a unitary $u \in M_R$ such that

$$\|\mathrm{Ad}\, u \circ \mathrm{diag}(\varphi(f), \varphi_0(f)) - \psi(f)\| < \varepsilon \tag{e 4.202}$$

for all $f \in \mathcal{F}$.

PROOF. Let $\delta > 0$, $\mathcal{P} \subset \underline{K}(A)$ be a finite subset, $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ be a finite subset, $\mathcal{H}_2 \subset A_{s.a.}$ be a finite subset, and N_0 be an integer as provided by Theorem 4.18 for $\varepsilon/4$ (in place of ε), $\mathcal{F}$, $(1/2)\Delta$, and A. Without loss of generality, we may assume that $\mathcal{H}_2 \subset A_+^{\mathbf{1}} \setminus \{0\}$. Let $\sigma_0 = \min\{\Delta(\hat{g}) : g \in \mathcal{H}_1 \cup \mathcal{H}_2\}$.

Let G denote the subgroup of $\underline{K}(A)$ generated by $\mathcal{P}$. Put $\mathcal{P}_0 = \mathcal{P} \cap K_0(A)$. We may also assume, without loss of generality, that $\mathcal{P}_0 = \{[p_1], [p_2], ..., [p_{m_1}]\} \cup \{z_1, z_2, ..., z_{m_2}\}$, where $p_1, p_2, ..., p_{m_1}$ are projections in $M_l(A)$ (for some integer l) and $z_j \in \ker\rho_A$, $j = 1, 2, ..., m_2$. Let $j \geq 1$ be an integer such that $K_0(A, \mathbb{Z}/j'\mathbb{Z}) \cap G = \varnothing$ for all $j' \geq j$. Put $J = j!$.

Let $N(\mathcal{P}_0) \geq 1$ denote the integer and $\mathcal{H}_3 \subset A_+^{\mathbf{1}} \setminus \{0\}$ the finite subset provided by Theorem 4.19 for $\mathcal{P}_0$. Let $p_s = (a_{i,j}^{(s)})_{l \times l}$, $s = 1, 2, ..., m_1$, and choose $\varepsilon_0 > 0$ and a finite subset $\mathcal{F}_0$ such that

$$[\psi']|_{\mathcal{P}} = [\psi'']|_{\mathcal{P}}, \tag{e 4.203}$$

whenever $\|\psi'(a) - \psi''(a)\| < \varepsilon_0$ for all $a \in \mathcal{F}_0$.

Put $\mathcal{F}_2 = \mathcal{F} \cup \mathcal{F}_1 \cup \mathcal{H}_2$ and put $\varepsilon_1 = \min\{\varepsilon/16, \varepsilon_0/2, \delta/2\}$. Let $K > 8((N(\mathcal{P}_0) + 1)(J+1)/\delta\sigma_0)$ be an integer. Let $\mathcal{H}_0 = \mathcal{H}_1 \cup \mathcal{H}_3$. Let $N_1 \geq 1$ (in place of N), $\delta_1 > 0$ (in place of δ), $\mathcal{H}_4 \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), and $\mathcal{H}_5 \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be the constants and finite subsets provided by Lemma 4.11 for ε_1 (in place of ε), $\mathcal{F}_2$ (in place of $\mathcal{F}$), $\mathcal{H}_0$, and K. Let $L = K(K+1)$, let $\mathcal{H} = \mathcal{H}_4 \cup \mathcal{H}_0$ and let $\alpha = 15/16$. Suppose that φ and ψ satisfy the assumption (e 4.201) for the above L and $\mathcal{H}$.

Then, by Lemma 4.11, there are mutually orthogonal projections $e_0, e_1, e_2, ..., e_K \in M_R$ such that $e_0 \lesssim e_1$ and e_i is equivalent to e_1, $i = 1, 2, ..., K$, a unital homomorphism $\psi_0 : A \to e_0 M_R e_0$, and a unital homomorphism $\psi_1 : A \to e_1 M_R e_1$ such that

$$\|\psi(a) - (\psi_0(a) \oplus \mathrm{diag}(\overbrace{\psi_1(a), \psi_1(a), ..., \psi_1(a)}^{K}))\| < \varepsilon_1 \tag{e 4.204}$$

for all $a \in \mathcal{F}_2$ and

$$\mathrm{tr} \circ \psi_1(g) \geq (15/16)\frac{\Delta(\hat{g})}{K} \text{ for all } g \in \mathcal{H}_0. \tag{e 4.205}$$

Put $\Psi = \psi_0 \oplus \mathrm{diag}(\overbrace{\psi_1(a), \psi_1(a), ..., \psi_1(a)}^{K})$). We compute that

$$[\Psi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}. \tag{e 4.206}$$

Let $R_0 = \mathrm{rank}(e_1)$. Then, by (e 4.205),

$$\begin{aligned} R_0 &= R\mathrm{tr} \circ \psi_1(1_A) \\ &\geq Lk(15/16)\frac{\Delta(\widehat{1_A})}{K} \geq k(K+1)(15/16)\Delta(\widehat{1_A}) && \text{(e 4.207)} \\ &\geq k(15/16)8N(\mathcal{P}_0)(J+1)/\delta. && \text{(e 4.208)} \end{aligned}$$

Moreover,

$$\mathrm{tr}' \circ \psi_1(\hat{g}) \geq (15/16)\Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_3, \tag{e 4.209}$$

where tr′ is the tracial state of M_{R_1}. It follows from Lemma 4.19 that there exists a unital homomorphism $h_0 : A \to M_{R_0-k}$ such that

$$(\varphi \oplus h_0)_{*0}|_{\mathcal{P}_0} = (\psi_1)_{\mathcal{P}_0}. \tag{e 4.210}$$

Put

$$h_1 = h_0 \oplus \mathrm{diag}(\overbrace{\varphi \oplus h_0, \varphi \oplus h_0, ..., \varphi \oplus h_0}^{J-1}) \text{ and } \psi_2 = \mathrm{diag}(\overbrace{\psi_1, \psi_1, ..., \psi_1}^{J}).$$

Then

$$[\varphi \oplus h_1]|_{\mathcal{P}} = [\psi_2]|_{\mathcal{P}}. \tag{e 4.211}$$

Put $\Psi' = \mathrm{diag}(\overbrace{\psi_1, \psi_1, ..., \psi_1}^{K-J}) = \psi_I \otimes 1_{K-J}$. Let $\varphi_0 = h_1 \oplus \psi_0 \oplus \Psi'$. Then (see (e 4.204))

$$[\varphi \oplus \varphi_0]|_{\mathcal{P}} = [\psi_0 \oplus \psi_2 \oplus \Psi']|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}. \tag{e 4.212}$$

Since $J/K < \delta/4$, by (e 4.204),

$$|\mathrm{tr} \circ (\varphi(g) \oplus \varphi_0(g)) - \mathrm{tr} \circ \psi(g)| < 2J/K + \varepsilon_1 < \delta \text{ for all } g \in \mathcal{H}_2.$$

Then, by (e 4.201), applying Theorem 4.18, we obtain a unitary $u \in M_R$ such that

$$\|\mathrm{Ad}\, u \circ (\varphi(f) \oplus \varphi_0(f)) - \psi(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}. \tag{e 4.213}$$

□

5. Almost Multiplicative maps to Finite Dimensional C^*-algebras

In the following statement, n is given and $(\mathcal{G}, \delta)$ depends on n : the proof is a standard compactness argument.

LEMMA 5.1. *Let $n \geq 1$ be an integer and let A be a unital separable C^*-algebra. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that, for any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\varphi : A \to M_n$, there exists a unital homomorphism $\psi : A \to M_n$ such that*

$$\|\varphi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

PROOF. Suppose that the conclusion is not true for a certain finite set $\mathcal{F} \subset A$ and $\varepsilon_0 > 0$. Let $\{\mathcal{G}_k\}_{k=1}^{\infty}$ be a sequence of finite subsets of A with $\mathcal{G}_k \subset \mathcal{G}_{k+1}$ and $\overline{\bigcup_k \mathcal{G}_k} = A$ and let $\{\delta_k\}$ be a decreasing sequence of positive numbers with $\delta_k \to 0$. Since the conclusion is assumed not to be true, there are unital $\mathcal{G}_k$-δ_k-multiplicative completely positive linear maps $\varphi_k : A \to M_n$ such that

$$\text{(e 5.1)} \quad \inf\{\max_{a \in \mathcal{F}} \|\varphi_k(a) - \psi(a)\| : \psi : A \to M_n \text{ a homomorphisms}\} \geq \varepsilon_0.$$

For each pair (i, j) with $1 \leq i, j \leq n$, let $l^{i,j} : M_n \to \mathbb{C}$ be the map defined by taking the matrix $a \in M_n$ to the entry of the i^{th} row and j^{th} column of a. Let $\varphi_k^{i,j} = l^{i,j} \circ \varphi_k : A \to \mathbb{C}$. Note that the unit ball of the dual space of A (as a Banach space) is weak* compact. Since A is separable, there is a subsequence (instead of subnet) of $\{\varphi_k\}$ (still denoted by φ_k) such that $\{\varphi_k^{i,j}\}$ is weak* convergent for all i, j. In other words, $\{\varphi_k\}$ converges pointwise. Let ψ_0 be the the limit. Then ψ_0 is a homomorphism and for k large enough, we have

$$\|\varphi_k(a) - \psi_0(a)\| < \varepsilon_0, \quad \text{for all } a \in \mathcal{F}.$$

This is in contradiction with (e 5.1) above. $\square$

LEMMA 5.2 (cf. Lemma 4.5 of [66]). *Let A be a unital C^*-algebra arising from a locally trivial continuous field of C^*-algebras isomorphic to M_n over a compact metric space X. Let T be a finite subset of tracial states on A. For any finite subset $\mathcal{F} \subset A$ and for any $\varepsilon > 0$ and any $\sigma > 0$, there are an ideal $J \subset A$ such that $\|\tau|_J\| < \sigma$ for all $\tau \in T$, a finite dimensional C^*-subalgebra $C \subset A/J$, and a unital homomorphism π_0 from A/J such that*

$$\text{(e 5.2)} \quad \mathrm{dist}(\pi(x), C) < \varepsilon \text{ for all } x \in \mathcal{F} \text{ and } \pi_0(A/J) = \pi_0(C) \cong C,$$

where $\pi : A \to A/J$ is the quotient map.

PROOF. This follows from Lemma 4.5 of [66]. In fact, the only difference is the existence of π_0. We will keep the notation of the proof of Lemma 4.5 of [66]. Let B_j and F_j be as in the proof of Lemma 4.5 of [66]. Note that in the proof

Lemma 4.5 of [66], $\varphi(\{g_{ij}\}) \cong D_\xi \cong M_n$. In other words, in the proof of Lemma 4.5 of [66], $(B_j)|_{F_j} \cong M_n$. Recall that

$$\begin{aligned} J &= \{f \in A : f(\zeta) = 0 \text{ for all } \zeta \in F\}, \\ F &= \bigsqcup_{i=1}^{k} F_i, \text{ and } F_j \subset B(\xi_j, \delta_j),\ j = 1, 2, ..., k, \end{aligned}$$

as in the proof of 4.5 of [66]. Recall also $D = \bigoplus_{i=1}^{k} f_j B_j$ and $\pi(D) = C$, where $f_j|_{F_j} = 1$ and $f_j|_{F_i} = 0$, if $j \neq i$. Choose $x_j \in F_j$, $j = 1, 2, ..., k$. One defines $\pi_0' = \bigoplus_{j=1}^{k} \pi_{x_j}$. Then

$$\pi_0'(D) = \bigoplus_{j=1}^{l} \pi_{x_j}(B_j) \cong \bigoplus_{j=1}^{l} M_n \cong C.$$

Note that $\xi_j \in F_j \subset F$, $j = 1, 2, ..., k$. Therefore, each π_{ξ_j} induces a homomorphism ψ_j of A/J such that $\psi_j(\pi(a)) = \pi_{\xi_j}(a)$ for all $a \in A$. In particular, $\psi_j(A/J) \cong M_n$, $j = 1, 2, ..., k$. Put $\pi_0 = \bigoplus_{j=1}^{k} \psi_j$. Then $\pi_0(A/J) = \pi_0'(A) = \bigoplus_{j=1}^{k} \pi_{\xi_j}(A) = \bigoplus_{j=1}^{k} \pi_{\xi_j}(D) = \pi_0'(D) = \pi_0(\pi(D)) = \pi_0(C) = \pi_0'(D) \cong C$. □

LEMMA 5.3 (cf. Lemma 4.7 of [66]). *Let A be a unital separable subhomogeneous C^*-algebra. Let $T \subset T(A)$ be a finite subset. For any finite subset $\mathcal{F} \subset A$, $\varepsilon > 0$ and $\sigma > 0$, there are an ideal $J \subset A$ such that $\|\tau|_J\| < \sigma$ for all $\tau \in T$, a finite dimensional C^*-subalgebra $C \subset A/J$, and a unital homomorphism π_0 from A/J such that*

$$\text{(e 5.3)}\quad \mathrm{dist}(\pi(x), C) < \varepsilon \text{ for all } x \in \mathcal{F} \text{ and } \pi_0(A/J) = \pi_0(C) \cong C,$$

where $\pi : A \to A/J$ is the quotient map.

PROOF. The proof is in fact contained in that of Lemma 4.7 of [66]. Each time Lemma 4.5 of [66] is applied, one can apply Lemma 5.2 above. Let us assume A and m to be as in Lemma 4.7 of [66] and keep the notation of the proof of Lemma 4.7 of [66]. Let us point out what the map π_0 is. When $m = 1$, π_0 is the map of Lemma 5.2.

Otherwise we proceed as in the proof of Lemma 4.7 of [66] to where π_1 and π_2 are constructed. Instead of applying Lemma 4.5 of [66], we apply Lemma 5.2 above to obtain a homomorphism $\pi_0^{(1)}$ (in place of π_0) from $\pi_2 \circ \pi_1(A)$ and a finite dimensional C^*-algebra $C_1 = \pi_2 \circ \pi_1(D_1) = \pi_2 \circ \pi_1(D_1')$ such that $\pi_0^{(1)}(\pi_2 \circ \pi_1(A)) = \pi_0^{(1)}(C_1) \cong C_1$.

In the proof of Lemma 4.7 of [66], the second time Lemma 4.5 of [66] is applied, we again, instead, apply Lemma 5.2 to obtain a homomorphism $\pi_0^{(2)}$ (in place of π_0) from $\pi_4 \circ \pi_3(A)$ and $C_2 = \pi_4 \circ \pi_3(D_2) = \pi_4 \circ \pi_3(D_2')$ such that $\pi_0^{(2)}(\pi_4 \circ \pi_3(A)) = \pi_0^{(2)}(C_2) \cong C_2$. Note, in the proof of Lemma 4.7 of [66],

$\pi(A) = \pi_2 \circ \pi_1(A) \oplus \pi_4 \circ \pi_3(A)$ and $C = \pi(D_1' \oplus D_2') = \pi(D_1') \oplus \pi(D_2') = \pi_2 \circ \pi_1(D_1') \oplus \pi_4 \circ \pi_3(D_2') = C_1 \oplus C_2$. Define $\pi_0 = \pi_0^{(1)} \oplus \pi_0^{(2)}$ from $\pi(A)$. Then $\pi_0(\pi(A)) = \pi_0^{(1)}(\pi_2 \circ \pi_1(A)) \oplus \pi_0^{(2)}(\pi_4 \circ \pi_3(A)) = \pi_0^{(1)}(C_1) \oplus \pi_0^{(2)}(C_2) = \pi_0(\pi(C)) = \pi_0^{(1)}(C_1) \oplus \pi_0^{(2)}(C_2) \cong C_1 \oplus C_2 = C$. ☐

LEMMA 5.4. *Let A be a unital separable subhomogeneous C^*-algebra. Let $\varepsilon > 0$, let $\mathcal{F} \subset A$ be a finite subset, and let $\sigma_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following condition: Suppose that $\varphi : A \to M_n$ (for some integer $n \geq 1$) is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map. Then, there exist a projection $p \in M_n$ and a unital homomorphism $\varphi_0 : A \to pM_np$ such that*

$$\|p\varphi(a) - \varphi(a)p\| < \varepsilon \text{ for all } a \in \mathcal{F}, \tag{e 5.4}$$

$$\|\varphi(a) - [(1-p)\varphi(a)(1-p) + \varphi_0(a)]\| < \varepsilon \text{ for all } a \in \mathcal{F}, \text{ and} \tag{e 5.5}$$

$$\mathrm{tr}(1-p) < \sigma_0, \tag{e 5.6}$$

where tr *is the normalized trace on M_n.*

PROOF. Assume that the conclusion is false. Then there exist $\varepsilon_0 > 0$, a finite subset $\mathcal{F}_0$, a positive number $\sigma_0 > 0$, an increasing sequence of finite subsets $\mathcal{G}_n \subset A$ such that $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ and such that $\bigcup_{n=1} \mathcal{G}_n$ is dense in A, a decreasing sequence of positive numbers $\{\delta_n\}$ with $\sum_{n=1}^{\infty} \delta_n < \infty$, a sequence of integers $\{m(n)\}$, and a sequence of unital $\mathcal{G}_n$-δ_n-multiplicative completely positive linear maps $\varphi_n : A \to M_{m(n)}$ satisfying the following condition:

$$\inf\{\max\{\|\varphi_n(a) - [(1-p)\varphi_n(a)(1-p) + \varphi_0(a)]\| : a \in \mathcal{F}_0\}\} \geq \varepsilon_0, \tag{e 5.7}$$

where the infimum is taken among all projections $p \in M_{m(n)}$ with $\mathrm{tr}_n(1-p) < \sigma_0$, and

$$\|p\varphi_n(a) - \varphi_n(a)p\| < \varepsilon_0,$$

where tr_n is the normalized trace on $M_{m(n)}$, and all possible unital homomorphisms $\varphi_0 : A \to pM_{m(n)}p$. By virtue of 5.1, one may also assume that $m(n) \to \infty$ as $n \to \infty$.

Note that $\{\mathrm{tr}_n \circ \varphi_n\}$ is a sequence of (not necessarily tracial) states of A. Let t_0 be a weak * limit of $\{\mathrm{tr}_n \circ \varphi_n\}$. Since A is separable, there is a subsequence (instead of a subnet) of $\{\mathrm{tr}_n \circ \varphi_n\}$ converging to t_0. Passing to a subsequence, we may assume that $\mathrm{tr}_n \circ \varphi_n$ converges to t_0. By the $\mathcal{G}_n$-δ_n-multiplicativity of φ_n, the limit t_0 is a tracial state on A.

Consider the ideal $\bigoplus_{n=1}^{\infty} M_{m(n)}$, where

$$\bigoplus_{n=1}^{\infty} M_{m(n)} = \{\{a_n\} : a_n \in M_{m(n)} \text{ and } \lim_{n\to\infty} \|a_n\| = 0\}.$$

Denote by Q the quotient $\prod_{n=1}^{\infty} M_{m(n)}/\bigoplus_{n=1}^{\infty} M_{m(n)}$. Let $\pi_\omega : \prod_{n=1}^{\infty} M_{m(n)} \to Q$ be the quotient map. Let $A_0 = \{\pi_\omega(\{\varphi_n(f)\}) : f \in A\}$ which is a subalgebra of Q. Denote by Ψ the canonical unital homomorphism from A to Q with $\Psi(A) = A_0$. If $a \in A$ has zero image in $\pi_\omega(A_0)$, i.e., $\varphi_n(a) \to 0$, then $t_0(a) = \lim_{n\to\infty} \mathrm{tr}_n(\varphi_n(a)) = 0$. So we may view t_0 as a state on $A_0 = \Psi(A)$.

It follows from Lemma 5.3 that there are an ideal $I \subset \Psi(A)$ and a finite dimensional C^*-subalgebra $B \subset \Psi(A)/I$ and a unital homomorphism $\pi_{00} : \Psi(A)/I \to B$ such that

$$\mathrm{dist}(\pi_I \circ \Psi(f), B) < \varepsilon_0/16 \text{ for all } f \in \mathcal{F}_0, \tag{e 5.8}$$

$$\|(t_0)|_I\| < \sigma_0/2 \text{ and } \pi_{00}|_B = \mathrm{id}_B. \tag{e 5.9}$$

Note that π_{00} can be regarded as map from $\Psi(A)$ to B with $\ker\pi_{00} \supset I$. There is, for each $f \in \mathcal{F}_0$, an element $b_f \in B$ such that

$$\|\pi_I \circ \Psi(f) - b_f\| < \varepsilon_0/16. \tag{e 5.10}$$

Put $C' = B + I$ and $I_0 = \Psi^{-1}(I)$ and $C_1 = \Psi^{-1}(C')$. For each $f \in \mathcal{F}_0$, there exists $a_f \in C_1 \subset A$ such that

$$\|f - a_f\| < \varepsilon_0/16 \text{ and } \pi_I \circ \Psi(a_f) = b_f. \tag{e 5.11}$$

Let $a \in (I_0)_+$ be a strictly positive element and let $J = \overline{\Psi(a)Q\Psi(a)}$ denote the hereditary C^*-subalgebra of Q generated by $\Psi(a)$. Since Q has real rank zero, so does J (see [10]). Put $C_2 = \Psi(C_1) + J$. Then J is a (σ-unital) ideal of C_2. Denote by $\pi_J : C_2 \to B$ the quotient map. Since Q has real rank zero, J is a hereditary C^*-subalgebra of Q, and $C_2/J = B$ has finite dimension, by Lemma 5.2 of [59], C_2 has real rank zero and projections in B lifts to a projection in C_2. It follows (see Theorem 9.8 of [25]) that the extension

$$0 \to J \to C_2 \to B \to 0$$

splits and is a quasidiagonal (see also the proof of Theorem 5.3 of [59]). As in Lemma 4.9 of [66], there are a projection $P \in J$ and a unital homomorphism $\psi_0 : B \to (1-P)C_2(1-P)$ such that

$$\|P\Psi(a_f) - \Psi(a_f)P\| < \varepsilon_0/8 \text{ and } \|\Psi(a_f) - [P\Psi(a_f)P + \psi_0 \circ \pi_J \circ \Psi(a_f)]\| < \varepsilon_0/8$$

for all $f \in \mathcal{F}_0$. Let $H : A \to \psi_0(B)$ be defined by $H = \psi_0 \circ \pi_{00} \circ \pi_I \circ \Psi$. One estimates that

$$\|P\Psi(f) - \Psi(f)P\| < \varepsilon_0/2 \text{ and} \tag{e 5.12}$$

$$\|\Psi(f) - [P\Psi(f)P + H(f)]\| < \varepsilon_0/2 \tag{e 5.13}$$

for all $f \in \mathcal{F}_0$. Note that $\dim H(A) < \infty$, and that $H(A) \subset Q$. There is a homomorphism $H_1 : H(A) \to \prod_{n=1}^{\infty} M_{m(n)}$ such that $\pi_\omega \circ H_1 \circ H = H$. One may write $H_1 = \{h_n\}$, where each $h_n : H(A) \to M_{m(n)}$ is a (not necessarily unital) homomorphism, $n = 1, 2, \ldots.$ There is also a sequence of projections $q_n \in M_{m(n)}$ such that $\pi_\omega(\{q_n\}) = P$. Let $p_n = 1 - q_n$, $n = 1, 2, \ldots.$ Then, for sufficiently large n, by (e 5.12) and (e 5.13),

$$\|(1-p_n)\varphi_n(f) - \varphi_n(f)(1-p_n)\| < \varepsilon_0 \text{ and} \tag{e 5.14}$$

$$\|\varphi_n(f) - [(1-p_n)\varphi_n(f)(1-p_n) + h_n \circ H(f)]\| < \varepsilon_0 \tag{e 5.15}$$

for all $f \in \mathcal{F}_0$. Moreover, since $P \in J$, for any $\eta > 0$, there is $b \in I_0$ with $0 \le b \le 1$ such that

$$\|\Psi(b)P - P\| < \eta.$$

However, by (e 5.9),

$$0 \le t_0(\Psi(b)) < \sigma_0/2 \text{ for all } b \in I_0 \text{ with } 0 \le b \le 1. \tag{e 5.16}$$

By choosing sufficiently small η, for all sufficiently large n, we have

$$\mathrm{tr}_n(1-p_n) < \sigma_0.$$

This, together with (e 5.14) and (e 5.15), contradicts (e 5.7). □

COROLLARY 5.5. *Let A be a unital subhomogeneous C^*-algebra. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset, and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following condition: Suppose that φ, $\psi : A \to M_n$ (for some integer $n \ge 1$) are two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps. Then, there exist projections $p, q \in M_n$ with* rank(p) = rank(q) *and unital homomorphisms $\varphi_0 : A \to pM_np$ and $\psi_0 : A \to qM_nq$ such that, for $a \in \mathcal{E}$,*

$$\|p\varphi(a) - \varphi(a)p\| < \eta, \quad \|q\psi(a) - \psi(a)q\| < \eta$$

$$\|\varphi(a) - [(1-p)\varphi(a)(1-p) + \varphi_0(a)]\| < \eta, \quad \|\psi(a) - [(1-q)\psi(a)(1-q) + \psi_0(a)]\| < \eta,$$
$$\text{and } \mathrm{tr}(1-p) = \mathrm{tr}(1-q) < \eta_0,$$

where tr *is the normalized trace on M_n.*

For convenience in future use, we have used η, η_0 and $\mathcal{E}$ to replace the ε, σ_0, and $\mathcal{F}$ of 5.4.

PROOF. By Lemma 5.4, we can obtain such decompositions for φ and ψ separately. Thus the only missing part is that rank(p) = rank(q). Let $\{z_1, z_2, \ldots, z_m\}$ be the set of ranks of irreducible representations of A and let T be the number given by 4.1 corresponding to $\{z_1, z_2, \ldots, z_m\}$. We apply 5.4 to $\eta_0/2$ instead of σ_0 (and, η and $\mathcal{E}$ in places of ε and $\mathcal{F}$). By Lemma 5.1, we can assume the size n of the matrix algebra M_n is large enough that $T/n < \eta_0/2$. By Lemma 4.1, we may take sub-representations out of φ_0 and ψ_0 (one of them has size at most T) so that the remainder of φ_0 and ψ_0 have same size—that is for rank(new p) = rank(new q), and tr$(1 - (\text{new } p))$ = tr$(1 - (\text{new } q)) < \eta_0/2 + T/n < \eta_0$. □

LEMMA 5.6. *Let $A \in \overline{\mathcal{D}}_s$ be an infinite dimensional unital C^*-algebra, let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\varepsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map.*

Suppose that $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ is a finite subset, $\varepsilon_1 > 0$ is a positive number and $K \geq 1$ is an integer. There exist $\delta > 0$, $\sigma > 0$, and a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{H}_2 \subset A_+^{\mathbf{1}} \setminus \{0\}$ satisfying the following condition: Suppose that $L_1, L_2 : A \to M_n$ (for some integer $n \geq 1$) are unital $\mathcal{G}$-δ-multiplicative completely positive linear maps such that

$$tr \circ L_1(h) \geq \Delta(\hat{h}) \quad \textit{for all } h \in \mathcal{H}_2, \textit{ and} \tag{e 5.17}$$

$$|tr \circ L_1(h) - tr \circ L_2(h)| < \sigma \text{ for all } \ h \in \mathcal{H}_2. \tag{e 5.18}$$

Then there exist mutually orthogonal projections $e_0, e_1, e_2, ..., e_K \in M_n$ such that $e_1, e_2, ..., e_K$ are pairwise equivalent, $e_0 \lesssim e_1$, $tr(e_0) < \varepsilon_1$, and $e_0 + \sum_{i=1}^K e_i = 1$, and there exist unital $\mathcal{G}_0$-ε_0-multiplicative completely positive linear maps $\psi_1, \psi_2 : A \to e_0 M_n e_0$, a unital homomorphism $\psi : A \to e_1 M_n e_1$, and a unitary $u \in M_n$ such that

$$\|L_1(f) - \mathrm{diag}(\psi_1(f), \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K})\| < \varepsilon \ \textit{ and} \tag{e 5.19}$$

$$\|u^* L_2(f) u - \mathrm{diag}(\psi_2(f), \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K})\| < \varepsilon \tag{e 5.20}$$

for all $f \in \mathcal{F}$, where tr *is the tracial state of M_n. Moreover,*

$$\mathrm{tr}(\psi(g)) \geq \frac{\Delta(\hat{g})}{2K} \ \textit{ for all } g \in \mathcal{H}_1. \tag{e 5.21}$$

PROOF. Let $\varepsilon > 0$, $\varepsilon_0 > 0$, $\mathcal{F}$ and $\mathcal{G}_0$ and Δ be given as stated. First note that the following statement is evident. For any C^*-algebra A, any finite subset $\mathcal{G}_0 \subset A$ and $\varepsilon_0 > 0$, there are a finite subset $\mathcal{F}' \subset A$ which contains $\mathcal{F}$ and $\varepsilon' > 0$ which is smaller than $\min\{\varepsilon/2, \varepsilon_0/2\}$ satisfying the following condition. If $L : A \to B$ is a unital $\mathcal{F}'$-ε'-multiplicative completely positive linear map, $p_0, p_1 \in B$ are projections with $p_0 + p_1 = 1_B$, and $L_0' : A \to p_0 B p_0$, $L_1' : A \to p_1 B p_1$ are unital completely positive linear maps with

$$\|L(f) - \mathrm{diag}(L_0'(f), L_1'(f))\| < \varepsilon' \text{ for all } f \in \mathcal{F}',$$

then both L_0' and L_1' are $\mathcal{G}_0$-ε_0-multiplicative.

Let $\varepsilon_1 > 0$, $\mathcal{H}_1$ and K be given as in the statement of the lemma. Choose an integer $k_0 \geq 1$ such that $1/k_0 < \varepsilon_1$ and let $K_1 = k_0 K$. Put

$$\varepsilon_2 = \min\{\varepsilon/16, \varepsilon'/16, \varepsilon_1/2, 1/2\}. \tag{e 5.22}$$

Let $\Delta_1 = (3/4)\Delta$. Let $\delta_1 > 0$ (in place of δ), $\mathcal{H}_{1,0} \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), and $\mathcal{H}_{2,0} \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be the constant and finite subsets provided by Lemma 4.11 (see also Remark 4.12) for ε_2 (in place of ε), $\mathcal{F}'$ (in place of $\mathcal{F}$), $2K_1$ (in place of K), $\mathcal{H}_1$ (in place of $\mathcal{H}_0$), Δ_1, and A, as well as $\alpha = 3/4$. It is clear that, without loss of generality, we may assume that $\mathcal{H}_{2,0}$ is in the self-adjoint part of the unit ball of A. Since every $h \in \mathcal{H}_{2,0}$ can be written as $h = (|h|+h)/2 + (h-|h|)/2$, choosing an even smaller δ_1 (half the size), without loss of generality, we may assume that $\mathcal{H}_{2,0} \subset A_+^{\mathbf{1}} \setminus \{0\}$.

Let $\eta_0 = \min\{\delta_1/16, \varepsilon_2/16, \min\{\Delta(\hat{h}) : h \in \mathcal{H}_{1,0}/16\}\}$. Let $\delta_2 > 0$ (in place of δ) and let $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be the constant and the finite subset provided by 5.5 for $\eta = \eta_0 \cdot \min\{\varepsilon_2, \delta_1/4\}$, η_0, and $\mathcal{E} = \mathcal{F}' \cup \mathcal{H}_{1,0} \cup \mathcal{H}_{2,0} \cup \mathcal{H}_1$. Let $\delta = \eta_0 \cdot \min\{\delta_2/2, \delta_1/2, \varepsilon_0/2\}$, let $\sigma = \min\{\eta_0/2, \eta/2\}$, let $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{F} \cup \mathcal{F}' \cup \mathcal{E}$, and let $\mathcal{H}_2 = \mathcal{H}_{1,0} \cup \mathcal{H}_{2,0} \cup \mathcal{H}_1$. Recall that we have assumed that $\mathcal{F}' \supset \mathcal{F}$ and $\varepsilon' < \min\{\varepsilon/2, \varepsilon_0/2\}$.

Now suppose that L_1 and L_2 satisfy the assumptions of the lemma with respect to the δ, σ and $\mathcal{G}$, $\mathcal{H}_2$ above. It follows from Corollary 5.5 that there exist a projection $p \in M_n$, two unital homomorphisms $\varphi_1, \varphi_2 : A \to pM_np$, and a unitary $u_1 \in M_n$ such that

$$\|u_1^* L_1(a) u_1 - ((1-p)u_1^* L_1(a) u_1 (1-p) + \varphi_1(a))\| < \eta, \tag{e 5.23}$$

$$\|L_2(a) - ((1-p)L_2(a)(1-p) + \varphi_2(a)\| < \eta \tag{e 5.24}$$

for all $a \in \mathcal{E}$, and

$$\mathrm{tr}(1-p) < \eta_0, \tag{e 5.25}$$

where tr is the tracial state on M_n.

We compute that

$$\mathrm{tr} \circ \varphi_1(g) \geq \Delta(\hat{g}) - \eta - \eta_0 \geq (3/4)\Delta(\hat{g}) \ \text{ for all } \ g \in \mathcal{H}_{1,0} \ \text{ and} \tag{e 5.26}$$

$$|\mathrm{tr} \circ \varphi_1(g) - \tau \circ \varphi_2(g)| < 2\eta + 2\eta_0 + \sigma < \delta_1 \ \text{ for all } \ g \in \mathcal{H}_{2,0}. \tag{e 5.27}$$

It follows from Lemma 4.11 (and Remark 4.12) that there exist mutually orthogonal projections $q_0, q_1, ..., q_{2K} \in pM_np$ such that $q_0 \lesssim q_1$ and q_i is equivalent to q_1 for all $i = 1, 2, ..., 2K_1$, two unital homomorphisms $\varphi_{1,0}, \varphi_{2,0} : A \to q_0 M_n q_0$, a unital homomorphism $\psi' : A \to q_1 M_n q_1$, and a unitary $u_2 \in pM_np$ such that

$$\|u_2^* \varphi_1(a) u_2 - (\varphi_{1,0}(a) \oplus \mathrm{diag}(\overbrace{\psi'(a), \psi'(a), ..., \psi'(a)}^{2K_1}))\| < \varepsilon_2 \tag{e 5.28}$$

$$\text{and } \ \|\varphi_2(a) - (\varphi_{2,0}(a) \oplus \mathrm{diag}(\overbrace{\psi'(a), \psi'(a), ..., \psi'(a)}^{2K_1}))\| < \varepsilon_2 \tag{e 5.29}$$

for all $a \in \mathcal{F} \cup \mathcal{F}'$. Moreover,

$$\tau \circ \psi'(g) \geq (3/4)^2 \Delta(\hat{g})/2K_1 \ \text{ for all } \ g \in \mathcal{H}_1, \tag{e 5.30}$$

where τ is the tracial state of pM_np. Let $u = u_1((1-p)+u_2)$, $e_0 = (1-p)\oplus q_0$, $e_i = \sum_{j=1}^{2k_0} q_{2k_0(i-1)+j}$, $i = 1, 2, ..., K$, let $\psi_1(a) = (1-p)u_1^*L_1(a)u_1(1-p)\oplus\varphi_{1,0}$, $\psi_2(a) = (1-p)L_2(a)(1-p)\oplus\varphi_{2,0}$ and $\psi(a) = \mathrm{diag}(\overbrace{\psi'(a), \psi'(a), ..., \psi'(a)}^{2k_0})$ for $a \in A$. Then

$$\|u^*L_1(f)u - (\psi_1(f)\oplus \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K}))\| < \eta + \varepsilon_2 < \varepsilon' < \varepsilon$$
$$\text{and } \|L_2(f) - (\psi_2(f)\oplus \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K}))\| < \eta + \varepsilon_2 < \varepsilon' < \varepsilon$$

for all $f \in \mathcal{F}\cup\mathcal{F}'$. It follows (by the choice of ε' and $\mathcal{F}'$) that ψ_1 and ψ_2 are $\mathcal{G}_0$-ε_0-multiplicative. Moreover,

$$\mathrm{tr}(e_0) = \mathrm{tr}(1-p) + \mathrm{tr}(q_0) < \eta_0 + 1/(2k_0K+1) < \varepsilon_1.$$

Further,

$$\mathrm{tr}\circ\psi(g) = \mathrm{tr(p)}\tau\circ\psi(\mathrm{g})$$
$$\geq (1-\eta_0)\tau\circ\psi(g) \tag{e 5.31}$$
$$\geq (1-\frac{1}{32})(9/16)(2k_0)\frac{\Delta(\hat{g})}{2k_0K} \geq \frac{\Delta(\hat{g})}{2K} \text{ for all } g \in \mathcal{H}_1. \tag{e 5.32}$$

□

LEMMA 5.7 (9.4 of [71]). *Let A be a unital separable C^*-algebra. For any $\varepsilon > 0$ and any finite subset $\mathcal{H} \subset A_{s.a.}$, there exist a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ satisfying the following condition: Suppose that $\varphi : A \to B$ (for some unital C^*-algebra B) is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map and $t \in T(B)$ is a tracial state of B. Then, there exists a tracial state $\tau \in T(A)$ such that*

$$|t\circ\varphi(h) - \tau(h)| < \varepsilon \text{ for all } h \in \mathcal{H}. \tag{e 5.33}$$

PROOF. This follows from the same proof of Lemma 9.4 of [71] □

THEOREM 5.8. *Let $A \in \overline{\mathcal{D}}_s$ be a unital C^*-algebra. Let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map.*

Let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exist a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$, and $\sigma > 0$ satisfying the following condition: Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps such that

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}, \tag{e 5.34}$$
$$\mathrm{tr}\circ L_1(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1, \text{ and} \tag{e 5.35}$$
$$|\mathrm{tr}\circ L_1(h) - \mathrm{tr}\circ L_2(h)| < \sigma \text{ for all } h \in \mathcal{H}_2. \tag{e 5.36}$$

Then there exists a unitary $u \in M_k$ *such that*

$$\|\mathrm{Ad}\, u \circ L_1(f) - L_2(f)\| < \varepsilon \ \textit{for all} \ f \in \mathcal{F}. \tag{e 5.37}$$

PROOF. The proof is almost the same as that of 4.18.

If A is finite dimensional, then A is semiprojective. Therefore, it is easy to see that the general case can be reduced to the case that both L_1 and L_2 are unital homomorphisms. Then we return to the situation of 4.18.

So we now assume that A is infinite dimensional.

Let $\Delta_1 = (1/3)\Delta$. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. Let $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$), $\mathcal{P} \subset \underline{K}(A)$, $\mathcal{H}_1' \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}$) be the finite subsets and $K \geq 1$ be the integer provided by Lemma 4.15 (see also Remark 4.16) for $\varepsilon/2$ (in place of ε), $\mathcal{F}$, Δ_1, and A.

Let $\mathcal{G}_0 = \mathcal{G}_1$ and $\varepsilon_0 = \delta_1/2$. Without loss of generality, we may assume the following: for any two $\mathcal{G}_0$-ε_0-multiplicative contractive completely positive linear maps $\Phi_1, \Phi_2 : A \to C$ (for any unital C^*-algebra C), $[\Phi_1]|_{\mathcal{P}}$ and $[\Phi_2]|_{\mathcal{P}}$ are well defined, and, if $\|\Phi_1(a) - \Phi_2(a)\| < \varepsilon_0$ for all $a \in \mathcal{G}_0$, then

$$[\Phi_1]|_{\mathcal{P}} = [\Phi_2]|_{\mathcal{P}}. \tag{e 5.38}$$

Let $\delta_2 > 0$ (in place of δ), $\sigma_1 > 0$ (in place of σ), a finite subset $\mathcal{G}_2 \subset A$ (in place of $\mathcal{G}$) and a finite subset $\mathcal{H}_2' \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_2$) be as given by Lemma 5.6 for $\mathcal{H}_1'$ (in place of $\mathcal{H}_1$), K, ε_0, $\mathcal{G}_0$ $\delta_1/4$ (in place of ε), $\mathcal{G}_1 = \mathcal{G}_0$ (in place of $\mathcal{F}$), Δ, and A.

Let $\delta = \min\{\delta_1/4, \delta_2, \varepsilon/4\}$, $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{F} \cup \mathcal{G}_2$, $\mathcal{H}_1 = \mathcal{H}_2'$ and $\mathcal{H}_2 = \mathcal{H}_2'$.

Now let L_1, $L_2 : A \to M_k$ be two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps which satisfy the assumption for the above $\mathcal{H}_1$, $\mathcal{G}$, δ, σ, $\mathcal{P}$, and $\mathcal{H}_2$.

By 5.6, we obtain a unitary $u_1 \in M_k$, mutually orthogonal non-zero projections $e_0, e_1, e_2, ..., e_K \in M_k$ with $\sum_{i=0}^K e_i = 1_{M_k}$, $e_0 \lesssim e_1$, e_i equivalent to e_1, $i = 1, 2, ..., K$, unital $\mathcal{G}_0$-ε_0-multiplicative completely positive linear maps $\Phi_1, \Phi_2 : A \to e_0 M_k e_0$, and a unital homomorphism $\psi : A \to e_1 M_k e_1$ such that

$$\|u_1^* \circ L_1(f) u_1 - (\Phi_1(f) \oplus \Psi(f))\| < \delta_1/4 \ \text{for all} \ f \in \mathcal{G}_1, \tag{e 5.39}$$

$$\|L_2(f) - (\Phi_2(f) \oplus \Psi(f))\| < \delta_1/4 \ \text{for all} \ f \in \mathcal{G}_1, \ \text{and} \tag{e 5.40}$$

$$\tau \circ \psi(g) \geq \Delta(\hat{g})/2K \ \text{for all} \ g \in \mathcal{H}_1', \tag{e 5.41}$$

where $\Psi(a) = \mathrm{diag}(\overbrace{\psi(a), \psi(a), ..., \psi(a)}^{K})$ for all $a \in A$ and τ is the tracial state on M_n.

Let τ_1 be the tracial state of $e_1 M_k e_1$. Then (e 5.41) implies

$$\tau_1 \circ \Psi(g) \geq \Delta(\hat{g})/2 \geq \Delta_1(\hat{g}) \ \text{for all} \ g \in \mathcal{H}_1. \tag{e 5.42}$$

By the choice of ε_0 and $\mathcal{G}_0$, by (e 5.39) and (e 5.40), one has, for all $x \in \mathcal{P}$,

$$[\Phi_1](x) + [\Psi](x) = [\Phi_2](x) + [\Psi](x) \tag{e 5.43}$$

in the group $\underline{K}(A)$. It follows that, for all $x \in \mathcal{P}$,

$$[\Phi_1](x) = [\Phi_2](x). \tag{e 5.44}$$

Now, by (e 5.44), (e 5.43), and the choice of K, on applying Lemma 4.15, one obtains a unitary $u_2 \in M_k$ such that

$$\|u_2^*(\Phi_1(a) \oplus \Psi(a))u_2 - (\Phi_2(a) \oplus \Psi(a))\| < \varepsilon/2 \text{ for all } a \in \mathcal{F}. \tag{e 5.45}$$

Choose $u = u_1 u_2$. Then, by (e 5.39) and (e 5.45), for all $a \in \mathcal{F}$,

$$\begin{aligned} &\|u^* L_1(a)u - L_2(a)\| \\ &\quad \leq \|u_2^*(u_1^* L_1(a)u_2 - (\Phi_1(a) \oplus \Psi(a))\| \\ &\qquad + \|u_2^*(\Phi_1(a) \oplus \Psi(a))u_2 - (\Phi_1(a) \oplus \Psi(a))\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

6. Homotopy Lemma in Finite Dimensional C^*-algebras

LEMMA 6.1. *Let S be a subset of M_k (for some integer $k \geq 1$), and let $u \in M_k$ be a unitary such that*

$$ua = au \text{ for all } a \in S. \tag{e 6.1}$$

Then there exists a continuous path of unitaries $\{u_t : t \in [0,1]\} \subset M_k$ such that

$$u_0 = u, \;\; u_1 = 1, \;\; u_t a = a u_t \text{ for all } a \in S \tag{e 6.2}$$

and for all $t \in [0,1]$, and moreover,

$$\text{length}(\{u_t\}) \leq \pi. \tag{e 6.3}$$

PROOF. There is a continuous function h from $sp(u)$ to $[-\pi, \pi]$ such that

$$\exp(ih(u)) = u. \tag{e 6.4}$$

Because h is a continuous function of u,

$$ah(u) = h(u)a \text{ for all } a \in S. \tag{e 6.5}$$

Note that $h(u) \in (M_k)_{s.a.}$ and $\|h(u)\| \leq \pi$. Define $u_t = \exp(i(1-t)h(u))$ $(t \in [0,1])$. Then $u_0 = u$ and $u_1 = 1$. Also,

$$u_t a = a u_t$$

for all $a \in S$ and $t \in [0,1]$. Moreover, one has

$$\text{length}(\{u_t\}) \leq \pi,$$

as desired. □

LEMMA 6.2. *Let $A \in \bar{\mathcal{D}}_s$ be a unital C^*-algebra, let $\mathcal{H} \subset (A \otimes C(\mathbb{T}))_{s.a.}$ be a finite subset, let $1 > \sigma > 0$ be a positive number, and let $\Delta : (A \otimes C(\mathbb{T}))_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\varepsilon > 0$, let $\mathcal{G}_0 \subset A \otimes C(\mathbb{T})$ be a finite subset, let $\mathcal{P}_0, \mathcal{P}_1 \subset \underline{K}(A)$ be finite subsets, and write $\mathcal{P} = \mathcal{P}_0 \cup \beta(\mathcal{P}_1) \subset \underline{K}(A \otimes C(\mathbb{T}))$. There exist $\delta > 0$, a finite subset $\mathcal{G} \subset A \otimes C(\mathbb{T})$, and a finite subset $\mathcal{H}_1 \subset (A \otimes C(\mathbb{T}))_+^{\mathbf{1}} \setminus \{0\}$ satisfying the following condition: Suppose that $L : A \otimes C(\mathbb{T}) \to M_k$ (for some integer $k \geq 1$) is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map such that*

$$\text{(e 6.6)} \qquad \operatorname{tr} \circ L(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1, \text{ and}$$

$$\text{(e 6.7)} \qquad [L]|_{\boldsymbol{\beta}(\mathcal{P}_1)} = 0,$$

where tr *is the tracial state of M_n. Then there exists a unital $\mathcal{G}_0$-ε-multiplicative completely positive linear map $\psi : A \otimes C(\mathbb{T}) \to M_k$ such that $u = \psi(1 \otimes z)$ is a unitary,*

$$\text{(e 6.8)} \qquad u\psi(a \otimes 1) = \psi(a \otimes 1)u \text{ for all } a \in A$$

$$\text{(e 6.9)} \qquad [L]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and,}$$

$$\text{(e 6.10)} \qquad |\operatorname{tr} \circ L(h) - \operatorname{tr} \circ \psi(h)| < \sigma \text{ for all } h \in \mathcal{H}.$$

PROOF. Let $\mathcal{H}$ and σ, ε and $\mathcal{G}_0$ be given. It is clear that, without loss of generality, we may assume that $\mathcal{H} \subset (A \otimes C(\mathbb{T}))_{s.a.}^{\mathbf{1}}$. By writing $h = h_+ - h_-$, where $h_+ = (|h| + h)/2$ and $h_- = (|h| - h)/2$, and choose a smaller σ, to simplify notation, without loss of generality, we may assume that $\mathcal{H} \subset (A \otimes C(\mathbb{T}))_+^{\mathbf{1}} \setminus \{0\}$. We may also assume that

$$\mathcal{G}_0 = \{g \otimes f : g \in \mathcal{G}_{0A} \text{ and } f \in \mathcal{G}_{1T}\},$$

where $1_A \subset \mathcal{G}_{0A} \subset A$ and $\mathcal{G}_{1T} \subset C(\mathbb{T})$ are finite subsets. To simplify matters further, we may assume, without loss of generality, that $\mathcal{G}_{1T} = \{1_{C(\mathbb{T})}, z\}$, where $z \in C(\mathbb{T})$ is the standard unitary generator.

We may assume that $\mathcal{G}_{0A}$ is sufficiently large and ε is sufficiently small that for any unital $\mathcal{G}_0$-ε-multiplicative completely positive linear map L, $[L]|_{\mathcal{P}}$ is well defined, any unital $\mathcal{G}_{0A}$-ε-multiplicative completely positive linear map Φ from A, $[\Phi]|_{\mathcal{P}_0}$ is well defined, and for any unital $\mathcal{G}_0$-ε-multiplicative completely positive linear maps L_1 and L_2 with

$$L_1 \approx_\varepsilon L_2 \quad \text{on } \mathcal{G}_0,$$

we have

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}},$$

and, furthermore, for any unital $\mathcal{G}_{0A}$-ε-multiplicative completely positive linear maps Φ_1 and Φ_2 (from A) with

$$\Phi_1 \approx_\varepsilon \Phi_2 \text{ on } \mathcal{G}_{0A},$$

we have

$$[\Phi_1]|_{\mathcal{P}_0} = [\Phi_2]|_{\mathcal{P}_0}.$$

There is $\delta_0 > 0$ satisfying the following condition: if $L', L'' : A \otimes C(\mathbb{T}) \to C$ (for any unital C^*-algebra C) are unital $\mathcal{G}_0$-δ_0-multiplicative completely positive linear maps such that $\|L'(a) - L''(a)\| < \delta_0$ for all $a \in \mathcal{G}_0$, then $\|L'(h) - L''(h)\| < \sigma/4$ for all $h \in \mathcal{H}$.

Let n be an integer such that $1/n < \sigma/2$. Note that $A \otimes C(\mathbb{T}) \in \overline{\mathcal{D}}_s$. Put $\varepsilon_0 = \min\{\sigma/2, \varepsilon/2, \delta_0/2\}$.

Let $\delta > 0$, $\sigma_0 > 0$ (in place of σ) $\mathcal{G} \subset A \otimes C(\mathbb{T})$, and $\mathcal{H}_1 \subset A \otimes C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_2$) be the constants and finite subsets provided by Lemma 5.6 for $A \otimes C(\mathbb{T})$ (in place of A), ε_0 (in place of ε), $\mathcal{G}_0$ (in place of $\mathcal{F}$), $\mathcal{H}$ (in place of $\mathcal{H}_1$), and Δ. Without loss of generality, we may assume that $\delta < \varepsilon$, and by choosing larger a $\mathcal{G}$ if necessary, we may assume that $\mathcal{G}_0 \subset \mathcal{G}$. Now suppose that $L : A \otimes C(\mathbb{T}) \to M_k$ satisfies the assumption for the above δ, $\mathcal{G}$, and $\mathcal{H}_1$. In particular, $[L]|_{\mathcal{P}}$ is well defined. It follows from Lemma 5.6 (for $L_1 = L_2 = L$) that there is a projection $e_0 \in M_k$ and a unital $\mathcal{G}_0$-ε_0-multiplicative completely positive linear map $\psi_0 : A \otimes C(\mathbb{T}) \to e_0 M_k e_0$ and a unital homomorphism $\psi_1 : A \otimes C(\mathbb{T}) \to (1 - e_0) M_k (1 - e_0)$ such that

$$\mathrm{tr}(e_0) < 1/n < \sigma, \tag{e 6.11}$$

$$\|L(a) - \psi_0(a) \oplus \psi_1(a)\| < \varepsilon_0 \text{ for all } a \in \mathcal{G}_0. \tag{e 6.12}$$

Define $\psi : A \otimes C(\mathbb{T}) \to M_k$ by $\psi(a) = \psi_0(a) \oplus \psi_1(a)$ for all $a \in A$ (see 2.26) and $\psi(1 \otimes z) = e_0 \oplus \psi_1(1 \otimes z)$. Put $u = \psi(1 \otimes z)$. Consider $\Phi_1(a) = L(a \otimes 1)$ and $\Phi_2(a) = \psi(a \otimes 1)$. Then, by (e 6.12) and the choices of $\mathcal{G}_0$ and ε,

$$[L]|_{\mathcal{P}_0} = [\psi]|_{\mathcal{P}_0}. \tag{e 6.13}$$

On the other hand, define $\Psi_0 : A \otimes C(\mathbb{T}) \to e_0 M_k e_0$ by $\Psi_0(a \otimes f) = \psi_0(a)(f(1)e_0)$ for all $a \in A$ and $f \in C(\mathbb{T})$ (and where 1 is the point on the unit circle). Then

$$[\psi]|_{\boldsymbol{\beta}(\mathcal{P}_1)} = [\Psi_0]|_{\boldsymbol{\beta}(\mathcal{P}_1)} + [\psi_1]|_{\boldsymbol{\beta}(\mathcal{P}_1)}.$$

Since $\Psi_0(1 \otimes z) = e_0$, one concludes that $[\Psi_0]|_{\boldsymbol{\beta}(\mathcal{P}_1)} = 0$. On the other hand, ψ_1 is a homomorphism from $A \otimes C(\mathbb{T})$ into M_k, and so $[\psi_1]|_{\boldsymbol{\beta}(\mathcal{P}_1)} = 0$ (this also follows from the first part of Lemma 6.1). Thus, by (e 6.7) and (e 6.13),

$$[L]|_{\mathcal{P}} = [L]|_{\mathcal{P}_0 \cup \boldsymbol{\beta}(\mathcal{P}_1)} = [\psi]|_{\mathcal{P}}.$$

Since $\mathcal{H} \subset \mathcal{G}_0$ and $\varepsilon_0 < \sigma$, by (e 6.12) and (e 6.11), the inequality (e 6.10) also holds. $\square$

DEFINITION 6.3. Let A be a unital C^*-algebra, let X be a compact metric space, and let $B = A\otimes C(X)$. We identify B with $C(X, A)$. Let $b \in C(X, A)_+^{\mathbf{1}} \setminus \{0\}$. Choose $x_0 \in X$ such that $\|b(x_0)\| > 0$ and choose $\varepsilon = \|b(x_0)\|/4$. Then $b(x_0) > b(x_0) - \varepsilon > 0$. Since $b \in C(X, A)_+$, there exists a neighborhood $N(x_0)$ of X such that $b(x) > b(x_0) - \varepsilon$. Choose $f \in C(X)_+$ with $0 \le f \le 1$ such that its support lies in $N(x_0)$. Then $b(x) > (b(x_0) - \varepsilon) \otimes f$. Note $(b(x_0) - \varepsilon) \otimes f \ge 0$.

Let $\Delta : (A \otimes C(X))_+^{q,\mathbf{1}} \to (0, 1)$ be an order preserving map. Then,

$$\Delta_0(\hat{h}) = \sup\{\Delta(\widehat{h_1 \otimes h_2}) : h_1 \otimes h_2 \le h, h_1 \in A_+^{\mathbf{1}} \setminus \{0\}, h_2 \in C(X)_+^{\mathbf{1}} \setminus \{0\}\} > 0$$

for any $h \in (A \otimes C(X))_+^{\mathbf{1}} \setminus \{0\}$. Note if $h \ge h'$, then

$$\Delta_0(\hat{h}) \ge \Delta_0(\hat{h'}).$$

In other words, $\Delta_0 : (A \otimes C(X))_+^{q,\mathbf{1}} \to (0, 1)$ is an order preserving map. Moreover, $\Delta_0(\hat{h}) \le \Delta(\hat{h})$ for all $h \in (A \otimes C(X))_+^{\mathbf{1}} \setminus \{0\}$. If $h = h_1 \otimes h_2$ for some $h_1 \in A_+^{\mathbf{1}} \setminus \{0\}$ and $h_2 \in C(X)_+^{\mathbf{1}} \setminus \{0\}$, then $\Delta_0(\hat{h}) = \Delta(\hat{h})$.

LEMMA 6.4. *Let $A \in \overline{\mathcal{D}}_s$ be a unital C^*-algebra and let $\Delta : (A\otimes C(\mathbb{T}))_+^{q,\mathbf{1}}\setminus\{0\} \to (0, 1)$ be an order preserving map. Let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exist a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$ and a finite subset $\mathcal{P} \subset \underline{K}(A)$ such that, if $L : A\otimes C(\mathbb{T}) \to M_k$ (for some integer $k \ge 1$) is a unital $\mathcal{G}'$-δ-multiplicative completely positive linear map where $\mathcal{G}' = \{g \otimes f : g \in \mathcal{G}, f = \{1, z, z^*\}\}$, and $u \in M_k$ is a unitary such that*

$$\|L(1 \otimes z) - u\| < \delta, \tag{e 6.14}$$

$$[L]|_{\beta(\mathcal{P})} = 0 \ \textit{and} \tag{e 6.15}$$

$$\mathrm{tr} \circ L(h_1 \otimes h_2) \ge \Delta(\widehat{h_1 \otimes h_2}) \tag{e 6.16}$$

for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, where tr is the tracial state of M_k, then there exists a continuous path of unitaries $\{u_t : t \in [0, 1]\} \subset M_k$ with $u_0 = u$ and $u_1 = 1$ such that

$$\|L(f \otimes 1)u_t - u_tL(f \otimes 1)\| < \varepsilon \ \textit{for all} \ f \in \mathcal{F} \tag{e 6.17}$$

and $t \in [0, 1]$. Moreover, $\{u_t\}$ can be chosen such that

$$\mathrm{length}(\{u_t\}) \le \pi + \varepsilon. \tag{e 6.18}$$

PROOF. Let Δ_0 be as associated with Δ in Definition 6.3. Let $\Delta_1 = (1/2)\Delta_0$, let $\mathcal{F}_0 = \{f \otimes 1 : 1 \otimes z : f \in \mathcal{F}\}$ and let $B = A \otimes C(\mathbb{T})$. Then $B \in \bar{\mathcal{D}}_s$. Let $\mathcal{H}' \subset B_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\mathcal{H}_0 \subset B_{s.a.}$ (in place of $\mathcal{H}_2$), $\sigma_0 > 0$ (in place of

σ), $\mathcal{G}_1 \subset A\otimes C(\mathbb{T})$ (in place of $\mathcal{G}$), $\delta_1 > 0$ (in place of δ), and $\mathcal{P}' \subset \underline{K}(B)$ (in place of $\mathcal{P}$) be the finite sets and constants provided by Theorem 5.8 (for B instead of A) for $\varepsilon/16$ (in place of ε), $\mathcal{F}_0$ (in place of $\mathcal{F}$), and Δ_1. It is clear that, without loss of generality, we may assume that every element of $\mathcal{H}_0$ has norm no more than 1. If $h \in \mathcal{H}_0$, then one may write $h = h_+ - h_-$, where $h_+, h_- \in B_+$ and $\|h_+\|$, $\|h_-\| \leq 1$. Therefore, choosing σ_0 even smaller, without loss of generality, we may assume that $\mathcal{H}_0 \subset B_+^{\mathbf{1}} \setminus \{0\}$. Since the elements of the form $\sum_{i=1}^{m} \alpha_i a_i$, where $\alpha_i \geq 0$, and $a_i = a^{(i)} \otimes b^{(i)}$ with $a_i \in A_+^{\mathbf{1}}$ and $b_i \in C(\mathbb{T})_+^{\mathbf{1}}$, are dense in $B_+^{\mathbf{1}}$, choosing an even smaller σ_0, without loss of generality, we may further assume that $\mathcal{H}_0 = \{h_1 \otimes h_2 : h_1 \in \mathcal{H}_1' \text{ and } h_2 \in \mathcal{H}_2'\}$, where $\mathcal{H}_1' \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_2' \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ are finite subsets. Similarly, choosing even smaller σ_0 and δ_1, we may assume that $\mathcal{G}_1 = \{g \otimes f : g \in \mathcal{G}_1' \text{ and } f \in \{1, z, z^*\}\}$ for some finite subset $\mathcal{G}' \subset A$.

By the definition of Δ_0 (see 6.3), for each $h \in \mathcal{H}'$, there exist $a_h \in A_+^{\mathbf{1}} \setminus \{0\}$ and $b_h \in C(X)_+^{\mathbf{1}} \setminus \{0\}$ such that $\hat{h} \geq \widehat{a_h \otimes b_h}$ and

$$\Delta_0(\hat{h}) \leq (16/15)\Delta(\widehat{a_h \otimes b_h}). \tag{e 6.19}$$

Choose finite subsets $\mathcal{H}_A \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_T \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ such that, for each $h \in \mathcal{H}'$, there are $a_h \in \mathcal{H}_A$ and $b_h \in \mathcal{H}_T$ such that (e 6.19) holds for the triple h, a_h and b_h. Put $\mathcal{H}'' = \{a \otimes b : a \in \mathcal{H}_A \text{ and } b \in \mathcal{H}_T\}$. Replacing both $\mathcal{H}_1'$ and $\mathcal{H}_A$ by $\mathcal{H}_1' \cup \mathcal{H}_A$, and both $\mathcal{H}_2'$ and $\mathcal{H}_T$ by $\mathcal{H}_2' \cup \mathcal{H}_A$, without loss of generality, we may assume that $\mathcal{H}_0 = \mathcal{H}''$. Let

$$\sigma = (1/4)\min\{\min\{\Delta_1(\hat{h}) : h \in \mathcal{H}'\}, \sigma_0\}. \tag{e 6.20}$$

Without loss of generality, we may assume that

$$\mathcal{P}' = \mathcal{P}_0 \sqcup \mathcal{P}_1, \tag{e 6.21}$$

where $\mathcal{P}_0 \subset \underline{K}(A)$ and $\mathcal{P}_1 \subset \boldsymbol{\beta}(\underline{K}(A))$ are finite subsets. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset such that $\boldsymbol{\beta}(\mathcal{P}) = \mathcal{P}_1$.

Let $\delta_2 > 0$ (in place of δ) with $\delta_2 < \varepsilon/16$, the finite subset $\mathcal{G}_2 \subset A \otimes C(\mathbb{T})$ (in place of $\mathcal{G}$), and the finite subset $\mathcal{H}_3 \subset (A \otimes C(\mathbb{T}))_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$) be as provided by Lemma 6.2 for σ, $(3/4)\Delta_0$ (in place of Δ), $\mathcal{H}''$ (in place of $\mathcal{H}$), $\min\{\varepsilon/16, \delta_1/2\}$ (in place of ε), $\mathcal{G}_1$ (in place of $\mathcal{G}_0$), and $\mathcal{P}_0$ and $\mathcal{P}$ (in place of $\mathcal{P}_0$ and $\mathcal{P}_1$). Choosing a smaller δ_2, we may also assume that

$$\mathcal{G}_2 = \{g \otimes f : g \in \mathcal{G}_2' \text{ and } f \in \{1, z, z^*\}\}$$

for a finite set $\mathcal{G}_2' \subset A$. Let

$$\mathcal{H}_3' = \{h_1 \otimes h_2 : h_1 \in \mathcal{H}_4 \text{ and } h_2 \in \mathcal{H}_5\}$$

for finite subsets $\mathcal{H}_4 \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_5 \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ be such that, if $h \in \mathcal{H}_3$, then there are $a_h \in \mathcal{H}_4$ and $b_h \in \mathcal{H}_5$ such that $\widehat{h} \geq \widehat{a_h \otimes b_h}$ and

$$\Delta_0(\widehat{h}) \leq (16/15)\Delta(\widehat{a_h \otimes b_h}). \tag{e 6.22}$$

Let $\mathcal{G} = \mathcal{F} \cup \mathcal{G}_1' \cup \mathcal{G}_2'$, $\delta = \min\{\delta_1/2, \delta_2/2, \varepsilon/16\}$, $\mathcal{H}_1 = \mathcal{H}_A \cup \mathcal{H}_4$, and $\mathcal{H}_2 = \mathcal{H}_T \cup \mathcal{H}_5$.

Now suppose that one has a unital completely positive linear map $L : A \otimes C(\mathbb{T}) \to M_k$ and a unitary $u \in M_k$ satisfying the assumptions ((e 6.14) to (e 6.16)) with the above $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{G}$, $\mathcal{P}$, δ and σ. In particular, by (e 6.22) and by the assumption (e 6.16), if $h \in \mathcal{H}_3$, there are $a_h \in \mathcal{H}_4 \subset \mathcal{H}_1$ and $b_h \in \mathcal{H}_5 \subset \mathcal{H}_2$ such that

$$\mathrm{tr}(L(\hat{h})) \geq \mathrm{tr}(L(\widehat{a_h \otimes b_h})) \geq \Delta(\widehat{a_h \otimes b_h}) \geq (15/16)\Delta_0(\widehat{h}) \geq (3/4)\Delta_0(\widehat{h}).$$

It follows from Lemma 6.2 (on using also (e 6.15)) that there is a unital $\mathcal{G}_1$-$\min\{\varepsilon/16, \delta_1/2\}$-multiplicative completely positive linear map $\psi : A \otimes C(\mathbb{T}) \to M_k$ such that $w = \psi(1 \otimes z)$ is a unitary,

$$w\psi(g \otimes 1) = \psi(g \otimes 1)w \text{ for all } g \in A, \tag{e 6.23}$$

$$[\psi]|_{\mathcal{P}'} = [L]|_{\mathcal{P}'}, \text{ and} \tag{e 6.24}$$

$$|\mathrm{tr} \circ L(g) - \mathrm{tr} \circ \psi(g)| < \sigma \text{ for all } g \in \mathcal{H}''. \tag{e 6.25}$$

It follows that, for $h \in \mathcal{H}''$, there are $a_h \in \mathcal{H}_A$ and $b_h \in \mathcal{H}_T$ such that $h = a_h \otimes b_h$ and (by (e 6.16)),

$$\mathrm{tr} \circ \psi(h) \geq \mathrm{tr} \circ L(h) - \sigma = \mathrm{tr}(L(a_h \otimes b_h)) - \sigma \tag{e 6.26}$$

$$\geq \Delta(\widehat{a_h \otimes b_h}) - \sigma \geq (3/4)\Delta(\widehat{a_h \otimes b_h}) = (3/4)\Delta(\hat{h}). \tag{e 6.27}$$

If $h \in \mathcal{H}'$, there is $h' \in \mathcal{H}''$ such that $h \geq h'$ and

$$(15/16)\Delta_0(\hat{h}) \leq \Delta(\widehat{h'}) \tag{e 6.28}$$

(see (e 6.22)). It follows that

$$\mathrm{tr} \circ \psi(h) \geq \mathrm{tr} \circ \psi(h') \geq (3/4)\Delta(\hat{h'}) \tag{e 6.29}$$

$$\geq (3/4)(15/16)\Delta_0(\hat{h}) \geq \Delta_1(\hat{h}) \text{ for all } h \in \mathcal{H}'. \tag{e 6.30}$$

Note that we have assumed that $\mathcal{H}_0 = \mathcal{H}''$. Combining (e 6.24), (e 6.30), and (e 6.25), and applying Theorem 5.8, one obtains a unitary $U \in M_k$ such that

$$\|\mathrm{Ad}\, U \circ \psi(f) - L(f)\| < \varepsilon/16 \text{ for all } f \in \mathcal{F}_0. \tag{e 6.31}$$

Let $w_1 = \mathrm{Ad}\, U \circ \varphi(1 \otimes z)$. Then

$$\|u - w_1\| \leq \|u - L(1 \otimes z)\| + \|L(1 \otimes z) - \mathrm{Ad}\, U \circ \psi(1 \otimes z)\| \tag{e 6.32}$$

$$< \delta + \varepsilon/16 < \varepsilon/8. \tag{e 6.33}$$

Thus there is $h \in (M_k)_{s.a.}$ with $\|h\| < \varepsilon\pi/8$ such that $uw_1^* = \exp(ih)$. It follows that there is a continuous path of unitaries $\{u_t \in [0, 1/2]\} \subset M_k$ such that

$$\|u_t - u\| < \varepsilon/8, \;\; \|u_t - w_1\| < \varepsilon/8, \;\; u_0 = u, \;\; u_{1/2} = w_1, \tag{e 6.34}$$

$$\text{and } \operatorname{length}(\{u_t : t \in [0, 1/2]\}) < \varepsilon\pi/8. \tag{e 6.35}$$

It follows from Lemma 6.1 that there exists a continuous path of unitaries $\{u_t : t \in [1/2, 1]\} \subset M_k$ such that

$$u_{1/2} = w_1, \;\; u_1 = 1 \;\text{ and }\; u_t(\operatorname{Ad} U \circ \psi(f \otimes 1)) = (\operatorname{Ad} U \circ \psi(f \otimes 1))u_t$$

for all $t \in [1/2, 1]$ and $f \in A \otimes 1$. Moreover, we may assume that

$$\operatorname{length}(\{u_t : t \in [1/2, 1]\}) \leq \pi. \tag{e 6.36}$$

It follows that

$$\operatorname{length}(\{u_t : t \in [0, 1]\}) \leq \pi + \varepsilon\pi/6. \tag{e 6.37}$$

Furthermore,

$$\|u_t L(f \otimes 1) - L(f \otimes 1)u_t\| < \varepsilon \;\text{ for all }\; f \in \mathcal{F} \tag{e 6.38}$$

and $t \in [0, 1]$. □

LEMMA 6.5. *Let $A \in \bar{\mathcal{D}}_s$ be a unital C^*-algebra, let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ and let $\mathcal{H}_2 \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ be finite subsets. For any order preserving map $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$, there exists a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H}_1' \subset A_+^{\mathbf{1}} \setminus \{0\}$, and $\delta > 0$ satisfying the following condition: for any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\varphi : A \to M_k$ (for some integer $k \geq 1$) and any unitary $u \in M_k$ such that*

$$\|u\varphi(g) - \varphi(g)u\| < \delta \;\textit{ for all }\; g \in \mathcal{G} \;\textit{ and} \tag{e 6.39}$$

$$\operatorname{tr} \circ \varphi(h) \geq \Delta(\hat{h}) \;\textit{ for all }\; h \in \mathcal{H}_1', \tag{e 6.40}$$

there exists a continuous path of unitaries $\{u_t : t \in [0, 1]\} \subset M_k$ such that

$$u_0 = u, \;\; u_1 = w, \;\; \|u_t\varphi(f) - \varphi(f)u_t\| < \varepsilon \;\textit{ for all }\; f \in \mathcal{F} \;\textit{ and }\; t \in [0, 1],$$

$$\textit{and } \operatorname{tr} \circ L(h_1 \otimes h_2) \geq \Delta(\widehat{h_1})\tau_m(h_2)/4$$

for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, where $L : A \otimes C(\mathbb{T}) \to M_k$ is a unital $\mathcal{F}_1$-ε-multiplicative completely positive linear map such that

$$\|L(f \otimes 1_{C(\mathbb{T})}) - \varphi(f)\| < \varepsilon \;\textit{ for all }\; f \in \mathcal{F}, \;\textit{ and }\; \|L(1 \otimes z) - w\| < \varepsilon,$$

and τ_m is the tracial state on $C(\mathbb{T})$ induced by the Lebesgue measure on the circle, where $\mathcal{F}_1 = \{f \otimes g : f \in \mathcal{F} \cup \{1\}, g \in \{1_{C(\mathbb{T})}, z, z^*\}\}$. Moreover, $\{u_t\}$ can be chosen such that

$$\operatorname{length}(\{u_t\}) \leq 2\pi + \varepsilon. \tag{e 6.41}$$

PROOF. Without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of A and if $f \in \mathcal{F}$, then $f^* \in \mathcal{F}$.

There exists an integer $n \geq 1$ such that

$$(1/n)\sum_{j=1}^{n} f(e^{\theta+j2\pi i/n}) \geq (63/64)\tau_m(f) \tag{e 6.42}$$

for all $f \in \mathcal{H}_2$ and for any $\theta \in [-\pi, \pi]$. We may also assume that $16\pi/n < \varepsilon$.

Let

$$\sigma_1 = (1/2^{10})\inf\{\Delta(\hat{h}) : h \in \mathcal{H}_1\} \cdot \inf\{\tau_m(g) : g \in \mathcal{H}_2\}.$$

Let $\mathcal{H}_{1a} \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_2$) be a finite subset as provided by Corollary 4.13 for $\min\{\varepsilon/32, \sigma_1/16\}$ (in place of ε), $\mathcal{F} \cup \mathcal{H}_1$ (in place of $\mathcal{F}$), $\mathcal{H}_1$ (in place of $\mathcal{H}$), $(1 - 1/2^{12})\Delta$ (in place of Δ), $\sigma_1/16$ (in place of σ), and the integer n (and for A).

Put $\mathcal{H}_1' = \mathcal{H}_{1a} \cup \mathcal{H}_1$. Put $\sigma = (1/2^{10})\inf\{\Delta(\hat{h}) : h \in \mathcal{H}_1'\} \cdot \inf\{\tau_m(g) : g \in \mathcal{H}_2\}$. Note that $A \otimes C(\mathbb{T})$ is a subhomogeneous C^*-algebra. Let $\mathcal{F}' = \{f \otimes 1_{C(\mathbb{T})}, f \otimes z : f \in \mathcal{F} \cup \mathcal{H}_1'\}$. Let $\delta_1 > 0$ (in place of δ) and $\mathcal{G}_1 \subset A \otimes C(\mathbb{T})$ (in place of $\mathcal{G}$) be as provided by Lemma 5.4 for $\varepsilon_0 := \min\{\varepsilon/64, \sigma/16\}$ (in place of ε), $\mathcal{F}'$ (in place of $\mathcal{F}$), and $\sigma/16$ (in place of σ_0). Choosing a smaller δ_1 if necessary, without loss of generality, one may assume that, for a finite set $\mathcal{G}_2 \subset A$,

$$\mathcal{G}_1 = \{g \otimes 1, 1 \otimes z : g \in \mathcal{G}_2\}.$$

We may also assume that if $g \in \mathcal{G}_2$, then $g^* \in \mathcal{G}_2$. Put

$$\mathcal{H}' = \{h_1 \otimes h_2, h_1 \otimes 1, 1 \otimes h_2 : h_1 \in \mathcal{H}_1' \text{ and } h_2 \in \mathcal{H}_2\}.$$

Let $\mathcal{G}_3 = \mathcal{G}_2 \cup \mathcal{H}_1'$. To simplify notation, without loss of generality, let us assume that $\mathcal{G}_3$ is in the unit ball of A and $\mathcal{F}'$ is in the unit ball of $A \otimes C(\mathbb{T})$, respectively. Let $\delta_2 = \min\{\varepsilon/64, \delta_1/2, \sigma/16\}$.

Let $\mathcal{G}_4 \subset A$ be a finite subset (in place of $\mathcal{G}$) and let δ_3 (in place of δ) be a positive number as provided by 2.13 for $\mathcal{G}_3$ (in place of $\mathcal{F}_0$), $\mathcal{F}'$ (in place of $\mathcal{F}$), and δ_2 (in place of ε).

Let $\mathcal{G} = \mathcal{G}_4 \cup \mathcal{G}_3 \cup \mathcal{F}$ and $\delta = \min\{\delta_1/4, \delta_2/2, \delta_3/2\}$. Now let $\varphi : A \to M_k$ be a unital $\mathcal{G}$-δ-multiplicative completely positive linear map, and let $u \in M_k$ be a unitary such that (e 6.39) and (e 6.40) hold for the above δ, $\mathcal{G}$, and $\mathcal{H}_1'$.

By Lemma 2.13, there exists a unital $\mathcal{G}_3$-δ_2-multiplicative completely positive linear map $L_1 : A \otimes C(\mathbb{T}) \to M_k$ such that

$$\|L_1(g \otimes 1_{C(\mathbb{T})}) - \varphi(g)\| < \delta_2 \text{ for all } g \in \mathcal{G}_3 \text{ and } \|L_1(1 \otimes z) - u\| < \delta_2.$$

Then

$$\begin{aligned} \mathrm{tr} \circ L_1(h \otimes 1) &\geq \mathrm{tr} \circ \varphi(h) - \delta_2 && \text{(e 6.43)} \\ &\geq \Delta(\hat{h}) - \sigma/16 \geq (1 - 1/2^{14})\Delta(\hat{h}) && \text{(e 6.44)} \end{aligned}$$

for all $h \in \mathcal{H}_1'$. It follows from Lemma 5.4 that there exist a projection $p \in M_k$ and a unital homomorphism $\psi : A \otimes C(\mathbb{T}) \to pM_kp$ such that

$$\|pL_1(f) - L_1(f)p\| < \varepsilon_0 \text{ for all } f \in \mathcal{F}', \tag{e 6.45}$$

$$\|L_1(f) - ((1-p)L_1(f)(1-p) + \psi(f))\| < \varepsilon_0 \text{ for all } f \in \mathcal{F}', \tag{e 6.46}$$

$$\text{and } \operatorname{tr}(1-p) < \sigma/16. \tag{e 6.47}$$

Note that $pM_kp \cong M_m$ for some $m \le k$. It follows from (e 6.44), (e 6.46), and (e 6.47) that

$$\operatorname{tr}' \circ \psi(h) \ge \operatorname{tr} \circ \psi(h) \ge (1 - 1/2^{14})\Delta(\hat{h}) - \sigma/16 - \sigma/16 \ge (1 - 1/2^{12})\Delta(\hat{h})$$

for all $h \in \mathcal{H}_1'$, where tr' is the normalized trace on $pM_kp \cong M_m$. It follows from 6.1 that there is a continuous path of unitaries $\{u_t : t \in [1/4, 1/2]\} \subset pM_kp$ such that $u_{1/4} = \psi(1 \otimes z)$, $u_{1/2} = p$, and

$$u_t\psi(f \otimes 1_{C(\mathbb{T})}) = \psi(f \otimes 1_{C(\mathbb{T})})u_t \text{ for all } f \in A \text{ and } \text{ for all } t \in [1/4, 1/2],$$

and $\text{length}(\{u_t : t \in [1/4, 1/2]\}) \le \pi$.

By Corollary 4.13, there are mutually orthogonal projections $e_0, e_1, e_2, ..., e_n \in pM_kp$ such that $e_1, e_2, ..., e_n$ are mutually equivalent and $\sum_{i=0}^n e_i = p$, and there are unital homomorphisms $\psi_0 : A \to e_0M_ke_0$ and $\psi_1 : A \to e_1M_ke_1$ such that

$$\operatorname{tr}'(e_0) < \sigma_1/16 \quad \text{and} \tag{e 6.48}$$

$$\begin{aligned}\|\psi(f \otimes 1_{C(\mathbb{T})}) - (\psi_0(f) \quad \oplus \quad & \operatorname{diag}(\overbrace{\psi_1(f \otimes 1_{C(\mathbb{T})}), ..., \psi_1(f \otimes 1_{C(\mathbb{T})})}^{n}))\| \\ & < \min\{\varepsilon/32, \sigma_1/16\}\end{aligned}$$

for all $f \in \mathcal{F} \cup \mathcal{H}_1$, where we identify $(\sum_{i=1}^n e_i)M_k(\sum_{i=1}^n e_i)$ with $M_n(e_1M_ke_1)$ (using the convention introduced in 2.26). Moreover,

$$\operatorname{tr}'(\psi_1(h \otimes 1)) \ge (1 - 1/2^{12})\Delta(\hat{h})/2n \text{ for all } h \in \mathcal{H}_1. \tag{e 6.49}$$

Note this implies that, by (e 6.47),

$$\operatorname{tr}(\psi_1(h \otimes 1)) \ge (1 - \sigma/16)(1 - 1/2^{12})\Delta(\hat{h})/2n \text{ for all } h \in \mathcal{H}_1. \tag{e 6.50}$$

Let $w_{0,j} = \exp(i(2\pi j/n))e_j$, $j = 1, 2, ..., n$. Define

$$w_{00} := \sum_{j=1}^n w_{0,j} = \operatorname{diag}(w_{0,1}, w_{0,2}, ..., w_{0,n}),$$

and define

$$w_0' = e_0 \oplus w_{00} = e_0 \oplus \mathrm{diag}(w_{0,1}, w_{0,2}, ..., w_{0,n}). \tag{e 6.51}$$

Then w_0' commutes with $\psi_0(f \otimes 1_{C(\mathbb{T})}) \oplus \mathrm{diag}(\psi_1(f \otimes 1_{C(\mathbb{T})}), \psi_1(f \otimes 1_{C(\mathbb{T})}), ..., \psi_1(f \otimes 1_{C(\mathbb{T})}))$ for all $f \in A$. As in 6.1, there exists a continuous path $\{u_t : t \in [1/2, 1]\} \subset pM_kp$ such that $u_{1/2} = p$, $u_1 = w_0'$ and u_t commutes with $\psi_0(f) \oplus \mathrm{diag}(\psi_1(f), \psi_1(f), ..., \psi_1(f))$ for all $f \in A$, and moreover, $\mathrm{length}(\{u_t : t \in [1/2, 1]\} \le \pi$.

There is a unitary $w_0'' \in (1-p)M_k(1-p)$ such that (see (**??**) and (e 6.45))

$$\|w_0'' - (1-p)L_1(1 \otimes z)(1-p)\| < \varepsilon/16. \tag{e 6.52}$$

For $f \in \mathcal{F}$, by (e 6.52), (e 6.45), and the fact that L_1 is $\mathcal{G}_3$-δ_2 multiplicative,

$$\begin{aligned}
& w_0''(1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p) \\
&\approx_{\varepsilon/16} (1-p)L_1(1 \otimes z)(1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p) && \text{(e 6.53)}\\
&\approx_{\varepsilon_0} (1-p)L_1(1 \otimes z)L_1(f \otimes 1_{C(\mathbb{T})})(1-p) && \text{(e 6.54)}\\
&\approx_{\delta_2} (1-p)L_1((f \otimes 1_{C(\mathbb{T})})(1 \otimes z))(1-p) && \text{(e 6.55)}\\
&\approx_{\delta_2} (1-p)L_1(f \otimes 1_{C(\mathbb{T})})L_1(1 \otimes z))(1-p) && \text{(e 6.56)}\\
&\approx_{\varepsilon_0} (1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p)(1-p)L_1(1 \otimes z))(1-p) && \text{(e 6.57)}\\
&\approx_{\varepsilon/16} (1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p)w_0''. && \text{(e 6.58)}
\end{aligned}$$

In other words, for all $f \in \mathcal{F}$,

$$\begin{aligned}
&\|w_0''(1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p) - (1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p)w_0''\| \\
&< \varepsilon/16 + 2\varepsilon_0 + 2\delta_2 + \varepsilon/16 < \varepsilon/16 + \varepsilon/32 + \varepsilon/32 + \varepsilon/16 = 3\varepsilon/16. && \text{(e 6.59)}
\end{aligned}$$

Note also, by (**??**) and (e 6.46),

$$\begin{aligned}
&\|\varphi(f) - ((1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p) + \psi(f \otimes 1_{C(\mathbb{T})}))\| \\
&< \delta_2 + \varepsilon_0 < \varepsilon/32 \ \text{ for all } \ f \in \mathcal{F}. && \text{(e 6.60)}
\end{aligned}$$

Put $u_0' = w_0'' \oplus \psi(1 \otimes z)$. Then u_0' is a unitary and

$$\begin{aligned}
\|u - u_0'\| &\le \|u - L_1(1 \otimes z)\| + \|L_1(1 \otimes z) - u_0'\| && \text{(e 6.61)}\\
&\le \delta_2 + \varepsilon/16 < \varepsilon/8. && \text{(e 6.62)}
\end{aligned}$$

As above, we obtain a continuous path of unitaries $\{w_t \in [0, 1/4]\} \subset M_k$ such that $w_0 = u$ and $w_{1/4} = u_0'$ and $\mathrm{cel}(\{w_t : t \in [0, 1/4]\}) < 2\arcsin(\varepsilon/16)$. Define $w_t = w_0'' \oplus u_t$ for $t \in [1/4, 1]$. Then $w_{1/4} = w_0'' \oplus \psi(1 \otimes z) = u_0'$, $w_1 = w_0'' \oplus w_0'$.

From the construction, $\{w_t : t \in [0,1]\}$ is a continuous path of unitaries in M_k such that

$$w_0 = u, \ \ w_1 = w_0'' \oplus w_0' \ \text{ and } \ \mathrm{cel}(\{w_t : t \in [0,1]\}) \le 2\pi + \varepsilon. \tag{e 6.63}$$

By (e 6.39) and the choice of δ, for $t \in [0, 1/4]$ and $f \in \mathcal{F}$,

$$\|w_t\varphi(f) - \varphi(f)w_t\| < \varepsilon \ \text{ for all } \ f \in \mathcal{F}. \tag{e 6.64}$$

For $t \in [1/4, 1/2]$, by (e 6.60) and by (e 6.59), for all $f \in \mathcal{F}$,

$$\begin{aligned} w_t\varphi(f) &= (w_0'' \oplus u_t)\varphi(f) \\ &\approx_{\varepsilon/32} (w_0'' \oplus u_t)((1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p) \oplus \psi(f \otimes 1_{C(\mathbb{T})})) \\ &\approx_{3\varepsilon/16} ((1-p)L_1(f \otimes 1_{C(\mathbb{T})})(1-p) \oplus \psi(f \otimes 1_{C(\mathbb{T})}))(w_0'' \oplus u_t) \\ &\approx_{\varepsilon/32} \varphi(f)w_t. \end{aligned}$$

Similarly, since u_t commutes with $\psi_0(f) \oplus \mathrm{diag}(\psi_1(f), \psi_1(f), ..., \psi_1(f))$ for all $f \in A$, one also has, for $t \in [1/2, 1]$,

$$\|w_t\varphi(f) - \varphi(f)w_t\| < \varepsilon \ \text{ for all } \ f \in \mathcal{F}. \tag{e 6.65}$$

Therefore, (e 6.65) holds for all $t \in [0, 1]$. Define $L : A \otimes C(\mathbb{T}) \to M_k$ by

$$\begin{aligned} L(a \otimes f) &= (1-p)L_1(a \otimes f)(1-p) \\ &\quad \oplus(\mathrm{diag}(\psi_0(a\otimes 1_{C(\mathbb{T})}), \overbrace{\psi_1(a\otimes 1_{C(\mathbb{T})}), ..., \psi_1(a\otimes 1_{C(\mathbb{T})})}^{n})f(w_0')) \end{aligned}$$

for all $a \in A$ and $f \in C(\mathbb{T})$. Then

$$\|L(f \otimes 1) - \varphi(f)\| < \varepsilon \ \text{ for all } \ f \in \mathcal{F}. \tag{e 6.66}$$

Moreover (see 2.26),

$$\begin{aligned} L(1 \otimes z) &= (1-p)L_1(1 \otimes z)(1-p) \oplus (\mathrm{diag}(\psi_0(1), \overbrace{\psi_1(1), ..., \psi_1(1)}^{n})(w_0')) \\ &= (1-p)L_1(1 \otimes z)(1-p) \oplus pw_0' \approx_{\varepsilon/16} w_0'' \oplus w_0' = w_1. \end{aligned} \tag{e 6.67}$$

One then verifies that L is $\mathcal{F}_1$-ε-multiplicative, where $\mathcal{F}_1 = \{f \otimes g : f \in \mathcal{F} \cup \{1\}, g \in \{1_{C(\mathbb{T})}, z, z^*\}\}$.

In the next few lines, for $h_1 \in A_+$ and $h_2 \in C(\mathbb{T})_+$, we write $h_1' := h_1 \otimes 1_{C(\mathbb{T})}$ and $h_2' := 1 \otimes h_2$. Also, we view h_2 as a positive function on $\mathbb{T}$. Recall that $w_{0,j} = \exp(i(2\pi j/n))e_j$ is a scalar multiple of e_j. Therefore, the element $h_2(w_{0,j}) = h_2(e^{i2\pi j/n})e_j$ is also a scalar multiple of e_j. Moreover, $h_2(w_{00})$ may be written as $\mathrm{diag}(h_2(e^{i2\pi/n}), h_2(e^{i2\pi 2/n}), ..., h_2(e^{i2\pi n/n}))$. Finally, we estimate, for for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, keeping in mind of 2.26, by (e 6.51),

$$\begin{aligned}
\mathrm{tr} \circ L(h_1 \otimes h_2) &\geq \mathrm{tr}(\psi_0(h_1')) + \mathrm{tr}(\mathrm{diag}(\overbrace{\psi_1(h_1'), ..., \psi_1(h_1')}^{n})h_2(w_{00}))\\
&\geq \mathrm{tr}(\mathrm{diag}(\overbrace{\psi_1(h_1'), ..., \psi_1(h_1')}^{n})h_2(w_{00}))\\
&= \sum_{j=1}^{n} \mathrm{tr}(\psi_1(h_1')h_2(e^{i2\pi j/n})) = \sum_{j=1}^{n} \mathrm{tr}(\psi_1(h_1'))(h_2(e^{i2\pi j/n}))\\
&\geq ((1-\sigma_2/16)(1-1/2^{12})\Delta(\hat{h})/3n)(\sum_{j=1}^{n} h_2(e^{i2\pi j/n}))\\
&\qquad \text{(by (e 6.50))}\\
&\geq (1-1/2^{10})\Delta(\widehat{h_1})(63/64)\tau_m(h_2)/3 \geq \Delta(\widehat{h_1}) \cdot \tau_m(h_2)/4\\
&\qquad \text{(by (e 6.42))}
\end{aligned}$$

□

DEFINITION 6.6. Let A be a unital C^*-algebra with $T(A) \neq \varnothing$ and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\tau_m : C(\mathbb{T}) \to \mathbb{C}$ denote the tracial state given by normalized Lebesgue measure. Define $\Delta_1 : (A \otimes C(\mathbb{T}))_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ by

$$\Delta_1(\hat{h}) = \sup\{\frac{\Delta(h_1)\tau_m(h_2)}{4} : \hat{h} \geq \widehat{h_1 \otimes h_2},\ h_1 \in A_+ \setminus \{0\},\ h_2 \in C(\mathbb{T})_+ \setminus \{0\}\}.$$

LEMMA 6.7. *Let $A \in \bar{\mathcal{D}}_s$ be a unital C^*-algebra. Let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$, and a finite subset $\mathcal{P} \subset \underline{K}(A)$ satisfying the following condition: For any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\varphi : A \to M_k$ (for some integer $k \geq 1$), and any unitary $v \in M_k$, such that*

(e 6.68) $$tr \circ \varphi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H},$$

(e 6.69) $$\|\varphi(g)v - v\varphi(g)\| < \delta \text{ for all } g \in \mathcal{G}, \text{ and}$$

(e 6.70) $$\mathrm{Bott}(\varphi, v)|_{\mathcal{P}} = 0,$$

there exists a continuous path of unitaries $\{u_t : t \in [0,1]\} \subset M_k$ such that

(e 6.71) $$u_0 = v,\ u_1 = 1,\ \text{and } \|\varphi(f)u_t - u_t\varphi(f)\| < \varepsilon$$

for all $t \in [0,1]$ *and* $f \in \mathcal{F}$, *and, moreover,*

$$\mathrm{length}(\{u_t\}) \le 2\pi + \varepsilon. \tag{e 6.72}$$

PROOF. Let Δ_1 be as in Definition 6.6 (associated with the given Δ). Let $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_2 \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$), and $\mathcal{P} \subset \underline{K}(A)$, be finite subsets and $\delta_1 > 0$ (in place of δ) the constant, as provided by 6.4 for $\varepsilon/4$ (in place of ε), $\mathcal{F}$ and Δ_1. We may assume that $1 \in \mathcal{F}$. Without loss of generality, we may assume that $(\delta_1, \mathcal{G}_1, \mathcal{P})$ is a KL-triple (see 2.12) Moreover, we may assume that $\delta_1 < \delta_{\mathcal{P}}$ and $\mathcal{G}_1 \supset \mathcal{F}_{\mathcal{P}}$ (see 2.14).

Let $\mathcal{G}_2 \subset A$ (in place of $\mathcal{G}$) and $\mathcal{H}_1' \subset A_+^{\mathbf{1}} \setminus \{0\}$ denote the finite subsets, and $\delta_2 > 0$ (in place of δ) the constant, provided by Lemma 6.5 for $\min\{\varepsilon/16, \delta_1/2\}$ (in place of ε), $\mathcal{G}_1 \cup \mathcal{F}$ (in place of $\mathcal{F}$), and $\mathcal{H}_1$ and $\mathcal{H}_2$ as well as Δ. Let $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_1 \cup \mathcal{F}$, let $\mathcal{H} = \mathcal{H}_1'$, and let $\delta = \min\{\delta_2, \varepsilon/16\}$.

Now suppose that $\varphi : A \to M_k$ is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map and $u \in M_k$ is a unitary which satisfy the assumptions (in particular, (e 6.68), (e 6.69), and (e 6.70) hold) for the above $\mathcal{H}$, δ, $\mathcal{G}$, and $\mathcal{P}$.

Applying Lemma 6.5, one obtains a continuous path of unitaries $\{u_t : t \in [0, 1/2]\} \subset M_k$ such that

$$u_0 = u, \;\; u_1 = w, \;\; \|u_t\varphi(g) - \varphi(g)u_t\| < \min\{\delta_1, \varepsilon/4\} \tag{e 6.73}$$

for all $g \in \mathcal{G}_1 \cup \mathcal{F}$ and $t \in [0, 1/2]$. Moreover, there is a unital $\mathcal{G}'$-$\min\{\delta_1/2, \varepsilon/16\}$-multiplicative completely positive linear map $L : A \otimes C(\mathbb{T}) \to M_k$, where $\mathcal{G}' = \{g \otimes f : g \in \mathcal{G}_1 \cup \mathcal{F} \text{ and } g \in \{1_{C(\mathbb{T})}, z, z^*\}\}$, such that

$$\|L(g \otimes 1) - \varphi(g)\| < \min\{\delta_1/2, \varepsilon/16\} \text{ for all } g \in \mathcal{G}_1 \cup \mathcal{F}, \tag{e 6.74}$$

$$\|L(1 \otimes z) - w\| < \min\{\delta_1/2, \varepsilon/16\}, \tag{e 6.75}$$

$$\text{and } \mathrm{tr} \circ L(h_1 \otimes h_2) \ge \Delta(h_1)\tau_m(h_2)/4 \tag{e 6.76}$$

for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$. Furthermore,

$$\mathrm{length}(\{u_t : t \in [0, 1/2]\}) \le \pi + \varepsilon/16. \tag{e 6.77}$$

Note that (see 2.14)

$$[L]|_{\beta(\mathcal{P})} = \mathrm{Bott}(\varphi, w)|_{\mathcal{P}} = \mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0. \tag{e 6.78}$$

By (e 6.74), (e 6.75), (e 6.78), and (e 6.76), applying Lemma 6.4, one obtains a continuous path of unitaries $\{u_t \in [1/2, 1]\} \subset M_k$ such that

$$u_{1/2} = w, \;\; u_1 = 1, \;\; \|u_t\varphi(f) - \varphi(f)u_t\| < \varepsilon/4 \text{ for all } f \in \mathcal{F}, \tag{e 6.79}$$

$$\text{and } \mathrm{length}(\{u_t : t \in [1/2, 1]\}) \le \pi + \varepsilon/4. \tag{e 6.80}$$

Therefore, $\{u_t : t \in [0,1]\} \subset M_k$ is a continuous path of unitaries in M_k with $u_0 = u$ and $u_1 = 1$ such that

$$\|u_t\varphi(f) - \varphi(f)u_t\| < \varepsilon \text{ for all } f \in \mathcal{F} \text{ and } \mathrm{length}(\{u_t : t \in [0,1]\}) \le 2\pi + \varepsilon.$$

□

7. An Existence Theorem for Bott Maps

Lemma 7.1. *Let A be a unital amenable separable residually finite dimensional C^*-algebra which satisfies the UCT, let $G = \mathbb{Z}^r \oplus \mathrm{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$, and let $J_0, J_1 \geq 0$ be integers.*

For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$, and any finite subset $\mathcal{P} \subset \underline{K}(A)$ with $[1_A] \in \mathcal{P}$ and $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N_0, N_1, ..., N_k$ and unital homomorphisms $h_j : A \to M_{N_j}$, $j = 1, 2, ..., k$, satisfying the following condition:

For any $\kappa \in Hom_\Lambda(\underline{K}(A), \underline{K}(\mathcal{K}))$, with $|\kappa([1_A])| \leq J_1$ (note that $K_0(\mathcal{K}) = \mathbb{Z}$) and

$$J_0 = \max\{|\kappa(g_i)| : g_i = (\overbrace{0, ..., 0}^{i-1}, 1, 0, ..., 0) \in \mathbb{Z}^r : 1 \leq i \leq r\}, \tag{e 7.1}$$

there exists a unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\Phi : A \to M_{N_0 + \kappa([1_A])}$ such that

$$[\Phi]|_{\mathcal{P}} = (\kappa + [h_1] + [h_2] + \cdots + [h_k])|_{\mathcal{P}}. \tag{e 7.2}$$

(Note that, as Φ is unital, $N_0 = \sum_{i=1}^k N_i$. The notation J_0 is for later use.)

Proof. It follows from 6.1.11 of [63] (see also [60] and [21]) that, for each such κ, there is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for some integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_{\mathcal{P}} = (\kappa + [h_\kappa])|_{\mathcal{P}}, \tag{e 7.3}$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_{\mathcal{P}}$ such that (e 7.1) holds and $|\kappa([1_A])| \leq J_1$, say $\kappa_1, \kappa_2, ..., \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, ..., k$. Let $N_i = N_{\kappa_i}$, $i = 1, 2, ...,k$. Note that $N_i = \kappa_i([1_A]) + n(\kappa_i)$, $i = 1, 2, ..., k$. Define

$$N_0 = \sum_{i=1}^k N_i.$$

If $\kappa = \kappa_i$, the map $\Phi : A \to M_{N_0 + \kappa([1_A])}$ defined by

$$\Phi = L_{\kappa_i} + \sum_{j \neq i} h_j$$

satisfies the requirements. □

Lemma 7.2. *Let A be a unital C^*-algebra as in 7.1 and let $G = \mathbb{Z}^r \oplus \mathrm{Tor}(G)$ with $[1_A] \in G$ be also as in 7.1. There exist $\Lambda_i \geq 0$, $i = 1, 2, ..., r$, such that the following statement holds: For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$, and any finite subset $\mathcal{P} \subset \underline{K}(A)$ with $[1_A] \in \mathcal{P}$ and $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N(\delta, \mathcal{G}, \mathcal{P}, i) \geq 1$, $i = 1, 2, ..., r$, satisfying the following condition:*

Let $\kappa \in Hom_\Lambda(\underline{K}(A), \underline{K}(\mathcal{K}))$ *and* $S_i = \kappa(g_i)$, *where* $g_i = (\overbrace{0, ..., 0}^{i-1}, 1, 0, ..., 0) \in \mathbb{Z}^r$. *There exist a unital* $\mathcal{G}$-δ-*multiplicative completely positive linear map* $L : A \to M_{N_1}$ *and a homomorphism* $h : A \to M_{N_1}$ *such that*

$$[L]|_{\mathcal{P}} = (\kappa + [h])|_{\mathcal{P}}, \tag{e 7.4}$$

where $N_1 = \sum_{i=1}^{r}(N(\delta, \mathcal{G}, \mathcal{P}, i) + \mathrm{sign}(S_i) \cdot \Lambda_i) \cdot |S_i|$.

PROOF. Let $\psi_i^+ : G \to \mathbb{Z}$ be the homomorphism defined by $\psi_i^+(g_i) = 1$, $\psi_i^+(g_j) = 0$ if $j \neq i$, and $\psi_i^+|_{\mathrm{Tor}(G)} = 0$, and similarly let $\psi_i^-(g_i) = -1$, $\psi_i^-(g_j) = 0$ if $j \neq i$, and $\psi_i^-|_{\mathrm{Tor}(G)} = 0$, $i = 1, 2, ..., r$. Note that $\psi_i^- = -\psi_i^+$, $i = 1, 2, ..., r$. Let $\Lambda_i = |\psi_i^+([1_A])|$, $i = 1, 2, ..., r$.

Let $\kappa_i^+, \kappa_i^- \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(\mathcal{K}))$ be such that $\kappa_i^+|_G = \psi_i^+$ and $\kappa_i^-|_G = \psi_i^-$, $i = 1, 2, ..., r$. Let $N_0(i) \geq 1$ (in place of N_0) be as provided by 7.1 for δ, $\mathcal{G}$, $J_0 = 1$, and $J_1 = \Lambda_i$. Define $N(\delta, \mathcal{G}, \mathcal{P}, i) = N_0(i)$, $i = 1, 2, ..., r$.

Let $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(\mathcal{K}))$. Then $\kappa|_G = \sum_{i=1}^{r} S_i\psi_i^+$, where $S_i = \kappa(g_i)$, $i = 1, 2, ..., r$. Applying Lemma 7.1, one obtains unital $\mathcal{G}$-δ-multiplicative completely positive linear maps $L_i^{\pm} : A \to M_{N_0(i)+\kappa_i^{\pm}([1_A])}$ and homomorphisms $h_i^{\pm} : A \to M_{N_0(i)}$ such that

$$[L_i^{\pm}]|_{\mathcal{P}} = (\kappa_i^{\pm} + [h_i^{\pm}])|_{\mathcal{P}}, \ \ i = 1, 2, ..., r. \tag{e 7.5}$$

Define $L = \sum_{i=1}^{r} L_i^{\mathrm{sign}(S_i), |S_i|}$, where $L_i^{\mathrm{sign}(S_i), |S_i|} : A \to M_{|S_i|N_0(i)}$ is defined by

$$L_i^{\mathrm{sign}(S_i),|S_i|}(a) = \mathrm{diag}(\overbrace{L_i^{\mathrm{sign}(S_i)}(a), ..., L_i^{\mathrm{sign}(S_i)}(a)}^{|S_i|}) = L_i^{\mathrm{sign}(S_i)}(a) \otimes 1_{|S_i|}$$

for all $a \in A$. One checks that the map $L : A \to M_{N_1}$, where $N_1 = \sum_{i=1}^{r} |S_i|(\Lambda_i' + N(\delta, \mathcal{G}, \mathcal{P}, i))$ with $\Lambda_i' = \psi_i^+([1_A])$ if $S_i > 0$, or $\Lambda_i' = -\psi_i^+([1_A])$ if $S_i < 0$, is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map and

$$[L]|_{\mathcal{P}} = (\kappa + [h])|_{\mathcal{P}}$$

for some homomorphism $h : A \to M_{N_1}$. □

LEMMA 7.3. *Let* $A \in \bar{\mathcal{D}}_s$ *be a unital* C^*-*algebra and let* $\mathcal{P} \subset \underline{K}(A)$ *be a finite subset. Denote by* $G \subset \underline{K}(A)$ *the group generated by* $\mathcal{P}$, *and write* $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus (\mathrm{Tor}(K_1(A) \cap G)$. *Let* $\mathcal{F} \subset A$ *be a finite subset, let* $\varepsilon > 0$, *and let* $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ *be an order preserving map.*

There exist $\delta > 0$, *a finite subset* $\mathcal{G} \subset A$, *a finite subset* $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$, *and an integer* $N \geq 1$ *satisfying the following condition: Let* $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ *and put*

$$K = \max\{|\kappa(\boldsymbol{\beta}(g_i))| : 1 \leq i \leq r\} \ \ \textit{(see 2.14 for the definition of } \boldsymbol{\beta}), \tag{e 7.6}$$

where $g_i = (\overbrace{0,...,0}^{i-1}, 1, 0, ..., 0) \in \mathbb{Z}^r$. *Then for any unital* $\mathcal{G}$-δ-*multiplicative completely positive linear map* $\varphi : A \to M_R$ *such that* $R \geq N(K+1)$ *and*

$$\mathrm{tr} \circ \varphi(h) \geq \Delta(\hat{h}) \ \textit{for all} \ h \in \mathcal{H}, \tag{e 7.7}$$

where tr *is the tracial state of* M_R, *there exists a unitary* $u \in M_R$ *such that*

$$\|[\varphi(f),\, u]\| < \varepsilon \ \textit{for all} \ f \in \mathcal{F} \ \textit{and} \tag{e 7.8}$$

$$\mathrm{Bott}(\varphi,\, u)|_{\mathcal{P}} = \kappa \circ \boldsymbol{\beta}|_{\mathcal{P}}. \tag{e 7.9}$$

Proof. To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ is a subset of the unit ball. Let $\Delta_1 = (1/8)\Delta$ and $\Delta_2 = (1/16)\Delta$.

Let $\varepsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset satisfying the following condition: If $\varphi' : A \to B$ (for any unital C^*-algebra B) is a unital $\mathcal{G}_0$-ε_0-multiplicative completely positive linear map and $u' \in B$ is a unitary such that

$$\|\varphi'(g)u' - u'\varphi'(g)\| < 4\varepsilon_0 \ \text{for all} \ g \in \mathcal{G}_0, \tag{e 7.10}$$

then $\mathrm{Bott}(\varphi',\, u')|_{\mathcal{P}}$ is well defined (see 2.14). Moreover, if $\varphi' : A \to B$ is another unital $\mathcal{G}_0$-ε_0-multiplicative completely positive linear map such that

$$\|\varphi'(g) - \varphi''(g)\| < 4\varepsilon_0 \ \text{and} \ \|u' - u''\| < 4\varepsilon_0 \ \text{for all} \ g \in \mathcal{G}_0, \tag{e 7.11}$$

then $\mathrm{Bott}(\varphi',\, u')|_{\mathcal{P}} = \mathrm{Bott}(\varphi'',\, u'')|_{\mathcal{P}}$. We may assume that $1_A \in \mathcal{G}_0$. Let

$$\mathcal{G}_0' = \{g \otimes f : g \in \mathcal{G}_0\} \ \text{and} \ f = \{1_{C(\mathbb{T})}, z, z^*\},$$

where z is the identity function on the unit circle $\mathbb{T}$. We also assume that if $\Psi' : A \otimes C(\mathbb{T}) \to C$ (for a unital C^*-algebra C) is a unital $\mathcal{G}_0'$-ε_0-multiplicative completely positive linear map, then there exists a unitary $u' \in C$ such that

$$\|\Psi'(1 \otimes z) - u'\| < 4\varepsilon_0. \tag{e 7.12}$$

Without loss of generality, we may assume that $\mathcal{G}_0$ is contained in the unit ball of A. Let $\varepsilon_1 = \min\{\varepsilon/64, \varepsilon_0/512\}$ and $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_0$.

Let $\mathcal{H}_0 \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}$) be the finite subset and $L \geq 1$ the integer provided by Lemma 4.20 for ε_1 (in place of ε) and $\mathcal{F}_1$ (in place of $\mathcal{F}$) as well as Δ_2 (in place of Δ).

Let $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$), $\delta_1 > 0$ (in place of δ), $\mathcal{P}_1 \subset \underline{K}(A)$ (in place of $\mathcal{P}$), $\mathcal{H}_2 \subset A_{s.a.}$, and $1 > \sigma > 0$ be as provided by Theorem 5.8 for ε_1 (in place of ε), $\mathcal{F}_1$ (in place of $\mathcal{F}$), and Δ_1. We may assume that $[1_A] \in \mathcal{P}_2$, $\mathcal{H}_2$ is in the unit ball of A, and $\mathcal{H}_0 \subset \mathcal{H}_1$.

Without loss of generality, we may assume that $\delta_1 < \varepsilon_1/16, \sigma < \varepsilon_1/16$, and $\mathcal{F}_1 \subset \mathcal{G}_1$. Put $\mathcal{P}_2 = \mathcal{P} \cup \mathcal{P}_1$.

Denote by $\{r_1, r_2, ..., r_k\}$ the set of all ranks of irreducible representations of A. Fix an irreducible representation $\pi_0 : A \to M_{r_1}$. Let $N(p) \geq 1$ (in place of $N(\mathcal{P}_0)$) and $\mathcal{H}_0' \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}$) denote the integer and finite subset provided by Lemma 4.19 for $\{1_A\}$ (in place of $\mathcal{P}_0$) and $(1/16)\Delta$. Let $\mathcal{H}_1' = \mathcal{H}_1 \cup \mathcal{H}_0'$.

Let $G_0 = G \cap K_0(A)$ and write $G_0 = \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \mathrm{Tor}(G_0)$, where $\mathbb{Z}^{s_2} \oplus \mathrm{Tor}(G_0) \subset \ker\rho_A$. Let $x_j = (\overbrace{0, ..., 0}^{j-1}, 1, 0, ..., 0) \in \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2}$, $j = 1, 2, ..., s_1 + s_2$. Note that $A \otimes C(\mathbb{T}) \in \overline{\mathcal{D}}_s$ and $A \otimes C(\mathbb{T})$ has irreducible representations of ranks $r_1, r_2, ..., r_k$. Let

$$\bar{r} = \max\{|(\pi_0)_{*0}(x_j)| : 0 \leq j \leq s_1 + s_2\}.$$

Let $\mathcal{P}_3 \subset \underline{K}(A \otimes C(\mathbb{T}))$ be a finite subset containing $\mathcal{P}_2$, $\{\boldsymbol{\beta}(g_j) : 1 \leq j \leq r\}$, and a finite subset which generates $\boldsymbol{\beta}(\mathrm{Tor}(G_1))$. Choose $\delta_2 > 0$ and a finite subset

$$\overline{\mathcal{G}} = \{g \otimes f : g \in \mathcal{G}_2, \ \ f \in \{1, z, z^*\}\}$$

in $A \otimes C(\mathbb{T})$, where $\mathcal{G}_2 \subset A$ is a finite subset such that, for any unital $\overline{\mathcal{G}}$-δ_2-multiplicative completely positive linear map $\Phi' : A \otimes C(\mathbb{T}) \to C$ (for any unital C^*-algebra C with $\mathrm{Tor}(K_0(C)) = \mathrm{Tor}(K_1(C)) = \{0\}$), $[\Phi']|_{\mathcal{P}_3}$ is well defined and

$$[\Phi']|_{\mathrm{Tor}(G_0) \oplus \boldsymbol{\beta}(\mathrm{Tor}(G_1))} = 0. \tag{e 7.13}$$

We may assume $\mathcal{G}_2 \supset \mathcal{G}_1 \cup \mathcal{F}_1$.

Let $\sigma_1 = \min\{\Delta_2(\hat{h}) : h \in \mathcal{H}_1'\}$. Note $K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \boldsymbol{\beta}(K_1(A))$ and $\underline{K}(A \otimes C(\mathbb{T})) = \underline{K}(A) \oplus \boldsymbol{\beta}(\underline{K}(A))$. Consider the subgroup of $K_0(A \otimes C(\mathbb{T}))= K_0(A) \oplus \boldsymbol{\beta}(K_1(A))$ given by

$$\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \mathbb{Z}^{r} \oplus \mathrm{Tor}(G_0) \oplus \boldsymbol{\beta}(\mathrm{Tor}(G_1)).$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Let $N(\delta_3, \overline{\mathcal{G}}, \mathcal{P}_3, i)$ and Λ_i, $i = 1, 2, ..., s_1 + s_2 + r$, be as provided by Lemma 7.2 for $A \otimes C(\mathbb{T})$. Choose an integer $n_1 > N(p)$ such that

$$\frac{(\sum_{i=1}^{s_1+s_2+r} N(\delta_3, \overline{\mathcal{G}}, \mathcal{P}_3, i) + 1 + \Lambda_i)N(p)}{n_1 - 1} < \min\{\sigma/16, \sigma_1/2\}. \tag{e 7.14}$$

Choose $n > n_1$ such that

$$\frac{n_1 + 2}{n} < \min\{\sigma/16, \sigma_1/2, 1/(L + 1)\}. \tag{e 7.15}$$

Let $\varepsilon_2 > 0$ and let $\mathcal{F}_2 \subset A$ be a finite subset such that $[\Psi]|_{\mathcal{P}_2}$ is well defined for any $\mathcal{F}_2$-ε_2-multiplicative contractive completely positive linear map $\Psi : A \to B$ (for any C^*-algebra B). Let $\varepsilon_3 = \min\{\varepsilon_2/2, \varepsilon_1\}$ and $\mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2$.

Denote by $\delta_4 > 0$ (in place of δ), $\mathcal{G}_3 \subset A$ (in place of $\mathcal{G}$), $\mathcal{H}_3 \subset A_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_2$) the constant and finite sets provided by Lemma 5.6 for ε_3 (in place of ε), $\mathcal{F}_3 \cup \mathcal{H}_1'$ (in place of $\mathcal{F}$), $\delta_3/2$ (in place of ε_0), $\mathcal{G}_2$ (in place of $\mathcal{G}_0$), Δ, $\mathcal{H}_1'$ (in place of $\mathcal{H}$), $\min\{\sigma/16, \sigma_1/2\}$ (in place of ε_1), and n^2 (in place of K) (with $L_1 = L_2$ so no σ is needed in 5.6).

Set $\mathcal{G} = \mathcal{F}_3 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, set $\delta = \min\{\varepsilon_3/16, \delta_4, \delta_3/16\}$, and set $\mathcal{G}_5 = \{g \otimes f : g \in \mathcal{G}_4,\ \ f \in \{1, z, z^*\}\}$.

Let $\mathcal{H} = \mathcal{H}_1' \cup \mathcal{H}_3$. Define $N_0 = (n+1)^2 N(p)(\sum_{i=1}^{s_1+s_2+r} N(\delta_3, \mathcal{G}_0, \mathcal{P}_3, i) + \Lambda_i + 1)$ and define $N = N_0 + N_0 \bar{r}$. Fix any $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ with

$$K = \max\{|\kappa(\boldsymbol{\beta}(g_i)| : 1 \le j \le r\}.$$

Let $R > N(K+1)$. Suppose that $\varphi : A \to M_R$ is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map such that

$$\text{(e 7.16)} \qquad \operatorname{tr} \circ \varphi(h) \ge \Delta(\hat{h}) \ \text{ for all } \ h \in \mathcal{H}.$$

Then, by 5.6, there exist mutually orthogonal projections $e_0, e_1, e_2, ..., e_{n^2} \in M_R$ such that $e_1, e_2, ..., e_{n^2}$ are equivalent, $e_0 \lesssim e_1$, $\operatorname{tr}(e_0) < \min\{\sigma/64, \sigma_1/4\}$ and $e_0 + \sum_{i=1}^{n^2} e_i = 1_{M_R}$, and there exist a unital $\mathcal{G}_2$-$\delta_3/2$-multiplicative completely positive linear map $\psi_0 : A \to e_0 M_R e_0$ and a unital homomorphism $\psi : A \to e_1 M_R e_1$ such that

$$\text{(e 7.17)} \quad \|\varphi(f) - (\psi_0(f) \oplus \overbrace{\psi(f), \psi(f), ..., \psi(f)}^{n^2})\| < \varepsilon_3 \ \text{ for all } \ f \in \mathcal{F}_3 \ \text{ and}$$

$$\text{(e 7.18)} \quad \operatorname{tr} \circ \psi(h) \ge \Delta(\hat{h})/2n^2 \ \text{ for all } \ h \in \mathcal{H}_1'.$$

Let $\alpha \in \operatorname{Hom}_\Lambda(\underline{K}(A \otimes C(\mathbb{T})), \underline{K}(M_r))$ be defined as follows: $\alpha|_{\underline{K}(A)} = [\pi_0]$ and $\alpha|_{\boldsymbol{\beta}(\underline{K}(A))} = \kappa|_{\boldsymbol{\beta}(\underline{K}(A))}$. Note that

$$\max\{\max\{|\kappa \circ \boldsymbol{\beta}(g_i)| : 1 \le i \le r\}, \max\{|\pi_0(x_j)| : 1 \le j \le s_1 + s_2\}\} \le \max\{K, \bar{r}\}.$$

Applying Lemma 7.2, we obtain a unital $\mathcal{G}$-δ_3-multiplicative completely positive linear map $\Psi : A \otimes C(\mathbb{T}) \to M_{N_1'}$, where $N_1' \le N_1 := \sum_{j=1}^{s_1+s_2+r} (N(\delta_3, \mathcal{G}_0, \mathcal{P}_3, j) + \Lambda_i) \max\{K, \bar{r}\}$, and a homomorphism $H_0 : A \otimes C(\mathbb{T}) \to H_0(1_A) M_{N_1'} H_0(1_A)$ such that

$$\text{(e 7.19)} \qquad [\Psi]|_{\mathcal{P}_3} = (\alpha + [H_0])|_{\mathcal{P}_3}.$$

Note that (since H_0 is a homomorphism with finite dimensional range)

$$\text{(e 7.20)} \qquad [\Psi]|_{\boldsymbol{\beta}(\mathcal{P})} = \kappa|_{\boldsymbol{\beta}(\mathcal{P})}.$$

In particular, since $[1_A] \in \mathcal{P}_2 \subset \mathcal{P}_3$, $\operatorname{rank}(\Psi(1_A)) = r_1 + \operatorname{rank}(H_0)$. Note that

$$\text{(e 7.21)} \qquad \frac{N_1' + N(p)}{R} \le \frac{N_1 + N(p)}{N(K+1)} < 1/(n+1)^2.$$

Let $R_1 = \operatorname{rank} e_1$. Then $R_1 \geq R/(n+1)^2$. Hence by (e 7.21), $R_1 \geq N_1 + N(p)$. In other words, $R_1 - N_1' \geq N(p) > 0$. Note that, by (e 7.18),

$$\operatorname{tr}' \circ \psi(\hat{g}) \geq (1/3)\Delta(\hat{g}) \geq \Delta_2(\hat{g}) \ \text{ for all } \ g \in \mathcal{H}_0',$$

where tr' is the tracial state on M_{R_1}. Note that $n \geq N(p)$. Applying Lemma 4.19 to the pair $\pi_0 \oplus H_0|_{A \otimes 1_{C(\mathbb{T})}}$ (in place of φ) and $\tilde{\psi}$ (in place of ψ), where $\tilde{\psi}$ is an amplification of ψ with ψ repeated n times, and $\mathcal{P}_0 = \{[1_A]\}$, we obtain a unital homomorphism $h_0 : A \otimes C(\mathbb{T}) \to M_{nR_1 - N_1'}$ such that $h_0(1 \otimes 1) = 1_{M_{nR_1-N_1'}}$. Define $\psi_0' : A \otimes C(\mathbb{T}) \to e_0 M_R e_0$ by $\psi_0'(a \otimes f) = \psi_0(a) \cdot (f(1) \cdot e_0)$ for all $a \in A$ and $f \in C(\mathbb{T})$, where $1 \in \mathbb{T}$. Define $\psi' : A \otimes C(\mathbb{T}) \to e_1 M_R e_1$ by $\psi'(a \otimes f) = \psi(a) \cdot (f(1) \cdot e_1)$ for all $a \in A$ and $f \in C(\mathbb{T})$. Note that

$$[\psi']|_{\boldsymbol{\beta}(\mathcal{P})} = [\psi_0']|_{\boldsymbol{\beta}(\mathcal{P})} = \{0\}. \tag{e 7.22}$$

Put $E_1 = \operatorname{diag}(e_1, e_2, ..., e_{nn_1})$. Define $L_1 : A \to E_1 M_R E_1$ by $L_1(a) = \pi_0(a) \oplus H_0|_A(a) \oplus h_0(a \otimes 1) \oplus (\overbrace{\psi(a), ..., \psi(a)}^{n(n_1-1)})$ for $a \in A$, and define $L_2 : A \to E_1 M_R E_1$ by $L_2(a) = \Psi(a \otimes 1) \oplus h_0(a \otimes 1) \oplus (\overbrace{\psi(a), ..., \psi(a)}^{n(n_1-1)})$ for $a \in A$. Note that

$$[L_1]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1}, \tag{e 7.23}$$

$$\operatorname{tr} \circ L_1(h) \geq \Delta_1(\hat{h}), \ \ \operatorname{tr} \circ L_2(h) \geq \Delta_1(\hat{h}) \ \text{ for all } \ h \in \mathcal{H}_1, \ \text{ and} \tag{e 7.24}$$

$$|\operatorname{tr} \circ L_1(g) - \operatorname{tr} \circ L_2(g)| < \sigma \ \text{ for all } \ g \in \mathcal{H}_2. \tag{e 7.25}$$

It follows from Theorem 5.8 that there exists a unitary $w_1 \in E_1 M_R E_1$ such that

$$\|\operatorname{Ad} w_1 \circ L_2(a) - L_1(a)\| < \varepsilon_1 \ \text{ for all } \ a \in \mathcal{F}_1. \tag{e 7.26}$$

Define $E_2 = (e_1 + e_2 + \cdots + e_{n^2})$ and define $\Phi : A \to E_2 M_R E_2$ by

$$\Phi(f)(a) = \operatorname{diag}(\overbrace{\psi(a), \psi(a), ..., \psi(a)}^{n^2}) \ \text{ for all } \ a \in A. \tag{e 7.27}$$

Then

$$\operatorname{tr} \circ \Phi(h) \geq \Delta_2(\hat{h}) \ \text{ for all } \ h \in \mathcal{H}_0. \tag{e 7.28}$$

By (e 7.15), one has $n/(n_1+2) > L+1$. Applying Lemma 4.20, we obtain a unitary $w_2 \in E_2 M_R E_2$ and a unital homomorphism $H_1 : A \to (E_2 - E_1) M_R (E_2 - E_1)$ such that

$$\|\operatorname{ad} w_2 \circ \operatorname{diag}(L_1(a), H_1(a)) - \Phi(a)\| < \varepsilon_1 \ \text{ for all } \ a \in \mathcal{F}_1. \tag{e 7.29}$$

Put

$$w = (e_0 \oplus w_1 \oplus (E_2 - E_1))(e_0 \oplus w_2) \in M_R.$$

Define $H_1' : A \otimes C(\mathbb{T}) \to (E_2 - E_1)M_R(E_2 - E_1)$ by $H_1'(a \otimes f) = H_1(a) \cdot f(1) \cdot (E_2 - E_1)$ for all $a \in A$ and $f \in C(\mathbb{T})$. Define $\Psi_1 : A \to M_R$ by

$$\Psi_1(f) = \psi_0'(f) \oplus \Psi(f) \oplus h_0(f) \oplus (\overbrace{\psi'(f), ..., \psi'(f)}^{n(n_1-1)}) \oplus H_1'(f) \text{ for all } f \in A \otimes C(\mathbb{T}).$$

By (e 7.20), (e 7.22),

$$[\Psi_1]|_{\boldsymbol{\beta}(\mathcal{P})} = \kappa|_{\boldsymbol{\beta}(\mathcal{P})}. \tag{e 7.30}$$

It follows from (e 7.26), (e 7.29), and (e 7.17) that

$$\|\varphi(a) - w^*\Psi_1(a \otimes 1)w\| < \varepsilon_1 + \varepsilon_1 + \varepsilon_3 \text{ for all } a \in \mathcal{F}. \tag{e 7.31}$$

Now pick a unitary $v \in M_R$ such that

$$\|\Psi_1(1 \otimes z) - v\| < 4\varepsilon_1. \tag{e 7.32}$$

Put $u = w^*vw$. Then, we estimate that

$$\|[\varphi(a), u]\| < \min\{\varepsilon, \varepsilon_0\} \text{ for all } a \in \mathcal{F}_1. \tag{e 7.33}$$

Moreover, by (e 7.30) and by the choice of ε_0, one has

$$\mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = \mathrm{Bott}(\Psi_1|_A, \Psi_1(1 \otimes z))|_{\mathcal{P}} = [\Psi_1] \circ \boldsymbol{\beta}|_{\mathcal{P}} = \kappa \circ \boldsymbol{\beta}|_{\mathcal{P}}.$$

□

8. A Uniqueness Theorem for C^*-algebras in $\mathcal{D}_s$

The main goal of this section is to prove Theorem 8.3.

DEFINITION 8.1. Let A be a unital C^*-algebra, $C = C(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$, $\pi_i : C \to F_2$ $(i = 0, 1,)$, and $\pi_e : C \to F_1$ be as in Definition 3.1. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\varphi_0 \circ L_e = \pi_0 \circ L \text{ and } \varphi_1 \circ L_e = \pi_1 \circ L. \tag{e 8.1}$$

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and L is $\mathcal{G}$-δ-multiplicative, then L_e is also $\mathcal{G}$-δ-multiplicative.

LEMMA 8.2. *Let A be a unital C^*-algebra and let $C = C(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ be as in 3.1. Let $L_1, L_2 : A \to C$ be unital completely positive linear maps, let $\varepsilon > 0$, and let $\mathcal{F} \subset A$ be a finite subset. Suppose that there are unitaries $w_0 \in \pi_0(C) \subset F_2$ and $w_1 \in \pi_1(C) \subset F_2$ such that*

$$\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| < \varepsilon \quad \textit{and} \tag{e 8.2}$$

$$\|w_1^* \pi_1 \circ L_1(a) w_1 - \pi_1 \circ L_2(a)\| < \varepsilon \ \textit{for all}\ a \in \mathcal{F}. \tag{e 8.3}$$

Then there exists a unitary $u \in F_1$ such that

$$\|\varphi_0(u)^* \pi_0 \circ L_1(a) \varphi_0(u) - \pi_0 \circ L_2(a)\| < \varepsilon \quad \textit{and} \tag{e 8.4}$$

$$\|\varphi_1(u)^* \pi_1 \circ L_1(a) \varphi_1(u) - \pi_1 \circ L_2(a)\| < \varepsilon \ \textit{for all}\ a \in \mathcal{F}. \tag{e 8.5}$$

PROOF. Write $F_1 = M_{n(1)} \oplus M_{n(2)} \oplus \cdots \oplus M_{n(k)}$ and $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(l)}$. We may assume that $\ker\varphi_0 \cap \ker\varphi_1 = \{0\}$ (see 3.1).

We may assume that there are $k(0)$ and $k(1)$ such that $\varphi_0|_{M_{n(j)}}$ is injective, $j = 1, 2, ..., k(0)$, with $k(0) \le k$, $\varphi_0|_{M_{n(j)}} = 0$ if $j > k(0)$, and $\varphi_1|_{M_{n(j)}}$ is injective, $j = k(1), k(1)+1, ..., k$, with $k(1) \le k$, $\varphi_1|_{M_{n(j)}} = 0$, if $j < k(1)$. Write $F_{1,0} = \bigoplus_{j=1}^{k(0)} M_{n(j)}$ and $F_{1,1} = \bigoplus_{j=k(1)}^{k} M_{n(j)}$. Note that $k(1) \le k(0) + 1$, and $\varphi_0|_{F_{1,0}}$ and $\varphi_1|_{F_{1,1}}$ are injective. Note $\varphi_0(F_{1,0}) = \varphi_0(F_1) = \pi_0(C)$ and $\varphi_1(F_{1,1}) = \varphi_1(F_1) = \pi_1(C)$.

For each fixed $a \in A$, since $L_i(a) \in C$ $(i = 1, 2)$, there are elements

$$g_{a,i} = g_{a,i,1} \oplus g_{a,i,2} \oplus \cdots \oplus g_{a,i,k(0)} \oplus \cdots \oplus g_{a,i,k} \in F_1,$$

such that $\varphi_0(g_{a,i}) = \pi_0 \circ L_i(a)$ and $\varphi_1(g_{a,i}) = \pi_1 \circ L_i(a)$, $i = 1, 2$, where $g_{a,i,j} \in M_{n(j)}$, $j = 1, 2, ..., k$ and $i = 1, 2$. Note that such $g_{a,i}$ is unique since $\ker\varphi_0 \cap \ker\varphi_1 = \{0\}$. Since $w_0 \in \pi_0(C) = \varphi_0(F_1)$, there is a unitary

$$u_0 = u_{0,1} \oplus u_{0,2} \oplus \cdots \oplus u_{0,k(0)} \oplus \cdots \oplus u_{0,k}$$

such that $\varphi_0(u_0) = w_0$. Note that the first $k(0)$ components of u_0 are uniquely determined by w_0 (since φ_0 is injective on this part) and the components after the $k(0)$'th component can be chosen arbitrarily (since $\varphi_0 = 0$ on this part). Similarly there exists

$$u_1 = u_{1,1} \oplus u_{1,2} \oplus \cdots \oplus u_{1,k(1)} \oplus \cdots \oplus u_{1,k}$$

such that $\varphi_1(u_1) = w_1$.

Now by (e 8.2) and (e 8.3), we have

$$\|\varphi_0(u_0)^* \varphi_0(g_{a,1}) \varphi_0(u_0) - \varphi_0(g_{a,2})\| < \varepsilon \ \text{and} \tag{e 8.6}$$

$$\|\varphi_1(u_1)^* \varphi_1(g_{a,1}) \varphi_1(u_1) - \varphi_1(g_{a,2}))\| < \varepsilon \ \text{for all}\ a \in \mathcal{F}. \tag{e 8.7}$$

Since φ_0 is injective on M_{n_j} for $j \le k(0)$ and φ_1 is injective on $M_{n(j)}$ for $j > k(0)$ (note that we use $k(1) \le k(0)+1$), we have

$$\|(u_{0,j})^*(g_{a,1,j})u_{0,j} - (g_{a,2,j})\| < \varepsilon \quad \text{for all } j \le k(0), \text{ and} \tag{e 8.8}$$

$$\|(u_{1,j})^*(g_{a,1,j})u_{1,j} - (g_{a,2,j})\| < \varepsilon \quad \text{for all } j > k(0), \tag{e 8.9}$$

and all $a \in \mathcal{F}$. Let $u = u_{0,1} \oplus \cdots \oplus u_{0,k(0)} \oplus u_{1,k(0)+1} \oplus \cdots \oplus u_{1,k} \in F_1$—that is, for the first $k(0)$ components of u, we use u_0's corresponding components, and for the last $k - k(0)$ components of u, we use u_1's. From (e 8.8) and (e 8.9), we have

$$\|u^* g_{a,1} u - g_{a,2}\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

Once we apply φ_0 and φ_1 to the above inequality, we get (e 8.4) and (e 8.5) as desired. □

Let us very briefly describe the proof of Theorem 8.3. The key ingredients are Theorem 5.8, Lemma 6.7, and Lemma 7.3. First fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. We note that the result has been established when the target algebra is finite dimensional (Theorem 5.8). So we may reduce the general case to the case that the target algebra is infinite dimensional, and has only a single direct summand (minimal) in $\mathcal{C}$. So we write the target algebra C as $A(F_1, F_2, h_0, h_1)$ and note $\lambda : C \to C([0,1], F_2)$ as given in 3.1 is injective (as C is minimal).

The first idea is to consider a partition $0 = t_0 < t_1 < \cdots t_n = 1$ such that $\|\pi_t \circ \varphi(f)(t) - \pi_{t_i} \circ \varphi(f)(t_i)\|$ is very small if $t \in [t_i, t_{i+1}]$, where $\pi_{t_i} : C \to F_2$ is the point evaluation of C at t_i. We will then consider the pair $\varphi_i = \pi_{t_i} \circ \varphi$ and $\psi_i = \pi_{t_i} \circ \psi$. At each point t_i, we apply 5.8 to obtain the unitary w_i. We then connect these w_i to obtain the unitary in C that we need to find. In other words, the continuous path from w_i to w_{i+1} given by u should change $\{\pi_t \circ \varphi(f)(t) : f \in \mathcal{F}, t \in [t_i, t_{i+1}]\}$ very little from $w_i(\pi_t \circ \varphi(f))w_i^*$. One then observes that $w_i w_{i+1}^*$ almost commutes with $\{\pi_{t_i} \circ \varphi(f) : f \in \mathcal{F}\}$ (and hence $\{\varphi(f)(t) : f \in \mathcal{F}, t \in [t_i, t_{i+1}]\}$). A basic homotopy lemma (such as Lemma 6.7) would provide a path $v(t)$ from $w_i w_{i+1}^*$ to 1 which also almost commutes with the set. Then one considers $v(t)w_{i+1}$ which starts as w_i and ends at w_{i+1} which would be a path as desired. However, the basic homotopy lemma may have an obstacle, the class $\mathbf{Bott}(\varphi_i, w_i w_{i+1}^*)$. We will then consider $w_i z_i$, where z_i is as provided by Lemma 7.3, so that z_i almost commutes with $\{\varphi(f)(t_i) : f \in \mathcal{F}\}$ and the class $\mathbf{Bott}(\varphi_i, w_i z_i (w_{i+1} z_{i+1})^*)$ will be zero. This is possible because of the condition (e 8.13). To simplify the process, we kill torsion elements in $\underline{K}(A)$ by repeating φ (and ψ) N times. Much of the proof is to make sure the idea can actually be carried out. One also needs to exercise special care at the endpoints, applying Lemma 8.2.

THEOREM 8.3. *Let $A \in \bar{\mathcal{D}}_s$ be a unital C^*-algebra with finitely generated $K_i(A)$ ($i = 0, 1$). Let $\mathcal{F} \subset A$ be a finite subset, let $\varepsilon > 0$ be a positive number, and let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. There exist a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset*

$\mathcal{P} \subset \underline{K}(A)$, *a finite subset* $\mathcal{H}_2 \subset A$, *a finite subset* $\mathcal{U} \subset J_c(K_1(A))$ (see Definition 2.16) *for which* $[\mathcal{U}] \subset \mathcal{P}$, *and* $N \in \mathbb{N}$ *satisfying the following condition : For any pair of unital* $\mathcal{G}$-δ-*multiplicative completely positive linear maps* $\varphi, \psi : A \to C$, *for some* $C \in \mathcal{C}$, *such that*

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{e 8.10}$$

$$\tau(\varphi(a)) \geq \Delta(a), \;\; \tau(\psi(a)) \geq \Delta(a), \quad \textit{for all } \tau \in T(C) \textit{ and } a \in \mathcal{H}_1, \tag{e 8.11}$$

$$|\tau \circ \varphi(a) - \tau \circ \psi(a)| < \gamma_1 \;\; \textit{for all } a \in \mathcal{H}_2, \;\; \text{and} \tag{e 8.12}$$

$$\mathrm{dist}(\varphi^{\ddagger}(u), \psi^{\ddagger}(u)) < \gamma_2 \;\; \textit{for all } u \in \mathcal{U}, \tag{e 8.13}$$

there exists a unitary $W \in C \otimes M_N$ *such that*

$$\|W(\varphi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \varepsilon, \;\; \textit{for all } f \in \mathcal{F}. \tag{e 8.14}$$

PROOF. If A is finite dimensional, then it is semiprojective. So, with sufficiently large $\mathcal{G}$ and sufficiently small δ, both φ and ψ are close to homomorphisms to within $\varepsilon/2$ and $\mathcal{F}$. Therefore the general case is reduced to the case that both φ and ψ are homomorphisms. Since two homomorphisms from A to a C^*-algebra with stable rank one are unitarily equivalent if and only if they induce the same map on the ordered K_0 groups (see, for instance, Lemma 7.3.2 (ii) of [108]), the theorem holds (with $N = 1$ and no requirements on $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{U}$, γ_1, and γ_2). So, it remains to consider the case that A is infinite dimensional.

Since $K_*(A)$ is finitely generated, there is n_0 such that $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(C))$ is determined by its restriction to $K(A, \mathbb{Z}/n\mathbb{Z})$, $n = 0, ..., n_0$. Set $N = n_0!$.

Let $\mathcal{H}_1' \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}$), $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$), and $\mathcal{P}_0 \subset \underline{K}(A)$ (in place of $\mathcal{P}$) be the finite subsets and constant be as provided by Lemma 6.7 for $\varepsilon/32$ (in place of ε), $\mathcal{F}$ and Δ. We may assume that $\delta_1 < \min\{\varepsilon/32, 1/64\}$ and $(2\delta_1, \mathcal{G}_1)$ is a KK-pair (see the end of Definition 2.12).

Moreover, we may assume that δ_1 is sufficiently small that if $\|uv - vu\| < 3\delta_1$, then the Exel formula

$$\tau(\mathrm{bott}_1(u, v)) = \frac{1}{2\pi\sqrt{-1}}(\tau(\log(u^*vuv^*))$$

holds for any pair of unitaries u and v in any unital C^*-algebra C and any $\tau \in T(C)$ (see Lemma 3.1 and 3.2 of [40], also, Theorem 3.7 of [52]). Furthermore (see 2.11 of [73] and 2.2 of [39]), we may assume that, if $\|uv_i - v_iu\| < 3\delta_1$, $i = 1, 2, 3$, then

$$\mathrm{bott}_1(u, v_1v_2v_3) = \sum_{i=1}^{3} \mathrm{bott}_1(u, v_i). \tag{e 8.15}$$

Fix a decomposition $K_1(A) = \mathbb{Z}^{k(A)} \oplus \mathrm{Tor}(K_1(A))$, where $k(A)$ is a positive integer. Choose $g_1, g_2, ..., g_{k(A)} \in U(M_{m(A)}(A))$ (for the some integer $m(A) \geq 1$)

such that $\{\bar{g_1}, \bar{g_2}, ..., \bar{g}_{k(A)}\} \subset J_c(K_1(A))$ and $[g_i] = (\overbrace{0, ..., 0}^{i-1}, 1, 0, ..., 0) \in \mathbb{Z}^{k(A)}$. Set $\mathcal{U} = \{\bar{g_1}, \bar{g_2}, ..., \bar{g}_{k(A)}\} \subset J_c(K_1(A))$.

Let $\mathcal{U}_0 \subset A$ be a finite subset such that

$$\{g_1, g_2, ..., g_{k(A)}\} \subset \{(a_{i,j}) : a_{i,j} \in \mathcal{U}_0\}.$$

Let $\delta_u = \min\{1/256m(A)^2, \delta_1/24m(A)^2\}$, $\mathcal{G}_u = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{U}_0$ and let $\mathcal{P}_u = \mathcal{P}_0 \cup \{[g_1], [g_2], ..., [g_{k(A)}]\}$. Let $\delta_2 > 0$ (in place of δ), $\mathcal{G}_2 \subset A$ (in place of $\mathcal{G}$), $\mathcal{H}'_2 \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}$), and $N_1 \geq 1$ (in place of N) be the finite subsets and constants provided by Lemma 7.3 for the data δ_u (in place of ε), $\mathcal{G}_u$ (in place of $\mathcal{F}$), $\mathcal{P}_u$ (in place of $\mathcal{P}$), and Δ, and with $[g_j]$ (in place of g_j), $j = 1, 2, ..., k(A)$ (with $k(A) = r$).

Let $\varepsilon_1 = \min\{1/192N_1m(A)^2, \delta_u/2, \delta_2/2m(A)^2\}$.

Let $\mathcal{H}'_3 \subset A^{\mathbf{1}}_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\delta_4 > 0$ (in place of δ), $\mathcal{G}_3 \subset A$ (in place of $\mathcal{G}$), $\mathcal{H}'_4 \subset A_{s.a.}$ (in place of $\mathcal{H}_2$), $\mathcal{P}_1 \subset \underline{K}(A)$ (in place of $\mathcal{P}$), and $\sigma > 0$ be the finite subsets and constants provided by Theorem 5.8 with respect to $\varepsilon_1/4$ (in place of ε), $\mathcal{G}_u$ (in place of $\mathcal{F}$), and Δ.

Choose $\mathcal{H}'_5 \subset A^{\mathbf{1}}_+ \setminus \{0\}$ and $\delta_5 > 0$ and a finite subset $\mathcal{G}_4 \subset A$ such that, for any $m \in \mathbb{N}$ and any unital $\mathcal{G}_4$-δ_5-multiplicative completely positive linear map $L' : A \to M_m$, if $\mathrm{tr} \circ L'(h) > \Delta(\hat{h})$ for all $h \in \mathcal{H}'_5$, then $m \geq 4N_1$. This is possible because we can apply Lemma 5.6 (taking $K = 4N_1$, $L_1 = L_2 = L'$), since we now assume that A is infinite dimensional.

Put $\delta = \min\{\varepsilon_1/16, \delta_4/4m(A)^2, \delta_5/4m(A)^2\}$, $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_u \cup \mathcal{G}_3 \cup \mathcal{G}_4$, and $\mathcal{P} = \mathcal{P}_u \cup \mathcal{P}_1$. Put

$$\mathcal{H}_1 = \mathcal{H}'_1 \cup \mathcal{H}'_2 \cup \mathcal{H}'_3 \cup \mathcal{H}'_4 \cup \mathcal{H}'_5$$

and let $\mathcal{H}_2 = \mathcal{H}'_4$. Let $\gamma_1 = \sigma$ and let

$$0 < \gamma_2 < \min\{1/64m(A)^2N_1, \delta_u/9m(A)^2, 1/256m(A)^2\}.$$

We assume that δ is sufficiently small and $\mathcal{G}$ is sufficiently large that for any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\Phi : A \to B$, where B is a unital C^*-algebra (so that $\Phi \otimes \mathrm{id}_{M_{m(A)}}$ is approximately multiplicative), one has that

$$\|(\Phi \otimes \mathrm{id}_{M_{m(A)}})(g_j) - \langle(\Phi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle\| < \varepsilon_1, \quad j = 1, 2, ..., k(A). \tag{e 8.16}$$

Now suppose that $C \in \mathcal{C}$ and $\varphi, \psi : A \to C$ are two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps satisfying the condition of the theorem for the given Δ, $\mathcal{H}_1$, δ, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}_2$, γ_1, γ_2, and $\mathcal{U}$.

We write $C = A(F_1, F_2, h_0, h_1)$, $F_1 = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_{F(1)}}$, and $F_2 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_{F(2)}}$. For each $t \in [0, 1]$, we will write $\pi_t : C \to C([0, 1], F_2)$ for the point evaluation at t as defined in 3.1. Note that, when C is finite dimensional, the theorem holds by Theorem 5.8. So, we may assume that C is infinite dimensional. It is also clear that the general case can be reduced to the case that C is minimal (see 3.1). As in Definition 3.1, then, we may assume that $\mathrm{ker} h_0 \cap \mathrm{ker} h_1 = \{0\}$, and $\lambda : C \to C([0,1], F_2)$ defined by $\lambda(f,a) = f$ is unital and injective.

Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition of $[0,1]$ such that

$$\|\pi_t \circ \varphi(g) - \pi_{t'} \circ \varphi(g)\| < \varepsilon_1/16 \ \text{ and } \ \|\pi_t \circ \psi(g) - \pi_{t'} \circ \psi(g)\| < \varepsilon_1/16 \tag{e 8.17}$$

for all $g \in \mathcal{G}$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, ..., n$.

Set $V_{i,j} = \langle \pi_{t_i} \circ \varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j)\rangle$, $j = 1, 2, ..., k(A)$ and $i = 0, 1, 2, ..., n$.

Applying Theorem 5.8, one obtains a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_n \in h_1(F_1)$, if $i = n$, such that

$$\|w_i \pi_{t_i} \circ \varphi(g) w_i^* - \pi_{t_i} \circ \psi(g)\| < \varepsilon_1/16 \ \text{ for all } \ g \in \mathcal{G}_u. \tag{e 8.18}$$

It follows from Lemma 8.2 that we may assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$. Since we assume that λ is injective, we also have

$$\|w_e(\pi_e \circ \varphi(g))w_e^* - \pi_e \circ \psi(g)\| < \varepsilon_1/16 \ \text{ for all } \ g \in \mathcal{G}_u. \tag{e 8.19}$$

By (e 8.13), one has (see Definition 2.16; see also Proposition 3.23)

$$\mathrm{dist}(\overline{\langle \varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle \langle (\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle}, \overline{1}) < \gamma_2.$$

It follows that there is a unitary $\theta_j \in M_{m(A)}(C)$ such that $\theta_j \in CU(M_{m(A)}(C))$ (see Definition 2.17) and, for $j = 1, 2, ..., k(A)$,

$$\|\langle (\varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle \langle (\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle - \theta_j\| < \gamma_2. \tag{e 8.20}$$

By Theorem 3.16, we can write

$$\theta_j = \prod_{l=1}^{4} \exp(\sqrt{-1} a_j^{(l)})$$

for self-adjoint elements $a_j^{(l)} \in M_{m(A)}(C)$, $l = 1, 2, ..., 4$, $j = 1, 2, ..., k(A)$. Write

$$a_j^{(l)} = (a_j^{(l,1)}, a_j^{(l,2)}, ..., a_j^{(l,F(2))}) \ \text{ and } \ \theta_j = (\theta_{j,1}, \theta_{j,2}, ..., \theta_{j,F(2)})$$

in $C([0,1], M_{m(A)} \otimes F_2) = C([0,1], M_{m(A)n_1}) \oplus \cdots \oplus C([0,1], M_{m(A)n_{F(2)}})$, where $\theta_{j,s} = \prod_{l=1}^{4} \exp(\sqrt{-1} a_j^{(l,s)})$, $s = 1, 2, ..., F(2)$. Then (see Lemma 3.10)

$$\sum_{l=1}^{4} \frac{n_s(\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t))}{2\pi} \in \mathbb{Z}, \quad t \in [0,1],$$

where tr_{n_s} is the normalized trace on M_{n_s}, $s = 1, 2, ..., F(2)$. In particular,

$$\begin{aligned} &\sum_{l=1}^{4} n_s(\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t)) \\ &= \sum_{l=1}^{4} n_s(\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{m(A)})(a_j^{(l,s)}(t')) \text{ for all } t, t'' \in [0,1]. \end{aligned} \tag{e 8.21}$$

Let $W_i = w_i \otimes 1_{M_{m(A)}}$, $i = 0, 1, ..., n$ and $W_e = w_e \otimes 1_{M_{m(A)}}$. Then, by (e 8.16), (e 8.18), the choice of $\mathcal{G}_5$, (e 8.16) again, (e 8.20), and the choices of ε_1 and γ_2,

$$\begin{aligned} &\|\pi_{t_i}(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle)W_i(\pi_{t_i}(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_i^* - \theta_j(t_i)\| \\ &< \varepsilon_1 + \|\pi_{t_i}(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle)W_i(\pi_{t_i}(\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j))W_i^* - \theta_j(t_i)\| \\ &< \varepsilon_1 + m(A)^2\varepsilon_1/16 + \|\pi_{t_i}(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle)\pi_{t_i}(\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j) - \theta_j(t_i)\| \\ &< \varepsilon_1 + m(A)^2\varepsilon_1/16 + \varepsilon_1 \\ &\quad + \|\pi_{t_i}(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle)\pi_{t_i}(\langle(\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle) - \theta_j(t_i)\| \\ &< \varepsilon_1 + m(A)^2\varepsilon_1/16 + \varepsilon_1 + \gamma_2 < 3m(A)^2\varepsilon_1 + \gamma_2 \ (\leq 1/32). \end{aligned} \tag{e 8.22}$$

Similarly (but using (e 8.19) instead of (e 8.18), we also have (with $\varphi_e = \pi_e \circ \varphi$)

$$\begin{aligned} &\|\langle(\varphi_e \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle W_e(\langle(\varphi_e \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_e^* - \pi_e(\theta_j)\| \\ &< 3m(A)^2\varepsilon_1 + \gamma_2 (\leq 1/32). \end{aligned} \tag{e 8.23}$$

It follows from (e 8.22) that, for $j = 1, ..., k(A)$ and $i = 0, ..., n$, there exist self-adjoint elements $b_{i,j} \in M_{m(A)}(F_2)$ and $b_{e,j} \in M_{m(A)}(F_1)$ such that

$$\begin{aligned} &\exp(\sqrt{-1}b_{i,j}) = \\ &\quad \theta_j(t_i)^*(\pi_{t_i}(\langle\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle)W_i(\pi_{t_i}(\langle\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_i^*, \end{aligned} \tag{e 8.24}$$

$$\begin{aligned} &\exp(\sqrt{-1}b_{e,j}) = \\ &\quad \pi_e(\theta_j)^*(\pi_e(\langle\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j^*)\rangle)W_e(\pi_e(\langle\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_e^*, \end{aligned} \tag{e 8.25}$$

and, for $j = 1, 2, ..., k(A)$ and $i = 0, 1, ..., n, e$.

$$\|b_{i,j}\| < 2\arcsin(3m(A)^2\varepsilon_1/2 + \gamma_2/2). \tag{e 8.26}$$

Note that (recall $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$)

$$(h_0 \otimes \mathrm{id}_{M(A)})(b_{e,j}) = b_{0,j} \text{ and } (h_1 \otimes \mathrm{id}_{M(A)})(b_{e,j}) = b_{n,j}. \tag{e 8.27}$$

Write

$$b_{i,j} = (b_{i,j}^{(1)}, b_{i,j}^{(2)}, ..., b_{i,j}^{F(2)}) \in M_{m(A)}(F_2)$$

and

$$b_{e,j} = (b_{e,j}^{(1)}, b_{e,j}^{(2)}, ..., b_{e,j}^{(F(1))}) \in M_{m(A)}(F_1).$$

Note also that, for $i = 0, 1, ..., n, e$,

$$\begin{aligned}(\pi_{t_i}(\langle\varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle))W_i(\pi_{t_i}(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)W_i^* \\ = \pi_{t_i}(\theta_j)\exp(\sqrt{-1}b_{i,j}),\end{aligned} \tag{e 8.28}$$

$1 \le j \le k(A)$. Then, for $s = 1, 2, ..., F(2)$, $j = 1, 2, ..., k(A)$, and $i = 0, 1, ..., n$,

$$\frac{n_s}{2\pi}(\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z}, \tag{e 8.29}$$

where tr_{n_s} is the normalized trace on M_{n_s}. We also have

$$\frac{m_s}{2\pi}(\mathrm{tr}_{m_s} \otimes \mathrm{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z}, \tag{e 8.30}$$

where tr_{m_s} is the normalized trace on M_{m_s}, $s = 1, 2, ..., F(1)$, $j = 1, 2, ..., k(A)$. Put

$$\lambda_{i,j}^{(s)} = \frac{n_s}{2\pi}(\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z}, \tag{e 8.31}$$

$s = 1, 2, ..., F(2)$, $j = 1, 2, ..., k(A)$, and $i = 0, 1, ..., n$, and put

$$\lambda_{e,j}^{(s)} = \frac{m_s}{2\pi}(\mathrm{tr}_{m_s} \otimes \mathrm{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z}, \tag{e 8.32}$$

$s = 1, 2, ..., F(1)$ and $j = 1, 2, ..., k(A)$. Set

$$\lambda_{i,j} = (\lambda_{i,j}^{(1)}, \lambda_{i,j}^{(2)}, ..., \lambda_{i,j}^{(F(2))}) \in \mathbb{Z}^{F(2)} \quad \text{and} \quad \lambda_{e,j} = (\lambda_{e,j}^{(1)}, \lambda_{e,j}^{(2)}, ..., \lambda_{e,j}^{(F(1))}) \in \mathbb{Z}^{F(1)}. \tag{e 8.33}$$

We have, by (e 8.26) and (e 8.31), that, for $1 \le s \le F(2)$, $1 \le j \le k(A)$, $0 \le i \le n$,

$$\begin{aligned}|\frac{\lambda_{i,j}^{(s)}}{n_s}| &< \frac{1}{2\pi}m(A)\|b_{i,j}^{(s)}\| < \frac{1}{\pi}m(A)\arcsin(3m(A)^2\varepsilon_1/2 + \gamma_2/2) \\ &< \frac{1}{2\pi\cos(3m(A)^2\varepsilon_1 + \gamma_2)}m(A)(3m(A)^2\varepsilon_1 + \gamma_2) \\ &< \frac{1}{2\pi\cos(1/32)}(1/64N_1 + 1/64N_1) < 1/4N_1.\end{aligned} \tag{e 8.34}$$

Similarly (using (e 8.26) and (e 8.32)),

$$|\frac{\lambda_{e,j}^{(s)}}{m_s}| < 1/4N_1, \;\; s = 1, 2, ..., F(1), \;\; j = 1, 2, ..., k(A). \tag{e 8.35}$$

Note that $K_1(A) = \mathbb{Z}^{k(A)} \oplus \mathrm{Tor}(K_1(A))$. Define $\alpha_i^{(0,1)} : K_1(A) \to \mathbb{Z}^{F(2)}$ by $\alpha_i^{(0,1)}|_{\mathrm{Tor}(K_1(A))} = 0$ and by sending $[g_j]$ to $\lambda_{i,j}$, $j = 1, 2, ..., k(A)$, $i = 0, 1, 2, ..., n$, and define $\alpha_e^{(0,1)} : K_1(A) \to \mathbb{Z}^{F(1)}$ by $\alpha_e^{(0,1)}|_{\mathrm{Tor}(K_1(A))} = 0$ and by sending $[g_j]$ to $\lambda_{e,j}$, $j = 1, 2, ..., k(A)$. We write $K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \boldsymbol{\beta}(K_1(A)))$ (see 2.14 for the definition of $\boldsymbol{\beta}$). Define $\alpha_i : K_*(A \otimes C(\mathbb{T})) \to K_*(F_2)$ as follows: On $K_0(A \otimes C(\mathbb{T}))$, define

$$\alpha_i|_{K_0(A)} = [\pi_i \circ \varphi]|_{K_0(A)}, \;\; \alpha_i|_{\boldsymbol{\beta}(K_1(A))} = \alpha_i \circ \boldsymbol{\beta}|_{K_1(A)} = \alpha_i^{(0,1)},$$

and on $K_1(A \otimes C(\mathbb{T}))$, define

$$\alpha_i|_{K_1(A \otimes C(\mathbb{T}))} = 0, \;\; i = 0, 1, 2, ..., n.$$

Also define $\alpha_e \in \mathrm{Hom}(K_*(A \otimes C(\mathbb{T})), K_*(F_1))$, by

$$\alpha_e|_{K_0(A)} = [\pi_e \circ \varphi]|_{K_0(A)}, \;\; \alpha_e|_{\boldsymbol{\beta}(K_1(A))} = \alpha_e \circ \boldsymbol{\beta}|_{K_1(A)} = \alpha_e^{(0,1)} \tag{e 8.36}$$

on $K_0(A \otimes C(\mathbb{T}))$, and $(\alpha_e)|_{K_1(A \otimes C(\mathbb{T}))} = 0$. Note that (see (e 8.27))

$$(h_0)_* \circ \alpha_e = \alpha_0 \;\text{ and }\; (h_1)_* \circ \alpha_e = \alpha_n. \tag{e 8.37}$$

Since $A \otimes C(\mathbb{T})$ satisfies the UCT, the map α_e can be lifted to an element of $KK(A \otimes C(\mathbb{T}), F_1)$ which will still be denoted by α_e. Then define

$$\alpha_0 = \alpha_e \times [h_0] \;\text{ and }\; \alpha_n = \alpha_e \times [h_1] \tag{e 8.38}$$

in $KK(A \otimes C(\mathbb{T}), F_2)$. For $i = 1, ..., n-1$, also pick a lifting of α_i in $KK(A \otimes C(\mathbb{T}), F_2)$, and still denote it by α_i. Combining (e 8.17) and (e 8.18), we compute that

$$\|(w_i^* w_{i+1})\pi_{t_i} \circ \varphi(g) - \pi_{t_i} \circ \varphi(g)(w_i^* w_{i+1})\| < \varepsilon_1/4 \;\text{ for all }\; g \in \mathcal{G}_u, \tag{e 8.39}$$

$i = 0, 1, ..., n-1$.

Recall that $V_{i,j} = \langle \pi_{t_i} \circ \varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j) \rangle$, $j = 1, 2, ..., k(A)$ and $i = 0, 1, 2, ..., n$, and

$$V_{i,j} = (V_{i,j,1}, V_{i,j,2}, ..., V_{i,j,F(2)}) \in M_{m(A)}(F_2), \;\; j = 1, 2, ..., k(A), \;\; i = 0, 1, 2, ..., n.$$

Similarly, write

$$W_i = (W_{i,1}, W_{i,2}, ..., W_{i,F(2)}) \in M_{m(A)}(F_2), \;\; i = 0, 1, 2, ..., n.$$

By (e 8.39) and (e 8.16), we have, for $i = 0, 1, ..., n$,

$$\|W_i^* W_{i+1} V_{i,j} - V_{i,j} W_i^* W_{i+1}\| < m(A)^2 \varepsilon/4 + 2\varepsilon_1 \le (9/4) m(A)^2 \varepsilon_1 < \delta_1, \tag{e 8.40}$$

and by (e 8.17) and (e 8.16), for any $j = 1, 2, ..., k(A)$, $i = 0, 1, ..., n-1$,

$$\|V_{i,j} - V_{i+1,j}\| < \varepsilon_1 + \varepsilon_1 m(A)^2/16 + \varepsilon_1 \le 3m^2(A)\delta_u/2 < \delta_1/12. \tag{e 8.41}$$

Thus, combining with (e 8.40),

$$\begin{aligned} & \|W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* - 1\| \\ \approx_{\delta_1/6} \;\; & \|W_i V_{i,j}^* W_i^* V_{i,j} V_{i,j}^* W_{i+1} V_{i,j} W_{i+1}^* - 1\| \tag{e 8.42} \\ = \;\; & \|W_i V_{i,j}^* W_i^* W_{i+1} V_{i,j} W_{i+1}^* - 1\| \approx_{\delta_1} 0. \tag{e 8.43} \end{aligned}$$

By (e 8.41), there is a continuous path $Z(t)$ of unitaries such that $Z(0) = V_{i,j}$, $Z(1) = V_{i+1,j}$, and

$$\|Z(t) - Z(1)\| < \delta_1/6, \quad t \in [0, 1]. \tag{e 8.44}$$

We also write

$$Z(t) = (Z_1(t), Z_2(t), ..., Z_{F(2)}(t)) \in F_2 \;\text{ and }\; t \in [0, 1].$$

We obtain a continuous path

$$W_i V_{i,j}^* W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^*$$

which is in $CU(M_{m(A)}(F_2))$ for all $t \in [0, 1]$, and by (e 8.44), and combining (e 8.42) and (e 8.43),

$$\|W_i V_{i,j}^* W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^* - 1\| < 2\delta_1/6 + \delta_1/16 + 3\delta_1 < 1/8$$

for $t \in [0, 1]$. It follows that the integer

$$(1/2\pi\sqrt{-1})(\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{M_{m(A)}})[\log(W_{i,s} V_{i,j,s}^* W_{i,s}^* V_{i,j,s} Z_s(t)^* W_{i+1,s} Z_s(t) W_{i+1,s}^*)]$$

is independent of $t \in [0, 1]$, $s = 1, 2, ..., F(2)$. In particular, with $T = (\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{Mm(A)})$,

$$\begin{aligned} & (1/2\pi\sqrt{-1}) T(\log(W_{i,s} V_{i,j,s}^* W_{i,s}^* W_{i+1,s} V_{i,j,s} W_{i+1,s}^*)) \tag{e 8.45} \\ & \quad = (1/2\pi\sqrt{-1}) T(\log(W_{i,s} V_{i,j,s}^* W_{i,s}^* V_{i,j} V_{i+1,j,s}^* W_{i+1,s} V_{i+1,j,s} W_{i+1,s}^*)) \end{aligned}$$

for $s = 1, 2, ..., F(2)$. One also has (by (e 8.24))

$$\begin{aligned} & W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* \\ & = (\theta_j(t_i)\exp(\sqrt{-1}b_{i,j}))^* \theta_j(t_{i+1})\exp(\sqrt{-1}b_{i+1,j}) \\ & = \exp(-\sqrt{-1}b_{i,j})\theta_j(t_i)^*\theta_j(t_{i+1})\exp(\sqrt{-1}b_{i+1,j}). \end{aligned} \tag{e 8.46}$$

Note that, by (e 8.20), (e 8.16), (e 8.17), and (e 8.20), for $t \in [t_i, t_{i+1}]$, $1 \le i \le n-1$, $1 \le j \le k(A)$,

$$\begin{aligned} & \|\theta_j(t_i)^*\theta_j(t) - 1\| \\ & \approx_{\gamma_2} \|\theta_j(t_i)^*(\pi_t(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle\langle(\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j)\rangle)) - 1\| \\ & \approx_{2\varepsilon_1} \|\theta_j(t_i)^*\pi_t((\varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)(\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j)) - 1\| \\ & \approx_{m(A)^2\varepsilon_1/8} \|\theta_j(t_i)^*\pi_{t_i}((\varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)(\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j)) - 1\| \\ & \approx_{2\varepsilon_1} \|\theta_j(t_i)^*\pi_{t_i}(\langle(\varphi \otimes \mathrm{id}_{M_{m(A)}}(g_j^*)\rangle\langle(\psi \otimes \mathrm{id}_{M_{m(A)}})(g_j))\rangle - 1\| \\ & \approx_{\gamma_2} \|\theta_j(t_i)^*\theta_j(t_i) - 1\| = 0. \end{aligned} \tag{e 8.47}$$

Note that $m(A)^2\varepsilon_1/8 + 2\varepsilon_1 + 2\gamma_2 < \delta_1 < 1/32$. By Lemma 3.5 of [82],

$$(\mathrm{tr}_{n_s} \otimes \mathrm{Tr}_{m(A)})(\log(\theta_{j,s}(t_i)^*\theta_{j,s}(t_{i+1}))) = 0. \tag{e 8.48}$$

It follows that (by the Exel formula (see 3.1 and 3.2 of [40] and 3.7 of [52]), using (e 8.40), (e 8.46), (e 8.46), (e 8.47), Lemma 2.11 of [73], (e 8.48), and the choice of δ_1), we have

$$\begin{aligned} & (t \otimes \mathrm{Tr}_{m(A)})(\mathrm{bott}_1(V_{i,j}, W_i^*W_{i+1})) \\ = \;& (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(V_{i,j}^* W_i^* W_{i+1} V_{i,j} W_{i+1}^* W_i)) \\ = \;& (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* W_{i+1} V_{i,j} W_{i+1}^*)) \\ = \;& (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^*)) \\ = \;& (\frac{1}{2\pi\sqrt{-1}})(t \otimes \mathrm{Tr}_{m(A)})(\log(\exp(-\sqrt{-1}b_{i,j})\theta_j(t_i)^*\theta_j(t_{i+1})\exp(\sqrt{-1}b_{i+1,j}))) \\ = \;& (\frac{1}{2\pi\sqrt{-1}})[(t \otimes \mathrm{Tr}_{m(A)})(-\sqrt{-1}b_{i,j}) + (t \otimes \mathrm{Tr}_{m(A)})(\log(\theta_j(t_i)^*\theta_j(t_{i+1})) \\ & \qquad\qquad + (t \otimes \mathrm{Tr}_{m(A)})(\sqrt{-1}b_{i+1,j})] \\ = \;& \frac{1}{2\pi}(t \otimes \mathrm{Tr}_{m(A)})(-b_{i,j} + b_{i+1,j}) \end{aligned}$$

for all $t \in T(F_2)$. By (e 8.31), (e 8.32), and (e 8.33), one has

$$\mathrm{bott}_1(V_{i,j}, W_i^* W_{i+1})) = -\lambda_{i,j} + \lambda_{i+1,j} \tag{e 8.49}$$

$j = 1, 2, ..., m(A)$, $i = 0, 1, ..., n-1$.

Note that φ is $\mathcal{G}_2$-δ_2-multiplicative, and, by (e 8.11), for $h \in \mathcal{H}_2' \cup \mathcal{H}_5' \subset \mathcal{H}_1$,

$$(\mathrm{tr} \circ \pi_{t_i})(\varphi(h)) \geq \Delta(\hat{h}) \text{ for all } \mathrm{tr} \in T(F_2) \tag{e 8.50}$$

$$\text{and } (\mathrm{tr}_e \circ \pi_e) \circ \varphi(h)) \geq \Delta(\hat{h}) \text{ for all } \mathrm{tr}_e \in T(F_1). \tag{e 8.51}$$

Recall that these inequalities imply that $n_s \geq 4N_1$ and $m_s \geq 4N_1$ (by the choice of $\mathcal{H}_5'$). Then, by (e 8.34),

$$\begin{aligned} n_s/N_1 \geq n_s/2N_1 + n_s/2N_1 &\geq 1 + \max\{|\lambda_{i,j}^{(s)}| : j = 1, 2, ..., k(A)\} \\ &\geq (1 + \max\{|\alpha_e(\boldsymbol{\beta}(g_j))| : j = 1, 2, ..., k(A)\}). \end{aligned} \tag{e 8.52}$$

Similarly (by (e 8.35)),

$$m_s/N_1 \geq (1 + \max\{|\alpha_e(\boldsymbol{\beta}(g_j))| : j = 1, 2, ..., k(A)\}). \tag{e 8.53}$$

Applying 7.3 (using (e 8.52), (e 8.53), (e 8.50) and (e 8.50)), we obtain unitaries $z_i \in F_2$, $i = 1, 2, ..., n-1$, and $z_e \in F_1$ such that

$$\|[z_i, \pi_{t_i} \circ \varphi(g)]\| < \delta_u \text{ and } \|[z_e, \pi_e \circ \varphi(g)]\| < \delta_u \text{ for all } g \in \mathcal{G}_u, \tag{e 8.54}$$

$$\mathrm{Bott}(z_i, \pi_{t_i} \circ \varphi) = \alpha_i, \text{ and } \mathrm{Bott}(z_e, \pi_e \circ \varphi) = \alpha_e. \tag{e 8.55}$$

Put $z_0 = h_0(z_e)$ and $z_n = h_1(z_e)$. Recall that $\pi_{t_0} \circ \varphi = h_0(\pi_e \circ \varphi)$ and $\pi_{t_n} \circ \varphi = h_1(\pi_e \circ \varphi)$. One verifies (by (e 8.38)) that

$$\mathrm{Bott}(z_0, \pi_{t_0} \circ \varphi) = \alpha_0 \text{ and } \mathrm{Bott}(z_n, \pi_{t_n} \circ \varphi) = \alpha_n. \tag{e 8.56}$$

Let $U_{i,i+1} = z_i w_i^* w_{i+1} z_{i+1}^*$, $i = 0, 1, 2, ..., n-1$. Then, by (e 8.54), (e 8.39), and (e 8.17), one has, for $i = 0, 1, 2, ..., n-1$,

$$\|[U_{i,i+1}, \pi_{t_i} \circ \varphi(g)]\| < 2\varepsilon_1 + 2\delta_u < \delta_1/2 \text{ for all } g \in \mathcal{G}_u. \tag{e 8.57}$$

Moreover, for $0 \leq i \leq n-1$, $1 \leq j \leq k(A)$, by (e 8.49) (and by (e 8.15), and (e 8.54), (e 8.40)),

$$\begin{aligned} \mathrm{bott}_1(U_{i,i+1}, \pi_{t_i} \circ \varphi)([g_j]) &= \mathrm{bott}_1(z_i, \pi_{t_i} \circ \varphi)([g_j]) + \mathrm{bott}_1(w_i^* w_{i+1}, \pi_{t_i} \circ \varphi)([g_j]) \\ &\quad + \mathrm{bott}_1(z_{i+1}^*, \pi_{t_i} \circ \varphi)([g_j]) \\ &= (\lambda_{i,j}) + (-\lambda_{i,j} + \lambda_{i+1,j}) + (-\lambda_{i+1,j}) = 0. \end{aligned}$$

Note that for any $x \in \bigoplus_{*=0,1}\bigoplus_{k=1}^{n_0} K_*(A\otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z})$, one has $Nx = 0$. Therefore,

(e 8.58)
$$\mathrm{Bott}((\underbrace{U_{i,i+1},...,U_{i,i+1}}_{N}), (\underbrace{\pi_{t_i}\circ\varphi,...,\pi_{t_i}\circ\varphi}_{N}))|_{\mathcal{P}} = N\mathrm{Bott}(U_{i,i+1},\pi_{t_i}\circ\varphi)|_{\mathcal{P}} = 0,$$

$i = 0,1,2,...,n-1$. Note that, by the assumption (e 8.11),

(e 8.59)
$$t_{n_s}\circ\pi_t\circ\varphi(h) \ge \Delta(\hat{h}) \text{ for all } h\in\mathcal{H}_1',$$

where t_{n_s} is the normalized trace on M_{n_s}, $1\le s\le F(2)$.

By Lemma 6.7 and our choice of $\mathcal{H}_1'$, δ_1, $\mathcal{G}_1$, $\mathcal{P}_0$, in view of (e 8.59), (e 8.57), and (e 8.58), there exists a continuous path of unitaries, $\{\tilde{U}_{i,i+1}(t) : t\in[t_i,t_{i+1}]\}\subset F_2\otimes M_N(\mathbb{C})$, such that

(e 8.60)
$$\tilde{U}_{i,i+1}(t_i) = 1_{F_2\otimes M_N(\mathbb{C})},\ \ \tilde{U}_{i,i+1}(t_{i+1}) = (z_iw_i^*w_{i+1}z_{i+1}^*)\otimes 1_{M_N(\mathbb{C})},$$

and

(e 8.61)
$$\|\tilde{U}_{i,i+1}(t)(\underbrace{\pi_{t_i}\circ\varphi(f),...,\pi_{t_i}\circ\varphi(f)}_{N})\tilde{U}_{i,i+1}(t)^* - (\underbrace{\pi_{t_i}\circ\varphi(f),...,\pi_{t_i}\circ\varphi(f)}_{N})\| < \varepsilon/32$$

for all $f\in\mathcal{F}$ and for all $t\in[t_i,t_{i+1}]$. Define $W = (W(t),\pi_e(W))\in C\otimes M_N$ by

(e 8.62)
$$W(t) = (w_iz_i^*\otimes 1_{M_N})\tilde{U}_{i,i+1}(t) \text{ for all } t\in[t_i,t_{i+1}],$$

$i = 0,1,...,n-1$, and $\pi_e(W) = w_ez_e^*\otimes 1_{M_N}$. Note that $W(t_i) = w_iz_i^*\otimes 1_{M_N}$, $i = 0,1,...,n$. Note also that

$$W(0) = w_0z_0^*\otimes 1_{M_N} = h_0(w_ez_e^*)\otimes 1_{M_N}$$

and

$$W(1) = w_nz_n^*\otimes 1_{M_N} = h_1(w_ez_e^*)\otimes 1_{M_N}.$$

So $W\in C\otimes M_N$. One then checks that, by (e 8.17), (e 8.61), (e 8.54), (e 8.18), and (e 8.17) again,

$$\begin{aligned}
&\|W(t)((\pi_t\circ\varphi)(f)\otimes 1_{M_N})W(t)^* - (\pi_t\circ\psi)(f)\otimes 1_{M_N}\| \\
&\quad < \|W(t)((\pi_t\circ\varphi)(f)\otimes 1_{M_N})W(t)^* - W(t)((\pi_{t_i}\circ\varphi)(f)\otimes 1_{M_N})W^*(t)\| \\
&\qquad +\|W(t)(\pi_{t_i}\circ\varphi)(f)W(t)^* - W(t_i)(\pi_{t_i}\circ\varphi)(f)W(t_i)^*\| \\
&\qquad +\|W(t_i)((\pi_{t_i}\circ\varphi)(f)\otimes 1_{M_N})W(t_i)^* - (w_i(\pi_{t_i}\circ\varphi)(f)w_i^*)\otimes 1_{M_N}\| \\
&\qquad +\|w_i(\pi_{t_i}\circ\varphi)(f)w_i^* - \pi_{t_i}\circ\psi(f)\| + \|\pi_{t_i}\circ\psi(f) - \pi_t\circ\varphi(f)\| \\
&\quad < \varepsilon_1/16 + \varepsilon/32 + \delta_u + \varepsilon_1/16 + \varepsilon_1/16 < \varepsilon
\end{aligned}$$

for all $f \in \mathcal{F}$ and for $t \in [t_i, t_{i+1}]$, $i = 0, 1, ..., n-1$.

Since $\lambda : C \to C([0,1], F_2)$ is assumed to be injective, this implies that

$$\|W(\varphi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \varepsilon \text{ for all } f \in \mathcal{F}. \tag{e 8.63}$$

□

REMARK 8.4. Although it will not be needed in this paper, it is perhaps worth pointing out that with some modification, the proof also works without assuming that $K_*(A)$ is finitely generated. In Theorem 8.3, the multiplicity N only depends on $\underline{K}(A)$ as $\underline{K}(A)$ is finitely generated. However, if $K_*(A)$ is not finitely generated, the multiplicity N then will depend on $\mathcal{F}$ and ε. On the other hand, if $K_1(A)$ is torsion free, or if $K_1(C) = 0$, then the multiplicity N can always be chosen to be 1. This also will not be needed here.

COROLLARY 8.5. *The statement of Theorem* 8.3 *holds if A is replaced by $M_m(A)$ for any integer $m \geq 1$.*

9. C^*-algebras in $\mathcal{B}_1$

DEFINITION 9.1. Let A be a unital simple C^*-algebra. We say $A \in \mathcal{B}_1$ if the following property holds: Let $\varepsilon > 0$, let $a \in A_+ \setminus \{0\}$, and let $\mathcal{F} \subset A$ be a finite subset. There exist a non-zero projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{C}$ with $1_C = p$ such that

$$\|xp - px\| < \varepsilon \text{ for all } x \in \mathcal{F},$$

$$\operatorname{dist}(pxp, C) < \varepsilon \text{ for all } x \in \mathcal{F}, \text{ and} \tag{e 9.1}$$

$$1 - p \lesssim a. \tag{e 9.2}$$

If C as above can always be chosen in $\mathcal{C}_0$, that is, with $K_1(C) = \{0\}$, then we say that $A \in \mathcal{B}_0$.

DEFINITION 9.2. Let A be a unital simple C^*-algebra. We say A has generalized tracial rank at most one, if the following property holds:

Let $\varepsilon > 0$, let $a \in A_+ \setminus \{0\}$ and let $\mathcal{F} \subset A$ be a finite subset. There exist a non-zero projection $p \in A$ and a unital C^*-subalgebra C which is a subhomogeneous C^*-algebra, with at most one dimensional spectrum, in particular, a finite dimensional C^*-algebra with $1_C = p$ such that

$$\|xp - px\| < \varepsilon \text{ for all } x \in \mathcal{F}, \tag{e 9.3}$$

$$\operatorname{dist}(pxp, C) < \varepsilon \text{ for all } x \in \mathcal{F}, \text{ and} \tag{e 9.4}$$

$$1 - p \lesssim a. \tag{e 9.5}$$

In this case, we write $gTR(A) \leq 1$.

REMARK 9.3. It follows from 3.21 that $gTR(A) \le 1$ if and only if $A \in \mathcal{B}_1$.

Let $\mathcal{D}$ be a class of unital C^*-algebras.

DEFINITION 9.4. Let A be a unital simple C^*-algebra. We say A is tracially approximately in $\mathcal{D}$, denoted by $A \in \mathrm{TA}\mathcal{D}$, if the following property holds:

For any $\varepsilon > 0$, any $a \in A_+ \setminus \{0\}$ and any finite subset $\mathcal{F} \subset A$, there exist a non-zero projection $p \in A$ and a unital C^*-subalgebra $C \in \mathcal{D}$, with $1_C = p$ such that

$$\|xp - px\| < \varepsilon \ \text{ for all } \ x \in \mathcal{F}, \tag{e 9.6}$$

$$\mathrm{dist}(pxp, C) < \varepsilon \ \text{ for all } \ x \in \mathcal{F}, \ \text{ and} \tag{e 9.7}$$

$$1 - p \lesssim a. \tag{e 9.8}$$

(see Definition 2.2 of [36]). Note that $\mathcal{B}_0 = \mathrm{TA}\mathcal{C}_0$ and $\mathcal{B}_1 = \mathrm{TA}\mathcal{C}$. If in the above definition, only (e 9.6) and (e 9.7) hold, then we say A has the property $(L_{\mathcal{D}})$.

The property $(L_{\mathcal{D}})$ is a generalization of Popa's property in Theorem 1.2 of [98] (also see Definition 1.2 of [11] and Definition 3.2 of [62]).

An earlier version of the following proposition first appeared in an unpublished paper of the second named author distributed in 1998 (see Corollary 6.4 of [62] and 5.1 of [71]).

PROPOSITION 9.5. *Let A be a unital simple C^*-algebra which has the property $(L_{\mathcal{D}})$. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist a projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{D}$ with $1_C = p$ such that*

$$\|[x, p]\| < \varepsilon \ \textit{ for all } \ f \in \mathcal{F}, \tag{e 9.9}$$

$$\mathrm{dist}(pxp, C) < \varepsilon, \ \textit{ and} \tag{e 9.10}$$

$$\|pxp\| \ge \|x\| - \varepsilon \ \textit{ for all } \ x \in \mathcal{F}. \tag{e 9.11}$$

PROOF. Fix $\varepsilon \in (0,1)$ and a finite subset $\mathcal{F} \subset A$. Without loss of generality, we may assume $1_A \in \mathcal{F}$. It follows from Proposition 2.2 of [3] that there is a unital separable simple C^*-subalgebra $B \subset A$ which contains $\mathcal{F}$. By Definition 9.4, there exist a sequence of C^*-subalgebra $C_n \in \mathcal{D}$ and a sequence of non-zero projections $p_n = 1_{C_n}$ such that

$$\lim_{n\to\infty} \|p_n b - bp_n\| = 0, \ \lim_{n\to\infty} \mathrm{dist}(p_n bp_n, C_n) = 0, \ \text{ and} \tag{e 9.12}$$

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \ \text{ for all } \ a, \in b \in B, \tag{e 9.13}$$

where $L_n : B \to p_n B p_n$ is defined by $L_n(b) = p_n b p_n$ for all $b \in B$. Consider the map $L : B \to \prod_{n=1}^{\infty} p_n B p_n$ which is a unital completely positive linear map.

Let $\pi : \prod_{n=1}^{\infty} p_nBp_n \to \prod_{n=1}^{\infty} p_nBp_n/\bigoplus_{n=1}^{\infty} p_nBp_n$ be the quotient map. Set $\varphi = \pi \circ L$. Then φ is a unital homomorphism. Since B is a simple, φ is an isometry. It follows that there exists a subsequence $\{n_k\}$ such that

$$\|p_{n_k}xp_{n_k}\| \geq \|x\| - \varepsilon \text{ for all } x \in \mathcal{F}. \tag{e 9.14}$$

Note that we have $\lim_{k\to\infty} \|p_{n_k}x - xp_{n_k}\| = 0$ and $\lim_{k\to\infty} \operatorname{dist}(p_{n_k}xp_{n_k}, C_{n_k}) = 0$ for all $x \in \mathcal{F}$. Choosing $p := p_{n_k}$ and $C = C_{n_k}$ for some sufficiently large k, the conclusion of the lemma holds.

□

THEOREM 9.6. *Let A be a unital simple separable C^*-algebra in* TA$\mathcal{D}$, *where $\mathcal{D}$ is a class of unital C^*-algebras. Then either A is locally approximated by subalgebras in $\mathcal{D}$, or A has the property (SP). In the case that $\mathcal{D}$ is a class of semiprojective C^*-algebras, then, when A does not have (SP), A is an inductive limit of C^*-algebras in $\mathcal{D}$ (with not necessarily injective maps).*

PROOF. This follows from Definition 9.1 immediately. Let $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n, ...$ be a sequence of increasing finite subsets of the unit ball of A whose union is dense in the unit ball. If A does not have property (SP), then there is a non-zero positive element $a \in A$ such that $\overline{aAa} \neq A$ and $\overline{aAa}$ has no non-zero projection. Then, for each $n \geq 1$, there is a projection $1_A - p_n \lesssim a$ and a C^*-subalgebra $C_n \in \mathcal{D}$ such that $1_{C_n} = p_n$ and

$$\|p_nx - xp_n\| < 1/n \text{ and } \operatorname{dist}(p_nxp_n, C_n) < 1/n \text{ for all } x \in \mathcal{F}_n. \tag{e 9.15}$$

Since $\overline{aAa}$ does not have any non-zero projection, one has $1_A - p_n = 0$. In other words, $1_A = p_n$ and

$$\operatorname{dist}(x, C_n) < 1/n \text{ for all } x \in \mathcal{F}_n,\ n = 1, 2, ..., \tag{e 9.16}$$

as asserted. In the case that the C^*-algebras in $\mathcal{D}$ are semiprojective, A is in fact an inductive limit of C^*-algebras in $\mathcal{D}$ (with not necessarily injective maps). □

THEOREM 9.7. *Let $A \in \mathcal{B}_1$. Then A has stable rank one.*

PROOF. This follows from Proposition 3.3 and Theorem 3.3 of [41] (see also 4.3 of [36]). □

LEMMA 9.8. *Let $\mathcal{D}$ be a family of unital separable C^*-algebras which are residually finite dimensional. Any unital separable simple C^*-algebra with property $(L_{\mathcal{D}})$ can be embedded in $\prod M_{r(n)}/\bigoplus M_{r(n)}$ for some sequence of integers $\{r(n)\}$.*

PROOF. Let A be a unital separable simple C^*-algebra with property $(L_{\mathcal{D}})$. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_i \subset \cdots$ be an increasing sequence of finite subsets of

A with union dense in A. Since A has property $(L_{\mathcal{D}})$, for each n, there are a projection $p_n \in A$ and $C_n \subset A$ with $1_{C_n} = p_n$ and $C_n \in \mathcal{D}$ such that

$$\|p_nf - fp_n\| < 1/2^{n+2},\ \|p_nfp_n\| \geq \|f\| - 1/2^{n+2},\ \text{and}\ p_nfp_n \in_{1/2^{n+2}} C_n \tag{e 9.17}$$

for all $f \in \mathcal{F}_n$. For each $a \in \mathcal{F}_n$, there exists $c(a) \in C_n$ such that $\|p_nap_n - c(a)\| < 1/2^{n+2}$. There are unital homomorphisms $\pi_n' : C_n \to B_n$, where B_n is a finite dimensional C^*-subalgebra such that

$$\begin{aligned} \|\pi_n'(c(a))\| &= \|c(a)\| \geq \|p_nap_n\| - 1/2^{n+2} \qquad \text{(e 9.18)} \\ &\geq \|a\| - (1/2^{n+2} + 1/2^{n+2}) = \|a\| - 1/2^{n+1} \\ &\text{for all } \in \mathcal{F}_n \quad n = 1, 2, ... \qquad \text{(e 9.19)} \end{aligned}$$

There is an integer $r(n) \geq 1$ such that B_n is unitally embedded into $M_{r(n)}$. Denote by $\pi_n : C_n \to M_{r(n)}$ the composition of π_n' and the embedding. Note $C_n \subset p_nAp_n$. Then there is a unital completely positive linear map $\Phi_n' : p_nAp_n \to M_{r(n)}$ such that

$$\Phi_n'|_{C_n} = \pi_n. \tag{e 9.20}$$

Define $\Phi_n : A \to M_{r(n)}$ by $\Phi_n(a) = \Phi_n'(p_nap_n)$ for all $a \in A$. It is a unital completely positive linear map. Moreover,

$$\|\Phi_n(p_nap_n) - \Phi_n(c(a))\| < 1/2^{n+1} \text{ for all } a \in \mathcal{F}_n, \tag{e 9.21}$$

$n = 1, 2, ...,$. Combining with (e 9.17), we obtain that

$$\|\Phi_n(f)\| \geq \|f\| - 1/2^n \text{ for all } f \in \mathcal{F}_n, \quad n = 1, 2, ... \tag{e 9.22}$$

Define $\Phi : A \to \prod_{n=1}^{\infty} M_{r(n)}$ by $\Phi(a) = \{\Phi_n(a)\}$ for all $a \in A$. Let

$$\Pi : \prod_{n=1}^{\infty} M_{r(n)} \to \prod_{n=1}^{\infty} M_{r(n)} / \bigoplus_{n=1}^{\infty} M_{r(n)}$$

be the quotient map. Put $\Psi = \Pi \circ \Phi$. One easily checks that Ψ is in fact a unital homomorphism. Since A is simple, Ψ is a monomorphism. □

Theorem 9.9. *Let $A \in \mathcal{B}_1$ (or $A \in \mathcal{B}_0$). Then, for any projection $p \in A$, one has $pAp \in \mathcal{B}_1$ (or $pAp \in \mathcal{B}_0$).*

Proof. Let us assume $p \neq 0$. Let $1/4 > \varepsilon > 0$, let $a \in (pAp)_+ \setminus \{0\}$, and let $\mathcal{F} \subset pAp$ be a finite subset. Without loss of generality, we may assume that

$\|x\| \le 1$ for all $x \in \mathcal{F}$. Since A is unital and simple, there are $x_1, x_2, ..., x_m \in A$ such that

$$\sum_{i=1}^{m} x_i^* p x_i = 1_A. \tag{e 9.23}$$

Since $x_i^* p x_i \le 1_A$, $\|px_i\| \le 1$. By replacing x_i by px_i, we may assume that $\|x_i\| \le 1$. Put $\mathcal{F}_1 = \{p, x_1, x_2, ..., x_m, x_1^*, x_2^*, ..., x_m^*\} \cup \mathcal{F}$. Since $A \in \mathcal{B}_1$, there is a projection $e \in A$ and a unital C^*-subalgebra $C_1 \in \mathcal{C}$ (or $C_1 \in \mathcal{C}_0$) with $1_{C_1} = e$ such that

$$\begin{aligned} \|xe - ex\| &< \varepsilon/64(m+1) \text{ for all } x \in \mathcal{F}_1, && \text{(e 9.24)}\\ \operatorname{dist}(exe, C_1) &< \varepsilon/64(m+1) \text{ for all } x \in \mathcal{F}_1, \text{ and} && \text{(e 9.25)}\\ 1 - e &\lesssim a. && \text{(e 9.26)} \end{aligned}$$

Put $\eta = \varepsilon/64(m+1)$. Then $0 < \eta < 1/2^8$. Since $p \in \mathcal{F}_1$, we have $\|epe - (epe)^2\| < \eta$ and $\|pep - (pep)^2\| < \eta$. Moreover, there is $c(p) \in (C_1)_{s.a.}$ such that

$$\|epe - c(p)\| < \eta. \tag{e 9.27}$$

One estimates that, since $0 < \eta < 1/2^8$,

$$\begin{aligned} &\sqrt{1-4\eta} > 1 - 2\eta - 4\eta^2 = 1 - (2+4\eta)\eta \text{ and } \sqrt{1-4\eta} < 1 - 2\eta,\\ &\text{or } (1 - \sqrt{1-4\eta})/2 < (1+2\eta)\eta \text{ and } (1+\sqrt{1-4\eta})/2 > 1 - \eta. \end{aligned} \tag{e 9.28}$$

One then computes $\operatorname{sp}(epe)$, $\operatorname{sp}(pep) \subset [0, (1+2\eta)\eta] \sqcup [1-\eta, 1]$. By the functional calculus, one obtains a projection $q_1 \in pAp$ such that

$$\|q_1 - pep\| < (1+2\eta)\eta. \tag{e 9.29}$$

If $\lambda \in ((1+2\eta)\eta, 1-\eta)$, then

$$\begin{aligned} &\|e - (\lambda e - epe)^{-1}(\lambda e - c(p))\|\\ \le\ & \|(\lambda e - epe)^{-1}\|\|(\lambda e - epe) - (\lambda e - c(p)\| && \text{(e 9.30)}\\ <\ & \frac{\eta}{\min\{\lambda - (1+2\eta)\eta, (1-\eta) - \lambda\}}. && \text{(e 9.31)} \end{aligned}$$

It follows that $(\lambda e - epe)^{-1}(\lambda e - c(p)$ is invertible in eAe when the expression in (e 9.31) is less than 1 in which case $\lambda e - c(p)$ is invertible in eAe (or in C_1). Thus,

$$\operatorname{sp}(c(p)) \subset [-\eta, (2+2\eta)\eta] \sqcup [1-2\eta, 1+\eta]. \tag{e 9.32}$$

Consequently (by the functional calculus), there is a projection $q \in C_1$ such that

$$\|epe - q\| < (2+2\eta)\eta + \eta = (3+2\eta)\eta. \tag{e 9.33}$$

Moreover, there are $y_1, y_2, ..., y_m \in C_1$ such that $\|y_i - ex_ie\| < \eta$. Then

$$y_i^*qy_i \approx_{\eta(1+\eta)} y_i^*qex_ie \approx_\eta ex_i^*eqex_ie \approx_{(3+2\eta)\eta} ex_i^*epex_ie \approx_{2\eta} ex_i^*px_ie. \tag{e 9.34}$$

Therefore (by also (e 9.23)),

$$\|\sum_{i=1}^m y_i^*qy_i - e\| < m(\eta(1+\eta) + (3+2\eta)\eta + 3\eta) = m\eta(7+3\eta) < \varepsilon < 1/4. \tag{e 9.35}$$

Therefore q is full in C_1. It follows from 3.19 that $qC_1q \in \mathcal{C}$ (or $qC_1q \in \mathcal{C}_0$). Note that

$$\|pep - epe\| \le \|pep - epep\| + \|epep - epe\| < 2\eta. \tag{e 9.36}$$

From (e 9.29), (e 9.36), and (e 9.33),

$$\begin{aligned} \|q_1 - q\| &< \|q_1 - pep\| + \|pep - epe\| + \|epe - q\| \\ &< (1+2\eta)\eta + 2\eta + (3+2\eta)\eta = (6+4\eta)\eta < 1. \end{aligned} \tag{e 9.37}$$

Hence, there is a unitary $u \in A$ such that $u^*qu = q_1 \le p$. Put $C = u^*qC_1qu$. Then $C \in \mathcal{C}$ (or $C \in \mathcal{C}_0$) and $1_C = q_1$. We also have, by (e 9.29) and (e 9.33)),

$$\|epe - q_1\| < (1+2\eta)\eta + (3+2\eta)\eta = (4+4\eta)\eta. \tag{e 9.38}$$

If $x \in \mathcal{F}$, then

$$\|q_1x - xq_1\| \le 2\|(q_1 - epe)x\| + \|epex - xepe\| \tag{e 9.39}$$

$$< 2(4+4\eta)\eta + 3\eta = (11+8\eta)\eta < \varepsilon \tag{e 9.40}$$

$$\text{for all } x \in \mathcal{F}.$$

Similarly, we estimate that

$$\mathrm{dist}(q_1xq_1, C) < \varepsilon \text{ for all } x \in \mathcal{F}. \tag{e 9.41}$$

We also have (by (e 9.29))

$$\|(p - q_1) - (p - pep)\| = \|q_1 - pep\| < (1+2\eta)\eta. \tag{e 9.42}$$

Put $\sigma = (1+2\eta)\eta < 1/16$. Let $f_\sigma(t) \in C_0((0,\infty))$ be as in 2.5. Then, by 2.2 of [104],

$$p - q_1 = f_\sigma(p - q_1) \lesssim p - pep \lesssim 1 - e \lesssim a. \tag{e 9.43}$$

This shows that $pAp \in \mathcal{B}_1$. □

PROPOSITION 9.10. *Let $\mathcal{D}$ denote the class of unital separable amenable C^*-algebras with faithful tracial states. Let A be a non-zero unital simple separable C^*-algebra which is $TA\mathcal{D}$. Then $QT(A) = T(A) \neq \varnothing$.*

PROOF. One may assume that A is infinite dimensional. Since A is a unital infinite dimensional simple C^*-algebra, there are n mutually orthogonal non-zero positive elements, for any integer $n \geq 1$ (see, for example, 1.11.45 of [63]). By repeatedly applying Lemma 3.5.4 of [63] (see also Lemma 2.3 of [57]), one finds a sequence of positive elements $\{b_n\}$ which has the following property: $b_{n+1} \lesssim b_{n,1}$, where $b_{n,1}, b_{n,2}, ..., b_{n,n}$ are mutually orthogonal non-zero positive elements in $\overline{b_nAb_n}$ such that $b_nb_{n,i} = b_{n,i}b_n = b_{n,i}$, $i = 1, 2, ..., n$, and $b_{n,i} \sim b_{n,1}$, $1 \leq i \leq n$, $n = 1, 2, ...$ Note that

$$\lim_{n\to\infty} \sup\{\tau(b_n) : \tau \in QT(A)\} = 0. \tag{e 9.44}$$

One obtains two sequences of unital C^*-subalgebras $A_{0,n} := e_nAe_n$, D_n of A, where $D_n \in \mathcal{D}$ with $1_{D_n} = (1 - e_n)$, two sequences of unital completely positive linear maps $\varphi_{0,n} : A \to A_{0,n}$ (defined by $\varphi_{0,n}(a) = e_nae_n$ for all $a \in A$) and $\varphi_{1,n} : A \to D_n$ with $A_{0,n} \perp D_n$ satisfying the following conditions:

$$\lim_{n\to\infty} \|\varphi_{i,n}(ab) - \varphi_{i,n}(a)\varphi_{i,n}(b)\| = 0 \ \text{ for all } \ a, b \in A, \tag{e 9.45}$$

$$\lim_{n\to\infty} \|a - (\varphi_{0,n} + \varphi_{1,n})(a)\| = 0 \ \text{ for all } \ a \in A, \tag{e 9.46}$$

$$(1 - e_n) \lesssim b_n, \text{ and} \tag{e 9.47}$$

$$\lim_{n\to\infty} \|\varphi_{1,n}(x)\| = \|x\| \ \text{ for all } \ x \in A. \tag{e 9.48}$$

Since quasitraces are norm continuous (Corollary II 2.5 of [5]), by (e 9.46),

$$\lim_{n\to\infty} (\sup|\tau(a) - \tau((\varphi_{0,n} + \varphi_{1,n})(a))| : \tau \in QT(A)\}) = 0 \ \text{ for all } \ a \in A. \tag{e 9.49}$$

Since $\varphi_{0,n}(a)\varphi_{1,n}(a) = \varphi_{1,n}(a)\varphi_{0,n}(a) = 0$, for any $\tau \in QT(A)$ we have

$$\tau((\varphi_{0,n} + \varphi_{1,n})(a)) = \tau(\varphi_{0,n}(a)) + \tau(\varphi_{1,n}(a)) \ \text{ for all } \ a \in A. \tag{e 9.50}$$

Note that, by (e 9.47) and (e 9.44),

$$\lim_{n\to\infty} \sup\{\tau(\varphi_{0,n}(a)) : \tau \in QT(A)\} = 0 \ \text{ for all } \ a \in A. \tag{e 9.51}$$

Therefore

$$\lim_{n\to\infty} (\sup\{|\tau(a) - \tau \circ \varphi_{1,n}(a)| : \tau \in \mathrm{QT}(A)\}) = 0 \ \text{ for all } \ a \in A. \tag{e 9.52}$$

It follows $\lim_{n\to\infty}\|\tau\circ\varphi_{1,n}\|=\|\tau\|=1$ for all $\tau\in \mathrm{QT}(A)$.

For any $\tau\in QT(A)$, let $t_n=(\|\tau\circ\varphi_{1,n}\|^{-1})\tau\circ\varphi_{1,n}$. Note that $t_n|_{D_n}$ is a tracial state. Therefore t_n is a state of A. Let t_0 be a weak* limit of $\{t_n\}$. Then, as A is unital, t_0 is a state of A.

It follows from (e 9.45) and the fact that $t_n|_{D_n}$ is a tracial state that t_0 is a trace. Then, by (e 9.52), for every $\tau\in \mathrm{QT}(A)$,

$$\tau(a+b)=\tau(a)+\tau(b)\ \text{ for all }\ a,b\in A.$$

It follows that τ is a trace. Therefore $QT(A)=T(A)$.

To see that $T(A)\neq\varnothing$, let $\tau_n\in T(D_n)$ be a faithful tracial state, $n=1,2,....$ Define $t_n=(\|\tau_n\circ\varphi_{1,n}\|^{-1})\tau_n\circ\varphi_{1,n}$. Let t_0 be a weak * limit of $\{t_n\}$. As above, t_0 is a tracial state of A.

□

THEOREM 9.11. *Let $\mathcal{D}$ be a class of unital C^*-algebras which is closed under tensor products with a finite dimensional C^*-algebra and which has the strict comparison property for positive elements (see* 2.6*). Let A be a unital simple separable C^*-algebra in the class* $\mathrm{TA}\mathcal{D}$*. Then A has strict comparison for positive elements. In particular, if $A\in\mathcal{B}_1$, then A has strict comparison for positive elements and $K_0(A)$ is weakly unperforated.*

PROOF. Note that from 9.10, we have $QT(A)=T(A)\neq\varnothing$. By a result of Rørdam (see, for example, Corollary 4.6 of [107]; note also that the exactness is only used to get $QT(A)=T(A)\neq\varnothing$ there), to show that A has strict comparison for positive elements, it is enough to show that $W(A)$ is almost unperforated, i.e., for any positive elements a,b in a matrix algebra over A, if $(n+1)[a]\le n[b]$ for some $n\in\mathbb{N}$, then $[a]\le[b]$.

Let a,b be such positive elements. Since any matrix algebra over A is still in $\mathrm{TA}\mathcal{D}$, let us assume that $a,b\in A$.

First we consider the case that A does not have (SP) property. In this case, by the proof of 9.6, $A=\overline{\bigcup_{n=1}^{\infty}A_n}$, where $A_n\in\mathcal{D}$ ($\{A_n\}$ may not be increasing).

Without loss of generality, we may assume that $0\le a,\ b\le 1$. Let $\varepsilon>0$. It follows from an argument of Rørdam (see Lemma 5.6 of [94]) that there exist an integer $m\ge 1$, and positive elements $a',b'\in A_m$ such that

$$\|a'-a\|<\varepsilon/2,\ \ \|b'-b\|<\varepsilon/2,\ \ b'\lesssim b\ \text{ and} \tag{e 9.53}$$

$$\mathrm{diag}(\overbrace{f_{\varepsilon/2}(a'),f_{\varepsilon/2}(a'),...,f_{\varepsilon/2}(a')}^{n+1})\lesssim \mathrm{diag}(\overbrace{b',b',...,b'}^{n})\ \text{ in }\ A_m.$$

Since A_m has strict comparison (see part (b) of Theorem 3.18), one has

$$f_{\varepsilon/2}(a')\lesssim b'\ \text{ in }\ A_m. \tag{e 9.54}$$

Note that $\|a-(a'-\varepsilon/2)_+\|\le\|a-a'\|+\|a'-(a'-\varepsilon/2)\|<\varepsilon$. It follows, on using 2.2 of [104], that

$$f_{2\varepsilon}(a)\lesssim(a'-\varepsilon/2)_+\lesssim f_{\varepsilon/2}(a')\lesssim b'\lesssim b \tag{e 9.55}$$

for every $\varepsilon > 0$. It follows that $a \lesssim b$.

Now we assume that A has (SP). Let $1/4 > \varepsilon > 0$. We may further assume that $\|b\| = 1$. Since A has (SP) and is simple, by Lemma 3.5.6 and Lemma 3.5.7 of [63] (also see Theorem I of [120]), there are mutually orthogonal and mutually equivalent non-zero projections $e_1, e_2, ..., e_{n+1} \in \overline{f_{3/4}(b)Af_{3/4}(b)}$. Put $E = e_1 + e_2 + \cdots + e_{n+1}$. By 2.4 of [104], we also have that

$$(n+1)[f_{\varepsilon/4}(a)] \le n[f_\delta(b)] \tag{e 9.56}$$

for some $\varepsilon > \delta > 0$. Put $0 < \eta < \min\{\varepsilon/4, \delta/4, 1/8\}$. It follows from Definition 9.1 that there are a C^*-subalgebra $C = pAp \oplus S$ with $S \in \mathcal{D}$ and $a', b', E', e_i' \in C$ $(i = 1, 2, ..., n+1)$ such that $0 \le a', b' \le 1$ and E', e_i' are projections in C,

$$||a - a'|| < \eta, \ b' \lesssim f_\delta(b), \|f_{1/2}(b')E' - E'\| < \eta, \tag{e 9.57}$$

$$E' = \sum_{i=1}^{n+1} e_i', \ \ \|e_i - e_i'\| < \eta \ \text{ and } \ \|E - E'\| < \eta < 1, \tag{e 9.58}$$

and

$$\operatorname{diag}(\overbrace{f_{\varepsilon/2}(a'), f_{\varepsilon/2}(a'), ..., f_{\varepsilon/2}(a')}^{n+1}) \lesssim \operatorname{diag}(\overbrace{b', b', ..., b'}^{n}) \ \text{ in } C \tag{e 9.59}$$

(see Lemma 5.6 of [94]). Moreover, the projection p can be chosen so that $p \lesssim e_1$. From (e 9.57), there is a projection e_i'', $E'' \in \overline{f_{1/2}(b')Cf_{1/2}(b')}$ $(i = 1, 2, ..., n+1)$ such that $\|E' - E''\| < 2\eta$, $\|e_i'' - e_i'\| < 2\eta$, $i = 1, 2, ..., n+1$, and $E'' = \sum_{i=1}^{n+1} e_i''$ (we also assume that $e_1'', e_2'', ..., e_{n+1}''$ are mutually orthogonal). Note that e_i' and e_i'' are equivalent. Choose a function $g \in C_0((0,1])_+$ with $g \le 1$ such that $g(b')f_{1/2}(b') = f_{1/2}(b')$ and $[g(b')] = [b']$ in $W(C)$. In particular, $g(b')E'' = E''$.

Write

$$a' = a_0' \oplus a_1', \quad g(b') = b_0' \oplus b_1', \ e_i'' = e_{i,0} \oplus e_{i,1}, \text{ and } \quad E'' = E_0' \oplus E_1'$$

with $a_0', b_0', E_0', e_{i,0} \in pAp$ and $a_1', b_1', e_{i,1}, E_1' \in S$, $i = 1, 2, ..., n+1$. Note that $E_1'b_1' = b_1'E_1' = E_1'$. This, in particular, implies that

$$\tau(b_1') \ge (n+1)\tau(e_{1,1}) \ \text{ for all } \ \tau \in T(S). \tag{e 9.60}$$

It follows from (e 9.59) that

$$d_\tau(f_{\varepsilon/2}(a_1')) \le \frac{n}{n+1} d_\tau(b_1'), \ \text{ for all } \ \tau \in \mathrm{T}(S).$$

Note that $(b_1' - e_{1,1})e_{1,1} = 0$ and $b_1' = (b_1' - e_{1,1}) + e_{1,1}$. For all $\tau \in T(S)$,

$$d_\tau((b_1' - e_{1,1})) = d_\tau(b_1') - \tau(e_{1,1}) > d_\tau(b_1') - \frac{1}{n+1} d_\tau(b_1') \ge d_\tau(f_{\varepsilon/2}(a_1'))).$$

Since S has the strict comparison, one has

$$f_{\varepsilon/2}(a_1') \lesssim (b_1' - e_{1,1}).$$

Just as in the calculation of (e 9.55), $f_\varepsilon(a) \lesssim (a' - \varepsilon/4)_+ \sim f_{\varepsilon/2}(a')$ as $\eta < \varepsilon/4$. Consequently,

$$f_\varepsilon(a) \lesssim f_{\varepsilon/2}(a') \lesssim p \oplus f_{\varepsilon/2}(a_1') \lesssim p \oplus (b_1' - e_{1,1}) \tag{e 9.61}$$

$$\lesssim e_1 \oplus (b_1' - e_{1,1}) \lesssim e_1 \oplus (b_1' - e_{1,1}) + (b_0' - e_{1,0}) \tag{e 9.62}$$

$$\sim e_1'' \oplus (g(b') - e_1'') \sim g(b') \sim b' \lesssim b. \tag{e 9.63}$$

Since ε is arbitrary, one has that $a \lesssim b$.

Hence one always has that $a \lesssim b$, and therefore $W(A)$ is almost unperforated.

□

The following fact is known to experts. We include it here for the reader's convenience.

LEMMA 9.12. *Let A be a unital simple C^*-algebra with $T(A) \neq \varnothing$. Let $a, b \in A_+$ and let $1 > \varepsilon > 0$ such that $|\tau(a) - \tau(b)| < \varepsilon$ for all $\tau \in T(A)$. Then there are $x_1, x_2, ..., x_n \in A$ such that*

$$\|\sum_{i=1}^n x_i^* x_i - a\| < 2\varepsilon \text{ and } \|\sum_{i=1}^n x_i x_i^* - b\| < \varepsilon. \tag{e 9.64}$$

PROOF. This follows from results in [17]. Let $\delta = \max\{|\tau(a) - \tau(b)| : \tau \in T(A)\}$. Then $0 \le \delta < \varepsilon$. Let $\eta = \varepsilon - \delta$. By Theorem 9.2 of [71], there exists $c \in A_{s.a.}$ with $\|c\| < \delta + \eta/4$ such that $\tau(c) = \tau(b) - \tau(a)$ for all $\tau \in T(A)$. Consider $a_1 = a + c + \|c\|$ and $b_1 = b + \|c\|$. Note that $a_1 \ge 0$. Then $\tau(a_1) = \tau(b_1)$ for all $\tau \in T(A)$. It follows from (iii) of Theorem 2.9 of [17] that $a_1 - b_1 \in A_0$ (in the notation of [17]). It follows from Theorem 5.2 of [17] that $a_1 \sim b_1$ (in the notation of [17]; see the lower half of page136 of [17]). Thus, there are $x_1, x_2, ..., x_n \in A$ such that

$$\|a_1 - \sum_{i=1}^n x_i^* x_i\| < \eta/4 \text{ and } \|b_1 - \sum_{i=1}^n x_i x_i^*\| < \eta/4.$$

It follows that

$$\|a - \sum_{i=1}^n x_i^* x_i\| \le \eta/4 + \|a + \|a\|\| \le \eta/4 + 2\delta < 2\varepsilon \text{ and}$$

$$\|b - \sum_{i=1}^n x_i x_i^*\| \le \eta/4 + \|a\| \le \eta/4 + \delta < \varepsilon. \tag{e 9.65}$$

□

Lemma 9.13. *Let $\mathcal{D}$ be a class of unital amenable C^*-algebras, let A be a separable unital C^*-algebra which is $TA\mathcal{D}$, and let C be a unital amenable C^*-algebra.*

Let $\mathcal{F}, \mathcal{G} \subset C$ be finite subsets, let $\varepsilon > 0$ and $\delta > 0$ be positive numbers. Let $\mathcal{H} \subset C_+^{\mathbf{1}}$ be a finite subset, and let $T : C_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ and $N : C_+ \setminus \{0\} \to \mathbb{N}$ be maps. Let $\Delta : C_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\mathcal{H}_1 \subset C_+^{\mathbf{1}} \setminus \{0\}$, $\mathcal{H}_2 \subset C_+$ and $\mathcal{U} \subset U(M_k(C))/CU(M_k(C))$ (for some $k \geq 1$) be finite subsets. Let $\sigma_1 > 0$ and $\sigma_2 > 0$ be constants. Let $\varphi, \psi : C \to A$ be unital $\mathcal{G}$-δ-multiplicative completely positive linear maps such that

(1) *φ and ψ are $T \times N$-$\mathcal{H}$-full (see the definition* 2.23*),*
(2) *$\tau \circ \varphi(c) > \Delta(\hat{c})$ and $\tau \circ \psi(c) > \Delta(\hat{c})$ for any $\tau \in T(A)$ and for any $c \in \mathcal{H}_1$,*
(3) *$|\tau \circ \varphi(c) - \tau \circ \psi(c)| < \sigma_1$ for any $\tau \in T(A)$ and any $c \in \mathcal{H}_2$,*
(4) *$\mathrm{dist}(\varphi^{\ddagger}(u), \psi^{\ddagger}(u)) < \sigma_2$ for any $u \in \mathcal{U}$.*

Then, for any finite subset $\mathcal{F}' \subset A$ and $\varepsilon' > 0$, there exists a C^*subalgebra $D \subset A$ with $D \in \mathcal{D}$ such that if $p = 1_D$, then, for any $a \in \mathcal{F}'$,

(a) *$\|pa - ap\| < \varepsilon'$,*
(b) *$pap \in_{\varepsilon'} D$, and*
(c) *$\tau(1-p) < \varepsilon'$, for any $\tau \in T(A)$.*

There are also a (completely positive) linear map $j_0 : A \to (1-p)A(1-p)$ and a unital completely positive linear map . $j_1 : A \to D$ such that

$$j_0(a) = (1-p)a(1-p) \ \text{ for all } \ a \in A \ \text{ and}$$
$$\|j_1(a) - pap\| < 3\varepsilon' \ \text{ for all } \ a \in \mathcal{F}.$$

Moreover, define

$$\varphi_0 = j_0 \circ \varphi, \ \psi_0 = j_0 \circ \psi, \varphi_1 = j_1 \circ \varphi \quad \text{and} \quad \psi_1 = j_1 \circ \psi.$$

With a sufficiently large $\mathcal{F}'$ and small enough ε', one has that φ_0, ψ_0, φ_1 and ψ_1 are $\mathcal{G}$-2δ-multiplicative and

(5) *$\|\varphi(c) - (\varphi_0(c) \oplus \varphi_1(c))\| < \varepsilon$ and $\|\psi(c) - (\psi_0(c) \oplus \psi_1(c))\| < \varepsilon$, for any $c \in \mathcal{F}$,*
(6) *φ_0, ψ_0 and φ_1, ψ_1 are $2T \times N$-$\mathcal{H}$-full,*
(7) *$\tau \circ \varphi_1(c) > \Delta(\hat{c})/2$ and $\tau \circ \psi_1(c) > \Delta(\hat{c})/2$ for any $c \in \mathcal{H}_1$ and for any $\tau \in T(D)$,*
(8) *$|\tau \circ \varphi_1(c) - \tau \circ \psi_1(c)| < 3\sigma_1$ for any $\tau \in \mathrm{T}(D)$ and any $c \in \mathcal{H}_2$, and*
(9) *$\mathrm{dist}(\varphi_i^{\ddagger}(u), \psi_i^{\ddagger}(u)) < 2\sigma_2$ for any $u \in \mathcal{U}$, $i = 0, 1$.*

If, furthermore, $\mathcal{P} \subset \underline{K}(C)$ is a finite subset and $[L]|_{\mathcal{P}}$ is well defined for any $\mathcal{G}$-2δ-multiplicative contractive completely positive linear map L, and $[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}$, then, we may require, with possibly smaller ε' and larger $\mathcal{F}'$, that

(10) $[\varphi_i]|_{\mathcal{P}} = [\psi_i]|_{\mathcal{P}}$, $i = 0, 1$.

PROOF. Without loss of generality, one may assume that each element of $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}_2$, or $\mathcal{F}'$ has norm at most one and that $1_A \in \mathcal{F}'$.

Since φ and ψ are $T \times N$-$\mathcal{H}$-full, for each $h \in \mathcal{H}$, there are $a_{1,h}, ..., a_{N(h),h}$ and $b_{1,h}, ..., b_{N(h),h}$ in A with $\|a_{i,h}\|, \|b_{i,h}\| \le T(h)$ such that

$$\sum_{i=1}^{N(h)} a_{i,h}^*\varphi(h)a_{i,h} = 1_A \quad \text{and} \quad \sum_{i=1}^{N(h)} b_{i,h}^*\psi(h)b_{i,h} = 1_A. \tag{e 9.66}$$

Put $d_0 = \min\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}$. By (9.13), It follows from 9.12 that there are, for each $c \in \mathcal{H}_2$, $x_{1,c}, x_{2,c}, ..., x_{t(c),c} \in A$ such that

$$\|\sum_{i=1}^{t(c)} x_{i,c}^*x_{i,c} - \varphi(c)\| \le \sigma_3 \ \text{and}\ \|\sum_{i=1}^{t(c)} x_{i,c}x_{i,c}^* - \psi(c)\| \le \sigma_3 \tag{e 9.67}$$

for some $0 < \sigma_3 < 6\sigma_1/5$. Let

$$t(\mathcal{H}_2) = \max\{(\|x_{i,c}\| + 1)(t(c) + 1) : 1 \le i \le t(c) : c \in \mathcal{H}_2\}.$$

For the given finite subset $\mathcal{F}' \subset A$, and given $\varepsilon' > 0$, since A can be tracially approximated by the C^*-algebras in the class $\mathcal{D}$, there exists a C^*-subalgebra $D \subset A$ with $D \in \mathcal{D}$ such that if $p = 1_D$, then, for any $a \in \mathcal{F}'$,

(i) $\|pa - ap\| < \varepsilon'$,
(ii) $pap \in_{\varepsilon'} D$, and
(iii) $\tau(1-p) < \varepsilon'$, for any $\tau \in T(A)$.

On the way to making $\mathcal{F}'$ large and ε' small, we may assume that $\mathcal{F}'$ contains $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, $\varphi(\mathcal{G} \cup \mathcal{F})$, $\psi(\mathcal{G} \cup \mathcal{F})$, $\varphi(\mathcal{H})$, $\psi(\mathcal{H})$, $\mathcal{H}_1$, $\mathcal{H}_2$, $x_{i,c}, x_{i,c}^*$, $i = 1, 2, ..., t(c)$ and $c \in \mathcal{H}_2$, as well as $a_{i,h}, a_{i,h}^*, b_{i,h}, b_{i,h}^*$, $i = 1, 2, ..., N(h)$ and $h \in \mathcal{H}$, and

$$\varepsilon' < \frac{\min\{\min\{1/64(T(h)+1)(N(h)+1) : h \in \mathcal{H}\}, \varepsilon, \delta, d_0, \sigma_1, \sigma_2, (2\sigma_1 - \sigma_3)\}}{64(t(\mathcal{H}_2)+1)^2}.$$

For each $a \in \mathcal{F}'$, choose $d_a \in D$ such that $\|pap - d_a\| < \varepsilon'$ (choose $d_{1_A} = 1_D$). Consider the finite subset $\{d_a d_b : \ a, b \in \mathcal{F}'\} \subset D$. Since D is an amenable C^*-subalgebra of pAp, by 2.3.13 of [63] (with $D = B = C$ and $\varphi = \mathrm{id}_D$), there is unital completely positive linear map $L : pAp \to D$ such that

$$\|L(d_a d_b) - d_a d_b\| < \varepsilon', \quad a, b \in \mathcal{F}'.$$

Define $j_1 : A \to D$ by $j_1(a) = L(pap)$. Then, for any $a \in \mathcal{F}'$, one has

$$\|j_1(a) - pap\| \ = \ \|L(pap) - pap\| = \|L(d_a) - d_a\| + 2\varepsilon' = 2\varepsilon'. \tag{e 9.68}$$

Note that j_0 and j_1 are $\mathcal{F}'$-$7\varepsilon'$ -multiplicative, and

$$\|a - j_0(a) \oplus j_1(a)\| < 4\varepsilon' \ \text{for all}\ a \in \mathcal{F}'. \tag{e 9.69}$$

Define

$$\varphi_0 = j_0 \circ \varphi,\ \psi_0 = j_0 \circ \psi, \varphi_1 = j_1 \circ \varphi \quad \text{and} \quad \psi_1 = j_1 \circ \psi.$$

Then (by the choices of $\mathcal{F}'$ and ε'), the maps φ_0, ψ_0, φ_1, and φ_1 are $\mathcal{G}$-2δ-multiplicative, and for any $c \in \mathcal{F}$,

$$\|\varphi(c) - (\varphi_0(c) \oplus \varphi_1(c))\| < \varepsilon \quad \text{and} \quad \|\psi(c) - (\psi_0(c) \oplus \psi_1(c))\| < \varepsilon. \tag{e 9.70}$$

So (5) holds. Apply j_1 to both sides of both equations in (e 9.66). One obtains two invertible elements

$$e_h := \sum_{i=1}^{N(h)} j_1(a_i^*)\varphi_1(h) j_1(a_i)$$

and

$$f_h := \sum_{i=1}^{N(h)} j_1(b_i^*)\psi_1(h) j_1(b_i)$$

such that $|\|e_h^{-\frac{1}{2}}\| - 1| < 1$ and $|\|f_h^{-\frac{1}{2}}\| - 1| < 1$. Note that

$$\sum_{i=1}^{N(h)} e_h^{-\frac{1}{2}} j_1(a_i^*)\varphi_1(h) j_1(a_i) e_h^{-\frac{1}{2}} = 1_D, \sum_{i=1}^{N(h)} f_h^{-\frac{1}{2}} j_1(b_i^*)\psi_1(h) j_1(b_i) f_h^{-\frac{1}{2}} = 1_D,$$

$$\|j_1(a_i)e_h^{-\frac{1}{2}}\| < 2T(h), \quad \text{and} \quad \|j_1(b_i)f_h^{-\frac{1}{2}}\| < 2T(h).$$

Therefore, φ_1 and ψ_1 are $2T \times N$-$\mathcal{H}$-full. The same calculation also shows that φ_0 and ψ_0 are $2T \times N$-$\mathcal{H}$-full. This proves (6).

To see (8), one notes that

$$\|\sum_{i=1}^{t(c)} d_{x_{i,c}}^* d_{x_{i,d}} - \varphi_1(c)\| < \sigma_3 + t(\mathcal{H}_2)\varepsilon' < 7\sigma_1/5 \text{ and}$$

$$\|\sum_{i=1}^{t(c)} d_{x_{i,c}} d_{x_{i,d}}^* - \psi_1(c)\| < 7\sigma_1/5$$

for all $c \in \mathcal{H}_2$. Then (8) also holds.

Let us show (7) holds for sufficiently large $\mathcal{F}'$ and ε'. Since A is separable, there are an increasing sequence of finite subsets $\mathcal{F}_1' \subset \mathcal{F}_2' \subset \cdots$ such that $\bigcup \mathcal{F}_n'$ is dense in the unit ball of A. Set $\epsilon_n' = \frac{1}{n}$. Suppose (7) were not true, for each $\mathcal{F}_n'$ and each ϵ_n', there are C*-subalgebra $D_n \in \mathcal{D}$ and $j_{1,n} : A \to D_n$ as constructed above (in place of j_1), and there is $\tau_n \in \mathrm{T}(D_n)$ such that there is $c \in \mathcal{H}_1$

$$\tau_n \circ \varphi_{1,n}(c) \le \Delta(\hat{c})/2 \quad \text{or} \quad \tau_n \circ \psi_{1,n}(c) \le \Delta(\hat{c})/2$$

(where $\varphi_{1,n}$ and $\psi_{1,n}$ are as φ_1 and ψ_1 corresponding to $\mathcal{F}' = \mathcal{F}'_n$ and $\varepsilon' = \varepsilon'_n = 1/n$ for all large n). Passing to a subsequence, one may assume that

$$\tau_n \circ \varphi_{1,n}(c) \leq \Delta(\hat{c})/2. \tag{e 9.71}$$

Consider $\tau_n \circ j_{1,n} : A \to \mathbb{C}$, and pick an accumulating point τ of $\{\tau_n \circ j_{1,n} : n \in \mathbb{N}\}$. Since $j_{1,n}$ is $7\epsilon'_n$-$\mathcal{F}'_n$-multiplicative, it is straightforward to verify that τ is actually a tracial state of A. Passing to a subsequence, we may assume that $\tau(a) = \lim_{n\to\infty} \tau_n \circ j_{1,n}(a)$ for all $a \in A$. By (e 9.71), for any $0 < \eta < d_0/4$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\tau \circ \varphi(c) \leq \tau_n \circ j_{1,n} \circ \varphi(c) + \tau_n \circ \varphi_{2,n}(c) + \eta \leq \Delta(\hat{c})/2 + d_0/4 + \eta < \Delta(\hat{c}),$$

which contradicts to the assumption (2).

Let us show that (9) holds with sufficiently large $\mathcal{F}'$ and sufficiently small ϵ'.

Choose unitaries $u_1, u_2, ..., u_n \in M_k(C)$ such that $\mathcal{U} = \{\overline{u_1}, \overline{u_2}, ..., \overline{u_n}\}$. Pick unitaries $w_1, w_2, ..., w_n \in M_k(A)$ such that each w_i is a commutator and

$$\mathrm{dist}(\langle\varphi(u_i)\rangle \langle\psi(u_i^*)\rangle, w_i) < \sigma_2.$$

Choose $\mathcal{F}'$ sufficiently large and ϵ' sufficiently small such that there are commutators $w'_1, w_2, ..., w'_n \in CU(M_k(D))$ and commutators $w''_1, w''_2, ..., w''_n \in (1-p)A(1-p) \otimes M_k$ satisfying

$$\|j_1(w_i) - w'_i\| < \sigma_2/2 \text{ and } \|j_0(w_i) - w''_i\| < \sigma_2/2, \quad 1 \leq i \leq n,$$

(see Appendix of [81]) and

$$\|\langle\varphi_k(u_i)\rangle \langle\psi_k(u_i^*)\rangle - j_k(\langle\varphi(u_i)\rangle \langle\psi(u_i^*)\rangle)\| < \sigma_2/2, \quad 1 \leq i \leq n \text{ and } k = 0, 1.$$

(Recall we use φ, ψ, j_0, and j_1 for $\varphi \otimes \mathrm{id}_{M_k}$, $\psi \otimes \mathrm{id}_{M_k}$, $j_0 \otimes \mathrm{id}_{M_k}$ and $j_1 \otimes \mathrm{id}_{M_k}$, respectively.) Then

$$\begin{aligned}
& \|\langle\varphi_k(u_i)\rangle \langle\psi_k(u_i^*)\rangle - w'_i\| \\
\leq\ & \|\langle\varphi_k(u_i)\rangle \langle\psi_k(u_i^*)\rangle - j_k(w_i)\| + \|j_k(w_i) - w'_i\| \qquad \text{(e 9.72)} \\
\leq\ & \|\langle\varphi_k(u_i)\rangle \langle\psi_k(u_i^*)\rangle - j_k(\langle\varphi(u_i)\rangle \langle\psi(u_i^*)\rangle)\| \qquad \text{(e 9.73)} \\
& +\|j_k(\langle\varphi(u_i)\rangle \langle\psi(u_i^*)\rangle) - j_k(w_i)\| + \sigma_2/2 \leq 2\sigma_2, \ \ k = 0, 1. \qquad \text{(e 9.74)}
\end{aligned}$$

This proves (9).

To see the last part of the lemma, let $\mathcal{P}_0$ be a finite subset of projections of C, let $q_\varphi, q_\psi \in A$ be projections, and let $v_q \in A$ be a partial isometry (for each $q \in \mathcal{P}_0$) such that $v_q^* v_q = q_\varphi$, $v_q v_q^* = q_\psi$, $\|q_\varphi - \varphi(q)\| < \delta'/2 < 1/4$, and $\|q_\psi - \psi(q)\| < \delta'/2 < 1/4$ for some $\delta' > 0$.

As in part (d) of Lemma 2.19, with sufficiently small ε' and large $\mathcal{F}'$, one obtains projections $q_{\varphi,0}, q_{\psi,0} \in (1-p)A(1-p)$ and $q_{\varphi,1}, q_{\psi,1} \in D$, and partial isometries $w_{q,0,} \in (1-p)A(1-p)$ and $w_{q,1} \in D$ such that

$$w_{q,i}^* w_{q,i} = q_{\varphi,i}, w_{q,i} w_{q,i}^* = q_{\psi,i}, \tag{e 9.75}$$

$$\|\varphi_i(q) - q_{\varphi,i}\| < \delta', \text{ and } \|\psi_i(q) - q_{\psi,i}\| < \delta' \tag{e 9.76}$$

for all $q \in \mathcal{P}_0$. This implies that $[\varphi_i(q)] = [\psi_i(q)]$ for all $q \in \mathcal{P}_0$, $i = 0, 1$.

Suppose that $\mathcal{U}_0$ is a finite subset of $U(C)$ and $\langle\varphi(u)\rangle = z_u\langle\psi(u)\rangle$, where $z_u = \prod_{k=1}^{l(u)} \exp(ih_{u,k})$ and where $h_{u,i} \in A_{s.a.}$, for each $u \in \mathcal{U}_0$. By virtue of part (d) of Lemma 2.19, there are $h_{u,k,0} \in (1-p)A(1-p)_{s.a.}$ and $h_{u,k,1} \in D_{s.a.}$ such that $\langle\varphi_s(u)\rangle = (\prod_{k=1}^{l(u)} \exp(ih_{u,k,s}))\langle\psi_s(u)\rangle$, $u \in \mathcal{U}_0$, $s = 0, 1$. This implies that $[\varphi_s]|_{\mathcal{U}_0} = [\psi_s]|_{\mathcal{U}_0}$, $s = 0, 1$.

If $\mathcal{P}_1$ is a finite subset of projections and unitaries in $M_m(A)$ for some integer $m \geq 1$, by considering $\varphi \otimes \mathrm{id}_{M_m}$ and $\psi \otimes \mathrm{id}_{M_m}$, with sufficiently small ε' and large $\mathcal{F}'$, we conclude that we can require that $[\varphi_i]|_{\mathcal{P}_1} = [\psi_i]|_{\mathcal{P}_1}$, $i = 0, 1$.

In general, consider a finite subset $\mathcal{P}_k \subset K_0(A \otimes B_0)$ for some $B_0 = \tilde{B}$, where B is a commutative C^*-algebra such that $K_0(B) = \mathbb{Z}/k\mathbb{Z}$ and $K_1(B) = \{0\}$, $k = 2, 3, ...$ Also consider $\tilde{\varphi} = \varphi \otimes \mathrm{id}_{B_0}$ and $\tilde{\psi} = \psi \otimes \mathrm{id}_{B_0}$. We will replace $(1-p)A(1-p)$ by A_0, where $A_0 := (1 - p \otimes 1_{B_0})(A \otimes B_0)(1 - p \otimes 1_{B_0})$, and D by $D \otimes B_0$ in the above argument. We will also consider $(j_i \otimes \mathrm{id}_{B_0}) \circ \tilde{\varphi}$ and $(j_i \otimes \mathrm{id}_{B_0}) \circ \tilde{\psi}$. Note that $1 - (p \otimes 1_{B_0})$ almost commutes with $\tilde{\varphi}$ and $\tilde{\psi}$ on a given finite subset provided that ε' is sufficiently small and $\mathcal{F}'$ is sufficiently large. Thus, as above, with sufficiently small ε' and $\mathcal{F}'$, $[\tilde{\varphi}]|_{\mathcal{P}_k} = [\tilde{\psi}]|_{\mathcal{P}_k}$. This implies that the last part, (10) of the lemma holds. ☐

PROPOSITION 9.14. $\mathcal{B}_1 \neq \mathcal{B}_0$.

PROOF. It follows from Theorem 1.4 of [88] that there is a unital separable simple C^*-algebra A which is an inductive limit of dimension drop circle algebras such that A has a unique tracial state, $(K_0(A), K_0(A)_+, [1_A]) = (\mathbb{Z}, \mathbb{Z}_+, 1)$, and $K_1(A) = \mathbb{Z}/3\mathbb{Z}$. Note that dimension drop circle algebras are in $\mathcal{C}$ (see 3.22).

Note also that A is a unital projectionless C^*-algebra. If A were in $\mathcal{B}_0$, since A does not have (SP), by Theorem 9.6, A would be an inductive limit (with not necessarily injective maps) of C^*-algebras in $\mathcal{C}_0$. However, every C^*-algebra in $\mathcal{C}_0$ has trivial K_1. This would imply that $K_1(A) = \{0\}$, a contradiction. Thus, $A \notin \mathcal{B}_0$. ☐

REMARK 9.15. The C^*-algebra A in the proof of Proposition 9.14 is rationally of tracial rank zero (see [84]), since it has a unique tracial state. Later we will see that there are many other C^*-algebras which are in $\mathcal{B}_1$ but not in $\mathcal{B}_0$.

PROPOSITION 9.16. *Let A be a C^*-algebra in $\mathcal{B}_1$ and let U be an infinite dimensional UHF-algebra. Then $A \otimes U$ has the property (SP).*

PROOF. It suffices to show that, for any non-zero positive element $b \in A \otimes U$, there exists a non-zero projection $e \in A \otimes U$ such that $e \lesssim b$. Without loss of generality, we may assume that $0 \le b \le 1$. Let $\sigma = \inf\{\tau(b) : \tau \in T(A)\} > 0$. Then there is a non-zero projection $e \in A \otimes U$ with the form $1_A \otimes p$, where $p \in U$ is a non-zero projection, such that

$$\tau(e) < \sigma. \tag{e 9.77}$$

By strict comparison for positive elements (Theorem 9.11), $e \lesssim b$, which implies that $\overline{bAb}$ has a projection equivalent to e. $\square$

10. $\mathcal{Z}$-stability

LEMMA 10.1. *Let $A \in \mathcal{B}_1$ (or $\mathcal{B}_0$) be a unital infinite dimensional simple C^*-algebra. Then, for any $\varepsilon > 0$, any $a \in A_+ \setminus \{0\}$, any finite subset $\mathcal{F} \subset A$ and any integer $N \ge 1$, there exist a projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{C}$ (or $\in \mathcal{C}_0$) with $1_C = p$ that satisfy the following conditions:*

(1) $\dim(\pi(C)) \ge N^2$ for every irreducible representation π of C,
(2) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,
(3) $\mathrm{dist}(pxp, C) < \varepsilon$ for all $x \in \mathcal{F}$, and
(4) $1 - p \lesssim a$.

PROOF. Since A is an infinite dimensional simple C^*-algebra, there are $N+1$ mutually orthogonal non-zero positive elements $a_1, a_2, ..., a_{N+1}$ in A. Since A is simple, there are $x_{i,j} \in A$, $j = 1, 2, ..., k(i)$, $i = 1, 2, ..., N+1$, such that

$$\sum_{j=1}^{k(i)} x_{i,j}^* a_i x_{i,j} = 1_A.$$

Let

$$K = (N+1)\max\{\|x_{i,j}\| + 1 : 1 \le j \le k(i),\ \ 1 \le i \le N+1\}.$$

Let $a_0 \in A_+ \setminus \{0\}$ be such that $a_0 \lesssim a_i$ for all $1 \le i \le N+1$, and furthermore $a_0 \lesssim a$. Since $\overline{a_0 A a_0}$ is also an infinite dimensional simple C^*-algebra, one obtains $a_{01}, a_{02} \in \overline{a_0 A a_0}$ which are mutually orthogonal and non-zero. One then obtains a non-zero element $b \in \overline{a_{01} A a_{01}}$ such that $b \lesssim a_{02}$.

Let

$$\mathcal{F} = \{a_i : 1 \le i \le N+1\} \cup \{x_{i,j} : 1 \le j \le k(i), 1 \le i \le N+1\} \cup \{a\}.$$

Now since $A \in \mathcal{B}_1$, there are a projection $p \in A$ and $C \in \mathcal{C}$ with $1_C = p$ such that

(1) $\|xp - px\| < \min\{1/2, \varepsilon\}/2K$ for all $x \in \mathcal{F}$,
(2) $\mathrm{dist}(pxp, C) < \min\{1/2, \varepsilon\}/2K$ for all $x \in \mathcal{F}$, and

(3) $1-p \lesssim b$.

Then, with a standard computation, we obtain mutually orthogonal non-zero positive elements $b_1, b_2, ..., b_{N+1} \in C$ and $y_{i,j}, \in C$ $(1 \le j \le k(i))$, $i = 1, 2, ..., N+1$, such that

$$\|\sum_{j=1}^{k(i)} y_{i,j}^* b_i y_{i,j} - p\| < \min\{1/2, \varepsilon/2\}. \tag{e 10.1}$$

For each i, we find another element $z_i \in C$ such that

$$\sum_{j=1}^{k(i)} z_i^* y_{i,j} b_i y_{i,j} z_i = p. \tag{e 10.2}$$

Let π be an irreducible representation of C. Then by (e 10.2),

$$\sum_{j=1}^{k(i)} \pi(z_i^* y_{i,j})\pi(b_i)\pi(y_{i,j} z_i) = \pi(p). \tag{e 10.3}$$

Therefore, $\pi(b_1), \pi(b_2), ..., \pi(b_{N+1})$ are mutually orthogonal non-zero positive elements in $\pi(A)$. Then (e 10.3) implies that $\pi(C) \cong M_n$ with $n \ge N+1$. This proves the lemma. ☐

COROLLARY 10.2. *Let $A \in \mathcal{B}_1$ (or $A \in \mathcal{B}_0$) be an infinite dimensional unital simple C^*-algebra. Then, for any $\varepsilon > 0$ and $f \in \mathrm{Aff}(T(A))^{++}$, there exist a C^*-subalgebra $C \in \mathcal{C}$ (or $C \in \mathcal{C}_0$) in A and an element $c \in C_+$ such that*

$$\mathrm{dim}\pi(C) \ge (4/\varepsilon)^2 \text{ for each irreducible representation } \pi \text{ of } C, \tag{e 10.4}$$

$$0 < \tau(f) - \tau(c) < \varepsilon/2 \text{ for all } \tau \in T(A). \tag{e 10.5}$$

PROOF. By 9.3 of [71], there is an element $x \in A_+$ such that $\tau(x) = \tau(f)$ for all $\tau \in T(A)$. Let $N \ge 16(\|x\|+1)/\varepsilon$. As in the beginning of the proof of Proposition 9.10, one finds a non-zero element $a \in A_+$ such that $N[a] \le 1$. Apply Lemma 10.1 to $\varepsilon/(16(\|x\|+1))$, N and $a \in A_+$ and $\mathcal{F} = \{x, 1\}$, to get C and $p = 1_C$ as in that lemma. Then C satisfies (e 10.4) and, by (4) of Lemma 10.1, $\tau(1-p) < 1/N$ for all $\tau \in T(A)$. It follows that $0 < \tau(x) - \tau(pxp) < 2\|x\|/N < \varepsilon/8$ for all $\tau \in T(A)$. Then choose $c' \in C$ with $\|pxp - c'\| < \varepsilon/16(\|x\|+1)$. Replacing c' by $(c' + (c')^*)/2$, we may assume that $c' \in C_{s.a.}$. Since $pxp \ge 0$, we obtain a positive element $c \in C$ such that $\|pxp - c\| < \varepsilon/8$. We have

$$0 < \tau(f) - \tau(c) = \tau(x) - \tau(c) < \varepsilon/2 \text{ for all } \tau \in T(A), \tag{e 10.6}$$

as desired. ☐

The following is known. In the following statement, we identify $[0,1]$ with the space of extreme points of $T(M_n(C[0,1]))$.

LEMMA 10.3. *Let $C = M_n(C([0,1]))$ and $g \in \mathrm{LAff}_b(T(C))_+$ with $0 \le g(t) \le 1$. Then there exists $a \in C_+$ with $0 \le a \le 1$ such that*

$$0 \le g(t) - d_t(a) \le 1/n \text{ for all } t \in [0,1],$$

where $d_t(a) = \lim_{k\to\infty} \mathrm{tr}(a^{1/k}(t))$ for all $t \in [0,1]$, where tr is the tracial state of M_n.

PROOF. We will use the proof of Lemma 5.2 of [12]. For each $0 \le i \le n-1$, define

$$X_i = \{t \in [0,1] : g(t) > i/n\}.$$

Since g is lower semi-continuous, X_i is open in $[0,1]$. There is a continuous function $g_i \in C([0,1])_+$ with $0 \le g_i \le 1$ such that

$$\{t \in [0,1] : g_i(t) \ne 0\} = X_i, \ \ i = 0,1,...,n-1.$$

Let $e_1, e_2, ..., e_n$ be n mutually orthogonal rank one projections in $C = M_n(C([0,1])$. Define

$$a = \sum_{i=1}^{n-1} g_i e_i \in C. \tag{e 10.7}$$

Then $0 \le a \le 1$. Put $Y_i = \{t \in [0,1] : (i+1)/n \ge g_i(t) > i/n\} = X_i \setminus \bigcup_{j>i} X_j$, $i = 0,1,2,...,n-1$. These are mutually disjoint sets. Note that

$$[0,1] = ([0,1] \setminus X_0) \cup \bigcup_{i=0}^{n-1} Y_i.$$

If $x \in ([0,1] \setminus X_0) \cup Y_0$, then $d_t(a) = 0$. So $0 \le g(t) - d_t(a)(t) \le 1/n$ for all such t. If $t \in Y_j$,

$$d_t(a) = j/n. \tag{e 10.8}$$

Then

$$0 \le g(t) - d_t(a) \le 1/n \text{ for all } t \in Y_j. \tag{e 10.9}$$

It follows that

$$0 \le g(t) - d_t(a) \le 1/n \text{ for all } t \in [0,1]. \tag{e 10.10}$$

□

LEMMA 10.4. *Let F_1 and F_2 be two finite dimensional C^*-algebras such that each simple direct summand of F_1 and F_2 has rank at least k, where $k \geq 1$ is an integer. Let $\varphi_0, \varphi_1 : F_1 \to F_2$ be unital homomorphisms. Let $C = A(\varphi_0, \varphi_1, F_1, F_2)$. Then, for any $f \in \mathrm{LAff}_b(T(C))_+$ with $0 \leq f \leq 1$, there exists a positive element $a \in M_2(C)$ such that*

$$\max_{\tau \in T(C)} |d_\tau(a) - f(\tau)| \leq 2/k.$$

PROOF. Let $I = \{(g, a) \in C \subset C([0,1], F_2) \oplus F_1 : g(0) = g(1) = 0 = a\}$. Note that $C/I \cong F_1$. Let

$$T = \{\tau \in T(C) : \ker\tau \supset I\}.$$

Then T may be identified with $T(C/I) = T(F_1)$. Let $f \in \mathrm{LAff}_b(T(C))_+$ with $0 \leq f \leq 1$. There exists $b \in (C/I)_+$ such that

$$0 \leq f(\tau) - d_\tau(b) \leq 1/k \text{ for all } \tau \in T, \tag{e 10.11}$$

and furthermore, if $f(\tau) > 0$, then $f(\tau) - d_\tau(b) > 0$. To see this, write $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$. Note that $R(s) \geq k$, $s = 1, 2, \ldots$. Denote by $\tau_{q(s)}$ the tracial state of $M_{R(s)}$, $s = 1, 2, \ldots, l$. One can find an integer $J_s \geq 0$ such that

$$\frac{J_s}{R(s)} \leq f(\tau_{q(s)}) \leq \frac{J_s + 1}{R(s)}, \quad s = 1, 2, \ldots, l. \tag{e 10.12}$$

Moreover, if $f(\tau_{q(s)}) > 0$, we may assume that $J_s/R(s) < f(\tau_{q(s)})$. Let $b \in F_1 = C/I$ be a projection with rank J_s in $M_{R(s)}$, $s = 1, 2, \ldots, l$. For such a choice, since b is a projection, we have $d_\tau(b) = \tau(b)$ for all $\tau \in T$. Then, by (e 10.12), (e 10.11) holds. Moreover, if $f(\tau) > 0$, $f(\tau) - d_\tau(b) > 0$. In particular, if $d_\tau(b) > 0$, then

$$f(\tau) - d_\tau(b) > 0. \tag{e 10.13}$$

Recall that $b = (b_1, b_2, \ldots, b_l) \in C/I = F_1$. Let $b^0 = \varphi_0(b) \in F_2$ and $b^1 = \varphi_1(b) \in F_2$. Write $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(m)}$. Write $b^0 = b_{0,1} \oplus b_{0,2} \oplus \cdots \oplus b_{0,r(m)}$ and $b^1 = b_{1,1} \oplus b_{1,2} \oplus \cdots \oplus b_{1,r(m)}$, where $b_{0,j}$, $b_{1,j} \in M_{r(j)}$, $j = 1, 2, \ldots, m$. Let $\tau_{t,j} = \mathrm{tr}_j \circ \Psi_j \circ \pi_t$, where tr_j is the normalized trace on $M_{r(j)}$, $\Psi_j : F_2 \to M_{r(j)}$ is the quotient map and $\pi_t : A \to F_2$ is the point evaluation at $t \in (0, 1)$.

Denote by $\tau_{0,j}$ the tracial state of C defined by $\tau_{0,j}((g, a)) = \mathrm{tr}_j(g(0)) = \mathrm{tr}_j(\varphi_0(a))$ for all $(g, a) \in C$, and denote by $\tau_{1,j}$ the tracial state of C defined by $\tau_{1,j}((g, a)) = \mathrm{tr}_j(g(1)) = \mathrm{tr}_j(\varphi_1(a))$ for all $(g, a) \in C$, respectively, $j = 1, 2, \ldots, m$. It follows from the third paragraph of Subsection 3.8 that $\tau_{t,j} \to \tau_{0,j}$ in $T(A)$ if $t \to 0$ in $[0, 1]$ and $\tau_{t,j} \to \tau_{1,j}$ in $T(A)$ if $t \to 1$ in $[0, 1]$.

Recall from Section 3 (see 3.8 and 3.4), we have

$$\mathrm{tr}_j(b_{0,j}) = \frac{1}{r(j)}\sum_{s=1}^{l} a_{js}R(s)\cdot\tau_{q(s)}(b_s) \text{ and}$$

(e 10.14)

$$\mathrm{tr}_j(b_{1,j}) = \frac{1}{r(j)}\sum_{s=1}^{l} b_{js}R(s)\cdot\tau_{q(s)}(b_s),$$

where $\{a_{js}\}$ and $\{b_{js}\}$ are matrices of non-negative integers given by the maps $(\varphi_0)_{*0} : K_0(F_1) = \mathbb{Z}^l \to K_0(F_2) = \mathbb{Z}^m$ and $(\varphi_1)_{*0} : K_0(F_1) = \mathbb{Z}^l \to K_0(F_2) = \mathbb{Z}^m$.

If $g \in \mathrm{LAff}_b(T(A))_+$, then, by (e 3.6) and (e 3.7) in Section 3,

$$g(\tau_{0,j}) = \sum_{s=1}^{l} a_{js}R(s)\cdot g(\tau_{q(s)}) \text{ and } g(\tau_{1,j}) = \frac{1}{r(j)}\sum_{s=1}^{l} b_{js}R(s)\cdot g(\tau_{q(s)}),$$

Recall that b is a projection, since f is lower semicontinuous on $T(C)$, we have, by (e 10.11) and (e 10.13),

$$\begin{aligned}\liminf_{t\to 0} f(\tau_{t,j}) \ge f(\tau_{0,j}) &= \sum_{s=1}^{l} a_{js}R(s)\cdot f(\tau_{q(s)})\\ &\ge \frac{1}{r(j)}\sum_{s=1}^{l} a_{js}R(s)\tau_{q(s)}(b_s) = \mathrm{tr}_j(b_{0,j})\\ \text{and } \liminf_{t\to 1} f(\tau_{t,j}) &\ge \frac{1}{r(j)}\sum_{s=1}^{l} b_{js}R(s)\cdot f(\tau_{q(s)}) \ge \mathrm{tr}_j(b_{1,j}).\end{aligned}$$

Note that, since φ_0 is unital, $\frac{1}{r(j)}\sum_{s=1}^{l} a_{js}R(s) = 1$. Therefore, if $\mathrm{tr}_j(b_{0,j}) > 0$, then

$$f(\tau_{0,j}) = f(\sum_{s=1}^{l}\frac{1}{r(j)}a_{js}R(s)\tau_{q(s)}) > \sum_{s=1}^{l}\frac{1}{r(j)}a_{js}R(s)\tau_{q(s)}(b_s) = \mathrm{tr}_j(b_{0,j}),$$

$$f(\tau_{1,j}) = f(\sum_{s=1}^{l}\frac{1}{r(j)}a_{js}R(s)\tau_{q(s)}) > \sum_{s=1}^{l}\frac{1}{r(j)}b_{js}R(s)\tau_{q(s)}(b_s) = \mathrm{tr}_j(b_{1,j}).$$

Hence, if $\mathrm{tr}_j(b_{0,j}) > 0$ (or $\mathrm{tr}_j(b_{1,j}) > 0$), then $\liminf_{t\to 0} f(\tau_{t,j}) > \mathrm{tr}_j(b_{0,j})$ (or $\liminf_{t\to 1} f(\tau_{t,j}) > \mathrm{tr}_j(b_{1,j})$). Therefore, there exists $1/8 > \delta > 0$ such that

(e 10.15) $\quad f(\tau_{t,j}) \ge \mathrm{tr}_j(b_{0,j})$ for all $t \in (0, 2\delta)$ and

(e 10.16) $\quad f(\tau_{t,j}) \ge \mathrm{tr}_j(b_{1,j})$ for all $t \in (1-2\delta, 1)$, $j = 1, 2, ..., l$.

Let

$$c(t) = (\frac{\delta - t}{\delta})b_0 \text{ if } t \in [0, \delta), \tag{e 10.17}$$

$$c(t) = 0 \text{ if } t \in [\delta, 1-\delta], \text{ and} \tag{e 10.18}$$

$$c(t) = (\frac{t-1+\delta}{\delta})b_1 \text{ for all } t \in (1-\delta, 1]. \tag{e 10.19}$$

Note that $c \in A$. Define

$$g_j(0) = 0, \tag{e 10.20}$$

$$g_j(t) = f(\tau_{t,j}) - \mathrm{tr}_j(b_{0,j}) \text{ for all } t \in (0, \delta], \tag{e 10.21}$$

$$g_j(t) = f(\tau_{t,j}) \text{ for all } t \in (\delta, 1-\delta), \tag{e 10.22}$$

$$g_j(t) = f(\tau_{t,j}) - \mathrm{tr}_j(b_{1,j}) \text{ for all } t \in [1-\delta, 1), \text{ and} \tag{e 10.23}$$

$$g_j(1) = 0. \tag{e 10.24}$$

One verifies that g_j is lower semicontinuous on $[0,1]$. It follows from Lemma 10.3 that there exists $a_1 \in C([0,1], F_2)_+$ such that

$$0 \le g_j(t) - d_{\mathrm{tr}_{t,j}}(a_1) \le 1/r(j) \le 1/k \text{ for all } t \in [0,1]. \tag{e 10.25}$$

Note that $a_1(0) = 0$ and $a_1(1) = 0$. Therefore $a_1 \in I \subset C$. Now let $a = c \oplus a_1 \in M_2(C)$. Note that

$$\begin{aligned} d_\tau(a) &= d_\tau(c) + d_\tau(a_1) = d_\tau(b) \text{ if } t \in T, \\ d_{\mathrm{tr}_{t,j}}(a) &= d_{\mathrm{tr}_{t,j}}(c) + d_{\mathrm{tr}_{t,j}}(a_1) = d_{t,j}(b_0) + d_{\mathrm{tr}_{t,j}}(a_1) \text{ for all } t \in (0, \delta), \\ d_{\mathrm{tr}_{t,j}}(a) &= d_{\mathrm{tr}_{t,j}}(a_1) \text{ for all } t \in [\delta, 1-\delta], \text{ and} \\ d_{\mathrm{tr}_{t,j}}(a) &= d_{\mathrm{tr}_{t,j}}(c) + d_{\mathrm{tr}_{t,j}}(a_1) = d_{t,j}(b_1) + d_{\mathrm{tr}_{t,j}}(a_1) \text{ for all } t \in (1-\delta, 1). \end{aligned}$$

Hence, by (e 10.11), (e 10.17), (e 10.18), (e 10.19), (e 10.20), (e 10.21), (e 10.22), (e 10.23) and (e 10.25),

$$0 \le f(\tau) - d_\tau(a) \le 2/k \tag{e 10.26}$$

for all $\tau \in T$ and $\tau = \mathrm{tr}_{t,j}$, $j = 1, 2, ..., l$, $t \in (0,1)$. Since $T \cup \{\mathrm{tr}_{t,j} : 1 \le j \le l, \text{ and } t \in (0,1)\}$ contains all the extreme points of $T(C)$, we conclude that

$$0 \le f(\tau) - d_\tau(a) \le 2/k \text{ for all } \tau \in T(C). \tag{e 10.27}$$

□

THEOREM 10.5. *Let $A \in \mathcal{B}_1$ be an infinite dimensional unital simple C^*-algebra. Then the map $W(A) \to V(A) \sqcup \mathrm{LAff}_b(A)_{++}$ is surjective.*

PROOF. The proof follows the lines of the proof of Theorem 5.2 of [12]. It suffices to show that the map $a \mapsto d_\tau(a)$ is surjective from $W(A)$ onto $\mathrm{LAff}_b(T(A))$. Let $f \in \mathrm{LAff}_b(A)_+$ with $f(\tau) > 0$ for all $\tau \in T(A)$. We may assume that $f(\tau) \le 1$ for all $\tau \in T(A)$. As in the proof of 5.2 of [12], it suffices to find a sequence of elements $a_i \in M_2(A)_+$ such that $a_i \lesssim a_{i+1}$, $[a_n] \ne [a_{n+1}]$ (in $W(A)$), and

$$\lim_{n\to\infty} d_\tau(a_n) = f(\tau) \ \text{ for all } \ \tau \in T(A).$$

Using the semicontinuity of f, we find a sequence $f_n \in \mathrm{Aff}(T(A))_{++}$ such that

$$f_n(\tau) < f_{n+1}(\tau) \ \text{ for all } \ \tau \in T(A), \ \ n = 1, 2, ..., \tag{e 10.28}$$

$$\lim_{n\to\infty} f_n(\tau) = f(\tau) \ \text{ for all } \ \tau \in T(A). \tag{e 10.29}$$

Since $f_{n+1} - f_n$ is continuous and strictly positive on the compact set T, there is $1 >\varepsilon_n > 0$ such that $(f_{n+1} - f_n)(\tau) > \varepsilon_n$ for all $\tau \in T(A)$, $n = 1, 2, ...$ By Corollary 10.2, for each n, there is a C^*-subalgebra C_n of A with $C_n \in \mathcal{C}$ and an element $b_n \in (C_n)_+$ such that

$$\dim\pi(C_n) \ \ \ge \ \ (16/\varepsilon_n)^2 \tag{e 10.30}$$

for each irreducible representation,

$$0 < \tau(f_n) - \tau(b_n) < \varepsilon_n/8 \ \text{ for all } \ \tau \in T(A).$$

Applying Lemma 10.4, one obtains an element $a_n \in M_2(C_n)_+$ such that

$$0 < t(b_n) - d_t(a_n) < \frac{2}{k} < \varepsilon_n/4 \ \text{ for all } \ t \in T(C_n), \tag{e 10.31}$$

where k is the minimal rank of all irreducible representations of C_n and $k \ge 16/\varepsilon_n$. It follows that

$$0 < \tau(f_n) - d_\tau(a_n) < \varepsilon_n/2 \ \text{ for all } \ \tau \in T(A). \tag{e 10.32}$$

One then checks that $\lim_{n\to\infty} d_\tau(a_n) = f(\tau)$ for all $\tau \in T(A)$. Moreover, $d_\tau(a_n) < d_\tau(a_{n+1})$ for all $\tau \in T(A)$, $n = 1, 2, ...$ It follows from Theorem 9.11 that $a_n \lesssim a_{n+1}$, $[a_n] \ne [a_{n+1}]$, $n = 1, 2, ...$ This ends the proof. □

THEOREM 10.6. *Let $A \in \mathcal{B}_1$ be an infinite dimensional unital simple C^*-algebra. Then $W(A)$ is 0-almost divisible.*

PROOF. Let $a \in M_n(A)_+ \setminus \{0\}$ and $k \ge 1$ be an integer. We need to show that there exists an element $x \in M_{m'}(A)_+$ for some $m' \ge 1$ such that

$$k[x] \le [a] \le (k+1)[x] \tag{e 10.33}$$

in $W(A)$. It follows from Theorem 10.5 that, since $kd_\tau(a)/(k^2+1) \in \mathrm{LAff}_b(T(A))$, there is $x \in M_{2n}(A)_+$ such that

$$d_\tau(x) = kd_\tau(a)/(k^2+1) \text{ for all } \tau \in T(A). \tag{e 10.34}$$

Then,

$$kd_\tau(x) < d_\tau(a) < (k+1)d_\tau(x) \text{ for all } \tau \in T(A). \tag{e 10.35}$$

It follows from Theorem 9.11 that

$$k[x] \le [a] \le (k+1)[x]. \tag{e 10.36}$$

□

THEOREM 10.7. *Let $A \in \mathcal{B}_1$ be an infinite dimensional unital separable simple amenable C^*-algebra. Then $A \otimes \mathcal{Z} \cong A$.*

PROOF. Since $A \in \mathcal{B}_1$, A has finite weak tracial nuclear dimension (see 8.1 of [81]). By Theorem 9.11, A has the strict comparison property for positive elements. Note that, by Theorem 9.9, every unital hereditary C^*-subalgebra of A is in $\mathcal{B}_1$. Thus, by Theorem 10.6, its Cuntz semigroup also has 0-almost divisibility. It follows from 8.3 of [81] that A is $\mathcal{Z}$-stable. □

11. The Unitary Groups

THEOREM 11.1 (cf. Theorem 6.5 of [71]). *Let $K \in \mathbb{N}$ be an integer and let $\mathcal{B}$ be a class of unital C^*-algebras which has the property that $\mathrm{cer}(B) \le K$ for all $B \in \mathcal{B}$. Let A be a unital simple C^*-algebra which is* TA$\mathcal{B}$ *(see Definition 9.4) and let $u \in U_0(A)$. Suppose that A has the cancellation property for projections, in particular, if $e, q \in A$ are projections and $v \in A$ such that $v^*v = e$ and $vv^* = q$, then there is a $w \in A$ such that $w^*w = 1 - e$ and $ww^* = 1 - q$. Then, for any $\varepsilon > 0$, there exist unitaries $u_1, u_2 \in A$ such that u_1 has exponential length no more than 2π, u_2 has exponential rank K, and*

$$\|u - u_1u_2\| < \varepsilon.$$

Moreover, $\mathrm{cer}(A) \le K + 2 + \varepsilon$.

PROOF. Suppose that A is locally approximated by C^*-subalgebras $\{A_n\}$ in $\mathcal{B}$. One may write (for some integer $m \ge 1$) $u = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_m)$, where $h_i \in A_{s.a.}$, $i = 1, 2, ..., m$. Without loss of generality, we may assume that, for some large n, $h_i \in A_n$, $i = 1, 2, ..., m$. In other words, we may assume that $u \in U_0(A_n)$. Then the conclusion of the lemma follows with $u_1 = 1$, since $\mathrm{cer}(A_n) \le K$ as assumed. This also follows from the proof of Theorem 6.5 of [71] with $p = 1$.

Now as in the proof of 9.6 we may assume that $A \in \mathrm{TA}\mathcal{B}$ and A has property (SP). Without loss of generality, we may assume that A is an infinite dimensional C^*-algebra. Let n be a positive integer.

As in the beginning of the proof of 9.10, one can find mutually orthogonal non-zero positive elements $a, b_1, b_2, ..., b_{2(n+1)} \in A$ such that $a \lesssim b_i$ for all $i \in \{1, 2, \cdots, 2(n+1)\}$. Hence, with a non-zero projection e such that $e \lesssim a$, there is a projection $q \in \overline{aAa}$ and $v \in A$ such that $v^*v = e$ and $vv^* = q$. Note that $b_1, b_2, ..., b_{2(n+1)} \in (1-q)A(1-q)$. It follows that $2(n+1)[e] \le [1-q]$, or, $2(n+1)[1-p] \le [p]$, where $p = 1-e$. By the assumption that A has cancellation of projections, $[1-q] = [p]$. So, if we apply the property that A is $TA\mathcal{B}$, as in the proof of Theorem 6.5 of [71], (3) in that proof holds as $(1-p) \lesssim a$. With this fact in mind, the rest of proof is exactly the same as that of Theorem 6.5 of [71]. $\square$

Corollary 11.2. *Any C^*-algebra in the class $\mathcal{B}_1$ has exponential rank at most $5+\epsilon$.*

Proof. By Theorem 3.16, C^*-algebras in $\mathcal{C}$ have exponential rank at most $3+\epsilon$. Moreover, by 9.7, A has stable rank one. Therefore projections of A have cancellation. Therefore, by Theorem 11.1, any C^*-algebra in $\mathcal{B}_1$ has exponential rank at most $5+\epsilon$. $\square$

Theorem 11.3. *Let $L > 0$ be a positive number and let $\mathcal{B}$ be a class of unital C^*-algebras such that* $\mathrm{cel}(v) \le L$ *for every unitary v in their closure of commutator subgroups. Let A be a unital simple C^*-algebra which is tracially in $\mathcal{B}$ and let $u \in CU(A)$. Suppose that A has cancellation property for projections. Then $u \in U_0(A)$ and* $\mathrm{cel}(u) \le 2\pi + L + \varepsilon$.

Proof. Let $1 > \varepsilon > 0$. There are $v_1, v_2, ..., v_k \in U(A)$ such that

(e 11.1) $$\|u - v_1 v_2 \cdots v_k\| < \varepsilon/16$$

and $v_i = a_i b_i a_i^* b_i^*$, where $a_i, b_i \in U(A)$. Let N be an integer as in Lemma 6.4 of [71] (with $8\pi k + \varepsilon$ playing the role of L there). We further assume that $(8\pi k+1)\pi/N < \frac{\varepsilon}{4}$. Since A is tracially in $\mathcal{B}$, there are a projection $p \in A$ and a unital C^*-subalgebra B in $\mathcal{B}$ with $1_B = p$ such that

(e 11.2) $$\|a_i - (a_i' \oplus a_i'')\| < \varepsilon/32k, \ \ \|b_i - (b_i' \oplus b_i'')\| < \varepsilon/32k, \ \ i = 1, 2, ..., k, \ \text{ and}$$

(e 11.3) $$\|u - \prod_{i=1}^{k} (a_i' b_i' (a_i')^* (b_i')^* \oplus a_i'' b_i'' (a_i'')^* (b_i'')^* \| < \varepsilon/8,$$

where $a_i', b_i' \in U((1-p)A(1-p))$, $a_i'', b_i'' \in U_0(B)$ and $6N[1-p] \le [p]$ (if A is locally approximated by C^*-algebras in $\mathcal{B}$, we can choose $p = 1_A$; otherwise, we can apply the argument in the proof of Theorem 11.1). Put

(e 11.4) $$w = \prod_{i=1}^{k} a_i' b_i' (a_i')^* (b_i')^* \ \text{ and } \ z = \prod_{i=1}^{k} a_i'' b_i'' (a_i'') (b_i'')^*.$$

Then $z \in CU(B)$. Therefore $\mathrm{cel}_B(z) \leq L$ in $B \subset pAp$. It is standard to show that then

$$a_i'b_i'(a_i')^*(b_i{}')^* \oplus (1-p) \oplus (1-p)$$

is in $U_0(M_3((1-p)A(1-p)))$ and it has exponential length no more than $4(2\pi)+2\varepsilon/16k$. This implies

$$\mathrm{cel}(w \oplus (1-p) \oplus (1-p)) \leq 8\pi k + \varepsilon/4$$

in $U(M_3((1-p)A(1-p)))$. Since $6N[1-p] \leq [p]$, there is a projection $q \in M_3(pAp)$ such that $(1-p) \oplus (1-p) \oplus q$ is Murray von Neumann equivalent to p and $N[(1-p) \oplus (1-p) \oplus (1-p)] \leq [q]$. View $w \oplus (1-p) \oplus (1-p) \oplus q$ as a unitary in

$$\big((1-p) \oplus (1-p) \oplus (1-p) \oplus q\big) M_3(A) \big((1-p) \oplus (1-p) \oplus (1-p) \oplus q\big).$$

It follows from Lemma 6.4 of [71] that

$$\text{(e 11.5)} \quad \mathrm{cel}(w \oplus (1-p) \oplus (1-p) \oplus q) \leq 2\pi + \frac{8\pi k + \varepsilon/4}{N}\pi < 2\pi + \varepsilon/4.$$

It follows that

$$\mathrm{cel}((w \oplus p)((1-p) \oplus z)) < 2\pi + \varepsilon/4 + L + \varepsilon/16.$$

Therefore $\mathrm{cel}(u) \leq 2\pi + L + \varepsilon$. ☐

Remark: The cancellation property of projections in both Lemma 11.1 and Theorem 11.3 can be replaced by requiring that every unital hereditary C^*-subalgebra of A is in $\mathrm{TA}\mathcal{B}$. We do not need this though.

COROLLARY 11.4. *Let A be a unital simple C^*-algebra in $\mathcal{B}_1$, and let $u \in CU(A)$. Then $u \in U_0(A)$ and $\mathrm{cel}(u) \leq 7\pi$.*

PROOF. This follows from Lemma 3.14 and Theorem 11.3. ☐

LEMMA 11.5. *Let A be a unital C^*-algebra. Then the group $U_0(A)/CU(A)$ is divisible. Let U be a UHF-algebra of infinite type, and let $B = A \otimes U$. Then the group $U_0(B)/CU(B)$ is torsion free.*

PROOF. It is well known that $U_0(A)/CU(A)$ is always divisible. Indeed, pick $\bar{u} \in U_0(A)/CU(A)$ for some $u \in U_0(A)$, and pick $k \in \mathbb{N}$. Since $u \in U_0(A)$, there are self-adjoint elements $h_1, h_2, ..., h_n \in A$ such that

$$u = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_n)).$$

Then the unitary $w := \exp(ih_1/k)\exp(ih_2/k)\cdots\exp(ih_n/k)$ satisfies $(\bar{w})^k = \bar{u}$ in $U_0(A)/CU(A)$, and thus $U_0(A)/CU(A)$ is divisible.

Now, consider $B = A \otimes U$, where U is an infinite dimensional UHF-algebra. By Corollary. 6.6 and Theorem 6.8 of [103], B is either purely infinite, or B has stable rank one. If B is purely infinite, by Corollary. 2.7 of [46], $U_0(B)/CU(B)$ is zero, whence $U_0(B)/CU(B)$ is torsion free. If B has stable rank one, by Corollary. 3.11 of [46], the map from $U_0(B)/CU(B)$ to $U_0(M_n(B))/CU(M_n(B))$ is an isomorphism for all $n \geq 1$. Therefore, by Theorem 3.2 of [112], the group $U_0(B)/CU(B)$ is isomorphic to $\mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$. Let $D = K_0(U)$. Then we may view D as a dense subgroup of $\mathbb{Q}$. It follows that, for any $x \in K_0(B)$ and $r \in D$, $r\rho_B(x) \in \rho_B(K_0(B))$. Therefore $\overline{\rho_B(K_0(B))}$ is divisible. Hence $U_0(B)/CU(B)$ is torsion free. □

THEOREM 11.6. *Let A be a unital C^*-algebra such that there is a number $K > 0$ such that* $\mathrm{cel}(u) \leq K$ *for all $u \in CU(A)$. Suppose that $U_0(A)/CU(A)$ is torsion free and suppose that $u, v \in U(A)$ such that $u^*v \in U_0(A)$. Suppose also that there is $k \in \mathbb{N}$ such that* $\mathrm{cel}((u^k)^*v^k) \leq L$ *for some $L > 0$. Then*

$$\mathrm{cel}(u^*v) \leq K + L/k. \tag{e 11.6}$$

PROOF. It follows from [101] that, for any $\varepsilon > 0$, there are $a_1, a_2, ..., a_N \in A_{s.a.}$ such that

$$(u^k)^*v^k = \prod_{j=1}^{N} \exp(\sqrt{-1}a_j) \text{ and } \sum_{j=1}^{N} \|a_j\| \leq L + \varepsilon/2. \tag{e 11.7}$$

Choose $w = \prod_{j=1}^{N} \exp(-\sqrt{-1}a_j/k)$. Then $(u^*vw)^k \in CU(A)$. Since $U_0(A)/CU(A)$ is assumed to be torsion free, it follows that

$$u^*vw \in CU(A). \tag{e 11.8}$$

Thus, $\mathrm{cel}(u^*vw) \leq K$. Note that $\mathrm{cel}(w) \leq L/k + \varepsilon/2k$. It follows that

$$\mathrm{cel}(u^*v) \leq K + L/k + \varepsilon/2k.$$

□

COROLLARY 11.7. *Let A be a unital simple C^*-algebra in $\mathcal{B}_1$, and let $B = A \otimes U$, where U is a UHF-algebra of infinite type. Then*

(1) *$U_0(B)/CU(B)$ is torsion free and divisible; and*
(2) *if $u, v \in U(B)$ with* $\mathrm{cel}((u^*)^kv^k) \leq L$ *for some integer $k > 0$, then*

$$\mathrm{cel}(u^*v) \leq 7\pi + L/k.$$

PROOF. The lemma follows from Lemma 11.5, Corollary 11.4, and Theorem 11.6. □

The following lemma consists of some standard perturbation results in the same spirit as that of Lemma 2.19. Some of these have been used in some of the proofs before. We present them here for convenience.

LEMMA 11.8. *Let A be a unital C^*-algebra in $\mathcal{B}_1$. Let $e \in A$ be a projection with $[1-e] = K[e]$ in $K_0(A)$ for some positive integer K. Suppose that $u \in U_0(eAe)$ and $w = u + (1-e)$ with $\mathrm{dist}(\bar{w}, \bar{1}) \le \eta < 2$. Suppose that $\mathcal{F} \subset A$ is a finite set, R is a positive integer, and $\varepsilon > 0$. Then there are a non-zero projection $p \in A$ and a C^*-subalgebra $D \in \mathcal{C}$ with $1_D = p$ such that*

(1) $\|[p, x]\| < \varepsilon$ *for all* $x \in \mathcal{F} \cup \{u, w, e, (1-e)\}$,
(2) $pxp \in_\varepsilon D$ *for* $x \in \mathcal{F} \cup \{u, w, e, (1-e)\}$,
(3) *there are a projection* $q \in D$, *a unitary* $z_1 \in qDq$, *and* $c_1 \in CU(D)$ *such that* $\|q - pep\| < \varepsilon$, $\|z_1 - quq\| < \varepsilon$, $\|z_1 \oplus (p-q) - pwp\| < \varepsilon$, *and* $\|z_1 \oplus (p-q) - c_1\| < \varepsilon + \eta$,
(4) *there are a projection* $q_0 \in (1-p)A(1-p)$, *a unitary* $z_0 \in q_0Aq_0$, *and* $c_0 \in CU((1-p)A(1-p))$ *such that* $\|q_0 - (1-p)e(1-p)\| < \varepsilon$, $\|z_0 - (1-p)u(1-p)\| < \varepsilon$, $\|z_0 \oplus (1-p-q_0) - (1-p)w(1-p)\| < \varepsilon$, *and* $\|z_0 \oplus (1-p-q_0) - c_0\| < \varepsilon + \eta$,
(5) $[p-q] = K[q]$ *in* $K_0(D)$, $[(1-p) - q_0] = K[q_0]$ *in* $K_0(A)$,
(6) $R[1-p] < [p]$ *in* $K_0(A)$, *and*
(7) $\mathrm{cel}_{(1-p)A(1-p)}(z_0 \oplus (1-p-q_0)) \le \mathrm{cel}_A(w) + \varepsilon$.

PROOF. Without loss of generality, we may assume that $\varepsilon < \frac{1}{2}\min\{\eta, 2-\eta, 1/2\}$.

Since $[1-e] = K[e]$ and A has stable rank one (see 9.7), there are mutually orthogonal projections $e_1, e_2, \cdots, e_K \in A$ and partial isometries $s_1, s_2, \cdots, s_K \in A$ satisfying $s_i^* s_i = e$, $s_i s_i^* = e_i$ for all $i = 1, 2, \cdots, K$ and $1 - e = \sum_{i=1}^K e_i$.

Let $\mathrm{cel}_A(w) = L$. Then (see the last line of 2.15) there are $h_1, h_2, ..., h_M \in A_{s.a.}$ such that $w = \prod_{j=1}^M \exp(ih_j)$ and $\sum_{j=1}^M \|h_j\| < L + \varepsilon/4$. Choose $\delta_0 > 0$ such that, if $h' \in A_{s.a.}$ (for any unital C^*-algebra A) with $\|h'\| \le L+1$, $p' \in A$ is a projection and $\|p'h' - h'p'\| < \delta_0$, then

$$\|p' \exp(ih') - p' \exp(ip'h'p')p'\| < \varepsilon/(64N(L+1)(M+1))). \tag{e 11.9}$$

Since $\mathrm{dist}(\overline{w}, \bar{1}) \le \eta < 2$, there exist unitaries $\{u_i, v_i\}_{i=1}^N \subset A$ such that

$$\|w - \Pi_{i=1}^N u_i v_i u_i^* v_i^*\| < \eta + \frac{\varepsilon}{4}. \tag{e 11.10}$$

Let $\mathcal{F}'$ be the set

$$\mathcal{F} \cup \{u, w, e, 1-e;\ \ e_k, s_k : 1 \le k \le K;\ \ h_j : 1 \le j \le M;\ \ u_i, v_i : 1 \le i \le N\}.$$

Let δ ($< \varepsilon/16N(L+1)(M+1)$) be the positive number δ in Lemma 2.19 for the positive integer K and $\min\{\delta_0, \varepsilon/16N(L+1)(M+1)\}$ (in place of ε). Recall that (by 9.7) A has stable rank one. As in the proof of Theorem 11.1, choose any positive element $a \in A_+$ such that $1 - p \lesssim a$ implies $R[1-p] < [p]$ in $K_0(A)$.

Since $A \in \mathcal{B}_1$, there are a non-zero projection $p \in A$ and a C^*-subalgebra $D \in \mathcal{C}$ with $1_D = p$ such that, for all $x \in \mathcal{F}'$,

$$(i)\ \ \|[p, x]\| < \delta, \quad (ii)\ \ pxp \in_\delta D, \quad \text{and} \quad (iii)\ \ 1 - p \lesssim a. \tag{e 11.11}$$

Then (1) follows from (i), and (2) follows from (ii). If we do not require the existence of c_1 in (3) and c_0 in (4) and the estimates involving c_1 and c_0, then (3) (the existence of q, z_1 and all estimates not involving c_1) and (4) (the existence of q_0, z_0 and all estimates not involving c_0) follow from part (a) and part (b) of Lemma 2.19, and (5) follows from part (c) of Lemma 2.19. Furthermore, (6) follows from (iii) above by the choice of a. We emphasize that, since δ is the number δ for $\frac{\varepsilon}{16N(L+1)(M+1)} < \frac{\varepsilon}{16N(L+1)} < \frac{\varepsilon}{16N}$ (instead of ε) in Lemma 2.19, we have

$$\|z_1 \oplus (p-q) - pwp\| < \frac{\varepsilon}{16N} \quad \text{and} \tag{e 11.12}$$

$$\|z_0 \oplus (1-p-q_0) - (1-p)w(1-p)\| < \frac{\varepsilon}{16N(L+1)}.$$

It remains to show that (7) holds and the existence of c_0 and c_1. Note, since $\delta < \delta_0$,

$$\|ph_j - h_jp\| < \delta_0, \ \ j = 1, 2, ..., M. \tag{e 11.13}$$

By part (a) and part (b) of Lemma 2.19, there are untaries $\{u_i', v_i'\}_{i=1}^N \subset D \subset pAp$, $\{u_i'', v_i''\}_{i=1}^N \subset (1-p)A(1-p)$ such that

$$\|u_i' - pu_ip\| < \frac{\varepsilon}{16N}, \quad \|v_i' - pv_ip\| < \frac{\varepsilon}{16N}, \tag{e 11.14}$$

$$\|u_i'' - (1-p)u_i(1-p)\| < \frac{\varepsilon}{16N}, \ \ \|v_i'' - (1-p)v_i(1-p)\| < \frac{\varepsilon}{16N}, \tag{e 11.15}$$

for all $i = 1, 2, \cdots, N$, and such that, by the choice of δ_0 and by (e 11.9),

$$\|(1-p)w(1-p) - (1-p)(\prod_{j=1}^M \exp(i(1-p)h_j(1-p)))(1-p)\| < \frac{\varepsilon}{64N(L+1)}. \tag{e 11.16}$$

Put $w_0 = (1-p)\prod_{j=1}^M \exp(i(1-p)h_j(1-p))(1-p)$ and $h_j' = (1-p)h_j(1-p)$, $j = 1, 2, ..., M$. Then $w_0 \in U_0((1-p)A(1-p))$. In $(1-p)A(1-p)$, $w_0 = \prod_{j=1}^M \exp(ih_j')$. Note that $\sum_{j=1}^M \|h_j'\| \le \sum_{j=1}^M \|h_j\| < L + \varepsilon/4$. From (e 11.12) and (e 11.16) we have

$$\begin{aligned}\|z_0 \oplus (1-p-q_0) - w_0\| &\le \|z_0 \oplus (1-p-q_0) - (1-p)w(1-p)\| \\ &\quad + \|(1-p)w(1-p) - w_0\| \\ &< \frac{\varepsilon}{8(L+1)}.\end{aligned}$$

There is $h_0' \in ((1-p)A(1-p))_{s.a.}$ such that

$$\|h_0\| < 2\arcsin(\varepsilon/16(L+1)) \text{ and } \tag{e 11.17}$$

$$z_0 \oplus (1-p-q_0) = w_0 \exp(ih_0) = \prod_{j=0}^{M} \exp(ih_j').$$

Then $\mathrm{cel}_{(1-p)A(1-p)}(z_0 \oplus (1-p-q_0)) < L + \varepsilon/4 + \varepsilon/2 < \mathrm{cel}(w) + \varepsilon$. So (7) holds.

Let $c_1 = \Pi_{i=1}^N u_i' v_i' (u_i')^* (v_i')^*$ and $c_0 = \Pi_{i=1}^N u_i'' v_i'' (u_i'')^* (v_i'')^*$. Then by (e 11.12), (e 11.10), (i) of (e 11.11), and (e 11.14), one gets

$$\begin{aligned}\|z_1 \oplus (p-q) - c_1\| &< \|z_1 \oplus (p-q) - pwp\| + \|pwp - p(\Pi_{i=1}^N u_i v_i u_i^* v_i^*)p\| \\ &+ \|p(\Pi_{i=1}^N u_i v_i u_i^* v_i^*)p - \Pi_{i=1}^N (pu_ipv_ipu_i^*pv_i^*p)\| + \|\Pi_{i=1}^N (pu_ipv_ipu_i^*pv_i^*p) - c_0\| \\ &< \frac{\varepsilon}{16N} + (\eta + \frac{\varepsilon}{4}) + (4N\delta) + (4N\frac{\varepsilon}{16N}) < \eta + \varepsilon.\end{aligned}$$

Similarly, $\|z_0 \oplus (1-p-q_0) - c_0\| < \varepsilon + \eta$. That is, (3) and (4) hold.

□

LEMMA 11.9. *Let $K \geq 1$ be an integer. Let A be a unital simple C^*-algebra in $\mathcal{B}_1$. Let $e \in A$ be a projection and let $u \in U_0(eAe)$. Suppose that $w = u + (1-e) \in U_0(A)$ and let $\eta \in (0,2]$. Suppose that*

$$[1-e] \leq K[e] \;\; \textit{in} \;\; K_0(A) \;\; \textit{and} \;\; \mathrm{dist}(\bar{w}, \bar{1}) \leq \eta. \tag{e 11.18}$$

Then, if $\eta < 2$, one has

$$\mathrm{dist}(\bar{u}, \bar{e}) < (K + 9/8)\eta \;\; \textit{and} \;\; \mathrm{cel}_{eAe}(u) < (\frac{(K+1)\pi}{2} + 1/16)\eta + 6\pi,$$

and if $\eta = 2$, one has

$$\mathrm{cel}_{eAe}(u) < (\frac{9K}{8} + 1)\mathrm{cel}(w) + 1/16 + 6\pi.$$

PROOF. We assume that (e 11.18) holds. Note that $\eta \leq 2$. Put $L = \mathrm{cel}(w)$. We first consider the case that $\eta < 2$. There is a projection $e' \in M_m(A)$ (for some integer m) such that

$$[(1-e) + e'] = K[e].$$

Note $M_m(A) \in \mathcal{B}_1$. Replacing A by $(1_A + e')M_m(A)(1_A + e')$ (which is in $\mathcal{B}_1$ by Theorem 9.9) and w by $w + e'$, without loss of generality, we may now assume that

$$[1-e] = K[e] \text{ and } \mathrm{dist}(\bar{w}, \bar{1}) < \eta. \tag{e 11.19}$$

There is $R_1 > 1$ such that $\max\{L/R_1, 2/R_1, \eta\pi/R_1\} < \min\{\eta/64, 1/16\pi\}$.

For any $\eta/32(K+1)^2\pi > \varepsilon > 0$ with $\varepsilon + \eta < 2$, since $gTR(A) \leq 1$, by Lemma 11.8, there exist a non-zero projection $p \in A$ and a C^*-subalgebra $D \in \mathcal{C}$ with $1_D = p$ such that

(1) $\|[p,\, x]\| < \varepsilon$ for $x \in \{u, w, e, (1-e)\}$,
(2) $pwp, pup, pep, p(1-e)p \in_\varepsilon D$,
(3) there are a projection $q \in D$, a unitary $z_1 \in qDq$, and $c_1 \in CU(D)$ such that $\|q-pep\| < \varepsilon$, $\|z_1-quq\| < \varepsilon$, $\|z_1 \oplus (p-q)-pwp\| < \varepsilon$, and $\|z_1 \oplus (p-q)-c_1\| < \varepsilon + \eta$,
(4) there are a projection $q_0 \in (1-p)A(1-p)$ and a unitary $z_0 \in q_0Aq_0$ such that
$\|q_0-(1-p)e(1-p)\| < \varepsilon$, $\|z_0-(1-p)u(1-p)\| < \varepsilon$, $\|z_0 \oplus (1-p-q_0) - (1-p)w(1-p)\| < \varepsilon$ and $\|z_0 \oplus (1-p-q_0)-c_0\| < \varepsilon + \eta$,
(5) $[p-q] = K[q]$ in $K_0(D)$, $[(1-p)-q_0] = K[q_0]$ in $K_0(A)$,
(6) $2(K+1)R_1[1-p] < [p]$ in $K_0(A)$, and
(7) $\mathrm{cel}_{(1-p)A(1-p)}(z_0 \oplus (1-p-q_0)) \le L + \varepsilon$,

where $c_1 \in CU(D)$ and $c_0 \in CU((1-p)A(1-p))$.

By Lemma 3.10, one has that $\det(\psi_D(c_1)) = 1$ for every irreducible representation ψ_D of D. Since $\varepsilon + \eta < 2$, there is $h \in D_{s.a}$ with $\|h\| < 2\arcsin((\varepsilon+\eta)/2)$ such that (by (3) above)

$$(z_1 \oplus (p-q))\exp(ih) = c_1. \tag{e 11.20}$$

By (5) above, since D has stable rank one, we may write $D = M_{K+1}(D_0)$, where $D_0 \cong qDq$. In particular, $D_0 \in \mathcal{C}$ (see 3.19). Let $\{e_{ij}\}$ be a system of matrix units for M_{K+1}. Define $h_i = e_{ii}he_{ii}$, $i = 1, 2, ..., K+1$, and $h_0 = \sum_{j=1}^{K+1} e_{1i}he_{i1}$. Then $\|h_0\| < 2(K+1)\arcsin((\varepsilon+\eta)/2)$. Note $\tau(h_0) = \tau(\sum_{j=1}^{K+1} h_j) = \tau(h)$ for all $\tau \in T(D)$. We may identify h_0 with an element in $(qDq)_{s.a.}$. Put $\omega_0 = \exp(ih_0) \in U_0(qDq)$. Then, $\det(\psi_D((\omega_0 \oplus (p-q))\exp(-ih)) = 1$ for every irreducible representation ψ_D of D. It follows by (e 11.20) that, for every irreducible representation ψ_D of D,

$$\det(\psi_D(z_1\omega_0 \oplus (p-q))) = 1. \tag{e 11.21}$$

By Lemma 3.10, this implies $z_1w_0 \oplus (p-q) \in CU(M_{K+1}(qDq))$. Since qDq has stable rank one (by Proposition 3.3), by Corollary 3.11 of [46], $zw_0 \in CU(qDq)$. It follows that, if $2(K+1)\arcsin(\frac{\varepsilon+\eta}{2}) < \pi$,

$$\begin{aligned} \mathrm{dist}(\overline{z_1}, \overline{q}) &= \mathrm{dist}(\overline{\omega_0}, \overline{q}) \\ &< 2\sin((K+1)\arcsin(\frac{\varepsilon+\eta}{2})) \le (K+1)(\varepsilon+\eta) \end{aligned} \tag{e 11.22}$$

(see (e 2.33)). If $2(K+1)\arcsin(\frac{\varepsilon+\eta}{2}) \ge \pi$, then $2(K+1)(\frac{\varepsilon+\eta}{2})\frac{\pi}{2} \ge \pi$. It follows that

$$(K+1)(\varepsilon+\eta) \ge 2 \ge \mathrm{dist}(\overline{z_1}, \overline{q}). \tag{e 11.23}$$

By combining both (e 11.23) and (e 11.23), one obtains that

$$\mathrm{dist}(\overline{z_1}, \overline{q}) \le (K+1)(\varepsilon+\eta) \le (K+1)\eta + \frac{\eta}{32(K+1)\pi}. \tag{e 11.24}$$

It follows from Lemma 3.14 that

$$\begin{aligned} \mathrm{cel}_{qDq}(z_1) &\le (K+1)(\varepsilon+\eta)\frac{\pi}{2}+4\pi \\ &\le ((K+1)\frac{\pi}{2}+\frac{1}{64(K+1)})\eta+4\pi. \end{aligned} \tag{e 11.25}$$

By (5) and (6) above,

$$(K+1)[q]=[p-q]+[q]=[p]>2(K+1)R_1[1-p].$$

By Theorem 9.11, $K_0(A)$ is weakly unperforated. Hence,

$$2R_1[1-p]<[q]. \tag{e 11.26}$$

There is a unitary $v\in A$ such that $v^*(1-p-q_0)v\le q$. Put $v_1=q_0\oplus(1-p-q_0)v$. Then

$$v_1^*(z_0\oplus(1-p-q_0))v_1=z_0\oplus v^*(1-p-q_0)v. \tag{e 11.27}$$

Note that (by (4))

$$\|(z_0\oplus v^*(1-p-q_0)v)v_1^*c_0^*v_1-q_0\oplus v^*(1-p-q_0)v\|<\varepsilon+\eta. \tag{e 11.28}$$

Moreover, by (7) above,

$$\mathrm{cel}(z_0\oplus v^*(1-p-q_0)v)\le L+\varepsilon. \tag{e 11.29}$$

It follows from (e 11.26) and Lemma 6.4 of [71] that

$$\mathrm{cel}_{(q_0+q)A(q_0+q)}(z_0\oplus q)\le 2\pi+(L+\varepsilon)/R_1. \tag{e 11.30}$$

Therefore, on combining this with (e 11.25),

$$\begin{aligned} &\mathrm{cel}_{(q_0+q)A(q_0+q)}(z_0+z_1) \\ &\le 2\pi+(L+\varepsilon)/R_1+((K+1)\frac{\pi}{2}+\frac{1}{64(K+1)})\eta+4\pi. \end{aligned} \tag{e 11.31}$$

By (e 11.29), (e 11.26), and Lemma 3.1 of [75], in $U_0((q_0+q)A(q_0+q))/CU((q_0+q)A(q_0+q))$,

$$\mathrm{dist}(\overline{z_0+q},\overline{q_0+q})<\frac{(L+\varepsilon)}{R_1}. \tag{e 11.32}$$

Therefore, by (e 11.32), (e 11.24) (and by the line below (e 2.33)),

$$\begin{aligned} &\mathrm{dist}(\overline{z_0\oplus z_1},\overline{q_0+q}) \\ &< \frac{(L+\varepsilon)}{R_1}+(K+1)\eta+\frac{\eta}{32(K+1)\pi}<(K+\frac{17}{16})\eta. \end{aligned} \tag{e 11.33}$$

We note that

$$\|e-(q_0+q)\|<2\varepsilon \text{ and } \|u-(z_0+z_1)\|<2\varepsilon. \tag{e 11.34}$$

It follows that

$$\mathrm{dist}(\bar{u},\bar{e})<4\varepsilon+(K+\frac{17}{16})\eta<(K+\frac{9}{8})\eta. \tag{e 11.35}$$

Similarly, by (e 11.31),

$$\begin{aligned}\mathrm{cel}_{eAe}(u) &\le 4\varepsilon\pi+2\pi+(L+\varepsilon)/R_1+((K+1)\frac{\pi}{2}+\frac{1}{64(K+1)})\eta+4\pi\\ &< ((K+1)\frac{\pi}{2}+1/16)\eta+6\pi.\end{aligned}$$

This proves the case that $\eta<2$.

Now suppose that $\eta=2$. Define an integer $R=[\mathrm{cel}(w)+1]\ge 4$. Note that $\mathrm{cel}(w)/R<1$. There is a projection $e'\in M_{R'}(A)$ (for some integer R') such that

$$[e']=2R[e]+2R[1-e]=2R[1_A].$$

It follows from Lemma 3.1 of [75] that

$$\mathrm{dist}(\overline{w\oplus e'},\overline{1_A+e'})<\frac{\mathrm{cel}(w)}{R}. \tag{e 11.36}$$

Put $K_1=2R(K+1)+K$. Then

$$\begin{aligned}([1-e]+[e'])&\le K[e]+2R[e]+2R[1-e]\\ &\le (K+2R+2RK)[e]=K_1[e].\end{aligned} \tag{e 11.37}$$

It follows from the first part of the lemma that

$$\begin{aligned}\mathrm{cel}_{eAe}(u) &< (\frac{(K_1+1)\pi}{2}+\frac{1}{16})\frac{\mathrm{cel}(w)}{R}+6\pi\\ &= (K+1+K/2R+1/16R)\mathrm{cel}(w)+6\pi\\ &\le (K+1+K/8)\mathrm{cel}(w)+(1/16)+6\pi\\ &= (9K/8+1)\mathrm{cel}(w)+1/16+6\pi.\end{aligned}$$

□

THEOREM 11.10. *(Theorem 4.6 of [46]) Let A be a unital simple C^*-algebra of stable rank one and let $e\in A$ be a non-zero projection. Then the map $u\mapsto u+(1-e)$ induces an isomorphism from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$.*

PROOF. This is Theorem 4.6 of [46]. □

COROLLARY 11.11. *Let A be a unital simple C^*-algebra of stable rank one. Then the map $j: a\to \mathrm{diag}(a,\overbrace{1,1,...,1}^{n-1})$ from A to $M_n(A)$ induces an isomorphism from $U(A)/CU(A)$ onto $U(M_n(A))/CU(M_n(A))$ for any integer $n\ge 1$.*

PROOF. This follows from 11.10 but also follows from 3.11 of [46]. □

12. A Uniqueness Theorem for C^*-algebras in $\mathcal{B}_1$

The following is taken from Definition 2.1 of [45].

DEFINITION 12.1. Let $\mathbf{r} : \mathbb{N} \to \mathbb{N} \cup \{0\}$, $\mathbf{T} : \mathbb{N}^2 \to \mathbb{N}$, and $\mathbf{E} : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ be maps, and $\mathbf{k}, \mathbf{R} \in \mathbb{N}$ be two positive integers. Let A be a unital C^*-algebra.

(1) We say that A has K_0-$\mathbf{r}$-cancellation if $p \oplus \mathbf{1}_{M_{\mathbf{r}(n)}(A)} \sim q \oplus \mathbf{1}_{M_{\mathbf{r}(n)}(A)}$ for any two projections $p, q \in M_n(A)$ with $[p] = [q]$ in $K_0(A)$.

(2) We say that A has K_1-$\mathbf{r}$-cancellation if $u \oplus \mathbf{1}_{M_{\mathbf{r}(n)}(A)}$ and $v \oplus \mathbf{1}_{M_{\mathbf{r}(n)}(A)}$ are in the same connected component of $U(M_{\mathbf{r}(n)}(A))$ for any pair $u, v \in M_n(A)$ with $[u] = [v]$ in $K_1(A)$.

(3) We say that A has K_1-stable rank at most $\mathbf{k}$ if $U(M_{\mathbf{k}}(A))$ is mapped surjectively to $K_1(A)$.

(4) We say that A has stable exponential rank at most $\mathbf{R}$ if $\mathrm{cer}(M_m(A)) \leq \mathbf{R}$ for all m.

(5) We say that A has K_0-divisible rank $\mathbf{T}$ if for any $x \in K_0(A)$ and any pair $(n, k) \in \mathbb{N} \times \mathbb{N}$,

$$-n[\mathbf{1}_A] \leq kx \leq n[\mathbf{1}_A]$$

implies that

$$-\mathbf{T}(n,k)[\mathbf{1}_A] \leq x \leq \mathbf{T}(n,k)[\mathbf{1}_A].$$

(6) We say that A has K_1-divisible rank $\mathbf{T}$ if $k[x] = [u]$ in $K_1(A)$ for some unitary $u \in M_n(A)$ implies that $[x] = [v]$ for some unitary $v \in M_{\mathbf{T}(n,k)}(A)$.

(7) We say that A has exponential length divisible rank $\mathbf{E}$ if $u \in U_0(M_n(A))$ with $\mathrm{cel}(u^k) \leq L$ implies that $\mathrm{cel}(u) \leq \mathbf{E}(L, k)$.

(8) Let $\mathbf{C}_{\mathbf{R},\mathbf{r},\mathbf{T},\mathbf{E}}$ be the class of unital C^* algebras which have K_i-$\mathbf{r}$-cancellation, K_i-divisible rank $\mathbf{T}$, exponential length divisible rank $\mathbf{E}$, and stable exponential rank at most $\mathbf{k}$.

REMARK 12.2. If A has stable rank one, then, by a standard result of Rieffel ([100]), A has K_i-0-cancellation, K_1-stable rank 1 and K_1-divisible rank $\mathbf{T}$ for any $\mathbf{T}$ (since any element in $K_1(A)$ can be realized by a unitary $u \in A$). If $K_0(A)$ is weakly unperforated, then by Proposition 2.2 (5) of [45], A has K_0-divisible rank $\mathbf{T}$ for $\mathbf{T}(n, k) = n + 1$.

The following theorem follows from Theorem 7.1 of [75]. We refer to 12.1 for some of the notation used below. Recall that, as before, if $L : A \to B$ is a map, we continue to use L for the extension $L \otimes \mathrm{id}_{M_n} : A \otimes M_n \to B \otimes M_n$ (see also 2.11). Recall also that, when A is unital, in the following statement and its proof, we always assume that $\varphi(1_A)$ and $\psi(1_A)$ are projections (see the 5th paragraph of 2.12). We also use the convention $\langle \varphi(v) \rangle$ for $\langle \varphi(v) \rangle + (1 - \varphi(1_A))$ when φ is a approximately multiplicative map and v is a unitary.

THEOREM 12.3. (cf. Theorem 7.1 of [75]) *Let A be a unital separable amenable C^*-algebra which satisfies the UCT, let $T \times N : A_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$ be a map and let $\mathbf{L} : U(M_\infty(A)) \to \mathbb{R}_+$ be another map. For any $\varepsilon > 0$ and any* *finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{U} \subset \bigcup_{m=1} U(M_m(A))$, a finite subset $\mathcal{P} \subset \underline{K}(A)$, and an integer $n > 0$ satisfying the following condition: for any unital separable simple C^*-algebra C in $\mathcal{B}_1$, if $\varphi, \psi, \sigma : A \to B = C \otimes U$, where U is a UHF-algebra of infinite type, are three $\mathcal{G}$-δ-multiplicative completely positive linear maps and σ is unital and $T \times N$-$\mathcal{H}$-full (see* 2.23*) with the properties that*

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \ \text{and} \ \mathrm{cel}(\langle\psi(v)\rangle^*\langle\varphi(v)\rangle) \le \mathbf{L}(v) \tag{e 12.1}$$

for all $v \in \mathcal{U}$, then there exists a unitary $u \in M_{n+1}(B)$ such that

$$\|u^*\mathrm{diag}(\varphi(a), \overline{\sigma}(a))u - \mathrm{diag}(\psi(a), \overline{\sigma}(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$, where

$$\overline{\sigma}(a) = \mathrm{diag}(\overbrace{\sigma(a), \sigma(a), ..., \sigma(a)}^{n}) = \sigma(a) \otimes 1_n \ \text{ for all } a \in A.$$

Moreover, the conclusion holds if one only assumes $B \in \mathbf{C}_{\mathbf{k},\mathbf{r},\mathbf{T},\mathbf{E}}$ for certain $\mathbf{k}, \mathbf{r}, \mathbf{T}, \mathbf{E}$ as in Definition 12.1*.*

Furthermore, if $K_1(A)$ is finitely generated, and $\mathcal{U}_0 \in U(M_m(A))$ is any finite set such that $\{[u] \in K_1(A) : u \in \mathcal{U}_0\}$ generates the whole group $K_1(A)$, then, in the above statement, we can always choose the set $\mathcal{U}$ to be this fixed set $\mathcal{U}_0$ for any $\mathcal{F}$ and ε.

PROOF. By Theorem 9.7, $B\,(= C \otimes U \in \mathcal{B}_1)$ has stable rank one. By Corollary 11.2, $\mathrm{cer}(M_m(B)) \le 6$ for all $m \in \mathbb{N}$. By part (2) of Corollary of 11.7, A has exponential length divisible rank $\mathbf{E}(L, k) = 7\pi + L/k$ (see (7) of 12.1). By Theorem 9.11, $K_0(B)$ is weakly unperforated (in particular, by Remark 12.2, B has K_0-divisible rank $\mathbf{T}(L, k) = L + 1$). In other words, $B = C \otimes U \in \mathbf{C}_{\mathbf{R},\mathbf{r},\mathbf{T},\mathbf{E}}$ for $\mathbf{R} = 6$, $\mathbf{r}(n) = 0$, $\mathbf{T}(L, k) = L + 1$ and $\mathbf{E}(L, k) = 7\pi + L/k$. Therefore, it suffices to prove the "Moreover" part (and "Furthermore" part) of the statement. Note the case that both φ and ψ are unital follows from Theorem 7.1 of [75]. However, the following argument also works for the non-unital case. Denote by $\mathbf{C}$ the class $\mathbf{C}_{\mathbf{R},\mathbf{r},\mathbf{T},\mathbf{E}}$ of unital C^*-algebras for $\mathbf{R} = 6$, $\mathbf{r}(n) = 0$, $\mathbf{T}(L, k) = L+1$, and $\mathbf{E}(L, k) = 7\pi + L/k$.

Suppose that the conclusion of the theorem is false. Then there exists $\varepsilon_0 > 0$ and a finite subset $\mathcal{F} \subset A$ such that there are a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \searrow 0$, an increasing sequence $\{\mathcal{G}_n\}$ of finite subsets of A such that $\bigcup_n \mathcal{G}_n$ is dense in A, an increasing sequence $\{\mathcal{P}_n\}$ of finite subsets of $\underline{K}(A)$ with $\bigcup_{n=1} \mathcal{P}_n = \underline{K}(A)$, an increasing sequence $\{\mathcal{U}_n\}$of finite subsets of $U(M_\infty(A))$ such that $\{[u] \in K_1(A) : u \in \bigcup_{n=1}^{\infty} \mathcal{U}_n\}$ generates $K_1(A)$ (for the case that $K_1(A)$ is generated by the single finite set $\mathcal{U}_0 \in M_m(A)$, we set $\mathcal{U}_n = \mathcal{U}_0$), an increasing sequence $\{\mathcal{H}_n\}$ of finite subsets of $A_+^{\mathbf{1}} \setminus \{0\}$ such that if $a \in \mathcal{H}_n$ and $f_{1/2}(a) \neq 0$, then $f_{1/2}(a) \in \mathcal{H}_{n+1}$, and $\bigcup_{n=1} \mathcal{H}_n$ is dense in $A_+^{\mathbf{1}}$, a sequence of positive integers $\{k(n)\}$ with $\lim_{n\to\infty} k(n) = \infty$, a sequence of unital C^*-algebras

$B_n \in \mathbf{C}$, sequences of $\mathcal{G}_n$-δ_n-multiplicative contractive completely positive linear maps $\varphi_n, \psi_n : A \to B_n$, such that

$$[\varphi_n]|_{\mathcal{P}_n} = [\psi_n]|_{\mathcal{P}_n} \text{ and } \mathrm{cel}(\langle \varphi_n(u)\rangle \langle \psi_n(u)\rangle^*) \le \mathbf{L}(u) \tag{e 12.2}$$

for all $u \in \mathcal{U}_n$, and a sequence of unital $\mathcal{G}_n$-δ_n-multiplicative completely positive linear maps, $\sigma_n : A \to B_n$, which are $T \times N$-$\mathcal{H}_n$-full, such that for any $n \in \mathbb{N}$,

$$\begin{aligned}\inf \Big\{ &\{\sup \|{v_n}^* \mathrm{diag}(\varphi_n(a), S_n(a)) v_n \\ &-\mathrm{diag}(\psi_n(a), S_n(a))\| : a \in \mathcal{F}\} : v_n \in U(M_{k(n)+1}(B_n))\Big\} \ge \varepsilon_0,\end{aligned} \tag{e 12.3}$$

where

$$S_n(a) = \mathrm{diag}(\overbrace{\sigma_n(a), \sigma_n(a), ..., \sigma_n(a)}^{k(n)}) = \sigma_n(a) \otimes 1_{k(n)} \text{ for all } a \in A.$$

Let $C_0 = \bigoplus_{n=1}^{\infty} B_n$, $C = \prod_{n=1}^{\infty} B_n$, $Q(C) = C/C_0$, and $\pi : C \to Q(C)$ be the quotient map. Define $\Phi, \Psi, S : A \to C$ by $\Phi(a) = \{\varphi_n(a)\}$, $\Psi(a) = \{\psi_n(a)\}$, and $S(a) = \{\sigma_n(a)\}$ for all $a \in A$. Note that $\pi \circ \Phi$, $\pi \circ \Psi$ and $\pi \circ S$ are homomorphisms.

As in the proof of 7.1 of [75], since $B_n \in \mathbf{C}$, one computes that

$$[\pi \circ \Phi] = [\pi \circ \Psi] \text{ in } KL(A, Q(C)) \tag{e 12.4}$$

as follows.

For each fixed element $x \in K_1(A)$, there are integers m, r such that $\mathcal{U}_m \subset M_r(A)$ and such that x is in the group generated by $\{[u] \in K_1(A) : u \in \mathcal{U}_m\}$. In the case that $K_1(A)$ is finitely generated, x is in the group generated by $\mathcal{U}_0$. Then, for $n \ge m$, we have that $\mathrm{cel}(\langle \varphi_n(u)\rangle \langle \psi_n(u)\rangle^*) \le \mathbf{L}(u)$. Hence by Lemma 1.1 of [45], there are a constant $\mathbf{K} = K(\mathbf{L}(u))$ and $U_n(t) \in M_r(B_n)$ such that $U_n(0) = \langle \varphi_n(u)\rangle$, $U_n(1) = \langle \psi_n(u)\rangle$, and $\|U_n(t) - U_n(t')\| \le \mathbf{K}|t - t'|$. Therefore, $\{U_n\}_{n=m}^{\infty} \in C([0,1], M_r(\prod_{n=m}^{\infty} B_n))$ and consequently

$$\begin{aligned}(\pi \circ \Phi)_{*1}([u]) &= [\{\langle \varphi_n(u)\rangle\}_n] = [\{\langle \psi_n(u)\rangle\}_n] \\ &= (\pi \circ \Psi)_{*1}([u]) \text{ in } K_1(Q(C)) \text{ for all } u \in \mathcal{U}_m.\end{aligned}$$

Thus, $(\pi \circ \Phi)_{*1}(x) = (\pi \circ \Phi)_{*1}(x)$. Note that this includes the case that $\mathcal{U}_m = \mathcal{U}_0$ when $K_1(A)$ is generated by $\{[u] : u \in \mathcal{U}_0\}$. In other words, $[\pi \circ \Phi]_{K_1(A)} = [\pi \circ \Psi]_{K_1(A)} : K_1(A) \to K_1(Q(C))$.

Since all B_n have K_0-$\mathbf{r}$-cancellation, by (1) (see (5) also) of Proposition 2.1 of page 992 in [45], $K_0(\prod B_n) = \prod_b K_0(B_n)$ (see page 990 of [45] for the definition of $\prod_b$). Hence by 5.1 of [43], $K_0(Q(C)) = \prod_b K_0(B_n)/\bigoplus K_0(B_n)$. Since all B_n have K_i-$\mathbf{r}$-cancellation, K_i-divisible rank $\mathbf{T}$, exponential length divisible rank $\mathbf{E}$, and stable exponential rank at most $\mathbf{R}$, by Theorem 2.1 and (3) of Proposition 2.1 on page 994 of [45], we have

$$\begin{aligned} K_i\Big(\prod_n B_n, \mathbb{Z}/k\mathbb{Z}\Big) &\subset \prod_n K_i(B_n, \mathbb{Z}/k\mathbb{Z}), K_i(Q(C), \mathbb{Z}/k\mathbb{Z}) \\ &\subset \prod_n K_i(B_n, \mathbb{Z}/k\mathbb{Z})/\bigoplus_n K_i(B_n, \mathbb{Z}/k\mathbb{Z}). \end{aligned}$$

That is, (2) of Proposition 2.2 of [45] holds for B_n, even though we do not assume that B_n have stable rank one. Consequently, by (e 12.2), we have $[\pi\circ\Phi]_{K_0(A)} = [\pi\circ\Psi]_{K_0(A)} : K_0(A) \to K_0(Q(C))$, and, for $i = 0, 1$, $[\pi\circ\Phi]_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = [\pi\circ\Psi]_{K_i(A,\mathbb{Z}/k\mathbb{Z})} : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_i(Q(C), \mathbb{Z}/k\mathbb{Z})$, since $\bigcup_{n=1} \mathcal{P}_n = \underline{K}(A)$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, $n = 1, 2, ...$

Since A satisfies UCT, we have $[\pi\circ\Phi] = [\pi\circ\Psi]$ in $KL(A, Q(C))$, i.e., (e 12.4) holds.

By Proposition 2.24, $\pi\circ S$ is a full homomorphism. It follows from Theorem 4.14 that there exist an integer $K \geq 1$ and a unitary $U \in M_{K+1}(Q(C))$ such that

$$\begin{aligned} &\|U^*\mathrm{diag}(\pi\circ\Phi(a), \Sigma(a))U \\ &\qquad -\mathrm{diag}(\pi\circ\Psi(a), \Sigma(a))\| < \varepsilon_0/4 \ \text{ for all } \ a \in \mathcal{F}, \end{aligned} \tag{e 12.5}$$

where $\Sigma(a) = \mathrm{diag}(\overbrace{(\pi\circ S)(a), (\pi\circ S)(a), ..., (\pi\circ S)(a)}^{K}) = (\pi\circ S)(a) \otimes 1_K$. It follows that there exist a unitary $V = \{v_n\} \in C$ and an integer $N \geq 1$ such that, for any $n \geq N$ with $k(n) \geq K$,

$$\|v_n^*\mathrm{diag}(\varphi_n(a), \overline{\sigma}_n(a))v_n - \mathrm{diag}(\psi_n(a), \overline{\sigma}_n(a))\| < \varepsilon_0/2 \tag{e 12.6}$$

for all $a \in \mathcal{F}$, where

$$\overline{\sigma}_n(a) = \mathrm{diag}(\overbrace{\sigma_n(a), \sigma_n(a), ..., \sigma_n(a)}^{K}) \ \text{ for all } \ a \in A.$$

This contradicts (e 12.3). $\square$

REMARK 12.4. (1) Note that φ and ψ are not assumed to be unital. Thus Theorem 4.15 can also be viewed as a special case of Theorem 12.3.

(2) If A has K_1-stable rank k, then in the proof of Theorem 12.3, we can choose the matrix size $r = k$ (for $u \in \mathcal{U}_m \subset M_r(A)$ to represent an element $x \in K_1(A)$). Suppose that there exists an integer $n_0 \geq 1$ such that $U(M_{n_0}(A))/U_0(M_{n_0}(A)) \to U(M_{n_0+k}(A))/U_0(M_{n_0+k}(A))$ is an isomorphism for all $k \geq 1$. Then A has K_1-stable rank n_0, and the map $\mathbf{L}$ may be replaced by a map from $U(M_{n_0}(A))$ to $\mathbb{R}_+$, and $\mathcal{U}$ can be chosen in $U(M_{n_0}(A))$.

Moreover, for the case $B = C \otimes U$ for $C \in \mathcal{B}_1$ in the theorem, by Corollary 11.4, the condition that $\mathrm{cel}(\langle\varphi(u)\rangle\langle\psi(u)\rangle^*) \leq \mathbf{L}(u)$ may, in practice, be replaced by the stronger condition that, for all $\bar{u} \in \overline{\mathcal{U}}$,

$$\mathrm{dist}(\varphi^{\ddagger}(\bar{u}), \psi^{\ddagger}(\bar{u})) < \mathbf{L}, \tag{e 12.7}$$

where $\overline{\mathcal{U}} \subset U(M_m(A))/CU(M_m(A))$ is a finite subset and where $\mathbf{L} < 2$ is a given constant, and where $\varphi^{\ddagger}$ and $\psi^{\ddagger}$ are maps from $U(M_m(A))/CU(M_m(A))$ to $U(M_m(B))/CU(M_m(B))$ (see also 2.17).

To see this, let $\mathcal{U}$ be a finite subset of $U(M_m(A))$ for some large m whose image in the group $U(M_m(A))/CU(M_m(A))$ is $\overline{\mathcal{U}}$. Then (e 12.7) implies that

$$\|\langle\varphi(u)\rangle\langle\psi(u)\rangle^* - v\| < 2 \tag{e 12.8}$$

for some $v \in CU(M_m(B))$, provided that δ is sufficiently small and $\mathcal{G}$ is sufficiently large. By 11.4, $\mathrm{cel}(v) \leq 7\pi$. Also from (e 12.8), one gets $(\langle\varphi(u)\rangle\langle\psi(u)\rangle^*)v^* = \exp(ih)$ with $\|h\| < \pi$. We conclude that

$$\mathrm{cel}(\langle\varphi(u)\rangle\langle\psi(u^*)\rangle) \leq \pi + 7\pi \tag{e 12.9}$$

for all $u \in \mathcal{U}$, and take $\mathbf{L} : U_\infty(A) \to \mathbb{R}_+$ to be constant equal to 8π. Furthermore, we may assume

$$\overline{\mathcal{U}} \subset U(M_{n_0}(A))/CU(M_{n_0}(C)),$$

if $K_1(C) = U(M_{n_0}(C))/U_0(M_{n_0}(C))$.

LEMMA 12.5. *Let A be a unital separable simple C^*-algebra with $T(A) \neq \varnothing$. Then there exists an order preserving map $\Delta_0 : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ with the following property: For any finite subset $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$, there exist a finite subset $\mathcal{G} \subset A$ and $\delta > 0$ such that, for any unital C^*-algebra B with $T(B) \neq \varnothing$ and any unital $\mathcal{G}$-δ-multiplicative completely positive linear map $\varphi : A \to B$, one has*

$$\tau \circ \varphi(h) \geq \Delta_0(\hat{h})/2 \ \textit{for all}\ h \in \mathcal{H} \tag{e 12.10}$$

and for all $\tau \in T(B)$, and moreover, one may assume that $\Delta_0(\widehat{1_A}) = 3/4$.

PROOF. Define, for each $h \in A_+^{\mathbf{1}} \setminus \{0\}$,

$$\Delta_0(\hat{h}) = \min\{3/4, \inf\{\tau(h) : \tau \in T(A)\}\}. \tag{e 12.11}$$

Let $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$ be a finite subset. Define

$$d = \min\{\Delta_0(\hat{h})/4 : h \in \mathcal{H}\} > 0. \tag{e 12.12}$$

Let $\delta > 0$ and let $\mathcal{G} \subset A$ be a finite subset as provided by 5.7 for $\varepsilon = d$ and $\mathcal{F} = \mathcal{H}$.

Suppose that $\varphi : A \to B$ is a unital $\mathcal{G}$-δ-multiplicative completely positive linear map. Then, by 5.7, for each $t \in T(B)$, there exists $\tau \in T(A)$ such that

$$|t \circ \varphi(h) - \tau(h)| < d \text{ for all } h \in \mathcal{H}. \tag{e 12.13}$$

It follows that $t \circ \varphi(h) > \tau(h) - d$ for all $h \in \mathcal{H}$. Thus,

$$t \circ \varphi(h) > \Delta_0(\hat{h}) - d > \Delta_0(\hat{h})/2 \tag{e 12.14}$$

for $h \in \mathcal{H}$ and $t \in T(B)$. □

LEMMA 12.6. *Let C be a unital C^*-algebra, and let $\Delta : C_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a map $T \times N : C_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$ with the following property: For any finite subset $\mathcal{H} \subset C_+^{\mathbf{1}} \setminus \{0\}$ and any unital C^*-algebra A with strict comparison of positive elements, if $\varphi : C \to A$ is a unital completely positive linear map satisfying*

$$\tau \circ \varphi(h) \geq \Delta(\hat{h}) \textit{ for all } h \in \mathcal{H} \textit{ for all } \tau \in \mathrm{T}(A), \tag{e 12.15}$$

then φ is $(T \times N)$-$\mathcal{H}$-full.

PROOF. For each $\delta \in (0,1)$, let $g_\delta : [0,1] \to [0,+\infty)$ be the continuous function defined by

$$g_\delta(t) = \begin{cases} 0 & \text{if } t \in [0, \delta/4], \\ f_{\delta/2}(t)/t & \text{otherwise,} \end{cases}$$

where $f_{\delta/2}$ is as defined in 2.5. Note that

$$g_\delta(t)t = f_{\delta/2}(t) \text{ for all } t \in [0,1]. \tag{e 12.16}$$

Let $h \in C_+^{\mathbf{1}} \setminus \{0\}$. Then define

$$T(h) = \|(g_{\Delta(\widehat{h})})^{\frac{1}{2}}\| = \sqrt{\frac{2}{\Delta(\widehat{h})}} \text{ and } N(h) = \lceil \frac{2}{\Delta(\widehat{h})} \rceil.$$

Then the function $T \times N$ has the property of the lemma.

Indeed, let $\mathcal{H} \subset C_+^1 \setminus \{0\}$ be a finite subset. Let A be a unital C^*-algebra with strict comparison for positive elements, and let $\varphi : C \to A$ be a unital positive linear map satisfying

$$\tau \circ \varphi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H} \text{ for all } \tau \in \mathrm{T}(A). \tag{e 12.17}$$

Put $\varphi(h)^- = (\varphi(h) - \frac{\Delta(\hat{h})}{2})_+$. Then, by (e 12.17), one has that, since $0 \leq h \leq 1$,

$$d_\tau(\varphi(h)^-) = d_\tau((\varphi(h) - \frac{\Delta(\hat{h})}{2})_+) \geq \tau((\varphi(h) - \frac{\Delta(\hat{h})}{2})_+) \geq \frac{\Delta(\hat{h})}{2} \text{ for all } \tau \in \mathrm{T}(A).$$

This shows in particular that $\varphi(h)^- \neq 0$. Since A has strict comparison for positive elements, one has $K\langle(\varphi(h)^-\rangle > \langle 1_A\rangle$, where $K = \lceil \frac{2}{\Delta(\hat{h})}\rceil$ and where $\langle x\rangle$ denotes the class of x in $W(A)$.

Therefore there is a partial isometry $v = (v_{ij})_{K\times K} \in \mathrm{M}_K(A)$ such that

$$vv^* = 1_A \quad \text{and} \quad v^*v \in \overline{(\varphi(h)^- \otimes 1_K)M_K(A)(\varphi(h)^- \otimes 1_K)}.$$

Note that $c(f_{\delta/2}(\varphi(h)) \otimes 1_K) = (f_{\delta/2}(\varphi(h)) \otimes 1_K)c = c$ for all

$$c \in \overline{(\varphi(h)^- \otimes 1_K)M_K(A)(\varphi(h)^- \otimes 1_K)},$$

where $\delta = \Delta(\hat{h})$, and therefore

$$v(f_{\delta/2}(\varphi(h)) \otimes 1_K)v^* = vv^* = 1_A.$$

Considering the upper-left corner of $M_K(A)$, one has $\sum_{i=1}^K v_{1,i}f_{\delta/2}(\varphi(h))v_{1,i}^* = 1_A$, and therefore, by (e 12.16), one has

$$\sum_{i=1}^K v_{1,i}(g_{\Delta(\hat{h})}(\varphi(h)))^{\frac{1}{2}}\varphi(h)(g_{\Delta(\hat{h})}(\varphi(h)))^{\frac{1}{2}}v_{1,i}^* = 1_A.$$

Since v is a partial isometry, one has $\|v_{i,j}\| \leq 1$, $i, j = 1, ..., K$, and therefore

$$\|v_{1,i}(g_{\Delta(\hat{h})}(\varphi(h)))^{\frac{1}{2}}\| \leq \|(g_{\Delta(\hat{h})}(\varphi(h)))^{\frac{1}{2}}\| \leq \|(g_{\Delta(\hat{h})})^{\frac{1}{2}}\| = T(h).$$

Hence the map φ is $T \times N$-$\mathcal{H}$-full, as desired. $\square$

THEOREM 12.7. *Let C be a unital C^*-algebra in $\bar{\mathcal{D}}_s$ (see 4.8) with finitely generated $K_i(C)$ $(i = 0, 1)$. Let $\mathcal{F} \subset C$ be a finite subset, let $\varepsilon > 0$ be a positive number and let $\Delta : C_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exist a finite subset $\mathcal{H}_1 \subset C_+^{\mathbf{1}} \setminus \{0\}$, constants $\gamma_1 > 0$, $1 > \gamma_2 > 0$, and $\delta > 0$, a finite subset $\mathcal{G} \subset C$, a finite subset $\mathcal{P} \subset \underline{K}(C)$, a finite subset $\mathcal{H}_2 \subset C_{s.a.}$, and a finite subset $\mathcal{U} \subset U(M_{n_0}(C))/CU(M_{n_0}(C))$ (for some $n_0 \geq 1$) for which $[\mathcal{U}] \subset \mathcal{P}$ satisfying the following condition: For any unital $\mathcal{G}$-δ-multiplicative completely positive linear maps $\varphi, \psi : C \to A$, where $A = A_1 \otimes U$ for some $A_1 \in \mathcal{B}_1$ and a UHF-algebra U of infinite type, satisfying*

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{e 12.18}$$

$$\tau(\varphi(a)) \geq \Delta(\hat{a}), \ \ \tau(\psi(a)) \geq \Delta(\hat{a}) \tag{e 12.19}$$

for all $\tau \in T(A)$ and for all $a \in \mathcal{H}_1$,

$$|\tau \circ \varphi(a) - \tau \circ \psi(a)| \ < \ \gamma_1 \ \text{for all} \ a \in \mathcal{H}_2, \ \text{and} \tag{e 12.20}$$

$$\mathrm{dist}(\varphi^\ddagger(z), \psi^\ddagger(z)) \ < \ \gamma_2 \ \text{for all} \ z \in \mathcal{U}, \tag{e 12.21}$$

there exists a unitary $W \in A$ such that

$$\|W^*\varphi(f)W - \psi(f)\| < \varepsilon \ \text{for all} \ f \in \mathcal{F}. \tag{e 12.22}$$

PROOF. Let $T' \times N : C_+ \setminus \{0\} \to \mathbb{R}_+\{0\} \times \mathbb{N}$ be the map of Lemma 12.6 with respect to C and $\Delta/4$. Let $T = 2T'$.

Define $\mathbf{L} = 1$. Let $\delta_0 > 0$ (in place of δ), $\mathcal{G}_0 \subset C$ (in place of $\mathcal{G}$), $\mathcal{H}_0 \subset C_+ \setminus \{0\}$ (in place of $\mathcal{H}$), $\mathcal{V}_0 \subset U(M_{n_0}(C))$ (in place of $\mathcal{U}$), and $\mathcal{P}_0 \subset \underline{K}(C)$ (in place of $\mathcal{P}$) be finite subsets and n_1 (in place of n) be an integer as provided by Theorem 12.3 (with the modification as in (2) of 12.4, and with the second inequality of (e 12.2) replaced by (e 12.7)) with respect to C (in place of A), $T \times N$, $\mathbf{L}$, $\mathcal{F}$, and $\epsilon/2$. Put $\mathcal{U}_0 = \{\bar{v} \in U(M_{n_0}(C))/CU(M_{n_0}(C)) : v \in \mathcal{V}_0\}$.

Let $\mathcal{H}_{1,1} \subset C_+^1 \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\mathcal{H}_{1,2} \subset A$ (in place of $\mathcal{H}_2$), $1 > \gamma_{1,1} > 0$ (in place of γ_1), $1 > \gamma_{1,2} > 0$ (in place of γ_2), $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset C$ (in place of $\mathcal{G}$), $\mathcal{P}_1 \subset \underline{K}(C)$ (in place of $\mathcal{P}$), $\mathcal{U}_1 \subset J_c(K_1(C))$ (in place of $\mathcal{U}$), and n_2 (in place of N) be the finite subsets and constants provided by Theorem 8.3 with respect to C (in place of A), $\Delta/4$, $\mathcal{F}$, and $\epsilon/4$.

Put $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$, $\delta = \min\{\delta_0/4, \delta_1/4\}$, $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$, $\mathcal{H}_1 = \mathcal{H}_{1,1}$, $\mathcal{H}_2 = \mathcal{H}_{1,2}$, $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$, $\gamma_1 = \gamma_{1,1}/2$, and $\gamma_2 = \gamma_{1,2}/2$. We assert that these are the desired finite subsets and constants (for $\mathcal{F}$ and ϵ). We may assume that $\gamma_2 < 1/4$.

In fact, let $A = A_1 \otimes U$, where $A \in \mathcal{B}_1$ and U is a UHF-algebra of infinite type. Let $\varphi, \psi : C \to A$ be $\mathcal{G}$-δ-multiplicative maps satisfying (e 12.18) to (e 12.21) for the above-chosen $\mathcal{G}$, $\mathcal{H}_1$, $\mathcal{P}$, $\mathcal{H}_2$, $\mathcal{U}$, γ_1, and γ_2. Applying Lemma 12.6, and by the choice of $T' \times N$ at the beginning of the proof, we know that both φ and ψ are $T' \times N$-$\mathcal{H}_1$-full.

Since $A = A_1 \otimes U$, $A \cong A \otimes U$. Moreover, the map $j \circ \imath : A \to A$ is approximately inner, where $\imath : A \to A \otimes U$ is defined by $a \mapsto a \otimes 1_U$ and $j : A \otimes U \to A$ is some isomorphism. Thus, we may assume that $A = A_1 \otimes U \otimes U = A_2 \otimes U$, where $A_2 = A_1 \otimes U$. Moreover, without loss of generality, we may assume that the images of φ and ψ are in A_2. Since $A_2 \in \mathcal{B}_1$, for every finite subset $\mathcal{G}'' \subset A_2$, $\delta' > 0$, and integer $m \geq 1$, there are a projection $p \in A_2$ and a C^*-subalgebra $D \in \mathcal{C}_1$ with $p = 1_D$ such that

(1) $\|pg - gp\| < \delta'$ for any $g \in \mathcal{G}''$,
(2) $pgp \in_{\delta'} D$, and
(3) $\tau(1-p) < \min\{\delta', \gamma_1/4, 1/8m\}$ for any $\tau \in \mathrm{T}(A)$.

Define $j_0 : A_2 \to (1-p)A_2(1-p)$ by $j_0(a) = (1-p)a(1-p)$ for all $a \in A_2$. With sufficiently small δ' and large $\mathcal{G}''$, applying Lemma 9.13, one obtains a unital completely positive linear map $j_1 : A_2 \to D$ with $\|j_1(a) - pap\| < \delta'$ for all $a \in \mathcal{F}$ such that $j_i \circ \varphi$ and $j_i \circ \psi$ are $\mathcal{G}$-2δ-multiplicative, $i = 0, 1$, and

(4) $\|\varphi(c) - (j_0 \circ \varphi(c) \oplus j_1 \circ \varphi(c))\| < \varepsilon/16$ and $\|\psi(c) - (\psi_0(c) \oplus \psi_1(c))\| < \varepsilon/16$, for any $c \in \mathcal{F}$,
(5) $j_0 \circ \varphi, j_0 \circ \psi$ and $j_1 \circ \varphi, j_1 \circ \psi_1$ are $2T' \times N$-$\mathcal{H}_1$-full,
(6) $\tau \circ (j_1 \circ \varphi(c) > \Delta(\hat{c})/2$ and $\tau \circ j_1 \circ \psi(c) > \Delta(\hat{c})/2$ for any $c \in \mathcal{H}_1$ and for any $\tau \in T(D)$,
(7) $|\tau \circ \varphi_1(c) - \tau \circ \psi_1(c)| < 2\gamma_1$ for any $\tau \in T(D)$ and any $c \in \mathcal{H}_2$,
(8) $\mathrm{dist}((j_i \circ \varphi)^{\ddagger}(u), (j_i \circ \psi)^{\ddagger}(u)) < 2\gamma_2$ for any $u \in \mathcal{U}$, $i = 0, 1$, and
(9) $[j_0 \circ \varphi]|_{\mathcal{P}} = [j_0 \circ \psi]|_{\mathcal{P}}$ and $[j_1 \circ \varphi]|_{\mathcal{P}} = [j_1 \circ \psi]|_{\mathcal{P}}$.

Choose an integer $m \ge 2(n_1+1)n_2$ and mutually orthogonal and mutually equivalent projections $e_1, e_2..., e_m \in U$ with $\sum_{i=1}^m e_i = 1_U$. Define $\varphi_i', \psi_i' : C \to A \otimes U$ by $\varphi_i'(c) = \varphi(c) \otimes e_i$ and $\psi_i'(c) = \psi(c) \otimes e_i$ for all $c \in C$, $i = 1, 2, ..., m$. Note that

$$[\varphi_1']|_{\mathcal{P}} = [\varphi_i']|_{\mathcal{P}} = [\psi_1']|_{\mathcal{P}} = [\psi_i']|_{\mathcal{P}}, \tag{e 12.23}$$

$i = 1, 2, ..., m$. Note also that $\varphi_i', \psi_i' : C \to e_iAe_i$ are $\mathcal{G}$-δ-multiplicative.

Write $m = kn_2 + r$, where $k \ge n_1 + 1$ and $r < n_2$ are integers. Define $\tilde{\varphi}, \tilde{\psi} : C \to (1-p)A_2(1-p) \oplus \bigoplus_{i=kn_2+1}^m A_2 \otimes e_i$ by

$$\tilde{\varphi}(c) = j_0 \circ \varphi(c) \oplus \sum_{i=kn_2+1}^{m} j_1 \circ \varphi(c) \otimes e_i \ \text{ and} \tag{e 12.24}$$

$$\tilde{\psi}(c) = j_0 \circ \psi(c) \oplus \sum_{i=kn_2+1}^{m} j_1 \circ \psi(c) \otimes e_i \tag{e 12.25}$$

for all $c \in C$. With sufficiently large $\mathcal{G}''$ and small δ', we may assume that $\tilde{\varphi}$ and $\tilde{\psi}$ are $\mathcal{G}$-2δ-multiplicative and, by (e 12.23),

$$[\tilde{\varphi}]|_{\mathcal{P}} = [\tilde{\psi}]|_{\mathcal{P}}. \tag{e 12.26}$$

Moreover, by (8) above, we have

$$\mathrm{dist}((j_i \circ \varphi)^{\ddagger}(\bar{v}), (j_i \circ \psi)^{\ddagger}(\bar{v})) < 2\gamma_2 \le \gamma_{1,2} \le \mathbf{L} \ \text{ for } \ i = 1, 2, \ \text{ and } \ \bar{v} \in \mathcal{U},$$

which implies

$$\mathrm{dist}(\tilde{\varphi}^{\ddagger}(\bar{v}), \tilde{\psi}^{\ddagger}(\bar{v})) < \mathbf{L} \tag{e 12.27}$$

for all $\bar{v} \in \mathcal{U}$. Define $\varphi_i^1, \psi_i^1 : C \to D \otimes e_i$ by $\varphi_i^1(c) = j_1 \circ \varphi(c) \otimes e_i$ and $\psi_i^1(c) = j_1 \circ \psi(c) \otimes e_i$. By (7) and (8) above, we have

$$\tau \circ \varphi_i^1(h) \ge \Delta(\hat{h})/2 \ \text{ and } \ \tau(\psi_i^1(h)) \ge \Delta(\hat{h})/2 \ \text{ for all } \ h \in \mathcal{H}_1, \tag{e 12.28}$$

$$|\tau \circ \varphi_i^1(c) - \tau \circ \psi_i^1(c)| < \gamma_{1,1} \ \text{ for all } \ c \in \mathcal{H}_2 \tag{e 12.29}$$

and for all $\tau \in T(pAp \otimes e_i)$, and, furthermore,

$$\mathrm{dist}((\varphi_i^1)^{\ddagger}(\bar{v}), (\psi_i^1)^{\ddagger}(\bar{v})) < \gamma_{1,2} \ \text{ for all } \ \bar{v} \in \mathcal{U}. \tag{e 12.30}$$

By (5) above, we have that φ_i^1 and ψ_i^1 are $T \times N$-$\mathcal{H}_1$-full, since $T = 2T'$. Moreover, we may also assume that φ_i^1 and ψ_i^1 are $\mathcal{G}$-2δ-multiplicative, and, by (12),

$$[\varphi_i^1]|_{\mathcal{P}} = [\psi_i^1]|_{\mathcal{P}}. \tag{e 12.31}$$

Define $\Phi, \Psi : C \to \bigoplus_{i=1}^{kn_2} D \otimes e_i$ by

$$\Phi(c) = \bigoplus_{i=1}^{kn_2} \varphi_i^1(c) \text{ and } \Psi(c) = \bigoplus_{i=1}^{kn_2} \psi_i^1(c) \tag{e 12.32}$$

for all $c \in C$. By (e 12.31), (e 12.28), (e 12.29), (e 12.30), and by Theorem 8.3, there exists a unitary $W_1 \in (\sum_{i=1}^{kn_2} p \otimes e_i)(A_2 \otimes U)(\sum_{i=1}^{kn_2} p \otimes e_i)$ such that

$$\|W_1^*\Phi(c)W_1 - \Psi(c)\| < \varepsilon/4 \text{ for all } c \in \mathcal{F}. \tag{e 12.33}$$

Note that

$$\tau(1-p) + \sum_{kn_2+1}^{m} \tau(e_i) < (1/m) + (r/m) \le n_2/m \tag{e 12.34}$$

for all $\tau \in T(A)$. Note also that $k \ge n_1$. By (e 12.26) and (e 12.27), since ψ_i^1 is $T \times N$-$\mathcal{H}_1$-full, on applying Theorem 12.3, one obtains a unitary $W_2 \in A$ such that

$$\|W_2^*(\tilde{\varphi}(c) \oplus \Psi(c))W_1 - (\tilde{\psi}(c) \oplus \Psi(c))\| < \varepsilon/2 \tag{e 12.35}$$

for all $c \in \mathcal{F}$. Set

$$W = (\mathrm{diag}(1-p, e_{kn_2+1}, e_{kn_2+2}, ..., e_m) \oplus W_1)W_2.$$

Then we compute that

$$\|W^*(\tilde{\varphi}(c) \oplus \Phi(c))W - (\tilde{\psi}(c) \oplus \Psi(c))\| < \varepsilon/2 + \varepsilon/4 \tag{e 12.36}$$

for all $c \in \mathcal{F}$. By (12), we have

$$\|W^*\varphi(c)W - \psi(c)\| < \varepsilon \text{ for all } c \in \mathcal{F} \tag{e 12.37}$$

as desired. □

Theorem 12.7 can be strengthened as follows.

COROLLARY 12.8. *(1) Theorem* 12.7 *still holds, if, in* (e 12.19)*, only one of the two inequalities holds.*

(2) In Theorem 12.7*, one can choose* $\mathcal{U} \subset U(M_{n_0}(C))/CU(M_{n_0}(C))$ *to be a finite subset of a torsion free subgroup of* $J_c(K_1(C))$ *(see* 2.16*). Furthermore, if* C *has stable rank* k*, then* $\mathcal{U} \subset J_c(K_1(C))$ $(= J_c(U(M_k(C))/U_0(M_k(C))))$ *may be chosen to be a finite subset of a free subgroup of (the abelian group)* $U(M_k(C)/CU(M_k(C))$*. In the case that* C *has stable rank one, then* $\mathcal{U}$ *may be assumed to be a subset of a free subgroup of* $U(C)/CU(C)$*. In the case that* $C = C' \otimes C(\mathbb{T})$ *for some* C' *with stable rank one, then the stable rank of* C *is no more than* 2*. Therefore, in this case,* $\mathcal{U}$ *may be assumed to be a finite subset of a free subgroup of* $U(M_2(C))/CU(M_2(C))$*.*

PROOF. For part (1), let Δ be given. Choose $\Delta_1 = (1/2)\Delta$. Suppose $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}_1$ and $\mathcal{H}_2$, δ, γ_1, and γ_2 are as provided by Theorem 12.7 for the given ε, $\mathcal{F}$, and Δ_1 (instead of Δ).

Set $\sigma = \min\{\Delta(\hat{a}) : a \in \mathcal{H}_1\}$ and $\gamma_1' = \min\{\gamma_1, \sigma/2\}$. Choose $\mathcal{H}_3 = \mathcal{H}_1 \cup \mathcal{H}_2$. Now suppose that

$$\text{(e 12.38)} \qquad \begin{aligned} &\varphi(a) \geq \Delta(\hat{a}) \text{ for all } a \in \mathcal{H}_1 \text{ and} \\ &|\tau(\varphi(b)) - \tau(\psi(b))| < \gamma_1' \text{ for all } b \in \mathcal{H}_3. \end{aligned}$$

Then

$$\text{(e 12.39)} \qquad \psi(a) \geq \Delta(\hat{a}) - \gamma_1' \geq \Delta_1(\hat{a}) \text{ for all } a \in \mathcal{H}_1.$$

This shows that, replacing $\mathcal{H}_2$ by $\mathcal{H}_3$ (or by choosing $\mathcal{H}_2 \supset \mathcal{H}_1$) and replacing γ_1 by γ_1', we only need one inequality in (e 12.19). This proves part (1).

For part (2), let $\mathcal{U}' \in U(M_{n_0}(C))/CU(M_{n_0}(C))$ be a finite subset and $\gamma_2' > 0$ be given. Suppose that $\mathcal{U} \subset U(M_{n_0}(C))/CU(M_{n_0}(C))$ is another finite subset such that $\mathcal{U}' \subset G(\mathcal{U})$, the subgroup generated by $\mathcal{U}$. Then, it is routine to check that, there exist a sufficiently small $\gamma_2 > 0$ and a sufficiently large finite subset $\mathcal{G} \subset C$ and small $\delta > 0$ that, for any $\mathcal{G}$-δ-multiplicative completely positive contractions $\varphi,\ \psi : C \to B$, if

$$\operatorname{dist}(\varphi^{\ddagger}(u), \psi^{\ddagger}(z)) < \gamma_2 \text{ for all } z \in \mathcal{U},$$

then

$$\operatorname{dist}(\varphi^{\ddagger}(u), \psi^{\ddagger}(z)) < \gamma_2' \text{ for all } z \in \mathcal{U}'.$$

This shows that, we may replace a finite subset $\mathcal{U}$ by any generating subset of the subgroup $G(\mathcal{U})$ (with possibly larger $\mathcal{G}$ and smaller δ and γ_2).

Note that, since $K_0(C)$ is finitely generated, $\rho_C^{n_0}(K_0(C)) = \rho_C(K_0(C))$ if n_0 is chosen large enough (see Definition 2.16). By Definition 2.16, this implies that $U(M_\infty(C))/CU(M_\infty(C)) \cong \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \cong U(M_{n_0}(C)/CU(M_{n_0}(C))$. Since $K_i(C)$ is finitely generated, on choosing a larger n_0 if necessary, $K_1(C)$ is generated by images of unitaries in $U(M_{n_0}(C))$. Write

$$\begin{aligned} U(M_{n_0}(C))/CU(M_{n_0}(C)) &= U(M_\infty(C))/CU(M_\infty(C)) \\ &= \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \bigoplus J_c(K_1(C)) \end{aligned}$$

as in 2.16. We may write $K_1(C) = G_f \oplus G_t$, where G_f is a finitely generated free abelian group and G_t is a finite group. By the previous paragraph, we may assume that $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_f \sqcup \mathcal{U}_t$, where $\mathcal{U}_0 \subset U_0(M_{n_0}(C))/CU(M_{n_0}(C))$, $\mathcal{U}_f \subset J_c(G_f)$, and $\mathcal{U}_t \subset J_c(G_t)$.

Suppose, in Theorem 12.7, that $\mathcal{U}$ has been chosen as above. We then enlarge $\mathcal{P}$ so that $\mathcal{P}$ contains G_t. Then, if $z \in \mathcal{U}_t$, by the assumption $[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}$,

$\kappa_1^C(\varphi^\ddagger(z)-\psi^\ddagger(z))=0$, where $\kappa_1^C : U(M_\infty(C))/CU(M_\infty(C)) \to K_1(C)$ is the quotient map defined in 2.16. It follows that

$$\varphi^\ddagger(z)-\psi^\ddagger(z)\in U_0(M_{n_0}(A))/CU(M_{n_0}(A))=\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}.$$

Also, since $J_c(G_t)$ is a torsion group, there is an $m>0$ such that $m\big(\varphi^\ddagger(z)-\psi^\ddagger(z)\big)=0$. Since $A=A_1\otimes U$ and $A_1\in\mathcal{B}_1$, by 11.7, $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ is torsion free. Therefore $\varphi^\ddagger(z)-\psi^\ddagger(z)=0$. In other words, $[\varphi]|_{\mathcal{P}}=[\psi]|_{\mathcal{P}}$ implies that $\varphi^\ddagger|_{\mathcal{U}_t}=\psi^\ddagger|_{\mathcal{U}_t}$ when $G_t\subset\mathcal{P}$, which means we may assume that $\mathcal{U}=\mathcal{U}_0\sqcup\mathcal{U}_f$ (by choosing $\mathcal{P}\supset G_t$).

Next, fix the finite subset $\mathcal{U}_0$. If $\gamma_2>0$ is given, by Lemma 2.18, if $\mathcal{H}_2$ and $\mathcal{G}$ are sufficiently large and δ and γ_1 are sufficiently small, then

$$\mathrm{dist}(\varphi^\ddagger(z),\psi^\ddagger(z))<\gamma_2 \text{ for all } z\in\mathcal{U}_0. \tag{e 12.40}$$

In other words, (e 12.40) follows from (e 12.20), provided that $\mathcal{H}_2$ and $\mathcal{G}$ are sufficiently large and δ and γ_1 are sufficiently small. Thus the first part of (2) of the corollary follows.

The case that C has stable rank k also follows since we have $K_1(C)=U(M_k(C))/U_0(M_k(C))$. In particular, if $k=1$, then $K_1(C)=U(C)/U_0(C)$. If $C=C'\otimes C(\mathbb{T})$ for some C' which has stable rank one, then C has stable rank at most two (see Theorem 7.1 of [99]). So the last statement also follows.

□

Corollary 12.9. *Let $\varepsilon>0$ be a positive number and let $\Delta: C(\mathbb{T})_+^{q,\mathbf{1}}\setminus\{0\}\to(0,1)$ be an order preserving map. There exist a finite subset $\mathcal{H}_1\subset C(\mathbb{T})_+^{\mathbf{1}}\setminus\{0\}$, $\gamma_1>0$, $1>\gamma_2>0$, and a finite subset $\mathcal{H}_2\subset C(\mathbb{T})_{s.a.}$ satisfying the following condition: For any two unitaries u_1 and u_2 in a unital separable simple C^*-algebra $A=A'\otimes U$ with $A'\in\mathcal{B}_1$ and U a UHF-algebra of infinite type such that*

$$[u_1]=[u_2]\in K_1(A),\ \tau(f(u_1)),\ \tau(f(u_2))\geq\Delta(\hat{f}) \tag{e 12.41}$$

for all $\tau\in T(C)$ and for all $f\in\mathcal{H}_1$, and

$$|\tau(g(u_1))-\tau(g)(u_2)|<\gamma_1 \text{ for all } g\in\mathcal{H}_2 \text{ and } \mathrm{dist}(\bar{u_1},\bar{u_2})<\gamma_2,$$

there exists a unitary $W\in C$ such that

$$\|W^*u_1W-u_2\|<\varepsilon. \tag{e 12.42}$$

Lemma 12.10. *Let A be a C^*-algebra and X be a compact metric space. Suppose that $y\in(A\otimes C(X))_+\setminus\{0\}$. Then there exist $a(y)\in A_+\setminus\{0\}$, $f(y)\in C(X)_+\setminus\{0\}$, and $r_y\in A\otimes C(X)$ such that $\|a(y)\|\leq\|y\|$, $\|f(y)\|\leq 1$, and $r_y^*yr_y=a(y)\otimes f(y)$.*

PROOF. Identify $A\otimes C(X)$ with $C(X,A)$. Let $x_0\in X$ be such that $\|y(x_0)\|=\|y\|$. There is $\delta>0$ such that $\|y(x)-y(x_0)\|<\|y\|/16$ for $x\in B(x_0,2\delta)$. Let $Y=\overline{B(x_0,\delta)}\subset X$. Let $z(x)=(y(x_0)-\|y\|/4)_+$ for all $x\in Y$. Note $z\neq 0$. By 2.2 and (iv) of 2.4 of [104], there exist $r\in C(Y,A)$ such that $r^*(y|_Y)r=z$. Choose $g\in C(X)_+\setminus\{0\}$ such that $0\le g\le 1$, $g(x)=0$ if $\mathrm{dist}(x,x_0)\ge\delta$, and $g(x)=1$ if $\mathrm{dist}(x,x_0)\le\delta/2$. Since g is a zero outside Y, one may view $rg^{1/2}, zg\in C(X,A)$. Put $r_y=rg^{1/2}$, $a(y)=(y(x_0)-\|y\|/4)_+$ and $f(y)=g$. Then

$$r_y^*yr_y=zg=a(y)\otimes f.$$

□

THEOREM 12.11. *Part (a). Let $A\in\mathcal{B}_1$ be a unital simple C^*-algebra which satisfies the UCT. For any $\varepsilon>0$, and any finite subset $\mathcal{F}\subset A$, there exist $\delta>0$, a finite subset $\mathcal{G}\subset A$, $\sigma_1,\sigma_2>0$, a finite subset $\mathcal{P}\subset\underline{K}(A)$, a finite subset $\mathcal{U}\subset U(A)/CU(A)$ (see 12.12), and a finite subset $\mathcal{H}\in A_{s.a}$ satisfying the following condition:*

Let $B'\in\mathcal{B}_1$, let $B=B'\otimes U$ for some UHF-algebra U of infinite type, and let $\varphi,\psi:A\to B$ be two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps such that

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{e 12.43}$$

$$|\tau\circ\varphi(a)-\tau\circ\psi(a)| < \sigma_1 \text{ for all } a\in\mathcal{H}, \text{and} \tag{e 12.44}$$

$$\mathrm{dist}(\varphi^{\ddagger}(\bar{u}),\psi^{\ddagger}(\bar{u})) < \sigma_2 \text{ for all } \bar{u}\in\mathcal{U}. \tag{e 12.45}$$

Then there exists a unitary $u\in U(B)$ such that

$$\|\mathrm{Ad}\,u\circ\varphi(f)-\psi(f)\|<\varepsilon \text{ for all } f\in\mathcal{F}. \tag{e 12.46}$$

Part (b). Let $A_1\in\mathcal{B}_1$ be a unital simple C^*-algebra which satisfies the UCT, and let $A=A_1\otimes C(\mathbb{T})$. For any $\varepsilon>0$, any finite subset $\mathcal{F}\subset A$, and any order preserving map $\Delta:C(\mathbb{T})_+^{\mathbf{1}}\setminus\{0\}\to(0,1)$, there exist $\delta>0$, a finite subset $\mathcal{G}\subset A$, $\sigma_1,\sigma_2>0$, a finite subset $\mathcal{P}\subset\underline{K}(A)$, a finite subset $\mathcal{H}_1\subset C(\mathbb{T})_+^{\mathbf{1}}\setminus\{0\}$, a finite subset $\mathcal{U}\subset U(M_2(A))/CU(M_2(A))$ (see 12.12), and a finite subset $\mathcal{H}_2\in A_{s.a}$ satisfying the following condition:

Let $B'\in\mathcal{B}_1$, let $B=B'\otimes U$ for some UHF-algebra U of infinite type and let $\varphi,\psi:A\to B$ be two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps such that

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{e 12.47}$$

$$\tau\circ\varphi(1\otimes h) \ge \Delta(\hat{h}) \text{ for all } h\in\mathcal{H}_1 \text{ and } \tau\in T(B), \tag{e 12.48}$$

$$|\tau\circ\varphi(a)-\tau\circ\psi(a)| < \sigma_1 \text{ for all } a\in\mathcal{H}_2, \text{and} \tag{e 12.49}$$

$$\mathrm{dist}(\varphi^{\ddagger}(\bar{u}),\psi^{\ddagger}(\bar{u})) < \sigma_2 \text{ for all } \bar{u}\in\mathcal{U}. \tag{e 12.50}$$

Then there exists a unitary $u\in U(B)$ such that

$$\|\mathrm{Ad}\,u\circ\varphi(f)-\psi(f)\|<\varepsilon \text{ for all } f\in\mathcal{F}. \tag{e 12.51}$$

PROOF. We only prove part (b); part(a) is simpler. Let $1 > \varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Without loss of generality, we may assume that $1_A \in \mathcal{F}$,

$$\mathcal{F} = \{a \otimes f : a \in \mathcal{F}_1 \text{ and } f \in \mathcal{F}_2\},$$

where $\mathcal{F}_1 \subset A$ is a finite subset and $\mathcal{F}_2 \subset C(\mathbb{T})$ is also a finite subset. We further assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are in the unit ball of A and $C(\mathbb{T})$, respectively.

Let $\Delta : C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $T' \times N' : C(\mathbb{T})_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$ be the map as given by 12.6 with respect to $3\Delta/16$. Since A_1 is a unital separable simple C^*-algebra, the identity map on A_1 is $T'' \times N''$-full for some $T'' \times N'' : (A_1)_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$.

Define a map $T \times N : A_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$ as follows: For any $y \in A_+ \setminus \{0\}$, by 12.10, choose $a(y) \in (A_1)_+ \setminus \{0\}$, $f(y) \in C(\mathbb{T})_+ \setminus \{0\}$, and $r_y \in A \otimes C(\mathbb{T})$ such that

$$r_y^* y r_y = a(y) \otimes f(y), \quad \|a(y)\| \le \|y\|, \quad \text{and} \quad \|f(y)\| \le 1. \tag{e 12.52}$$

There are $x_{a(y),1}, x_{a(y),2}, ..., x_{a(y),N''(a(y))} \in A_1$ with $\max\{\|x_{a(y),i}\| : 1 \le i \le N''(a(y))\} = T''(a(y))$ such that

$$\sum_{i=1}^{N''(a(y))} x_{a(y),i}^* a(y) x_{a(y),i} = 1_{A_1}. \tag{e 12.53}$$

Then define

$$\begin{aligned} &(T \times N)(y) \\ &\quad = \quad (1 + \max\{T''(a(y)), T'(f(y))\} \cdot 2\max\{1, \|y\|, \|r_y\|\}, \\ &\qquad\qquad N''(a(y)) \cdot N'(f(y))). \end{aligned} \tag{e 12.54}$$

Let $\mathbf{L} = 1$. Let $\varepsilon/16 > \delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$), $\mathcal{H}_0 \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}$), $\mathcal{U}_1 \subset U(M_2(A))/CU(M_2(A))$ (in part (a), one can choose $\mathcal{U}_1 \subset U(A)/CU(A)$) (in place of $\mathcal{U}$—see (2) of 12.4) (Note that $A \in \mathcal{B}_1$ has stable rank 1 in part (a), $A = A_1 \otimes C(\mathbb{T})$ has stable rank at most 2 in part (b)), $\mathcal{P}_1 \subset \underline{K}(A)$ (in place of $\mathcal{P}$), and $n_0 \ge 1$ (in place of n) be the finite subsets and constants provided by 12.3 for A, $\mathbf{L}$= 1, $\varepsilon/16$ (in place of ε), $\mathcal{F}$, and $T \times N$. Note, here we refer to the inequality (e 12.7) instead of the inequality (e 12.1). Moreover, this also implies that $[L']|_{\mathcal{P}_1}$ is well defined, for any $\mathcal{G}_1$-δ_1-multiplicative contractive completely positive linear map $L' : A \to B'$ (for any C^*-algebra B').

Without loss of generality, we may assume that

$$\mathcal{G}_1 = \{a \otimes g : a \in \mathcal{G}_1' \text{ and } g \in \mathcal{G}_1''\}, \tag{e 12.55}$$

where $\mathcal{G}_1' \subset A_1$ and $\mathcal{G}_1'' \subset C(\mathbb{T})$ are finite subsets. We may further assume that $\mathcal{F}_1 \subset \mathcal{G}_1'$ and $\mathcal{F}_2 \subset \mathcal{G}_1''$, and that $\mathcal{H}_0, \mathcal{G}_1'$, and $\mathcal{G}_1''$ are all in the unit balls, respectively. In particular, $\mathcal{F} \subset \mathcal{G}_1$. Let

$$\begin{aligned} \overline{\mathcal{H}_0} &= \{a(y) \otimes f(y) : y \in \mathcal{H}_0\} \\ &= \{a \otimes f : a \in \mathcal{H}_0' \text{ and } f \in \mathcal{H}_0''\}, \end{aligned} \tag{e 12.56}$$

where $a(y)$, $f(y)$ are defined in (e 12.52), $\mathcal{H}_0' \subset (A_1)_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_0'' \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ are finite subsets. We may assume that $1_{A_1} \in \mathcal{H}_0'$ and $1_{C(\mathbb{T})} \in \mathcal{H}_0''$. For convenience, let us further assume that, for the above integer $n_0 \geq 1$,

$$1/n_0 < \inf\{\Delta(h) : h \in \mathcal{H}_0''\}/16. \tag{e 12.57}$$

Set $M_0 = \sup\{((T'(h) + 1) \cdot N'(h) : h \in \mathcal{H}_0''\}$ and choose $n \geq n_0$ such that $K_0(U) \subset \mathbb{Q}$ is divisible by n, i.e., $r/n \in K_0(U)$ for all $r \in K_0(U)$.

Let $\mathcal{U}_1 = \{\bar{v_1}, \bar{v_2}, ..., \bar{v_K}\}$, where $v_1, v_2, ..., v_K \in U(M_2(A))$. Put $\mathcal{U}_0 = \{v_1, v_2, ..., v_K\}$. Choose a finite subset $\mathcal{G}_u \subset A$ such that

$$v_j \in \{(a_{i,j})_{1 \leq i,j \leq 2} : a_{i,j} \in \mathcal{G}_u \text{ for all } v_j \in \mathcal{U}_0. \tag{e 12.58}$$

Choose a small enough $\delta_1' > 0$ and a large enough finite subset $\mathcal{G}_v \subset A_1$ that the following condition holds: If $p \in A_1$ is a projection such that

$$\|px - xp\| < \delta_1' \text{ for all } x \in \mathcal{G}_v,$$

then there are unitaries $w_j \in (\mathrm{diag}(p,p) \otimes 1_{C(\mathbb{T})})M_2(A)(\mathrm{diag}(p,p) \otimes 1_{C(\mathbb{T})})$ such that (for $1 \leq j \leq K$)

$$\begin{aligned} &\|(\mathrm{diag}(p,p) \otimes 1_{C(\mathbb{T})})v_j(\mathrm{diag}(p,p) \otimes 1_{C(\mathbb{T})}) - w_j\| \\ &\quad < \delta_1/16n \text{ for all } v_j \in \mathcal{U}_0. \end{aligned} \tag{e 12.59}$$

Let

$$\mathcal{G}_2' = \mathcal{F}_1 \cup \mathcal{G}_1' \cup \mathcal{H}_0' \cup \mathcal{G}_v \cup \{a(y), x_{a(y),j}, x_{a(y),j}^*, r_y, r_y^* : y \in \mathcal{H}_0\} \subset A_1, \text{ and}$$

$$M_1 = (\max\{\|x_{a(y),j}\| : y \in \mathcal{H}_0'\} + 1) \cdot \max\{N(y) : y \in \mathcal{H}_0\}. \tag{e 12.60}$$

Put

$$\delta_1'' = \min\{\delta_1', \delta_1\}/(2^{16}(n+1)M_1^2M_0^2). \tag{e 12.61}$$

Since $A_1 \in \mathcal{B}_1$, there exist mutually orthogonal projections $p_0', p_1' \in A_1$, a C^*-subalgebra (of A_1) $C \in \mathcal{C}$ with $1_C = p_1'$, and unital $\mathcal{G}_2'$-$\delta_1''/16$-multiplicative completely positive linear maps $\imath_{00}' : A_1 \to p_0'A_1p_0'$ and $\imath_{01}' : A_1 \to C$ such that

$$\begin{aligned} &\|p_1'x - xp_1'\| < \delta_1''/2, \ \mathrm{diag}(\overbrace{p_0', p_0', ..., p_0'}^{n+1}) \lesssim p_1', \\ &\text{and } \|x - \imath_{00}'(x) \oplus \imath_{01}'(x)\| < \delta_1'' \end{aligned} \tag{e 12.62}$$

for all $x \in \tilde{\mathcal{G}}_2'$, where $p_0'+p_1' = 1_{A_1}$, $\imath_{00}'(a) = p_0'ap_0'$ for all $a \in A_1$, and $\imath_{01}'$ factors through the map $a \mapsto p_1'ap_1'$, and $\tilde{\mathcal{G}}_2' = \{xy : x, y \in \mathcal{G}_2'\} \cup \mathcal{G}_2'$ (see the lines around (e 9.68) and (e 9.69)). Define $p_0 = p_0' \otimes 1_{C(\mathbb{T})}$, $p_1 = p_1' \otimes 1_{C(\mathbb{T})}$. Without loss of generality, we may assume that $p_0' \neq 0$. Since A_1 is simple, there is an integer $N_0 > 1$ such that

$$(N_0-1)[p_0'] \geq [p_1'] \text{ in } W(A_1). \tag{e 12.63}$$

This also implies that

$$(N_0-1)[p_0] \geq [p_1]. \tag{e 12.64}$$

Define $\imath_{00} : A \to p_0Ap_0$ by $\imath_{00}(a \otimes f) = \imath_{00}'(a) \otimes f$ and $\imath_{01} : A \to C \otimes C(\mathbb{T})$ by $\imath_{01}(a \otimes f) = \imath_{01}'(a) \otimes f$ for all $a \in A_1$ and $f \in C(\mathbb{T})$. Define $L_0 : A \to A$ by

$$L_0(a) = \imath_{00}(a) \oplus \imath_{01}(a) \text{ for all } a \in A.$$

For each $v_j \in \mathcal{U}_0$, there exists a unitary $w_j \in M_2(p_0Ap_0)$ such that

$$\|\mathrm{diag}(p_0,p_0)v_j\mathrm{diag}(p_0,p_0) - w_j\| < \delta_1/16n, \ j = 1, 2, ..., K. \tag{e 12.65}$$

Since $C \subset A_1$, $C_{\mathbb{T}} := C \otimes C(\mathbb{T}) \subset A_1 \otimes C(\mathbb{T}) = A$.

Let $\imath_0 : C \to A_1$ be the natural embedding of C as a unital C^*-subalgebra of $p_1'A_1p_1'$. Let $\imath_0^\sharp : C^q \to A_1^q$ be defined by $\imath_0^\sharp(\hat{c}) = \hat{c}$ for $c \in C$. Let $\Delta_0 : A_1^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be the map given by Lemma 12.5 and define

$$\Delta_1(\hat{h}) = \sup\{\Delta_0(\imath_0^\sharp(\hat{h}_1))\Delta(\hat{h}_2)/8 : h \geq h_1 \otimes h_2,$$

$h_1 \in C \setminus \{0\}$ and $h_2 \in C(\mathbb{T})_+ \setminus \{0\}\}$ for all $h \in (C_X)_+^{\mathbf{1}} \setminus \{0\}$.

Let $\mathcal{G}_3' = \imath_{0,1}'(\tilde{\mathcal{G}}_2')$. (Note that $\mathcal{G}_1' \subset \mathcal{G}_2' \subset \tilde{\mathcal{G}}_2'$.) Let $\mathcal{G}_3 = \{a \otimes f : a \in \mathcal{G}_3'$ and $f \in \mathcal{G}_1''\}$. Let $\mathcal{H}_3 \subset (C_{\mathbb{T}})_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\gamma_1' > 0$ (in place of γ_1), $\gamma_2' > 0$ (in place of γ_2), $\delta_2 > 0$ (in place of δ), $\mathcal{G}_4 \subset C_{\mathbb{T}}$ (in place of $\mathcal{G}$), $\mathcal{P}_2 \subset \underline{K}(C_{\mathbb{T}})$ (in place of $\mathcal{P}$), $\mathcal{H}_4' \subset (C_{\mathbb{T}})_{s.a.}$ (in place of $\mathcal{H}_2$), and $\overline{\mathcal{U}_2} \subset U(M_2(C_{\mathbb{T}}))/CU(M_2(C_{\mathbb{T}}))$ (in place of $\mathcal{U}$—see Corollary 12.8) be the finite subsets and constants provided by Theorem 12.7 for $\delta_1/16$ (in place of ε), $\mathcal{G}_3$ (in place of $\mathcal{F}$), $\Delta_1/2$ (in place of Δ), and $C_{\mathbb{T}}$ (in place of A).

Let $\mathcal{U}_2 \subset U(M_2(C_{\mathbb{T}}))$ be a finite subset which has a one-to-one correspondence to its image in $U(M_2(C_{\mathbb{T}}))/CU(M_2(C_{\mathbb{T}}))$ which is exactly $\overline{\mathcal{U}_2}$. We also assume that $\{[u] : u \in \mathcal{U}_2\} \subset \mathcal{P}_2$.

Without loss of generality, we may assume that

$$\mathcal{H}_3 = \{h_1 \otimes h_2 : h_1 \in \mathcal{H}_3' \text{ and } h_2 \in \mathcal{H}_3''\}, \tag{e 12.66}$$

where $\mathcal{H}_3' \subset C_+^{\mathbf{1}} \setminus \{0\}$ and $\mathcal{H}_3'' \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ are finite subsets, $1_C \in \mathcal{H}_3'$ and $1_{C(\mathbb{T})} \in \mathcal{H}_3''$, and

$$\mathcal{G}_4 = \{a \otimes f : a \in \mathcal{G}_4' \text{ and } f \in \mathcal{G}_4''\} \cup \{p_0, p_1\}, \tag{e 12.67}$$

where $\mathcal{G}_4' \subset C$ and $\mathcal{G}_4'' \subset C(\mathbb{T})$ are finite subsets. Set

$$\sigma_0 = \min\{\inf\{\Delta_1(\hat{h}) : h \in \mathcal{H}_3\}, \inf\{\Delta(\hat{h}) : h \in \mathcal{H}_0''\}\} > 0. \tag{e 12.68}$$

Note that Δ_0 is the map given by Lemma 12.5 for the simple C^* algebra A_1. For $\mathcal{H}_3'$ (in place of $\mathcal{H}$), there are $\delta_3 > 0$ (in place of δ) and $\mathcal{G}_5 \subset A_1$ (in place of $\mathcal{G}$) as the constant and finite subset provided by Lemma 12.5. Set

$$\delta_0 = \frac{\min\{1/16, \varepsilon/16, \delta_1, \delta_1'', \delta_2, \delta_3, \sigma_0, \gamma_1', \gamma_2'\}}{128(N_0+2)(n+1)} \tag{e 12.69}$$

and set

$$\mathcal{G}_6 = \{x, xy : x, y \in \tilde{\mathcal{G}_2'} \cup L_0(\tilde{\mathcal{G}_2'}) \cup \imath_{01}'(\tilde{\mathcal{G}_2'})\} \cup \imath_{01}'(\tilde{\mathcal{G}_2'})\} \cup \mathcal{G}_4' \cup \mathcal{G}_5$$

and

$$\begin{aligned} \mathcal{G}_0 = \{a \otimes f : a \in \mathcal{G}_6 \text{ and } f \in \mathcal{G}_1'' \cup \mathcal{H}_0'' \cup \mathcal{G}_4''\} \\ \cup\{p_j : 0 \le j \le 1\} \cup \{v_j, w_j : 1 \le j \le K\}. \end{aligned} \tag{e 12.70}$$

To simplify notation, without loss of generality, let us assume that $\mathcal{G}_0 \subset A^{\mathbf{1}}$. Let

$$\mathcal{P} = \mathcal{P}_1 \cup \{p_j : 0 \le j \le 1\} \cup [\imath](\mathcal{P}_2) \cup [\imath_{00}](\mathcal{P}_1), \tag{e 12.71}$$

where $\imath : C_{\mathbb{T}} \to A$ is the embedding. Let $\mathcal{H}_1 = \mathcal{H}_0'' \cup \mathcal{H}_3''$.

By (e 12.64) and (e 12.62), we may assume that $\mathcal{G}_0$ is sufficiently large (with even smaller δ_0, if necessary), that any $\mathcal{G}_0$-δ_0-multiplicative contractive completely positive linear map L' from A (to any unital C^*-algebra B') has the properties

$$(N_0-1)([L'(p_0)]) \ \ \ge \ \ [L'(p_1)], \ (n+1)[L'(p_0)] \le [L'(p_1)], \text{ and} \tag{e 12.72}$$

$$\tau(L'(1-p_1)) \ \ < \ \ 16/15n \text{ for all } \tau \in T(B'). \tag{e 12.73}$$

Let $\mathcal{U}_2' = \{\mathrm{diag}(1-p_1, 1-p_1) + w : w \in \mathcal{U}_2\}$ and let $\mathcal{U}_0'' = \{w_j + \mathrm{diag}(p_1, p_1) : 1 \le j \le K\}$. Let $\mathcal{U} = \{\bar{v} : v \in \mathcal{U}_1 \cup \mathcal{U}_2' \cup \mathcal{U}_0''\}$ and let $\mathcal{H}_2 = \mathcal{H}_4'$. Let $\sigma_1 = \min\{\frac{1}{4n}, \frac{\gamma_1'}{16n(N_0+2)}\}$ and $\sigma_2 = \min\{\frac{1}{16n(N_0+2)}, \frac{\gamma_2'}{16n(N_0+2)}\}$.

We then choose a finite $\mathcal{G} \supset \mathcal{G}_0$ and a positive number $0 < \delta < \delta_0/64$ with the following property: If $L' : A \to B'$ (for any unital C^*-algebra B') is any unital $\mathcal{G}$-δ-multiplicative completely positive linear map then there exist a projection $q' \in B'$ such that $\|L'(p_1) - q'\| < \delta_0/64$, an element $b_0' \in (1-q')B'(1-q')$ such that

$$\|b_0' - (1-q')\| < \delta_0/16 \ \text{ and } \ b_0' L'(1-p_1) b_0' = 1 - q', \tag{e 12.74}$$

and an element $b_1' \in q'Bq'$ such that

$$\|b_1' - q'\| < \delta_0/16 \quad \text{and} \quad b_1'L'(p_1)b_1' = q', \tag{e 12.75}$$

with the elements b_0' and b_1', cutting L' down approximately as follows: $\|b_0'L'(x)b_0' - L'(x)\| < \delta_0/4$ for all $x \in (1-p_1)A(1-p_1)$ with $\|x\| \le 1$, and $\|b_1'L'(x)b_1' - L'(x)\| < \delta_0/4$ for all $x \in p_1Ap_1$ with $\|x\| \le 1$.

Now let us assume that B is as in the statement of the theorem, and $\varphi,\ \psi : A \to B$ are two unital $\mathcal{G}$-δ-multiplicative completely positive linear maps satisfying the assumption for δ, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}_1$, $\mathcal{U}$, $\mathcal{H}_2$, σ_1, and σ_2 (as defined above).

Note that B' is in $\mathcal{B}_1$ and $B = B' \otimes U$. We may also write $B = B_1 \otimes U$, where $B_1 = B' \otimes U$, since U is strongly self absorbing. Without loss of generality, by the fact that U is strongly self absorbing, we may assume that the image of both φ and ψ are in $B_1 \otimes 1_U$.

As mentioned two paragraphs above, there are two projections $q_\varphi, q_\psi \in B_1$, unital completely positive linear maps $\varphi_0' : (1-p_1)A(1-p_1) \to ((1-q_\varphi)B_1(1-q_\varphi))\otimes 1_U$, $\psi_0' : (1-p_1)A(1-p_1) \to (1-q_\psi)B_1(1-q_\psi)\otimes 1_U$, and unital completely positive linear maps $\varphi_1' : pAp \to (q_\varphi B_1 q_\varphi)\otimes 1_U$ and $\psi_1' : p_1Ap_1 \to (q_\psi B q_\psi)\otimes 1_U$ such that

$$\|\varphi_0' - \varphi|_{(1-p_1)A(1-p_1)}\| < \delta_0/4,\ \|\psi_0' - \psi|_{(1-p_1)A(1-p_1)}\| < \delta_0/4, \tag{e 12.76}$$

$$\|\varphi_1' - \varphi|_{p_1Ap_1}\| < \delta_0/4 \text{ and } \|\psi_1' - \psi|_{p_1Ap_1}\| < \delta_0/4. \tag{e 12.77}$$

Note since $[\varphi(p_1)] = [\psi(p_1)]$ and B_1 has stable rank one (see 9.7), there exists a unitary $u_0 \in B_1$ such that $u_0^*(q_\psi)u_0 = q_\psi$. Then $\|u^*\psi(p_1)u - q_\varphi\| < \delta$. Thus, without loss of generality, by replacing ψ by $\mathrm{Ad}\,(u_0 \otimes 1_U)\circ\psi$, we may assume that $q := q_\varphi = q_\psi$. Note, by (e 12.72),

$$n\tau(1-q) < \tau(q) \quad \text{and} \quad N_0\tau(1-q) > \tau(q) \tag{e 12.78}$$

for $\tau \in T(B)$. Since $K_0(U)$ is divisible by n, there are mutually orthogonal and unitarily equivalent projections $e_1, ..., e_n \in U$ such that $\sum_{j=1}^n e_j = 1_U$. Define $\varphi_{00}', \psi_{00}' : A \to (1-q)B_1(1-q)\otimes 1_U$ by

$$\varphi_{00}'(a) = \varphi_0' \circ \imath_{00}(a) \otimes 1_U \quad \text{and} \quad \psi_{00}'(a) = \psi_0' \circ \imath_{00}(a)\otimes 1_U \tag{e 12.79}$$

for all $a \in A$. Define $\Phi_A, \Psi_A : A \to qB_1q \otimes 1_U$ by

$$\Phi_A(a) = \varphi_1' \circ \imath_{01}(a) \otimes 1_U \quad \text{and} \quad \Psi_A(a) = \psi_1' \circ \imath_{01}(a)\otimes 1_U, \tag{e 12.80}$$

and define $\Phi_{i,A}$: $A \to qB_1q \otimes e_i$ by $\Phi_{i,A}(a) = \Phi_A(a)e_i$ for all $a \in A$ $(1 \le i \le n)$. Define $\Phi_C, \Psi_C : C_{\mathbb{T}} \to (qB_1q)\otimes 1_U$ by

$$\Phi_C = \varphi_1' \circ \imath \text{ and } \Psi_C = \psi_1' \circ \imath. \tag{e 12.81}$$

Note, by the choice of δ and $\mathcal{G}$, $\varphi|_{A_1\otimes 1_{C(\mathbb{T})}}$ and $\psi|_{A_1\otimes 1_{C(\mathbb{T})}}$ are $\mathcal{G}_5$-δ_3-multiplicative. By (e 12.77), and then by Lemma 12.5 (see also (e 12.69) and (e 12.68)),

$$\begin{aligned}\tau(\Phi_C(h\otimes 1_{C(\mathbb{T})})) &\geq \tau(\varphi(h\otimes 1_{C(\mathbb{T})}))-\delta_0/4 \geq \Delta_0(\hat{h})/2-\delta_0/4\\ &\geq \Delta_0(\hat{h})/2-\sigma_0/16 \geq \Delta_0(\hat{h})/3 \text{ for all } h\in\mathcal{H}_3'.\end{aligned}$$

Similarly, by the assumption (e 12.48), we also have that

$$\begin{aligned}\tau(\Phi_C(1_{A_1}\otimes h)) &\geq \tau(\varphi(\imath(1_{A_1}\otimes h))-\delta_0/4 \geq \Delta(\hat{h})-\sigma_0/16\\ &\geq \Delta(\hat{h})/2 \text{ for all } h\in\mathcal{H}_3''.\end{aligned}$$

It follows that (see the definition of Δ_1)

(e 12.82)
$$\tau(\Phi_C(h))\geq \Delta_1(\hat{h})/2 \text{ for all } h\in\mathcal{H}_3.$$

By the assumption (e 12.49), and (e 12.77), for all $\tau\in T(B_1)$,

(e 12.83)
$$|\tau(\Phi_C(c))-\tau(\Psi_C(c))|<\sigma_1-\delta_0/4 \text{ for all } c\in\mathcal{H}_4'=\mathcal{H}_2.$$

Therefore, by (e 12.78), for all $t\in T(qB_1q)$,

(e 12.84)
$$\begin{aligned}|t(\Phi_C(c))-t(\Psi_C(c))| &< \frac{n}{n+1}(\sigma_1-\delta_0/4)\\ &< (1/2)(\sigma_1-\delta_0/4)<\gamma_1' \text{ for all } c\in\mathcal{H}_4'.\end{aligned}$$

Since $[\imath](\mathcal{P}_2)\subset\mathcal{P}$, by the assumptions, one also has

(e 12.85)
$$[\Phi_C]|_{\mathcal{P}_2}=[\Psi_C]|_{\mathcal{P}_2}.$$

By (e 12.50), (e 12.77), and applying Lemma 11.9 (with $K=1$), since $\mathcal{U}_2'\subset\mathcal{U}$, one has

(e 12.86)
$$\begin{aligned}&\mathrm{dist}_{M_2(qB_1q)}(\Phi_C^{\ddagger}(\bar{v})\Psi_C^{\ddagger}(\bar{v}^*),\mathrm{diag}(q,q))\\ &\qquad < (1+9/8)(\sigma_1+\delta_0/2<\gamma_2' \text{ for all } v\in\mathcal{U}_2.\end{aligned}$$

By the choices of δ, $\mathcal{G}_4$, γ_1', γ_2', $\mathcal{P}_2$, $\mathcal{H}_3$, $\mathcal{H}_4'$, and $\overline{\mathcal{U}_2}$, and by applying Theorem 12.7 (see also Corollary 12.8), there exists a unitary $u_1\in qB_1q$ such that

(e 12.87)
$$\|u_1^*\Psi_C(c)u_1-\Phi_C(c)\|<\delta_1/16 \text{ for all } c\in\mathcal{G}_3.$$

Thus, since $\imath_{01}'(\mathcal{G}_1'))\subset\mathcal{G}_3'$, by (e 12.55), and by (e 12.62), one obtains

(e 12.88)
$$\|u_1^*\Psi_A(a)u_1-\Phi_A(a)\|<\delta_1/16+\delta'' \text{ for all } a\in\mathcal{G}_1.$$

Note that, by (e 12.69) and (e 12.68), $\delta_0<\frac{\sigma_0}{128(N_0+2)}<\frac{\inf\{\Delta(\hat{h}):h\in\mathcal{H}_0''\}}{128(N_0+2)}$. One has, by (e 12.77), (e 12.73), the assumption (e 12.48), (e 12.57), and the choice of δ_0,

$$
\begin{aligned}
&\tau(\Phi_A(1_C\otimes h))\\
&\quad\geq\ \tau(\varphi(p_1'\otimes h))-\delta_0/4\\
&\qquad=\tau(\varphi(p_1(1\otimes h)))-\delta_0/4\\
&\qquad=\tau(\varphi(1_{A_1}\otimes h))-\tau(\varphi((1-p_1)\otimes h))-\delta_0/4\\
&\quad\geq\ \tau(\varphi(1_{A_1}\otimes h))-\tau(\varphi(1-p_1))-\delta_0/4\\
&\quad\geq\ \tau(\varphi(1_{A_1}\otimes h))-16/15n-\delta_0/4\\
&\quad\geq\ \Delta(\hat{h})-16/15n-\delta_0/4\geq 13\Delta(\hat{h})/15\\
&\qquad\ \text{for all } h\in\mathcal{H}_0'' \text{ and for all } \tau\in T(B_1).
\end{aligned}
$$

By Lemma 12.6, it follows that $\Phi_A|_{1_{A_1}\otimes C(\mathbb{T})}:\ C(T)\to qB_1q\otimes 1_U$ is $T'\times N'$-$\mathcal{H}_0''$-full. In other words, for any $h=f(y)\in\mathcal{H}_0''$ (where $y\in\mathcal{H}_0$), there are $z_1,z_2,...,z_m\in qB_1q$ with $\|z_i\|\leq T'(h)$ and $m\leq N'(h)$ such that $\sum_{j=1}^m z_i^*\Phi_C(1_{A_1}\otimes h)z_i=q\otimes 1_U$. Note that $\imath_{01}'$ is $\tilde{\mathcal{G}}_2'$-$\delta''/16$-multiplicative, and $\mathcal{G}_0\subset\mathcal{G}$ (see (e 12.70)). By(e 12.77), the fact that φ is $\mathcal{G}$-δ-multiplicative (used several times), (e 12.77) again, the fact that φ is $\mathcal{G}$-δ-multiplicative again, the fact that $\imath_{01}'$ is $\tilde{\mathcal{G}}_2'$-$\delta''/16$-multiplicative, (e 12.53), and then using the linearity of the maps involved, one has, that for $y\in\mathcal{H}_0$,

$$
\begin{aligned}
&\sum_{i=1}^{N''(a)}\Phi_A(x_{a(y),i}^*\otimes 1_{C(\mathbb{T})})\Phi_A(a(y)\otimes f(y))\Phi_A(x_{a(y),i}\otimes 1_{C(\mathbb{T})})\\
&\approx_{(M_1^2+M_1)\frac{\delta_0}{4}}\ \sum_{i=1}^{N''(a)}\Phi_A(x_{a(y),i}^*\otimes 1_{C(\mathbb{T})})\varphi((\imath_{01}'(a(y))\otimes 1_{C(\mathbb{T})})\\
&\quad\cdot(1_C\otimes f(y)))\varphi(\imath_{01}'(x_{a(y),i})\otimes 1_{C(\mathbb{T})})\\
&\quad\approx_{M_1\delta}\ \sum_{i=1}^{N''(a(y))}\Phi_A(x_{a(y),i}^*\otimes 1_{C(\mathbb{T})})\varphi((\imath_{01}'(a(y))\otimes 1_{C(\mathbb{T})})\\
&\quad\cdot(1_C\otimes f(y))(\imath_{01}'(x_{a(y),i})\otimes 1_{C(\mathbb{T})}))\\
&\quad\ \approx_{M_1\delta}\ \sum_{i=1}^{N''(a)}\Phi_A(x_{a(y),i}^*\otimes 1_{C(\mathbb{T})})\varphi(\imath_{01}'(a(y))\imath_{01}'(x_{a(y),i})\otimes 1_{C(\mathbb{T})})\varphi(1_C\otimes f(y))
\end{aligned}
$$

$$\approx_{(M_1^2+M_1)\frac{\delta_0}{4}} \sum_{i=1}^{N''(a(y))} \varphi(\imath'_{01}(x^*_{a(y),i}) \otimes 1_{C(\mathbb{T})}))\varphi(\imath'_{01}(a(y))\imath'_{01}(x_{a(y),i}) \otimes 1_{C(\mathbb{T})}))$$

$$\cdot\Phi_A(1_C \otimes f(y))$$

$$\approx_{M_1\delta} \sum_{i=1}^{N''(a(y))} \varphi((\imath'_{01}(x^*_{a(y),i})\imath'_{01}(a(y))\imath'_{01}(x_{a(y),i})) \otimes 1_{C(\mathbb{T})})\Phi_A(1_C \otimes f(y))$$

$$\approx_{2M_1(\delta_1''/16)} \sum_{i=1}^{N''(a(y))} \varphi(\imath'_{01}(x^*_{a(y),i}a(y)x_{a(y),i}) \otimes 1_{C(\mathbb{T})}))\Phi_A(1_C \otimes f(y))$$

$$\approx_{\delta_0/4} (q \otimes 1_U)\Phi_A(1_C \otimes f(y)) = \Phi_A(1_C \otimes f(y)).$$

From these estimates, it follows that, for any $y \in \mathcal{H}_0$ with $f(y) = h \in \mathcal{H}_0''$ (note $M_1 \le M_1^2$ and $\delta_0/4 < \delta_1''/16$)

$$\|\sum_{j=1}^{m} z_j^*(\sum_{i=1}^{N''(a(y))} \Phi_A(x^*_{a(y),i} \otimes 1_{C(\mathbb{T})})\Phi_A(r_y^*)\Phi_A(y)\Phi_A(r_y)\Phi_A(x_{a(y),i}$$

$$\otimes 1_{C(\mathbb{T})}))z_j - q \otimes 1_U\|$$

$$< 2mT'(h)^2T''(a(y))^2N''(a(y))(\delta_0/2 + \delta_1''/16)$$

$$+\|\sum_{j=1}^{m} z_j^*(\sum_{i=1}^{N''(a)} \Phi_A(x^*_{a(y),i} \otimes 1_{C(\mathbb{T})})\Phi_A(a(y) \otimes f(y))$$

$$\Phi_A(x_{a(y),i} \otimes 1_{C(\mathbb{T})})))z_j - q \otimes 1_U\|$$

$$< 2M_0^2M_1^2(\delta_0/2 + \delta_1''/16) + mT'(h)(10M_1^2)(\delta_1''/16)$$

$$+\|\sum_{j=1}^{m} z_j^*\Phi_A(1_C \otimes h)z_j - q \otimes 1_U\|$$

$$= 2M_0^2M_1^2(\delta_0/2 + \delta_1''/16) + mT'(h)^2(10M_1^2)(\delta_1''/16)$$

$$\le 2M_0^2M_1^2(\delta_0/2) + 12M_0^2M_1^2(\delta_1''/16) < 1/64(n+1).$$

Note also the image of Φ_A is in $qB_1q \otimes 1_U$. There is $c_y \in (qBq)_+ \otimes 1_U$ with $\|c_y - q \otimes 1_U\| < 1/64(n+1)$ such that

$$\sum_{j=1}^{m} c_y z_j^*(\sum_{i=1}^{N''(a(y))} \Phi_A(x^*_{a(y),i})\Phi_A(r_y^*)\Phi_A(y)\Phi(r_y)\Phi_A(x_{a(y),i}))z_jc_y = q\otimes 1_U.$$

By the definition of $T\times N$ (see (e 12.54)), one then computes that Φ_A is $T\times N$-$\mathcal{H}_0$-full. It follows that $\Phi_{i,A}$ is $T \times N$-$\mathcal{H}_0$-full (as a map to $qB_1q \otimes e_i$), $i = 1, 2, ..., n$. Since $[\imath_{00}](\mathcal{P}_1) \subset \mathcal{P}$,

$$[\psi'_{00}]|_{\mathcal{P}_1} = [\varphi'_{00}]|_{\mathcal{P}_1}. \tag{e 12.89}$$

By (e 12.50), since $\mathcal{U}_0'' \subset \mathcal{U}$,

$$\begin{aligned}\text{(e 12.90)} \qquad & \operatorname{dist}(\varphi^{\ddagger}(\overline{w_j + \operatorname{diag}(p_1, p_1)}),\\ & \qquad \psi^{\ddagger}(\overline{w_j + \operatorname{diag}(p_1, p_1)})) < \sigma_2,\ 1 \le j \le K.\end{aligned}$$

Put $w_j^{\sim} = w_j + \operatorname{diag}(p_1, p_1)$, $J = 1, 2, ..., K$. By (e 12.77), one has

$$\|\operatorname{diag}(1-q, 1-q)(\varphi \otimes \operatorname{id}_{M_2})(w_j^{\sim})\operatorname{diag}(1-q, 1-q) - (\varphi_{00}' \otimes \operatorname{id}_{M_2})(w_j)\| < \delta_0$$

and

$$\|\operatorname{diag}(1-q, 1-q)(\varphi \otimes \operatorname{id}_{M_2})(w_j^{\sim})\operatorname{diag}(1-q, 1-q) - (\varphi_{00}' \otimes \operatorname{id}_{M_2})(w_j)\| < \delta_0.$$

It follows from (e 12.90) and the above inequalities, that for $1 \le j \le K$,

$$\begin{aligned}\text{(e 12.91)} \qquad & \operatorname{dist}(\overline{(\varphi_{00}' \otimes \operatorname{id}_{M_2})(w_j) + \operatorname{diag}(q, q)},\\ & \qquad \overline{(\psi_{00}' \otimes \operatorname{id}_{M_2})(w_j) + \operatorname{diag}(q, q)}) < 2\delta_0 + \sigma_2.\end{aligned}$$

It follows from (e 12.78) that $N_0[1-q] > [q]$. By (e 12.65) (with N_0 in place of K and $1-q$ in place of e), (e 12.64), and by Lemma 11.9, in $\operatorname{diag}(1-q, 1-q)M_2(B)\operatorname{diag}(1-q, 1-q)$, one has

$$\begin{aligned}& \operatorname{dist}((\varphi_{00}')^{\ddagger}(\bar{v}_j), (\psi_{00}')^{\ddagger}(\bar{v}_j))\\ & \quad < \quad \delta_1/8n + \operatorname{dist}((\varphi_{00}')^{\ddagger}(\bar{w}_j), (\psi_{00}')^{\ddagger}(\bar{w}_j))\\ & \quad < \quad \delta_1/8n + (N_0 + 9/8)(2\delta_0 + \sigma_2) < 1/2 < \mathbf{L},\ j = 1, 2, ..., K.\end{aligned}$$

Note that, by (e 12.78), since B has strict comparison, $[(1-q) \otimes 1_U] \le [q \otimes e_1] = [q \otimes e_i]$ $(1 \le i \le n)$. Note also that $\Phi_{i,A}$ is unitarily equivalent to $\Phi_{1,A}$, $1 \le i \le n$. It follows from Theorem 12.3 and Remark 12.4, by (e 12.89) and (e 12.92), that there exists a unitary $u_2 \in B$ such that

$$\|u_2^*(\varphi_{00}'(a) \oplus \Phi_{1,A}(a) \oplus \cdots \oplus \Phi_{n,A}(a))u_2 - (\psi_{00}'(a) \oplus \Phi_{1,A}(a) \oplus \cdots \oplus \Phi_{n,A}(a))\| < \varepsilon/16$$

for all $a \in \mathcal{F}$. In other words,

$$\text{(e 12.92)} \qquad \|u_2^*(\varphi_{00}'(a) \oplus \Phi_A(a))u_2 - \psi_{00}'(a) \oplus \Phi_A(a)\| < \varepsilon/16$$

for all $a \in \mathcal{F}$.Thus, by (e 12.88),

$$\begin{aligned}\text{(e 12.93)} \qquad & \|u_2^*(\varphi_{00}'(a) \oplus \Phi_A(a))u_2 - (\psi_{00}'(a) \oplus u_1^*\Psi_A(a)u_1)\|\\ & \qquad < \varepsilon/16 + \delta_1/16 + \delta'' \text{ for all } a \in \mathcal{F}.\end{aligned}$$

Let $u = u_2(q + u_1^*) \in U(B)$. 0 Then, for all $a \in \mathcal{F}$,

$$\begin{aligned} \|u^*(\varphi_{00}'(a) \oplus \Phi_A(a))u - (\psi_{00}'(a) \oplus \Psi_A(a))\| \\ < \varepsilon/16 + \delta_1/8 + \delta''. \end{aligned} \tag{e 12.94}$$

It then follows from (e 12.77), (e 12.79), and (e 12.80) that, for all $a \in \mathcal{F}$,

$$\begin{aligned} \|u^*(\varphi \circ \imath_{00}(a) + \varphi \circ \imath_{01}(a))u \quad &- \quad (\psi \circ \imath_{00}(a) + \psi \circ \imath_{01}(a))\| \\ &< \quad \varepsilon/16 + \delta_1/8 + \delta'' + 4(\delta_0/4). \end{aligned}$$

Then, by (e 12.62), finally, one has

$$\|u^*\varphi(a)u - \psi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}, \tag{e 12.95}$$

as desired. ☐

REMARK 12.12. As in Corollary 12.8, the condition $\mathcal{U} \subset U(M_2(A))/CU(M_2(A))$ can be replaced by $\mathcal{U} \subset J_c(U(M_2(A))/U_0(M_2(A)))$, which generates a torsion free subgroup. Moreover, for part (a), or equivalently, the case $A \in \mathcal{B}_1$, we may take $\mathcal{U} \subset J_c(U(A)/U_0(A))$, so that generates a torsion free subgroup. (Note that A has stable rank one.)

In the case that $A = \overline{\bigcup_{n=1}^{\infty} A_n} \in \mathcal{B}_1$, in the theorem above, one may choose $\mathcal{U}$ to be in $U(A_n)/CU(A_n)$ for some sufficiently large n.

13. The Range of the Invariant

NOTATION 13.1. In this section we will use the concept of set with multiplicity. Fix a set $\mathcal{X}$. A subset with multiplicities in $\mathcal{X}$ is a collection of elements in $\mathcal{X}$ which may be repeated finitely many times. Therefore $X_1 = \{x, x, x, y\}$ is different from $X_2 = \{x, y\}$. Let $x \in \mathcal{X}$ be a single element. Denote by $x^{\sim k}$ the subset with multiplicities $\{\underbrace{x, x, ..., x}_{k}\}$ (see 1.1.7 of [43]). Let $X = \{x_1^{\sim i_1}, x_2^{\sim i_2}, ..., x_n^{\sim i_n}\}$ and $Y = \{x_1^{\sim j_1}, x_2^{\sim j_2}, ..., x_n^{\sim j_n}\}$ (some of the i_k's (or j_k's) may be zero which means the element x_k does not appear in the set X (or Y)). If $i_k \le j_k$ for all $k = 1, 2, ..., n$, then we say that $X \subset Y$ (see 3.21 of [43]). We define

$$X \cup Y := \{x_1^{\sim \max(i_1, j_1)}, x_2^{\sim \max(i_2, j_2)}, ..., x_n^{\sim \max(i_n, j_n)}\} \text{ and} \tag{e 13.1}$$

$$X \sqcup_{mult} Y = \{x_1^{\sim (i_1 + j_1)}, x_2^{\sim (i_2 + j_2)}, \cdots, x_n^{\sim (i_n + j_n)}\}. \tag{e 13.2}$$

We may also use the convention $\{X, Y\} := X \sqcup_{mult} Y$. For example, $\{x^{\sim 2}, y^{\sim 3}\} = \{x, x, y, y, y\}$ and $\{x^{\sim 2}, y^{\sim 3}, z^{\sim 1}\} = \{x, x, y, y, y, z\}$. By $X^{\sim k}$ we mean the set $\{x_1^{\sim k i_1}, x_2^{\sim k i_2}, ..., x_n^{\sim k i_n}\}$. Note

$$\{X^{\sim k}, Y^{\sim n}\} = X^{\sim k} \sqcup_{mult} Y^{\sim n}.$$

For example,

$$\{x, \{x, y\}^{\sim 2}, \{y, z\}, x, w\} = \{x^{\sim 4}, y^{\sim 3}, z, w\} = \{x, x, x, x, y, y, y, z, w, \}.$$

NOTATION 13.2. Let A be a unital subhomogeneous C^*-algebra; that is, the maximal dimension of irreducible representations of A is finite.

Let us use $Sp(A)$ to denote the set of equivalence classes of all irreducible representations of A. The set $Sp(A)$ will serve as base set when we talk about a finite set with (finite) multiplicities.

Since A is of type I, the set $Sp(A)$ has a one-to-one correspondence to the set of primitive ideals of A. Let $X \subset Sp(A)$ be a closed subset, then X corresponds to the ideal $I_X = \bigcap_{\psi \in X} \ker \psi$. In this section, let us use $A|_X$ to denote the quotient algebra A/I_X. If $\varphi : B \to A$ is a homomorphism, then we will use $\varphi|_X : B \to A|_X$ to denote the composition $\pi \circ \varphi$, where $\pi : A \to A|_X$ is the quotient map. As usual, if B_1 is a subset of B, we will also use $\varphi|_{B_1}$ to denote the restriction of φ to B_1. These two notations will not be confused, since it will be clear from the context which notation we refer to.

If $\varphi : A \to B$ is a homomorphism, then we write $Sp(\varphi) = \{x \in Sp(A) : \ker \varphi \subset \ker x\}$. Denote by $RF(A)$ the set of equivalence classes of all (not necessarily irreducible) finite dimensional representations. An element $[\pi] \in RF(A)$ will be identified with the set with multiplicity $\{[\pi_1], [\pi_2], \cdots, [\pi_n]\}$ in $Sp(A)$, where $\pi_1, \pi_2, \cdots, \pi_n$ are irreducible representations and $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n$. For $X, Y \in RF(A)$, as sets with multiplicities, we write $X \subset Y$ in the sense of 13.1, if and only if the representation corresponding to X is equivalent to a sub-representation of that corresponding to Y. Any finite subset of $RF(A)$ also defines an element of $RF(A)$ which is the equivalence class of the direct sum of all corresponding representations in the set with the correct multiplicities. For example, the subset $\{X, Y\} \subset RF(A)$ defines an element $X \oplus Y \in RF(A)$ which is direct sum of two representations corresponding to X and Y (see the notation $\{X, Y\}$ in 13.1).

When we write $X = \{z_1^{\sim k_1}, z_2^{\sim k_2}, ..., z_m^{\sim k_m}\}$, we do not insist that z_i be itself in $Sp(A)$. In other words, z_i could be itself a subset with multiplicity of $Sp(A)$. It might be a list of several elements of $Sp(A)$—that is, we do not insist that z_i should be irreducible (but as we know, it can always be decomposed into irreducibles). So, in this notation, we do not differentiate between $\{x\}$ and x; both give the same element of $RF(A)$ and the same set with multiplicities whose elements are in $Sp(A)$.

If $\varphi : A \to M_m$ is a homomorphism, let us use $SP(\varphi)$ to denote the corresponding equivalence class of φ in $RF(A)$.

If $\varphi : A \to M_m$ is a homomorphism and if $SP(\varphi) = \{x_1^{\sim k_1}, x_2^{\sim k_2}, ..., x_i^{\sim k_i}\}$, with $x_1, x_2, ..., x_i$ irreducible representations and $k_j > 0$, $j = 1, 2, \cdots, i$, then $Sp(\varphi) = \{x_1, x_2, ..., x_i\} \subset Sp(A)$. So $Sp(\varphi)$ is an ordinary set which is a subset of $Sp(A)$, while $SP(\varphi)$ is a set with multiplicities, whose elements are also elements of $Sp(A)$.

NOTATION 13.3. Let us recall some notation from Definition 4.8. Suppose that

$A = A_m \in \mathcal{D}_m$ (see the end of 4.8) is as constructed in the following sequence:

$$A_0 = P_0C(X_0, F_0)P_0, \quad A_1 = P_1C(X_1, F_1)P_1 \oplus_{Q_1C(Z_1,F_1)Q_1} A_0,$$
$$..., A_m = P_mC(X_m, F_m)P_m \oplus_{Q_mC(Z_m,F_m)Q_m} A_{m-1}, \tag{e 13.3}$$

where $F_j = M_{s(j,1)} \oplus M_{s(j,2)} \oplus \cdots \oplus M_{s(j,t_j)}$ is a finite dimensional C^*-algebra, $P_j \in C(X_j, F_j)$ is a projection, $j = 0, 1, ..., m$, and

$$P_k(C(X_k, F_k))P_k = \bigoplus_{i=1}^{t_k} P_{k,i}C(X_k, M_{s(k,i)})P_{k,i},\ k = 1, 2, ..., m,$$

are as in Definition 4.8. Let $\Lambda : A \to \bigoplus_{k=0}^{m} P_kC(X_k, F_k)P_k$ be the inclusion as in Definition 4.8. Note that $Z_0 = \varnothing$. Let $\pi_{(x,j)}$, where $x \in X_k$ and the positive integer j refers to the j-th block of $\bigoplus_{i=1}^{t_k} P_{k,i}C(X_k, F_k)P_{k,i}$ (and $P_{k,j}(x) \neq 0$), be the finite dimensional representations of A as in 4.8. According to 13.2, one has that $\pi_{(x,j)} \in RF(A)$. If $x \in X_k \setminus Z_k$, then $\pi_{(x,j)} \in Sp(A)$, i.e., it is irreducible. In fact, $Sp(A) = \{\pi_{(x,j)} : x \in X_k \setminus Z_k, k = 0, 1, ..., m, \text{ and } P_k(x) \neq 0,\ j = 1, 2, \cdots, t_k\}$. Recall that all X_k are compact metric spaces. For $k \leq m$ and $j \leq t_k$, we use $X_{k,j}$ to denote all the (not necessary irreducible) non-zero representations $\pi_{(x,j)}$ for $x \in X_k$ (see 4.8). Note that, as a set, $X_{k,j} = \{x \in X : P_{k,j}(x) \neq 0\}$. Set $Z_{k,j} = \{\pi_{(x,j)} \in X_{k,j} : x \in Z_k\}$. Note that $X_{k,j}$ (and $Z_{k,j}$, respectively) has a natural metric induced from X_k—i.e., $\mathrm{dist}(\pi_{(x,j)}, \pi_{(y,j)}) = \mathrm{dist}(x, y)$ for $x, y \in X_k$. In what follows, if $\theta \in RF(A)$ and $\theta = \pi_{(x,j)}$ for some $\pi_{(x,j)} \in X_{k,j}$, then we will write $\theta \in X_{k,j}$.

The usual topology on $Sp(A_m)$ is in general not Hausdorff. For each $\theta \in Sp(A)$ and $\delta > 0$, the subset $B_\delta(\theta)$ of $Sp(A)$, called the δ-neighborhood of θ, is defined as follows:

An irreducible representation $\Theta \in Sp(A_m)$ is in $B_\delta(\theta)$ if there is a sequence of finite dimensional representations $\theta = \sigma_0, \theta_0, \sigma_1, \theta_1, \sigma_2, \theta_2, \cdots, \sigma_k, \theta_k = \Theta$ such that

1. For each $i = 1, 2, \cdots, k$, $\theta_{i-1} \subset \sigma_i$ (see 13.2);
2. For each $i = 0, 1, \cdots, k$, there exists a pair (l_i, j_i) such that $\sigma_i \in X_{l_i,j_i}$ and $\theta_i \in X_{l_i,j_i} \setminus Z_{l_i,j_i}$;
3. $\sum_{i=0}^{k} \mathrm{dist}_{X_{l_i,j_i}}(\sigma_i, \theta_i) < \delta$.

Note that, since $\theta_{i-1} \in X_{l_{i-1},j_{i-1}} \setminus Z_{l_{i-1},j_{i-1}}$, $\sigma_i \in X_{l_i,j_i}$, and $\theta_{i-1} \subset \sigma_i$, one must have $l_i \geq l_{i-1}$, $i = 1, 2, ..., k$. Note also that we borrow the concept of δ-neighborhood from metric space theory, but there is no metric on $Sp(A)$. In fact $\Theta \in B_\delta(\theta)$ does not imply that $\theta \in B_\delta(\Theta)$. Note that if $\Theta \in B_{\delta_1}(\Theta_1)$ and $\Theta_1 \in B_{\delta_2}(\theta)$, then $\Theta \in B_{\delta_1+\delta_2}(\theta)$.

Let $\varphi : A \to M_\bullet$ be a homomorphism. We shall say that $SP(\varphi)$ is δ-dense in $Sp(A)$ if for each irreducible representation θ of A, there is $\Theta \in B_\delta(\theta)$ such that $\Theta \subset SP(\varphi)$. For any algebra in $\mathcal{D}_1$ (see the end of 4.8), including all Elliott-Thomsen building blocks, δ-density of $SP(\varphi)$ means for each irreducible representation θ of A, either $\theta \subset SP(\varphi)$, or there are two points $x, y \in X_1$ (for the case of an Elliott-Thomsen building block, $X_1 = [0,1]$) and j (a single j) such that $\mathrm{dist}(x,y) < \delta$, and such that $\theta \subset \pi_{(y,j)}$ and $\pi_{(x,j)} \subset SP(\varphi)$. This will be used in this section and the next.

If $X_k = [0,1]$, then we use the standard metric of the interval $[0,1]$.

LEMMA 13.4. *Let $A \in \mathcal{D}_m$.*

(1) If $f \in A$ and $\theta \in Sp(A)$ such that $\theta(f) \neq 0$, then there exist $\delta > 0$ and $d_0 > 0$ such that $\|\Theta(f)\| \geq d_0$ for all $\Theta \in B_\delta(\theta)$.

(2) For any $\delta > 0$, there is a finite set $\mathcal{F} \subset A_+ \setminus \{0\}$ satisfying the following condition. For any irreducible representation $\theta \in Sp(A)$, there is an element $f \in \mathcal{F}$ such that if $\Theta \in Sp(A)$ satisfies $\Theta(f) \neq 0$, then $\Theta \in B_\delta(\theta)$. (We do not require that $\theta(f) \neq 0$ for the element f corresponding to θ.)

PROOF. In what follows we will keep the notation introduced in 13.3.

For part (1), we will prove it by induction. It is clear that part (1) holds when $A \in \mathcal{D}_0$. Assume that part (1) holds for all $A \in \mathcal{D}_k$ with $k < m$.

Let $A \in \mathcal{D}_m$ and let $\theta(f) \neq 0$. Write $A_m = P_m(C(X_m, F_m)P_m \oplus_{Q_mC(Z_k,F_m)Q_m} A_{m-1}$ and $f = (g,h)$, where $g \in P_m(C(X_m,F_m)P_m$ and $h \in A_{m-1}$. If $\theta \in \{\pi_{(x,j)} : x \in X_m \setminus Z_m$ and $P_{m,j}(x) \neq 0\}$, for some j, then, by continuity of g at x in the jth block, one obtains $\delta > 0$ such that $\|\pi_{(y,j)}(f)\| \geq d_0$ for all $(y,j) \in X_{m,j}$ such that $\mathrm{dist}(y,x) < \delta$, where $d_0 > 0$. In other words, $\|\Theta(f)\| \geq d_0$ for all $\Theta \in B_\delta(\theta)$.

Suppose $\theta \in \{\pi_{(x,j)} : x \in X_k \setminus Z_k$ and $P_{k,j}(x) \neq 0\}$ for some j and for some $0 < k < m$. Thus, one may view θ as a point in A_{m-1} and $\theta(h) \neq 0$. By the induction assumption, there are $\delta_0 > 0$ and $d_{00} > 0$ such that, for any $\Theta' \in B^{m-1}_{\delta_0}(\theta)$, $\|\Theta'(h)\| \geq d_{00}$ (where $B^{m-1}_\eta(\theta)$ is a η-neighborhood of $Sp(A_{m-1})$ for any $\eta > 0$). Note that we may also view $Sp(A_{m-1})$ as a subset of $Sp(A)$. For each $j = 1, 2, ..., t_m$, let

$$Y_{m,j} = \{\pi_{(z,j)} \subset Z_{m,j} : \exists\ \theta' \in B^{m-1}_{\delta_0/2}(\theta) \text{ such that } \ \theta' \subset \pi_{(z,j)}\}.$$

If $\pi_{(x,j)} \in Y_{m,j}$, then $\|\pi_{(x,j)}(f)\| \geq \|\theta'(f)\| \geq d_{00}$. Note that $g|_{Z_m} = \Lambda_m(h)$. In the jth block, $\|\pi_{(x,j)}(f)\| \geq d_{00}$ for all $\pi_{(x,j)} \in \overline{Y_{m,j}}$, the closure of $Y_{m,j}$. Since $\overline{Y_{m,j}}$ is compact, there is $\delta_j > 0$ such that $\|\pi_{(x,j)}(f)\| \geq d_{00}/2$ for all $\pi_{(x,j)} \in X_{m,j}$ such that $\mathrm{dist}(x, Y_{m,j}) < \delta_j$. If $Y_{m,j_0} = \varnothing$, put $\delta_{j_0} = 1$. Choose $\delta = \min\{1/2, \delta_0/2, \delta_j : 1 \leq j \leq t_m\}$.

We claim that $\|\Theta(f)\| \geq d_{00}/2$ for all $\Theta \in B_\delta(\theta)$. Let $\Theta \in X_{k,j}$ for some j. Then, by the definition of $B_\delta(\theta)$, there exist finite dimensional representations $\theta = \sigma_0, \theta_0, \sigma_1, \theta_1, \sigma_2, \theta_2, \cdots, \sigma_n, \theta_n = \Theta$ (for some integer $n \geq 1$) such that

1. For each $i = 1, 2, \cdots, n$, $\theta_{i-1} \subset \sigma_i$;
2. For each $i = 0, 1, \cdots, n$, there exist a pair (l_i, j_i) such that $\sigma_i, \theta_i \in X_{l_i,j_i}$ $\theta_i \in X_{l_i,j_i} \setminus Z_{l_i,j_i}$; and
3. $\sum_{i=1}^n \mathrm{dist}_{X_{l_i,j_i}}(\sigma_i, \theta_i) < \delta$.

Note, by 13.3, if $\Theta \in B_\delta(\theta) \cap Sp(A_{m-1})$, then all $l_i < m$. It follows that $\Theta \in B^{m-1}_{\delta_0}(\theta)$. Then, by the choice of δ_0, $\|\Theta(f)\| \geq d_{00}$.

Otherwise, $\Theta \in X_{m,j}$. Choose the largest i such that $l_i < m$. Then $\theta_i \in X_{l_i,j_i} \setminus Z_{l_i,j_i}$ and $\sigma_{i+1} \in X_{m,j_{i+1}}$ for some j_{i+1}. Therefore, by the definition of $B_\delta^{m-1}(\theta)$, $\theta_i \in B_\delta^{m-1}(\theta)$. Note that, for any $i' > i$, $l_{i'} = m$. It follows that $\sigma_{i'}$, $\theta_{i'} \in X_{m,j_{i'}}$. On the other hand, since $\theta_n = \Theta \in X_{m,j}$, $\sigma_n \in X_{m,j}$. Since $\theta_{n-1} \subset \sigma_n$, either $l_{n-1} < m$, in which case, $i = n-1$, or $\theta_{n-1} = \sigma_n$ as $\theta_{n-1} \in X_{l_{n-1},j_{n-1}} \setminus Z_{l_{n-1},j_{n-1}}$, in which case $l_{n-1} = m$ and $j_{n-1} = j$. By repeatedly using 1,2,3, above, one concludes that $j_{i'} = j$ for $i' > i$. In other words, $\sigma_{i+1} \in X_{m,j}$. Note that $\theta_i \subset \sigma_{i+1}$ and $\theta_i \in B_\delta^{m-1}(\theta)$. It follows that $\sigma_{i+1} \in Y_{m,j}$. One checks (by 3 above) that $\mathrm{dist}(\Theta, \sigma_{i+1}) = \mathrm{dist}(\theta_n, \sigma_{i+1}) < \delta < \delta_0/2$. By the choice of δ, one obtains $\|\Theta(f)\| \geq d_{00}/2$. This completes the induction and part (1) holds.

We will also prove part (2) by induction. If

$$A = A_0 = \bigoplus_{j=1}^{t_0} P_{0,j} C(X_0, M_{s(0,j)}) P_{0,j} \in \mathcal{D}_0,$$

then it is easy to see that this reduces to the case that $A = A_0 = P_0 C(X_0, M_s) P_0$, where $s \geq 1$ is an integer and $P_0 \in C(X_0, M_s)$ is a projection. Given $\delta > 0$, let $\{U_i : 1 \leq i \leq m\}$ be an open cover of X_0 with the diameter of each U_i smaller than $\delta/2$. Consider a partition of unity $\mathcal{F} = \{f_i \in C(X_0) : 1 \leq i \leq m\}$ subordinate to the open cover $\{U_i : 1 \leq i \leq m\}$. Then, clearly $\mathcal{F}$ satisfies the requirements. Let us assume that $A = A_m \in \mathcal{D}_m$ with $m \geq 1$ and that the conclusion of the lemma is true for algebras in $\mathcal{D}_{m-1}$.

Write $A_m = P_m C(X_m, F_m) P_m \oplus_{Q_m C(Z_m, F_m) Q_m} A_{m-1}$ with $\Gamma_m : A_{m-1} \to Q_m C(Z_m, F_m) Q_m$, where $Q_m = P_m|_{Z_m}$ (see 4.8). By the induction assumption, there is a finite subset $\mathcal{F}_1 \subset (A_{m-1})_+ \setminus \{0\}$ with the following property: For any irreducible representation $\theta \in Sp(A_{m-1})$, there is an element $f \in \mathcal{F}_1$ such that if $\Theta \in Sp(A_{m-1})$ satisfies $\Theta(f) \neq 0$, then $\Theta \in B_{\delta/2}^{m-1}(\theta)$ (notation from the proof of part (1)). For each $f \in \mathcal{F}_1$, by the Tietze Extension Theorem, there is an

$$h = (h_1, h_2, \cdots, h_{t_m}) \in P_m C(X_m, F_m) P_m = \bigoplus_{j=1}^{t_m} P_{m,j} C(X_m, M_{s(m,j)}) P_{m,j}$$

such that $\Gamma_m(f) = h|_{Z_m}$. For each $j \leq t_m$, let $\Omega_{m,j} \subset Z_m$ $(\subset X_m)$ be the closure of the set

$$\{z \in Z_m : \ \exists\, \theta' \in Sp(A_{m-1}) \ \text{ such that } \ \theta' \subset \pi_{(z,j)} \ \text{ and } \ \theta'(f) \neq 0\},$$

where $\theta \subset \pi_{(x,j)}$ as subsets of $RF(A)$ with multiplicities. Choose, in the case $\Omega_{m,j} \neq \varnothing$, a continuous function $\chi_j : X_m \to [0,1]$ such that

$$\chi_j(x) = 1 \ \text{ if } \ x \in \Omega_{m,j} \quad \text{and} \quad \chi_j(x) = 0 \ \text{ if } \ \mathrm{dist}(x, \Omega_{m,j}) \geq \delta/3,$$

and, in the case that $\Omega_{m,j} = \varnothing$, let $\chi_j = 0$. Set $h' = (\chi_1 \cdot h_1, \chi_2 \cdot h_2, \cdots, \chi_{t_m} \cdot h_{t_m})$. Then $h'|_{Z_m} = h|_{Z_m} = \Gamma_m(f)$. Thus, $\tilde{f} = (h', f) \in P_m C(X_m, F_m) P_m \oplus A_{m-1}$ defines an element of $A_m = P_m C(X_m, F_m) P_m \oplus_{Q_m C(Z_m, F_m) Q_m} A_{m-1}$.

Now let $\Theta \in Sp(A_m)$ be such that $\Theta(\tilde{f}) \neq 0$. If $\Theta \in X_{m,j} \setminus Z_{m,j}$ for some j, then $\Omega_{m,j} \neq \varnothing$ and $\mathrm{dist}(\Theta, \Omega_{m,j}) < \delta/3$. Therefore, there exists $\sigma \in Z_{m,j}$ such that $\mathrm{dist}(\sigma, \Theta) \leq \delta/3$ and $\Theta' \in Sp(A_{m-1})$ such that $\Theta' \subset \sigma$ and $\Theta'(f) \neq 0$. By the inductive assumption, $\Theta' \in B_{\delta/2}(\theta)$. By the definition of $B_\delta(\theta)$, this implies that $\Theta \in B_\delta(\theta)$. If $\Theta \in Sp(A_{m-1})$, then $\Theta \in B^{m-1}_{\delta/2}(\theta)$, which also implies that $\Theta \in B_{\delta/2}(\theta)$. Let $\tilde{\mathcal{F}}_1 = \{\tilde{f} : f \in \mathcal{F}_1\}$.

Choose a finite subset $\Xi := \{x_1, x_2, \cdots, x_\bullet\} \subset X_m$ which is $\delta/4$-dense in X_m—that is, if $x \in X_m$, then there is x_i such that $\mathrm{dist}(x, x_i) < \delta/4$. We need to modify the set Ξ so that $\Xi \subset X_m \setminus Z_m$. If $x_i \in Z_m$ and

$$W := \{x \in (X_m \setminus Z_m) : \mathrm{dist}(x, x_i) < \delta/4\} \neq \varnothing,$$

then replace x_i by any element of W; if W is the empty set, then simply delete x_i. After the modification, we have $\Xi \subset X_m \setminus Z_m$ and Ξ is $\delta/2$ (instead of $\delta/4$) dense in $X_m \setminus Z_m$ (instesd of X_m). For each $x_i \in \Xi$, choose an open set $U_i \ni x_i$ such that $U_i \subset X_m \setminus Z_m$ and $\mathrm{dist}(x, x_i) < \delta/2$ for any $x \in U_i$. For each $x_i \in \Xi$, choose a function $g_{x_i} : X_m \to [0,1]$ such that $g_{x_i}(x_i) = 1$ and $g_{x_i}(x) = 0$ for $x \notin U_i$. For $j \leq t_m$ let

$$g_{x_i,j} = (\underbrace{0, \cdots, 0}_{j-1}, g_{x_i} \cdot P_{m,j}, \underbrace{0, \cdots, 0}_{t_m - j}) \oplus 0 \in \bigoplus_{j=1}^{t_m} P_{m,j} C(X_m, M_{s(m,j)}) P_{m,j} \oplus A_{m-1},$$

which defines an element (still denoted by $g_{x_i,j}$) of A_m. Define $\mathcal{G} = \{g_{x_i,j} : x_i \in \Xi, j \leq t_m\}$. Set $\mathcal{F} = \mathcal{G} \cup \tilde{\mathcal{F}}_1$.

Now fix $\theta \in Sp(A)$. If $\theta \in Sp(A_{m-1})$, then choose $f \in \mathcal{F}_1$ with the property $\Theta(f) \neq 0$ implies $\Theta \in B^{m-1}_{\delta/2}(\theta)$. Now consider $\tilde{f}$. By what has been proved, if $\Theta(\tilde{f}) \neq 0$, then $\Theta \in B_\delta(\theta)$.

If $\theta \in X_{m,j} \setminus Z_{m,j}$, then there is $x_i \in \Xi$, such that $\mathrm{dist}(\theta, x_i) < \delta/2$. Then $g_{x_i,j} \in \mathcal{F}$ and $\pi_{(x_i,j)}(g_{x_i,j}) \neq 0$. Moreover, by the construction of $g_{x_i,j}$, if $\Theta \in Sp(A)$ with $\Theta(g_{x_i,j}) \neq 0$, then $\Theta \in B_\delta(\theta)$. This ends the induction.

□

The special case of the following lemma for AH-algebras can be found in [22]. In the following statement, we use notation introduced in 13.2.

PROPOSITION 13.5. *Let $A = \varinjlim(A_n, \varphi_{n,m})$ be a unital inductive limit of C^*-algebras, where $A_n \in \mathcal{D}_{l(n)}$ (for some $l(n)$) and where each $\varphi_{n,m}$ is injective. Then the limit C^*-algebra is simple if and only if the inductive system satisfies the following condition: for any $n > 0$ and $\delta > 0$, there is an integer $m > n$ such that for any $\sigma \in Sp(A_m)$, $SP(\varphi_{n,m}|_\sigma)$ is δ-dense in $Sp(A_n)$—equivalently, for any $m' \geq m$ and any $\sigma' \in Sp(A_{m'})$, $SP(\varphi_{n,m'}|_{\sigma'})$ is δ-dense in $Sp(A_n)$.*

PROOF. Note that, since A is unital and each A_n is unital, without loss of generality, we may assume that all $\varphi_{n,m}$ are unital. The proof of this proposition is standard. Suppose that the condition holds. For any non-zero element $f \in A_n$, there is $\theta \in Sp(A_n)$ such that $\theta(f) \neq 0$. Consequently, by part (1) of 13.4, there is $B_\delta(\theta)$ (for some $\delta > 0$) such that $\Theta(f) \neq 0$ for any $\Theta \in B_\delta(\theta)$. Then, by the given condition, there is an integer m such that for any $m' \geq m$ and any irreducible representation σ of $A_{m'}$, one has $Sp(\varphi_{n,m'}|_\sigma) \cap B_\delta(\theta) \neq \varnothing$. Let $\Theta \in Sp(\varphi_{n,m'}|_\sigma) \cap B_\delta(\theta)$. Then, $\Theta \circ \varphi_{n,m'}(f) \neq 0$. Consequently, $\|\sigma \circ \varphi_{n,m'}(f)\| \geq \|\Theta \circ \varphi_{n,m'}(f)\| > 0$. It follows that the ideal I generated by $\varphi_{n,m'}(f))$ in $A_{m'}$ equals $A_{m'}$—otherwise, any irreducible representation σ of $A_{m'}/I \neq 0$ (which is also an irreducible representation of $A_{m'}$) satisfies $\sigma(\varphi_{n,m'}(f)) = 0$, and this contradicts the fact that $\sigma(\varphi_{n,m'}(f)) \neq 0$ for every irreducible representation σ. Fix $m' \geq m$. Then $\varphi_{n,m'}(f)$ is full. Since each $\varphi_{n,m}$ is unital, it follows that $\varphi_{m',\infty} \circ \varphi_{n,m'}(f) = \varphi_{n,\infty}(f)$ is full. In other words (since f above is arbitrary), for any proper ideal I of A, $\varphi_{n,\infty}(A_n) \cap I = \{0\}$. It is standard to show that this implies that A is simple. In fact, for any $a \in \varphi_{n,\infty}(A_n)$, $\pi_I(\|a^*a\|) = \|a^*a\|$, where $\pi_I : A \to A/I$ is the quotient map. It follows that $\|\pi_I(a)\| = \|a\|$. Note that $\bigcup_{n=1}^\infty (A_n)$ is dense in A. The quotient map π_I is thus isometric, and so $I = \{0\}$. (This proof is due to Dixmier (see the proof of Theorem 1.4 of [24]).) Therefore A is simple.

Suppose the unital limit algebra A is simple. For any A_n and $\delta > 0$, let $\mathcal{F}_n \subset (A_n)_+ \setminus \{0\}$ be as in the Lemma 13.4. Since A is simple and $\varphi_{n,m}$ is injective, for any $f \in \mathcal{F}_n$, the ideal generated by $\varphi_{n,\infty}(f) \in A$ contains $\mathbf{1}_A$. Hence there is $m_f > n$ such that if $m' \geq m_f$ then the ideal generated by $\varphi_{n,m'}(f)$ in $A_{m'}$ contains $\mathbf{1}_{A_{m'}}$ and therefore is the whole of $A_{m'}$. Let $m = \max\{m_f : f \in \mathcal{F}_n\}$. For any $\sigma \in Sp(A_{m'})$ (with $m' \geq m$) and $f \in \mathcal{F}_n$, we have $\sigma(\varphi_{n,m'}(f)) \neq 0$.

We are going to verify that for any $\sigma \in Sp(A_{m'})$, the set $SP(\varphi_{n,m'}|_\sigma)$ is δ-dense in $Sp(A_n)$. For any $\theta \in Sp(A_n)$, there is an $f \in \mathcal{F}_n$ such that, if $\Theta(f) \neq 0$, then $\Theta \in B_\delta(\theta)$. From $\sigma(\varphi_{n,m'}(f)) \neq 0$, one then concludes that there is an irreducible representation $\Theta \subset SP(\varphi_{n,m'}|_\sigma)$ such that $\Theta(f) \neq 0$. Hence $\Theta \in B_\delta(\theta)$ by Lemma 13.4. This implies that $SP(\varphi_{n,m'}|_\sigma)$ is δ-dense. $\square$

DEFINITION 13.6. Denote by $\mathcal{N}_0$ the class of those unital simple C^*-algebras A in $\mathcal{N}$ for which $A \otimes U \in \mathcal{N} \cap \mathcal{B}_0$ for any UHF-algebra U of infinite type (see 2.9 for the definition of the class $\mathcal{N}$).

Denote by $\mathcal{N}_1$ the class of those unital C^*-algebras A in $\mathcal{N}$ for which $A \otimes U \in \mathcal{N} \cap \mathcal{B}_1$ for any UHF-algebra of infinite type. In Section 19, we will show that $\mathcal{N}_1 = \mathcal{N}_0$.

Also denote by $\mathcal{N}_0^{\mathcal{Z}}$ (respectively, $\mathcal{N}_1^{\mathcal{Z}}$) the class of all $\mathcal{Z}$-stable C^*-algebras in $\mathcal{N}_0$ (respectively, $\mathcal{N}_1$).

13.7. Let (G, G_+, u) be a scaled ordered abelian group (G, G_+) with order unit $u \in G_+ \setminus \{0\}$, with the scale given by $\{g \in G_+ : g \leq u\}$. Sometimes we will also call u the scale of the group. Let $S(G) := S_u(G)$ be the state space of G. Suppose that $((G, G_+, u), K, \Delta, r)$ is a weakly unperforated Elliott invariant—that is, (G, G_+, u) is a simple scaled ordered countable group, K is a countable abelian group, Δ is a metrizable Choquet simplex, and $r : \Delta \to S(G)$

is a surjective affine map such that for any $x \in G$,

$$\text{(e 13.4)} \quad x \in G_+ \setminus \{0\} \text{ if and only if } r(\tau)(x) > 0 \text{ for all } \tau \in \Delta.$$

Condition (e 13.4) above is also called weak unperforation for the simple ordered group. (Note that this condition is equivalent to the condition that $x \in G_+ \setminus \{0\}$ if and only if for any $f \in S(G)$, $f(x) > 0$. The latter condition does not mention Choquet simplex Δ.) In this paper, we only consider the Elliott invariant for stably finite simple unital nuclear C^*-algebras and therefore Δ is not empty. Evidently, the above weak unperforation condition implies the condition that $x > 0$ if $nx > 0$ for some positive integer n. The converse follows from Proposition 3.2 of [104].

In this section, we will show that for any weakly unperforated Elliott invariant $((G, G_+, u), K, \Delta, r)$, there is a unital simple C^*-algebra A in the class $\mathcal{N}_0^{\mathcal{Z}}$ such that

$$((K_0(A), K_0(A)_+, [\mathbf{1}_A]), K_1(A), T(A), r_A) \cong ((G, G_+, u), K, \Delta, r).$$

A similar general range theorem was presented by Elliott in [31]. To obtain our version, we will modify the construction given by Elliott in [31]. Our modification will reveal more details in the construction and will also ensure that the algebras constructed are actually in the class $\mathcal{N}_0^{\mathcal{Z}}$. One difference is that, at an important step of the construction, we will use a finite subset Z_i of a certain space X_i instead of a one-dimensional subspace of X_i.

13.8. Our construction will be a modification of the Elliott construction mentioned above. As a matter of fact, for the case that $K = \{0\}$ and G is torsion free, our construction uses the same building blocks, in $\mathcal{C}_0$, as in [31]. We will repeat a part of the construction of Elliott for this case. There are two steps in Elliott's construction:

Step 1. Construct an inductive limit

$$C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C$$

with inductive limit of ideals

$$I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I$$

such that the non-simple limit C has the prescribed Elliott invariant and the quotient C/I is a simple AF algebra. For the case $K = \{0\}$ and G torsion free, we will use the notation C_n and C for the construction, and reserve A_n and A for the general case.

Step 2. Modify the above inductive limit to make C (or A in the general case) simple without changing the Elliott invariant of C (or A).

For the reader's convenience, we will repeat Step 1 of Elliott's construction with minimum modification. For Step 2, we are not able to reconstruct the argument of [31] and in particular we do not know how to make the eigenvalue patterns given on pages 81–82 [31] satisfy the required boundary conditions. We will use a different way of modifying the inductive limit (see 13.46 for the details). This new way will be also more suitable for our purpose—i.e., to construct an inductive limit $A \in \mathcal{N}_0^{\mathcal{Z}}$ in the general case, with possibly nontrivial K_1 and nontrivial $\mathrm{Tor}(K_0(A))$.

13.9. Suppose that $((G, G_+, u), K, \Delta, r)$ is a weakly unperforated Elliott invariant as defined in 13.7. Let $\rho : G \to \mathrm{Aff}(\Delta)$ be the dual map of $r : \Delta \to S(G)$. That is, for every $g \in G,\ \tau \in \Delta$,

$$\rho(g)(\tau) = r(\tau)(g) \in \mathbb{R}.$$

Since G is weakly unperforated, one has that $g \in G_+ \setminus \{0\}$ if and only if $\rho(g)(\tau) > 0$ for all $\tau \in \Delta$. Note that $\mathrm{Aff}(\Delta)$ is an ordered vector space with the strict (pointwise) order, i.e., $f \in \mathrm{Aff}(\Delta)_+ \setminus \{0\}$ if and only if $f(\tau) > 0$ for all $\tau \in \Delta$. Note that Δ is a metrizable compact convex set, and $\mathrm{Aff}(\Delta)$ is a norm closed subspace of $C_{\mathbb{R}}(\Delta)$, the real Banach space of real continuous functions on Δ. Consequently $\mathrm{Aff}(\Delta)$ is separable. We assume that $G \neq \{0\}$ and $\Delta \neq \varnothing$. Therefore there is a countable dense subgroup $G^1 \subset \mathrm{Aff}(\Delta)$. Put $H = G \oplus G^1$ and define $\tilde{\rho} : H \to \mathrm{Aff}(\Delta)$ by $\tilde{\rho}((g, f))(\tau) = \rho(g)(\tau) + f(\tau)$ for all $(g, f) \in G \oplus G^1$ and $\tau \in \Delta$. Define $H_+ \setminus \{0\}$ to be the set of elements $(g, f) \in G \oplus G^1$ with $\tilde{\rho}((g, f))(\tau) > 0$ for all $\tau \in \Delta$. The order unit (or scale) $u \in G_+$, regarded as $(u, 0) \in G \oplus G^1 = H$, is an order unit for H_+ (still denote it by u).

Then (H, H_+, u) is a simple ordered group with the Riesz interpolation property. With the strict order, (H, H_+, u) is a simple ordered group. Since Δ is a simplex, by Corollary II.3.11 of [1], $\mathrm{Aff}(\Delta)$ has the weak Riesz interpolation property. Since $\tilde{\rho}(H)$ is dense, it is straightforward to prove that (H, H_+, u) is a Riesz group. Let us give a brief proof of this fact. Let $a_1, a_2, b_1, b_2 \in H$ with $a_i < b_j$ for $i, j \in \{1, 2\}$. Then, since $\mathrm{Aff}(\Delta)$ has the weak Riesz interpolation property (see page 90 of [1]), there exists $f \in \mathrm{Aff}(\Delta)$ such that $\tilde{\rho}(a_i) < f < \tilde{\rho}(b_j)$, $i, j \in \{1, 2\}$. Let

$$\begin{aligned} d \;\; = \;\; & \min\{\min\{\tau(f) - \tau(a_i) : \tau \in \Delta,\ 1 \leq i \leq 2\}, \\ & \quad \min\{\tau(b_j) - \tau(f) : \tau \in \Delta,\ 1 \leq j \leq 2\}\}. \end{aligned}$$

Then $d > 0$. Since $\tilde{\rho}(H)$ is dense in $\mathrm{Aff}(\Delta)$, there exists $h \in H$ such that $\|\tilde{\rho}(h) - f\| < d/2$. Then $\tilde{\rho}(a_i) < \tilde{\rho}(h) < \tilde{\rho}(b_j)$, $i, j \in \{1, 2\}$. Consequently, $a_i < h < b_j$ for $i, j \in \{1, 2\}$.

As a direct summand of H, the subgroup G is *relatively divisible* subgroup of H; that is, if $g \in G$, $m \in \mathbb{N} \setminus \{0\}$, and $h \in H$ such that $g = mh$, then there is $g' \in G$ such that $g = mg'$. Note that $G \subsetneqq H$ since $\Delta \neq \varnothing$.

Now we assume, until 13.23, that G is torsion free and $K = 0$. Then H is also torsion free. Therefore, H is a simple dimension group.

REMARK 13.10. (1) There is a unital simple AF-algebra B such that $(K_0(B), K_0(B)_+, [1_B]) = (H, H_+, u)$ and $S_u(H) = \Delta = T(B)$, where $S_u(H)$ is the state space of H and $T(B)$ is the tracial state space of B. In fact, by [26], there is a unital simple AF-algebra B such that $(K_0(B), K_0(B)_+, [1_B]) = (H, H_+, u)$. It follows from Theorem III.1.3 of [5] that the state space $S_u(H)$ of $K_0(B) = H$ is $T(B)$ (with the topology of pointwise convergence on $S_u(H)$ and the weak* topology of $T(B)$). On the other hand, evaluation at a point of Δ also gives a state of H. Therefore Δ is a closed convex subset of $S_u(H)$. Since $x < y$ in H if and only if $\tilde{\rho}(x)(s) < \tilde{\rho}(y)(s)$ for all $s \in \Delta$, by Lemma 2.9 of [8], $\Delta = S_u(H) = T(B)$. Furthermore, the map $\rho_B : K_0(B) = H \to \mathrm{Aff}(T(B)) = \mathrm{Aff}(\Delta)$ is the same as $\tilde{\rho} : H \to \mathrm{Aff}(\Delta)$.

(2) Suppose that A is a unital C^*-algebra and $\varphi : A \to B$ is a unital homomorphism (where B is as in part (1)). Suppose that $(K_0(A), K_0(A)_+, [1_A]) = (G, G_+, u)$, and $T(A) \cong T(B)$ and suppose that the induced maps $\varphi_{*0} : K_0(A) \to K_0(B)$ and $T(\varphi) : T(B) \to T(A)$ (of the homomorphism φ) are the inclusion from G to H and the affine homeomorphism between $T(B)$ and $T(A)$. Then the map $\rho_A : K_0(A) = G \to \mathrm{Aff}(T(A)) = \mathrm{Aff}(\Delta)$ is the same as $\rho : G \to \mathrm{Aff}(\Delta)$, under the identification of $T(A) = T(B) = \Delta$. This is true because $\rho = \tilde{\rho} \circ \iota_{G,H} : G \to \mathrm{Aff}(\Delta)$, where $\iota_{G,H} : G \to H$ is the inclusion.

13.11. In 13.9 we can choose the dense subgroup $G^1 \subset \mathrm{Aff}(\Delta)$ to contain at least three elements $x, y, z \in \mathrm{Aff}(\Delta)$ such that x, y and z are $\mathbb{Q}$-linearly independent. With this choice, when we write H as the inductive limit of a sequence

$$H_1 \longrightarrow H_2 \longrightarrow \cdots$$

of finite direct sums of copies of the ordered group $(\mathbb{Z}, \mathbb{Z}_+)$ as in Theorem 2.2 of [26], we can assume all H_n have at least three copies of $\mathbb{Z}$.

Note that the homomorphism

$$\gamma_{n,n+1} : \ H_n = \mathbb{Z}^{p_n} \longrightarrow H_{n+1} = \mathbb{Z}^{p_{n+1}}$$

is given by a $p_{n+1} \times p_n$, matrix $\mathfrak{c} = (c_{ij})$ of nonnegative integers, where $i = 1, 2, ..., p_{n+1}$, $j = 1, 2, ..., p_n$, and $c_{ij} \in \mathbb{Z}_+ := \{0, 1, 2, ...\}$. For $M > 0$, if all $c_{ij} \geq M$, then we will say $\gamma_{n,n+1}$ is at least M-large or has multiplicity at least M. Note that since H is a simple ordered group, passing to a subsequence, we may assume that at each step $\gamma_{n,n+1}$ is at least M_n-large for an arbitrary choice of M_n depending on our construction up to step n.

13.12. Recall that with the dimension group H as in 13.9 and 13.11, we have $G \subset H$ with $G_+ = H_+ \cap G$, and both G and H share the same order unit $u \in G \subset H$. As in 13.11, write H as the inductive limit of H_n—a sequence of finite direct sums of three or more copies of the ordered group $(\mathbb{Z}, \mathbb{Z}_+)$ (with connecting maps with large multiplicities). Let $G_n = \gamma_{n,\infty}^{-1}(\gamma_{n,\infty}(H_n) \cap G)$, where $\gamma_{n,\infty} : \ H_n \to H$ is the canonical map to the limit. There is a $k_0 \in \mathbb{N}$ such that $u \in G_{k_0}$, and without loss of generality, we may assume that $k_0 = 1$. In other words, we may assume $u \in G_n \subset H_n$ for each n. Since $G_+ = H_+ \cap G$, if we confer an order structure on G_n by setting $(G_n)_+ \colon = (H_n)_+ \cap G_n$, then we have $G_+ = \lim_{n\to\infty}(G_n)_+$.

We claim that G_n is a relatively divisible subgroup of H_n. Let us suppose that $g \in G_n$ and $g = mh$ for $h \in H_n$ and for some integer $m \geq 1$. Since G is relatively divisible in H, there is a $g' \in G$ such that $\gamma_{n,\infty}(g) = mg'$. Noting that $\gamma_{n,\infty}(g) = m\gamma_{n,\infty}(h)$, we have $m(g' - \gamma_{n,\infty}(h)) = 0$. Hence $\gamma_{n,\infty}(h) = g' \in G$ which implies that $h \in \gamma_{n,\infty}^{-1}(\gamma_{n,\infty}(H_n) \cap G)$. That is, $h \in G_n$. Since G_n is a relatively divisible subgroup of H_n and H_n is torsion free, the quotient H_n/G_n is a torsion-free finitely generated abelian group and therefore a direct sum of copies of $\mathbb{Z}$, denoted by $\mathbb{Z}^{l_n}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
G_1 \arrow[r, "\gamma_{12}|_{G_1}"] \arrow[d, hook] & G_2 \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & G \arrow[d, hook] \\
H_1 \arrow[r, "\gamma_{12}"] \arrow[d] & H_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_1/G_1 \arrow[r, "\tilde{\gamma}_{12}"] & H_2/G_2 \arrow[r] & \cdots \arrow[r] & H/G.
\end{tikzcd}$$

Let $H_n = \left(\mathbb{Z}^{p_n}, \mathbb{Z}_+^{p_n}, u_n\right)$, where $u_n{:=}([n,1],[n,2],...,[n,p_n]) \in (\mathbb{Z}_+ \setminus \{0\})^{p_n}$. Then H_n can be realized as the K_0-group of $F_n = \bigoplus_{i=1}^{p_n} M_{[n,i]}$—that is,

$$(K_0(F_n), K_0(F_n)_+, [1_{F_n}]) = (H_n, (H_n)_+, u_n).$$

If there are infinitely many n such that the inclusion maps $G_n \to H_n$ are isomorphisms, then, passing to a subsequence, we have that $G \to H$ is also an isomorphism which contradicts $G \subsetneqq H$ in 13.9. Hence, without loss of generality, we may assume that for all n, $G_n \subsetneqq H_n$, and therefore $H_n/G_n \neq 0$.

To construct a C^*-algebra with K_0 equal to $(G_n, (G_n)_+, u_n)$, we consider the quotient map $\pi\colon\ H_n \to H_n/G_n$ as a map (still denoted in the same way)

$$\pi\colon\ \mathbb{Z}^{p_n} \longrightarrow \mathbb{Z}^{l_n},$$

as in [31]. We emphasize that $l_n > 0$ for all n (as $H_n/G_n \neq \{0\}$). Such a map can be realized as difference of two maps

$$\mathfrak{b}_0,\ \mathfrak{b}_1\colon\ \mathbb{Z}^{p_n} \longrightarrow \mathbb{Z}^{l_n}$$

corresponding to two $l_n \times p_n$ matrices of strictly positive integers $\mathfrak{b}_0 = (b_{0,ij})$ and $\mathfrak{b}_1 = (b_{1,ij})$. That is,

$$\pi\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{p_n} \end{pmatrix} = \left(\mathfrak{b}_1 - \mathfrak{b}_0\right)\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{p_n} \end{pmatrix} \in \mathbb{Z}^{l_n}, \qquad \text{for any} \qquad \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{p_n} \end{pmatrix} \in \mathbb{Z}^{p_n}.$$

Note that $u_n = ([n,1],[n,2],...,[n,p_n]) \in G_n$ and hence $\pi(u_n) = 0$. Consequently,

$$\mathfrak{b}_1 \begin{pmatrix} [n,1] \\ [n,2] \\ \vdots \\ [n,p_n] \end{pmatrix} = \mathfrak{b}_0 \begin{pmatrix} [n,1] \\ [n,2] \\ \vdots \\ [n,p_n] \end{pmatrix} =: \begin{pmatrix} \{n,1\} \\ \{n,2\} \\ \vdots \\ \{n,p_n\} \end{pmatrix},$$

i.e.,

$$\{n,i\} := \sum_{j=1}^{p_n} b_{1,ij}[n,j] = \sum_{j=1}^{p_n} b_{0,ij}[n,j].$$

Let $E_n = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}$. Choose any two homomorphisms $\beta_0, \beta_1 : F_n \to E_n$ such that $(\beta_0)_{*0} = \mathfrak{b}_0$ and $(\beta_1)_{*0} = \mathfrak{b}_1$. Then define

$$\begin{aligned} C_n &:= \Big\{(f,a) \in C([0,1],E_n) \oplus F_n;\ f(0) = \beta_0(a), f(1) = \beta_1(a)\Big\} \\ &= C([0,1],E_n) \oplus_{\beta_0,\beta_1} F_n, \end{aligned}$$

which is $A(F_n, E_n, \beta_0, \beta_1)$ as in the definition of 3.1. By Proposition 3.5 and the fact that the map $\pi = (\beta_1)_{*0} - (\beta_0)_{*0}$ (playing the role of $\varphi_{1*0} - \varphi_{0*0}$ there) is surjective (see definition of π, $\mathfrak{b}_0$, and $\mathfrak{b}_1$ above), we have $K_1(C_n) = 0$ and

$$\text{(e 13.5)} \qquad \Big(K_0(C_n), K_0(C_n)_+, \mathbf{1}_{C_n}\Big) = \Big(G_n, (G_n)_+, u_n\Big).$$

As in (e 3.3), the map $K_0(F_n) = \mathbb{Z}^{p_n} \to K_0(E_n) = K_1(C_0\big((0,1),E_n\big)) = \mathbb{Z}^{l_n}$ is given by $\mathfrak{b}_1 - \mathfrak{b}_0 \in M_{l_n \times p_n}(\mathbb{Z})$, which is surjective, as π is the quotient map $H_n\,(=\mathbb{Z}^{p_n}) \to H_n/G_n\,(=\mathbb{Z}^{l_n})$. *In particular, all* $C_n \in \mathcal{C}_0$.

As observed in [31], in the construction of C_n with (e 13.5), we have the freedom to choose the pair of the K_0-maps $(\beta_0)_{*0} = \mathfrak{b}_0$ and $(\beta_1)_{*0} = \mathfrak{b}_1$ as long as the difference is the same map $\pi :\ H_n\,(=\mathbb{Z}^{p_n}) \longrightarrow H_n/G_n\,(=\mathbb{Z}^{l_n})$. For example, if $(m_{ij}) \in M_{l_n \times p_n}\big(\mathbb{Z}_+ \setminus \{0\}\big)$ is any $l_n \times p_n$ matrix of positive integers, then we can replace $b_{0,ij}$ by $b_{0,ij} + m_{ij}$ and, at the same time, replace $b_{1,ij}$ by $b_{1,ij} + m_{ij}$. That is, we can assume that each entry of $\mathfrak{b}_0$ (and of $\mathfrak{b}_1$) is larger than any fixed integer M which may depend on C_{n-1} and $\psi_{n-1,n} : F_{n-1} \to F_n$. Also, we can make all the entries of one column (say, the third column) of both $\mathfrak{b}_0$ and $\mathfrak{b}_1$ much larger than all the entries of another column (say, the second), by choosing $m_{i3} \gg m_{j2}$ for all $i,\ j$.

13.13. Let us consider a slightly more general case than in 13.12. Let $G_n = \mathbb{Z}^{p_n^0} \oplus G'_n$ and $H_n = \mathbb{Z}^{p_n^0} \oplus H'_n$, with the inclusion map being the identity for the first p_n^0 copies of $\mathbb{Z}$. In this case, the quotient map $H_n\,(=\mathbb{Z}^{p_n}) \to H_n/G_n\,(=\mathbb{Z}^{l_n})$ given by the matrix $\mathfrak{b}_1 - \mathfrak{b}_0$ maps the first p_n^0 copies of $\mathbb{Z}$ to zero. For this case, it will be much more convenient to assume that the first p_n^0 columns of both the matrices $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are zero and each entry of the last $p_n^1 := p_n - p_n^0$ columns of them are larger than any previously given integer M. Now we have that the entries of the matrices $\mathfrak{b}_0, \mathfrak{b}_1$ are strictly positive integers except the ones in the first p_n^0 columns which are zero.

Consider the following diagram:

$$\begin{tikzcd}
\mathbb{Z}^{p_1^0} \oplus G_1' \arrow[r, "\gamma_{12}|_{G_1}"] \arrow[d, hook] & \mathbb{Z}^{p_2^0} \oplus G_2' \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & G \arrow[d, hook] \\
\mathbb{Z}^{p_1^0} \oplus H_1' \arrow[r, "\gamma_{12}"] \arrow[d] & \mathbb{Z}^{p_2^0} \oplus H_2' \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_1/G_1 \arrow[r, "\tilde{\gamma}_{12}"] & H_2/G_2 \arrow[r] & \cdots \arrow[r] & H/G\,.
\end{tikzcd}$$

That is, $G_n = \mathbb{Z}^{p_n^0} \oplus G_n'$ and $H_n = \mathbb{Z}^{p_n^0} \oplus H_n'$, with the inclusion map being the identity for the first p_n^0 copies of $\mathbb{Z}$. We are now assuming that the entries of the matrices $\mathfrak{b}_0, \mathfrak{b}_1$ are strictly positive integers for the last p_n^1 columns and are zeros for the first p_n^0 columns. Note that since $l_n > 0$ (see 13.12), we have $p_n^1 > 0$.

The inductive limits $H = \varinjlim(H_n, \gamma_{n,n+1})$ and $G = \varinjlim(G_n, \gamma_{n,n+1}|_{G_n})$ (with $G_n \subset H_n$) constructed in 13.12 are in fact special cases of the present construction when we assume that $p_n^0 = 0$. One notices that, for the case $G_n = \mathbb{Z}^{p_n^0} \oplus G_n'$ and $H_n = \mathbb{Z}^{p_n^0} \oplus H_n'$, with the inclusion map being the identity for the first p_n^0 copies of $\mathbb{Z}$, one could still use the construction of 13.12 to make all the entries of $\mathfrak{b}_0$ and $\mathfrak{b}_1$ (not only the entries of the last p_n^1) be strictly positive (of course with the first p_n^0 columns of the matrices $\mathfrak{b}_0$ and $\mathfrak{b}_1$ equal to each other). However, for the algebras with the property (SP), it is possible to assume that $p_n^0 \neq 0$ for all n, and to get a certain decomposition property that we will discuss in the next section. In fact, for the case that $p_n^0 \neq 0$ for all n, the construction is much simpler than the case that $p_n^0 = 0$ for all n.

If we write F_n above as $\bigoplus_{i=1}^{p_n^0} M_{[n,i]} \oplus F_n'$, where $F_n' = \bigoplus_{i=p_n^0+1}^{p_n} M_{[n,i]}$, then the maps β_0 and β_1 are zero on the part $\bigoplus_{i=1}^{p_n^0} M_{[n,i]}$. Moreover, the algebra $C_n = \Big\{(f,a) \in C([0,1], E_n) \oplus F_n;\ f(0) = \beta_0(a), f(1) = \beta_1(a)\Big\}$ can be written as $\bigoplus_{i=1}^{p_n^0} M_{[n,i]} \oplus C_n'$, where

$$\begin{aligned}
C_n' &= \Big\{(f,a) \in C([0,1], E_n) \oplus F_n';\ f(0) = \beta_0(a), f(1) = \beta_1(a)\Big\} \\
&= C([0,1], E_n) \oplus_{\beta_0|_{F_n'}, \beta_1|_{F_n'}} F_n'
\end{aligned}$$

as in 13.12. It should be remembered that 13.12 is a special case of 13.13.

13.14. Let us emphasize that once

$$(H_n, (H_n)_+, u_n) = \big(\mathbb{Z}^{p_n}, (\mathbb{Z}_+)^{p_n}, ([n,1],[n,2],...,[n,p_n])\big), \text{ and } \mathfrak{b}_0, \mathfrak{b}_1 : \mathbb{Z}^{p_n} \to \mathbb{Z}^{l_n},$$

are fixed, then the algebras $F_n = \bigoplus_{i=1}^{p_n} M_{[n,i]}$, $E_n = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}$ are fixed, where $\{n,i\}{:=}\sum_{j=1}^{p_n} b_{1,ij}[n,j] = \sum_{j=1}^{p_n} b_{0,ij}[n,j]$; and, by Proposition 3.6, the algebras $C_n = A(F_n, E_n, \beta_0, \beta_1)$, with $(\beta_i)_{*0} = \mathfrak{b}_i$ $(i = 0,1)$, are determined up to isomorphism.

To construct the inductive limit, we not only need to construct C_n's (later on A_n's for the general case) but also need to construct $\varphi_{n,n+1}$ which realizes the corresponding K-theory map—i.e., $(\varphi_{n,n+1})_{*0} = \gamma_{n,n+1}|_{G_n}$. In addition, we need to make the limit algebras have the desired tracial state space. In order to do all these things, we need some extra conditions on the maps $\mathfrak{b}_0, \mathfrak{b}_1$ for C_{n+1} (or for A_{n+1}) depending on C_n (or A_n). We will divide the construction into several steps with gradually stronger conditions on $\mathfrak{b}_0, \mathfrak{b}_1$ (for C_{n+1})—of course depending on C_n and the map $\gamma_{n,n+1} : H_n \to H_{n+1}$, to guarantee the construction can go through.

Let $G_n \subset H_n = \mathbb{Z}^{p_n}$, $G_{n+1} \subset H_{n+1} = \mathbb{Z}^{p_{n+1}}$, and $\gamma_{n,n+1} : H_n \to H_{n+1}$, with $\gamma_{n,n+1}(G_n) \subset G_{n+1}$, be as in 13.11 and 13.12 (also see 13.13). Then $\gamma_{n,n+1}$ induces a map $\tilde{\gamma}_{n,n+1} : H_n/G_n\,(= \mathbb{Z}^{l_n}) \to H_{n+1}/G_{n+1} = (\mathbb{Z}^{l_{n+1}})$. Let $\gamma_{n,n+1} : H_n\,(= \mathbb{Z}^{p_n}) \to H_{n+1}\,(= \mathbb{Z}^{p_{n+1}})$ be given by the matrix $\mathfrak{c} = (c_{ij}) \in M_{p_{n+1}\times p_n}(\mathbb{Z}_+ \setminus \{0\})$ and $\tilde{\gamma}_{n,n+1} : \mathbb{Z}^{l_n} \to \mathbb{Z}^{l_{n+1}}$ (as a map from $H_n/G_n \to H_{n+1}/G_{n+1}$) be given by the matrix $\mathfrak{d} = (d_{ij})$. Let $\pi_{n+1} : H_{n+1}(= \mathbb{Z}^{p_{n+1}}) \to H_{n+1}/G_{n+1}(= \mathbb{Z}^{l_{n+1}})$ be the quotient map.

Let us use $\mathfrak{b}_0', \mathfrak{b}_1' : \mathbb{Z}^{p_{n+1}} \to \mathbb{Z}^{l_{n+1}}$ to denote the maps required for the construction of C_{n+1}, and reserve $\mathfrak{b}_0, \mathfrak{b}_1$ for C_n. Of course, $\pi_{n+1} = \mathfrak{b}_1' - \mathfrak{b}_0'$.

We will prove that if $\mathfrak{b}_0', \mathfrak{b}_1'$ satisfy:

$$\tilde{b}_{0,ji},\ \tilde{b}_{1,ji} \ > \ \sum_{k=1}^{l_n} \big(|d_{jk}|+2\big)\max(b_{0,ki},\ b_{1,ki}) \qquad (\diamondsuit) \tag{e 13.6}$$

for all $i \in \{1,2,...,p_n\}$ and for all $j \in \{1,2,...,l_{n+1}\}$, where $\tilde{b}_{0,ji}$ and $\tilde{b}_{1,ji}$ are the entries of $\tilde{\mathfrak{b}}_0{:=}\mathfrak{b}_0' \cdot \mathfrak{c}$ and $\tilde{\mathfrak{b}}_1{:=}\mathfrak{b}_1' \cdot \mathfrak{c}$, respectively, then one can construct the homomorphism $\varphi_{n,n+1} : C_n \to C_{n+1}$ to realize the desired K-theory map (see 13.15 below). If $\mathfrak{b}_0', \mathfrak{b}_1'$ satisfy the stronger condition

$$\tilde{b}_{0,ji}, \tilde{b}_{1,ji} > 2^{2n}\left(\sum_{k=1}^{l_n}(|d_{jk}|+2)\{n,k\}\right) \qquad (\diamondsuit\diamondsuit) \tag{e 13.7}$$

for all $i \in \{1,2,...,p_n\}$ and for all $j \in \{1,2,...,l_{n+1}\}$, then we can prove the limit algebra constructed has the desired tracial state space (see the corresponding calculation in 13.17–13.19, which will be used in the proof of Theorem 13.42). (It follows from the fact that $\{n,k\}{:=}\sum_{i=1}^{p_n} b_{1,ki}[n,i] = \sum_{i=1}^{p_n} b_{0,ki}[n,i]$ that the inequality $(\diamondsuit\diamondsuit)$ is stronger than $(\diamondsuit)$.)

From the definition of $\tilde{\mathfrak{b}}_0$ and $\tilde{\mathfrak{b}}_1$, we have

$$\tilde{b}_{0,ji} = \sum_{k=0}^{p_{n+1}} b'_{0,jk}c_{ki} \qquad \text{and} \qquad \tilde{b}_{1,ji} = \sum_{k=0}^{p_{n+1}} b'_{1,jk}c_{ki}, \tag{e 13.8}$$

where $b'_{0,jk}$ and $b'_{1,jk}$ are the entries of $\mathfrak{b}'_0$ $\mathfrak{b}'_1$, respectively.

Let us also emphasize that when we modify inductive limit system to make it simple, we never change the algebras C_n (or A_n in the general case), what will be changed are the connecting homomorphisms.

Let A and B be C^*-algebras, $\varphi : A \to B$ be a homomorphism, and $\pi \in RF(B)$. For the rest of this section, we will use $\varphi|_\pi$ for the composition $\pi\circ\varphi$, in particular, in the following statement and its proof. This notation is consistent with 13.2.

In the following lemma we will give the construction of $\varphi_{n,n+1}$ in the case that $\mathfrak{b}'_0$ and $\mathfrak{b}'_1$ satisfy Condition ($\diamondsuit$) of 13.14—of course, the condition depends on the previous step. So this lemma provides the $(n+1)$st step of the construction. Again, we first have G, and then obtain H, H_n, and G_n as constructed in 13.9 and 13.11.

LEMMA 13.15. *Let*

$$(H_n, (H_n)_+, u_n) = \big(\mathbb{Z}^{p_n}, (\mathbb{Z}_+)^{p_n}, ([n,1],[n,2],...,[n,p_n])\big),$$

$$F_n = \bigoplus_{i=1}^{p_n} M_{[n,i]},\ \ \mathfrak{b}_0, \mathfrak{b}_1 : \mathbb{Z}^{p_n} \to \mathbb{Z}^{l_n},\ \ E_n = \bigoplus_{i=1}^{l_n} M_{\{n,i\}},\ \ \beta_0, \beta_1 : F_n \to E_n$$

with $(\beta_0)_{*0} = \mathfrak{b}_0 = (b_{0,ij})_{l_n\times p_n}, (\beta_1)_{*0} = \mathfrak{b}_1 = (b_{1,ij})_{l_n\times p_n}$, *and* $C_n = A(F_n, E_n, \beta_0, \beta_1)$ *with* $K_0(C_n) = G_n$ *be as in* 13.13, *which includes the case of* 13.12 *as the special case* $p_n^0 = 0$. *Let*

$$(H_{n+1}, H_{n+1}^+, u_{n+1}) = \big(\mathbb{Z}^{p_{n+1}}, (\mathbb{Z}_+)^{p_{n+1}}, ([n+1,1],[n+1,2],...,[n+1,p_{n+1}])\big),$$

let $\gamma_{n,n+1} : H_n \to H_{n+1}$ *be an order homomorphism with* $\gamma_{n,n+1}(u_n) = u_{n+1}$ (*as in* 13.12 *or* 13.13), *and let* $G_{n+1}\subset H_{n+1}$ *be a subgroup containing* u_{n+1} (*as in* 13.11) *and satisfying* $\gamma_{n,n+1}(G_n)\subset G_{n+1}$. *Let* $\pi_{n+1} : H_{n+1}\ (= \mathbb{Z}^{p_{n+1}}) \to H_{n+1}/G_{n+1}\ (= \mathbb{Z}^{l_{n+1}})$ *denote the quotient map, and let* $\gamma_{n,n+1}$ *be represented by the* $p_{n+1}\times p_n$*-matrix* (c_{ij}).

Suppose that the maps $\mathfrak{b}'_0 = (b'_{0,ij})$, $\mathfrak{b}'_1 = (b'_{1,ij}) : \mathbb{Z}^{p_{n+1}} \to \mathbb{Z}^{l_{n+1}}$ *satisfy* $\mathfrak{b}'_1 - \mathfrak{b}'_0 = \pi_{n+1}$ *and satisfy Condition* ($\diamondsuit$) *of* 13.14. *(As a convention, we assume that the entries of the first* p^0_{n+1} *columns of the matrices* $\mathfrak{b}'_0$ *and* $\mathfrak{b}'_1$ *are zeros, and the entries of the last* $p^1_{n+1} = p_{n+1}-p^0_{n+1}$ *columns are strictly positive. Note that* p^0_{n+1} *might be zero as in the special case* 13.12.*)*

Put $F_{n+1} = \bigoplus_{i=1}^{p_{n+1}} M_{[n+1,i]}$ *and* $E_{n+1} = \bigoplus_{i=1}^{l_{n+1}} M_{\{n+1,i\}}$ *with*

$$\{n+1,i\} = \sum_{j=1}^{p_{n+1}} b'_{1,ij}[n+1,j] = \sum_{j=1}^{p_{n+1}} b'_{0,ij}[n+1,j],$$

and pick unital homomorphisms $\beta_0', \beta_1' : F_{n+1} \to E_{n+1}$ *with*

$$(\beta_0')_{*0} = \mathfrak{b}_0' \quad \textit{and} \quad (\beta_1')_{*0} = \mathfrak{b}_1'.$$

Set

$$C_{n+1} = A(F_{n+1}, E_{n+1}, \beta_0', \beta_1').$$

Then there is a homomorphism $\varphi_{n,n+1} : C_n \to C_{n+1}$ *satisfying the following conditions:*

(1) $K_0(C_{n+1}) = G_{n+1}$ *as scaled ordered groups (as already verified in* 13.12*).*
(2) $(\varphi_{n,n+1})_{*0} : K_0(C_n) = G_n \to K_0(C_{n+1}) = G_{n+1}$ *satisfies* $(\varphi_{n,n+1})_{*0} = \gamma_{n,n+1}|_{G_n}$.
(3) $\varphi_{n,n+1}(C_0((0,1), E_n)) \subset C_0((0,1), E_{n+1})$.
(4) *Let* $\tilde{\varphi}_{n,n+1} : F_n \to F_{n+1}$ *be the quotient map induced by* $\varphi_{n,n+1}$ *(note from (3), we know that this quotient map exists); then* $(\tilde{\varphi}_{n,n+1})_{*0} = \gamma_{n,n+1} : K_0(F_n) = H_n \to K_0(F_{n+1}) = H_{n+1}$.
(5) *For each* $y \in Sp(C_{n+1})$, $Sp(F_n) \subset Sp(\varphi_{n,n+1}|_y)$.
(6) *For each pair* $j_0 \in \{1, 2, ..., l_{n+1}\}$, $i_0 \in \{1, 2, ..., l_n\}$, *one of the following properties holds:*
    (i) *for each* $t \in (0,1)_{j_0} \subset Sp(I_{n+1}) = \bigcup_{j=1}^{l_{n+1}} (0,1)_j \subset Sp(C_{n+1})$, $Sp(\varphi_{n,n+1}|_t) \cap (0,1)_{i_0}$ *contains* $t \in (0,1)_{i_0} \subset Sp(C_n)$*; or*
    (ii) *for each* $t \in (0,1)_{j_0} \subset Sp(I_{n+1}) = \bigcup_{j=1}^{l_{n+1}} (0,1)_j \subset Sp(C_{n+1})$, $Sp(\varphi_{n,n+1}|_t) \cap (0,1)_{i_0}$ *contains* $1 - t \in (0,1)_{i_0} \subset Sp(C_n)$.
(7) *The map* $\varphi_{n,n+1}$ *is injective.*
(8) *If* $X \subset Sp(C_{n+1})$ *is* δ*-dense, then* $\bigcup_{x \in X} Sp(\varphi_{n,n+1}|_x)$ *is* δ*-dense in* $Sp(C_n)$ *(see* 13.3*).*

Remark 13.16. Let $I_n = C_0((0,1), E_n)$ and $I_{n+1} = C_0((0,1), E_{n+1})$. If $\varphi_{n,n+1} : C_n \to C_{n+1}$ is as described in 13.15, then we have the following map between exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K_0(C_n) \arrow[r] \arrow[d, "\gamma_{n,n+1}|_{G_n}"'] & K_0(C_n/I_n) \arrow[r] \arrow[d, "\gamma_{n,n+1}"'] & K_1(I_n) \arrow[r] \arrow[d, "\tilde{\gamma}_{n,n+1}"'] & 0 \\
0 \arrow[r] & K_0(C_{n+1}) \arrow[r] & K_0(C_{n+1}/I_{n+1}) \arrow[r] & K_1(I_{n+1}) \arrow[r] & 0,
\end{tikzcd}$$

where $K_0(C_n)$ and $K_0(C_{n+1})$ are identified with G_n and G_{n+1}, $K_0(C_n/I_n)\,(= K_0(F_n))$ and $K_0(C_{n+1}/I_{n+1})\,(= K_0(F_{n+1}))$ are identified with H_n and H_{n+1}, $K_1(I_n)$ is identified with H_n/G_n, and $K_1(I_{n+1})$ is identified with H_{n+1}/G_{n+1}. Moreover, $\tilde{\gamma}_{n,n+1}$ is induced by $\gamma_{n,n+1} : H_n \to H_{n+1}$.

Consider the matrix $\mathfrak{c} = (c_{ij})_{p_{n+1} \times p_n}$ with $c_{ij} \in \mathbb{Z} \setminus \{0\}$ which is induced by the map $\gamma_{n,n+1} : H_n\,(= \mathbb{Z}^{p_n}) \to H_{n+1}\,(= \mathbb{Z}^{p_{n+1}})$ and consider the matrix

$\mathfrak{d} = (d_{ij})$ which is induced by the map $\tilde{\gamma}_{n,n+1} : \mathbb{Z}^{l_n} \to \mathbb{Z}^{l_{n+1}}$ (as a map $H_n/G_n \to H_{n+1}/G_{n+1}$). Note that $\pi_n : H_n (= \mathbb{Z}^{p_n}) \to H_n/G_n (= \mathbb{Z}^{l_n})$ is given by $\mathfrak{b}_1 - \mathfrak{b}_0$, a $l_n \times p_n$-matrix with entries in $\mathbb{Z}$. Here, in the situation of 13.13, we assume the first p_n^0 columns of both $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are zero and the last p_n^1 columns are strictly positive. Let $\pi_{n+1} : H_{n+1} (= \mathbb{Z}^{p_{n+1}}) \to H_{n+1}/G_{n+1} (= \mathbb{Z}^{l_{n+1}})$ be the quotient map. Write $G_{n+1} = \mathbb{Z}^{p^0_{n+1}} \oplus G'_{n+1}$, $H_{n+1} = \mathbb{Z}^{p^0_{n+1}} \oplus H'_{n+1}$. Then we can choose both $\mathfrak{b}'_0$ and $\mathfrak{b}'_1$, with the first p^0_{n+1} columns zero and the last $p^1_{n+1} = p_{n+1} - p^0_{n+1}$ columns strictly positive, so that $\pi_{n+1} = \mathfrak{b}'_1 - \mathfrak{b}'_0$ and Condition ($\diamondsuit$) is satisfied, i.e.,

$$\tilde{b}_{0,ji},\ \tilde{b}_{1,ji}\ >\ \sum_{k=1}^{l_n} \big(|d_{jk}| + 2\big) \max(b_{0,ki},\ b_{1,ki})$$

for all $i \in \{1, 2, ..., p_n\}$ and for all $j \in \{1, 2, ..., l_{n+1}\}$, where $\tilde{\mathfrak{b}}_0 = \mathfrak{b}'_0 \cdot \mathfrak{c} = (\tilde{b}_{0,ji})$ and $\tilde{\mathfrak{b}}_1 = \mathfrak{b}'_1 \cdot \mathfrak{c} = (\tilde{b}_{1,ji})$.

Indeed, note that $l_{n+1} > 0$ and $p^1_{n+1} > 0$. So we can make ($\diamondsuit$) hold by only increasing the last p^1_{n+1} columns of the the matrices $\mathfrak{b}'_0$ and $\mathfrak{b}'_1$—that is, the first p^0_{n+1} columns of the matrices are still kept to be zero, since all the entries in $\mathfrak{c}$ are strictly positive. Note that, even though the first p^0_{n+1} columns of $\mathfrak{b}'_0$ and $\mathfrak{b}'_1$ (as $l_{n+1} \times p_{n+1}$ matrices) are zero, all entries of $\tilde{\mathfrak{b}}_0$ and $\tilde{\mathfrak{b}}_1$ (as $l_{n+1} \times p_n$ matrices) have been made strictly positive. Again note that the case of 13.12 is the special case of 13.13 for $p^0_{n+1} = 0$, so one does not need to deal with this case separately.

PROOF OF 13.15. Suppose that $\mathfrak{b}'_0$ and $\mathfrak{b}'_1$ satisfy $\mathfrak{b}'_1 - \mathfrak{b}'_0 = \pi_{n+1}$ and (e 13.6) (and the first p^0_{n+1} columns are zero). Now let E_{n+1}, $\beta'_0, \beta'_1 : F_{n+1} \to E_{n+1}$, and $C_{n+1} = A(F_{n+1}, E_{n+1}, \beta'_0, \beta'_1)$ be as constructed in 13.12. We will define $\varphi_{n,n+1} : C_n \to C_{n+1}$ to satisfy the conditions (2)–(8) of 13.15 (the condition (1) is a property of C_{n+1} which is verified in 13.12).

As usual, let us use F_n^i (or E_n^i) to denote the i-th block of F_n (or E_n).

There exists a unital homomorphism $\tilde{\varphi}_{n,n+1} : F_n \to F_{n+1}$ such that

$$\text{(e 13.9)}\ (\tilde{\varphi}_{n,n+1})_{*0} = \gamma_{n,n+1} : K_0(F_n) = H_n \to K_0(F_{n+1}) = H_{n+1},$$

where $\gamma_{n,n+1}$ is as described in the hypotheses of Lemma 13.15 (see also 13.11 and 13.14). Note that $Sp(C_{n+1}) = \bigsqcup_{j=1}^{l_{n+1}} (0,1)_j \cup Sp(F_{n+1})$ (see §3). Write $Sp(F_{n+1}) = (\theta'_1, \theta'_2, ..., \theta'_{p_{n+1}})$ and $Sp(F_n) = (\theta_1, \theta_2, ..., \theta_{p_n})$. To define

$$\varphi_{n,n+1} : C_n \to C_{n+1},$$

we need to specify each map $\varphi_{n,n+1}|_y = e_y \circ \varphi_{n,n+1}$, i.e., for each $y \in Sp(C_{n+1})$, the composed map

$$\varphi_{n,n+1}|_y : C_n \longrightarrow C_{n+1} \xrightarrow{e_y} C_{n+1}|_y \quad ,$$

with e_y the point evaluation at y.

To actually construct $\varphi_{n,n+1}$, we first construct a homomorphism

$$\psi:\ C([0,1],E_n)\to C([0,1],E_{n+1}).$$

This can be done by defining the map

$$\psi^j:\ C([0,1],E_n)\to C([0,1],E^j_{n+1}), \tag{e 13.10}$$

for each $j=1,2,...,l_{n+1}$, as follows. Let $(f_1,f_2,...,f_{l_n})\in C([0,1],E_n)$. For any $k\in\{1,2,...,l_n\}$, if $d_{jk}>0$, then let

$$F_k(t)=\mathrm{diag}\big(\underbrace{f_k(t),f_k(t),...,f_k(t)}_{d_{jk}}\big)\in M_{d_{jk}\cdot\{n,k\}} \tag{e 13.11}$$

($F_k\in C\left([0,1],M_{d_{jk}\cdot\{n,k\}}\right)$); if $d_{jk}<0$, then let

$$F_k(t)=\mathrm{diag}\big(\underbrace{f_k(1-t),f_k(1-t),...,f_k(1-t)}_{|d_{jk}|}\big)\in M_{|d_{jk}|\cdot\{n,k\}} \tag{e 13.12}$$

($F_k\in C\left([0,1],M_{|d_{jk}|\cdot\{n,k\}}\right)$); and if $d_{jk}=0$, then let

$$F_k(t)=\mathrm{diag}\big(f_k(t),f_k(1-t)\big)\in C\left([0,1],M_{2\cdot\{n,k\}}\right). \tag{e 13.13}$$

With the above notation, define

$$\begin{aligned}&\psi^j(f_1,f_2,...,f_{l_n})(t)\\ &\quad=\ \mathrm{diag}(F_1(t),F_2(t),...,F_{l_n}(t))\in C\Big([0,1],M_{\left(\sum_{k=1}^{l_n}d'_k\cdot\{n,k\}\right)}\Big),\end{aligned} \tag{e 13.14}$$

where

$$d'_k=\begin{cases}|d_{jk}| & \text{if } d_{jk}\neq 0,\\ \\ 2 & \text{if } d_{jk}=0.\end{cases}$$

Note that

$$\begin{aligned}\{n+1,j\} &= \sum_{l=1}^{p_{n+1}} b'_{0,jl}[n+1,l]\\ &= \sum_{l=1}^{p_{n+1}}\sum_{i=1}^{p_n} b'_{0,jl}c_{li}[n,i]=\sum_{i=1}^{p_n}\tilde{b}_{0,ji}[n,i].\end{aligned} \tag{e 13.15}$$

Recall that $\tilde{\mathfrak{b}}_0=\mathfrak{b}'_0\cdot\mathfrak{c}=\big(\tilde{b}_{0,ji}\big)$. From (e 13.6), (e 13.15), $d'_k\le|d_{kj}|+2$, and $\{n,k\}=\sum b_{0,ki}[n,i]$, we deduce

$$\{n+1,j\}=\sum_{i=1}^{p_n}\tilde{b}_{0,ji}[n,i]>\sum_{i=1}^{p_n}\big(\sum_{k=1}^{l_n}\big(|d_{jk}|+2\big)b_{0,ki}\big)[n,i]\ge\sum_{k=1}^{l_n}d'_k\{n,k\}.$$

Hence the C^*-algebra $C\left([0,1], M_{\left(\sum_{k=1}^{l_n} d'_k\cdot\{n,k\}\right)}\right)$ can be regarded as a corner of the C^*-algebra $C([0,1], E^j_{n+1}) = C\left([0,1], M_{\{n+1,j\}}\right)$, and consequently, ψ^j can be regarded as a map from $C([0,1], E_n)$ into $C([0,1], E^j_{n+1})$. Putting all ψ^j together we get a map $\psi:\ C([0,1], E_n) \to C([0,1], E_{n+1})$ defined by $\psi(f) = (\psi^1(f), \psi^2(f), ..., \psi^{l_n}(f))$ for all $f \in C([0,1], E_n)$.

Define $\psi_0, \psi_1:\ C_n \to E_{n+1}$ to be

$$\psi_0(f) = \psi(f)(0) \ \text{ and } \ \psi_1(f) = \psi(f)(1)$$

for any $f \in C_n \subset C([0,1], E_n)$. Since $\psi_0(C_0((0,1), E_n)) = 0$ and $\psi_1(C_0((0,1), E_n)) = 0$, this defines maps $\alpha_0,\ \alpha_1:\ F_n \to E_{n+1}$. Note that for each $j \in \{1, 2, ..., l_{n+1}\}$, the maps $\alpha_0^j,\ \alpha_1^j:\ F_n \to E_{n+1} \to E^j_{n+1}$ have spectra

$$SP(\alpha_0^j) = \left\{\theta_1^{\sim i_1}, \theta_2^{\sim i_2}, ..., \theta_{p_n}^{\sim i_{p_n}}\right\} \ \text{ and } \ SP(\alpha_1^j) = \left\{\theta_1^{\sim i'_1}, \theta_2^{\sim i'_2}, ..., \theta_{p_n}^{\sim i'_{p_n}}\right\},$$

respectively (see 13.1 for the notation used here), where

$$i_l = \sum_{\{k:d_{jk}<0\}} |d_{jk}| b_{1,kl} + \sum_{\{k:d_{jk}>0\}} |d_{jk}| b_{0,kl} + \sum_{\{k:d_{jk}=0\}} (b_{0,kl} + b_{1,kl}),$$

$$\text{and} \qquad i'_l = \sum_{\{k:d_{jk}>0\}} |d_{jk}| b_{1,kl} + \sum_{\{k:d_{jk}<0\}} |d_{jk}| b_{0,kl} + \sum_{\{k:d_{jk}=0\}} (b_{0,kl} + b_{1,kl}).$$

Note that $i'_l - i_l = \sum_{k=1}^{l_n} d_{jk}(b_{1,kl} - b_{0,kl})$, and note that, if $l \le p_n^0$ in the case 13.13, then $b_{0,kl} = b_{1,kl} = 0$, and consequently, $i_l = i'_l = 0$. Put

$$\tilde{\alpha}_0 = \beta'_0 \circ \tilde{\varphi}_{n,n+1}:\ F_n \xrightarrow{\tilde{\varphi}_{n,n+1}} F_{n+1} \xrightarrow{\beta'_0} E_{n+1}$$

and

$$\tilde{\alpha}_1 = \beta'_1 \circ \tilde{\varphi}_{n,n+1}:\ F_n \xrightarrow{\tilde{\varphi}_{n,n+1}} F_{n+1} \xrightarrow{\beta'_1} E_{n+1} \quad .$$

Then, for each $j \in \{1, 2, ..., l_{n+1}\}$, the maps $\tilde{\alpha}_0^j,\ \tilde{\alpha}_1^j:\ F_n \to E_{n+1} \to E^j_{n+1}$ have spectra

$$SP(\tilde{\alpha}_0^j) = \left\{\theta_1^{\sim \bar{i}_1}, \theta_2^{\sim \bar{i}_2}, ..., \theta_{p_n}^{\sim \bar{i}_{p_n}}\right\} \ \text{ and } \ SP(\tilde{\alpha}_1^j) = \left\{\theta_1^{\sim \bar{i}'_1}, \theta_2^{\sim \bar{i}'_2}, ..., \theta_{p_n}^{\sim \bar{i}'_{p_n}}\right\},$$

where

$$\bar{i}_l = \sum_{k=1}^{p_{n+1}} b'_{0,jk} c_{kl} = \tilde{b}_{0,jl} \ \text{ and } \ \bar{i}'_l = \sum_{k=1}^{p_{n+1}} b'_{1,jk} c_{kl} = \tilde{b}_{1,jl}.$$

From (e 13.6), we have that $\bar{i}_l > i_l$ and $\bar{i}'_l > i'_l$. Furthermore, $\bar{i}'_l - \bar{i}_l = \sum_{k=1}^{p_{n+1}} (b'_{1,jk} - b'_{0,jk}) c_{kl}$. Since $(\mathfrak{b}'_1 - \mathfrak{b}'_0)\mathfrak{c} = \mathfrak{d}(\mathfrak{b}_1 - \mathfrak{b}_0)$, we have that $\bar{i}'_l - \bar{i}_l = i'_l - i_l$, and hence

$\bar{i}'_l - i'_l = \bar{i}_l - i_l{:=}r_l > 0$. Note that these numbers are defined for the homomorphisms $\alpha_0^j, \alpha_1^j, \tilde{\alpha}_0^j, \tilde{\alpha}_1^j : F_n \to E_{n+1}^j$. So, strictly speaking, $r_j > 0$ means $r_l^j > 0$.

Define a unital homomorphism $\Phi : C_n \to C([0,1], E_{n+1}) = \bigoplus_{j=1}^{l_{n+1}} C([0,1], E_{n+1}^j)$ by

$$\Phi^j(f, (a_1, a_2, ..., a_{p_n})) = \mathrm{diag}\big(\psi^j(f),\ a_1^{\sim r_1^j}, a_2^{\sim r_2^j}, ..., a_{p_n}^{\sim r_{p_n}^j}\big) \tag{e 13.16}$$

for all $f \in C([0,1], E_n)$ and $(a_1, a_2, ..., a_{p_n}) \in F_n$. Again, define the maps $\Phi_0,\ \Phi_1 : C_n \to E_{n+1}$ by

$$\Phi_0(F) = \Phi(F)(0) \text{ and } \Phi_1(F) = \Phi(F)(1),$$

for $F = (f_1, f_2, ..., f_{l_n}; a_1, a_2, ..., a_{p_n}) \in C_n$. These two maps induce two quotient maps

$$\tilde{\tilde{\alpha}}_0,\ \tilde{\tilde{\alpha}}_1,\ : F_n \to E_{n+1},$$

as $\Phi_0(I_n) = 0$ and $\Phi_1(I_n) = 0$.

From our calculation, for each $j \in \{1, 2, ..., l_{n+1}\}$, the map $\tilde{\tilde{\alpha}}_0^j$ (resp. $\tilde{\tilde{\alpha}}_1^j$) has the same spectrum (with multiplicities) as $\tilde{\alpha}_0^j$ (resp. $\tilde{\alpha}_1^j$) does. That is, $(\tilde{\tilde{\alpha}}_0^j)_{*0} = (\tilde{\alpha}_0^j)_{*0}$ and $(\tilde{\tilde{\alpha}}_1^j)_{*0} = (\tilde{\alpha}_1^j)_{*0}$. There are unitaries $U_0,\ U_1 \in E_{n+1}$ such that $\mathrm{Ad}\, U_0 \circ \tilde{\tilde{\alpha}}_0^j = \tilde{\alpha}_0^j$ and $\mathrm{Ad}\, U_1 \circ \tilde{\tilde{\alpha}}_1^j = \tilde{\alpha}_1^j$. Choose a unitary path $U \in C([0,1], E_{n+1})$ such that $U(0) = U_0$ and $U(1) = U_1$. Finally, set $\varphi_{n,n+1} : C_n \to C([0,1], E_{n+1})$ to be defined as

$$\varphi_{n,n+1} = \mathrm{Ad}\, U \circ \Phi. \tag{e 13.17}$$

From the construction, we have that $\psi(C_0\big((0,1), E_n\big)){\subset}C_0\big((0,1), E_{n+1}\big)$ and consequently, $\Phi(C_0\big((0,1), E_n\big)){\subset}C_0\big((0,1), E_{n+1}\big)$, i.e., $\varphi_{n,n+1}(I_n) \subset I_{n+1}$. So (3) follows.

Since $\mathrm{Ad}\, U(0) \circ \tilde{\tilde{\alpha}}_0^j = \tilde{\alpha}_0^j = \beta'_0 \circ \tilde{\varphi}_{n,n+1}$ and $\mathrm{Ad}\, U(1) \circ \tilde{\tilde{\alpha}}_1^j = \tilde{\alpha}_1^j = \beta'_1 \circ \tilde{\varphi}_{n,n+1}$ we have that $\varphi_{n,n+1}(C_n) \subset C_{n+1}$ and furthermore, the quotient map from $C_n/I_n \to C_{n+1}/I_{n+1}$ induced by $\varphi_{n,n+1}$ is the same as $\tilde{\varphi}_{n,n+1}$ (see definition of $\tilde{\alpha}_0^j$ and $\tilde{\alpha}_1^j$). Condition (4) follows from (e 13.9), and condition (2) follows from condition (4) and the following commutative diagram

$$\begin{tikzcd}
K_0(C_n) = G_n \arrow[r] \arrow[d, "(\varphi_{n,n+1})_{*0}"'] & K_0(C_n/I_n) = H_n \arrow[d, "(\tilde{\varphi}_{n,n+1})_{*0}"'] \\
K_0(C_{n+1}) = G_{n+1} \arrow[r] & K_0(C_{n+1}/I_{n+1}) = H_{n+1}.
\end{tikzcd}$$

If $x \in Sp(F_{n+1}) \subset Sp(C_{n+1})$, then $Sp(F_n) \subset Sp(\varphi_{n,n+1}|_x)\,(= Sp(\tilde{\varphi}_{n,n+1}|_x))$, by the fact that all entries of $\mathfrak{c}$ are strictly positive. If

$$x \in (0,1)_j = Sp(C_0((0,1), E^j_{n+1}),$$

then each θ_i, as the only element in $Sp(F^i_n)\,(\subset Sp(F_n))$, appears $r^j_i > 0$ times in $Sp(\varphi_{n,n+1}|_x)$ (see (e 13.16). Consequently, we also have $Sp(F_n) \subset Sp(\varphi_{n,n+1}|_x)$. Hence condition (5) holds.

To see (6), we will use (e 13.16) and (e 13.14). Note each $F_k(t)$ appears in (e 13.14). Therefore if $d_{jk} > 0$, then, by (e 13.11), the point

$$t \in (0,1)_{n,k} \subset Sp(C([0,1], E^k_n))$$

is in $Sp(\varphi_{n,n+1}|_t)$ for $t \in (0,1)_{n+1,j} \subset Sp(C([0,1], E^j_{n+1}))$; if $d_{jk} < 0$, then, by (e 13.12), the point $1-t \in (0,1)_{n,k} \subset Sp(C([0,1], E^k_n))$ is in the $Sp(\varphi_{n,n+1}|_t)$ for $t \in (0,1)_{n+1,j} \subset Sp(C([0,1], E^j_{n+1}))$; and if $d_{jk} = 0$, then, by (e 13.13), both points t and $1-t$ from $(0,1)_{n,k} \subset Sp(C([0,1], E^k_n))$ are in the $Sp(\varphi_{n,n+1}|_t)$ for $t \in (0,1)_{n+1,j} \subset Sp(C([0,1], E^j_{n+1}))$. Hence (6) follows.

From (5), we know $Sp(F_n) \subset Sp(\varphi_{n,n+1}|_y)$ for any $y \in Sp(C_{n+1})$. From (6) and $l_{n+1} > 0$, we know that $\bigcup_{i=1}^{l_n}(0,1)_{n,i} \subset \bigcup_{t\in(0,1)_{n+1,1}} Sp(\varphi_{n,n+1}|_t)$. Thus $Sp(C_n) \subset Sp(\varphi_{n,n+1})$. By the definition of $Sp(\varphi_{n,n+1})$, this implies $\ker\varphi_{n,n+1} \subset \bigcap_{x\in Sp(C_n)}\{\ker x\} = \{0\}$. It follows that φ is injective. Hence (7) holds. Note that if $X \subset (0,1)_{n+1,1}$ is δ-dense, then the sets $\{x : x \in X\} = X$ and $\{1-x, x \in X\}$ are δ-dense in $(0,1)$. Hence $\bigcup_{t\in X} Sp(\varphi_{n,n+1}|_t)$ is δ-dense in $\bigcup_{i=1}(0,1)_{n,i}$. Since $Sp(C_n) = Sp(F_n) \cup \bigcup_{i=1}(0,1)_{n,i}$, combining with (5), we get (8). This finishes the proof. $\square$

13.17. Let $\varphi:\ C_n \to C_{n+1}$ be as in the proof above. We will calculate the contractive linear map

$$\varphi^\sharp_{n,n+1}:\ \mathrm{Aff}(T(C_n)) \to \mathrm{Aff}(T(C_{n+1}))$$

which also preserves the order (i.e., maps $\mathrm{Aff}(T(C_n))_+$ to $\mathrm{Aff}(T(C_{n+1}))_+$). Recall from 3.8 (see 3.17 also) that $\mathrm{Aff}(T(C_n))$ is the subset of $\bigoplus_{i=1}^{l_n} C([0,1]_i, \mathbb{R}) \oplus \mathbb{R}^{p_n}$ consisting of the elements $(f_1, f_2, ..., f_{l_n};\ h_1, h_2, ..., h_{p_n})$ which satisfy the conditions

$$\begin{aligned} f_i(0) &= \frac{1}{\{n,i\}}\sum b_{0,ij}h_j\cdot[n,j] \text{ and} \\ f_i(1) &= \frac{1}{\{n,i\}}\sum b_{1,ij}h_j\cdot[n,j], \end{aligned} \tag{e 13.18}$$

and $\mathrm{Aff}(T(C_{n+1}))$ is the subset of $\bigoplus_{i=1}^{l_{n+1}} C([0,1]_i, \mathbb{R}) \oplus \mathbb{R}^{p_{n+1}}$ consisting of the elements

$(f_1', f_2', ..., f_{l_{n+1}}'; h_1', h_2', ..., h_{p_{n+1}}')$ which satisfy

$$\begin{aligned} f_i'(0) &= \frac{1}{\{n+1,i\}}\sum b_{0,ij}' h_j' \cdot [n+1,j] \text{ and} \\ f_i'(1) &= \frac{1}{\{n+1,i\}}\sum b_{1,ij}' h_j' \cdot [n+1,j]. \end{aligned} \tag{e 13.19}$$

Recalling that $(c_{ij})_{p_{n+1}\times p_n}$ is the matrix corresponding to $(\tilde{\varphi}_{n,n+1})_{*0} = \gamma_{n,n+1}$ for $\tilde{\varphi}_{n,n+1} : F_n \to F_{n+1}$, and noting that since $\tilde{\varphi}_{n,n+1}$ is unital, one has

$$\sum_{j=1}^{p_n} c_{ij}[n,j] = [n+1,j].$$

Let

$$\varphi^{\sharp}_{n,n+1}(f_1, f_2, ..., f_{l_n}; h_1, h_2, ..., h_{p_n}) = (f_1', f_2', ..., f_{l_{n+1}}'; h_1', h_2', ..., h_{p_{n+1}}').$$

Then

$$h_i' = \frac{1}{[n+1,i]}\sum_{j=1}^{p_n} c_{ij} h_j [n,j].$$

Combining this with (e 13.8), we have

$$\begin{aligned} f_i'(0) &= \frac{1}{\{n+1,i\}}\sum \tilde{b}_{0,il} h_l [n,l] \text{ and} \\ f_i'(1) &= \frac{1}{\{n+1,i\}}\sum \tilde{b}_{1,il} h_l [n,l] \end{aligned} \tag{e 13.20}$$

Also note that

$$\begin{aligned} f_i'(t) = &\frac{1}{\{n+1,i\}}\left\{\sum_{d_{ik}>0} d_{ik} f_k(t)\{n,k\}\right. \\ &+ \sum_{d_{ik}<0} |d_{ik}| f_k(1-t)\{n,k\} + \sum_{d_{ik}=0} (f_k(t)+f_k(1-t))\{n,k\} \\ &\left. + \sum_{l=1}^{p_n} r_l^i h_l [n,l]\right\}, \end{aligned} \tag{e 13.21}$$

where

$$\begin{aligned} r_l^i &= \sum_{k=1}^{p_{n+1}} b_{0,ik}' c_{kl} - \left(\sum_{d_{ik}<0} |d_{ik}| b_{1,kl} + \sum_{d_{ik}>0} |d_{ik}| b_{0,kl} + \sum_{d_{ik}=0} (b_{0,kl}+b_{1,kl})\right) \\ &= \sum_{k=1}^{p_{n+1}} b_{1,ik}' c_{kl} - \left(\sum_{d_{ik}>0} |d_{ik}| b_{1,kl} + \sum_{d_{ik}<0} |d_{ik}| b_{0,kl} + \sum_{d_{ik}=0} (b_{0,kl}+b_{1,kl})\right). \end{aligned}$$

It follows from the last paragraph of 13.12 and from 13.13 that, when we define C_{n+1}, we can always increase the entries of the last $p^1_{n+1} = p_{n+1} - p^0_{n+1}$ columns of the matrices $\mathfrak{b}'_0 = (b'_{0,ik})$ and $\mathfrak{b}'_1 = (b'_{1,ik})$ by adding an arbitrary (but the same for $\mathfrak{b}'_0$ and $\mathfrak{b}'_1$) matrix $(m_{ik})_{l_{n+1}\times p^1_{n+1}}$, with each $m_{ik} > 0$ sufficiently large, to the last p^1_{n+1} columns of the matrices. In particular we can strengthen the requirement $(\diamondsuit)$ (see (e 13.6)) to condition $(\diamondsuit\diamondsuit)$, i.e.,

$$\tilde{b}_{0,il}\left(=\sum_{k=1}^{p_{n+1}} b'_{0,ik}c_{kl}\right),\ \tilde{b}_{1,il}\left(=\sum_{k=1}^{p_{n+1}} b'_{1,ik}c_{kl}\right) > 2^{2n}\left(\sum_{k=1}^{l_n}(|d_{ik}|+2)\{n,k\}\right) \tag{e 13.22}$$

for all $i\in\{1,...,l_{n+1}\}$. This condition and (e 13.22) (and note that $b_{0,kl} \le \{n,k\}$, $b_{1,kl} \le \{n,k\}$ for any $k \le l_n$) imply

$$r^i_l \ge \frac{2^{2n}-1}{2^{2n}}\tilde{b}_{0,il}, \qquad \text{or equivalently}$$

$$0 \le \tilde{b}_{0,il} - r^i_l < \frac{1}{2^{2n}}\tilde{b}_{0,il}. \tag{e 13.23}$$

Recall that $\varphi^\sharp_{n,n+1} : \mathrm{Aff}(T(C_n)) \to \mathrm{Aff}(T(C_{n+1}))$ and $\tilde{\varphi}^\sharp_{n,n+1} : \mathrm{Aff}(T(F_n)) \to \mathrm{Aff}(T(F_{n+1}))$ are the maps induced by the homomorphisms $\varphi_{n,n+1}$ and $\tilde{\varphi}_{n,n+1}$; and the same for $\pi^\sharp_n : \mathrm{Aff}(T(C_n)) \to \mathrm{Aff}(T(F_n))$ and $\pi^\sharp_{n+1} :\ \mathrm{Aff}(T(C_{n+1})) \to \mathrm{Aff}(T(F_{n+1}))$, which are the maps induced by the quotient maps $\pi_n : C_n \to F_n$ and $\pi_{n+1} : C_{n+1} \to F_{n+1}$, respectively. Since $\tilde{\varphi}_{n,n+1}\circ\pi_n = \pi_{n+1}\circ\varphi_{n,n+1}$, we have

$$\pi^\sharp_{n+1}\circ\varphi^\sharp_{n,n+1} = \tilde{\varphi}^\sharp_{n,n+1}\circ\pi^\sharp_n :\ \mathrm{Aff}(T(C_{n+1})) \to \mathrm{Aff}(T(F_{n+1})).$$

13.18. For each n, we will now define a map $\Gamma_n :\ \mathrm{Aff}(T(F_n)) \to \mathrm{Aff}(T(C_n))$ which is a right inverse of $\pi^\sharp_n :\ \mathrm{Aff}(T(C_n)) \to \mathrm{Aff}(T(F_n))$—that is, $\pi^\sharp_n\circ\Gamma_n = \mathrm{id}|_{\mathrm{Aff}(T(F_n))}$.

Recall that $C_n = A(F_n, E_n, \beta_0, \beta_1)$ with unital homomorphisms $\beta_0, \beta_1 : F_n \to E_n$ whose K-theory maps satisfy $(\beta_0)_{*0} = \mathfrak{b}_0 = (b_{0,ij})$ and $(\beta_1)_{*0} = \mathfrak{b}_1 = (b_{1,ij})$. Let $\beta^\sharp_i : \mathrm{Aff}(T(F_n)) \to \mathrm{Aff}(T(E_n))$ be the contractive linear order preserving map induced by the homomorphism β_i, $i = 0, 1$. For each $h \in \mathrm{Aff}(T(F_n))$, consider the function

$$\Gamma'_n(h)(t) = t\cdot\beta^\sharp_1(h) + (1-t)\cdot\beta^\sharp_0(h), \tag{e 13.24}$$

an element of $C([0,1],\mathbb{R}^{l_n})=\bigoplus_{i=1}^{l_n}C([0,1]_i,\mathbb{R})$. Finally, define the map

$$\Gamma_n:\ \ \mathrm{Aff}(T(F_n))=\mathbb{R}^{p_n}\to\mathrm{Aff}(T(C_n))\subset\bigoplus_{j=1}^{l_n}C([0,1]_j,\mathbb{R})\oplus\mathbb{R}^{p_n}\ \ \text{by}$$

$$\Gamma_n(h)=(\Gamma'_n(h),h)\in\bigoplus_{j=1}^{l_n}C([0,1]_j,\mathbb{R})\oplus\mathbb{R}^{p_n}. \tag{e 13.25}$$

Note that $\Gamma_n(h)\in\mathrm{Aff}(T(C_n))$ (see (e 13.18) in 13.17). One verifies that the map Γ_n is a contractive linear and order preserving map from $\mathrm{Aff}(T(F_n))$ to $\mathrm{Aff}(T(C_n))$. Evidently, $\pi_n^\sharp\circ\Gamma_n=\mathrm{id}|_{\mathrm{Aff}(T(F_n))}$.

LEMMA 13.19. *If Condition* ($\diamondsuit\diamondsuit$) *(see* (e 13.22)*) holds, then for any* $f\in\mathrm{Aff}(T(C_n))$ *with* $\|f\|\le 1$, *and* $f':=\varphi^\sharp_{n,n+1}(f)\in\mathrm{Aff}(T(C_{n+1}))$, *we have*

$$\|\Gamma_{n+1}\circ\pi^\sharp_{n+1}(f')-f'\|<\frac{2}{2^{2n}}.$$

PROOF. As in 13.17 (see 3.8 also), one can write

$$\begin{aligned} f&=(f_1,f_2,...,f_{l_n};\,h_1,h_2,...,h_{p_n})\in\mathrm{Aff}(T(C_n))\ \text{ and}\\ f'&=(f'_1,f'_2,...,f'_{l_{n+1}};\,h'_1,h'_2,...,h'_{p_{n+1}})\in\mathrm{Aff}(T(C_{n+1})).\end{aligned} \tag{e 13.26}$$

(Note that the element f satisfies (e 13.18) and (e 13.18); and the element f' satisfies (e 13.19) and (e 13.20).) Also, we have

$$\|f_i\|,\|h_j\|\le\|f\|\le 1\ \text{ for }\ 1\le i\le l_n,1\le j\le p_n,\ \text{ and}$$

$$\|f'_i\|,\|h'_j\|\le\|f'\|\le 1\ \text{ for }\ 1\le i\le l_{n+1},1\le j\le p_{n+1}.$$

Since $\pi^\sharp_{n+1}\circ\Gamma_{n+1}=\mathrm{id}|_{\mathrm{Aff}(T(F_{n+1}))}$, one has

$$\Gamma_{n+1}\circ\pi^\sharp_{n+1}(f'):=g':=(g'_1,g'_2,...,g'_{l_{n+1}};\,h'_1,h'_2,...,h'_{p_{n+1}});$$

that is, f' and g' have the same boundary value $(h'_1,h'_2,...,h'_{p_{n+1}})$.

Note that, from (e 13.19), the evaluation of f' at zero, $(f'_1(0),f'_2(0),...,f'_{l_{n+1}}(0))$, and the evaluation at one, $(f'_1(1),f'_2(1),...,f'_{l_{n+1}}(1))$, are completely determined by $h'_1, h'_2, ..., h'_{p_{n+1}}$. By (e 13.20) and (e 13.22), we also have

$$\text{(e 13.27)} \qquad \begin{aligned} f'_i(t) - f'_i(0) \;=\; & \frac{1}{\{n+1, i\}} \Bigg(\sum_{d_{ik}>0} d_{ik} f_k(t) \{n, k\} \\ & + \sum_{d_{ik}<0} |d_{ik}| f_k(1-t) \{n, k\} \\ & + \sum_{d_{ik}=0} (f_k(t) + f_k(1-t)) \{n, k\} \\ & - \sum_{l=1}^{p_n} (\tilde{b}_{0,il} - r_l^i) h_l [n, l] \Bigg). \end{aligned}$$

From Condition ($\diamondsuit\diamondsuit$) (see (e 13.22)) and $\|f\| \le 1$, one has

$$\text{(e 13.28)} \qquad \begin{aligned} & \Bigg| \sum_{\{k : d_{ik}>0\}} d_{ik} f_k(t) \{n, k\} + \sum_{\{k : d_{ik}<0\}} |d_{ik}| f_k(1-t) \{n, k\} \\ & \quad + \sum_{\{k : d_{ik}=0\}} (f_k(t) + f_k(1-t)) \{n, k\} \Bigg| \\ & \le \sum_{k=1}^{l_n} (|d_{ik}| + 2) \{n, k\} \|f_k\| \qquad \text{(since } |d_{ik}| \le |d_{ik}| + 2,\ 2 \le |d_{ik}| + 2) \\ & \le \frac{1}{2^{2n}} \tilde{b}_{0,il} \qquad \text{(by (e 13.22) and } \|f_k\| \le \|f\| \le 1) \\ & < \frac{1}{2^{2n}} \cdot \{n+1, i\} \qquad \text{(by (e 13.15) and } [n, j] \ge 1). \end{aligned}$$

By (e 13.23), (e 13.15), and $\|h_l\| \le \|f\| \le 1$, one obtains

$$\text{(e 13.29)} \qquad \left| \sum_{l=1}^{p_n} (\tilde{b}_{0,il} - r_l^i) h_l [n, l] \right| \le \frac{1}{2^{2n}} \tilde{b}_{0,il} \|h_l\| [n, l] \le \frac{1}{2^{2n}} \{n+1, i\}.$$

Combining (e 13.27), (e 13.28), and (e 13.29), one has

$$|f'_i(t) - f'_i(0)| < \frac{2}{2^{2n}}.$$

Similarly, one has

$$|f'_i(t) - f'_i(1)| < \frac{2}{2^{2n}}.$$

But, by the definition of Γ_{n+1} (with $n+1$ in place of n) in (e 13.24) and (e 13.25), we have

$$g_i'(t) = tg_i'(1) + (1-t)g_i'(0).$$

Combining this with $g_i'(0) = f_i'(0)$ and $g_i'(1) = f_i'(1)$, we have

$$|g_i'(t) - f_i'(t)| < \frac{2}{2^{2n}} \text{ for all } i,$$

as desired. $\square$

THEOREM 13.20. *Let $((G, G_+, u), K, \Delta, r)$ be a weakly unperforated Elliott invariant as defined in* 13.7 *with G torsion free and $K = 0$. Let (H, H_+, u) be the simple ordered group with $H \supset G$ defined in* 13.9 *Let C_n, $\varphi_{n,n+1} : C_n \to C_{n+1}$, F_n, and $\bar{\varphi}_{n,n+1} : F_n \to F_{n+1}$ be as in Lemma* 13.15*, and let I_n be as in Remark* 13.16*. Then the inductive limit $C = \lim(C_n, \varphi_n)$ has the property $((K_0(C), K_0(C)_+, \mathbf{1}_C), K_1(C)) = ((G, G_+, u), K,)$. In particular, $K_1(C) = \{0\}$. Moreover, $I = \lim_{n\to\infty}(I_n, \varphi_{n,n+1}|_I)$ is an ideal of C such that $C/I = F = \lim_{n\to\infty}(F_n, \bar{\varphi}_{n,n+1})$ with $K_0(F) = H$ and $T(C/I) = \Delta$. (If we further assume that Condition $(\diamondsuit)$ in* (e 13.6) *is replaced by the stronger Condition $(\diamondsuit\diamondsuit)$ in* (e 13.22)*, then we have $T(C) = T(C/I) = \Delta$. We will not use this. However, it follows from the proof of Theorem* 13.42*.)*

PROOF. Recall, from 13.12, that $C_n = C([0,1], E_n) \oplus_{\beta_0,\beta_1} F_n$ with ideal $I_n = C_0((0,1), E_n)$ and quotient $C_n/I_n = F_n$. Hence, we have the following infinite commutative diagram:

$$\begin{tikzcd}
I_1 \arrow[r] \arrow[d] & I_2 \arrow[r] \arrow[d] & I_3 \arrow[r] \arrow[d] & \cdots I \\
C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & \cdots C \\
F_1 \arrow[r] & F_2 \arrow[r] & F_3 \arrow[r] & \cdots C/I\ .
\end{tikzcd}$$

Also from the construction, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
K_0(C_1) = G_1 \arrow[r] \arrow[d, "\iota_1"] & K_0(C_2) = G_2 \arrow[r] \arrow[d, "\iota_2"] & K_0(C_3) = G_3 \arrow[r] \arrow[d, "\iota_3"] & \cdots \\
K_0(F_1) = H_1 \arrow[r] \arrow[d] & K_0(F_2) = H_2 \arrow[r] \arrow[d] & K_0(F_3) = H_3 \arrow[r] \arrow[d] & \cdots \\
K_1(I_1) \arrow[r] \arrow[d] & K_1(I_2) \arrow[r] \arrow[d] & K_1(I_3) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots ,
\end{tikzcd}$$

where the inclusion maps $\iota_n : K_0(C_n) = G_n \to K_0(F_n) = H_n$ are induced by the homomorphisms $\pi_i : C_n \to F_n$. So $K_0(C) = \lim(G_n, \gamma_{n,n+1}|_{G_n}) = G$. Let $\iota : K_0(C) \to K_0(C/I)$ be the homomorphism given by the above diagram. Note that the AF algebra C/I has $K_0(C/I) = \lim(H_n, \gamma_{n,n+1}) = H$ as scaled ordered group. Hence by the part (1) of Remark 13.10, we have $T(C/I) = \Delta$. $\square$

13.21. Let us fix some notation. Recall that $C_n = C([0,1], E_n) \bigoplus_{(\beta_{n,0},\beta_{n,1})} F_n$, where $E_n = \bigoplus_{i=1}^{l_n} E_n^i = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}$, $F_n = \bigoplus_{i=1}^{p_n} F_n^i = \bigoplus_{i=1}^{p_n} M_{[n,i]}$, and $\beta_{n,0}, \beta_{n,1} : F_n \to E_n$ are unital homomorphisms. (Here we use $\beta_{n,0}, \beta_{n,1}$ instead of β_0, β_1 to distinguish the maps for different n.)

In the rest of this section, we will use $t_{n,j}$ to denote the representation

$$C \ni (f_1, f_2, \cdots, f_{l_n};\ a_1, a_2, \cdots, a_{p_n}) \mapsto f_j(t) \in E_n^j = M_{\{n,j\}},$$

and $\theta_{n,i}$ to denote the representation

$$C \ni (f_1, f_2, \cdots, f_{l_n};\ a_1, a_2, \cdots, a_{p_n}) \mapsto a_i \in F_n^i = M_{[n,j]},$$

for $t \in [0,1]$, $j = 1, 2, \cdots l_n$, and $i = 1, 2, \cdots, p_n$.

Note that $\theta_{n,i} \in Sp(C_n)$ and $t_{n,j} \in Sp(C_n)$ for $t \in (0,1)$, but in general, $0_{n,j}, 1_{n,j} \notin Sp(C_n)$; they may not be irreducible. In $RF(C_n)$, in the notation of 13.2, we have

$$0_{n,j} = \{\theta_{n,1}^{\sim b_{0,j1}}, \theta_{n,2}^{\sim b_{0,j2}}, \cdots, \theta_{n,p_n}^{\sim b_{0,jp_n}}\} \text{ and } 1_{n,j} = \{\theta_{n,1}^{\sim b_{1,j1}}, \theta_{n,2}^{\sim b_{1,j2}}, \cdots, \theta_{n,p_n}^{\sim b_{1,jp_n}}\},$$

where $(b_{0,ji})_{l_n \times p_n}$ and $(b_{1,ji})_{l_n \times p_n}$ are the matrices corresponding to $(\beta_{n,0})_*, (\beta_{n,1})_* : \mathbb{Z}^{p_n} \to \mathbb{Z}^{l_n}$. Let us denote the set of all points $t_{n,j}$ for $t \in [0,1]$ by $[0,1]_{n,j}$. Hence $Sp(A_n) = \{\theta_{n,1}, \theta_{n,2}, \cdots, \theta_{p_n}\} \cup \bigcup_{j=1}^{l_n} (0,1)_{n,j}$.

In the previous construction, we only describe how to construct C_{n+1} from C_n for a fixed n. Now we need to let n move. Accordingly, we need to change some notation. For example, $\mathfrak{c} = (c_{ij})$ of 13.11 will be denoted by $\mathfrak{c}_{n,n+1} = (c_{ij}^{n,n+1})$ as it is the matrix of $\gamma_{n,n+1} : \ H_n = \mathbb{Z}^{p_n} \longrightarrow H_{n+1} = \mathbb{Z}^{p_{n+1}}$; $\mathfrak{b}_0 = (b_{0,ji})$ and $\mathfrak{b}_1 = (b_{1,ji})$ of 13.12 (and $\mathfrak{b}'_0 = (b'_{0,ji})$ and $\mathfrak{b}'_1 = (b'_{1,ji})$ of 13.14) will be denoted by $\mathfrak{b}_{n,0} = (b^n_{0,ji})$ and $\mathfrak{b}_{n,1} = (b^n_{1,ji})$ (and $\mathfrak{b}_{n+1,0} = (b^{n+1}_{0,ji})$ and $\mathfrak{b}_{n+1,1} = (b^{n+1}_{1,ji})$, respectively); $\mathfrak{d} = (d_{ij})$ of 13.14 will be denoted by $\mathfrak{d}_{n,n+1} = (d_{ij}^{n,n+1})$ as it is the matrix of $\tilde{\gamma}_{n,n+1} : \ H_n/G_n = \mathbb{Z}^{l_n} \to H_{n+1}/G_{n+1} = \mathbb{Z}^{l_{n+1}}$; $\tilde{\mathfrak{b}}_0 = (\tilde{b}_{0,ji})$ and $\tilde{\mathfrak{b}}_1 = (\tilde{b}_{1,ji})$ will be denoted by $\tilde{\mathfrak{b}}_{(n,n+1),0} = (\tilde{b}^{n,n+1}_{0,ji})$ and $\tilde{\mathfrak{b}}_{(n,n+1),1} = (\tilde{b}^{n,n+1}_{1,ji})$ (hence, the relations $\tilde{\mathfrak{b}}_0 = \mathfrak{b}'_0 \cdot \mathfrak{c}$ and $\mathfrak{b}_1 = \mathfrak{b}'_1 \cdot \mathfrak{c}$ there will be $\tilde{\mathfrak{b}}_{(n,n+1),0} = \mathfrak{b}_{n+1,0} \cdot \mathfrak{c}_{n,n+1}$ and $\tilde{\mathfrak{b}}_{(n,n+1),1} = \mathfrak{b}_{n+1,1} \cdot \mathfrak{c}_{n,n+1}$).

13.22. In this paper, a projection $p \in M_l(C(X))$ is called trivial if there is a unitary $u \in M_l(C(X))$ such that $u^*(x)p(x)u(x) = \mathrm{diag}(1, 1, \cdots, 1, 0, \ldots, 0)$. By part 2 of Remark 3.26 of [32], if X is a finite CW complex with dimension at most three, then the above statement is equivalent to the statement that p is Murray-von Neumann equivalent to $\mathrm{diag}(1, 1, \cdots, 1, 0, \cdots, 0)$. A projection $p \in QM_l(C(X))Q$ is trivial if it is trivial when regarded as a projection in $M_l(C(X))$.

Let X be a connected finite CW complex with dimension at most three with the base point x_0. Then $K_0(C(X)) = K_0(C_0(X \setminus \{x_0\})) \oplus K_0(\mathbb{C}) = K_0(C_0(X \setminus \{x_0\})) \oplus \mathbb{Z}$.

We will use the fact $K_0(C(X))_+ \setminus \{0\} = \{(x, n) \in K_0(C_0(X \setminus \{x_0\})) \oplus \mathbb{Z},\ n > 0\}$. It is clear that $K_0(C(X))_+ \setminus \{0\} \subset \{(x, n) \in K_0(C_0(X \setminus \{x_0\})) \oplus \mathbb{Z},\ n > 0\}$. Let $(x, n) \in \{(x, n) \in K_0(C_0(X \setminus \{x_0\})) \oplus \mathbb{Z},\ n > 0\}$. Choose $p \in M_m(C(X))$ such that $[p] - [1_k] = (x, n)$ for some integers $0 \le k \le m$. Then p has rank $n + k$. Since $n \ge 1 \ge (3 + 1)/2 - 1$, it follows from Theorem 9.12 of [50] that 1_k is unitarily equivalent to a subprojection of p. Therefore there is a projection $q \le p$ such that $[q] = [1_k]$. Then $(x, n) = [p - q] = [p] - [q] \in K_0(A)_+ \setminus \{0\}$. But $[p] - [q] = (x, n)$.

The following theorem is in Section 3 of [32] (see Proposition 3.16 and Theorem 3.10 there).

PROPOSITION 13.23. *Let X and Y be path-connected finite CW complexes of dimension at most three, with base points $x_0 \in X$ and $y_0 \in Y$, such that the cohomology groups $H^3(X)$ and $H^3(Y)$ are finite. Let $\alpha_0 : \ K_0(C(X)) \to K_0(C(Y))$ be a homomorphism such that α_0 is at least 12-large (see the definition in Section 3 of* [32]*) and*

$$\alpha_0(K_0(C(X))_+ \setminus \{0\}) \subset K_0(C(Y))_+ \setminus \{0\}, \tag{e 13.30}$$

and let $\alpha_1 : \ K_1(C(X)) \to K_1(C(Y))$ be any homomorphism. Let $P \in M_\infty(C(X))$ be any non-zero projection and $Q \in M_\infty(C(Y))$ be a projection with $\alpha_0([P]) =$

$[Q]$ (such projections always exist; see the proof below). Then there exists a unital homomorphism $\varphi: PM_\infty(C(X))P \to QM_\infty(C(Y))Q$ such that $\varphi_{*0} = \alpha_0$ and $\varphi_{*1} = \alpha_1$, and such that

$$\varphi(PM_\infty(C_0(X \setminus \{x_0\}))P) \subset QM_\infty(C_0(Y \setminus \{y_0\}))Q.$$

That is, if $f \in PM_\infty(C(X))P$ satisfies $f(x_0) = 0$, then $\varphi(f)(y_0) = 0$. Moreover, if α_0 is at least 13-large and Y is not a single point, then one may further require that φ is injective.

Suppose that $B = \bigoplus_{i=1}^m B_i$ and $D = \bigoplus_{j=1}^n D_j$, where each B_i has the form $P_iM_\infty(C(X_i))P_i$ and each D_j has the form $Q_jM_\infty(C(Y_j))Q_j$, where X_i, Y_j are connected finite CW complexes with at least one Y_i not a single point, and P_i and Q_j are as in the first part of the statement. Fix the base points $x_i \in X_i$ and $y_j \in Y_j$. Let $I_i = \{f \in B_i : f(x_i) = 0\}$ and $J_j = \{g \in D_j : g(x_j) = 0\}$. Let $\alpha_0 : K_0(B) \to K_0(D)$ be an order preserving homomorphism with $\alpha_0([1_B]) = [1_D]$ such that $\alpha_0|_{K_0(B_i)}$ is at least 13-large in each component of $K_0(D_j)$, and let $\alpha_1 : K_1(B) \to K_1(D)$ be any homomorphism. Then there is a unital injective homomorphism $\varphi : B \to D$ such that $(\varphi)_{*i} = \alpha_i$, $i = 0, 1$, and $\varphi(I) \subset J$, where $I = \bigoplus_{i=1}^m I_i$ and $J = \bigoplus_{j=1}^n J_j$.

PROOF. Let us first prove the first part of the lemma. Since $[P] \in K_0(C(X))_+ \setminus \{0\}$, by the condition (e 13.30), we have $\alpha_0([P]) \in K_0(C(Y))_+ \setminus \{0\}$. Hence there is a projection $Q \in M_\infty(C(Y))$ such that $\alpha_0([P]) = [Q]$. Similarly, there is a projection $q \in M_k(C(Y))$ (for some integer $k \geq 1$) such that $\alpha_0([1_{C(X)}]) = [q]$.

This proposition is a special case of Theorem 9.1 of [81]. To see this, first recall $K_0(C(X)) = K_0(C_0(X \setminus \{x_0\})) \oplus \mathbb{Z}$ and $K_0(C(Y)) = K_0(C_0(Y \setminus \{y_0\}) \oplus \mathbb{Z}$. By hypothesis, $\mathrm{rank}\, q \geq 12$. Note also $K_1(C(X)) = K_1(C_0(X \setminus \{x_0\}))$ and $K_1(C(Y)) = K_1(C_0(Y \setminus \{x_0\}))$. By 13.22, $K_0(C(X))_+ \setminus \{0\} = \{(z_1, z_2) \in K_0(C_0(X \setminus \{x_0\})) \oplus \mathbb{Z} : z_2 > 0\}$. One then computes that the condition (e 13.30) implies $\alpha_0(K_0(C_0(X \setminus \{x_0\}))) \subset K_0(C_0(Y \setminus \{y_0\}))$. Let $\kappa \in KK(C_0(X \setminus \{x_0\}), C_0(Y \setminus \{y_0\}))$ be such that $\kappa|_{K_0(C_0(X \setminus \{x_0\})} = (\alpha_0)|_{K_0(C_0(X \setminus \{x_0\}))}$ and $\kappa|_{K_1(C_0(X \setminus \{x_0\}))} = \alpha_1$. By the definition of $F_3K_*(C(X))$ in 3.7 of [32] and by 3.4.4 of [23], we know $F_3K_*(C(X)) = H^3(X)$, for a finite CW complex X of dimension at most three. Note that by our assumption on X and Y, we have $H^3(X) = \mathrm{Tor}(K_1(C(X)))$, and $H^3(Y) = \mathrm{Tor}(K_1(C(Y)))$, and so $\kappa_*(F_3K_*(C(X))) \subset F_3K_*(C(Y))$, since κ_* maps torsion elements to torsion elements.

Since $\mathrm{rank}\, q \geq 12 \geq 3(\mathrm{dim} Y + 1)$, we may write $q = q_0 \oplus e$, where e is zero, or e is a rank one trivial projection when $\mathrm{rank}\, q \geq 13$ (in the case α_0 is at least 13-large), by Theorem 9.1.2 of [50]. If $e \neq 0$, choose a surjective homotopy-trivial continuous map $s : Y \to X$, which induces an injective homomorphism $s^* : C(X) \to C(Y)$. Let $\alpha_0'(g, z) = \alpha_0(g) + z([q] - [e])$ (note that $\alpha_0(0, 1) = [q]$ as $(0, 1) = [1_C(X)] \in K_0(C(X))$) for all $(g, z) \in K_0(C_0(X \setminus \{x_0\}) \oplus \mathbb{Z} = K_0(C(X))$ as in 13.22. For α_0' and α_1, since α_0' is at least 12-large, there is a unital homomorphism $\psi_0 : C(X) \to q_0M_\infty(C(Y))q_0$ such that $(\psi_0)_{*0} = \alpha_0'$ and $(\psi_0)_{*1} = \alpha_1$. In both cases, by Theorem 9.1 of [81], we obtain a unital homomorphism $\psi_0 : C(X) \to q_0M_\infty(C(Y))q_0 \subset C(Y) \otimes \mathcal{K}$ such that $[\psi_0|_{C_0(X \setminus \{x_0\})}] = \kappa$.

In the case $e = 0$, $q = q_0$. Let $\psi = \psi_0$. If $e \neq 0$, define $\psi : C(X) \to qM_k(C(Y))q$ by $\psi(f) = \psi_0(f) \oplus s^*(f)$ for all $f \in C(X)$. Then, in this case, ψ is a unital injective homomorphism and we still have $[\psi|_{C_0(X\setminus\{x_0\})}] = \kappa$.

We still need to define $\varphi : PM_\infty(C(X))P \to QM_\infty(C(X))Q$. Without loss of generality, we may assume that $P \in M_m(C(X))$ for some integer $m \geq 1$. Then Q (with $[Q] = \alpha_0([P])$ and $\mathrm{rank}(Q) \geq 12$) can be regarded as a subprojection of $q \otimes 1_m$, since $[q \otimes 1_m] = \alpha_0([1_{M_m(C(X))}]) \geq \alpha_0([P])$. Hence we may assume that $Q \in M_m(q(M_\infty(C(Y)))q)$. Furthermore, the homomorphism $\psi \otimes \mathrm{id}_m : M_m(C(X)) \to M_m(q(M_\infty(C(Y)))q)$ satisfies $[\psi \otimes \mathrm{id}_m(P)] = \alpha_0([P]) = [Q]$. By 9.15 of [50], there is a unitary $u \in M_m(q(M_\infty(C(Y)))q)$ such that $u^*(\psi \otimes \mathrm{id}_m)(P)u = Q$. Define $\varphi = \mathrm{Ad}\, u \circ (\psi \otimes \mathrm{id}_m)|_{PM_m(C(X))P}$ (see 2.8). One then checks that φ meets all requirements including the injectivity when α_0 is at least 13-large.

The second part follows from the first part by considering each partial map $\alpha_0^{i,j} : K_0(B_i) \to K_0(D_j)$ and $\alpha_1^{i,j} : K_1(B_i) \to K_1(D_j)$ separately. Note that, since at least one of Y_j is not a single point, say Y_{j_0} is not a single point, then, each partial map $\varphi_{i,j_0} : B_i \to D_{j_0}$ can be chosen injective for each i. It follows that φ is injective. ☐

The following lemma is elementary.

LEMMA 13.24. *Let* $0 \to D \to H \to H/D \to 0$ *be a short exact sequence of countable abelian groups with* H/D *torsion free, and let*

$$\begin{tikzcd} H_1' \arrow[r, "\gamma_{1,2}'"] & H_2' \arrow[r, "\gamma_{2,3}'"] & H_3' \arrow[r, "\gamma_{3,4}'"] & \cdots \arrow[r] & H/D \end{tikzcd}$$

be an inductive system with limit H/D *such that each* H_i' *is a finitely generated free abelian group. Then there are an increasing sequence of finitely generated subgroups* $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots \subset D$ *with* $D = \bigcup_{i=1}^\infty D_i$, *and an inductive system*

$$\begin{tikzcd} D_1 \oplus H_1' \arrow[r, "\gamma_{1,2}"] & D_2 \oplus H_2' \arrow[r, "\gamma_{2,3}"] & D_3 \oplus H_3' \arrow[r, "\gamma_{3,4}"] & \cdots \arrow[r] & H \end{tikzcd}$$

with limit H *satisfying the following conditions:*

(i) $\gamma_{n,n+1}(D_n) \subset D_{n+1}$ *and* $\gamma_{n,n+1}|_{D_n}$ *is the inclusion from* D_n *to* D_{n+1}.

(ii) *If* $\pi_{n+1} : D_{n+1} \oplus H_{n+1}' \to H_{n+1}'$ *is the canonical projection, then* $\pi_{n+1} \circ \gamma_{n,n+1}|_{H_n'} = \gamma_{n,n+1}'$.

(Here, we do not assume $\gamma_{n,n+1}'$ *to be injective.)*

PROOF. Since D is countable, one can list all the elements of D as $D = \{e_i\}_{i=1}^\infty$. We will construct the system $(D_n \oplus H_n, \gamma_{n,n+1})$, inductively. Let us assume that we already have $D_n \subset D$ with $\{e_1, e_2, \cdots, e_n\} \subset D_n$ and a map $\gamma_{n,\infty} : D_n \oplus H_n' \to H$ such that $\gamma_{n,\infty}|_{D_n}$ is the inclusion and $\pi \circ \gamma_{n,\infty}|_{H_n'} = \gamma_{n,\infty}'$, where $\pi : H \to H/D$ is the quotient map. To begin the induction process we must start with $n = 0$, and $D_0 = H_0' = \{0\}$. Note that, since $\gamma_{n+1,\infty}'(H_{n+1}')$ is a finitely generated free abelian subgroup of H/D, one has a lifting map $\gamma_{n+1,\infty}: H'_{n+1} \to H$ such that $\pi \circ \gamma_{n+1,\infty} = \gamma'_{n+1,\infty}$. For each $h \in H'_n$, we have $\gamma(h) := \gamma_{n,\infty}(h) - \gamma_{n+1,\infty}(\gamma'_{n,n+1}(h)) \in D$. Let $D_{n+1} \subset D$ be the finitely generated subgroup generated by $D_n \cup \{e_{n+1}\} \cup \gamma(H'_n)$ and extend the map $\gamma_{n+1,\infty}$ on $D_{n+1} \oplus H'_{n+1}$ by defining it to be inclusion on D_{n+1}. And finally let $\gamma_{n,n+1}: D_n \oplus H'_n \to D_{n+1} \oplus H'_{n+1}$ be defined by $\gamma_{n,n+1}(e,h) = (e + \gamma(h), \gamma'_{n,n+1}(h)) \in D_{n+1} \oplus H'_{n+1}$ for each $(e,h) \in D_n \oplus H'_n$. Evidently, $\gamma_{n,\infty} = \gamma_{n+1,\infty} \circ \gamma_{n,n+1}$. □

13.25. We now fix an object $((G, G_+, u), K, \Delta, r)$ as described in 13.7, which is to be shown to be in the range of Elliott invariant. In general, G may have torsion and K may not be zero. Let $G^1 \subset \mathrm{Aff}(\Delta)$ be the dense subgroup with at least three $\mathbb{Q}$-linearly independent elements and let $H = G \oplus G^1$ be as in 13.9. The order unit $u \in G_+$ when regarded as $(u, 0) \in G \oplus G^1 = H$ is also an order unit of H_+. Note that $\mathrm{Tor}(H) = \mathrm{Tor}(G)$. We have the split short exact sequence

$$0 \longrightarrow G \longrightarrow H \longrightarrow H/G\,(= G^1) \longrightarrow 0$$

with $H/\mathrm{Tor}(H)$ a dimension group (see 13.9). Since $H/\mathrm{Tor}(H)$ is a simple dimension group, as in 13.11, we may write $H/\mathrm{Tor}(H)$ as an inductive limit

$$H'_1 \xrightarrow{\gamma'_{1,2}} H'_2 \xrightarrow{\gamma'_{2,3}} H'_3 \xrightarrow{\gamma'_{3,4}} \cdots \longrightarrow H/\mathrm{Tor}(H),$$

where $H'_n = \mathbb{Z}^{p_n}$ with standard positive cone $(H'_n)_+ = (\mathbb{Z}_+)^{p_n}$. Moreover, we may assume that $\gamma'_{n,n+1}$ is 2-large. Applying Lemma 13.24 to the short exact sequence $0 \to \mathrm{Tor}(H) \to H \overset{\pi}{\longrightarrow} H/\mathrm{Tor}(H) \to 0$, with $D = \mathrm{Tor}(H)$, we may write H as an inductive limit of finitely generated abelian groups,

$$H_1 \xrightarrow{\gamma_{1,2}} H_2 \xrightarrow{\gamma_{2,3}} H_3 \xrightarrow{\gamma_{3,4}} \cdots \longrightarrow H,$$

where $H_n = D_n \oplus H'_n = D_n \oplus \mathbb{Z}^{p_n}$, and (i) and (ii) of 13.24 hold. It follows that $\gamma_{n,n+1}(d,h) = (\gamma_{D,n,n+1}(d+h), \gamma'_{n,n+1}(h))$, where $\gamma_{D,n,n+1}: D_n \oplus H'_n \to D_{n+1}$ is a homomorphism. Note that $h \in H_+ \setminus \{0\}$ if and only if $\pi(h) > 0$. We may write $H_n = \bigoplus_{j=1}^{p_n} H_n^j$ with $H_n^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_n)$, and $H_n^i = \mathbb{Z}$ for all $i \geq 2$. Define

$$(H_n)_+ = \big((\mathbb{Z}_+ \setminus \{0\}) \oplus \mathrm{Tor}(H_n) \cup \{(0,0)\}\big) \oplus \mathbb{Z}_+^{p_n - 1}.$$

Since $\gamma'_{n,n+1}$ is positive and 2-large, one checks easily that each $\gamma_{n,n+1}$ is also positive with respect to the positive cones $(H_n)_+$ and $(H_{n+1})_+$. Let $H_{+'} = \bigcup_{n=1}^{\infty} \gamma_{n,\infty}((H_n)_+)$. If $x \in H_+ \setminus \{0\}$, then, as mentioned above, $\pi(x) > 0$. Suppose that $y = (y_D, y_H) \in H_n = D_n \oplus H'_n$ is such that $\gamma_{n,\infty}(y)) = x$. Then $\gamma_{n,m}(y) = (y'_D, \gamma'_{n,m}(y_H))$, where $\gamma'_{n,m} = \gamma'_{m-1,m} \circ \gamma'_{m-2,m-1} \circ \cdots \circ \gamma'_{n,n+1}$ and $m > n$. It follows that, for some large m, $\gamma'_{n,m}(y_H) \in (H'_n)_+ \setminus \{0\}$. Since each $\gamma'_{n,n+1}$

is 2-large, there is $m_1 > m$ such that $\gamma'_{m,m_1}(\gamma_{n,m}(y_{_H})) \in \{(z_1, z_2, ..., z_{p_{m_1}}) \in \mathbb{Z}_+^{p_{m_1}}$ and $z_1 > 0\}$. In other words, $y := \gamma_{n,m_1}(y) \in (H_n)_+\backslash\{0\}$. Since $\gamma_{m_1,\infty}(y) = x$, this implies $x \in H_{+'}$.

Conversely, if $x \in H_{+'} \setminus \{0\}$, then there exist $n > 0$ and $y_1 \in (H_n)_+$ such that $\gamma_{n,\infty}(y_1) = x$. It follows that $y = (z_1, \eta) \oplus (z_2, z_3, ..., z_{p_n})$ with $z_1 \in \mathbb{Z}_+ \setminus \{0\}$, $\eta \in D_n$, and $(z_2, z_3, ..., z_{p_n}) \in \mathbb{Z}_+^{p_n-1}$. By construction, $(z_1, z_2, ..., z_{p_n}) \in (H'_n)_+ \setminus \{0\}$. It follows that $\gamma'_{n,\infty}((z_1, z_2, ..., z_{p_n})) \in H_{+'} \setminus \{0\}$. By (ii) of 13.24, this implies that $\pi(x) > 0$. Therefore $x \in H_+$. This shows that $H_+ = H_{+'}$.

The notation H and $(H_n)_+$ above will be used later.

Let $\gamma'_{n,n+1} : H'_n \to H'_{n+1}$ be determined by the $p_n \times p_{n+1}$ matrix of positive intergers $\mathfrak{c}_{n,n+1}$ which we assume at least 2-large. We now represent $\gamma_{n,n+1}$ by a $p_{n+1} \times p_n$ matrix of homomorphisms $\tilde{\mathfrak{c}}_{n,n+1} = (\tilde{c}_{ij}^{n,n+1})$, where $\tilde{c}_{ij}^{n,n+1} : H_n^j \to H_{n+1}^i$ are described as follows. Note $\tilde{c}_{ij}^{n,n+1} = P_i \circ \gamma_{n,n+1}|_{H_n^j}$, where $P_i : H_{n+1} \to H_{n+1}^i$.

If $i > 1$, j>1, then define $\tilde{c}_{ij}^{n,n+1} = c_{ij}^{n,n+1}$ (recall $H_n^j = \mathbb{Z}$ and $H_n^i = \mathbb{Z}$, and $c_{ij}^{n,n+1}$ maps m to $c_{ij}^{n,n+1}m$).

If $i > 1, j = 1$, define $\tilde{c}_{i1}^{n,n+1} : H_n^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_n) \to H_{n+1}^i = \mathbb{Z}$ by $\tilde{c}_{i1}^{n,n+1}(z, t) = c_{i1}^{n,n+1}z$ for all $(z, t) \in \mathbb{Z} \oplus \mathrm{Tor}(H_n)$, and let us still denote this by $c_{i1}^{n,n+1}$.

If $i = 1$, let $Q_1 : H_{n+1}^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_{n+1}) \to \mathrm{Tor}(H_{n+1}) \subset H_{n+1}^1$ be a projection. Define $T_j^{n,n+1} = Q_1 \circ P_1 \circ \gamma_{n,n+1}|_{H_n^j}$. Viewing $c_{1j}^{n,n+1}$ as a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ as mentioned above (see 13.21), we define $\tilde{c}_{1j}^{n,n+1} = c_{1j}^{n,n+1} + T_j^{n,n+1}$ as follows. If $j \neq 1$, $\tilde{c}_{ij}^{n,n+1}(m) = c_{1j}^{n,n+1}m + T_j^{n,n+1}(m)$ for $m \in H_n^j = \mathbb{Z}$; if $j = 1$, $\tilde{c}_{ij}^{n,n+1}((m, t)) = c_{1j}^{n,n+1}m + T_j^{n,n+1}((m, t))$ for all $(m, t) \in \mathbb{Z} \oplus \mathrm{Tor}(H_n)$.

Since $\gamma_{n,n+1}$ satisfies $\gamma_{n,n+1}(\mathrm{Tor}(H_n)) \subset \mathrm{Tor}(H_{n+1})$, it induces the map

$$\gamma'_{n,n+1} :\ H_n/\mathrm{Tor}(H_n) = \mathbb{Z}^{p_n} \longrightarrow H_{n+1}/\mathrm{Tor}(H_{n+1}) = \mathbb{Z}^{p_{n+1}}.$$

(By (ii) of Lemma 13.24, this map is the same as the map $\gamma'_{n,n+1} : H'_n$ $(= H_n/\mathrm{Tor}(H_n)) \to H'_{n+1}$ $(= H_{n+1}/\mathrm{Tor}(H_{n+1}))$.) Thus, $\gamma'_{n,n+1}$ is given by the matrix $\mathfrak{c}_{n,n+1} = (c_{ij}^{n,n+1})$ with positive integer entries. Put $G'_k = G_k/\mathrm{Tor}(G_k)$ $(k = 1, 2, ...)$. Then $\gamma'_{n,n+1}(G'_n) \subset G'_{n+1}$. Note that, passing to a subsequence, we may always assume $c_{ij}^{n,n+1} > 2$.

13.26. Let $G_n = H_n \cap \gamma_{n,\infty}^{-1}(G)$ with $(G_n)_+ = (H_n)_+ \cap G_n$. Then the order unit
$u_n = (([n, 1], \tau_n), [n, 2], ..., [n, p_n])$ of $(H_n)_+$ is also an order unit for $(G_n)_+$.

Recall that $\mathrm{Tor}(G) = \mathrm{Tor}(H)$, which implies that $\gamma_{n,\infty}(\mathrm{Tor}(H_n)) \subset \mathrm{Tor}(H) \subset G$. Therefore, $\mathrm{Tor}(H_n) \subset H_n \cap \gamma_{n,\infty}^{-1}(G) = G_n$. Since G_n is a subgroup of H, $\mathrm{Tor}(H_n) = \mathrm{Tor}(G_n)$. Furthermore, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
G_1 \arrow[r, "\gamma_{12}|_{G_1}"] \arrow[d] & G_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & G \arrow[d] \\
H_1 \arrow[r, "\gamma_{12}"] \arrow[d] & H_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_1/G_1 \arrow[r, "\tilde{\gamma}_{12}"] \arrow[d] & H_2/G_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & H/G \arrow[d] \\
0 & 0 & & 0
\end{tikzcd}$$

where $\tilde{\gamma}_{n,n+1}$ is induced by $\gamma_{n,n+1}$. Note that the inductive limit of the quotient groups, $H_1/G_1 \to H_2/G_2 \to \cdots \to H/G$, has no obvious order structure. Since $\mathrm{Tor}(H_n) = \mathrm{Tor}(G_n)$, the quotient map $H_n \to H_n/G_n$ induces a surjective map $\pi_n : H_n/\mathrm{Tor}(H_n) = \mathbb{Z}^{p_n} \to H_n/G_n = \mathbb{Z}^{l_n}$, which will be used in the our construction later. We will reserve the notation π_n for this map from $\mathbb{Z}^{p_n}$ to $\mathbb{Z}^{l_n}$. The map $\pi_n : H_n \to H'_n = H_n/\mathrm{Tor}(H_n)$ which appeared in 13.24 (see π_{n+1} there) will be denoted by π_{H_n,H'_n} from now on. Note also $u'_n = \pi_{H_n,H'_n}(u_n) =([n,1],[n,2],...,[n,p_n])$. Denote by $\pi_{G,H/\mathrm{Tor}(H)}$ the composed map from G to H and then to $H/\mathrm{Tor}(H)$.

Also write the group K of 13.25 as an inductive limit,

$$K_1 \xrightarrow{\chi_{12}} K_2 \xrightarrow{\chi_{23}} K_3 \xrightarrow{\chi_{34}} \cdots \longrightarrow K\ ,$$

where each K_n is finitely generated.

13.27. Recall from [32] that the finite CW complex $T_{2,k}$ ($T_{3,k}$, respectively) is defined to be a 2-dimensional connected finite CW complex with $H^2(T_{2,k}) = \mathbb{Z}/k$ and $H^1(T_{2,k}) = 0$ (3-dimensional finite CW complex with $H^3(T_{3,k}) = \mathbb{Z}/k$ and $H^1(T_{3,k}) = 0 = H^2(T_{3,k})$, respectively). (In [32] the spaces are denoted by $T_{II,k}$ and $T_{III,k}$.) For each n, write

$$H_n^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_n) := \mathbb{Z} \oplus (\mathbb{Z}/k_1\mathbb{Z}) \oplus (\mathbb{Z}/k_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/k_i\mathbb{Z}) \quad \text{and}$$
$$K_n := \mathbb{Z}^l \oplus (\mathbb{Z}/m_1\mathbb{Z}) \oplus (\mathbb{Z}/m_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/m_j\mathbb{Z}).$$

Set

$$X'_n = \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{l} \vee T_{2,k_1} \vee T_{2,k_2} \vee \cdots \vee T_{2,k_i} \vee T_{3,m_1} \vee T_{3,m_2} \vee \cdots \vee T_{3,m_j}.$$

Then $K_0(C(X_n')) = H_n^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_n)$ and $K_1(C(X_n')) = K_n$. Let x_n^1 be the base point of X_n' which is the common point of all copies of the spaces S^1, $T_{2,k}$, $T_{3,k}$ appearing above in the wedge $\vee$. By 13.22, there is a projection $P_n \in M_\infty(C(X'_n))$ such that

$$[P_n] = ([n,1], \tau_n) \in K_0(C(X_n^{'})) = \mathbb{Z} \oplus \mathrm{Tor}(K_0(C(X_n^{'}))),$$

where $([n,1], \tau_n)$ is the first component of the unit $u_n \in H_n$.

Suppose that $P_n \in M_N(C(X_n'))$, where N is a large enough integer. Since $\mathrm{rank}(P_n) = [n,1]$, there is a unitary $u \in M_N$ such that

$$uP_n(x_n^1)u^* = \mathrm{diag}(\mathbf{1}_{[n,1]}, \underbrace{0, \cdots, 0}_{N-[n,1]}).$$

Replacing P_n by uP_nu^*, we may assume that $P_n(x_n^1) = \mathbf{1}_{M_{[n,1]}}$, where $M_{[n,1]}$ is identified with the upper left corner of M_N. Define $X_n = [0,1] \vee X_n'$ with $1 \in [0,1]$ identified with the base point $x_n^1 \in X_n'$. Let us label the point $0 \in [0,1]$ by the symbol x_n^0. So $[0,1]$ is identified with $[x_n^0, x_n^1]$—the convex combinations of x_n^0 and x_n^1. Under this identification, we have $X_n = [x_n^0, x_n^1] \vee X_n'$. It is convenient to write the point $(1-t)x_n^0 + tx_n^1 \in [x_n^0, x_n^1]$ as $x_n^0 + t$ for any $t \in [0,1]$. In particular, we have $x_n^1 = x_n^0 + 1$. The projection $P_n \in M_N(C(X_n'))$ can be extended to a projection in $M_N(C(X_n))$, still called P_n, by $P_n(x_n^0 + t) = \mathbf{1}_{M_{[n,1]}}$ for each $t \in (0,1)$. Let us choose x_n^0 as the base point of X_n. The (old) base point of X_n' is $x_n^0 + 1 = x_n^1$.

13.28. Note that the map

$$\pi_n : H_n/\mathrm{Tor}(H_n) \ (= \mathbb{Z}^{p_n}) \longrightarrow H_n/G_n \ (= \mathbb{Z}^{l_n})$$

may be written as $\pi_n = \mathfrak{b}_{n,0} - \mathfrak{b}_{n,1}$, where $\mathfrak{b}_{n,0}, \mathfrak{b}_{n,1} : \mathbb{Z}^{p_n} \to \mathbb{Z}^{l_n}$, are two order preserving homomorphisms. Note also that we may assume that $\mathfrak{b}_{n,0} = (b_{0,ij}^n)$ and $\mathfrak{b}_{n,0} = (b_{1,ij}^n)$ are two $l_n \times p_n$ matrices of non-negative integers which satisfy the conditions of 13.13 and 13.14.

Exactly as in 13.12 (in which we considered the special case of torsion free K_0 and trivial K_1), we can define

$$\{n,i\} := \sum_{j=1}^{p_n} b_{0,ij}^n [n,j] = \sum_{j=1}^{p_n} b_{1,ij}^n [n,j] \ . \tag{e 13.31}$$

As in 13.13, set $F_n = \bigoplus_{i=1}^{p_n} F_n^i = \bigoplus_{i=1}^{p_n} M_{[n,i]}$, $E_n = \bigoplus_{i=1}^{l_n} M_{\{n,j\}}$, and let $\beta_{n,0}, \beta_{n,1} : F_n \to E_n$ be homomorphisms such that $(\beta_{n,0})_{*0} = \mathfrak{b}_{n,0}$ and $(\beta_{n,1})_{*0} = \mathfrak{b}_{n,1}$. Since $P_n(x_n^0)$ has rank $[n,1]$, there is an isomorphism $j_n : F_n^1 = M_{[n,1]} \to (P_nM_N(C(X))P_n)|_{x_n^0}$. Let $\beta_{X_n} : F_n \to (P_nM_N(C(X))P_n)|_{x_n^0}$ be defined by $\beta_{X_n}(a_1, a_2, \cdots, a_{p_n}) = j_n(a_1) \in (P_nM_N(C(X))P_n)|_{x_n^0}$.

Now consider the algebra

$$A_n := \big\{(f,g,a) \in C([0,1],E_n) \oplus P_nM_N(C(X_n))P_n \oplus F_n; \quad f(0) = \beta_{n,0}(a), f(1) = \beta_{n,1}(a), g(x_n^0) = \beta_{X_n}(a)\big\}. \tag{e 13.32}$$

We may also write $A_n = \big(C([0,1],E_n) \oplus P_nM_N(C(X_n))P_n\big) \oplus_{\beta_{n,0},\beta_{n,1},\beta_{X_n}} F_n$. Denote by $\pi^e_{A_n} : A_n \to F_n$ the quotient map.

As in the construction of C_n (see 13.14), once $E_n = \bigoplus_{i=1}^{l_n} M_{\{n,j\}}, F_n = \oplus_{i=1}^{p_n} M_{[n,i]}$, and $P_nM_N(C(X_n))P_n$ with $P(x_n^0) = \mathrm{diag}(\mathbf{1}_{[n,1]},0,...,0) \in M_N$ are fixed, the algebra A_n depends only on $\mathfrak{b}_{n,0}, \mathfrak{b}_{n,1} : K_0(F_n) = \mathbb{Z}^{p_n} \to \mathbb{Z}^{l_n}$ up to isomorphism. Thus, once E_n,F_n, $\mathfrak{b}_{n,0}, \mathfrak{b}_{n,1}$, and $P_nM_N(C(X_n))P_n$ are specified, the construction of the algebra A_n is complete.

Note that by (e 13.31), $\beta_{n,0}$ and $\beta_{n,1}$ are unital homomorphisms and therefore A_n is a unital algebra. Note that this algebra, in general, is *not* the direct sum of a homogeneous algebra and an algebra in $\mathcal{C}_0$. It is easy to verify that $A_n \in \mathcal{D}_1$ in the sense of Definition 4.8, by writing $A_n = PC(X_n,F)P \oplus_\Gamma B$ with $B = F_n$, $X = [0,1] \sqcup X_n$, $Z = \{0,1,x_n^0\} \subset X$, and $F = E_n \oplus M_N$, with P defined by $P = \mathbf{1}_{E_n}$ on $[0,1]$, and $P = P_n$ on X_n, and, finally, $\Gamma : F \to P|_ZC(Z,F)P|_Z$ defined by $\beta_{n,0}$ for the point $0 \in Z \subset X$, $\beta_{n,1}$ for the point $1 \in Z \subset X$, and β_{X_n} for the point $x_n^0 \in Z \subset X$. Later, we will deal with a nicer special case in which A_n is in fact a direct sum of a homogeneous C^*-algebra and a C^*-algebra in $\mathcal{C}_0$.

13.29. Let $J_n = \{(f,g,a) \in A_n : f = 0,\ a = 0\}$ and $I_n = \{(f,g,a) \in A_n : g = 0,\ a = 0\}$. Denote the quotient algebra A_n/J_n by $A_{C,n}$, and A_n/I_n by $A_{X,n}$. Let $\pi_{J,n} : A_n \to A_n/J_n$ and $\pi_{I,n} : A_n \to A_n/I_n$ denote the quotient maps. Set $\bar{I}_n = \pi_{J,n}(I_n)$ and $\bar{J}_n = \pi_{I,n}(J_n)$. Since $I_n \cap J_n = \{0\}$, the map $\pi_{J,n}$ is injective on I_n, whence $\bar{I}_n \cong I_n$. Similarly, $\bar{J}_n \cong J_n$. Note that

$$\begin{aligned} A_{C,n} &= C([0,1],E_n) \oplus_{\beta_{n,0},\beta_{n,1}} F_n \\ &= \big\{(f,a) \in C([0,1],E_n) \oplus F_n;\ f(0) = \beta_{n,0}(a)\ f(1) = \beta_{n,1}(a)\big\}, \end{aligned} \tag{e 13.33}$$

and

$$A_{X,n} = P_nM_N(C(X_n))P_n \oplus_{\beta_{X_n}} F_n. \tag{e 13.34}$$

Denote by $\pi^e_{C,n} : A_{C,n} \to F_n$ and $\pi^e_{X,n} : A_{X,n} \to F_n$ the quotient maps, and let $\lambda_{C,n} : A_{C,n} \to C([0,1],E_n)$ be the homomorphism defined in (e 4.45) (as λ_k with $A_{k-1} = F_n$). Then $\bar{I}_n = C_0((0,1),E_n) = \ker\pi^e_{C,n}$ and $\bar{J}_n = \{g \in P_nM_N(C(X_n))P_n : g(\xi_n^0) = 0\} = \ker\pi^e_{X,n}$. Note that $\pi^e_{A_n} = \pi^e_{C,n} \circ \pi_{J,n} = \pi^e_{X,n} \circ \pi_{I,n}$.

Evidently, we can write

$$\begin{aligned} Sp(I_n) &= \bigcup_{j=1}^{l_n} (0,1)_{n,j}, \quad Sp(J_n) = X_n \setminus \{x_n^0\} \text{ and} \\ Sp(F_n) &= \{\theta_{n,1}, \theta_{n,2}, \cdots, \theta_{n,p_n}\}. \end{aligned}$$

We also have:

$Sp(A_{C,n}) = Sp(I_n) \sqcup Sp(F_n)$, $Sp(A_{X,n}) = Sp(J_n) \sqcup Sp(F_n)$ (disjoint union);

$$\begin{aligned} Sp(A_n) &= Sp(I_n) \sqcup Sp(A_{X,n}) \\ &= Sp(J_n) \sqcup Sp(A_{C,n}) \text{ (disjoint union); and} \end{aligned} \tag{e 13.35}$$

$$\begin{aligned} Sp(A_n \quad) &= Sp(A_{C,n}) \cup Sp(A_{X,n}) \text{ with} \\ & Sp(A_{C,n}) \cap Sp(A_{X,n}) = Sp(F_n). \end{aligned} \tag{e 13.36}$$

We can also see that $Sp(A_{X,n}) = X_n \cup Sp(F_n)$, with $\theta_{n,1} \in Sp(F_n)$ identified with $x_n^0 \in X_n$. Also note that for any $\theta \in Sp(A_n)$, by (e 13.35), we have

$$\theta \in Sp(A_{X,n}) \text{ if and only if } \theta|_{I_n} = 0. \tag{e 13.37}$$

The homomorphism $\varphi_{n,n+1} : A_n \to A_{n+1}$ to be constructed should satisfy the conditions $\varphi_{n,n+1}(I_n) \subset I_{n+1}$ and $\varphi_{n,n+1}(J_n) \subset J_{n+1}$, and therefore should induce two homomorphisms $\psi_{n,n+1} : A_n/I_n = A_{X,n} \to A_{n+1}/I_{n+1} = A_{X,n+1}$ and $\bar{\varphi}_{n,n+1} : A_n/J_n = A_{C,n} \to A_{n+1}/J_{n+1} = A_{C,n+1}$. Conversely, if two homomorphisms $\psi_{n,n+1} : A_{X,n} \to A_{X,n+1}$ (necessarily injective) and $\bar{\varphi}_{n,n+1} : A_{C,n} \to A_{C,n+1}$ satisfy the two conditions

(a) $\psi_{n,n+1}(\bar{J}_n) \subset \bar{J}_{n+1}$ and $\bar{\varphi}_{n,n+1}(\bar{I}_n) \subset \bar{I}_{n+1}$, and

(b) the homomorphism $\psi^q_{n,n+1} : A_{X,n}/\bar{J}_n = F_n \to A_{X,n+1}/\bar{J}_{n+1} = F_{n+1}$ induced by $\psi_{n,n+1}$ and the homomorphism $\varphi^q_{n,n+1} : A_{C,n}/\bar{I}_n = F_n \to A_{C,n+1}/\bar{I}_{n+1} = F_{n+1}$ induced by $\bar{\varphi}_{n,n+1}$ are the same, then, there is a unique (necessarily injective) homomorphism $\varphi_{n,n+1} : A_n \to A_{n+1}$ satisfying

(c) $\varphi_{n,n+1}(I_n) \subset I_{n+1}$, $\varphi_{n,n+1}(J_n) \subset J_{n+1}$, and $\varphi_{n,n+1}$ induces the homomorphisms $\psi_{n,n+1}$ and $\bar{\varphi}_{n,n+1}$.

To see this, define $\varphi_{n,n+1} : A_n \to C([0,1], E_{n+1}) \oplus P_{n+1}M_\infty(C(X_{n+1}))P_{n+1} \oplus F_{n+1}$ by

$$\begin{aligned} \varphi_{n,n+1}(x) = (&\lambda_{C,n+1}(\bar{\varphi}_{n,n+1}(\pi_{J,n}(x))), \\ &\lambda_{X,n+1}(\psi_{n,n+1}(\pi_{I,n}(x))), \psi^q_{n,n+1}(\pi^e_{A_n}(x))), \end{aligned} \tag{e 13.38}$$

for all $x \in A_n$. It is a unital homomorphism. Since $\psi^q_{n,n+1}$ is induced by $\psi_{n,n+1}$, one sees that

$$\begin{aligned} &\lambda_{X,n+1}(\psi_{n,n+1}(\pi_{I,n}(x)))(x^0_{n+1}) \\ &= \beta_{X,n+1}(\psi^q_{n,n+1}(\pi^e_{X,n}(\pi_{I,n}(x))) \\ &= \beta_{X_{n+1}}(\psi^q_{n,n+1}(\pi^e_{A_n}(x))) \end{aligned} \tag{e 13.39}$$

for all $x \in A_n$. On the other hand, since $\varphi^q_{n,n+1}$ is induced by $\bar{\varphi}_{n,n+1}$, also, by (b), one has

$$\begin{aligned} \pi_{t_i}(\lambda_{C,n+1}(\bar{\varphi}_{n,n+1}(\pi_{J,n}(x)))) &= \beta_{n+1,i}(\varphi^q_{n,n+1}(\pi^e_{C,n}(\pi_{J,n}(x)))) \\ &= \beta_{n+1,i}(\varphi^q_{n,n+1}(\pi^e_{A_n}(x))) = \beta_{n+1,i}(\psi^q_{n,n+1}(\pi^e_{A_n}(x))),\ i = 0, 1, \end{aligned} \tag{e 13.40}$$

for all $x \in A_n$, where $\pi_{t_i} : C([0,1], E_n) \to E_n$ is the point evaluation at t_i, where $t_0 = 0$ and $t_1 = 1$. By (e 13.39) and (e 13.40), one has $\varphi_{n,n+1}(x) \in A_{n+1}$ for all $x \in A_n$. Thus, $\varphi_{n,n+1}$ is a homomorphism from A_n into A_{n+1}. By definition, $\varphi_{n,n+1}$ induces $\bar{\varphi}_{n,n+1}$ as well as $\psi_{n,n+1}$. There is only one such map. Note also, if both $\psi_{n,n+1}$ and $\bar{\varphi}_{n,n+1}$ are injective, then $\varphi_{n,n+1}$ defined in (e 13.38) is also injective, as $J_n \cap I_n = \{0\}$.

Note that $A_{C,n}$ is the same as C_n in 13.12–13.20, with F_n and $\bar{I}_n = \pi_{J,n}(I_n)$ in place of F_n and I_n in 13.12–13.20. Therefore the construction of $\bar{\varphi}_{n,n+1}$ can be carried out as in 13.12–13.20, with the map $F_n \to F_{n+1}$ being given by the matrix $\mathfrak{c}_{n,n+1}$ as in 13.25 above—of course, we need to assume that the corresponding maps $\mathfrak{b}_{n,0}$ and $\mathfrak{b}_{n,1}$ in this case (see 13.28) satisfy ($\diamondsuit\diamondsuit$) (in place of $\mathfrak{b}_0$ and $\mathfrak{b}_1$). So, in what follows, we will focus on the construction of $\psi_{n,n+1}$. But before the construction, let us introduce the following notation.

Note that $A_{X,n} = (P_n M_\infty(C(X_n))P_n \oplus_{(\beta_{X_n})|_{F^1_n}} F^1_n) \oplus \bigoplus_{i\geq 2} F^i_n$. Since $F^1_n = M_{[n,1]}$ and P_n has rank $[n,1]$ (see 13.28), one verifies that

$$P_n M_\infty(C(X_n))P_n \oplus_{(\beta_{X_n})|_{F^1_n}} F^1_n = P_n M_\infty(C(X_n))P_n.$$

Hence $A_{X,n} = P_n M_\infty(C(X_n))P_n \oplus \bigoplus_{i=2}^{p_n} F^i_n$. Let us emphasize that we will write $A_{X,n} = \bigoplus_{i=1}^{p_n} A^i_{X,n}$ with $A^1_{X,n} = P_n M_\infty(C(X_n))P_n$ and $A^i_{X,n} = F^i_n$ for $i \geq 2$. Note that $\bar{J}_n \subset A^1_{X,n}$. Note that we may also write

$$A_n = C([0,1], E_n) \oplus_{\beta_{n,0}\circ\pi_{X,n},\beta_{n,1}\circ\pi_{X,n}} A_{X,n} \tag{e 13.41}$$

$$\begin{aligned} &= \{(f, g) : f \in C([0,1], E_n),\ g \in A_{X,n}, \\ &\quad f(0) = \beta_{n,0}(\pi_{X,n}(g)) \text{ and } f(1) = \beta_{n,1}(\pi_{X,n}(g))\}. \end{aligned} \tag{e 13.42}$$

Note that from $\varphi_{n,n+1}(I_n) \subset I_{n+1}$ for each n, we know that for any $m > n$, $\varphi_{n,m}(I_n) \subset I_m$. By (e 13.37), for any $m > n$ and $y \in Sp(A_{X,m})$, we have

$$Sp(\varphi_{n,m}|_y) \subset Sp(A_{X,n}) \subset Sp(A_n). \tag{e 13.43}$$

LEMMA 13.30. *Let* $(H_n, (H_n)_+, u_n)$, $G_n \subset H_n$, $(G_n)_+ = (H_n)_+ \cap G_n$, *and* $\pi_n : H_n/\mathrm{Tor}(H_n) = \mathbb{Z}^{p_n} \to H_n/G_n = \mathbb{Z}^{l_n}$ *be as in* 13.26. *Let* $\beta_{n,0}, \beta_{n,1} : H_n/\mathrm{Tor}(H_n) = \mathbb{Z}^{p_n} \to \mathbb{Z}^{l_n}$ *satisfy* $\beta_{n,1} - \beta_{n,0} = \pi_n$ *as in* 13.28. *Let* $F_n = \bigoplus_{i=1}^{p_n} M_{[n,i]}, E_n = \bigoplus_{j=1}^{l_n} M_{\{n,j\}}, A^1_{X,n} = P_n M_N(C(X_n))P_n$, $A_{X,n} = A^1_{X,n} \oplus_{i=2}^{p_n} M_{[n,i]}$, *and* $A_n = C([0,1], E_n) \oplus_{\beta_{n,0}\circ\pi_{X,n},\beta_{n,1}\circ\pi_{X,n}} A_{X,n}$ *be as in* (e 13.41).

(1) Suppose that

$$\begin{aligned}&(K_0(F_n), K_0(F_n)_+, [1_{F_n}])\\ &\quad= \quad (H_n/\mathrm{Tor}(H_n), (H_n/\mathrm{Tor}(H_n))_+, ([n,1],[n,2],\cdots[n,p_n]))\end{aligned}$$

(recall $H_n/\mathrm{Tor}(H_n) = \mathbb{Z}^{p_n}$*) and*

$$\begin{aligned}&(K_0(A^1_{X,n}), K_0(A^1_{X,n})_+, [1_{A^1_{X,n}}], K_1(A^1_{X,n}))\\ &\quad= \quad (\mathbb{Z}\oplus\mathrm{Tor}(H_n), (\mathbb{Z}_+\setminus\{0\})\oplus\mathrm{Tor}(H_n)\cup\{(0,0)\}, ([n,1],\tau_n), K_n).\end{aligned}$$

Then

$$(K_0(A_n), K_0(A_n)_+, [1_{A_n}], K_1(A_n)) = (G_n, (G_n)_+, u_n, K_n),$$

$K_0(\bar{J}_n) = \mathrm{Tor}(G_n) = \mathrm{Tor}(H_n)$*, and* $K_1(\bar{J}_n) = K_n$*.*

(2) Suppose that $\varphi_{n,n+1} : A_{C,n} \to A_{C,n+1}$ *and* $\psi_{n,n+1} : A_{X,n} \to A_{X,n+1}$ *satisfy the conditions (a) and (b) of* 13.29. *If* ${\psi_{n,n+1}}_{*0} = \gamma_{n,n+1} : K_0(A_{X,n}) = H_n \to K_0(A_{X,n+1}) = H_{n+1}$ *and* ${\psi_{n,n+1}}_{*1} = \chi_{n,n+1} : K_1(A_{X,n}) = H_n \to K_1(A_{X,n+1}) = H_{n+1}$*, then* ${\varphi_{n,n+1}}_{*0} = \gamma_{n,n+1}|_{G_n} : K_0(A_n) = G_n \to K_0(A_{n+1}) = G_{n+1}$ *and* ${\varphi_{n,n+1}}_{*1} = \chi_{n,n+1} : K_1(A_n) = K_n \to K_1(A_{n+1}) = K_{n+1}$*.*

PROOF. Part (1): As $\bar{J}_n$ is an ideal of $A^1_{X,n}$ with quotient $A^1_{X,n}|_{x^0_n} \cong M_{[n,1]}$, evidently, $K_0(\bar{J}_n) = \mathrm{Tor}(G_n) = \mathrm{Tor}(H_n)$, and $K_1(\bar{J}_n) = K_n$. Since $A_{X,n} = A^1_{X,n} \oplus \bigoplus_{i=2}^{p_n} F^i_n$, and this differs from F_n by replacing F^1_n by $A^1_{X,n}$, from the description of $K_0(A^1_{X,n})_+ = K_0(C(X_n))_+$ in 13.22 and the choice of X_n in 13.27, we have $(K_0(A_{X,n}), K_0(A_{X,n})_+, [1_{A_{X,n}}], K_1(A_{X,n})) = (H_n, (H_n)_+, u_n, K_n)$. On considering the six term exact sequence for the K-theory of the short exact sequence

$$0 \to I_n \to A_n \to A_{X,n} \to 0,$$

the proof of part (1) follows the lines of the proof of Proposition 3.5, with F_n replaced by $A_{X,n}$. (Note that $I_n \cong C_0((0,1), E_n)$, and the boundary map $K_0(A_{X,n}) \to K_1(I_n) = K_0(E_n)$ is given by

$$\begin{aligned}(\beta_{n,1}-\beta_{n,0})\circ\pi = \pi_n\circ\pi : \; & K_0(A_{X,n}) = H_n \xrightarrow{\pi} K_0(F_n)\\ = \quad & H_n/\mathrm{Tor}(H_n) = \mathbb{Z}^{p_n} \xrightarrow{\pi_n} K_0(E_n) = \mathbb{Z}^{l_n},\end{aligned}$$

and this map (playing the role of the map $\varphi_{1*0} - \varphi_{0*0}$ in 3.5) is surjective, since both π_n and π are surjective.) In particular, $(\pi_{I,n})_{*0} : K_0(A_n) = G_n \to K_0(A_{X,n}) = H_n$ is the inclusion map, and $(\pi_{I,n})_{*1} : K_1(A_n) = K_n \to K_1(A_{X,n}) = K_n$ is the identity map.

For part (2), we know that, in the commutative diagram

$$\begin{tikzcd} K_{*i}(A_n) \arrow[r, "(\pi_{I,n})_{*i}"] \arrow[d, "(\varphi_{n,n+1})_{*i}"'] & K_{*i}(A_{X,n}) \arrow[d, "(\psi_{n,n+1})_{*i}"'] \\ K_{*i}(A_{n+1}) \arrow[r, "(\pi_{I,n+1})_{*}"] & K_{*i}(A_{X,n+1}) \end{tikzcd}$$

$(i=0,1)$, the horizontal maps are injective. Thus, $(\varphi_{n,n+1})_*$ is uniquely determined by $(\psi_{n,n+1})_*$. $\square$

13.31. Recall that $(G, G_+, u, K, \Delta, r)$ is fixed, as in 13.25 and 13.26. The construction of A_{n+1} and $\varphi_{n,n+1}$ will be done by induction. Suppose that we already have the first part of the inductive sequence:

$$A_1 \xrightarrow{\varphi_{1,2}} A_2 \xrightarrow{\varphi_{2,3}} A_3 \xrightarrow{\varphi_{3,4}} \cdots \xrightarrow{\varphi_{n-1,n}} A_n,$$

satisfying the following four conditions: for each $m = 1, 2, \cdots n-1$,

(a) For subsequences G_{k_n} and H_{k_n} (of G_n and H_n), we have

$$(K_0(A_{X,n}), K_0(A_{X,n})_+, [\mathbf{1}_{A_{X,n}}], K_1(A_{X,n})) = (H_{k_n}, (H_{k_n})_+, u_{k_n}, K_{k_n}),$$

$$(K_0(F_n), K_0(F_n)_+, [\mathbf{1}_{F_n}]) = (H_{k_n}/\mathrm{Tor}(H_{k_n}), (H_{k_n}/\mathrm{Tor}(H_{k_n}))_+, \pi_{H_{k_n}, H'_{k_n}}(u_{k_n}))$$

and

$$(K_0(A_n), K_0(A_n)_+, [\mathbf{1}_{A_n}], K_1(A_n)) = (G_{k_n}, (G_{k_n})_+, u_{k_n}, K_{k_n}),$$

where $\pi_{H_{k_n}, H'_{k_n}}$ is the projection from H_{k_n} to $H'_{k_n} = H_{k_n}/\mathrm{Tor}(H_{k_n})$. (Without loss of generality, we may relabel k_n by n, and then $\gamma_{k_n,k_{n+1}} := \gamma_{k_{n+1}-1,k_{n+1}} \circ \gamma_{k_{n+1}-2,k_{n+1}-1} \circ \cdots \gamma_{k_n,k_n+1}$ and $\chi_{k_n,k_{n+1}} := \chi_{k_{n+1}-1,k_{n+1}} \circ \chi_{k_{n+1}-2,k_{n+1}-1} \circ \cdots \chi_{k_n,k_n+1}$ will become $\gamma_{n,n+1}$ and $\chi_{n,n+1}$, respectively.) Furthermore, $\varphi_{m,m+1}(I_m) \subset I_{m+1}$ and $\varphi_{m,m+1}(J_m) \subset J_{m+1}$, and therefore the map $\varphi_{m,m+1}$ induces three homomorphisms $\psi_{m,m+1} : A_{X,m}\ (= A_m/I_m) \to A_{X,m+1}$, $\bar{\varphi}_{m,m+1} : A_{C,m}\ (= A_m/J_m) \to A_{C,m+1}$, and $\psi^q_{m,m+1} = \varphi^q_{m,m+1} : F_m \to F_{m+1}$, where F_m arises as a quotient algebra in three ways: $F_m = A_m/(I_m \oplus J_m) = A_{X,m}/\bar{J}_m = A_{C,m}/\bar{I}_m$ (see 13.29);

(b) all the homomorphisms $\varphi_{m,m+1}$, $\psi_{m,m+1}$, and $\bar{\varphi}_{m,m+1}$ are injective; in particular, $\psi_{m,m+1}|_{\bar{J}_m} : \bar{J}_m \to \bar{J}_{m+1}$ is injective, and $(\varphi_{m,m+1})_{*0} = \gamma_{m,m+1}|_{G_n}$, $(\varphi_{m,m+1})_{*1} = \chi_{m,m+1}$, $(\psi_{m,m+1})_{*0} = \gamma_{m,m+1}$, and $(\psi_{m,m+1})_{*1} = \chi_{m,m+1}$;

(c) the induced map $\bar{\varphi}_{m,m+1} : A_{C,m} \to A_{C,m+1}$ satisfies the conditions (1)–(8) of 13.15 with m in place of n, with $A_{C,m}$ in place of C_n (and of course naturally with $A_{C,m+1}$ in place of C_{n+1}), with $G_m/\mathrm{Tor}(G_m)$ (or $G_{m+1}/\mathrm{Tor}(G_{m+1})$) and $H_m/\mathrm{Tor}(H_m)$ (or $H_{m+1}/\mathrm{Tor}(H_{m+1})$, all from 13.25, in place of G_n (or G_{n+1}) and H_n (or H_{n+1}), and with

$$\begin{aligned}(\varphi^q_{m,m+1})_{*0} = \gamma'_{m,m+1} &= (c^{m,m+1}_{i,j}) :\ H_m/\mathrm{Tor}(H_m)\\ &= \mathbb{Z}^{p_m} \longrightarrow H_{m+1}/\mathrm{Tor}(H_{m+1}) = \mathbb{Z}^{p_{m+1}}\end{aligned}$$

(satisfying $\gamma'_{m,m+1}(G_m/\mathrm{Tor}(G_m)) \subset G_{m+1}/\mathrm{Tor}(G_{m+1})$) from 13.25 in place of $\gamma_{n,n+1} :\ H_n \to H_{n+1}$ (satisfying $\gamma_{n,n+1}(G_n) \subset G_{n+1}$), respectively; moreover,

$$(\Diamond\Diamond)_1 \qquad c^{m,m+1}_{ij} > 13{\cdot}2^{2m}{\cdot}(M_m+1)L_m \qquad \text{for all}\ \ i,j, \qquad ,$$

where $M_m = \max\{b^m_{0,ij}: \ i=1,2,...,p_m, \ j=1,2,...,l_m\}$ and L_m is specified below,

(d) the matrices $\mathfrak{b}_{m+1,0}$ and $\mathfrak{b}_{m+1,1}$ for each A_{m+1} satisfy the condition

$$\begin{aligned}(\Diamond\Diamond) \quad \tilde{b}_{0,ji} \quad &:= \quad \sum_{k=1}^{p_{m+1}} b^{m+1}_{0,ik} \cdot c^{m,m+1}_{kl}, \ \tilde{b}_{1,ji} \\ &:= \quad \sum_{k=1}^{p_{m+1}} b^{m+1}_{1,ik} \cdot c^{m,m+1}_{kl} > 2^{2n}\left(\sum_{k=1}^{l_n}(|d^{n,n+1}_{jk}|+2)\{n,k\}\right),\end{aligned}$$

where $\tilde{\gamma}_{n,n+1} = (d^{n,n+1}_{ij}): H_n/G_n \to H_{n+1}/G_{n+1}$.

The number L_m above which was to be chosen after the m-th step will now be specified:

Choose a finite set $Y_m \subset X_m \setminus \{x^0_m\}$ (where x^0_m is the base point of X_m) such that for each $i<m$, $\bigcup_{y\in Y_m} Sp(\varphi_{i,m}|_y)$ is $\frac{1}{m}$-dense in X_i. This can be done since the corresponding map $\psi_{i,i+1}|_{\bar{J}_i}: \ \bar{J}_i(\subset A^1_{X,i}) \to \bar{J}_{i+1}(\subset A^1_{X,i+1})$ is injective for each $i<m$, by the induction assumption (see (b) above). Recall from 13.21 that we denote $t\in(0,1)_j \subset Sp\big(C([0,1],E^j_m)\big)$ by $t_{m,j}$ to distinguish the spectra of different direct summands of $C([0,1],E_m)$ and different n. Let $T_m \subset Sp(A_m)$ be defined by

$$T_m = \left\{(\frac{k}{m})_{m,j}; \ j=1,2,...,l_m; \ k=1,2,...,m-1\right\}.$$

Let $Y_m = \{y_1, y_2, ..., y_{L_{m,Y}}\} \subset X_m$ and let $L_m = l_m\cdot(m-1) + L_{m,Y} = \#(T_m \cup Y_m)$.

We will construct A_{n+1} now and later the homomorphism $\varphi_{n,n+1}$. As part of the induction assumption, suppose that the algebra $A_n = \big(C([0,1],E_n) \oplus P_nM_\infty(C(X_n))P_n\big) \oplus_{\beta_{n,0},\beta_{n,1},\beta_{X_n}} F_n$ is already constructed, with $(\beta_{n,0})_{*0} = \mathfrak{b}_{n,0} = (b^n_{0,ij})$ and $(\beta_{n,1})_{*0} = \mathfrak{b}_{n,1} = (b^n_{1,ij})$. So M_n and L_n can be chosen as described above.

As at the end of 13.25, choose $m' > n$ such that $\gamma'_{n,m}$ has multiplicity at least $13\cdot 2^{2m}\cdot(M_n+1)L_n$. Then, by renaming $H_{m'}$ as H_{n+1}, without loss of generality, we may assume that

$$c^{n,n+1}_{ij} > 13\cdot 2^{2n}\cdot(M_n+1)L_n \quad \text{for all} \quad i,j,$$

in order to satisfy $(\Diamond\Diamond)_1$ in condition (c).

Note that we still have 13.25 and 13.26 (passing to a subsequence). Since the scaled ordered group $(G_{n+1}, u_{n+1}) \subset (H_{n+1}, u_{n+1})$, with $u_{n+1} = \big(([n+1,1],\tau_{n+1}),[n+1,2],...,[n+1,p_n]\big)$ and $\mathrm{Tor}(G_{n+1}) = \mathrm{Tor}(H_{n+1})$, and the group K_{n+1} are now chosen, the algebra $F_{n+1} = \bigoplus_{i=1}^{p_{n+1}} M_{[n+1,i]}$ (with $K_0(F_{n+1}) = H_{n+1}/\mathrm{Tor}(H_{n+1})$ as scaled ordered group) can now be defined. Furthermore, the space X_{n+1} (with base point x^0_{n+1} and $K_0(X_{n+1}) = \mathbb{Z}\oplus\mathrm{Tor}(G_{n+1})$, $K_1(X_{n+1}) = K_{n+1}$) and the projection P_{n+1} (with $[P_{n+1}] = ([n{+}1, 1], \tau_{n+1}) \in \mathbb{Z}\oplus\mathrm{Tor}(G_{n+1})$), and the identification $\beta_{X_{n+1}}$ of F^1_{n+1} with $\big(P_{n+1}M_\infty(C(X_{n+1}))P_{n+1}\big)|_{x^0_{n+1}}$ can now be defined as in 13.27 and 13.28.

Let us modify $b_{n+1,0}$ and $b_{n+1,1}$ given by 13.28. Let

$$\Lambda_n = 2^{2n}\left(\sum_{k=1}^{l_n}(|d_{jk}|+2)\{n,k\}\right) \tag{e 13.44}$$

and let $k_0 = \max\{p^0_{n+1}+1, 3\}$. We replace b^{n+1}_{1,ik_0} and b^{n+1}_{1,ik_0} by $b^{n+1}_{0,ik_0}+\Lambda_n$, and by $b^{n+1}_{1,j,k_0}+\Lambda_n$, $i = 1, 2, ..., l_{n+1}$. Then, since $c^{n,n+1}_{ij}$ is at least 26, one easily sees that, with the new b^{n+1}_{0,ik_0}, $(\diamondsuit\diamondsuit)$ holds for $n+1$. Moreover, the first p^0_{n+1} columns of both $\mathfrak{b}_{n+1,0}$ and $\mathfrak{b}_{n+1,1}$ are still zero, and the last $p_{n+1}-p^0_{n+1}$ columns of both $\mathfrak{b}_{n+1,0}$ and $\mathfrak{b}_{n+1,1}$ are strictly positive (see 13.13).

With the choice of m' (as $n+1$) above, and the choice of $\mathfrak{b}_{n+1,0}$ and $\mathfrak{b}_{n+1,1}$ above, one defines A_{n+1} as in (e 13.32) which satisfies Conditions $(\diamondsuit\diamondsuit)_1$ and $(\diamondsuit\diamondsuit)$. Note that as A_{n+1} is constructed, we also obtain $A_{C,n+1}$.

Next, we will begin the construction of $\varphi_{n,n+1}$ by constructing $\bar\varphi_{n,n+1}$ and $\psi_{n,n+1}$ first. Note that we already have the map $\gamma_{n,n+1} = \tilde{\mathfrak{c}}_{n,n+1} = (\tilde c^{n,n+1}_{ij}) : H_n \to H_{n+1}$ with $\gamma_{n,n+1}(G_n) \subset G_{n+1}$ from 13.25. Recall that $H'_k = H_k/\mathrm{Tor}(G_k)$ in 13.25, $k = 1, 2, ...$ Denote by u'_k the image of u_k in H'_k. Therefore, $\gamma'_{n,n+1} : H'_n \to H'_{n+1}$ is also defined (with new subscripts). Note also that the algebras $A_{C,n}$ and $A_{C,n+1}$ have the same property as C_n and C_{n+1} of Lemma 13.15 with $(H_n, (H_n)_+, u_n)$ and $(H_{n+1}, (H_{n+1})_+, u_{n+1})$ replaced by $(H'_n, (H'_n)_+, u'_n)$ and $(H'_{n+1}, (H'_{n+1})_+, u'_{n+1})$, respectively. Moreover, $\gamma'_{n,n+1}(G'_n) \subset G'_{n+1}$. To apply 13.15, we also replace $\gamma_{n,n+1}$ by $\gamma'_{n,n+1}$, and G_n and G_{n+1} by G'_n and G'_{n+1}. Then, by Lemma 13.15, there is an injective homomorphism $\bar\varphi_{n,n+1}\colon A_{C,n} \to A_{C,n+1}$ (in place of the homomorphism $\varphi_{n,n+1}$) which satisfies (1)–(8) of Lemma 13.15. In particular, the homomorphism $\bar\varphi_{n,n+1}$ satisfies

(a'): $\bar\varphi_{n,n+1}(\bar I_n) \subset \bar I_{n+1}$ (this is (3) of Lemma 13.15) and

(b') the K-theory map $(\varphi^q_{n,n+1})_{*,0} : K_0(F_n) = H'_n \to K_0(F_{n+1}) = H'_{n+1}$, induced by the quotient map $\varphi^q_{n,n+1} : F_n \to F_{n+1}$, is the same as $\gamma'_{n,n+1} = \mathfrak{c}_{n,n+1} : K_0(F_n) = H'_n = \mathbb{Z}^{p_n} \to K_0(F_{n+1}) = H'_{n+1} = \mathbb{Z}^{p_{n+1}}$ (this is (4) of Lemma 13.15).

13.32. Recall that $(K_0(F_n), 1_{F_n}) = (H_n, u_n)$, $(K_0(F_{n+1}), [1_{F_{n+1}}]) = (H_{n+1}, u_{n+1})$, and the map $\tilde{\mathfrak{c}} = (\tilde c_{ij}) : H_n \to H_{n+1}$ are as in 13.25. Assume that $c_{ij} > 13$ for any i and j (which is a consequence of $(\diamondsuit\diamondsuit)_1$). We shall define the unital homomorphism $\psi_{n,n+1} : A_{X,n} \to A_{X,n+1}$ to satisfy the following conditions:

$$\begin{aligned}
(1)\quad (\psi_{n,n+1})_{*0} &= \gamma_{n,n+1} : K_0(A_{X,n})(= H_n) \longrightarrow K_0(A_{X,n+1})\,(= H_{n+1}),\ \text{and}\\
(\psi_{n,n+1})_{*1} &= \chi_{n,n+1} :\ K_1(A_{X,n})\,(= K_n) \longrightarrow K_1(A_{X,n+1})\,(= K_{n+1});
\end{aligned}$$

(2) $\psi_{n,n+1}(\bar{J}_n) \subset \bar{J}_{n+1}$, and the quotient map $\psi^q_{n,n+1}: F_n \to F_{n+1}$ induced by $\psi_{n,n+1}$ satisfies

$$(\psi^q_{n,n+1})_{*0} = \gamma'_{n,n+1} = \mathfrak{c}_{n,n+1} = (c^{n,n+1}_{ij}) : K_0(F_n)(=\mathbb{Z}^{p_n}) \to (F_{n+1})_{*0}(=\mathbb{Z}^{p_{n+1}}).$$

Denote by $F'_k = \bigoplus_{i\geq 2} F^i_k = \bigoplus_{i\geq 2} A^i_{X,k}$, $\pi''_k : F_k \to F'_k$, and $\pi^{-,1}_k : F_k \to F^1_k$ the projection maps, $k = 1, 2, \ldots$. Let $\pi_{x^0_n} : P_n M_\infty(C(X_n))P_n \to F^1_n$ be the point evaluation at x^0_n. Define $\pi^{\sim}_{x^0_n} : A_{X,n} = A^1_{X,n} \oplus F'_n \to F_n$ by $\pi^{\sim}_{x^0_n}(a,b) = (\pi_{x^0_n}(a), b)$ for all $(a,b) \in A^1_{X,n} \oplus F'_n$. It is well known (see [29]) that there is a homomorphism ${\psi^q_{n,n+1}}' : F_n \to F_{n+1}$ such that $({\psi^q_{n,n+1}}')_{*0} = \mathfrak{c}_{n,n+1}$. Then, define $\psi''_{n,n+1} : A_{X,n} \to F'_{n+1}$ by $\psi''_{n,n+1} = \pi''_{n+1} \circ {\psi^q_{n,n+1}}' \circ \pi^{\sim}_{x^0_n}$.

Recall from 13.25, $\gamma^{i,1}_{n,n+1}([\mathbf{1}_{A^i_{X,n}}]) = (c^{n,n+1}_{1,i} \cdot [n,i] + T^{n,n+1}_i([\mathbf{1}_{A^i_{X,n}}])) \in \mathbb{Z} \oplus \mathrm{Tor}(H_{n+1}) = K_0(A^1_{X,n+1})$, where $[1_{A^i_{X,n}}] = [n,i]$ if $i \geq 2$, or $[1_{A^i_{X,n}}] = [n,1] + \tau_n$ if $i = 1$. By 13.22, one can find projections $Q_1, Q_2, \ldots, Q_{p_n}$ such that $\gamma^{i,1}_{n,n+1}([\mathbf{1}_{A^i_{X,n}}]) = [Q_i] \in K_0(P_{n+1}M_\infty(C(X_{n+1}))P_{n+1})$. Since

$$\bigoplus_{i=1}^{p_n} \gamma^{i,1}_{n,n+1}([1_{A^i_{X,n}}]) = \gamma^{-,1}_{n,n+1}([1_{A_{X,n}}]) = [1_{A^1_{X,n+1}}] = [P_{n+1}],$$

by Remark 3.26 of [32], one can make $\{Q_i\}_{i=1}^{p_n}$ mutually orthogonal with

$$Q_1 + Q_2 + \cdots + Q_{p_n} = P_{n+1}.$$

Since $c_{ij} > 13$ for all i, j—i.e., $\mathrm{rank}(Q_i)/\mathrm{rank}(\mathbf{1}_{A^i_{X,n}}) = c_{1,i} > 13$—, by 13.23 (using the base point x^0_{n+1} (as y_0)), there are unital homomorphisms $\psi^{i,1}_{n,n+1} : A^i_{X,n} \to Q_i A^1_{X,n+1} Q_i$ which realize the K-theory map $\gamma^{i,1}_{n,n+1} : K_0(A^i_{X,n}) \to K_0(A^1_{X,n+1})$, and $\chi_{n,n+1} : K_1(A^1_{X,n}) (= K_n) \to K_1(A^1_{X,n+1}) (= K_{n+1})$ (note that $K_1(A^i_{X,n}) = 0$ for $i \geq 2$). Moreover, since, in the application of 13.23, we used the base point x^0_{n+1} (as y_0), we have $\psi^{1,1}_{n,n+1}(\bar{J}_n) \subset \bar{J}_{n+1}$ (see also the last line of 13.29). It follows that $\psi^{1,1}_{n,n+1}$ induces a (not necessarily unital) homomorphism $\psi^{1,1,q}_{n,n+1} : A^1_{X,n}/\bar{J}_n \to A^1_{X,n+1}/\bar{J}_{n+1}$. That is, $\psi^{1,1}_{n,n+1}$ induces a (not necessarily unital) homomorphism $\psi^{1,1,q}_{n,n+1} : F^1_n \to F^1_{n+1}$. Note that $K_0(A^1_{X,n}) = \mathbb{Z} \oplus \mathrm{Tor}(H_n)$ and $K_0(A^1_{X,n+1}) = \mathbb{Z} \oplus \mathrm{Tor}(H_{n+1})$. It is then clear that ${\psi^{1,1,q}_{n,n+1}}_{*0} = c^{n,n+1}_{1,1}$ (see 13.25). Define $\psi^{-,1}_{n,n+1} : A_{X,n} \to A^1_{X,n+1}$ by $\psi^{-,1}_{n,n+1}|_{A^i_{X,n}} = \psi^{i,1}_{n,n+1}$, $i = 1, 2, \ldots, p_n$. It follows that $\psi^{-,1}_{n,n+1}$ induces a unital homomorphism $\psi^{-,1,q}_{n,n+1} : A_{X,n}/\bar{J}_n \to A^1_{X,n+1}/\bar{J}_{n+1}$. One computes that $(\psi^{-,1,q}_{n,n+1})_{*0} = (\pi^{-,1}_{n+1})_{*0} \circ \gamma_{n,n+1}$. Now define $\psi_{n,n+1} : A_{X,n} \to A_{X,n+1} = A^1_{X,n+1} \oplus F'_n$ by $\psi_{n,n+1} = \psi^{-,1}_{n,n+1} \oplus \psi''_{n,n+1}$. One then checks that (1) above holds. Let $\psi^q_{n,n+1} : F_n \to F_{n+1}$ be the induced unital homomorphism. Then, as above, $(\pi^{-,1}_{n+1} \circ \psi^q_{n,n+1})_{*0} = (\pi^{-,1}_{n+1})_{*0} \circ \gamma'_{n,n+1}$ and $(\pi''_{n+1} \circ \psi^q_{n,n+1})_{*0} = (\pi''_{n+1} \circ {\psi^q_{n,n+1}}')_{*0} = (\pi''_{n+1})_{*0} \circ \gamma'_{n,n+1}$. Therefore, $(\psi^q_{n,n+1})_{*0} = \gamma'_{n,n+1}$.

13.33. In this subsection, we will describe $\psi^{i,1}_{n,n+1}$ for $i \geq 2$.

Recall that $\psi^{i,1}_{n,n+1}$ is a unital homomorphism from $A^i_{X,n}$ to $Q_iA^1_{X,n+1}Q_i$, $i = 1,2,...,p_n$. Since n is fixed, to simplify notation, in this subsection, let us use ψ^i for $\psi^i_{n,n+1}$. Note that $A^1_{X,n+1}|_{[x^0_{n+1},x^0_{n+1}+1]}$ may be identified with $M_{[n+1,1]}(C([0,1]))$. Without loss of generality, by fixing a system of matrix units, we may write

$$\text{(a)}\quad Q_i|_{[x^0_{n+1},x^0_{n+1}+1]} = \operatorname{diag}(\mathbf{0}_{c_{11}[n,1]},\mathbf{0}_{c_{12}[n,2]},...,\mathbf{0}_{c_{1\ i-1}[n,i-1]},\mathbf{1}_{c_{1i}[n,i]},\\ \mathbf{0}_{c_{1\ i+1}[n,i+1]},...,\mathbf{0}_{c_{1p_n}[n,p_n]}),$$

where $P_{n+1}|_{[x^0_{n+1},x^0_{n+1}+1]}$ is identified with

$$\mathbf{1}_{[n+1,1]} \in M_{[n+1,1]}(C[x^0_{n+1},x^0_{n+1}+1]).$$

Let $\{e_{kl}\}$ be the matrix units of $F^i_n = M_{[n,i]}$ and let $q = \psi^i(e_{11})$. The unital homomorphism $\varphi^i : M_{[n,i]} \to Q_iA^1_{X,n+1}Q_i$ allows us to write $Q_iA^1_{X,n+1}Q_i = qA^1_{X,n+1}q \otimes M_{[n,i]} = qM_\infty(C(X_{n+1}))q \otimes M_{[n,i]}$, and to write

$$\text{(e 13.45)}\qquad \psi^i\big((a_{ij})\big) = q \otimes (a_{ij}) \in qM_\infty(C(X_{n+1}))q \otimes M_{[n,i]}.$$

By Remark 3.26 of [32] (see also 13.22) we can write $q = q_1 + q_2 + \cdots + q_d + p$, where $q_1, q_2, \cdots, q_d$ are mutually equivalent trivial rank 1 projections and p is a (possibly non-trivial) rank 1 projection. Under the identification $Q_i = q\otimes\mathbf{1}_{M_{[n,i]}}$, we write $\hat{q}_j{:=}q_j \otimes \mathbf{1}_{M_{[n,i]}}$ and $\hat{p}{:=}p \otimes \mathbf{1}_{M_{[n,i]}}$.

(b) Projections $\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_d$, and $\hat{p}$ for ψ^i can be chosen such that $\hat{q}_1|_{[x^0_{n+1},x^0_{n+1}+1]}$, $\hat{q}_2|_{[x^0_{n+1},x^0_{n+1}+1]}$,..., $\hat{q}_d|_{[x^0_{n+1},x^0_{n+1}+1]}$, and $\hat{p}|_{[x^0_{n+1},x^0_{n+1}+1]}$ are diagonal matrices with $\mathbf{1}_{[n,i]}$ in the correct place (see below) when $Q_iM_\infty(C[x^0_{n+1},x^0_{n+1}+1])Q_i$ is identified with $M_{c_{i1}[n,i]}(C[x^0_{n+1},x^0_{n+1}+1])$. That is,

$$\hat{q}_j = \operatorname{diag}(\underbrace{\mathbf{0}_{[n,i]},...,\mathbf{0}_{[n,i]}}_{j-1},\mathbf{1}_{[n,i]},\mathbf{0}_{[n,i]},...,\mathbf{0}_{[n,i]})$$

and

$$\hat{p} = \operatorname{diag}(\underbrace{\mathbf{0}_{[n,i]},...,\mathbf{0}_{[n,i]}}_{d},\mathbf{1}_{[n,i]}).$$

LEMMA 13.34. *Let $i \geq 2$ and $\psi^i : A^i_{X,n} \to Q_iA^1_{X,n+1}Q_i$ be as in* 13.33 *above. Suppose $m \leq d = c^{n,n+1}_{1i} - 1$. Let $\Lambda : Q_iA^1_{X,n+1}Q_i \to M_m(Q_iA^1_{X,n+1}Q_i)$ be the amplification defined by $\Lambda(a) = a \otimes \mathbf{1}_m$. Then there is a projection $R^i \in M_m(Q_iA^1_{X,n+1}Q_i)$ such that*

(i) R^i *commutes with* $\Lambda(\psi^i(F_n^i))$ *(note that* $F_n^i = A_{X,n}^i$ *for* $i \geq 2$*), and*

(ii) $R^i(x_{n+1}^0) = Q_i(x_{n+1}^0) \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{diag}(\underbrace{Q_i(x_{n+1}^0), ..., Q_i(x_{n+1}^0)}_{m-1}, 0)) \in M_m(A_{X,n+1}^1|_{x_{n+1}^0})$*. Consequently, rank*$(R^i) = c_{i1}^{n,n+1}(m-1)[n,i] = (d+1)(m-1)[n,i]$*.*

Let

$$\begin{aligned}\pi := \pi_{X,n+1}^e|_{M_m(Q_iA_{X,n+1}^1Q_i)} & \otimes & \mathrm{id}_{M_m} : M_m(Q_iA_{X,n+1}^1Q_i) \\ & \to & M_m(\pi_{X,n+1}^e(Q_iA_{X,n+1}^1Q_i)).\end{aligned}$$

Then π *maps* $R^iM_m(Q_iA_{X,n+1}^1Q_i)R^i$ *onto*

$$M_{m-1}(\pi_{X,n+1}^e(Q_iA_{X,n+1}^1Q_i)) \subset M_m(\pi_{X,n+1}^e(Q_iA_{X,n+1}^1Q_i)).$$

Below we will use the same notation π *to denote the restriction of* π *to* $R^iM_m(Q_iA_{X,n+1}^1Q_i)R^i$*, whose codomain is* $M_{m-1}(\pi_{X,n+1}^e(Q_iA_{X,n+1}^1Q_i))$*.*

(iii) *There is a unital embedding*

$$\iota : M_{m-1}(\pi_{X,n+1}^e(Q_iA_{X,n+1}^1Q_i)) \hookrightarrow R^iM_m(Q_iA_{X,n+1}^1Q_i)R^i$$

such that $\pi \circ \iota = \mathrm{id}|_{M_{m-1}(\pi_{X,n+1}^e(Q_iA_{X,n+1}^1Q_i))}$ *and such that* $R^i(\Lambda(\psi^i(F_n^i)))R^i \subset \mathrm{Image}(\iota)$*.*

PROOF. In the proof of this lemma, $i \geq 2$ is fixed.

The homomorphism $\Lambda \circ \psi^i : A_{X,n}^i = F_n^i = M_{[n,i]} \to M_m(Q_iA_{X,n+1}^1Q_i)$ can be regarded as $\Lambda_1 \otimes \mathrm{id}_{F_n^i} = \Lambda_1 \otimes \mathrm{id}_{[n,i]}$, where $\Lambda_1 : \mathbb{C} \to M_m(qA_{X,n+1}^1q)$ is the unital homomorphism given by

$$\Lambda_1(c) = c \cdot (q \otimes \mathbf{1}_m), \tag{e 13.46}$$

and $q = \psi^i(e_{11})$ is as in 13.33, by identifying $M_m(Q_iA_{X,n+1}^1Q_i)$ with $M_m(qA_{X,n+1}^1q) \otimes F_n^i$.

Note that $q = q_1 + q_2 + \cdots + q_d + p$ with $\{q_i\}$ mutually equivalent rank one projections. Furthermore, $p|_{[x_n^0, x_n^0+1]}$ is also a rank one trivial projection. Let $r \in M_m(qA_{X,n+1}^1q) = qA_{X,n+1}^1q \otimes M_m$ be defined as follows:

$$\begin{aligned} r(x_n^0) &= q(x_n^0) \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= (q_1(x_n^0) + q_2(x_n^0) + \cdots + q_d(x_n^0) + p(x_n^0)) \\ &\quad \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{e 13.47}$$

$$r(x_n^0+1) = (q_1(x_n^0+1) + q_2(x_n^0+1) + \cdots + q_{m-1}(x_n^0+1)) \otimes \mathbf{1}_m + \\ + (q_m(x_n^0+1) + \cdots + q_d(x_n^0+1)) \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

In this definition, between x_n^0 and x_n^0+1, $r(t)$ can be defined to be any continuous path connecting the projections $r(x_n^0)$, and $r(x_n^0+1)$, both of rank $(d+1)(m-1) = (m-1)m + (d-m+1)(m-1)$. (Note that all $q_i(t)$ and $p(t)$ are constant on $[x_n^0, x_n^0+1]$.) Finally, for $x \in X'_{n+1} \subset X_{n+1}$, define

$$r(x) = (q_1(x)+q_2(x)+\cdots+q_{m-1}(x))\otimes\mathbf{1}_m+(q_m(x)+\cdots+q_d(x))\otimes\begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since all $\{q_k\}_{k=1}^d$ are trivial projections, $r|_{X'_{n+1}}$ is also a trivial projection. Hence r itself is also a trivial projection as the inclusion map $X'_{n+1} \to X_{n+1}$ is a homotopy equivalence.

Let $R^i = r \otimes \mathbf{1}_{[n,i]}$ under the identification of $M_m(Q_i A^1_{X,n+1} Q_i)$ with $M_m(qA^1_{X,n+1}q) \otimes \mathbf{1}_{[n,i]}$. Since $\Lambda_1 : \mathbb{C} \to M_m(qA^1_{X,n+1}q)$ sends $\mathbb{C}$ to the center of $M_m(qA^1_{X,n+1}q)$ (see (e 13.46)), we have that r commutes with $\Lambda_1(\mathbb{C})$, and consequently, $R^i = r \otimes \mathbf{1}_{[n,i]}$ commutes with $\Lambda(\psi^i(F_n^i))$ as $\Lambda \circ \psi^i = \Lambda_1 \otimes \mathrm{id}_{[n,i]}$. That is, condition (i) holds. Condition (ii) follows from the definition of $r(x_{n+1}^0)$ (see (e 13.47)) and $R^i(x_{n+1}^0) = r(x_{n+1}^0) \otimes \mathbf{1}_{[n,i]}$.

Note that $r \in M_m(qA^1_{X,n+1}q) = M_m(qM_\infty(C(X_{n+1}))q)$ is a trivial projection of rank $(d+1)(m-1)$ and $r(x_{n+1}^0) = q(x_{n+1}^0) \otimes \mathrm{diag}(\mathbf{1}_{m-1}, 0)$.

Note also that

$$\pi(r)M_m(\pi^e_{X,n+1}(qA^1_{X,n+1}q))\pi(r) = M_{m-1}(\pi^e_{X,n+1}(qA^1_{X,n+1}q)) \cong M_{(d+1)(m-1)}.$$

Let $r_{ij}^0, 1 \le i,j \le (d+1)(m-1)$ be a system of matrix units for $M_{(d+1)(m-1)}$. Since r is a trivial projection, one can construct $r_{ij} \in rM_m(qA^1_{X,n+1}q)r$, $1 \le i,j \le (d+1)(m-1)$, with $\pi(r_{ij}) = r_{ij}^0$ serving as a system of matrix units for $M_{(d+1)(m-1)} \subset rM_m(qA^1_{X,n+1}q)r \cong M_{(d+1)(m-1)}(C(X_{n+1}))$. Here, by matrix units, we mean $r_{ij}r_{kl} = \delta_{jk}r_{il}$ and $r = \sum_{i=1}^{(d+1)(m-1)} r_{ii}$. We define

$$\begin{aligned} \iota_1 : M_{m-1}(\pi^e_{X,n+1}(qA^1_{X,n+1}q)) \big(&\cong \pi(r)M_m(\pi^e_{X,n+1}(qA^1_{X,n+1}q)\pi(r))\big) \\ &\hookrightarrow rM_m(qA^1_{X,n+1}q)r \end{aligned}$$

by $\iota_1(r_{ij}^0) = r_{ij}$. Finally, we define $\iota = \iota_1 \otimes \mathrm{id}_{[n,i]}$. Since $\pi(r_{ij}) = r_{ij}^0$, we have $\pi \circ \iota = \mathrm{id}|_{M_{m-1}(\pi^e_{X,n+1}(Q_iA^1_{X,n+1}Q_i))}$. Note that

$$r(\Lambda_1(\mathbb{C}))r = \mathbb{C} \cdot r \subset \iota_1(M_{m-1}(\pi(qA^1_{X,n+1}q))).$$

On the other hand, using the identification $M_m(Q_iA^1_{X,n+1}Q_i) = M_m(qA^1_{X,n+1}q) \otimes F_n^i$ (recall that $F_n^i = M_{[n,i]}$), we have $R^i = r \otimes \mathbf{1}_{[n,i]}$ and $\Lambda \circ \psi^i = \Lambda_1 \otimes \mathbf{1}_{[n,i]}$. Hence $R^i(\Lambda(\psi^i(F_n^i)))R^i \subset \mathrm{Image}(\iota)$. So (iii) follows. □

13.35. Now we would like to choose $\psi^{1,1}_{n,n+1}$ in a specially simple form described below. We know that $\mathrm{rank}(Q_1) = c^{n,n+1}_{11}[n,1]$, where $[n,1] = \mathrm{rank}(P_n)$ for $A^1_{X,n} = P_n M_\infty(C(X_n))P_n$. Note that $c^{n,n+1}_{11} > 13$. Set $d = c^{n,n+1}_{11} - 13$. We may write $Q_1 = Q' \oplus \tilde{Q}$, where $\tilde{Q}$ is a trivial projection of rank d. Recall that as in 13.33, we use ψ^i for $\psi^{i,1}_{n,n+1}$. Then we can choose $\psi^1 := \psi^{1,1}_{n,n+1}$ satisfying the following conditions:

(a) $Q_1 = \mathbf{1}_{d[n,1]} \oplus \tilde{Q} := Q' \oplus \tilde{Q} \in M_\infty(C(X_{n+1}))$ such that $\tilde{Q}|_{[x^0_{n+1}, x^0_{n+1}+1]} = \mathbf{1}_{13[n,1]}$ (but in the lower right corner of $Q_1|_{[x^0_{n+1}, x^0_{n+1}+1]}$);

(b) $\psi^1 : A^1_{X,n} \to Q_1 A^1_{X,n+1} Q_1$ can be decomposed as $\psi^1 = \psi_1 \oplus \psi_2$, where $\psi_1 : A^1_{X,n} \to M_{d[n,i]}(C(X_{n+1})) = Q' M_\infty(C(X_{n+1}))Q'$ and $\psi_2 : A^1_{X,n} \to \tilde{Q} M_\infty(C(X_{n+1}))\tilde{Q}$ are as follows:

(b1) the unital homomorphism

$$\psi_1 : A^1_{X,n} \to M_{d[n,i]}(C(X_{n+1})) = Q' M_\infty(C(X_{n+1}))Q'$$

is defined by $\psi_1(f) = \mathrm{diag}(\underbrace{f(x^0_n), f(x^0_n), ..., f(x^0_n)}_{d})$ as a constant function on X_{n+1};

(b2) the unital injective homomorphism $\psi_2 : A^1_{X,n} \to \tilde{Q} M_\infty(C(X_{n+1}))\tilde{Q}$ is a homomorphism satisfying $(\psi_2)_{*0} = \tilde{c}^{n,n+1}_{11} - d = c^{n,n+1}_{11} - d + T^{n,n+1}_1$ (where $T^{n,n+1}_1 : H^1_n (= K_0(A^1_{X,n})) \to \mathrm{Tor}(H_{n+1}) \subset H^1_{n+1}$ is as in 13.25) and $(\psi_2)_{*1} = \chi_{n,n+1} : K_1(A^1_{X,n})\ (= K_n) \to K_1(A^1_{X,n+1})\ (= K_{n+1})$ (such ψ_2 exists since $\tilde{Q}$ has rank $13[n,1]$, by 13.23).

Moreover, we may write $\psi_2(f)(x^0_{n+1}+1) = \mathrm{diag}(\underbrace{f(x^0_n+1), f(x^0_n+1), ..., f(x^0_n+1)}_{13})$ for a fixed system of matrix units for $M_{13[n,1]}(C([x^0_{n+1}, x^0_{n+1}+1]))$. Then, we may change $\psi_2(f)|_{[x^0_{n+1}, x^0_{n+1}+1]}$ so that it also satisfies the following condition:

(b3) For $t \in [0, \frac{1}{2}]$,

$$\psi_2(f)(x^0_{n+1}+t) = \mathrm{diag}(\underbrace{f(x^0_n), f(x^0_n), ..., f(x^0_n)}_{13}),$$

and for $t \in [\frac{1}{2}, 1]$,

$$\psi_2(f)(x^0_{n+1}+t) = \mathrm{diag}(\underbrace{f(x^0_n+2t-1), f(x^0_n+2t-1), ..., f(x^0_n+2t-1)}_{13}).$$

Here, $f(x^0_n+s) \in P_n(x^0_n+s)M_\infty P_n(x^0_n+s)$ is regarded as an $[n,1]\times[n,1]$ matrix for each $s \in [0,1]$ by using the fact $P_n|_{[x^0_n, x^0_n+1]} = \mathbf{1}_{[n,1]}$.

Let us remark that $\psi_1 : A^1_{X,n} \to M_{d[n,1]}(C(X_{n+1}))(= Q'M_\infty(C(X_{n+1}))Q')$ factors through $F^1_n = M_{[n,1]}(\mathbb{C})$, and the restriction $\psi^1|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]}$ also factors through F^1_n, as

$$\psi^1(f)(x) = \mathrm{diag}(\underbrace{f(x^0_n), ..., f(x^0_n)}_{d+13}) \text{ for any } x \in [x^0_{n+1}, x^0_{n+1} + \frac{1}{2}].$$

Lemma 13.36. *Suppose that* Q_1 *and* $\psi^1 : A^1_{X,n} \to Q_1 A^1_{X,n+1} Q_1$ *satisfy conditions (a) and (b) above (including (b1), (b2), and (b3)). Suppose that* $13m \le d = c^{n,n+1}_{11} - 13$. *Let* $\Lambda : Q_1 A^1_{X,n+1} Q_1 \to M_m(Q_1 A^1_{X,n+1} Q_1)$ *be the amplification defined by* $\Lambda(a) = a \otimes \mathbf{1}_m$. *There is a projection* $R^1 \in M_m(Q_1 A^1_{X,n+1} Q_1)$ *satisfying the following conditions:*

(i) R^1 *commutes with* $\Lambda(\psi^1(A^1_{X,n}))$ *and*

(ii) $R^1(x^0_{n+1}) = Q_1(x^0_{n+1}) \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{diag}(\underbrace{Q_1(x^0_{n+1}), \cdots, Q_1(x^0_{n+1})}_{m-1}, 0) \in M_m(A^1_{X,n+1}|_{x^0_{n+1}})$. *Consequently,* $rank(R^1) = c^{n,n+1}_{11}(m-1)[n,1] = (d+13)(m-1)[n,1]$.

Let

$$\begin{aligned} \pi := \pi^e_{X,n+1}|_{M_m(Q_1 A^1_{X,n+1} Q_1)} \quad &\otimes \quad \mathrm{id}_{M_m} : M_m(Q_1 A^1_{X,n+1} Q_1) \\ &\to \quad M_m(\pi^e_{X,n+1}(Q_1 A^1_{X,n+1} Q_1)). \end{aligned}$$

Then π *maps* $R^1 M_m(Q_1 A^1_{X,n+1} Q_1) R^1$ *onto* $M_{m-1}(\pi^e_{X,n+1}(Q_1 A^1_{X,n+1} Q_1)) \subset M_m(\pi^e_{X,n+1}(Q_1 A^1_{X,n+1} Q_1))$.

Below we will use the same notation π *to denote the restriction of* π *to* $R^1 M_m(Q_1 A^1_{X,n+1} Q_1) R^1$, *whose codomain is* $M_{m-1}(\pi^e_{X,n+1}(Q_1 A^1_{X,n+1} Q_1))$.

(iii) *There is a unital embedding*

$$\iota : M_{m-1}(\pi^e_{X,n+1}(Q_1 A^1_{X,n+1} Q_1)) \hookrightarrow R^1 M_m(Q_1 A^1_{X,n+1} Q_1) R^1$$

such that $\pi \circ \iota = \mathrm{id}|_{M_{m-1}(\pi^e_{X,n+1}(Q_1 A^1_{X,n+1} Q_1))}$ *and such that* $R^1(\Lambda(\psi^1(A^1_{X,n})))R^1 \subset \mathrm{Image}(\iota)$.

The notation Λ, d, and m in the lemma above, and $q, q_1, q_2, ..., q_d, p, r$, and Λ_1 in the proof below, are also used in Lemma 13.34 and its proof for the case $i \ge 2$ (comparing with $i = 1$ here). Since they are used for the same purpose, we choose the same notation.

Proof. The map

$$\begin{aligned} \text{(e 13.48)} \qquad \psi_1 : A^1_{X,n} \xrightarrow{\pi} F^1_n \longrightarrow &M_{d[n,1]}(C(X_{n+1})) \\ &= Q'M_\infty(C(X_{n+1}))Q' \end{aligned}$$

(where $Q' = \mathbf{1}_{d[n,1]}$) can be written as $(\Lambda_1 \otimes \mathrm{id}_{[n,1]}) \circ \pi$, where $\Lambda_1 : \mathbb{C} \to M_d(C(X_{n+1}))$ is the map sending $c \in \mathbb{C}$ to $c \cdot \mathbf{1}_d$. We write $\Lambda_1(1) := q' = q_1 + q_2 + \cdots + q_d$, with each q_i a trivial constant projection of rank 1. Here q' is a constant subprojection of Q' with $Q' = q' \otimes \mathbf{1}_{[n,1]}$. Consider the map $\tilde{\psi}_2 := \psi_2|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]} : A^1_{X,n} \to \tilde{Q} A^1_{X,n+1} \tilde{Q}|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]}$, and $\tilde{\psi}^1 := \psi^1|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]} = (\psi_1 + \psi_2)|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]} : A^1_{X,n} \to Q_1 A^1_{X,n+1} Q_1|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]}$. As pointed out in 13.35, $\tilde{\psi}_2$ has the factorization

$$A^1_{X,n} \xrightarrow{\pi} F^1_n \longrightarrow M_{13[n,1]}(C[x^0_n, x^0_{n+1} + \tfrac{1}{2}]).$$

Hence $\tilde{\psi}^1$ has the factorization

$$\text{(e 13.49)} \quad A^1_{X,n} \xrightarrow{\pi} F^1_n \longrightarrow M_{(d+13)[n,1]}(C[x^0_{n+1}, x^0_{n+1} + \tfrac{1}{2}]).$$

The map $\tilde{\psi}^1$ can be written as $(\Lambda_2 \otimes \mathrm{id}_{[n,1]}) \circ \pi$, where $\Lambda_2 : \mathbb{C} \to M_{d+13}(C[x^0_{n+1}, x^0_{n+1} + \frac{1}{2}])$ is the map defined by sending $c \in \mathbb{C}$ to $c \cdot \mathbf{1}_{d+13}$. We write $\Lambda_2(1) := q = q_1 + q_2 + \cdots + q_d + p$ with each q_i the restriction of q_i appearing in the definition of $\Lambda_1(1)$ on $[x^0_{n+1}, x^0_{n+1} + \frac{1}{2}]$, and p a rank 13 constant projection. Here q is a constant projection on $[x^0_{n+1}, x^0_{n+1} + \frac{1}{2}]$ and $Q_1|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]} = q \otimes \mathbf{1}_{[n,1]}$. Let $r \in M_m(Q_1 A^1_{X,n+1} Q_1) = Q_1 A^1_{X,n+1} Q_1 \otimes M_m$ be defined as follows:

$$\text{(e 13.50)} \quad \begin{aligned} r(x^0_{n+1}) &= q(x^0_{n+1}) \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= (q_1(x^0_{n+1}) + q_2(x^0_{n+1}) + \cdots + q_d(x^0_{n+1}) + p(x^0_{n+1})) \\ &\quad \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}; \end{aligned}$$

for $t \in [\frac{1}{2}, 1]$,

$$\begin{aligned} r(x^0_{n+1} + t) = &\Big(q_1(x^0_{n+1} + t) + q_2(x^0_{n+1} + t) + \cdots + q_{13(m-1)}(x^0_{n+1} + t)\Big) \otimes \mathbf{1}_m \\ &+ \Big(q_{13(m-1)+1}(x^0_{n+1} + t) + \cdots + q_d(x^0_{n+1} + t)\Big) \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}; \end{aligned}$$

and, for $x \in X'_{n+1} \subset X_{n+1}$,

$$\begin{aligned} r(x) &= (q_1(x) + q_2(x) + \cdots + q_{13(m-1)}(x)) \otimes \mathbf{1}_m \\ &\quad + (q_{13(m-1)+1}(x) + \cdots + q_d(x)) \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

In the definition above, between x^0_{n+1} and $x^0_{n+1}+\frac{1}{2}$, $r(t)$ can be defined to be any continuous path connecting the projections $r(x^0_{n+1})$ and $r(x^0_{n+1}+\frac{1}{2})$; note that both have rank $(d+13)(m-1) = 13(m-1)m+(d-13(m-1))(m-1)$. (Note that all $q_i(x)$ are constant on $x \in X_{n+1} = [x^0_{n+1}, x^0_{n+1}+1] \vee X'_{n+1}$ and $p(t)$ is constant for $t \in [x^0_{n+1}, x^0_{n+1}+1]$.) Note that for $x \in [x^0_{n+1}+\frac{1}{2}, x^0_{n+1}+1] \vee X'_{n+1}$, $r(x)$ has the same form as $r(x^0_{n+1}+\frac{1}{2})$ which is a constant sub-projection of the constant projection $q' \otimes \mathbf{1}_m$. Hence r is a trivial projection. We will define R^1 to be $r \otimes \mathbf{1}_{[n,i]}$ under a certain identification described below. Note that the projection Q_1 is identified with $q \otimes \mathbf{1}_{[n,1]}$ only on the interval $[x^0_{n+1}, x^0_{n+1}+1]$, so the definition of R^1 will be divided into two parts. For the part on $[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]$, we use the identification of Q_1 with $q \otimes \mathbf{1}_{[n,1]}$, and for the part that $x \in [x^0_{n+1}+\frac{1}{2}, x^0_{n+1}+1] \vee X'_{n+1}$, we use the identification of $Q' = \mathbf{1}_{d[n,1]}$ with $q' \otimes \mathbf{1}_m$ (of course, we use the fact that r is a sub-projection of q' on this part). This is the only difference between the proof of this lemma and that of Lemma 13.34. The definition of $\iota : M_{m-1}(\pi^e_{X,n+1}(Q_1 A^1_{X,n+1} Q_1)) \hookrightarrow R^1 M_m(Q_1 A^1_{X,n+1} Q_1) R^1$ and the verification that ι and R^1 satisfy the conditions are exactly the same as in the proof of 13.34, with $(d+1)(m-1)$ replaced by $(d+13)(m-1)$. We will now give the details.

On $[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]$, $Q_1 \otimes \mathbf{1}_m \in M_m(Q_1(A^1_{X,n+1}|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]})Q_1)$ is identified with $(q \otimes \mathbf{1}_m) \otimes \mathbf{1}_{[n,i]}$ (as Q_1 is identified with $q \otimes \mathbf{1}_{[n,i]}$). As r is a sub-projection of $q \otimes \mathbf{1}_m$, we can define $R = r \otimes \mathbf{1}_{[n,i]}$ as a sub-projection of $Q_1 \otimes \mathbf{1}_m$ on $[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]$. Note that on $[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]$, R commutes with $\Lambda \circ \tilde{\psi}^1$ (recall $\tilde{\psi}^1 = \psi^1|_{[x^0_{n+1}, x^0_{n+1}+\frac{1}{2}]}$), since $\tilde{\psi}^1 = (\Lambda_2 \otimes \mathrm{id}_{[n,1]}) \circ \pi$, and $R = r \otimes \mathbf{1}_{[n,1]}$, and r commutes with $\mathrm{range}(\Lambda \circ \Lambda_2) = \mathbb{C} \cdot q \otimes \mathbf{1}_m$, as r is a sub-projection of $q \otimes \mathbf{1}_m$. On $[x^0_{n+1}+\frac{1}{2}, x^0_{n+1}+1] \vee X'_{n+1}$, $Q' \otimes \mathbf{1}_m \in M_m(Q_1 A^1_{X,n+1}|_{[x^0_{n+1}+\frac{1}{2}, x^0_{n+1}+1] \vee X'_{n+1}})Q_1$ is identified with $(q' \otimes \mathbf{1}_m) \otimes \mathbf{1}_{[n,i]}$ (as Q' is identified with $q' \otimes \mathbf{1}_{[n,i]}$). As r is a sub-projection of $q' \otimes \mathbf{1}_m$, we can define $R^1 = r \otimes \mathbf{1}_{[n,i]}$ as a sub-projection of $Q' \otimes \mathbf{1}_m$ on $[x^0_{n+1}+\frac{1}{2}, x^0_{n+1}+1] \vee X'_{n+1}$. On this part, R^1 commutes with $\Lambda \circ \psi_1$ since $\psi_1 = (\Lambda_1 \otimes \mathrm{id}_{[n,1]}) \circ \pi$ and $R^1 = r \otimes \mathbf{1}_{[n,1]}$ and r commutes with $\mathrm{range}(\Lambda \circ \Lambda_1) = \mathbb{C} \cdot q' \otimes \mathbf{1}_m$, as r is a sub-projection of $q' \otimes \mathbf{1}_m$. Note that R^1 is a sub-projection of $Q' \otimes \mathbf{1}_m$ and therefore is orthogonal to $\tilde{Q} \otimes \mathbf{1}_m$ and the range of $\Lambda \circ \psi_2$. Hence on this part, R^1 also commutes with $\Lambda \circ \psi^1$ as $\psi^1 = \psi_1 + \psi_2$. On combining this with the previous paragraph, (i) follows.

On the other hand, (ii) follows from the definition of R^1 and (e 13.50).

From the last paragraph, we know that $A^1_{X,n} \ni f \mapsto R^1(\Lambda \circ \psi^1(f))R^1 \in R^1(M_m(Q_1 A^1_{X,n+1} Q_1))R^1$ is a homomorphism. We denote it by Ξ. From (e 13.48) and (e 13.49), we know that Ξ factors through $F^1_n = A^1_{X,n}|_{x^0_n}$ as $\Xi = \Xi' \circ \pi$. Also when we identify $R^1 = r \otimes \mathbf{1}_{[n,i]}$, the map $\Xi' : M_{[n,1]} \to R^1(M_m(Q_1 A^1_{X,n+1} Q_1))R^1$ can be identified as $\xi \otimes \mathrm{id}_{[n,1]}$, where $\xi : \mathbb{C} \to r(M_m(Q_1 A^1_{X,n+1} Q_1))r$ is defined by $\xi(c) = c \cdot r$.

Let $r^0_{ij}, 1 \leq i,j \leq (d+13)(m-1)$, be the matrix units for $M_{(d+13)(m-1)}$. Since r is a trivial projection of rank $(d+13)(m-1)$, one can construct $r_{ij} \in rM_m(Q_1(A^1_{X,n+1} Q_1))r$, $1 \leq i,j \leq (d+13)(m-1)$, with $\pi(r_{ij}) = r^0_{ij}$ serving as a system of matrix units for $M_{(d+13)(m-1)} \subset rM_m(Q_1A^1_{X,n+1}Q_1)r \cong M_{(d+13)(m-1)}(C(X_{n+1}))$. Recall that $j_{n+1}: F^1_{n+1} \to \pi_{x^0_{n+1}}(A^1_{X,n+1})$ is an isomorphism (see 13.28). Note that $q(x^0_{n+1})$ is a projection in $\pi_{x^0_{n+1}}(A^1_{X,n+1})$. Set $\hat{q} = j^{-1}_{n+1}(q(x^0_{n+1}))$. Then $\hat{q}$ has rank $d+13$. Define

$$\iota_1 : M_{m-1}(\hat{q}F^1_{n+1}\hat{q}) \to rM_m(Q_1A^1_{X,n+1}Q_1)r$$

by $\iota_1(r^0_{ij}) = r_{ij}$. Note that $M_{m-1}(\pi^e_{X,n+1}(Q_1A^1_{X,n+1}Q_1)) = M_{m-1}(\hat{q}F_{n+1}\hat{q}) \otimes M_{[n,1]}$. Define

$$\iota = \iota_1 \otimes \mathrm{id}_{[n,1]} : M_{m-1}(\pi^e_{X,n+1}(Q_1A^1_{X,n+1}Q_1)) \to R(M_m(Q_1A^1_{X,n+1}Q_1))R.$$

Note that range $(\xi) = \mathbb{C} \cdot r \subset \iota_1(M_{m-1}(\hat{q}F^1_{n+1}\hat{q})) = \mathrm{range}(\iota_1)$. Since $\Xi' = \xi \otimes \mathrm{id}_{[n,i]}$ and $\iota = \iota_1 \otimes \mathrm{id}_{[n,1]}$, we have range $(\Xi') \subset \mathrm{range}(\iota)$. Hence

$$R(\Lambda(\psi^1(A^1_{X,n})))R = \Xi(A^1_{X,n}) = \Xi'(F^1_n) \subset \mathrm{range}(\iota).$$

□

13.37. Recall that we have constructed $A_{X,n+1}$ and the unital injective homomorphism $\psi_{n,n+1}$ (with the specific form described in 13.35). Note that, by the end of 13.31, we know that $A_{C,n+1}$ is fixed and the injective map $\bar{\varphi}_{n,n+1}: A_{C,n} \to A_{C,n+1}$ is defined. One also has A_{n+1} as defined in (e 13.32). As in 13.32, $(\varphi^q_{n,n+1})_{*0} = \gamma'_{n,n+1} = (\psi^q_{n,n+1})_{*0}$. Therefore, there exists a unitary $U \in F_{n+1}$ such that $\mathrm{Ad}\, U \circ \psi^q_{n,n+1} = \varphi^q_{n,n+1}$. In F_{n+1}, there exist $h \in (F_{n,n+1})_{s.a.}$ such that $U = \exp(ih)$. Let $H \in (A_{X,n+1})_{s.a.}$ be such that $\pi^e_{X,n+1}(H) = h$. Define $V = \exp(iH)$. Note that $V^*\bar{J}_{n+1}V \subset \bar{J}_{n+1}$. If we replace $\psi_{n,n+1}$ by $\mathrm{Ad}\, V \circ \psi_{n,n+1}$, then we still have (1) and (2) of 13.32. Moreover, 13.34 and 13.36 also hold (up to unitary equivalence in $A_{X,n+1}$). More importantly, $\psi^q_{n,n+1} = \varphi^q_{n,n+1}$. It follows from 13.29 that there is a unital injective homomorphism $\varphi_{n,n+1}: A_n \to A_{n+1}$ which satisfies (a), (b), and (c) of 13.29. Moreover, A_{n+1} satisfies (d) of 13.31. Therefore, by 13.30, one checks that $\varphi_{n,n+1}$ also satisfies (a), (b), and (c) of 13.31. This ends the induction of 13.31.

Now we have

$$A_1 \xrightarrow{\varphi_{1,2}} A_2 \xrightarrow{\varphi_{2,3}} A_3 \xrightarrow{\varphi_{3,4}} \cdots \xrightarrow{\varphi_{i-1,i}} A_i \cdots \longrightarrow A,$$

satisfying conditions (a)–(d) of 13.31. From (a) of 13.31, we obtain the two inductive limits

$$A_{C,1} \xrightarrow{\bar{\varphi}_{1,2}} A_{C,2} \xrightarrow{\bar{\varphi}_{2,3}} A_{C,3} \xrightarrow{\bar{\varphi}_{3,4}} \cdots \xrightarrow{\bar{\varphi}_{i-1,i}} A_{C,i} \cdots \longrightarrow A_C \text{ and}$$

$$A_{X,1} \xrightarrow{\psi_{1,2}} A_{X,2} \xrightarrow{\psi_{2,3}} A_{X,3} \xrightarrow{\psi_{3,4}} \cdots \xrightarrow{\psi_{i-1,i}} A_{X,i} \cdots \longrightarrow A_X.$$

Since $\psi_{n,n+1}(\bar{J}_n) \subset \bar{J}_{n+1}$, this procedure also gives an inductive limit of quotient algebras $F = \lim(F_n, \psi^q_{n,n+1})$, where $F_n = A_{X,n}/\bar{J}_n$. Evidently, F is an AF algebra with $K_0(F) = H/\mathrm{Tor}(H)$. (Note that $\psi^q_{n,n+1} = \bar{\varphi}^q_{n,n+1} : F_n \to F_{n+1}$.) Since $\varphi_{n,n+1}(I_n) \subset I_{n+1}$, $\varphi_{n,n+1}(J_n) \subset J_{n+1}$, $\{\pi_{I,n}\}$ and $\{\pi_{J,n}\}$ induce the quotient maps $\pi_I : A \to A_X$ and $\pi_J : A \to A_C$, respectively. Moreover, $\{\pi^e_{X,n}\}$ and $\{\pi^e_{C,n}\}$ induce the quotient maps $\pi^e_X : A_X \to F$, and $\pi^e_C : A_C \to F$, respectively. Finally, $\pi^e_A = \pi^e_X \circ \pi_I = \pi^e_C \circ \pi_J : A \to F$, and this map is induced by $\{\pi^e_{A_n}\}$.

Combining 13.34 and 13.36, we have the following theorem which will be used to conclude that the algebra A (which will be constructed later) has the property that $A \otimes U$ is in $\mathcal{B}_0$.

THEOREM 13.38. *Suppose that* $1 < m \leq \min\{(c_{11} - 13)/13, c_{12} - 1, c_{13} - 1, ..., c_{1p_n} - 1\}$. *Let* $\psi : A_{X,n} \to A^1_{X,n+1}$ *be the composition*

$$A_{X,n} \xrightarrow{\psi_{n,n+1}} A_{X,n+1} \xrightarrow{\pi^1} A^1_{X,n+1},$$

where π^1 *is the quotient map to the first block. Let* $\Lambda : A^1_{X,n+1} \to M_m(A^1_{X,n+1})$ *be the amplification defined by* $\Lambda(a) = a \otimes \mathbf{1}_m$. *There is a projection* $R \in M_m(A^1_{X,n+1}) = A^1_{X,n+1} \otimes M_m$ *and there is a unital inclusion homomorphism* $\iota : M_{m-1}(F^1_{n+1}) = F^1_{n+1} \otimes M_{m-1} \hookrightarrow RM_m(A^1_{X,n+1})R$, *satisfying the following three conditions:*

(i) R *commutes with* $\Lambda(\psi(A_{X,n}))$, *and*

(ii) $R(x^0_{n+1}) = \mathbf{1}_{F^1_{n+1}} \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}$.

Consequently, the map $\pi : A^1_{X,n+1} \to F^1_{n+1}$ *takes* $RM_m(A^1_{X,n+1})R$ *onto* $M_{m-1}(F^1_{n+1})$.

Below we will use the same notation π *to denote the restriction of* π *to* $RM_m(A^1_{X,n+1})R$, *whose codomain is* $M_{m-1}(F^1_{n+1})$.

(iii) $\pi \circ \iota = \mathrm{id}|_{M_{m-1}(F^1_{n+1})}$, *and* $R(\Lambda(\psi(A_{X,n})))R \subset \mathrm{range}(\iota)$.

PROOF. Choose $R = \bigoplus_{i=1}^{p_n} R^i \in M_m(A^1_{X,n+1})$, where $R^1 \in M_m(Q_1 A^1_{X,n+1} Q_1)$ is as described in Lemma 13.36 and $R^i \in M_m(Q_i A^1_{X,n+1} Q_i)$ (for $i \geq 2$) is as described in Lemma 13.34. Then the theorem follows. □

THEOREM 13.39. *Let* (H, H_+, u) *be as in* 13.25 *and* 13.26. *Then*

$$((K_0(A_X), K_0(A_X)_+, [\mathbf{1}_{A_X}]), K_1(A_X)) \cong ((H, H_+, u), K).$$

PROOF. Note that $\psi_{n,n+1}$ satisfies (1) in 13.32, and, consequently,

$$((K_0(A_X), K_0(A_X)_+, [\mathbf{1}_{A_X}]), K_1(A_X)) \cong ((H, H_+, u), K,).$$

□

Let $\pi^e_{X,n} : A_{X,n} \to F_n$ be as in 13.29. Then $(\pi^e_{X,n})^\sharp : \mathrm{Aff}(T(A_{X,n})) = C(X_n, \mathbb{R}) \oplus \mathbb{R}^{p_n-1} \to \mathrm{Aff}(T(F_n)) = \mathbb{R}^{p_n}$ is given by $(\pi^e_{X,n})^\sharp(g, h_2, h_3, ..., h_{p_n}) = (g(\theta_1), h_2, h_3, ..., h_{p_n})$. Define $\Gamma_n : \mathrm{Aff}(T(F_n)) = \mathbb{R}^{p_n} \to \mathrm{Aff}(T(A_{X,n})) = C(X_n, \mathbb{R}) \oplus \mathbb{R}^{p_n-1}$ to be the right inverse of $(\pi^e_{X,n})^\sharp$, given by

$$\Gamma_n(h_1, h_2, h_3, ..., h_{p_n}) = (g, h_2, h_3, ..., h_{p_n})$$

with g the constant function $g(x) = h_1$ for all $x \in X_n$. Then with the condition $c_{ij} > 13 \cdot 2^{2n}$, we have the following lemma:

LEMMA 13.40. *For any* $f \in \mathrm{Aff}(T(A_{X,n}))$ *with* $\|f\| \le 1$ *and* $f' {:=} \psi^\sharp_{n,n+1}(f) \in \mathrm{Aff}(T(A_{X,n+1}))$, *we have*

$$\|\Gamma_{n+1} \circ (\pi^e_{X,n+1})^\sharp(f') - f'\| < \frac{2}{2^{2n}}.$$

PROOF. Write $f = (g, h_2, ..., h_{p_n})$ and $f' = (g', h'_2, ..., h'_{p_{n+1}})$. Then $\Gamma_{n+1} \circ (\pi^e_{X,n+1})^\sharp(f') = (g'', h'_2, ..., h'_{p_{n+1}})$ with

$$g''(x) = g'(x^0_{n+1}) \quad \text{for all} \quad x \in X_{n+1}. \tag{e 13.51}$$

Recall that $\psi^{i,1}_{n,n+1}$ is denoted by $\psi^i : A_{X,n} \to Q_i A^1_{X,n+1} Q_i$ and $\psi^1 = \psi_1 + \psi_2$ with $\psi_1 : A^1_{X,n} \to Q' A^1_{X,n+1} Q'$, and $\psi_2 : A^1_{X,n} \to \tilde{Q} A^1_{X,n+1} \tilde{Q}$ as in 13.33 and 13.35. Note that

$$\frac{\mathrm{rank}(Q_i)}{\mathrm{rank}(P_{n+1})} = \frac{c_{i,1}}{\sum_{j=1}^{p_n} c_{j,1}}, \qquad \frac{\mathrm{rank}(Q')}{\mathrm{rank}(P_{n+1})} = \frac{c_{1,1} - 13}{\sum_{j=1}^{p_n} c_{j,1}},$$

and

$$\frac{\mathrm{rank}(\tilde{Q})}{\mathrm{rank}(P_{n+1})} = \frac{13}{\sum_{j=1}^{p_n} c_{j,1}}.$$

Hence

$$g' = \frac{c_{1,1} - 13}{\sum_{j=1}^{p_n} c_{j,1}} \psi_1^\sharp(g) + \frac{13}{\sum_{j=1}^{p_n} c_{j,1}} \psi_2^\sharp(g) + \sum_{i=2}^{p_n} \frac{c_{i,1}}{\sum_{j=1}^{p_n} c_{j,1}} (\psi^i)^\sharp(h_i).$$

Also from the construction in 13.33 and 13.35, we know that $\psi_1^\sharp(g)$ and $(\psi^i)^\sharp(h_i)$ $(i \ge 2)$ are constant. So we have

$$|g'(x) - g'(x^0_{n+1})| \le \frac{2 \times 13}{\sum_{j=1}^{p_n} c_{j,1}} < \frac{2}{2^{2n}}.$$

Then the lemma follows from from (e 13.51). □

Using Lemma 13.40 one can actually prove (see the proof of 13.42) that $((K_0(A_X), K_0(A_X)_+, [\mathbf{1}_{A_X}]), K_1(A_X), T(A_X), r_{A_X})$ is isomorphic to $((H, H_+, u), K, \Delta, r)$.

13.41. We will also compute the tracial state space for the C^*-algebra A in 13.37.

As in 13.17 (see (e 13.18)), the subspace

$$\mathrm{Aff}(T(A_n)) \subset \bigoplus_{i=1}^{l_n} C([0,1]_i, \mathbb{R}) \oplus C(X_n, \mathbb{R}) \oplus \mathbb{R}^{p_n-1}$$

consists of the elements $(f_1, f_2, ..., f_{l_n};\ g, h_2, ..., h_{p_n})$ (here we do not need h_1, since it is identified with $g(x_n^0)$) which satisfy the conditions

$$f_i(0) \quad = \quad \frac{1}{\{n,i\}}\big(b^n_{0,i1} g(x^0_n)[n,1] + \sum_{j=2}^{p_n} b^n_{0,ij} h_j \cdot [n,j]\big) \tag{e 13.52}$$

and

$$f_i(1) \quad = \quad \frac{1}{\{n,i\}}\big(b^n_{1,i1} g(x^0_n)[n,1] + \sum_{j=2}^{p_n} b^n_{1,ij} h_j \cdot [n,j]. \tag{e 13.53}$$

For $h = (h_1, h_2, ..., h_{p_n}) \in \mathrm{Aff}\, T(F_n)$, let $\Gamma'_n(h)(t) = t \cdot \beta^\sharp_{n,1}(h) + (1-t) \cdot \beta^\sharp_{n,0}(h)$ (see 13.18 and 13.19), which gives an element $C([0,1], \mathbb{R}^{l_n}) = \bigoplus_{i=1}^{l_n} C([0,1]_i, \mathbb{R})$. Let

$$\Gamma_n: \ \mathrm{Aff}(T(F_n)) = \mathbb{R}^{p_n} \to \mathrm{Aff}(T(A_n)) \subset \bigoplus_{j=1}^{l_n} C([0,1]_j, \mathbb{R}) \oplus C(X_n, \mathbb{R}) \oplus \mathbb{R}^{p_n-1}$$

be defined by $\Gamma_n(h_1, h_2, ..., h_{p_n}) = (\Gamma'_n(h_1, h_2, ..., h_{p_n}), g, h_2, ..., h_{p_n}) \in \mathrm{Aff}(T(A_n))$, where $g \in C(X_n, \mathbb{R})$ is the constant function $g(x) = h_1$.

Now we are ready to show that the Elliott invariant of A is as desired.

THEOREM 13.42. *Let A be as constructed. Then*

$$\big((K_0(A), K_0(A)_+, [1_A]),\ K_1(A), T(A), r_A\big) \cong \big((G, G_+, u),\ K, \Delta, r\big).$$

PROOF. Claim: For any $f \in \mathrm{Aff}(T(A_n))$ with $\|f\| \le 1$ and $f' := \varphi^\sharp_{n,n+1}(f) \in \mathrm{Aff}(T(A_{n+1}))$, we have

$$\|\Gamma_{n+1} \circ \pi^{e\ \sharp}_{A_{n+1}}(f') - f'\| < \frac{4}{2^{2n}},$$

where $(\pi^e_{A_{n+1}})^\sharp : \mathrm{Aff}(T(A_{n+1})) \to \mathrm{Aff}(T(F_{n+1}))$ is induced by $\pi^e_{A_{n+1}}$ (see 13.29).

Proof of the claim: For any $n \in \mathbb{Z}_+$, write

$$\Gamma_n = \Gamma_n^1 \circ \Gamma_n^2 : \mathrm{Aff}(T(F_n)) \xrightarrow{\Gamma_n^2} \mathrm{Aff}(T(A_{X,n})) \xrightarrow{\Gamma_n^1} \mathrm{Aff}(T(A_n))$$

with $\Gamma_n^2 : \mathrm{Aff}(T(F_n)) = \mathbb{R}^{p_n} \to \mathrm{Aff}(T(A_{X,n})) = C(X_n, \mathbb{R}) \oplus \mathbb{R}^{p_n-1}$ defined by

$$\Gamma_n^2(h_1, h_2, ..., h_{p_n}) = (g, h_2, ..., h_{p_n}),$$

where g is the constant function $g(x) = h_1$, and with $\Gamma_n^1 : \mathrm{Aff}(T(A_{X,n})) = C(X_n, \mathbb{R}) \oplus \mathbb{R}^{p_n-1} \to \mathrm{Aff}(T(A_n))$ defined by

$$\Gamma_n^1(g, h_2, ..., h_{p_n}) = (\Gamma'_n(g(x_n^0), h_2, ..., h_{p_n}), g, h_2, ..., h_{p_n}).$$

Define $\Gamma_{n,C} : \mathrm{Aff}(T(F_n)) \to \mathrm{Aff}(T(A_{C,n}))$ by

$$\Gamma_{n,C}(h_1, h_2, ..., h_{p_n}) = (\Gamma'_n(h_1, h_2, ..., h_{p_n}), h_1, h_2, ..., h_{p_n}).$$

Note also $\pi_{A_n}^{e\,\sharp} = \pi_{X,n}^{e\,\sharp} \circ \pi_{I,n}^{e\,\sharp}$ (see 13.29).

One has, for any $(f, g, h_2, h_3, ..., h_{p_n}) \in \mathrm{Aff}(T(A_n))$,

$$\begin{aligned}
&\pi_{J,n}^{\sharp} \circ \Gamma_n^1 \circ \pi_{I,n}^{\sharp}(f, g, h_2, h_3, ..., h_{p_n}) = \pi_{J,n}^{\sharp} \circ \Gamma_n^1(g, h_2, ..., h_{p_n})\\
&= \pi_{J,n}^{\sharp} \circ (\Gamma'_n(g(x_n^0), h_2, ..., h_{p_n}), g, h_2, ..., h_{p_n})\\
&= (\Gamma'_n(g(x_n^0), h_2, ..., h_{p_n}), g(x_n^0), h_2, ..., h_{p_n}) = \Gamma_{n,C}(g(x_n^0), h_2, ..., h_{p_n})\\
&= \Gamma_{n,C} \circ \pi_{C,n}^{e\,\sharp}(f, g(x_n^0), h_2, h_3, ..., h_{p_n}) = \Gamma_{n,C} \circ \pi_{C,n}^{e\,\sharp} \circ \pi_{J,n}(f, g, h_2, ..., h_{p_n}).
\end{aligned}$$

In other words, for all $n \in \mathbb{Z}_+$,

$$\pi_{J,n+1}^{\sharp} \circ \Gamma_{n+1}^1 \circ \pi_{I,n+1}^{\sharp} = \Gamma_{n+1,C} \circ \pi_{C,n+1}^{\sharp} \circ \pi_{J,n+1}^{\sharp}. \tag{e 13.54}$$

One also checks that

$$\pi_{I,n+1}^{\sharp} \circ \Gamma_{n+1}^1 \circ \pi_{I,n+1}^{\sharp} = \pi_{I,n+1}^{\sharp}. \tag{e 13.55}$$

For any $f \in \mathrm{Aff}(T(A_n))$ with $\|f\| \le 1$, write $f_1 = \pi_{I,n}^{\sharp}(f)$, $f' = \varphi_{n,n+1}^{\sharp}(f)$ and $f_1' = \psi_{n,n+1}^{\sharp}(f_1)$. By the condition $c_{ij} > 13 \cdot 2^{2n}$ and the claim in the proof of Theorem 13.39, we have

$$\|\Gamma_{n+1}^2 \circ \pi_{X,n+1}^{e\,\sharp}(f_1') - f_1'\| < \frac{2}{2^{2n}}. \tag{e 13.56}$$

Using condition ($\Diamond\Diamond$), applying Lemma 13.19 to the map $A_{C,n} \to A_{C,n+1}$ as $C_n \to C_{n+1}$ (note that $\Gamma_{n,C}$ is the same as Γ_n in 13.18 and $\bar{\varphi}_{n,n+1} : A_{C,n} \to A_{C,n+1}$ is the same as $\varphi_{n,n+1} : C_n \to C_{n+1}$), one has

$$\|\Gamma_{n+1,C} \circ \pi^{e\,\sharp}_{C,n+1} \circ \bar{\varphi}^{\,\sharp}_{n,n+1}(\pi^{\sharp}_{J,n}(f)) - \bar{\varphi}^{\,\sharp}_{n,n+1}(\pi^{\,\sharp}_{J,n}(f))\| < \frac{2}{2^{2n}}. \tag{e 13.57}$$

Note that $\bar{\varphi}^{\,\sharp}_{n,n+1} \circ \pi^{\sharp}_{J,n} = \pi^{\,\sharp}_{J,n+1} \circ \varphi^{\,\sharp}_{n,n+1}$. By (e 13.54),

$$\|\pi^{\,\sharp}_{J,n+1} \circ \Gamma^{1}_{n+1} \circ \pi^{\sharp}_{I,n+1}(f') - \pi^{\,\sharp}_{J,n+1}(f')\| < \frac{2}{2^{2n}}. \tag{e 13.58}$$

Combining this with (e 13.55), we have

$$\left(\Gamma^{1}_{n+1} \circ \pi^{\,\sharp}_{I,n+1}(f') - f'\right)|_{T(A_{X,n+1})} = 0$$

and

$$\|\left(\Gamma^{1}_{n+1} \circ \pi^{\,\sharp}_{I,n+1}(f') - f'\right)|_{T(A_{C,n+1})}\| < \frac{2}{2^{2n}}.$$

Recall that $Sp(A_{X,n+1}) \cup Sp(A_{C,n+1}) = Sp(A_{n+1})$ (see (e 13.36)). By Lemma 2.16 of [66], we know that any extreme trace of A_{n+1} is induced by either an extreme trace of $A_{X,n+1}$ or an extreme trace of $A_{C,n+1}$. It follows that

$$\|\Gamma^{1}_{n+1} \circ \pi^{\sharp}_{I,n+1}(f') - f'\| < \frac{2}{2^{2n}}. \tag{e 13.59}$$

Consequently (applying (e 13.56) and (e 13.59)), we obtain

$$\begin{aligned}
\|\Gamma_{n+1} \circ \pi^{e\,\sharp}_{A_{n+1}}(f') - f'\| &= \|\Gamma^{1}_{n+1} \circ \Gamma^{2}_{n+1} \circ \pi^{e\,\sharp}_{X,n+1} \circ \pi^{\sharp}_{I,n+1}(f') - f'\| \\
&= \|\Gamma^{1}_{n+1} \circ \Gamma^{2}_{n+1} \circ \pi^{e\,\sharp}_{X,n+1} \circ \psi^{\sharp}_{n,n+1} \circ \pi^{\sharp}_{I,n}(f) - f'\| \\
= \quad & \|\Gamma^{1}_{n+1} \circ \Gamma^{2}_{n+1} \circ \pi^{\sharp}_{X,n+1} \circ \psi^{\sharp}_{n,n+1}(f_1) - f'\| = \|\Gamma^{1}_{n+1} \circ \Gamma^{2}_{n+1} \circ \pi^{e\,\sharp}_{X,n+1}(f_1') - f'\| \\
< \quad & \|\Gamma^{1}_{n+1}(f_1') - f'\| + \frac{2}{2^{2n}} \qquad \text{(by (e 13.56))} \\
= \quad & \|\Gamma^{1}_{n+1} \circ \psi^{\sharp}_{n,n+1} \circ \pi^{\sharp}_{I,n}(f) - f'\| + \frac{2}{2^{2n}} \\
= \quad & \|\Gamma^{1}_{n+1} \circ (\pi_{I,n+1})^{\sharp} \circ \varphi^{\sharp}_{n,n+1}(f) - f'\| + \frac{2}{2^{2n}} \\
= \quad & \|\Gamma^{1}_{n+1} \circ (\pi_{I,n+1})^{\sharp}(f') - f'\| + \frac{2}{2^{2n}} < \frac{2}{2^{2n}} + \frac{2}{2^{2n}}.
\end{aligned}$$

This proves the claim.

Using the claim, one obtains the following approximate intertwining diagram:

$$\begin{tikzcd}
\mathrm{Aff}(T(A_1)) \arrow[r, "\varphi^{\sharp}_{1,2}"] \arrow[d, bend right, "\pi^{e\,\sharp}_{A_1}"'] & \mathrm{Aff}(T(A_2)) \arrow[r, "\varphi^{\sharp}_{2,3}"] \arrow[d, bend right, "\pi^{e\,\sharp}_{A_2}"'] & \mathrm{Aff}(T(A_3)) \arrow[r] \arrow[d, bend right, "\pi^{e\,\sharp}_{A_3}"'] & \cdots\mathrm{Aff}(T(A)) \\
\mathrm{Aff}(T(F_1)) \arrow[r, "\psi^{q,\sharp}_{1,2}"] \arrow[u, bend right, "\Gamma_1"'] & \mathrm{Aff}(T(F_2)) \arrow[r, "\psi^{q,\sharp}_{2,3}"] \arrow[u, bend right, "\Gamma_2"'] & \mathrm{Aff}(T(F_3)) \arrow[r] \arrow[u, bend right, "\Gamma_3"'] & \cdots\mathrm{Aff}(T(F)).
\end{tikzcd}$$

Recall that $\pi^e_A = \pi^e_X \circ \varphi_I = \pi^e_C \circ \pi_J : A \to F$ is induced by $\{\pi^e_{A_n}\}$. Thus, the above approximately intertwining diagram shows that $\pi^{e\,\sharp}_A : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(F))$ is an isometric isomorphism of Banach spaces which also preserves the order and the inverse also preserves the order. By 2.2, this implies the inverse $(\pi^{e\,\sharp}_A)^{-1}$ induces an affine homeomorphism from Δ onto $T(A)$. Note that we also have

$$\pi^{e\,\sharp}_A \circ \rho_A = \rho_F \circ (\pi^e_A)_{*0}. \tag{e 13.60}$$

By 13.37, $(\pi^e_A)_{*0} = \pi_{G,H/\mathrm{Tor}(H)}$. Therefore $\pi^{e\,\sharp}_A \circ \rho_A = \rho : G \to \mathrm{Aff}(\Delta)$ is given by the part (2) of Remark 13.10. It then follows from the fact that $\pi^{e\,\sharp}_A$ is an affine homeomorphism that

$$\big((K_0(A), K_0(A)_+, [1_A]),\ K_1(A), T(A), r_{_A}\big) \cong \big((G, G_+, u),\ K, \Delta, r\big).$$

□

13.43. The algebra A of 13.37 and 13.41 is not simple, and so we need to modify the homomorphisms $\varphi_{n,n+1}$ to make the limit algebra simple. Let us emphasize that every homomorphism $\varphi : A_n \to A_{n+1}$ is completely determined by $\varphi_x = \pi_x \circ \varphi$ for each $x \in Sp(A_{n+1})$, where the map $\pi_x :\ A_{n+1} \to A_{n+1}|_x$ is the corresponding irreducible representation.

Note that from the definition of $\varphi : A_n \to A_{n+1}$ and from the assumption that $c_{ij} > 13$ for each entry of $\mathfrak{c} = (c_{ij})$, we know that for any $x \in Sp(A_{n+1})$,

$$Sp(\varphi_{n,n+1}|_x) \supset Sp(F_n) = (\theta_{n,1}, \theta_{n,1}, ..., \theta_{n,p_n}). \tag{e 13.61}$$

(See 13.2 and 13.1 for notation.)

To make the limit algebra simple, we will change $\varphi_{n,n+1}$ to $\xi_{n,n+1}$ so that the set $Sp(\xi_{n,m}|_x)$ is sufficiently dense in $Sp(A_n)$, for any $x \in Sp(A_m)$, provided m is large enough.

Write $Sp(A_n) = \bigcup_{j=1}^{l_n}(0,1)_{n,j} \cup X_n \cup S_n$, where $S_n = Sp(\bigoplus_{i=2}^{p_n} M_{[n,i]})$. Let $0 < d < 1/2$ and let $Z \subset Sp(A_n)$ be a subset.

Recall from 13.3 *that* Z *is* d*-dense in* $Sp(A_n)$ *if the following (sufficient) condition holds:* $Z \cap (0,1)_{n,j}$ *is* d*-dense in* $(0,1)_{n,j}$ *with the usual metric,* $Z \cap X_n$ *is* d*-dense in* X_n *with a given metric of* X_n*, and* $Sp(F_n) \subset Z$.

For each fixed n, let $\{\mathcal{F}_{n,k} : 1 \le k < \infty\}$ be an increasing sequence of finite subsets of the unit ball of A_n such that $\bigcup_{k=1}^{\infty}\mathcal{F}_{n,k}$ is dense in the unit ball of A_n. Fix a sequence of positive numbers $\eta_1 > \eta_2 > \cdots > \eta_n \cdots > 1$ such that $\prod_{n=1}^{\infty}\eta_n < 2$. Now we will change $\varphi_{n,n+1}$ to a unital injectve homomorphism $\xi_{n,n+1} : A_n \to A_{n+1}$ satisfying:

(i) $(\xi_{n,n+1})_{*i} = (\varphi_{n,n+1})_{*i}$.

(ii) $\|\varphi^{\sharp}_{n,n+1}(\hat{f})) - \xi^{\sharp}_{n,n+1}(\hat{f}))\| \le \frac{1}{2^{2n-2}}$ for all $f \in \mathcal{G}_n$, where $\hat{f} \in \operatorname{Aff} T(A_n)$ is defined by $\hat{f}(\tau) = \tau(f)$, $\mathcal{G}_1 = \mathcal{F}_{1,1}$, $\mathcal{G}_k = \mathcal{F}_{k,k} \cup (\bigcup_{i=1}^{k-1}\varphi_{i,k}(\mathcal{F}_{i,k}\cup\mathcal{G}_i)) \cup (\bigcup_{i=1}^{k-1}\xi_{i,k}(\mathcal{F}_{i,k}\cup\mathcal{G}_i))$, $k = 2, 3, ..., n$.

(iii) for any $y \in Sp(A_{X,n+1})\,(\subset Sp(A_{n+1}))$, $Sp(\xi_{n,n+1}|_y) \supset Sp(\varphi_{n,n+1}|_y)$; and for any $y \in Sp(A_{n+1})$, $Sp(\xi_{n,n+1}|_y) \supset Sp(F_n)$.

(iv) For any $\delta > 0$ and any finite subset $S \subset Sp(A_{C,n+1}) \subset Sp(A_{n+1})$, if $S \supset Sp(F_{n+1})$ and S is d-dense in $Sp(A_{C,n+1})$, then $\left(\bigcup_{s\in S}Sp(\xi_{n,n+1}|_s)\right) \cap Sp(A_{C,n})$ is $\eta_n\delta$-dense in $Sp(A_{C,n})$.

(v) $Y_n \cup T_n \cup Sp(F_n) \subset Sp(\xi_{n,n+1}|_{\theta_{n+1,2}})$ (see 13.31 for Y_n and T_n) and consequently, $Sp(\xi_{n,n+1}|_{\theta_{n+1,2}})$ is $1/n$- dense in $Sp(A_n)$, where

$$\theta_{n+1,2} \in Sp(F^2_{n+1}) \subset Sp(F_{n+1}) = \{\theta_{n+1,1}, \theta_{n+1,2}, ..., \theta_{n+1,p_{n+1}}\}$$

is the second point of $\{\theta_{n+1,1}, \theta_{n+1,2}, ..., \theta_{n+1,p_{n+1}}\}$ (note that $\theta_{n+1,1}$ is identified with the base point x^0_{n+1} of $[x^0_{n+1}, x^0_{n+1}+1] \vee X'_{n+1} = X_{n+1} = Sp(A^1_{X,n})$, and we do not want to modify this one).

(vi) For any $x \in Sp(A_{X,n+1})$ satisfying $x \ne \theta_{n+1,2}$,

$$\varphi_{n,n+1}|_x = \xi_{n,n+1}|_x. \tag{e 13.62}$$

In particular we have, (vi') $\varphi_{n,n+1}|_{Sp(A^1_{X,n+1})} = \xi_{n,n+1}|_{Sp(A^1_{X,n+1})}$, or equivalently, for any $x \in X_{n+1} = Sp(A^1_{X,n+1})$,

$$\varphi_{n,n+1}|_x = \xi_{n,n+1}|_x. \tag{e 13.63}$$

Moreover, we have the following remarks:

(vii) Suppose that (iii) holds, for $n = 1, 2, ..., k$. Let $i < k$. For any $y \in Sp(A_{X,k+1})$, we have $Sp(\xi_{i,k+1}|_y) \supset Sp(\varphi_{i,k+1}|_y)$. Furthermore, $Sp(F_i) \subset SP(\xi_{i,k+1}|_y)$ for all $y \in Sp(A_{k+1})$.

The proof of (vii): We will prove it by reverse induction. From the assumption that (iii) holds for $n = k$, we know that both conclusions above are true for $i = k$. Let us assume both conclusions hold for $i = j \le k$ (and $j \ge 2$), i.e., for any $y \in Sp(A_{X,k+1})$, $Sp(\xi_{j,k+1}|_y) \supset Sp(\varphi_{j,k+1}|_y)$; and for any $y \in Sp(A_{k+1})$, $Sp(F_j) \subset Sp(\xi_{j,k+1}|_y)$. We will prove both conclusions for $i = j-1$. Assume that $y \in Sp(A_{k+1})$. Choose any $z_0 \in Sp(F_j) \subset Sp(\xi_{j,k+1}|_y)$ (the inclusion is true by the induction assumption). By the second part of (iii) for $n = j-1$, we have $Sp(F_{j-1}) \subset Sp(\xi_{j-1,j}|_{z_0})$, and consequently $Sp(F_{j-1}) \subset Sp(\xi_{j-1,j}|_{z_0}) \subset \bigcup_{z\in Sp(\xi_{j,k+1}|_y)} Sp(\xi_{j-1,j}|_z) = Sp(\xi_{j-1,k+1}|_y)$. That is, the second part of (vii) is true for $i = j-1$.

For the first statement of the claim for $i = j-1$, let $y \in Sp(A_{X,k+1})$. By the induction assumption,

$$Sp(\xi_{j,k+1}|_y) \supset Sp(\varphi_{j,k+1}|_y). \tag{e 13.64}$$

Hence we have

$$\begin{aligned} Sp(\varphi_{j-1,k+1}|_y) &= \bigcup\nolimits_{z \in Sp(\varphi_{j,k+1}|_y)} Sp(\varphi_{j-1,j}|_z) \\ &\subset \bigcup\nolimits_{z \in Sp(\varphi_{j,k+1}|_y)} Sp(\xi_{j-1,j}|_z) \qquad \text{(by (e 13.43) and by (iii) for } n = j-1) \\ &\subset \bigcup\nolimits_{z \in Sp(\xi_{j,k+1}|_y)} Sp(\xi_{j-1,j}|_z) \qquad \text{(by (e 13.64))} \\ &= Sp(\xi_{j-1,k+1}|_y). \end{aligned}$$

(viii) Suppose that (iii) holds, for $n = 1, 2, ..., k$. Let $i < k$. Then

$$\Big(\bigcup_{y \in Y_n} Sp(\xi_{i,n}|_y)\Big) \cap X_i$$

is $1/n$-dense in X_i for all $0 < i \le n$ (see the construction of Y_n from 13.31). Furthermore, $\big(\bigcup_{y \in Y_n \cup Sp(F_n)} SP(\xi_{i,n}|_y)\big) \cap Sp(A_{X,i})$ is $1/n$-dense in $Sp(A_{X,i})$.

Note that $\bigcup_{y \in Y_n} Sp(\varphi_{i,n}|_y)\ \cap X_i$ is $1/n$-dense in X_i, and $\bigcup_{y \in Y_n \cup Sp(F_n)} Sp(\varphi_{i,n}|_y)(\subset Sp(A_{X,i}))$ is $1/n$-dense in $Sp(A_{X,i})$. So (viii) follows from (vii).

(ix) Suppose that (iii), (iv), (v), and (viii) hold for $n = 1, 2, ..., k$. Then, for any $1 \le i \le k$,

$$Sp(\xi_{i,k+1}|_{\theta_{k+1,2}}) \ \text{ contains } Sp(F_i) \text{ and is } 2/k\text{- dense in } Sp(A_i).$$

The proof of (ix): First note that, by (v) (holds for $1 \le n \le k$), $Y_k \cup T_k \cup Sp(F_k) \subset Sp(\xi_{k,k+1}|_{\theta_{k+1,2}})$. It follows that $Sp(\xi_{k,k+1}|_{\theta_{k+1,2}})$ is $1/k$-dense in $Sp(A_k)$. That is, the statement holds for $i = k$ as $1/k$-dense implies $2/k$-dense. By (viii), $\big(\bigcup_{y \in Y_k \cup Sp(F_k)} Sp(\xi_{i,k}|_y)\big) \cap Sp(A_{X,i})$ is $1/k$-dense in $Sp(A_{X,i})$. Hence

$$\begin{aligned} &Sp(\xi_{i,k+1}|_{\theta_{k+1,2}}) \cap Sp(A_{X,i}) \\ &\qquad \left(\supset \bigcup_{y \in Y_k \cup Sp(F_k)} Sp(\xi_{i,k}|_y) \cap Sp(A_{X,i})\right) \text{ is } 1/k\text{-dense in } Sp(A_{X,i}). \end{aligned} \tag{e 13.65}$$

From the fact that $T_k \cup Sp(F_k)$ is $1/k$-dense in $Sp(A_{C,k})$ (see the construction of T_k in 13.31), by applying (iv) to $S = T_k \cup Sp(F_k) \subset Sp(A_{C,k})$ (and (iii), respectively), we know that $\big(\bigcup_{s \in T_k \cup Sp(F_k)} Sp(\xi_{k-1,k}|_s)\big) \cap Sp(A_{C,k-1})$ is η_{k-1}/k-dense in $Sp(A_{C,k-1})$ (and contains $Sp(F_{k-1})$, respectively). Hence $Sp(\xi_{k-1,k+1}|_{\theta_{k+1,2}}))\cap$

$Sp(A_{C,k-1})$ is η_{k-1}/k-dense in $Sp(A_{C,k-1})$. Now applying (iv) to $S' = \big(\cup_{t\in T_k\cup Sp(F_k)} Sp(\xi_{k-1,k}|_t)\big) \cap Sp(A_{C,k-1})$, we have $\big(\bigcup_{s\in S'} Sp(\xi_{k-2,k-1}|_s)\big) \cap Sp(A_{C,k-1})$ is $(\eta_{k-2}\eta_{k-1})/k$-dense in $Sp(A_{C,k-2})$. Since $S' \subset Sp(\xi_{k-1,k+1}|_{\theta_{k+1,2}})) \cap Sp(A_{C,k-1})$, $Sp(\xi_{k-2,k+1}|_{\theta_{k+1,2}})) \cap Sp(A_{C,k-2})$ is $(\eta_{k-2}\eta_{k-1})/k$-dense in $Sp(A_{C,k-2})$. Similarly, by induction (reversely), one gets that $Sp(\xi_{i,k+1}|_{\theta_{k+1,2}})) \cap Sp(A_{C,i})$ is $(\eta_i \cdots \eta_{k-2}\eta_{k-1})/k$-dense in $Sp(A_{C,i})$. Note that $\prod_{n=i}^{k-1} \eta_n \le \prod_{n=1}^{\infty} \eta_n < 2$. So $Sp(\xi_{i,k+1}|_{\theta_{k+1,2}})) \cap Sp(A_{C,i})$ is $2/k$-dense in $Sp(A_{C,i})$. Combining with (e 13.65), we get the desired conclusion.

(x): Suppose that $\varphi_{n,n+1}$ are constructed which satisfy (i)–(vi) for $n = 1, 2, ..., k+2$. For $i < k$, then, for any $y \in Sp(A_{k+2})$, we have

$$Sp(\xi_{i,k+2}|_y) \text{ is } 2/k\text{−dense in } Sp(A_i). \tag{e 13.66}$$

To see this, we note, by (iii), $\theta_{k+1,2} \in Sp(F_{k+1}) \subset Sp(\xi_{k+1,k+2}|_y)$ for any $y \in Sp(A_{k+2})$. Therefore $Sp(\xi_{i,k+2}|_y) \supset Sp(\xi_{i,k+1}|_{\theta_{k+1,2}})$. Combining this with (ix), we obtain (x).

Property (vi') will be used (together with 13.38) to prove that the limit algebra A has the property that $A \otimes U \in \mathcal{B}_0$ for any UHF-algebra U.

13.44. Suppose that we have constructed the finite sequence

$$A_1 \xrightarrow{\xi_{1,2}} A_2 \xrightarrow{\xi_{2,3}} \cdots \xrightarrow{\xi_{n-1,n}} A_n,$$

in such a way that for every $i \le n-1$, $\xi_{i,i+1}$ satisfies conditions (i) −(vi) (with i in place of n) in 13.43. We will construct the map $\xi_{n,n+1}: A_n \to A_{n+1}$. Write

$$A_n = \big\{(f,g) \in C([0,1],E_n) \oplus A_{X,n};\ f(0) = \beta_{n,0}(\pi^e_{X,n}(g)), f(1) = \beta_{n,1}(\pi^e_{X,n}(g))\big\}$$

with

$$A_{X,n} = P_n M_\infty(C(X_n))P_n \oplus \bigoplus_{i=2}^{p_n} M_{[n,i]},$$

where $\mathrm{rank}(P_n) = [n,1]$ (recall that $\pi^e_{X,n}: A_{X,n} \to F_n$ is from 13.29).

Also recall that we denote $t \in (0,1)_{n,j} \subset Sp\big(C([0,1],E^j_n)\big)$ by $t_{n,j}$ to distinguish the spectra from different direct summands of $C([0,1],E_n)$, and we denote $0 \in [0,1]_{n,j}$ by $0_{n,j}$, and $1 \in [0,1]_{n,j}$ by $1_{n,j}$. Note that $0_{n,j}$ and $1_{n,j}$ do not correspond to single irreducible representations. In fact, $0_{n,j}$ corresponds to the direct sum of irreducible representations for the set

$$\Big\{\theta_{n,1}^{\sim b^n_{0,j1}}, \theta_{n,2}^{\sim b^n_{0,j2}}, ...,\ \theta_{n,p_n}^{\sim b^n_{0,jp_n}}\Big\},$$

and $1_{n,j}$ corresponds to the set

$$\Big\{\theta_{n,1}^{\sim b^n_{1,j1}}, \theta_{n,2}^{\sim b^n_{1,j2}}, ...,\ \theta_{n,p_n}^{\sim b^n_{1,jp_n}}\Big\}.$$

Again recall from 13.31, $T_n \subset Sp(A_n)$ is defined by

$$T = \left\{ (\frac{k}{n})_{n,j};\ j = 1, 2, ..., l_n;\ k = 1, 2, ..., n-1 \right\}.$$

Recall that in the condition $(\Diamond\Diamond)_1$, $L_n = l_n \cdot (n-1) + L_{n,Y} = \#(T_n \cup Y_n)$ and $M = \max\{b^n_{0,ij} : i = 1, 2, ..., p_n;\ j = 1, 2, ..., l_n\}$, where we write $Y_n = \{y_1, y_2, ..., y_{L_{n,Y}}\} \subset X_n$.

13.45. First we define a unital homomorphism, $\xi_X : A_n \to A_{X,n+1}$. Denote, only in this subsection, by $\Pi' : A_{X,n+1} \to \bigoplus_{i \neq 2} A^i_{X,n+1} (= A^1_{X,n+1} \oplus \bigoplus_{i=3}^{p_{n+1}} F^i_{n+1})$ and $\Pi^{(2)} : A_{X,n+1} \to A^2_{X,n+1} (= F^2_{n+1})$ the quotient maps. Define $\xi'_X : A_n \to \bigoplus_{i \neq 2} A^i_{X,n+1}$ by $\xi'_X = \Pi' \circ \circ \pi_{I,n+1} \circ \varphi_{n,n+1}$. We note that, for $a \in A_n$ such that $\pi^1_{X,n}(a) \neq 0$, by the definition of $\psi_{n,n+1}$, $\xi'_X(a) \neq 0$.

Note that, by (a) of 13.31, $\varphi_{n,n+1}(I_n + J_n) \subset I_{n+1} + J_{n+1}$. Therefore,

$$SP(\varphi_{n,n+1}|_{\theta_{n+1,2}}) = \left\{\theta_{n,1}^{\sim c_{21}}, \theta_{n,2}^{\sim c_{22}}, ..., \theta_{n,p_n}^{\sim c_{2p_n}}\right\}. \tag{e 13.67}$$

(Here c_{jk} means $c^{n,n+1}_{jk}$.) Let $Y_n = \{y_1, y_2, ..., y_{L_{n,Y}}\}$ be as in 13.31. Since X_n is path connected, for each i, there is a continuous simple path $\{y_i(s) : s \in [0,1]\} \subset X_n$ such that $y_i(0) = y_i$ and $y_i(1) = x^0_n$. Note that $P_n(y_i) M_\infty P_n(y_i)$ can be identified with $M_{[n,1]} = F^1_n$. We will also identify $P_n(y_i(s)) M_\infty P_n(y_i(s))$ with $M_{[n,1]} = P_n(x^0_n) M_\infty P_n(x^0_n) = A_n|_{\theta_{n,1}} = F^1_n$—such an identification could be chosen to be continuously depending on s. (Here, we only use the fact that any projection (or vector bundle) over the interval is trivial to make such an identification. Since the projection P_n itself may not be trivial, it is possible that the paths for different y_i and y_j $(i \neq j)$ may intersect at $y_i(s_1) = y_j(s_2)$ and we may use a different identification of $P_n(y_i(s_1)) M_\infty P_n(y_i(s_2)) = P_n(y_j(s_2)) M_\infty P_n(y_j(s_2))$ with $M_{[n,1]}$ for i and j.) When we talk about $f(y_i(s))$ later, we will consider it to be an element of $M_{[n,1]}$ (rather than of $P_n(y_i(s))$ $M_\infty P_n(y_i(s))$). Define (recall $\pi_{I,n} : A_n \to A_{X,n}$ is the quotient map)

$$\Omega_{Y,s}(f) = \mathrm{diag}(\pi_{I,n}(f)(y_1(s)), ..., \pi_{I,n}(f)(y_{L_{n,Y}}(s)) \in M_{L_{n,Y}[n,1]}$$

for all $f \in A_n$ and $s \in [0,1]$. Note that $\Omega_{Y,s}(\mathbf{1}_{A_n}) = \mathbf{1}_{M_{L_{n,Y}[n,1]}}$ is independent of s and that $\Omega_{Y,1}(f) = (\theta_{n,1}^{\sim L_{n,Y}})(f)$ for all $f \in A_n$.

For each $(\frac{h}{n})_{n,j} \in T_n$, where $1 \leq h \leq n-1$ (see 13.31), there exists also a continuous path $\{g_{n,j,h}(s) : s \in [0,1]\} \subset [0,1]_{n,j}$ such that $g_{n,j,h}(0) = (\frac{h}{n})_{n,j}$ and $g_{n,j,h}(1) = 0_{n,j}$. Define, for each $f \in A_n$ and $s \in [0,1]$,

$$\begin{aligned}\Omega_{I,s}(f) \quad &= \quad \bigoplus_{j=1}^{l_n} \mathrm{diag}(\pi_{I,n}(f)(g_{n,j,1}(s)), \pi_{I,n}(f)(g_{n,j,2}(s)), ..., \pi_{I,n}(f)(g_{n,j,n-1}(s))) \\ &\qquad \in M_{\sum_{j=1}^{l_n}(n-1)\{n,j\}}.\end{aligned}$$

(Recall that $\{n,j\} = \sum_{k=1}^{p_n} b^n_{0,jk} = \sum_{k=1}^{p_n} b^n_{1,jk}$ is the rank of the representation of A_n corresponding to any point $t \in [0,1]_{n,j}$ as $E_n^j = M_{\{n,j\}}$.) Note that $\Omega_{I,s}(\mathbf{1}_{A_n}) \in M_{\sum_{j=1}^{l_n}(n-1)\{n,j\}}$ is independent of s. Then

$$\Omega_{I,0}(f) = \bigoplus_{j=1}^{l_n} \mathrm{diag}(\pi_{I,n}(f)((\frac{1}{n})_{n,j}, \pi_{I,n}(f)(\frac{2}{n})_{n,j}, ..., \pi_{I,n}(f)((\frac{(n-1)}{n})_{n,j})),$$

and, as $0_{n,j} = \left\{\theta_{n,1}^{\sim b^n_{0,j1}}, \theta_{n,2}^{\sim b^n_{0,j2}}, ..., \theta_{n,p_n}^{\sim b^n_{0,jp_n}}\right\}$,

$$SP(\Omega_{I,1}) = \left\{\theta_{n,1}^{\sim b_1}, \theta_{n,2}^{\sim b_2}, ..., \theta_{n,p_n}^{\sim b_{p_n}}\right\},$$

where $b_k = (n-1)(\sum_{j=1}^{l_n} b^n_{n,jk})$, $k = 1, 2, ..., p_n$. Put

$$\begin{aligned} a_1 &= c_{21} - L_{n,Y} - \left(\sum_{j=1}^{l_n} b^n_{0,j1}\right)(n-1), \\ a_k &= c_{2k} - \left(\sum_{j=1}^{l_n} b^n_{0,jk}\right)(n-1),\ k = 2, 3, ..., p_n. \end{aligned}$$

Let $\xi'_{X,s}$ be the finite dimensional representation of A_n defined by, for each $f \in A$ and $s \in [0,1]$,

$$\begin{aligned} \xi'_{X,s}(f) &= \mathrm{diag}(\theta_{n,1}^{\sim a_1}(f), \theta_{n,2}^{\sim a_2}(f), ..., \theta_{n,p_n}^{\sim a_{p_n}}(f)) \\ &\quad \oplus \Omega_{X,s}(f) \oplus \Omega_{I,s}(f). \end{aligned} \tag{e 13.68}$$

In particular,

$$SP(\xi'_{X,0}) = \left\{\theta_{n,1}^{\sim a_1}, \theta_{n,2}^{\sim a_2}, ..., \theta_{n,p_n}^{\sim a_{p_n}}\right\} \cup T_n \cup Y_n, \text{ and} \tag{e 13.69}$$

$$SP(\xi'_{X,1}) = \left\{\theta_{n,1}^{\sim c_{21}}, \theta_{n,2}^{\sim c_{22}}, ..., \theta_{n,p_n}^{\sim c_{2p_n}}\right\} = SP(\varphi_{n,n+1}|_{\theta_{n+1,2}}). \tag{e 13.70}$$

Since $\xi'_{X,0}$ is homotopic to $\xi'_{X,1}$, and $\xi'_{X,s}(1_{A_n}) = \xi'_{X,1}(1_{A_n})$ for all $s \in [0,1]$, up to unitary equivalence, we may view $\{\xi'_{X,s} : s \in [0,1]\}$ as a continuous path of unital homomorphisms from A_n into $A^2_{X,n+1} = F^2_{n+1}$. View

$$e_u := \mathrm{diag}(\theta_{n,1}^{\sim a_1}(1_{A_n}), \theta_{n,2}^{\sim a_2}(1_{A_n}), ..., \theta_{n,p_n}^{\sim a_{p_n}}(1_{A_n})) \text{ and} \tag{e 13.71}$$

$$e_c := \Omega_{Y,s}(1_{A_n}) \oplus \Omega_{I,s}(1_{A_n}) \tag{e 13.72}$$

as two projections in F^2_{n+1}, which do not depend on s. From $(\diamondsuit\diamondsuit)_1$ of 13.31, we know that

$$a_i \geq \frac{2^{2n}-1}{2^{2n}} c_{2i}. \tag{e 13.73}$$

Then, for the tracial state τ of F^2_{n+1},

$$\tau(e_c) < 1/2^{2n} \text{ and } \tau(e_u) > 1 - (1/2^{2n}). \tag{e 13.74}$$

Define $\xi_{X,s} : A_n \to A_{X,n+1}$ by $\xi_{X,s} = (\xi'_X \oplus \xi'_{X,s})$. By replacing $\xi_{X,s}$ by $\mathrm{Ad}\, U \circ \xi_{X,s}$ for a suitable unitary path $U_s = \oplus U^j_s \in \oplus A^j_{X,n+1}$ (with $U^j_s = 1$ if $j \neq 2$), we may assume that

$$\xi_{X,1} = \pi_{I,n+1} \circ \varphi_{n,n+1}. \tag{e 13.75}$$

Since $U^j_s = 1$ for $j \neq 2$, from the definition of ξ'_X, we get $\xi_{X,s}|_x = \varphi_{n,n+1}|_x$ for all $x \in Sp(A_{X,n+1})$ with $x \neq \theta_{n+1,2}$. It follows that

$$\xi_{X,1}(f) - \xi_{X,s}(f) = e_c(\xi_{X,1}(f) - \xi_{X,s}(f)) \tag{e 13.76}$$

for all $s \in [0,1]$ and $f \in A_n$. Define $\xi_X := \xi_{X,0}$. Therefore ξ_X is homotopic to $\pi_{I,n+1} \circ \varphi_{n,n+1}$ and $\xi_X|_x = \varphi_{n,n+1}|_x$ for all $x \in Sp(A_{X,n+1})$ with $x \neq \theta_{n+1,2}$. Since ξ_X is homotopic to $\pi_{I,n} \circ \varphi_{n,n+1}$, $(\xi_X)_{*i} = (\pi_{I,n} \circ \varphi_{n,n+1})_{*i}$, $i = 0, 1$. From (e 13.76) and (e 13.74), we also have

$$\begin{aligned} &|\tau(\pi_{I,n+1} \circ \varphi_{n,n+1}(f)) - \tau(\xi_X(f))| \\ &< \tau(e_c)\|f\| \leq (1/2^{2n-1})\|f\| \text{ for all } \tau \in T(A_{X,n+1}). \end{aligned} \tag{e 13.77}$$

From the first paragraph of this section and by (e 13.68), if $a \in A_n \setminus I_n$, then $\xi'_X(a) \neq 0$.

13.46. In 13.45, we have defined a unital homomorphism $\xi_X : A_n \to A_{X,n+1}$. In this subsection, we define the map $\xi_{n,n+1}$. We first define a unital homomorphism $\xi_E : A_n \to C([0,1], E_{n+1})$. Define

$$\xi_E|_{0_{n+1,j}} = \pi^j \circ \beta_{n+1,0} \circ \pi^e_{X,n+1} \circ \xi_X : \ A_n \to E^j_{n+1} \text{ and} \tag{e 13.78}$$

$$\xi|_E|_{1_{n+1,j}} = \pi^j \circ \beta_{n+1,1} \circ \pi^e_{X,n+1} \circ \xi_X : \ A_n \to E^j_{n+1}, \tag{e 13.79}$$

where $\pi^j : \ E_{n+1} \to E^j_{n+1}$. Now we need to connect $\xi_E|_{0_{n+1,j}}$ and $\xi_E|_{1_{n+1,j}}$ to obtain ξ_E.

Fix a finite subset $\mathcal{G}_n \subset A_n$ in (ii) of 13.43, There is $\delta > 0$ such that

$$\begin{aligned} &\|\bar\varphi_{n,n+1} \circ \pi_{J,n}(f)(s) \\ &-\bar\varphi_{n,n+1} \circ \pi_{J,n}(f)(s')\| < 1/2^{2n+1} \text{ for all } f \in \mathcal{G}_n, \end{aligned} \tag{e 13.80}$$

provided $|s - s'| < 3\delta$ and $s, s' \in [0,1]$, and such that $1/(1-2\delta) < \eta_n$. Let $h_0(t) = t/\delta$ for $t \in [0,\delta]$, $h_z(t) = (t-\delta)/(1-2\delta)$ for $t \in [\delta, 1-\delta]$ and $h_1(t) = (1-t)/\delta$ for $t \in [1-\delta, 1]$. Define, for each $f \in A_n$,

$$\begin{aligned} &\xi_E(f)|_{[0,\delta]_{n+1,j}}(t) \\ &= \pi^j \circ \beta_{n+1,0} \circ \pi^e_{X,n+1} \circ \xi_{X,h_0(t)}(f) \text{ for all } t \in [0,\delta], \end{aligned} \tag{e 13.81}$$

$$\begin{aligned}\xi_E(f)|_{(\delta,1-\delta]_{n+1,j}}(t) \\ = (\pi^j \circ \bar{\varphi}_{n,n+1} \circ \pi_{J,n}(f))(h_z(t)) \text{ for all } t \in (\delta, 1-\delta] \text{ and}\end{aligned} \tag{e 13.82}$$

$$\begin{aligned}\xi_E(f)|_{(1-\delta,1]_{n+1,j}}(t) \\ = \pi^j \circ \beta_{n+1,1} \circ \pi^e_{X,n+1} \circ \xi_{X,h_1(t)}(f) \text{ for all } t \in (1-\delta, 1],\end{aligned} \tag{e 13.83}$$

where $\bar{\varphi}_{n,n+1} : A_{C,n} \to A_{C,n+1} \subset C([0,1], E_{n+1})$ is the injective homomorphism given by 13.37. In particular, using the fact that $\varphi_{n,n+1}$ is a map from A_n to A_{n+1}, and (e 13.75) (note that $h_0(\delta) = 1 = h_1(1-\delta)$), for all $f \in A_n$, we have

$$\begin{aligned}&\pi^j \circ \bar{\varphi}_{n,n+1} \circ \pi_{J,n}(f)(h_z(\delta)) \\ &= (\pi^j \circ \bar{\varphi}_{n,n+1} \circ \pi_{J,n}(f))(0) \\ &= \pi^j \circ \beta_{n+1,0} \circ \pi^e_{X,n+1} \circ \varphi_{n,n+1}(f) \\ &= \pi^j \circ \beta_{n+1,0} \circ \pi^e_{X,n+1} \circ \xi_{X,h_0(\delta)}(f), \text{ and}\end{aligned} \tag{e 13.84}$$

$$\begin{aligned}(\pi^j \circ \bar{\varphi}_{n,n+1} \circ \pi_{J,n}(f))(h_z(1-\delta)) &= (\pi^j \circ \bar{\varphi}_{n,n+1} \circ \pi_{J,n}(f))(1) \\ &= \pi^j \circ \beta_{n+1,1} \circ \pi^e_{X,n+1} \circ \varphi_{n,n+1}(f) \\ &= \pi^j \circ \beta_{n+1,1} \circ \pi^e_{X,n+1} \circ \xi_{X,h_1(1-\delta)}(f).\end{aligned}$$

Thus, ξ_E defines a unital homomorphism from A_n into $C([0,1], E_{n+1})$ which is injective on I_n. Finally, define

$$\begin{aligned}&\xi_{n,n+1} : A_n \to A_{n+1} \\ &= C([0,1], E_{n+1}) \oplus_{\beta_{n+1,0}\circ\pi^e_{X,n+1}, \beta_{n+1,1}\circ\pi^e_{X,n+1}} A_{X,n+1}\end{aligned}$$

by

$$\xi_{n,n+1}(f) = (\xi_E(f), \xi_X(f)) \text{ for all } f \in A_n.$$

By (e 13.78) and (e 13.79), this is indeed a unital homomorphism from A_n into A_{n+1} (see (e 13.41) and (e 13.42)). Since ξ_E is injective, by the end of 13.45, $\xi_{n,n+1}$ is injective. Note that, by (the end of) 13.45, $(\pi_{I,n+1} \circ \xi_{n,n+1})_{*i} = (\xi_X)_{*i} = (\pi^e_{I,n+1} \circ \varphi_{n,n+1})_{*i}$, $i = 0, 1$. It follows from part (1) of 13.30 (also see the last sentence of the proof of part (1)) that $(\pi_{i,n+1})_{*i} : K_i(A_{n+1}) \to K_i(A_{X,n+1})$ is injective. Therefore $(\xi_{n,n+1})_{*i} = (\varphi_{n,n+1})_*$. So (i) of 13.43 holds. We have already mentioned that $\xi_X|_x = \varphi_{n,n+1}|_x$ for all $x \in Sp(A_{X,n+1})$ with $x \neq \theta_{n+1,2}$ in 13.45, and so (vi) of 13.43 holds. By the presence of the representations corresponding to Y_n, T_n, and $Sp(F_n)$ in (e 13.68), we also see that (v) holds.

For $y \in Sp(A_{X,n+1})$ and $y \neq \theta_{n+1,2}$, by (v), $Sp(\varphi_{n,n+1}|_y) = Sp(\xi_{n,n+1}|_y)$. For $y = \theta_{n+1,2}$, by (e 13.67), (e 13.68), and (e 13.73), if $x \in Sp(A_n)$ and $x \in$

$Sp(\varphi_{n,n+1}|_y)$, then $x \in Sp(\xi_{n,n+1}|_y)$.That is, the first part of (iii) of 13.43 holds. Moreover, $Sp(F_n) \subset Sp(\xi_{n,n+1}|_y)$ for all $y \in SP(A_{X,n+1})$. Therefore, $Sp(F_n) \subset Sp(\xi_{n,n+1}|_y)$ for all $y \in [0,\delta)_{n+1,j} \cup (1-\delta,1)_{n+1,j}$. By (c) of 13.31 and (5) of 13.15, $Sp(F_n) \subset SP(\bar{\varphi}_{n,n+1}|_s)$ for each $s \in (0,1)_{n+1,j}$. It follows from (e 13.84) that $Sp(F_n) \subset SP(\xi_{n,n+1}|_s)$ for all $s \in [\delta, 1-\delta]_{n+1,j}$. Therefore, $Sp(F_n) \subset SP(\xi_{n,n+1}|_y)$ for all $y \in Sp(A_{n+1})$. This proves the second part of (iii).

Claim: If a finite subset $Z \subset (0,1)$ is such that $Z \cup \{0,1\}$ is d-dense in $[0,1]$, then the finite subset $h_z(Z \cap (\delta, 1-\delta)) \cup \{0,1\}$ is $\eta_n d$-dense in $[0,1]$

PROOF. Let us order the set $Z = \{z_j\}_{j=1}^k$ as $0 < z_1 < z_2 < \cdots < z_k < 1$. Then $Z \cup \{0,1\}$ is d-dense in $[0,1]$ if and only if

$$z_1 < 2d, \quad 1 - z_k < 2d, \quad \text{and} \quad z_{j+1} - z_j < 2d \text{ for all } j = 1, 2, \cdots, k-1.$$

For convenience, let $z_0 = 0$ and $z_{k+1} = 1$. Let j_1 be the smallest index such that $z_{j_1} > \delta$ and j_2 be the largest index such that $z_{j_2} < 1 - \delta$. Then $Z \cap (\delta, 1-\delta) = \{z_j\}_{j=j_1}^{j_2}$ and the set $h_z(Z \cap (\delta, 1-dt)) \cup \{0,1\}$ can be listed as

$$h_z(z_{j_1-1}) = 0 < h_z(z_{j_1}) < h_z(z_{j_1+1}) < \cdots < h_z(z_{j_2}) < 1 = h_z(z_{j_2+1}).$$

The claim follows from the fact that

$$h_z(z_{j+1}) - h_z(z_j) \le (z_{j+1} - z_j)/(1-2\delta) < 2d/(1-2\delta) < 2\eta_n d.$$

□

Let us verify that (iv) of 13.43 holds. For any $d > 0$, let $S \subset Sp(A_{C,n+1})$ be d-dense in $Sp(A_{C,n+1})$, and satisfy that $S \supset Sp(F_{n+1})$ (note that $Sp(F_{n+1})$ is a subset of $Sp(A_{C,n+1})$). Let $Z = S \cap (0,1)_{n+1,1}$ and $Z_0 = S \cap (\delta, 1-\delta)_{n+1,1}$. It follows from the d-density of S in $Sp(A_{C,n+1})$, regarding $Z_0 \subset Z$ as subsets of the open interval $(0,1)$, that $Z \cup \{0,1\}$ is d-dense in $[0,1]$. Hence by the claim, $h_z(Z_0) \cup \{0,1\}$ is $\eta_n d$-dense in $[0,1]$. Then, by (c) of 13.31 and (6) of 13.15 (applied to all indices $i_0 = 1, 2, \cdots l_n$ and $j_0 = 1$), we know that $\left(\bigcup_{z \in h(Z_0)} Sp(\varphi_{n,n+1}|_z) \cap Sp(A_{C,n})\right) \cup Sp(F_n)$ is $\eta_n d$-dense in $Sp(A_{C,n})$. By (e 13.81), $\bigcup_{z \in Z_0} Sp(\xi_{n,n+1}|_z) = \bigcup_{z \in h(Z_0)} Sp(\varphi_{n,n+1}|_z)$. By (iii) of 13.43, we know that $\bigcup_{s \in S} Sp(\xi_{n,n+1}|_s) \supset Sp(F_n) \cup \left(\bigcup_{z \in Z_0} Sp(\xi_{n,n+1}|_z)\right)$. It follows that $\bigcup_{s \in S} Sp(\xi_{n,n+1}|_s) \cap Sp(A_{C,n})$ is $\eta_n d$-dense in $Sp(A_{C,n})$. Hence (iv) of 13.43 hold.

It remains to check (ii) of 13.43 holds. Note that, by 13.45, $Sp(\xi_{n,n+1}|_y) = Sp(\varphi_{n,n+1}|y)$ for all $y \in Sp(A^j_{X,n+1})$ for $j \neq 2$. Note that, for the tracial state t of E^j_{n+1}, the map

$$a \mapsto \frac{t(\pi^j \circ \beta_{n+1,0}(a))}{t(\pi^j \circ \beta_{n+1,0}(1_{F^2_{n+1}}))} \quad \text{for} \quad a \in F^2_{n+1}$$

is a tracial state of F_{n+1}^2. It follows from (e 13.74) that

$$\frac{t(\pi^j\circ\beta_{n+1,0}(e_c))}{t(\pi^j\circ\beta_{n+1,0}(1_{F_{n+1}^2}))}<1/2^{2n}. \tag{e 13.85}$$

Define $e'_{c,j}=\pi^j\circ\beta_{n+1,0}(e_c))$. Then $t(e'_c)<1/2^{2n}$ for $t\in E_{n+1}^j$. It follows that, for each $t\in[0,\delta]_{n+1,j}$, and $\tau\in T(E_{n+1}^j)$ (see also (e 13.76)),

$$\begin{aligned}&|\tau(\xi_{n,n+1}(f)(s))-\tau(\varphi_{n,n+1}(f)(0_{n+1,j}))|\\ &\quad=\ |\tau(e'_{c,j}((\xi_{n,n+1}(f)(s))-\varphi_{n,n+1}(f)(0_{n+1,j})))|<(1/2^{2n-1})\|f\|\\ &\qquad\text{for all } f\in A_n.\end{aligned} \tag{e 13.86}$$

Exactly the same computation shows that, for all $s\in[1-\delta,1]_{n+1,j}$ and for all $\tau\in T(E_{n+1}^j)$,

$$\begin{aligned}&|\tau(\xi_{n,n+1}(f)(s))-\tau(\varphi_{n,n+1}(f)(1_{n+1,j}))|\\ &<(1/2^{2n-1})\|f\| \text{ for all } f\in A_n.\end{aligned} \tag{e 13.87}$$

Note that $|s-h_z(s)|\le\delta<3\delta$ for all $s\in[0,1]$. By the choice of δ,

$$\begin{aligned}&\|\xi_{n,n+1}(f)(s)-\varphi_{n,n+1}(f)(s)\|\\ &\quad=\ \|(\bar\varphi_{n,n+1}\circ\pi_{J,n}(f))(h_z(s))-(\bar\varphi_{n,n+1}\circ\pi_{J,n}(f))(s)\|\\ &\quad<1/2^{2n+1} \text{ for all } f\in\mathcal{G}_n\end{aligned} \tag{e 13.88}$$

for all $s\in[\delta,1-\delta]_{n+1,j}$. Again, by the choice of δ, we also have, for all $f\in\mathcal{G}_n$,

$$\begin{aligned}&\|\varphi_{n,n+1}(f)(s)-\varphi_{n,n+1}(f)(0_{n+1,j})\|\\ &<1/2^{2n+1} \text{ for all } s\in[0,\delta]_{n+1,j} \text{ and}\end{aligned} \tag{e 13.89}$$

$$\begin{aligned}&\|\varphi_{n,n+1}(f)(s)-\varphi_{n,n+1}(f)(1_{n+1,j})\|\\ &<1/2^{2n+1} \text{ for all } s\in[1-\delta,1]_{n+1,j}.\end{aligned} \tag{e 13.90}$$

Combining (e 13.86), (e 13.89), (e 13.87), (e 13.90), (e 13.88), and (e 13.77),

$$\begin{aligned}&\|\xi_{n,n+1}^\sharp(\hat f)-\varphi_{n,n+1}^\sharp(\hat f)\|\\ &<1/2^{2n-1}+1/2^{2n+1}<1/2^{2n-2} \text{ for all } f\in\mathcal{G}_n.\end{aligned} \tag{e 13.91}$$

This proves (ii). By induction, this completes the construction of ξ_n.

13.47. Let $B = \lim(A_n, \xi_{n,n+1})$. Recall that $A = \lim(A_n, \varphi_{n,n+1})$. By (i) of 13.43, ${\xi_{n,n+1}}_{*i} = {\varphi_{n,n+1}}_{*i}$, $i = 0, 1$. It follows that

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

For each n and $\sigma > 0$, choose $m > n + 1$ with $2/m - 2 < \sigma$. Then, by (x) of 13.43, $Sp(\xi_{n,m}|_x)$ is $2/(m-2)$-dense in $Sp(A_n)$ for any $x \in Sp(A_m)$. It follows from Proposition 13.5 (see also the end of 13.3) that B is a simple C^*-algebra.

We will show, in fact,

$$\begin{aligned}(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B), r_B)\\ = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A).\end{aligned} \tag{e 13.92}$$

Consider the following non-commutative diagram:

$$\begin{tikzcd}
\mathrm{Aff}(T(A_1)) \arrow[r, "\varphi^\sharp_{1,2}"] \arrow[d, bend right, "\mathrm{id}^\sharp_1"'] & \mathrm{Aff}(T(A_2)) \arrow[r, "\varphi^\sharp_{2,3}"] \arrow[d, bend right, "\mathrm{id}^\sharp_2"'] & \mathrm{Aff}(T(A_3)) \arrow[r] \arrow[d, bend right, "\mathrm{id}^\sharp_3"'] & \cdots \mathrm{Aff}(T(A)) \\
\mathrm{Aff}(T(A_1)) \arrow[r, "\xi^\sharp_{1,2}"] \arrow[u, bend right, "\iota^\sharp_1"'] & \mathrm{Aff}(T(A_2)) \arrow[r, "\xi^\sharp_{2,3}"] \arrow[u, bend right, "\iota^\sharp_2"'] & \mathrm{Aff}(T(A_3)) \arrow[r] \arrow[u, bend right, "\iota^\sharp_3"'] & \cdots \mathrm{Aff}(T(B)),
\end{tikzcd}$$

where $\mathrm{id}_k : A_k \to A_k$ and $\iota_k : A_k \to A_k$ are both the identity maps—but we write them differently as they come from two different systems. Recall that, for each n, $\{\mathcal{F}_{n,k} : k \geq 1\}$ is an increasing sequence of finite subsets of A_n whose union is dense in the unit ball of A_n. Recall also that $\mathcal{G}_1 = \mathcal{F}_{1,1}$, $\mathcal{G}_n = \mathcal{F}_{n,n} \cup \bigcup_{i=1}^{n-1} \varphi_{i,n}(\mathcal{F}_{i,n} \cup \mathcal{G}_i)\} \cup \bigcup_{i=1}^{n-1} \xi_{i,n}(\mathcal{F}_{i,n} \cup \mathcal{G}_i)$. For each i and each $x \in A_i$, and for each $n > i$, there exist $j > i$ and $y \in \mathcal{F}_{i,j}$ such that $\|x - y\| < 1/2^{n+1}$. Then $\|\varphi_{i,j}(x) - \varphi_{i,j}(y)\| < 1/2^{n+1}$. Note $\varphi_{i,j}(y) \in \mathcal{G}_j$, Let $n_0 = \max\{j, n\}$. Denote by $\hat{x} \in \mathrm{Aff}(T(A_i))$ the function $\hat{x}(\tau) = \tau(x)$ for all $\tau \in T(A_i)$. Put $z = \varphi^\sharp_{i,n_0}(\hat{y})$. By (ii) of 13.43, for any $m > n_0$,

$$\|\xi^\sharp_{n_0,m} \circ \mathrm{id}^\sharp_{n_0}(z) - \mathrm{id}^\sharp_m \circ \varphi^\sharp_{n_0,m}(z)\| < 1/2^{2n_0-2}. \tag{e 13.93}$$

It follows that

$$\|\xi^\sharp_{n_0,m} \circ \mathrm{id}^\sharp_{n_0}(\varphi^\sharp_{i,n_0}(\hat{x})) - \mathrm{id}^\sharp_m \circ \varphi^\sharp_{n_0,m}(\varphi_{i,n_0}(\hat{x}))\| < 1/2^{n-1}. \tag{e 13.94}$$

By exactly the same reason, for any $m > n_0$,

$$\|\varphi^\sharp_{n_0,m} \circ \iota^\sharp_{n_0}(\xi^\sharp_{i,n_0}(\hat{x})) - \iota^\sharp_m \circ \xi^\sharp_{n_0,m}(\xi^\sharp_{i,n_0}(\hat{x}))\| < 1/2^{n-1}. \tag{e 13.95}$$

Note also that $\mathrm{id}^\sharp_k$ and $\iota^\sharp_k$ are isometric isomorphisms. It follows that the non-commutative diagram above is approximately intertwining, and the sequences of maps $\{\mathrm{id}_k^\sharp\}$ and $\{\iota_k^\sharp\}$ induce two isometric isomorphisms $j : \mathrm{Aff}(T(A)) \to \mathrm{Aff}\, T(B))$ and $\iota^\sharp : \mathrm{Aff}(T(B)) \to \mathrm{Aff}(T(A))$ between the Banach spaces $\mathrm{Aff}(T(A))$ and $\mathrm{Aff}(T(B))$ such that $j \circ \iota^\sharp = \mathrm{id}_{Aff(T(B))}$ and $\iota^\sharp \circ j = \mathrm{id}_{\mathrm{Aff}(T(A))}$. Moreover, since each $\mathrm{id}_k^\sharp$ and $\iota_k^\sharp$ are order preserving, j and $\iota^\sharp$ are order preserving. It follows from 2.2 that they induce affine homeomorphisms $j_T : T(A) \to T(B)$ and $\iota_T : T(B) \to T(A)$ such that $j_T \circ \iota_T = \mathrm{id}_{T(B)}$ and $\iota_T \circ j_T = \mathrm{id}_{T(A)}$.

It remains to show, by identifying $K_0(B)$ with $K_0(A)$, that $\iota^\sharp \circ \rho_B = \rho_A$. Let $x \in K_0(B)$. We may assume that there are an integer $i \geq 1$ and $y \in K_0(A_i)$ such that $(\xi_{i,\infty})_{*0}(y) = x$. By the approximate intertwining diagram above, there is an integer $k \geq i$ such that, for any $n > k$,

$$\|\iota_n^\sharp \circ \xi_{i,n}^\sharp \circ \rho_{A_i}(y) - \varphi_{k,n}^\sharp \circ \iota_k^\sharp \circ \xi_{i,k}^\sharp \circ \rho_{A_i}(y)\| < \varepsilon. \tag{e 13.96}$$

Note that $\iota^\sharp \circ \xi_{i,\infty}^\sharp(\rho_{A_i}(y)) = \lim_{n\to\infty} \varphi_{n,\infty}^\sharp \circ \iota_n^\sharp \circ \xi_{i,n}^\sharp(\rho_{A_i}(y))$. It follows that

$$\|\iota^\sharp \circ \xi_{i,\infty}^\sharp \circ \rho_{A_i}(y) - \varphi_{k,\infty}^\sharp \circ \xi_{i,k}^\sharp \circ \rho_{A_i}(y)\| \leq \varepsilon, \tag{e 13.97}$$

where we omit $\iota_k^\sharp$ since it is the identity map from $\mathrm{Aff}\, T(A_k)$ to itself. Since $(\xi_{i,k})_{*0} = (\varphi_{i,k})_{*0}$, for all $k > i$,

$$\varphi_{k,\infty}^\sharp \circ \xi_{i,k}^\sharp(\rho_{A_i}(y)) = \varphi_{k,\infty}^\sharp \circ \rho_{A_k} \circ (\xi_{i,k})_{*0}(y) \tag{e 13.98}$$

$$= \varphi_{k,\infty}^\sharp \circ \rho_{A_k} \circ (\varphi_{i,k})_{*0}(y) \tag{e 13.99}$$

$$= \rho_A \circ (\varphi_{i,\infty})_{*0}(y) = \rho_A(x), \text{ and} \tag{e 13.100}$$

$$\xi_{i,\infty}^\sharp \circ \rho_{A_i}(y = \rho_B \circ (\xi_{i,\infty})_{*0}(y) = \rho_B(x) \tag{e 13.101}$$

Therefore,

$$\|\iota^\sharp \circ \rho_B(x) - \rho_A(x)\| \leq \varepsilon \tag{e 13.102}$$

for any given $\varepsilon > 0$. This shows that $\iota^\sharp \circ \rho_B = \rho_A$, and completes the proof.

COROLLARY 13.48. *For any $m > 0$ and any A_i, there are an integer $n \geq i$ and a projection $R \in M_m(A_{n+1})$ such that the following statements are true:*

(1) *R commutes with $\Lambda \circ \xi_{n,n+1}(A_n)$, where $\Lambda : A_{n+1} \to M_m(A_{n+1})$ is the amplification map sending a to an $m \times m$ diagonal matrix: $\Lambda(a) = \mathrm{diag}(a, ..., a)$;*
(2) *Recall*

$$A_{C,n+1} = C([0,1], E_{n+1}) \oplus_{\beta_{n+1,0},\beta_{n+1,1}} F_{n+1} = A\left(F_{n+1}, E_{n+1}, \beta_{n+1,0}, \beta_{n+1,1}\right),$$

where $\beta_{n+1,0}, \beta_{n+1,1} : F_{n+1} \to E_{n+1}$ are as in the definition of A_{n+1} (see (e 13.33). There is a unital injective homomorphism

$$\iota : M_{m-1}(A_{C,n+1}) \longrightarrow RM_m(A_{n+1})R$$

such that $R\Lambda(\xi_{n,n+1}(A_n))R \subset \iota(M_{m-1}(A_{C,n+1}))$.

PROOF. Let $R_X := R \in M_m(A^1_{X,n+1})$ be as obtained in 13.38 (the definition is given by combining 13.34 and 13.36) with the property described in 13.38. Let $\iota_X := \iota : M_{m-1}(F^1_{n+1}) \to R_X M_m(A^1_{X,n+1})R_X$ be the unital injective homomorphism given by 13.38. Let $\pi^1_X : A_{X,n+1} \to A^1_{X,n+1}$. Then, since $\varphi_{n,n+1}(I_n) \subset I_{n+1}$, one has $\pi^1_X(\varphi_{n,n+1}(A_n)) = \pi^1_X(\psi_{n,n+1}(A_{X,n}))$. Since $\xi_{n,n+1}|_{Sp(A^1_{X,n+1})} = \varphi_{n,n+1}|_{Sp(A^1_{X,n+1})}$, one obtains

(i') R_X commutes with $(\pi^1_X \otimes \mathrm{id}_m) \circ \Lambda \circ \xi_{n,n+1}(A_n)$.

Moreover, one also has two additional properties
(ii')

$$R_X(\theta_{n+1,1}) = \mathbf{1}_{F^1_{n+1}} \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{e 13.103}$$

and consequently the point evaluation $\pi_{\theta_{n+1,1}} : M_m(A^1_{X,n+1}) \to M_{m(F^1_{n+1})}$ takes $R_X M_m(A^1_{X,n+1})R_X$ to $M_{m-1}(F^1_{n+1})$. Below we will use the same notation $\pi_{\theta_{n+1,1}}$ to denote the restriction of this map to $R_X M_m(A^1_{X,n+1})R_X$, whose codomain is $M_{m-1}(F^1_{n+1})$.
(iii') $\pi_{\theta_{n+1,1}} \circ \iota_X = \mathrm{id}_{M_{m-1}(F^1_{n+1})}$, and $R_X(\pi^1_X \otimes \mathrm{id}_{M_m})(\Lambda \circ \xi_{n,n+1}(A_n))R_X \subset \iota_X(M_{m-1}(F^1_{n+1}))$.

One extends the definition of R as follows. For each $x \in Sp(A_{n+1})\backslash Sp(A^1_{X,n+1})$, define

$$R(x) = \mathbf{1}_{A_{n+1}|_x} \otimes \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{e 13.104}$$

Then define $R(x) = R_X(x)$ for $x \in Sp(A^1_{X,n+1})$. By checking the boundary, one easily sees that $R \in A_{n+1}$ is a projection. Then, by (i'), (e 13.103), and (e 13.104), R commutes with $\Lambda(\xi_{n,n+1}(A_n))$.

Define $\iota : M_{m-1}(A_{C,n+1}) \to RM_m(A_{n+1})R$ by

$$\iota(f, a_1, a_2, ..., a_{p_{n+1}}) = (f, \iota_X(a_1), a_2, ..., a_{p_{n+1}}) \tag{e 13.105}$$

for $f \in M_{m-1}(C([0,1], E_{n+1}))$ and

$$(a_1, a_2, ..., a_{p_n}) \in M_{m-1}(F_{n+1}) = \bigoplus_{i=1}^{p_{n+1}} M_{m-1}(F^i_{n+1}).$$

By (iii'), $\pi_{\theta_{n+1,1}} \circ \iota_X(a_1) = a_1$. Since $(f, a_1, a_2, ..., a_{p_n}) \in M_{m-1}(A_{C,n+1})$, we have

$$\begin{aligned} f(0) &= \beta_{n+1,0}((a_1, a_2, ..., a_{p_{n+1}}) \text{ and} \\ f(1) &= \beta_{n+1,1}((a_1, a_2, ..., a_{p_{n+1}}). \end{aligned} \tag{e 13.106}$$

Note that

$$\begin{aligned} &\pi^e_{X,n+1}((\iota_X(a_1), a_2, ..., a_{p_{n+1}}) \\ &\quad = \;(\pi_{\theta_{n+1,1}}(\iota_X(a_1)), a_2, ..., a_{p_{n+1}}) = (a_1, a_2, ..., a_{p_{n+1}}). \end{aligned}$$

Thus,

$$f(0) = \beta_{n+1,0} \circ \pi^e_{X,n+1}((\iota(a_1), a_2, ..., a_{p_{n+1}})$$

and

$$f(1) = \beta_{n+1,1} \circ \pi^e_{X,n+1}(\iota(a_1), a_2, ..., a_{p_{n+1}}).$$

Since ι_X is unital and injective, one checks that ι just defined is also unital and injective. In other words, ι maps $M_{m-1}(A_{C,n+1})$ to $M_m(A_{n+1})$. Note that $\iota_X(M_{m-1}(F^1_{n+1})) \subset R_X(M_m(A^1_{X,n+1})R_X$. Then, by (e 13.103) and (e 13.104), $\iota(M_{m-1}(A_{C,n+1})) \subset RM_m(A_{n+1})R$.

By (iii'),

$$(\pi^1_{X,n+1} \otimes \mathrm{id}_{M_m})(R(\Lambda(\xi_{n,n+1}(A_n)))R) \subset (\pi^1_{X,n+1} \otimes \mathrm{id}_{M_m})(\iota(M_{m-1}(A_{C,n+1}))).$$

On the other hand, $(\pi^e_{C,n+1} \otimes \mathrm{id}_{M_m}) \circ \iota = \mathrm{id}_{M_{m-1}(A_{C,n+1})}$. By (e 13.104), $(\pi_{J,n+1} \otimes \mathrm{id}_{M_m})(R) = 1_{M_{m-1}(A_{C,n+1})}$. It follows that

$$\begin{aligned}(\pi_{J,n+1} \otimes \mathrm{id}_{M_m})(R(\Lambda(\xi_{n,n+1}(A_n))R)) &\subset M_{m-1}(A_{C,n+1})\\ &= (\pi_{J,n+1} \otimes \mathrm{id}_{M_m})(\iota(M_{m-1}(A_{C,n+1}))).\end{aligned}$$

Since $(\ker \pi^1_{X,n+1} \otimes \mathrm{id}_{M_m}) \cap (J_{n+1} \otimes \mathrm{id}_{A_m}) = \{0\}$, it follows from that $R(\Lambda(\xi_{n,n+1}(A_n))R \subset \iota(M_{m-1}(A_{C,n+1}))$. This completes the proof. □

COROLLARY 13.49. *Let B be as constructed above. Then $B \otimes U \in \mathcal{B}_0$ for every UHF-algebra U of infinite dimension.*

PROOF. Note that, by 13.12 (see lines below (e 13.5)), $A_{C,n+1} \in \mathcal{C}_0$. Thus, in 13.48, $M_{m-1}(A_{C,n+1})$ and $\iota(M_{m-1}(A_{C,n+1}))$ are in $\mathcal{C}_0$. Also, for each n and each $\tau \in T(M_m(A_{n+1}))$, we have $\tau(\mathbf{1} - R) = 1/m$. Fix an integer $k \geq 1$ and a finite subset $\mathcal{F} \subset B \otimes M_k$, and let $\mathcal{F}_1 \subset B$ be a finite subset such that $\{(f_{ij})_{k\times k} : f_{ij} \in \mathcal{F}_1\} \supset \mathcal{F}$. Now, by applying (13.48), one shows that the inductive limit algebra $B = \lim_{n\to\infty}(A_n, \xi_{n,m})$ has the following property: For any finite set $\mathcal{F}_1 \subset B$, $\varepsilon > 0$, $\delta > 0$, and any $m > 1/\delta$, there is a unital C^*-subalgebra $C \subset M_m(B)$ with different unit $\mathbf{1}_C$ which is in $\mathcal{C}_0$ such that

(i) $\|[\mathbf{1}_C, \mathrm{diag}\{\underbrace{f, ..., f}_{m}\}]\| < \varepsilon/k^2$, for all $f \in \mathcal{F}_1$,

(ii) $\mathrm{dist}(\mathbf{1}_C(\mathrm{diag}\{\underbrace{f, ..., f}_{m}\})\mathbf{1}_C, C) < \varepsilon/k^2$, for all $f \in \mathcal{F}_1$, and

(iii) $\tau(\mathbf{1}_{M_m(B)} - \mathbf{1}_C) = 1/m < \delta$ for all $\tau \in T(M_m(B))$.

Consequently,

(i') $\|[\mathbf{1}_{M_k(C)}, \mathrm{diag}\{\underbrace{f, ..., f}_{m}\}]\| < \varepsilon$, for all $f \in \mathcal{F}$,

(ii') $\mathrm{dist}(\mathbf{1}_{M_k(C)}(\mathrm{diag}\{\underbrace{f, ..., f}_{m}\})\mathbf{1}_{M_k(C)}, M_k(C)) < \varepsilon$, for all $f \in \mathcal{F}$, and (iii') $\tau(\mathbf{1}_{M_{mk}(B)} - \mathbf{1}_{M_k(C)}) = 1/m < \delta$ for all $\tau \in T(M_{mk}(B))$.

Now $B \otimes U$ can be written as $\lim_{n\to\infty}(B \otimes M_{k_n}, \iota_{n,m})$ with $k_1|k_2|k_3\cdots$ and $k_{n+1}/k_n \to \infty$, and $\Lambda_{n,n+1}$ is the amplification map by sending $f \in B \otimes M_{k_n}$ to $\mathrm{diag}(f, ..., f) \in B \otimes M_{k_{n+1}}$, where f repeated k_{n+1}/k_n times.

To show $B \otimes U \in \mathcal{B}_0$, let $\mathcal{F} \subset B \otimes U$ be a finite subset and let $a \in (B \otimes U)_+ \setminus \{0\}$. There is an integer $m_0 > 0$ such that $\tau(a) > 1/m_0$ for all $\tau \in B \otimes U$. Without loss of generality, we may assume that $\mathcal{F} \subset B \otimes M_{k_n}$ with $k_{n+1}/k_n > m_0$. Then by (i'), (ii'), and (iii') for $B \otimes M_{k_n}$ (i.e., $k = k_n$) with $m = k_{n+1}/k_n$ (and recall that $\iota_{n,n+1}$ is the amplification), there is a unital C^*-subalgebra $D := M_k(C) \subset B \otimes M_{k_{n+1}}$ with $C \in \mathcal{C}_0$ such that $\|[\mathbf{1}_D, \iota_{n,n+1}(f)]\| < \varepsilon$, for all $f \in \mathcal{F}$, such that $\mathrm{dist}(\mathbf{1}_D(\iota_{n,n+1}(f))\mathbf{1}_D, D) < \varepsilon$ for all $f \in \mathcal{F}$, and such that $\tau(\mathbf{1} - \mathbf{1}_D) = 1/m < \delta$ for all $\tau \in T(M_{k_{n+1}}(B))$. Then $\Lambda_{n+1,\infty}(D)$ is the desired subalgebra. (Note that $1 - \mathbf{1}_D \lesssim a$ follows from the strict comparison property of $B \otimes U$ (see 5.2 of [104]). It follows that $B \otimes U \in \mathcal{B}_0$. □

THEOREM 13.50. *For any simple weakly unperforated Elliott invariant* $\big((G, G_+, u),\ K, \Delta, r\big)$*, there is a unital simple* C^**-algebra* $A \in \mathcal{N}_0^{\mathcal{Z}}$ *which is an inductive limit of* $(A_n, \varphi_{n,m})$ *with* A_n *as described in* 13.28*, with* $\varphi_{n,m}$ *injective, such that*

$$\big((K_0(A), K_0(A)_+, \mathbf{1}_A),\ K_1(A), T(A), r_A\big) \cong \big((G, G_+, u),\ K, \Delta, r\big).$$

PROOF. By 13.49, $A \in \mathcal{N}_0$. Since A is a unital simple inductive limit of subhomogeneous C^*-algebras with no dimension growth, by Corollary 6.5 of [118], A is $\mathcal{Z}$-stable. □

COROLLARY 13.51. *For any simple weakly unperforated Elliott invariant* $\big((G, G_+, u),\ K, \Delta, r\big)$ *with* $K = \{0\}$ *and* G *torsion free, there is a unital* $\mathcal{Z}$*-stable simple* C^**-algebra which is an inductive limit of* $(A_n, \varphi_{n,m})$ *with* A_n *in* $\mathcal{C}_0$ *as described in* 3.1*, with* $\varphi_{n,m}$ *injective, such that*

$$\big((K_0(A), K_0(A)_+, \mathbf{1}_A),\ K_1(A), T(A), r_A\big) \cong \big((G, G_+, u),\ 0, \Delta, r\big).$$

PROOF. In the construction of A_n, just let all the spaces X_n involved be the space consisting of a single point.

□

14. Models for C^*-algebras in $\mathcal{N}_0$ with Property (SP) Let us recall some notation concerning the classes of C^*-algebras used in this section. $\mathcal{C}$ is the class of Elliott-Thomsen building blocks defined in Definition 3.1, and $\mathcal{C}_0$ consists of the C^*-algebras in $\mathcal{C}$ with zero K_1-group. $\mathcal{D}_k$ is a class of recursive subhomogenous algebras defined in Definition 4.8. $\mathcal{N}$ is the class of all separable amenable C^*-algebras which satisfy the Universal Coefficient Theorem (UCT). $\mathcal{B}_0$ and $\mathcal{B}_1$ are the classes of C^*-algebras defined in Definition 9.1 (roughly speaking $\mathcal{B}_0$ ($\mathcal{B}_1$, respectively) contains the C^* algebras which can be approximated by C^* algebras in the class $\mathcal{C}_0$ ($\mathcal{C}$, respectively) tracially). As in Definition 13.6, $\mathcal{N}_0$ is the class of unital simple C^*-algebras A in $\mathcal{N}$ for which $A \otimes U \in \mathcal{N} \cap \mathcal{B}_0$.

14.1. For technical reasons, in the construction of our model algebras, it is important for us to be able to decompose A_n into the direct sum of two parts: the homogeneous part which stores the information of $\operatorname{Inf} K_0(A)$ and $K_1(A)$ and the part of the algebra in $\mathcal{C}_0$ which stores the information of $K_0(A)/\operatorname{Inf} K_0(A)$, $T(A)$ and the pairing between these. This cannot be done in general for the algebras in $\mathcal{N}_0$ (see 13.6), but we will prove that this can be done if the Elliott invariant satisfies an extra condition, property (SP), described below.

Let $\big((G, G_+, u),\ K, \Delta, r\big)$ be a weakly unperforated Elliott invariant as in 13.7. We say that it has the property (SP) if for any real number $s > 0$, there is $g \in G_+ \setminus \{0\}$ such that $\tau(g) < s$ for any state τ on G, or equivalently, $r(\tau)(g) < s$ for any $\tau \in \Delta$. In this case, we will prove that the algebra in 13.50 can be chosen to be in the class $\mathcal{B}_0$ (rather than in the larger class $\mathcal{N}_0 = \{A\colon A \otimes M_{\mathfrak{p}} \in \mathcal{B}_0\}$). Roughly speaking, for each A_n, we will separate the part of the homogeneous algebra which will store all the information of the infinitesimal part of K_0 and K_1, and it will be in the corner $P_nA_nP_n$ with P_n small compared to $\mathbf{1}_{A_n}$ in the limit algebra. In fact, the construction of this case is much easier, since the homogeneous blocks can be separated from the part in $\mathcal{C}_0$—we will first write the group inclusion $G_n \hookrightarrow H_n$ as in 13.13.

Let us point out that if $A \in \mathcal{N}_0$ then the Elliott invariant of $A \otimes U$ has property (SP) for any infinite dimensional UHF algebra U, even though the Elliott invariant of A itself may not have the property. One can verify this fact as follows. As U is a UHF algebra of infinite dimension, $(K_0(U), K_0(U)_+, [\mathbf{1}_U]) = (\mathbb{P}, \mathbb{P} \cap \mathbb{R}_+, 1)$, where $\mathbb{P} \subset \mathbb{Q} \subset \mathbb{R}$ is a dense subgroup of $\mathbb{R}$. For any positive number s, by density of $\mathbb{P}$, we can choose a number $r \in (\mathbb{P} \cap \mathbb{R}_+) \setminus \{0\}$ such that $r < s$. Let $x \in U$ be a projection such that $[x] = r \in K_0(U)$. Then the projection $\mathbf{1}_A \otimes x \in A \otimes U$ satisfies $\tau(\mathbf{1}_A \otimes x) = r < s$ for all $\tau \in T(A \otimes U)$.

14.2. Let $((G, G_+, u), K, \Delta, r)$ be as given in 13.7 or 13.25. As in 13.9, let $\rho :\ G \to \operatorname{Aff} \Delta$ be dual to the map r. Denote the kernel of the map ρ by $\operatorname{Inf}(G)$—the infinitesimal part of G, i.e.,

$$\operatorname{Inf}(G) = \{g \in G : \rho(g)(\tau) = 0, \quad \text{for all } \tau \in \Delta\}.$$

Let $G^1 \subset \operatorname{Aff} \Delta$ be a countable dense subgroup which is $\mathbb{Q}$-linearly independent with $\rho(G)$—that is, if $g \in \rho(G) \otimes \mathbb{Q}$ and $g^1 \in G^1 \otimes \mathbb{Q}$ satisfy $g + g^1 = 0$, then both g and g^1 are zero. Note that such G^1 exists, since $\mathbb{Q}$ is a vector space, and the dimension of $\rho(G) \otimes \mathbb{Q}$ is countable, but the dimension of $\operatorname{Aff}(\Delta)$ is uncountable. Again as in 13.9, let $H = G \oplus G^1$ with $H_+ \setminus \{0\}$ the set of $(g, f) \in G \oplus G^1$ with

$$\rho(g)(\tau) + f(\tau) > 0 \ \text{ for all } \ \tau \in \Delta.$$

The scale $u \in G_+$ may be regarded as $(u, 0) \in G \oplus G^1 = H$ and so as the scale of H_+. Since $\rho(u)(\tau) > 0$, it follows that u is an order unit for H. Since G^1 is $\mathbb{Q}$-linearly independent of $\rho(G)$, we know $\operatorname{Inf}(G) = \operatorname{Inf}(H)$—that is, when we embed

G into H, it does not create more elements in the infinitesimal group. Evidently, $\mathrm{Tor}(G) = \mathrm{Tor}(H) \subset \mathrm{Inf}(G)$. Let $G' = G/\mathrm{Inf}(G)$ and $H' = H/\mathrm{Inf}(H)$. Then we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Inf}(G) \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, hook] & G' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathrm{Inf}(H) \arrow[r] & H \arrow[r] & H' \arrow[r] & 0.
\end{tikzcd}$$

Let ${G'}_+$ (or H'_+), and u' be the image of G_+ (or H_+) and u under the quotient map from G to G' (or from H to H'). Then (G', G'_+, u') is a weakly unperforated group without infinitesimal elements. Note that G and H share the same unit u, and therefore G' and H' share the same unit u'. Since $r(\tau)|_{\mathrm{Inf}(G)} = 0$ for any $\tau \in \Delta$, the map $r : \Delta \to S_u(G)$ induces a map $r' : \Delta \to S_{u'}(G')$. Hence $((G', G'_+, u'), \{0\}, \Delta, r')$ is a weakly unperforated Elliott invariant with trivial K_1 group and no infinitesimal elements in the K_0-group.

14.3. With the same argument as that of 13.12, we have the following diagram of inductive limits:

$$\begin{tikzcd}
{G'}_1 \arrow[r, "\alpha'_{12}"] \arrow[d, hook, "\iota_1"'] & {G'}_2 \arrow[r, "\alpha'_{23}"] \arrow[d, hook, "\iota_2"'] & \cdots \arrow[r] & G' \arrow[d, hook, "\iota"'] \\
{H'}_1 \arrow[r, "\gamma'_{12}"] & {H'}_2 \arrow[r, "\gamma'_{23}"] & \cdots \arrow[r] & H',
\end{tikzcd}$$

where each ${H'}_n = \mathbb{Z}^{p_n}$ is a direct sum of finitely many copies of the group $\mathbb{Z}$ with the positive cone $(H'_n)_+ = (\mathbb{Z}_+)^{p_n}$, $\alpha'_{n,n+1} = \gamma'_{n,n+1}|_{{G'}_n}$, and ${H'}_n/{G'}_n$ is a free abelian group. Note that not only is $(H'_n)_+$ finitely generated, but also $(G'_n)_+ := (H'_n)_+ \cap G'_n$ is finitely generated, by Theorem 3.15. As in 13.25, we may assume that all $\gamma'_{n,n+1}$ are at least 2-large.

By 13.24, we can construct an increasing sequence of finitely generated subgroups

$$\mathrm{Inf}_1 \subset \mathrm{Inf}_2 \subset \mathrm{Inf}_3 \subset \cdots \subset \mathrm{Inf}(G),$$

with $\mathrm{Inf}(G) = \bigcup_{i=1}^{\infty} \mathrm{Inf}_n$, and such that one has the inductive limit

$$\mathrm{Inf}_1 \oplus H'_1 \xrightarrow{\gamma_{1,2}} \mathrm{Inf}_2 \oplus H'_2 \xrightarrow{\gamma_{2,3}} \mathrm{Inf}_3 \oplus H'_3 \xrightarrow{\gamma_{3,4}} \cdots \longrightarrow H.$$

Put $H_n := \mathrm{Inf}_n \oplus H'_n$ and $G_n := \mathrm{Inf}_n \oplus G'_n$. Since G'_n is a subgroup of H'_n, the group G_n is also a subgroup of H_n. Define $\alpha_{n,n+1} : G_n \to G_{n+1}$ by $\alpha_{n,n+1} = \gamma_{n,n+1}|_{G_n}$, which is compatible with $\alpha'_{n,n+1}$ in the sense that (ii) of 13.24 holds. Hence we obtain the following diagram of inductive limits:

$$\begin{tikzcd}
G_1 \arrow[r, "\alpha_{12}"] \arrow[d, hook, "\iota_1"'] & G_2 \arrow[r, "\alpha_{23}"] \arrow[d, hook, "\iota_2"'] & \cdots \arrow[r] & G \arrow[d, hook, "\iota"'] \\
H_1 \arrow[r, "\gamma_{12}"] & H_2 \arrow[r, "\gamma_{23}"] & \cdots \arrow[r] & H,
\end{tikzcd}$$

with $\alpha_{n,n+1}(\mathrm{Inf}_n) \subset \mathrm{Inf}_{n+1}$ and $\alpha_{n,n+1}|_{\mathrm{Inf}_n}$ the inclusion map. *We will fix, for each n, a positive non-zero homomorphism $\lambda_n : H_n \to \mathbb{Z}$ such that $\lambda_n(x) > 0$ for any $x \in (H'_n)_+ \setminus \{0\}$* and such that $\lambda_n|_{\mathrm{Inf}_n} = 0$. Note that all notations discussed so far in this section will be used for the rest of this section.

LEMMA 14.4. *Let $(G, G_+, u) = \lim_n((G_n, (G_n)_+, u_n), \alpha_{n,m})$ and $(H, H_+, u) = \lim_n((H_n, (H_n)_+, u_n), \gamma_{n,m})$ be as above. Suppose that $\big((G, G_+, u),\ K, \Delta, r\big)$ has the property (SP). For any n with $G_n \overset{\iota_n}{\hookrightarrow} H_n$, and for any $D = \mathbb{Z}^k$ (for any positive integer k), there are positive maps $(\kappa_n, \mathrm{id}) : H_n \to D \oplus H_n$ and $(\kappa'_n, \mathrm{id}) : G_n \to D \oplus G_n$ such that, for any integer $L > 0$, there are an integer $m(n) > n$ and positive maps $\eta : D \oplus H_n \to H_{m(n)}$ and $\eta' : D \oplus G_n \to G_{m(n)}$ such that the following diagram commutes:*

$$\begin{tikzcd}
G_n \arrow[rr, "\alpha_{n,m}"] \arrow[dr, "{(\kappa_n{}',\mathrm{id})}"] \arrow[ddd, "\iota_n"'] & & G_m \arrow[ddd, "\iota_m"] \\
& D \oplus G_n \arrow[ur, "\eta'"'] \arrow[d, hook, "{(\mathrm{id},\iota_n)}"'] & \\
& D \oplus H_n \arrow[dr, "\eta"] & \\
H_n \arrow[ur, "{(\kappa_n,\mathrm{id})}"] \arrow[rr, "\gamma_{n,m}"'] & & H_{m(n)},
\end{tikzcd}$$

and such that the following statements are true:

(1) *The map $\kappa_n : H_n \to D = \mathbb{Z}^k$ is defined by $\kappa_n(x) = (\lambda_n(x), \lambda_n(x), ..., \lambda_n(x))$ for $x \in H_n$ and $\kappa'_n = \kappa_n|_{G_n}$, in particular, each component of $\kappa_n(u_n) = \kappa'_n(u_n)$ in $\mathbb{Z}^k$ is strictly positive.*

(2) *For any $\tau \in \Delta$,*

$$r(\tau)((\alpha_{m(n),\infty} \circ \eta')(\mathbf{1}_D))(= r(\tau)((\gamma_{m(n),\infty} \circ \eta)(\mathbf{1}_D))) < 1/L.$$

(3) *Each component of the map $\pi' \circ \eta|_{D \oplus H'_n} : D \oplus H'_n = \mathbb{Z}^k \oplus \mathbb{Z}^{p_n} \to H'_{m(n)} = \mathbb{Z}^{p_{m(n)}}$ (where $\pi' : H_m \to H'_m$ is the projection map) is strictly positive, and*

L-large—i.e., all entries in the $(k+p_n)\times p_{m(n)}$ *matrix corresponding to the map are larger than* L*. Moreover,* $\eta' = \eta\circ(\mathrm{id}, \iota_n)$ *and* $\pi'\circ\eta'(D\oplus G_n')\subset G_{n+1}'$*.*

(Note the maps (κ_n, id) *and* (κ_n', id) *are independent of* L*.)*

PROOF. We will use the following fact (which was pointed out in the first paragraph of 14.3) several times: the positive cone of G_n' (and that of H_n') is finitely generated (note that even though G_n and H_n are finitely generated, their positive cones may not be finitely generated). Note that H_n' is a subgroup of H_n so we will continue to use λ_n for $\lambda_n|_{H_n'}$.

We now fix n and an integer $k>0$. Define $\kappa_n : H_n \to D = \mathbb{Z}^k$ and $\kappa_n : G_n \to D$ by

$$\kappa_n(a) = (\underbrace{\lambda_n(a), ..., \lambda_n(a)}_{k}) \in D \ \text{ and } \ \kappa_n'(a) = (\underbrace{\lambda_n\circ\iota_n(a), ..., \lambda_n\circ\iota_n(a)}_{k}) \in D.$$

Since G has the property (SP) and since $(H_n')_+$ is finitely generated, there is $p'\in G_+\setminus\{0\}$ such that for any $a\in(H_n')_+$,

$$\gamma_{n,\infty}(a) - k\cdot\lambda_n(a)\cdot p' \in H_+.$$

Consequently,

$$\alpha_{n,\infty}(a) - k\cdot\lambda_n\circ\iota_n(a)\cdot p' \in G_+ \ \text{ for all } \ a\in(G_n')_+,$$

where the maps $\alpha_{n,\infty}$ and $\gamma_{n,\infty}$ are the homomorphisms from G_n to G and from H_n to H respectively. Moreover, for any positive integer L, one may require that

$$r(\tau)(\lambda_n(u_n)\cdot p') < 1/2kL \ \text{ for all } \ \tau\in\Delta. \tag{e 14.1}$$

Since $(G_n')_+$ and $(H_n')_+$ are finitely generated, there are an integer $m(n)\geq 1$ and $p\in(G_{m(n)})_+$ such that

$$\alpha_{m(n),\infty}(p) = p', \ \ \alpha_{n,m(n)}(a) - k\cdot\lambda_n\circ\iota_n(a)\cdot p \in (G_{m(n)})_+ \ \text{ for all } \ a\in(G_n')_+, \ \text{ and}$$
$$\gamma_{n,m(n)}(a) - k\cdot\lambda_n(a)\cdot p \in (H_{m(n)})_+ \ \text{ for all } \ a\in(H_n')_+.$$

Then define $\alpha_n'' : G_n \to G_{m(n)}$ and $\gamma_n'' : H_n \to H_{m(n)}$ by

$$\begin{aligned} &\alpha_n'' : G_n \ni a \mapsto \alpha_{n,m(n)}(a) - k\cdot\lambda_n\circ\iota_n(a)\cdot p \in G_{m(n)}, \ \text{ and} \\ &\gamma_n'' : H_n \ni a \mapsto \gamma_{n,m(n)}(a) - k\cdot\lambda_n(a)\cdot p \in H_{m(n)}. \end{aligned} \tag{e 14.2}$$

By the choice of p, the maps α_n'' and γ_n'' are positive. (Note that $\alpha_n''|_{\mathrm{Inf}_n} = \alpha_{n,m(n)}|_{\mathrm{Inf}_n}$ and $\gamma_n''|_{\mathrm{Inf}_n} = \gamma_{n,m(n)}|_{\mathrm{Inf}_n}$.)

A direct calculation shows the following diagram commutes (where $D = \mathbb{Z}^k$):

$$\begin{tikzcd}
G_n \arrow[rr, "\alpha_{n,m(n)}"] \arrow[dr, "({\kappa_n}'{,}\mathrm{id})"'] \arrow[ddd, "\iota_n"'] & & G_{m(n)} \arrow[ddd, "\iota_{m(n)}"] \\
& D \oplus G_n \arrow[ur, "\eta'"'] \arrow[d, hook, "(\mathrm{id}{,}\iota_n)"'] & \\
& D \oplus H_n \arrow[dr, "\eta"] & \\
H_n \arrow[ur, "(\kappa_n{,}\mathrm{id})"] \arrow[rr, "\gamma_{n,m(n)}"'] & & H_{m(n)},
\end{tikzcd}$$

$$\eta'((m_1, ..., m_k, g)) = (m_1 + \cdots + m_k)p + \alpha''_n(g), \text{ and}$$

$$\eta((m_1, ..., m_k, g)) = (m_1 + \cdots + m_k)p + \gamma''_n(g).$$

(Recall that

$$\kappa'_n(a) = (\underbrace{\lambda_n \circ \iota_n(a), ..., \lambda_n \circ \iota_n(a)}_{k}) \in D \text{ and } \kappa_n(a) = (\underbrace{\lambda_n(a), ..., \lambda_n(a)}_{k}) \in D.)$$

The order of $D \oplus G_n$ and $D \oplus H_n$ are the standard order on direct sums, i.e., $(a, b) \geq 0$ if and only if $a \geq 0$ and $b \geq 0$. Since the maps α''_n and γ''_n are positive, the maps η' and η are positive. Condition (1) follows from the construction; condition (2) follows from (e 14.1); and condition (3) follows from the fact that $\gamma'_{k,k+1}$ are 2-large, if one passes to a further stage (choose larger $m(n)$). $\square$

DEFINITION 14.5. A C^*-algebra is said to be in the class $\mathbf{H}$ if it is the direct sum of algebras of the form $P(C(X) \otimes M_n)P$, where $X = \{pt\}, [0, 1]$, S^1, S^2, $T_{2,k}$, or $T_{3,k}$ (see 13.27 for the definitions of $T_{2,k}$ and $T_{3,k}$). In addition, we assume that the rank of P is at least 13 when $X = T_{2,k}$ or $X = T_{3,k}$.

14.6. Write K (the K_1 part of the invariant) as the union of an increasing sequence of finitely generated abelian subgroups: $K_1 \subset K_2 \subset K_3 \subset \cdots \subset K$ with $K = \bigcup_{i=1}^{\infty} K_i$. Denote by $\chi_{n,n+1} : K_n \to K_{n+1}$ the embedding, $n = 1, 2, ...$ For a finitely generated abelian group G, we use $\operatorname{rank} G$ to denote the smallest cardinality of a set of generators of G—that is, G can be written as a direct sum of rank(G) cyclic groups (e.g., $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, $m \in \mathbb{N}$).

Let $d_n = \max\{2,\ 1 + \mathrm{rank}(\mathrm{Inf}_n) + \mathrm{rank}(K_n)\}$. Apply 14.4 repeatedly, with $k = d_n$ and $L > 13 \cdot 2^{2n}$ and $\tilde{\gamma}_{n,n+1} = (\kappa_n, \mathrm{id}) \circ \eta_n$, where κ_n and η_n are defined in

14.4, for each n. Passing to a subsequence if necessary, we obtain the following commutative diagram of inductive limits:

$$\begin{tikzcd}
\mathbb{Z}^{d_1} \oplus G_1 \arrow[r, "\tilde{\alpha}_{1,2}"] \arrow[d, hook, "(\mathrm{id},\iota_1)"'] & \mathbb{Z}^{d_2} \oplus G_2 \arrow[r, "\tilde{\alpha}_{2,3}"] \arrow[d, hook, "(\mathrm{id},\iota_2)"'] & \cdots \arrow[r] & G \arrow[d, hook, "\iota"'] \\
\mathbb{Z}^{d_1} \oplus H_1 \arrow[r, "\tilde{\gamma}_{1,2}"'] & \mathbb{Z}^{d_2} \oplus H_2 \arrow[r, "\tilde{\gamma}_{2,3}"'] & \cdots \arrow[r] & H,
\end{tikzcd}$$

where $\bar{G}_n := \mathbb{Z}^{d_n} \oplus G_n$ has the order unit $(\kappa_n'(u_n), u_n)$, $\mathbb{Z}^{d_n} \oplus H_n$ has the order unit $(\kappa_n(u_n), u_n)$, and $\tilde{\alpha}_{n,n+1} = \tilde{\gamma}_{n,n+1}|_{\mathbb{Z}^{d_n} \oplus G_n}$. Set $u_n^d := \kappa_n'(u_n) \in \mathbb{Z}^{d_n}$. Then $u_n^d = \kappa_n(u_n) = \kappa_n'(u_n)$. Write $\bar{u}_n = (\kappa_n(u_n), u_n)$ and $G_n'' = \mathbb{Z}^d \oplus \mathrm{Inf}_n$. So $\bar{G}_n = \mathbb{Z}^{d_n} \oplus G_n = G_n'' \oplus G_n'$. We may also write $\bar{u}_n = (u_n'', u_n')$, where $u_n'' \in G_n''$. Then $\tilde{\alpha}_{n,n+1}(\bar{u}_n) = \bar{u}_{n+1}$. Let $\rho_n' : \mathbb{Z}^{d_n} \oplus \mathrm{Inf}_n \oplus H_n' \to \mathbb{Z}^{d_n} \oplus H_n'$ be the quotient map, and let $\rho_{G',n}' := \rho_n'|_{\mathbb{Z}^{d_n} \oplus G_n}$ be the map which maps $\mathbb{Z}^{d_n} \oplus G_n = \mathbb{Z}^{d_n} \oplus \mathrm{Inf}_n \oplus G_n'$ to $\mathbb{Z}^{d_n} \oplus G_n'$. Put $\tilde{\gamma}_{n,n+1}' = \rho_n' \circ \tilde{\gamma}_{n,n+1}|_{\mathbb{Z}^{d_n} \oplus H_n'}$ and $\tilde{\alpha}_{n,n+1}' = \rho_{G',n+1}' \circ \tilde{\alpha}_{n,n+1}|_{\mathbb{Z}^{d_n} \oplus G_n'}$. Note also that, if we replace G_n by $\mathbb{Z}^{d_n} \oplus G_n$, and H_n by $\mathbb{Z}^{d_n} \oplus H_n$, respectively, then the limit ordered groups G and H do not change (see 14.4). In particular, we still have $\mathrm{Inf}(G) = \bigcup_{i=1}^{\infty} \tilde{\alpha}_{n,\infty}(\mathrm{Inf}_n)$. Moreover, by (2) of 14.4, $r(\tau)(\tilde{\alpha}_{n,\infty}(u_n'')) < 1/13 \cdot 2^{2n}$ for all $\tau \in \Delta$. For exactly the same reason one has the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}^{d_1} \oplus G_1' \arrow[r, "\tilde{\alpha}_{1,2}'"] \arrow[d, hook, "(\mathrm{id},\iota_1)"'] & \mathbb{Z}^{d_2} \oplus G_2' \arrow[r, "\tilde{\alpha}_{2,3}'"] \arrow[d, hook, "(\mathrm{id},\iota_2)"'] & \cdots \arrow[r] & G' \arrow[d, hook, "\iota"'] \\
\mathbb{Z}^{d_1} \oplus H_1' \arrow[r, "\tilde{\gamma}_{1,2}'"'] & \mathbb{Z}^{d_2} \oplus H_2' \arrow[r, "\tilde{\gamma}_{2,3}'"'] & \cdots \arrow[r] & H'.
\end{tikzcd} \tag{e 14.3}$$

Note that $\mathbb{Z}^{d_n} \oplus G_n'$ and $\mathbb{Z}^{d_n} \oplus H_n'$ share the order unit (u_n^d, u_n'). Let $\bar{H}_n = Z^{d_n} \oplus H_n = G_n'' \oplus H_n'$. Note that each $\tilde{\gamma}_{k,k+1}'$ is $13 \cdot 2^{2k}$-large; that is, $\tilde{\gamma}_{k,k+1}' = (c_{ij}^{k,k+1}) \in M_{(d_{k+1}+p_{k+1})\times(d_k+p_k)}(\mathbb{Z}_+)$ with $c_{ij}^{k,k+1} \geq 13 \cdot 2^{2k}$.

We now construct C^*-algebras $\{C_n\}$, $\{B_n\}$, $\{F_n\}$, and $\{A_n\}$, and unital injective homomorphisms $\varphi_{n,n+1} : A_n \to A_{n+1}$, inductively as in Section 13.

As in 13.13 (applied to $G_n' \subset H_n'$), one can find finite dimensional C^*-algebras $F_n = \bigoplus_{i=1}^{p_n} F_n^i$ and $E_n = \bigoplus_{j=1}^{l_n} E_n^j$, unital homomorphisms $\beta_{n,0},\ \beta_{n,1} : F_n \to E_n$, and form the C*-algebra

$$\begin{aligned}
C_n &= A(F_n, E_n, \beta_0, \beta_1) \\
&:= \{(f, a) \in C([0,1], E_n) \oplus F_n; f(0) = \beta_{n,0}(a), f(1) = \beta_{n,1}(a)\} \\
&:= C([0,1], E_n) \oplus_{\beta_{n,0},\beta_{n,1}} F_n,
\end{aligned}$$

such that

$$(K_0(F_n), K_0(F_n)_+, [\mathbf{1}_{F_n}]) = (H'_n, (H'_n)_+, u'_n),$$
$$(K_0(C_n), K_0(C_n)_+, \mathbf{1}_{C_n}) = (G'_n, (G'_n)_+, u'_n), \quad K_1(C_n) = \{0\},$$

and furthermore $K_0(C_n)$ is identified with

$$\ker((\beta_{n,1})_{*0} - (\beta_{n,0})_{*0}) = \{x \in K_0(F_n); ((\beta_0)_{*0} - (\beta_1)_{*0})(x) = 0 \in K_0(E_n)\}.$$

Write $G''_n = \bigoplus_{i=1}^{d_n} (G''_n)^i$, with $(G''_n)^i = \mathbb{Z}$ for $i \leq 1 + \mathrm{rank}(K_n)$, and $(G''_n)^i = \mathbb{Z} \oplus S_i$, where S_i is a cyclic group, for $1+\mathrm{rank}(K_n) < i \leq d_n$, and $\bigoplus_{i=2+\mathrm{rank}(K_n)} S_i = \mathrm{Inf}_n$. Here the positive cone of $(G''_n)^i$ is given by the strict positivity of the first coordinate for non-zero positive elements. An element in G''_n is positive if each of its components in $(G''_n)^i$ is positive. Define a unital C^*-algebra $B_n \in \mathbf{H}$ such that (see 13.22)

$$(K_0(B_n), K_0(B_n)_+, \mathbf{1}_{B_n}, K_n) = (G''_n, (G''_n)_+, u''_n, K_n). \tag{e 14.4}$$

More precisely, we have that $B_n = \bigoplus_{i=1}^{d_n} B_n^i$, with $K_0(B_n^i) = (G''_n)^i$, and $K_1(B_n^i)$ is either a cyclic group for the case $2 \leq i \leq 1 + \mathrm{rank}(K_n)$ or zero for the other cases. In particular, the algebra B_n^1 can be chosen to be a matrix algebra over $\mathbb{C}$, by the choice of d_n. We may also assume that, for at least one block B_n^2, the spectrum is not a single point (note that $d_n \geq 2$); otherwise, we will replace the single point spectrum by the interval $[0, 1]$. We may also write $B_n^i = P_{X,n,i} M_{m_i}(X_{n,i}) P_{X,n,i}$ as in 14.5 where $P_{X,n,i}$ has rank at least 13 and $X_{n,i}$ is connected. For each block B_n^i, choose a base point $x_{n,i} \in Sp(B_n^i)$. Denote by $\pi_{x_{n,i}} : B_n^i \to B_n^i(x_{n,i}) := F_{X,n}^i$ $(\cong M_{\mathrm{rank} P_{X,n,i}(x_{n,i})})$ the point evaluation at the point $\pi_{x_{n,i}}$. Let $F_{X,n} = \bigoplus_{i=1}^{d_n} F_{X,n}^i$ and let $\pi_{pts,n} : B_n \to F_{X,n}$ be the quotient map. Put $I_{B,n} = \ker \pi_{pts,n}$. Then $(K_0(F_{X,n}), K_0(F_{X,n})_+, [1_{F_{X,n}}]) = (\mathbb{Z}^{d_n}, \mathbb{Z}_+^{d_n}, [\pi_{pts,n}(1_{B_n}])$. Note that $[\pi_{pts,n}(1_{B_n})] = u_n^d$.

Let us give some obvious properties of the homomorphism $\tilde{\alpha}_{n,n+1} : G''_n \oplus G'_n \to G''_{n+1} \oplus G'_{n+1}$ as below. Let $\alpha_{n,n+1}^{G''_n, G''_{n+1}} : G''_n \to G''_{n+1}$, $\alpha_{n,n+1}^{G''_n, G'_{n+1}} : G''_n \to G'_{n+1}$, $\alpha_{n,n+1}^{G'_n, G''_{n+1}} : G'_n \to G''_{n+1}$ and $\alpha_{n,n+1}^{G'_n, G'_{n+1}} : G'_n \to G'_{n+1}$ be the corresponding partial maps. Define $\alpha_{n,n+1}^{\bar{G}_n, G'_{n+1}} : \bar{G}_n \to G'_{n+1}$ by $\alpha_{n,n+1}(g'', g') = \alpha_{n,n+1}^{G''_n, G'_{n+1}}(g'') + \alpha_{n,n+1}^{G'_n, G'_{n+1}}(g')$. Since $\mathrm{Inf}_{n+1} \cap G'_{n+1} = 0$ and $\tilde{\alpha}_{n,n+1}$ maps Inf_n to Inf_{n+1}, we know $\alpha_{n,n+1}^{G''_n, G'_{n+1}}(\mathrm{Inf}_n) = 0$. Hence $\alpha_{n,n+1}^{G''_n, G'_{n+1}}$ factors through G''_n / Inf_n as

$$\alpha_{n,n+1}^{G''_n, G'_{n+1}} : G''_n \xrightarrow{\pi} G''_n / \mathrm{Inf}_n \xrightarrow{\tilde{\alpha}''_{n,n+1}} G'_{n+1}.$$

In other words, we may write $\alpha_{n,n+1}^{G''_n, G'_{n+1}} = \tilde{\alpha}''_{n,n+1} \circ \pi$, where $\pi : G''_n \to G''_n / \mathrm{Inf}_n$ is the quotient map, and $\tilde{\alpha}''_{n,n+1} : G''_n / \mathrm{Inf}_n \to G'_{n+1}$ is the induced homomorphism. Also, since $\tilde{\alpha}_{n,n+1} = \tilde{\gamma}_{n,n+1}|_{G_n}$, $\alpha_{n,n+1}^{G'_n, G''_{n+1}}$ factors through H'_n as

$$\alpha_{n,n+1}^{G'_n, G''_{n+1}} : G'_n \xrightarrow{\iota_n|_{G'_n}} H'_n \xrightarrow{\gamma_{n,n+1}^{H'_n, G''_{n+1}}} G''_{n+1}.$$

That is, $\alpha_{n,n+1}^{G_n',G_{n+1}''} = \gamma_{n,n+1}^{H_n',G_{n+1}''} \circ \iota_n|_{G_n'}$, where $\gamma_{n,n+1}^{H_n',G_{n+1}''} : H_n' \to G_{n+1}''$ is the homomorphism induced by $\gamma_{n,n+1}'$. Note that $\tilde{\gamma}_{n,n+1}^{H_n',H_{n+1}'}$ has multiplicity at least $13 \cdot 2^{2n}$.

14.7. We can extend the maps $\beta_{n,0}$ and $\beta_{n,1}$ to $\beta_{n,0},\ \beta_{n,1} : B_n \oplus F_n \to E_n$, by defining them to be zero on B_n. Consider $A_n = B_n \oplus C_n$. Then the C*-algebra A_n can be written as

$$A_n = \{(f,a) \in C([0,1],E_n) \oplus (B_n \oplus F_n) : f(0) = \beta_{n,0}(a), f(1) = \beta_{n,1}(a)\}.$$

Then

$$(K_0(A), K_0(A_n)_+, [1_{A_n}], K_1(A)) = (\bar{G}_n, (\bar{G}_n)_+, \bar{u}_n, K_n).$$

Let $C_n' = F_{X,n} \oplus C_n$ (after C_n is defined) and let $\pi_{C_n',C_n} : C_n' \to C_n$ be the quotient map. Note that $C_n' = C([0,1],E_n) \oplus_{\beta_{n,0},\beta_{n,1}} F_{X,n} \oplus F_n$, where $\beta_{n,0}$ and $\beta_{n,1}$ are extended to maps from $F_{X,n} \oplus F_n$ which are zero on $F_{X,n}$. Let $I_n = C_0\big((0,1),E_n\big)$ be the ideal of C_n (also an ideal of A_n). Denote by $\pi_{I,n} : A_n \to B_n \oplus F_n$ and $\pi_{A_n,C_n} : A_n \to C_n$ the quotient maps. We also use $\pi_{A_n,B_n} : A_n \to B_n$ for the quotient map.

We will construct B_m, C_m, F_m, and A_m (in order for comparison with 13.31, we use subscripts m instead of n) as above together with a unital injective homomorphism $\varphi_{m,m+1} : A_m \to A_{m+1}$, an injective homomorphism $\bar{\varphi}_{m,m+1}' : C_m' \to C_{m+1}'$, and an injective homomorphism $\psi_{B,m,m+1} : B_m \to B_{m+1}$ satisfying the following conditions (similar to those of 13.31)

(a) $\varphi_{m,m+1}(I_m) \subset I_{m+1}$, $\varphi_{m,m+1}(I_{B,m}) \subset I_{B,m+1}$ (see 14.6 for $I_{B,m+1}$). (Hence $\varphi_{m,m+1}$ induces a homomorphism $\varphi_{m,m+1}^q : F_{X,m} \oplus F_m = A_m/(I_m \oplus I_{B,m}) \to F_{X,m+1} \oplus F_{m+1} = A_{m+1}/(I_{m+1} \oplus I_{B,m+1})$.) Furthermore

$$\pi_{C_{m+1}',C_{m+1}} \circ \bar{\varphi}_{m,m+1}'|_{C_m} = \varphi_{m,m+1}^{C_m,C_{m+1}},$$

where $\varphi_{m,m+1}^{C_m,C_{m+1}} := \pi_{A_{m+1},C_{m+1}} \circ \varphi_{m,m+1}|_{C_m}$ is the partial map of $\varphi_{m,m+1}$ from C_m to C_{m+1}, and $(\pi_{A_{m+1},B_{m+1}} \circ \varphi_{m,m+1})|_{B_m} = \psi_{B,m,m+1}$;

(b) $(\varphi_{m,m+1})_{*0} = \tilde{\alpha}_{m,m+1}$, $(\varphi_{m,m+1})_{*1} = \chi_{m,m+1}$, $(\bar{\varphi}_{m,m+1}')_{*0} = \tilde{\alpha}_{m,m+1}'$ and $(\pi_{C_{m+1}',C_{m+1}} \circ (\bar{\varphi}_{m,m+1}')|_{C_m})_{*0} = \alpha_{m,m+1}^{G_m',G_{m+1}'}$ and $(\psi_{B,m,m+1})_{*0} = \alpha_{m,m+1}^{G_m'',G_{m+1}''}$;

(c) (compare (c) of 13.31), the map $\bar{\varphi}_{m,m+1}'$ satisfies the conditions (1)–(8) of 13.15 (where C_m and C_{m+1} are replaced by C_m' and C_{m+1}', and H_m and G_m are replaced by $\mathbb{Z}^{d_m} \oplus H_m'$ and $\mathbb{Z}^{d_m} \oplus G_m'$), and satisfies condition $(\diamondsuit\diamondsuit)_1$ in (c) of 13.31 (with the number L_m as described below). In particular,

$$\begin{aligned}(\varphi_{m,m+1}^q)_{*0} = (c_{ij}^{m,m+1}) &: K_0(F_{X,m} \oplus F_m)\\ = \quad \mathbb{Z}^{d_m} \oplus H_m' &\to K_0(F_{X,m+1} \oplus F_{m+1}) = \mathbb{Z}^{d_{m+1}} \oplus H_{m+1}';\end{aligned}$$

(d) the matrices $\beta_{m+1,0}$ and $\beta_{m+1,1}$ for C'_{m+1} satisfy the condition $(\diamondsuit\diamondsuit)$ (where $(d_{ij}^{m,m+1}) : (\mathbb{Z}^{d_m} \oplus H'_m)/(\mathbb{Z}^{d_m} \oplus G'_m) = H'_m/G'_m \to H'_{m+1}/G'_{m+1} = (\mathbb{Z}^{d_{m+1}} \oplus H'_{m+1})/(\mathbb{Z}^{d_{m+1}} \oplus G'_{m+1})$ is the map induced by $\tilde{\gamma}'_{m,m+1}$).

The number L_m in $(\diamondsuit\diamondsuit)_1$ in (c) of 13.31 (see (c) above) which was to be chosen at the m-th step is described as follows: Set $\psi_k := \psi_{B,m-1,m} \circ \psi_{B,m-2,m-1} \circ \cdots \circ \psi_{B,k,k+1} : B_k \to B_m$. (Recall that $\psi_{B,i,i+1} = \pi_{A_{i+1},B_{i+1}} \circ \varphi_{i,i+1}|_{B_i}$ is the partial map from $B_i \subset A_i$ to $B_{i+1} \subset A_{i+1}$ of the homomorphism $\varphi_{i,i+1}$. There is a finite subset $Y_m \in Sp(B_m)$ such that, for each $k \le m-1$, the set $\bigcup_{x \in Y_m} Sp(\psi_k)_x$ is $1/m$-dense in $Sp(B_k)$, and Y_m is also $1/m$-dense $Sp(B_m)$. Let $T_m := \big\{ \big(\frac{k}{m}\big)_{n,j} \in Sp(E_m^j),\ \ 1 \le k \le m-1,\ 1 \le j \le l_m \big\}$, and write $\Omega_m = Y_m \cup T_m$. Choose

$$L_m > 13 \cdot 2^{2m} \cdot (\#(\Omega_m)) \cdot \mathcal{R} \tag{e 14.5}$$

where $\mathcal{R} = (\max\{\mathrm{rank}(\mathbf{1}_{B^l_m}), \mathrm{rank}(\mathbf{1}_{F^i_m}), \mathrm{rank}(\mathbf{1}_{E^j{}_m})\})$.

14.8. We begin to construct C_1 and F_1 in exactly the same way as in Section 13 (for G'_1 and H'_1). Suppose that $C_1, C_2, ..., C_n,\ F_1, F_2, ..., F_n,\ B_1, B_2, ..., B_n$, $\varphi_{k,k+1}$, $\bar{\varphi}'_{k,k+1}$ and $\psi_{B,k,k+1}$ (for $k \le n-1$) have been constructed. We will choose B_{n+1} as above. Since each $\varphi_{B,k,k+1}$ is injective ($1 \le k < n$), so also is ψ_k above. Therefore Y_n exists and L_n can be defined. Since each map $\tilde{\gamma}'_{n,n+1} : \mathbb{Z}^{d_n} \oplus H'_n \to \mathbb{Z}^{d_{n+1}} \oplus H'_{n+1}$ is strictly positive, exactly as in 13.31, (passing to larger $n+1$), we may assume that $(\diamondsuit\diamondsuit)_1$ holds. This determines the integer $n+1$ (which is originally denoted by some large integer greater than n). Then, as in 13.31, using Λ_n (defined in (e 13.44)), we can choose $\beta_{n+1,0}$ and $\beta_{n+1,1}$ so that $(\diamondsuit\diamondsuit)$ holds. Then we use these $\beta_{n+1,0}$ and $\beta_{n+1,1}$ (which are zero on $F_{X,n+1}$) to define C'_{n+1}. So C_{n+1} is also defined. Let $A_{n+1} = C_{n+1} \oplus B_{n+1}$. Note that (d) has been verified.

Let $P_n, Q_n \in A_{n+1}$ be projections such that $[P_n] = \tilde{\alpha}_{n,n+1}([\mathbf{1}_{C_n}])$, $[Q_n] = \tilde{\alpha}_{n,n+1}([\mathbf{1}_{B_n}])$, and $P_n + Q_n = \mathbf{1}_{A_{n+1}}$. Let $P_{n,C} = \pi_{{}_{A_{n+1},C_{n+1}}}(P_n)$, $P_{n,B} = \pi_{{}_{A_{n+1},B_{n+1}}}(P_n)$, $Q_{n,C} = \pi_{{}_{A_{n+1},C_{n+1}}}(Q_n)$, and $Q_{n,B} = \pi_{{}_{A_{n+1},B_{n+1}}}(Q_n)$. Note that $\tilde{\alpha}'_{n,n+1}(u^d_n, u'_n) = (u^d_{n+1}, u'_{n+1})$.

It follows by Lemma 13.15 that there is a unital injective homomorphism $\bar{\varphi}'_{n,n+1} : C'_n \to C'_{n+1}$ which satisfies the conditions (1)–(8) of 13.15. In particular, $(\bar{\varphi}'_{n,n+1})_{*0} = \tilde{\alpha}'_{n,n+1}$. Define a homomorphism $\varphi_{n,n+1}^{A_n,C_{n+1}} : A_n \to C_{n+1}$ by $\varphi_{n,n+1}^{A_n,C_{n+1}} = \pi_{C'_{n+1},C_{n+1}} \circ \bar{\varphi}'_{n,n+1} \circ \pi_{{}_{A_n,C'_n}}$. One checks that $(\varphi_{n,n+1}^{A_n,C_{n+1}})_{*0} = \alpha_{n,n+1}^{\bar{G}_n,G'_{n+1}}$, and $(\varphi_{n,n+1}^{A_n,C_{n+1}})_{*1} = 0$. Moreover, $(\varphi_{n,n+1}^{A_n,C_{n+1}})|_{I_{B,n}} = 0$, and $(\pi_{C'_{n+1},C_{n+1}} \circ \varphi_{n,n+1}^{A_n,C_{n+1}}|_{C_n})_{*0} = \alpha_{n,n+1}^{G'_n,G'_{n+1}}$. Since $\tilde{\gamma}'_{n,n+1}$ is $13 \cdot 2^{2n}$-large, so also is $\alpha_{n,n+1}^{G''_n,G''_{n+1}}$. It follows from the second part of 13.23 that there is an injective homomorphism $\psi_{B,n,n+1} : B_n \to Q_{n,B}B_{n+1}Q_{n,B}$ which maps $I_{B,n}$ into $I_{B,n+1}$ such that $(\psi_{B,n,n+1})_{*0} = \alpha_{n,n+1}^{G''_n,G''_{n+1}}$ and $(\psi_{B,n,n+1})_{*1} = \chi_{n,n+1}$. $(\varphi_{n,n+1}^{B_n,B_{n+1}})_{*1} = \chi_{n,n+1}$. Since $\alpha_{n,n+1}^{H'_n,G''_{n+1}}$ is at least 13-large, by the second part of 13.23 (with each X_i a point) again, there is a unital injective homomorphism $\bar{\varphi}_{n,n+1}^{F_n,B_{n+1}} : F_n \to P_{n,B}B_{n+1}P_{n,B}$ such that $(\bar{\varphi}_{n,n+1}^{F_n,B_{n+1}})_{*0} = \alpha_{n,n+1}^{H_n',G_{n+1}''}$. Define $\bar{\varphi}_{n,n+1}^{C_n,B_{n+1}} = (\bar{\varphi}_{n,n+1}^{F_n,B_{n+1}}) \circ \pi_{I,n}|_{C_n}$. Therefore $(\bar{\varphi}_{n,n+1}^{C_n,B_{n+1}})_{*0} = \alpha_{n,n+1}^{G_n',G_{n+1}''}$. Then define $\varphi_{n,n+1} : A_n \to A_{n+1}$ by, for all $(b,c) \in B_n \oplus C_n$,

$$\text{(e 14.6)}\quad \varphi_{n,n+1}(b,c) = \psi_{B,n,n+1}(b) \oplus \bar{\varphi}_{n,n+1}^{C_n,B_{n+1}}(c) \oplus \varphi_{n,n+1}^{A_n,C_{n+1}}(b,c).$$

One checks that $\varphi_{n,n+1}$ (together with the induced maps) also satisfies (a), (b), and (c). Let $A = \lim_{n\to\infty}(A_n, \varphi_{n,n+1})$.

14.9. For the algebra A constructed above, we have

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (G, G_+, u, K_1).$$

By (2) of 14.4, we also have

$$\text{(e 14.7)}\qquad \frac{\tau(\alpha_{n,\infty}(\mathbf{1}_{B_n}))}{\tau(\alpha_{n,\infty}(\mathbf{1}_{C_n}))} < \frac{1}{13\cdot 2^n - 1},$$

for all $\tau \in T(A)$. Let $F_n' = F_{X,n} \oplus F_n$, and define $\pi_{_{A_n,F_n'}} : A_n \to F_n'$ by $\pi_{A_n,F_n'}(b,c) = (\pi_{pts,n}(b), c)$ for $(b,c) \in B_n \oplus F_n$. By (a) of 14.8, $\varphi_{n,n+1}(I_n + I_{B,n}) \subset I_{n+1} + I_{B,n+1}$. Therefore, $\varphi_{n,n+1}$ induces a unital homomorphism $\varphi_{_{F',n,n+1}} : F_n' \to F_{n+1}'$. (The map $\varphi_{_{F',n,n+1}}$ is denoted by $\varphi_{n,n+1}^q$ in 14.7 to be consistence with section 13, see 13.31.) Note $K_0(F_n') = \mathbb{Z}^{d_n} \oplus G_n'$ and $(\varphi_{F',n,n+1})_{*0} = \tilde{\gamma}_{n,n+1}'$. Define $F' = \lim_{n\to\infty}(F_n', \varphi_{F,n,n+1}')$. Then F' is a unital AF-algebra. By (e 14.3), $(K_0(F'), K_0(F')_+, [1_F']) = (H', H_+', u')$. It follows that $F' \cong F$ by Elliott's classification. Therefore $T(F') = \Delta$. We also have the following commutative diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_{1,2}"] \arrow[d, "\pi_{_{A_1,F_1'}}"'] & A_2 \arrow[r, "\varphi_{2,3}"] \arrow[d, "\pi_{_{A_2,F_2'}}"'] & \cdots \arrow[r] & A \arrow[d] \\
F_1' \arrow[r, "\varphi_{_{F',1,2}}"'] & F_2' \arrow[r, "\varphi_{_{F',2,3}}"'] & \cdots \arrow[r] & F'.
\end{tikzcd}$$

As in Section 13, using $(\diamondsuit\diamondsuit)_1$, (14.9), and the above diagram, one shows that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A) = (G, G_+, u, K, \Delta, r).$$

Note that A is not simple. So we make one more modification just as in Section 13 (but much simpler as B_n is a direct summad of A_n). We describe this briefly as follows: Let $A_{n+1}^- = \bigoplus_{i=2}^{d_{n+1}} B_n^i \oplus C_{n+1}$, let $\pi_{B_{n+1}^1} : A_{n+1} \to B_{n+1}^1$, and $\pi_{B_{n+1}}^- : A_{n+1} \to \bigoplus_{i\geq 2} B_{n+1}^i$ be the quotient maps. We also view $\pi_{B_{n+1}^1}$ as the quotient map from B_{n+1} to B^1_{n+1}. Recall that $B^1_{n+1} \cong M_{R(n+1)}$ for integer $R(n+1) \geq 1$. We may write $\psi_{B,n,n+1} = \psi^-_{B,n,n+1} \oplus \psi^1_{B,n,n+1}$, where $\psi^-_{B,n,n+1} = \pi^-_{B_{n+1}} \circ \psi_{B,n,n+1}$ and $\psi^1_{B,n,n+1} = \pi_{B^1_{n+1}} \circ \psi_{B,n,n+1}$. One may write, using the notation of Section 13, and using $(\diamondsuit\diamondsuit)_1$,

(e 14.8) $$\psi^1_{B,n,n+1}(f) = (\pi^{\sim a_1}_{x_{n,1}}, \pi^{\sim a_2}_{x_{n,2}}, ..., \pi^{\sim a_{d_n}}_{x_{n,d_n}})(f) \text{ for all } f \in B_n,$$

where $a_i > \#(Y_n)$ $(1 \leq i \leq d_n)$. Let $q_{n+1} = \psi^1_{B,n,n+1}(1_{B_n})$. There is a continuous path $\psi^{1,t}_{B,n,n+1} : B_n \to q_{n+1}B_{n+1}q_{n+1}$ $(t \in [0,1])$ such that $\psi^{1,0}_{B,n,n+1} = \psi^1_{B,n,n+1}$, and $\psi^{1,1}_{B,n,n+1} := \xi^{B_n,B^1_{n+1}}$ satisfies

(e 14.9) $$Sp(\xi^{B_n,B^1_{n+1}}) \supset Y_n \cup Sp(F_{X,n}).$$

Define $\xi^{B_n,B_{n+1}} = \psi^-_{B,n,n+1} \oplus \xi^{B_n,B^1_{n+1}}$. Then $Sp(\xi^{B_n,B_{n+1}}) \supset Y_n \cup Sp(F_{X,n})$. $q^{\sim}_{n+1} = 1^1_{B_{n+1}} - q_{n+1} = \pi_{B^1_{n+1}} \circ \bar{\varphi}^{C_n,B_{n+1}}(1_{C_n})$. Consider the map $\pi_{B^1_{n+1}} \circ \bar{\varphi}^{C_n,B_{n+1}} : C_n \to B^1_{n+1}$. Exactly as in 13.45, since $(\diamondsuit\diamondsuit)_1$ holds, one has a continuous path of unital homomorphisms $\Omega_{I,s} : C_n \to q^{\sim}_{n+1}B_{n+1}q^{\sim}_{n+1}$ with $\Omega_{I,0} = \pi_{B^1_{n+1}} \circ \bar{\varphi}^{C_n,B_{n+1}}$ and $\Omega_{I,1} := \xi^{C_n,B^1_{n+1}}$ such that

(e 14.10) $$Sp(\xi^{C_n,B^1_{n+1}}) \supset T_n \cup Sp(F_n).$$

Put $\bar{\varphi}^{C_n,B^-_{n+1}} = \pi^-_{B_{n+1}} \circ \bar{\varphi}^{C_n,B_{n+1}}$. Define, for all $(b,c) \in A_n = B_n \oplus C_n$,

(e 14.11) $$\begin{aligned} &\xi_{n,n+1}(b,c) \\ &= \xi^{B_n,B_{n+1}}(b) \oplus \xi^{C_n,B^1_{n+1}}(c) \oplus \bar{\varphi}^{C_n,B^-_{n+1}}(c) \oplus \varphi^{A_n,C_{n+1}}_{n,n+1}(b,c). \end{aligned}$$

Then, by (e 14.9) and (e 14.10), we have $Sp(\xi|_{x_{n+1,1}}) \supset \Omega_n \cup Sp(F_n)$. Hence, by (c) of 14.7 (see (5) and (8) of 13.15), $Sp(\xi_{m,n+2}|_x)$ is $1/n$-dense in $Sp(A_m)$ for all $m \leq n$. Note that $\xi_{n,n+1}$ and $\varphi_{n,n+1}$ are homotopic. It follows that $(\xi_{n,n+1})_{*i} = (\varphi_{n,n+1})_{*i}$, $i = 0,1$. Let $B = \lim_{n\to\infty}(A_n, \xi_{n,n+1})$. Then

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_B], K).$$

It follows from 13.5 that B is a unital simple C^*-algebra. By (e 14.7),

$$\lim_{n\to\infty} \|\xi^{\sharp}_{n,n+1} - \varphi^{\sharp}_{n,n+1}\| = 0.$$

Therefore, as in 13.47 (but much more simply),

(e 14.12) $$\begin{aligned} &(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B), r_B) \\ &= (G, G_+, u, K, \Delta, r). \end{aligned}$$

At this point, we would like to point out that, if $\ker\rho_A = \{0\}$ *and* $K_1(A) = \{0\}$, *then we do not need the direct summand* B_n *in the construction above.*

We may summarize the construction above as follows:

THEOREM 14.10. *Let* $((G, G_+, u), K, \Delta, r)$ *be a sextuple consisting of the following objects:* (G, G_+, u) *is a countable weakly unperforated simple order-unit abelian group,* K *is a countable abelian group,* Δ *is a metrizable Choquet simplex, and* $r : \Delta \to \mathrm{S}_u(G)$ *is a surjective continuous affine map, where* $S_u(G)$ *is the compact convex set of states of* (G, G_+, u)*. Assume that* (G, G_+, u) *has the property (SP) in the sense (as above) that for any real number* $s > 0$*, there is* $g \in G_+ \setminus \{0\}$ *such that* $\tau(g) < s$ *for any state* τ *on* G*.*

Then there is a unital simple C-algebra* $A \in \mathcal{B}_0$ *which can be written as* $A = \varinjlim(A_i, \psi_{i,i+1})$ *with injective* $\psi_{i,i+1}$*, where* $A_i = B_i \oplus C_i$*,* $B_i \in \mathbf{H}$*, and* $C_i \in \mathcal{C}_0$*, in such a way that*

(1) $\lim_{i\to\infty} \sup\{\tau(\psi_{i,\infty}(1_{B_i})) : \tau \in T(A)\} = 0$,
(2) $\ker\rho_A \subset \bigcup_{i=1}^{\infty}(\psi_{i,\infty})_{*0}(\ker\rho_{B_i})$*, and*
(3) $\mathrm{Ell}(A) \cong ((G, G_+, u), K, \Delta, r)$.

Moreover, the inductive system (A_i, ψ_i) *can be chosen so that* $\psi_{i,i+1} = \psi^{(0)}_{i,i+1} \oplus \psi^{(1)}_{i,i+1}$ *with* $\psi^{(0)}_{i,i+1} : A_i \to A^{(0)}_{i+1}$*,* $\psi^{(0)}_{i,i+1}$ *is non-zero on each summand of* A_i*, and* $\psi^{(1)}_{i,i+1} : A_i \to A^{(1)}_{i+1}$ *for C*-subalgebras* $A^{(0)}_{i+1}$ *and* $A^{(1)}_{i+1}$ *of* A_{i+1} *with* $1_{A^{(0)}_{i+1}} + 1_{A^{(0)}_{i+1}} = 1_{A_{i+1}}$ *such that* $A^{(0)}_{i+1}$ *is a non-zero finite dimensional C*-algebra, and* $(\psi_{i,\infty})_{*1}$ *is injective.*

Furthermore, if $K = \{0\}$ *and* $\mathrm{Inf}(G)$*=0, which implies that* G *is torsion free, we can assume that* $A_i = C_i$*,* $i = 1, 2, ...$

PROOF. Condition (2) follows from the fact that $\mathrm{Inf}(G) = \bigcup_{i=1}^{\infty} \tilde{\alpha}_{n,\infty}(\mathrm{Inf}_n)$ and (e 14.4), and (1) follows from (e 14.7). Since A is a unital simple inductive limit of subhomogeneous C*-algebras with no dimension growth, it then follows from Corollary 6.5 of [118] that A is $\mathcal{Z}$-stable. Hence A has strict comparison for positive elements (and projections). By (1) above, the strict comparison property mentioned above, and the fact that $C_i \in \mathcal{C}_0$, we conclude that $A \in \mathcal{B}_0$. (This actually also follows from our construction immediately.) □

REMARK 14.11. Note that $A^{(0)}_{i+1}$ can be chosen to be the first block B^1_{i+1}, so we have

$$\lim_{i\to\infty} \tau(\psi_{i+1,\infty}(1_{A^{(0)}_{i+1}})) = 0$$

uniformly for $\tau \in T(A)$. Moreover, for any i, there exists $n \geq i$ such that

$$\lim_{n\to\infty} \sup_{\tau\in T(A)} \tau(\psi_{n+1,\infty} \circ \psi^{(0)}_{n,n+1} \circ \psi_{i,n}(1_{A_i})) = 0. \qquad \text{(e 14.13)}$$

REMARK 14.12. Let $\psi^B_{n,n+1} := \psi^{B_n,B_{n+1}}_{n,n+1} : B_n \to B_{n+1}$ be the partial map of $\psi_{n,n+1} : A_n \to A_{n+1}$, and let $\psi^B_{n,m} : B_n \to B_m$ be the corresponding composition $\xi^B_{m-1,m} \circ \xi^B_{m-2,m-1} \circ \cdots \circ \xi^B_{n,n+1}$. Let $e_n = \xi_{1,n}(\mathbf{1}_{B_1})$. Then, from the construction, we know that the algebra $B = \varinjlim(e_nB_ne_n, \psi^B_{n,m})$ is simple, as we know that $SP(\xi^B_{n,n+1}|_{x_{n+1,1}})$ is dense enough in $Sp(B_n)$. Note that the simplicity of B does not follow from the simplicity of A itself, since it is not a corner of A.

COROLLARY 14.13. *Let $B_1 \in \mathcal{B}_0$ be a unital separable C^*-algebra and set $B = B_1 \otimes U$, where U is a UHF-algebra of infinite type (i.e., $U \otimes U = U$). Then there exists a C^*-algebra A with all the properties described in Theorem* 14.10 *such that* $\mathrm{Ell}(A) = \mathrm{Ell}(B)$. *Moreover, A may be chosen such that $A \otimes U \cong A$ and* 14.11 *also is valid for A.*

PROOF. Let $((G, G_+, u), K, \Delta, r) = (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B), r_B)$. Since $A = A' \otimes U$, it has the property (SP). Then, by Theorem 14.10, there is a C^*-algebra $A' \in \mathcal{B}_0$ which has all the properties of A described in 14.10 such that $\mathrm{Ell}(A') = \mathrm{Ell}(B)$. Put $A = A' \otimes U$. It is easy to check $\mathrm{Ell}(A' \otimes U) = \mathrm{Ell}(B \otimes U)$. Since $B \otimes U \cong B$, we conclude that $\mathrm{Ell}(A) = \mathrm{Ell}(B)$. Write $U = \lim_{n\to\infty}(M_{k(n)}, \imath_n)$, where $k(n+1) = r(n)k(n)$ and $\imath_n : M_{k(n)} \to M_{k(n+1)}$ is defined by $\imath_n(a) = a \otimes 1_{M_{r(n)}}$. Write $A' = \lim_{n\to\infty}(B_n \oplus C_n, \psi_{n,n+1})$. Then one checks $A \cong \lim_{n\to\infty}(B_n \otimes M_{k(n)} \oplus C_n \otimes M_{k(n)}, \psi_{n,n+1} \otimes \imath_n)$. It follows that A has all the properties described in 14.10 and also the one described in 14.11. Since $A = A' \otimes U$ and U is a UHF-algebra of infinite type, $A \otimes U \cong A$. $\Box$

The following result is an analog of Theorem 1.5 of [65].

COROLLARY 14.14. *Let A_1 be a simple separable C*-algebra in $\mathcal{B}_1$, and let $A = A_1 \otimes U$ for an infinite dimensional UHF-algebra U. There exists an inductive limit algebra B as constructed in Theorem* 14.10 *such that A and B have the same Elliott invariant. Moreover, the C*-algebra B may be chosen to have the following properties:*

Let G_0 be a finitely generated subgroup of $K_0(B)$ with decomposition $G_0 = G_{00} \oplus G_{01}$, where G_{00} vanishes under all states of $K_0(A)$. Suppose $\mathcal{P} \subset \underline{K}(B)$ is a finite subset which generates a subgroup G such that $G_0 \subset G \cap K_0(B)$.

Then, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset B$, any $1 > r > 0$, and any positive integer K, there is an $\mathcal{F}$-ϵ-multiplicative map $L : B \to B$ such that:

(1) *$[L]|_{\mathcal{P}}$ is well defined.*
(2) *$[L]$ induces the identity maps on the infinitesimal part of $G \cap K_0(B)$, $G \cap K_1(B)$, $G \cap K_0(B, \mathbb{Z}/k\mathbb{Z})$ and $G \cap K_1(B, \mathbb{Z}/k\mathbb{Z})$ for $k = 1, 2, \cdots$.*
(3) *$\rho_B \circ [L](g) \le r\rho_B(g)$ for all $g \in G \cap K_0(B)$, where ρ_B is the canonical positive homomorphism from $K_0(B)$ to* $\mathrm{Aff}(\mathrm{S}(K_0(B), K_0(B)_+, [1_B]))$.
(4) *For any positive element $g \in G_{01}$, there exists $f \in K_0(B)_+$ with $g - [L](g) = Kf$.*

PROOF. As is pointed out in the last paragraph of 14.1, the Elliott invariant of $A_1 \otimes U$ has the property (SP). Replacing A_1 by $A_1 \otimes U$, we may assume that $\mathrm{Ell}(A_1)$ has property (SP).

Consider $\mathrm{Ell}(A_1)$, which satisfies the conditions of Theorem 14.10. Therefore, by the first part of Theorem 14.10, there is an inductive system $B_1 = \varinjlim(T_i \oplus S_i, \psi_{i,i+1})$ such that

(i) $T_i \in \mathbf{H}$ and $S_i \in \mathcal{C}_0$ with $K_1(S_i) = \{0\}$,
(ii) $\lim \tau(\varphi_{i,\infty}(1_{T_i})) = 0$ uniformly on $\tau \in T(B_1)$,
(iii) $\ker(\rho_{B_1}) = \bigcup_{i=1}^{\infty}(\psi_{i,\infty})_{*0}(\ker(\rho_{T_i}))$, and
(iv) $\mathrm{Ell}(B_1) = \mathrm{Ell}(A_1)$.

Put $B = B_1 \otimes U$. Then $\mathrm{Ell}(A) = \mathrm{Ell}(B)$. Let $\mathcal{P} \subset \underline{K}(B)$ be a finite subset, and let G be the subgroup generated by $\mathcal{P}$, which we may assume contains G_0. Then there is a positive integer M' such that $G \cap K_*(B, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ if $k > M'$. Put $M = M'!$. Then $Mg = 0$ for any $g \in G \cap K_*(B, \mathbb{Z}/k\mathbb{Z})$, $k = 1, 2, \dots$.

Let $\varepsilon > 0$, a finite subset $\mathcal{F} \subset B$, and $0 < r < 1$ be given. Choose a finite subset $\mathcal{G} \subset B$ and $0 < \varepsilon' < \varepsilon$ such that $\mathcal{F} \subset \mathcal{G}$ and for any $\mathcal{G}$-ϵ'-multiplicative map $L : B \to B$, the map $[L]_{\mathcal{P}}$ is well defined, and $[L]$ is a homomorphism on G (see 2.12).

Choosing a sufficiently large i_0, we may assume that $[\psi_{i_0,\infty}](\underline{K}(T_{i_0} \oplus S_{i_0})) \supset G$. In particular, we may assume, by (iii) above, that $G \cap \ker\rho_{B_1} \subset (\psi_{i_0,\infty})_{*0}(\ker\rho_{T_{i_0}})$. Let $G' \subset \underline{K}(T_{i_0} \oplus S_{i_0})$ be such that $[\psi_{i_0,\infty}](G') \supset G$. Since $B = B_1 \otimes U$, we may write $U = \varinjlim(M_{m(n)}, \imath_{n,n+1})$, where $m(n)|m(n+1)$ and $\imath_{n,n+1} : M_{m(n)} \to M_{m(n+1)}$ is defined by $a \mapsto a \otimes 1_{m(n+1)}$. One may assume that for each $f \in \mathcal{G}$, there exists $i > i_0$ such that

$$f = (f_0 \oplus f_1) \otimes 1_m \in (T_i' \oplus S_i') \otimes M_m \tag{e 14.14}$$

for some $f_0 \in T_i'$, $f_1 \in S_i'$, and $m > 2MK/r$, where $m = m(i+1)m(i+2)\cdots m(n)$, $T_i' = \psi_{i,\infty}'(T_i \otimes M_{m(i)})$, $S_i' = \psi_{i,\infty}'(S_i \otimes M_{m(i)})$, and where $\psi_{i,\infty} = \psi_{i,\infty} \otimes \imath_{i,\infty}$. Moreover, one may assume that $\tau(1_{T_i'}) < r/2$ for all $\tau \in T(A_1)$.

Choose a large n such that $m = M_0 + l$ with M_0 divisible by KM and $0 \leq l < KM$. Then define the map

$$L : (T_i' \oplus S_i') \otimes M_m \to (T_i' \oplus S_i') \otimes M_m$$

to be

$$L((f_{i,j} \oplus g_{i,j})_{m\times m}) = (f_{i,j})_{m\times m} \oplus E_l(g_{i,j})_{m\times m}E_l,$$

where $E_l = \mathrm{diag}(\underbrace{1_{S_i'}, 1_{S_i'}, \dots, 1_{S_i'}}_{l}, \underbrace{0, 0, \dots, 0}_{M_0})$, which is a contractive completely positive linear map from $(T_I' \oplus S_i') \otimes M_m$ to B, where we identify B with $B \otimes M_m$. We then extend L to a completely positive linear map

$$B \to (1_B - (\mathbf{1}_{M_m(S_i')} - E_l))B(1_B - (\mathbf{1}_{M_m(S_i')} - E_l)).$$

Also define

$$R : (T_i' \oplus S_i') \otimes M_m \to T_i' \oplus S_i'$$

to be

$$(\text{e}\,14.15)\qquad R\begin{pmatrix} f_{1,1} \oplus g_{1,1} & f_{1,2} \oplus g_{1,2} & \cdots & f_{1,m} \oplus g_{1,m} \\ f_{2,1} \oplus g_{2,1} & f_{2,2} \oplus g_{2,2} & \cdots & f_{,m} \oplus g_{2,m} \\ & \ddots & & \\ f_{m,1} \oplus g_{m,1} & f_{m,2} \oplus g_{m,2} & \cdots & f_{m,m} \oplus g_{m,m} \end{pmatrix} = g_{1,1},$$

where $f_{j,k} \in T_i'$ and $g_{j,k} \in S_i'$, and extend it to a contractive completely positive linear map $B \to B$, where $T_i' \oplus S_i'$ is regarded as a corner of $(T_i' \oplus S_i') \otimes M_m \subset B$. Then L and R are $\mathcal{G}$-ϵ'-multiplicative. Hence $[L]|_{\mathcal{P}}$ is well defined. Moreover,

$$\tau(L(1_A)) < \tau(1_{T_1}) + \frac{l}{m} < \frac{r}{2} + \frac{MK}{2MK/r} = r \text{ for all } \tau \in T(A).$$

Note that for any f in the form (e 14.14), if f is written in the form $(f_{jk} \oplus g_{jk})_{m\times m}$, then $g_{jj} = g_{11}$ and $g_{jk} = 0$ for $j \neq k$. Hence one has

$$f = L(f) + \overline{R}(f),$$

where $\overline{R}(f)$ may be written as

$$\overline{R}(f) = \text{diag}\{\underbrace{0, 0, ..., 0}_{l}, \underbrace{(0 \oplus g_{1,1}), ..., (0 \oplus g_{1,1})}_{M_0}\}.$$

Hence for any $g \in G$,

$$g = [L](g) + M_0[R](g).$$

Then, if $g \in (G_{0,1})_+ \subset (G_0)_+$, one has

$$g - [L](g) = M_0[R](g) = K((\frac{M_0}{K})[R](g)).$$

And if $g \in G \cap K_i(B, \mathbb{Z}/k\mathbb{Z})$ $(i = 0, 1)$, one also has

$$g - [L](g) = M_0[R](g).$$

Since $Mg = 0$ and $M|M_0$, one has $g - [L](g) = 0$.

Since L is the identity on $\psi_{i,\infty}'(T_i \otimes M_{m(i)})$ and $i > i_0$, by (iii), L is the identity map on $G \cap \ker \rho_B$. Since $K_1(S_i) = 0$ for all i, L induces the identity map on $G \cap K_1(B)$. It follows that L is the desired map. □

Related to the considerations above we have the following decomposition:

PROPOSITION 14.15. *Let A_1 be a separable amenable C^*-algebra in $\mathcal{B}_1$ (or $\mathcal{B}_0$) and let $A = A_1 \otimes U$ for some infinite dimensional UHF-algebra U. Let $\mathcal{G} \subset A$, $\mathcal{P} \subset \underline{K}(A)$ be finite subsets, $\mathcal{P}_0 \subset A \otimes \mathcal{K}$ be a finite subset of projections, and let $\epsilon > 0$, $0 < r_0 < 1$ and $M \in \mathbb{N}$ be arbitrary. Then there are a projection $p \in A$, a C^*-subalgebra $B \in \mathcal{C}$ (or $\mathcal{C}_0$) with $p = 1_B$ and $\mathcal{G}$-ϵ-multiplicative unital completely positive linear maps $L_1 : A \to (1-p)A(1-p)$ and $L_2 : A \to B$ such that:*

(1) $\|L_1(x) + L_2(x) - x\| < \varepsilon$ *for all* $x \in \mathcal{G}$;
(2) $[L_i]|_{\mathcal{P}}$ *is well defined,* $i = 1, 2$;
(3) $[L_1]|_{\mathcal{P}} + [\imath \circ L_2]|_{\mathcal{P}} = [\mathrm{id}]|_{\mathcal{P}}$;
(4) $\tau \circ [L_1](g) \le r_0 \tau(g)$ *for all* $g \in \mathcal{P}_0$ *and* $\tau \in T(A)$;
(5) *For any* $x \in \mathcal{P}$, *there exists* $y \in \underline{K}(B)$ *such that* $x - [L_1](x) = [\imath \circ L_2](x) = M[\imath](y)$; *and*
(6) *for any* $d \in \mathcal{P}_0$, *there exist positive element* $f \in K_0(B)_+$ *such that*

$$d - [L_1](d) = [\imath \circ L_2](d) = M[\imath](f),$$

where $\imath : B \to A$ is the embedding. Moreover, we can require that $1 - p \neq 0$.

PROOF. Since A is in $\mathcal{B}_1$ (or $\mathcal{B}_0$), there is a sequence of projections $p_n \in A$ and a sequence of C^*-subalgebras $B_n \in \mathcal{B}_1$ (or $\mathcal{B}_0$) with $1_{B_n} = p_n$ such that

$$\lim_{n\to\infty} \|(1-p_n)a(1-p_n) + p_n a p_n - a\| = 0, \tag{e 14.16}$$

$$\lim_{n\to\infty} \mathrm{dist}(p_n a p_n, B_n) = 0, \text{ and} \tag{e 14.17}$$

$$\lim_{n\to\infty} \max\{\tau(1-p_n) : \tau \in T(A)\} = 0 \tag{e 14.18}$$

for all $a \in A$. Since each B_n is amenable, one obtains easily a sequence of unital completely positive linear maps $\Psi_n : A \to B_n$ such that

$$\lim_{n\to\infty} \|p_n a p_n - \Psi_n(a)\| = 0 \text{ for all } a \in A. \tag{e 14.19}$$

In particular,

$$\lim_{n\to\infty} \|\Psi_n(ab) - \Psi_n(a)\Psi_n(b)\| = 0 \text{ for all } a, b \in A. \tag{e 14.20}$$

Let $j : A \to A \otimes U$ be defined by $j(a) = a \otimes 1_U$. There is a unital homomorphism $s : A \otimes U \to A$ and a sequence of unitaries $u_n \in A \otimes U$ such that

$$\lim_{n\to\infty} \|a - \mathrm{Ad}\, u_n \circ s \circ j(a)\| = 0 \text{ for all } a \in A. \tag{e 14.21}$$

There are non-zero projections $e_n' \in U$ and $e_n \in U$ such that

$$\lim_{n\to\infty} t(e_n) = 0 \text{ and } 1 - e_n = \mathrm{diag}(\overbrace{e_n', e_n', ..., e_n'}^{M}), \tag{e 14.22}$$

where $t \in T(U)$ is the unique tracial state on U. Choose $N \geq 1$ such that

$$0 < t(e_n) < r_0/2 \text{ and } \max\{\tau(1-p_n) : \tau \in T(A)\} < r_0/2.$$

Define $\Phi_n : A \to (1-p_n)A(1-p_n)$ by $\Phi_n(a) = (1-p_n)a(1-p_n)$ for all $a \in A$. Define $\Phi'_n(a) = \Phi_n(a) \oplus \mathrm{Ad}\, u_n \circ s(a \otimes e_n)$, and $\Psi'_n(a) = \mathrm{Ad}\, u_n \circ s(\Psi(a) \otimes (1-e_n))$ for all $n \geq N$. Note that $u_n^* s(B_n \otimes (1-e_n))u_n \in \mathcal{C}_1$ (or $\mathcal{C}_0$). It is then easy to verify that, if we choose a large n, the maps $L_1 = \Phi'_n$ and $L_2 = \Psi'_n$ meet the requirements. $\square$

15. **Positive Maps from the K_0-group of a C^*-algebra in $\mathcal{C}$**

This section contains some technical lemmas about positive homomorphisms from $K_0(C)$ for some $C \in \mathcal{C}$.

LEMMA 15.1 (cf. 2.8 of [61]). *Let $G \subset \mathbb{Z}^l$ (for some $l > 1$) be a subgroup. There is an integer $M > 0$ satisfying the following condition: Let $1 > \sigma_1, \sigma_2 > 0$ be any given numbers. Then, there is an integer $R > 0$ such that: if a set of l positive numbers $\alpha_i \in \mathbb{R}_+$ $(i = 1, 2, ..., l)$ satisfies $\alpha_i \geq \sigma_1$ for all i and satisfies*

$$\sum_{i=1}^{l} \alpha_i m_i \in \mathbb{Z} \text{ for all } (m_1, m_2, ..., m_l) \in G, \tag{e 15.1}$$

then for any integer $K \geq R$, there exists a set of positive rational numbers $\beta_i \in \frac{1}{KM}\mathbb{Z}_+$ $(i = 1, 2, ..., l)$ such that

$$\sum_{i=1}^{l} |\alpha_i - \beta_i| < \sigma_2 \text{ and } \tilde{\varphi}|_G = \varphi|_G, \tag{e 15.2}$$

where $\varphi((n_1, n_2, ..., n_l)) = \sum_{i=1}^{l} \alpha_i n_i$ and $\tilde{\varphi}((n_1, n_2, ..., n_l)) = \sum_{i=1}^{l} \beta_i n_i$ for all $(n_1, n_2, ..., n_l) \in \mathbb{Z}^l$.

PROOF. Denote by $e_j \in \mathbb{Z}^l$ the element having 1 in the j-th coordinate and 0 elsewhere. First we consider the case that $\mathbb{Z}^l/G$ is finite. In this case there is an integer $M \geq 1$ such that $Me_j \in G$ for all $j = 1, 2, ..., l$. It follows that $\varphi(Me_j) \in \varphi(G) \subset \mathbb{Z}$, $j = 1, 2, ..., l$. Hence $\alpha_j = \varphi(e_j) \in \frac{1}{M}\mathbb{Z}_+$. We choose $\beta_j = \alpha_j$, $j = 1, 2, ..., l$, and $\tilde{\varphi} = \varphi$. The conclusion of the lemma follows—that is, for any σ_1, σ_2, we can choose $R = 1$.

Now we assume that $\mathbb{Z}^l/G$ is not finite. Regard $\mathbb{Z}^l$ as a subset of $\mathbb{Q}^l$ and set H_0 to be the vector subspace of $\mathbb{Q}^l$ spanned by elements in G. Since G is a subgroup of $\mathbb{Z}^l$, it must be finitely generated and hence is isomorphic to $\mathbb{Z}^p$, where p is the (torsion–free) rank of G. Since the rank of $\mathbb{Z}^l/G$ is strictly positive (otherwise $\mathbb{Z}^l/G$ would be a finitely generated torsion group, hence finite), and using the fact that the rank is additive, one concludes that $0 \leq p < l$.

Let $g_1, g_2, ..., g_p \in G$ be indepdent generators of G. View them as elements in $H_0 \subset \mathbb{Q}^l$ and write

$$g_i = (g_{i,1}, g_{i,2}, ..., g_{i,l}), \;\; i = 1, 2, ..., p. \tag{e 15.3}$$

Define $L : \mathbb{Q}^p \to \mathbb{Q}^l$ to be $L = (f_{i,j})_{l\times p}$, where $f_{i,j} = g_{j,i}$, $i = 1, 2, ..., l$ and $j = 1, 2, ..., p$. Then $L^* = (g_{i,j})_{p\times l}$. We also view $L^* : \mathbb{Q}^l \to \mathbb{Q}^p$. Define $T = L^*L : \mathbb{Q}^p \to \mathbb{Q}^p$. This map is invertible. Note that $T = T^*$ and $(T^{-1})^* = T^{-1}$. Note also that the matrix representation $(a_{i,j})_{l\times p}$ of $L \circ T^{-1}$ is an $l \times p$ matrix with entries in $\mathbb{Q}$. There is an integer $M_1 \geq 1$ such that $a_{i,j} \in \frac{1}{M_1}\mathbb{Z}$, $i = 1, 2, ..., l$ and $j = 1, 2, ..., p$.

Let $H_{00} = \ker L^*$. It has dimension $l - p > 0$. Let $P : \mathbb{Q}^l \to H_{00}$ be an orthogonal projection which is a $\mathbb{Q}$-linear map. Represent P as an $l \times l$ matrix. Then its entries are in $\mathbb{Q}$. There is an integer $M_2 \geq 1$ such that all entries are in $\frac{1}{M_2}\mathbb{Z}$. We will use the fact that $L^* = L^*(1 - P)$.

It is important to note that M_1 and M_2 depend on G only and are independent of $\{\alpha_j : 1 \leq j \leq l\}$. Let $M = M_1M_2$.

Suppose that $\sigma_1, \sigma_2 \in (0, 1)$ are two positive numbers and the numbers $\alpha_i \geq \sigma_1$ ($i = 1, 2, ..., l$) satisfy condition (e 15.1). The condition (e 15.1) is equivalent to the condition that $b_i := \sum_{j=1}^{l} \alpha_j g_{i,j} \in \mathbb{Z}$, $i = 1, 2, ..., p$. Put $b = (b_1, b_2, ..., b_p)^T$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)^T$. Then $b = L^*\alpha$.

If we write

$$L(T^*)^{-1}b = c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_l \end{pmatrix}, \tag{e 15.4}$$

then, since $b \in \mathbb{Z}^p$, one has $c_i \in \frac{1}{M_1}\mathbb{Z}$. Choose an integer $R \geq 1$ such that $1/R < \sigma_1\sigma_2/(4l^2)$. Let $K \geq R$ be any integer. Note that

$$L^*c = L^*L(T^*)^{-1}b = L^*LT^{-1}b = L^*\alpha. \tag{e 15.5}$$

Thus $\alpha - c \in \ker L^*$ as a subspace of $\mathbb{R}^l$.

For the space $\mathbb{R}^l$, we use $\|\cdot\|_1$ and $\|\cdot\|_2$ to denote the l_1 norm and l_2 norm on it, respectively. Then we have

$$\|v\|_2 \leq \|v\|_1 \leq l\|v\|_2 \quad \text{ for all } v \in \mathbb{R}^l.$$

Since H_{00} is dense in the real subspace $\ker L^*$, there exists $\xi \in H_{00}$ such that

$$\|\alpha - c - \xi\|_2 \leq \|\alpha - c - \xi\|_1 < \sigma_1\sigma_2/(4l). \tag{e 15.6}$$

Pick $\eta \in \mathbb{Q}^l$ such that $\xi = P\eta$. Then there is $\eta_0 \in \mathbb{Q}^l$ such that $K\eta_0 \in \mathbb{Z}^l$ and

$$\|\eta_0 - \eta\|_2 \leq \|\eta_0 - \eta\|_1 < \sigma_1\sigma_2/(2l). \tag{e 15.7}$$

Since P has norm one with respect to the l_2 norm,

$$\|\alpha - c - P\eta_0\|_1 \le l\|\alpha - c - P\eta_0\|_2 \le l(\|\alpha - c - \xi\|_2 + \|P(\eta_0 - \eta)\|_2) < \sigma_1\sigma_2.$$

Put $\beta = c + P\eta_0 = (\beta_1, \beta_2, ..., \beta_l)^T$. Note that $M_2K(P\eta_0) \in \mathbb{Z}^l$, and that $M_1c \in \mathbb{Z}^l$.

We have $KM\beta \in \mathbb{Z}^l$, and

$$L^*\beta = L^*c = L^*\alpha \text{ and } \|\alpha - \beta\|_1 < \sigma_1\sigma_2. \tag{e 15.8}$$

Moreover, since $\alpha_i \ge \sigma_1$,

$$\beta_i > 0, \;\; i = 1, 2, ..., l. \tag{e 15.9}$$

Since $P\eta_0 \in H_{00}$, one has that $L^*\beta = L^*(1-P)\beta = L^*(1-P)c = L^*c = L^*\alpha = b$. Define $\tilde{\varphi} : \mathbb{Q}^l \to \mathbb{Q}$ by

$$\tilde{\varphi}(x) = \langle x, \beta\rangle \tag{e 15.10}$$

for all $x \in \mathbb{Q}^l$. Note $L^*e_i = g_i$, where e_i is the element in $\mathbb{Z}^p$ with i-th coordinate 1 and 0 elsewhere. So

$$\begin{aligned}\varphi(g_i) &= \langle Le_i, \alpha\rangle = \langle e_i, L^*\alpha\rangle = \langle e_i, L^*\beta\rangle \\ &= \langle Le_i, \beta\rangle = \langle g_i, \beta\rangle = \tilde{\varphi}(g_i),\end{aligned}$$

$i = 1, 2, ..., p$. It follows that $\tilde{\varphi}(g) = \varphi(g)$ for all $g \in G$. Hence $\tilde{\varphi}|_G = \varphi|_G$. Note that $\tilde{\varphi}(\mathbb{Z}^l) \subset \frac{1}{KM}\mathbb{Z}$, since $\beta_i \in \frac{1}{KM}\mathbb{Z}_+$, $i = 1, 2, ..., l$. □

If we do not need to approximate $\{\alpha_i : 1 \le i \le l\}$, then R can be chosen to be 1, with $M = M_1$, which only depends on G and l (by replacing β by c in the proof).

From the proof of 15.1, since L and $(T^*)^{-1}$ depend only on $g_1, g_2, ..., g_p$, we have the following result:

LEMMA 15.2. *Let $G \subset \mathbb{Z}^l$ be an ordered subgroup with order unit e, and let $g_1, g_2, ..., g_p$ ($p \le l$) be a set of free generators of G. For any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following condition: if $\varphi : G \to \mathbb{R}$ is a homomorphism such that*

$$|\varphi(g_i)| < \delta, \;\; i = 1, 2, ..., p,$$

then there is $\beta = (\beta_1, \beta_2, ..., \beta_l) \in \mathbb{R}^l$ with $|\beta_i| < \varepsilon$, $i = 1, 2, ..., l$, such that

$$\varphi(g) = \psi(g) \text{ for all } g \in G,$$

where $\psi : \mathbb{Z}^l \to \mathbb{R}$ is defined by $\psi((m_1, m_2, ..., m_l)) = \sum_{i=1}^l \beta_i m_i$ for all $m_i \in \mathbb{Z}$.

COROLLARY 15.3. *Let $G \subset \mathbb{Z}^l$ be an ordered subgroup. Then, there exists an integer $M \geq 1$ satisfying the following condition: for any positive homomorphism $\kappa : G \to \mathbb{Z}^n$ (for any integer $n \geq 1$) with every element in $\kappa(G)$ divisible by M, there is $R_0 \geq 1$ such that, for any integer $K \geq R_0$, there is a positive homomorphism $\tilde{\kappa} : \mathbb{Z}^l \to \mathbb{Z}^n$ such that $\tilde{\kappa}|_G = K\kappa$.*

PROOF. We first prove the case that $n = 1$.

Let $S \subset \{1, 2, ..., l\}$ be a subset and denote by $\mathbb{Z}^{(S)}$ the subset

$$\mathbb{Z}^{(S)} = \{(m_1, m_2, ..., m_l) : m_i = 0 \text{ if } i \notin S\}.$$

Let $\Pi_S : \mathbb{Z}^l \to \mathbb{Z}^{(S)}$ be the obvious projection and $G(S) = \Pi_S(G)$.

Let $M(S)$ be the integer (in place of M) as in 15.1 associated with $G(S) \subset \mathbb{Z}^{(S)}$. Put $M = \prod_{S \subset \{1,2,...,l\}} M(S)$.

Now assume that $\kappa : G \to \mathbb{Z}$ is a positive homomorphism with multiplicity M—that is, every element in $\kappa(G)$ is divisible by M.

By applying 2.8 of [61], we obtain a positive homomorphism $\beta : \mathbb{Z}^l \to \mathbb{R}$ such that $\beta|_G = \kappa$. Define $f_i = \beta(e_i)$, where e_i is the element in $\mathbb{Z}^l$ with 1 at the i-th coordinate and 0 elsewhere, $i = 1, 2, ..., l$. Then $f_i \geq 0$. Choose S such that $f_i > 0$ if $i \in S$ and $f_i = 0$ if $i \notin S$.

Evidently if $\xi_1, \xi_2 \in \mathbb{Z}^l$ satisfy $\Pi_S(\xi_1) = \Pi_S(\xi_2)$, then $\beta(\xi_1) = \beta(\xi_2)$, and if we further assume $\xi_1, \xi_2 \in G$ then $\kappa(\xi_1) = \kappa(\xi_2)$. Hence the maps β and κ induce maps $\beta' : \mathbb{Z}^{(S)} \to \mathbb{R}$ and $\kappa' : G(S) \to \mathbb{Z}$ such that $\beta = \beta' \circ \Pi_S$ and $\kappa = \kappa' \circ (\Pi_S)|_G$. In addition, we have $\beta'|_{G(S)} = \kappa'$.

Let $\sigma_1 = 2\sigma_2 = \frac{\min\{f_i:\ i \in S\}}{2M}$. By applying Lemma 15.1 to $\alpha_i = f_i/M > \sigma_1$ for $i \in S$ and to $G(S) \subset \mathbb{Z}^{(S)}$, we obtain the number $R(\kappa)$ (depending on σ_1 and σ_2 and therefore depending on κ) as in the lemma. For any $K \geq R(\kappa)$, it follows from the lemma that there are $\beta_i \in \frac{1}{KM}\mathbb{Z}_+$ (for $i \in S$) such that $\tilde{\kappa}'|_{G(S)} = \frac{1}{M}\kappa'$, where $\tilde{\kappa}' : \mathbb{Z}^{(S)} \to \mathbb{Q}$ is defined by $\tilde{\kappa}'(\{m_i\}_{i \in S}) = \sum_{i \in S} \beta_i m_i$. Evidently, $\tilde{\kappa} = KM(\tilde{\kappa}' \circ \Pi_S) : \mathbb{Z}^l \to \mathbb{Z}$ is as desired for this case.

This proves the case $n = 1$.

In general, let $s_i : \mathbb{Z}^n \to \mathbb{Z}$ be the projection onto the i-th direct summand, $i = 1, 2, ..., n$. Apply the case $n = 1$ to each of the the maps $\kappa_i := s_i \circ \kappa$ (for $i = 1, 2, ..., n$) to obtain $R(\kappa_i)$, and let $R_0 = \max_i R(\kappa_i)$. For any $K \geq R_0$, by what has been proved, we obtain $\tilde{\kappa}_i : \mathbb{Z}^l \to \mathbb{Z}$ such that

$$\tilde{\kappa}_i|_G = K s_i \circ \kappa|_G, \ \ i = 1, 2, ..., n. \tag{e 15.11}$$

Define $\tilde{\kappa} : \mathbb{Z}^l \to \mathbb{Z}^n$ by $\tilde{\kappa}(z) = (\tilde{\kappa}_1(z), \tilde{\kappa}_2(z), ..., \tilde{\kappa}_n(z))$. The lemma follows. □

LEMMA 15.4 (Lemma 3.2 of [36]). *Let G be a countable abelian unperforated ordered group such that G_+ is finitely generated and let $r : G \to \mathbb{Z}$ be a strictly positive homomorphism, i.e., $r(G_+ \setminus \{0\}) \subset \mathbb{Z}_+ \setminus \{0\}$. Then, for any order unit $u \in G_+$, there exists a natural number m such that if the map $\theta : G \to G$ is defined by $g \mapsto r(g)u$, then the positive homomorphism $\mathrm{id} + m\theta : G \to G$ factors through $\bigoplus_{i=1}^n \mathbb{Z}$ positively for some n.*

PROOF. Let u be an order unit of G, which exists as G is finitely generated, and define the map $\varphi : G \to G$ by $\varphi(g) = g + r(g)u$; that is, $\varphi = \mathrm{id} + \theta$. Define $G_n = G$ and $\varphi_n : G \to G$ by $\varphi_n(g) = \varphi(g)$ for all g and n. Consider the inductive limit

$$G \xrightarrow{\varphi} G \xrightarrow{\varphi} \cdots \longrightarrow \varinjlim G.$$

Then the ordered group $\varinjlim G$ has the Riesz decomposition property. In fact, let $a, b, c \in (\varinjlim G)_+$ such that

$$a \le b + c.$$

Without loss of generality, one may assume that $a \neq b + c$.

We may assume that there are $a', b', c' \in G_+$ for some n-th (finite stage) G such that $\varphi_{n,\infty}(a') = a$, $\varphi_{n,\infty}(b') = b$, and $\varphi_{n,\infty}(c') = c$, and furthermore

$$a' < b' + c'. \tag{e 15.12}$$

A straightforward calculation shows that for each k, there is $m(k) \in \mathbb{N}$ such that

$$\varphi_{n,n+k}(a') = a' + m(k)r(a')u, \ \varphi_{n,n+k}(b') = b' + m(k)r(b')u,$$

and

$$\varphi_{n,n+k}(c') = c' + m(k)r(c')u.$$

Moreover, the sequence $(m(k))$ is strictly increasing. Since r is strictly positive, combining with (e 15.12), we have that

$$r(a') < r(b') + r(c') \ \text{(in } \mathbb{Z}_+).$$

There are, for $i = 1, 2$, $l_i \in \mathbb{Z}_+$ such that

$$l_1 + l_2 = r(a'), \ l_1 \le r(b'), \ \text{and} \ l_2 \le r(c').$$

Without loss of generality, we may assume $d = r(b') - l_1 > 0$ (otherwise we let $d = r(c') - l_2$). Since u is an order unit, there is $m_1 \in \mathbb{Z}_+$ such that

$$m_1 du > a'.$$

Choose $k \ge 1$ such that $m(k) > m_1$. Let $a_1 = a' + m(k)l_1u$ and $a_2 = m(k)l_2u$. Then

$$a_2 = m(k)l_2u \le m(k)r(c')u \le c' + m(k)r(c')u = \varphi_{n,n+k}(c').$$

Moreover,

$$a_1 = a' + m(k)l_1u \le m(k)du + m(k)l_1u \le b' + m(k)r(b')u = \varphi_{n,n+k}(b').$$

Note

$$\varphi_{n,n+k}(a') = a_1 + a_2 \le \varphi_{n,n+k}(b') + \varphi_{n,n+k}(c').$$

These inequalities imply that

$$a = \varphi_{n+k,\infty}(a_1) + \varphi_{n+k,\infty}(a_2) \le b + c, \tag{e 15.13}$$

$$\varphi_{n+k,\infty}(a_1) \le b \text{ and } \varphi_{n+k,\infty}(a_2) \le c. \tag{e 15.14}$$

It follows that the limit group $\varinjlim G$ has the Riesz decomposition property. Since G is unperforated, so is $\varinjlim G$. It then follows from the Effros-Handelman-Shen Theorem ([26]) that the ordered group $\varinjlim G$ is a dimension group. Therefore, since G and G_+ are finitely generated, for sufficiently large k, the map $\varphi_{1,k} : G \to G$ factors positively through $\bigoplus_{i=1}^n \mathbb{Z}$ for some n. As already pointed out, the map $\varphi_{1,k}$ is of the desired form $\mathrm{id} + m\theta$ for some $m \ge 0$. The lemma follows. □

Let G_1 and G_2 be groups and $K \ge 1$ be an integer. A homomorphism $\gamma : G_1 \to G_2$ is said to have multiplicity K, if there is, for each $g_1 \in G_1$, $\gamma(g_1) = Kg_2$ for some $g_2 \in G_2$.

LEMMA 15.5. *Let (G, G_+, u) be an ordered group with order unit u such that the positive cone G_+ is generated by finitely many positive elements which are strictly smaller than u. Let $\lambda : G \to K_0(A)$ be an order preserving map such that $\lambda(u) = [1_A]$ and $\lambda(G_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$, where $A \in \mathcal{B}_1$ (or $A \in \mathcal{B}_0$). Let $a \in K_0(A)_+ \setminus \{0\}$ with $a \le [1_A]$. Let $\mathcal{P} \subset G_+ \setminus \{0\}$ be a finite subset. Suppose that there exists an integer $N \ge 1$ such that $N\lambda(x) > [1_A]$ for all $x \in \mathcal{P}$.*

Then, there are two positive homomorphisms $\lambda_0 : G \to K_0(A)$ and $\gamma : G \to K_0(S')$, with $S' \subset A$ and $S' \in \mathcal{C}$ (or $\mathcal{C}_0$) such that $\gamma(u) = [1_{S'}]$ and

$$\lambda = \lambda_0 + \lambda_1, \ \ 0 \le \lambda_0(u) < a \text{ and } \gamma(g) > 0 \tag{e 15.15}$$

for all $g \in G_+ \setminus \{0\}$, where $\lambda_1 = \imath_{*0} \circ \gamma$ and $\imath : S' \to A$ is the embedding. Moreover, $N\gamma(x) \ge \gamma(u)$ for all $x \in \mathcal{P}$. Furthermore, if $A = A_1 \otimes U$, where U is an infinite dimensional UHF-algebra, and $A_1 \in \mathcal{B}_1$ (or $\mathcal{B}_0$), then, for any integer $K \ge 1$, we can require that $S' = S \otimes M_K$ for some $S \in \mathcal{C}$ (or $\mathcal{C}_0$) and γ has multiplicity K.

PROOF. Let $\{g_1, g_2, ..., g_m\} \subset G_+$ be a set of generators of G_+ with $g_i < u$. Since A has stable rank one, it is easy to check that there are projections $q_1, q_2, ..., q_m \in A$ such that $\lambda(g_i) = [q_i]$, $i = 1, 2, ..., m$. To simplify notation, let us assume that $\mathcal{P} = \{g_1, g_2, ..., g_m\}$. Define

$$Q_i = \mathrm{diag}(\overbrace{q_i, q_i, ..., q_i}^{N}), \ \ i = 1, 2, ..., m.$$

By the assumption, there are $v_i \in M_N(A)$ such that

$$v_i^* v_i = 1_A \text{ and } v_i v_i^* \le Q_i, \ \ i = 1, 2, ..., m. \tag{e 15.16}$$

Since $A \in \mathcal{B}_1$ (or $\mathcal{B}_0$), there exist a sequence of projections $\{p_n\}$ in A, a sequence of C^*-subalgebras $S_n \in \mathcal{C}$ ($\mathcal{C}_0$) with $p_n = 1_{S_n}$, and a sequence of unital completely positive linear maps $L_n : A \to S_n$ such that

$$\lim_{n\to\infty} \|a - ((1-p_n)a(1-p_n) + p_n a p_n)\| = 0, \quad \lim_{n\to\infty} \|L_n(a) - p_n a p_n\| = 0,$$

and

$$\lim_{n\to\infty} \sup\{\tau(1-p_n) : \tau \in T(A)\} = 0, \tag{e 15.17}$$

for all $a \in A$. It is also standard to find, for each i, a projection $e'_{i,n} \in (1-p_n)A(1-p_n)$, a projection $e_{i,n} \in M_N(S_n)$, and a partial isometry $w_{i,n} \in M_N(S_n)$ such that

$$\lim_{n\to\infty} \|(1-p_n)q_i(1-p_n) - e'_{i,n}\| = 0, \tag{e 15.18}$$

$$w^*_{i,n} w_{i,n} = p_n, \quad w_{i,n} w^*_{i,n} \le e_{i,n}, \tag{e 15.19}$$

$$\lim_{n\to\infty} \|(L_n \otimes \mathrm{id}_{M_N})(v_i) - w_{i,n}\| = 0, \text{ and} \tag{e 15.20}$$

$$\lim_{n\to\infty} \|(L_n \otimes \mathrm{id}_{M_N})(Q_i) - e_{i,n}\| = 0. \tag{e 15.21}$$

Let $\Psi_n : A \to (1-p_n)A(1-p_n)$ be defined by $\Psi_n(a) = (1-p_n)a(1-p_n)$ for all $a \in A$. We will use $[\Psi_n] \circ \lambda$ for λ_0 and $[L_n] \circ \lambda$ for γ for some large n. Since G is finitely generated, choosing sufficiently large n, we may assume that λ_0 and γ are homomorphisms. To see that λ_0 is positive, we use (e 15.18) and the fact that G_+ is finitely generated. It follows from (e 15.19) and (e 15.20) that $N\gamma(x) \ge \gamma(u)$ for all $x \in \mathcal{P}$. Since we assume that the positive cone of G_+ is generated by $\mathcal{P}$, this also shows that $\gamma(x) > 0$ for all $x \in G_+ \setminus \{0\}$. By (**??**), we can choose large n so that $0 \le\lambda_0(u) < a$.

It should be noted that when A does not have (SP), one can choose $\lambda = \lambda_1$ and $\lambda_0 = 0$.

If $A = A_1 \otimes U$, then, without loss of generality, we may assume that $p_n \in A_1$ for all n. Choose a sequence of non-zero projections $e_n \in U$ such that $t(1-e_n) = r(n)/K$, where t is the unique tracial state on U and $r(n)$ are positive rational numbers such that $\lim_{n\to\infty} r(n) = 0$. Thus $S_n \otimes (1-e_n) \subset B_n$ where $B_n \cong S_n \otimes M_K$ and $p_n \otimes (1-e_n) = 1_{B_n}$. We check that the lemma follows if we replace Ψ_n by Ψ'_n, where $\Psi'_n(a) = (1-p_n)a \otimes 1_U + p_n a \otimes e_n$ ☐

LEMMA 15.6 (see Lemma 3.6 of [36] or Lemma 2.8 of [92]). *Let $G = K_0(S)$, where $S \in \mathcal{C}$. Let $H = K_0(A)$ for $A = A_1 \otimes U$, where $A_1 \in \mathcal{B}_1$ (or $\mathcal{B}_0$), and U is an infinite dimensional UHF-algebra. Let $M_1 \ge 1$ be a given integer and $d \in K_0(A)_+ \setminus \{0\}$. Then for any strictly positive homomorphism $\theta : G \to H$ with multiplicity M_1, and any integers $M_2 \ge 1$ and $K \ge 1$ such that $K\theta(x) > [1_A]$ for all $x \in G_+ \setminus \{0\}$, one has a decomposition $\theta = \theta_1 + \theta_2$, where θ_1 and θ_2 are* *positive homomorphisms from G to H such that the following diagrams commute:*

$$\begin{tikzcd}
G \arrow[rr, "\theta_1"] \arrow[dr, "\varphi_1"'] & & H & G \arrow[rr, "\theta_2"] \arrow[dr, "\varphi_2"'] & & H ,\\
& G_1 \arrow[ur, "\psi_1"'] & & & G_2 \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

where $\theta_1([1_S]) \leq d$, $G_1 \cong \bigoplus_n \mathbb{Z} := \bigoplus_{i=1}^n \mathbb{Z}$ for some natural number n and $G_2 = K_0(S')$ for some C^*-subalgebra S' of A which is in the class $\mathcal{C}$ (or in $\mathcal{C}_0$), φ_1, ψ_1 are positive homomorphisms and $\psi_2 = \imath_{*0}$, where $\imath : S' \to A$ is the embedding. Moreover, φ_1 has multiplicity M_1, φ_2 has multiplicity M_1M_2 and $2K\varphi_2(x) > \varphi_2([1_S])= [1_{S'}] > 0$ for all $x \in G_+ \setminus \{0\}$.

PROOF. Let $u = [1_S]$. Suppose that $S = A(F_1, F_2, \psi_0, \psi_1)$ with $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$. It is easy to find a strictly positive homomorphism $\eta_0 : K_0(F_1) \to \mathbb{Z}$. Define $r : G \to \mathbb{Z}$ by $r(g) = \eta_0 \circ (\pi_e)_{*0}$. By replacing S with $M_d(S)$ and A by $M_d(A)$ for some integer $d \geq 1$, without loss of generality, we may assume that S contains a finite subset of projections $\mathcal{P} = \{p_1, p_2, ..., p_l\}$ such that every projection $q \in S$ is equivalent to one of the projections in $\mathcal{P}$ and $\{[p_i] : 1 \leq i \leq l\}$ generates $K_0(S)_+$ (see 3.15). Let

$$\sigma_0 = \min\{\rho_A(d)(\tau) : \tau \in T(A)\}.$$

Note that since A is simple, one has $\sigma_0 > 0$.

Let

$$\sigma_1 = \inf\{\tau(\theta([p])) : p \in \mathcal{P},\ \ \tau \in T(A)\} > 0.$$

Since $A = A_1 \otimes U$, A has the (SP) property, there is a projection $f_0 \in A_+ \setminus \{0\}$ such that

$$0 < \tau(f_0) < \min\{\sigma_0, \sigma_1\}/8Kr(u) \text{ for all } \tau \in T(A). \tag{e 15.22}$$

Since $A = A_1 \otimes U$, we may choose f_0 so that $[f_0] = M_1\tilde{h}$ for some non-zero $\tilde{h} \in K_0(A)_+$. Define $\theta_0' : G \to K_0(A)$ by $\theta_0'(g) = r(g)\tilde{h}$ for all $g \in G$. Set $\theta' = M_1\theta_0'$. Then $2\theta'(x) < \theta(x)$ for all $x \in G_+ \setminus \{0\}$.

Since θ has multiplicity M_1, one has that $\theta(g) - \theta'(g)$ is divisible by M_1 for any $g \in G$. By the choice of σ_0, one checks that $\theta - \theta'$ is strictly positive. Moreover,

$$\begin{aligned} 2K\rho_A((\theta(x) - \theta'(x))(\tau) &> 2K\rho_A(\theta(x))(\tau) - K\rho_A(\theta(x))(\tau) \\ &= K\rho_A(\theta(x))(\tau) \geq \rho_A([1_A])(\tau) \text{ for all } \tau \in T(A). \end{aligned}$$

Applying 15.5, one obtains a C^*-subalgebra $S' \subset A$, a positive homomorphism $\tilde{\theta}_1 : G \to K_0(A)$ and strictly positive homomorphism $\varphi_2 : G \to K_0(S')$ such that

$$\theta - \theta' = \tilde{\theta}_1 + \imath_{*0} \circ \varphi_2, \tag{e 15.23}$$

(e 15.24)

$$0 \le \tau(\tilde{\theta}_1(u)) < \frac{\tau(\tilde{h})}{mM_1M_2}, \quad \tau \in T(A),\ 2K\varphi_2(x) > \varphi_2([1_S]),\ \varphi_2([1_S]) = [1_{S'}],$$

where m is from 15.4 and φ_2 has multiplicity M_1M_2, and where $\imath : S' \to A$ is the embedding. Put

$$\theta_2 = \imath_{*0} \circ \varphi_2, \quad \text{and} \quad \psi_2 = \imath_{*0}.$$

Since $\theta(g) - \theta'(g)$ is divisible by M_1 and any element in $\theta_2(G)$ is divisible by M_1, one has that any element in $\tilde{\theta}_1(G)$ is divisible by M_1. Therefore, the map $\tilde{\theta}_1$ can be decomposed further as $M_1\theta'_1$, and one has that $\theta - \theta' = M_1\theta'_1 + \theta_2$. Therefore, there is a decomposition

$$\theta = \theta' + M_1\theta'_1 + \theta_2 = M_1\theta'_0 + M_1\theta'_1 + \theta_2.$$

Put $\theta_1 = M_1(\theta'_0 + \theta'_1)$. Then,

$$\rho_A(\theta_1([1_S]))(\tau) < (1/2)\rho_A(d)(\tau) \ \text{ for all } \ \tau \in T(A).$$

We now show that θ_1 has the desired factorization property. For $\theta'_0 + \theta'_1$, one has the following further decomposition: for any $g \in G$,

$$\begin{aligned}\theta'_0(g) + \theta'_1(g) &= r(g)\tilde{h} + \theta'_1(g) = r(g)(\tilde{h} - m\theta'_1(u)) + r(g)m\theta'_1(u) + \theta'_1(g)\\ &= r(g)(\tilde{h} - m\theta'_1(u)) + \theta'_1(mr(g)u) + \theta'_1(g)\\ &= r(g)(\tilde{h} - m\theta'_1(u)) + \theta'_1(mr(g)u + g).\end{aligned}$$

By (e 15.24), $\tilde{h} - m\theta'_1(u) > 0$. By Lemma 15.4, $g \mapsto mr(g)u + g$ factors through $\bigoplus_n \mathbb{Z}(= \bigoplus_{i=1}^n \mathbb{Z})$ positively for some n. Therefore, the map $M_1(\theta'_0 + \theta'_1)$ factors though $\bigoplus_{(1+n)M_1} \mathbb{Z}$ positively. So there are positive homomorphisms $\varphi_1 : G \to \bigoplus_{(1+n)M_1} \mathbb{Z}$ and $\psi_1 : \bigoplus_{(1+n)M_1} \mathbb{Z} \to K_0(A)$ such that $\theta_1 = \psi_1 \circ \varphi_1$ and φ_1 has multiplicity M_1. □

16. Existence Theorems for Affine Maps for Building Blocks

The main purpose of this section is to present Theorem 16.10. We first consider the case that the target algebras are finite dimensional (Lemma 16.6). We then replace them by interval algebras via a path (Lemma 16.9). Then we establish Theorem 16.10 for the target algebras in $\mathcal{C}$. Methods used in this section may also be found later in [34].

LEMMA 16.1. *Let A be a unital separable C^*-algebra with $T(A) \neq \varnothing$ and let $\mathcal{H} \subset A$ be a finite subset. Then, for any $\sigma > 0$, there exist an integer $N > 0$ and a finite subset $E \subset \partial_e(T(A))$ with the following property: For any $\tau \in T(A)$ and* *any $k \geq N$, there are $t_1, t_2, ..., t_k$ (which are not necessarily distinct points) in E such that*

$$|\tau(h) - \frac{1}{k}\sum_{i=1}^{k} t_i(h)| < \sigma \ \ for\ \ all\ \ h \in \mathcal{H}. \tag{e 16.1}$$

(If τ is a (not necessarily normalized) trace on A with $\|\tau\| \leq 1$, then there are $t_1, t_2, ..., t_{k'}$ in E with $k' \leq k$ such that

$$|\tau(h) - \frac{1}{k}\sum_{i=1}^{k'} t_i(h)| < \sigma \ \ for\ \ all\ \ h \in \mathcal{H}.)$$

Suppose that A, in addition, is a subhomogeneous C^*-algebra. Then there are $\pi_1, \pi_2, ..., \pi_k$ in $\mathrm{Irr}(A)$ such that

$$|\tau(h) - \frac{1}{k}\left(\mathrm{tr}_1 \circ \pi_1(h) + \mathrm{tr}_2 \circ \pi_2(h) + \cdots + \mathrm{tr}_k \circ \pi_k(h)\right)| < \sigma \ \ for\ \ all\ \ h \in \mathcal{H}, \tag{e 16.2}$$

where $\pi_j \in E$ and tr_j is the unique tracial state of $\pi_j(A)$. Moreover, if the space $\hat{A}_l$ of irreducible representations of dimension exactly l has no isolated points for each l, then $\pi_1, \pi_2, ..., \pi_k$ can be chosen to be distinct.

Remark: Note that in (e 16.2), the π_i may not be distinct. But the subset E of irreducible representations can chosen to be independent of τ (but depending on σ and $\mathcal{H}$).

PROOF. Without loss of generality, one may assume that $\|f\| \leq 1$ for all $f \in \mathcal{H}$. Since $T(A)$ is weak*-compact, there are $\tau_1, \tau_2, ..., \tau_m \in T(A)$ such that, for any $\tau \in T(A)$, there is $j \in \{1, 2, ..., m\}$ such that

$$|\tau(f) - \tau_j(f)| < \sigma/4 \ \text{ for all } \ f \in \mathcal{H}. \tag{e 16.3}$$

By the Krein-Milman Theorem, there are $t_1', t_2', ..., t_n' \in \partial_e(T(A))$ and nonnegative numbers $\{\alpha_{i,j}\}$ such that, for each $j = 1, 2, ..., m$,

$$|\tau_j(f) - \sum_{i=1}^{n} \alpha_{i,j} t_i'(f)| < \sigma/8 \ \text{ and } \ \sum_{i=1}^{n} \alpha_{i,j} = 1. \tag{e 16.4}$$

Put $E = \{t_1', t_2', ..., t_n'\}$. Choose $N > 32mn/\sigma$. Let τ be a possibly unnormalized trace on A with $0 < \tau(1) \leq 1$. Suppose that j is chosen so that $|\tau(f)/\tau(1) - \tau_j(f)| < \sigma/4$ for all $f \in \mathcal{H}$ as in (e 16.3). Then, for any $k \geq N$, there exist positive rational numbers $r_{i,j}$ and positive integers $p_{i,j}$ ($1 \leq i \leq n$ and $1 \leq j \leq m$) such that

$$\sum_{i=1}^{n} r_{i,j} \leq 1, \ \text{ or } \ \sum_{i=1}^{n} r_{i,j} = 1 \ \text{ if } \ \tau(1) = 1, \ \text{ for } \ 1 \leq j \leq m \tag{e 16.5}$$

(e 16.6)
$$r_{i,j} = \frac{p_{i,j}}{k}, \quad \text{and} \quad |\tau(1)\alpha_{i,j} - r_{i,j}| < \frac{\sigma}{8n}, \quad \text{for } 1 \le i \le n, \text{ and } 1 \le j \le m.$$

Note that, by (e 16.5),

$$\sum_{i=1}^{n} p_{i,j} \le k, \quad \text{or} \quad \sum_{i=1}^{n} p_{i,j} = k \ \text{ if } \ \tau(1) = 1. \tag{e 16.7}$$

Then, by (e 16.6),

$$|\tau_j(f) - \sum_{i=1}^{n} (\frac{p_{i,j}}{k}) t_i'(f)| < \sigma/4 + \sigma/8 = 3\sigma/8 \ \text{ for all } \ f \in \mathcal{H}. \tag{e 16.8}$$

It is then clear that (e 16.1) holds on repeating each t_i' $p_{i,j}$ times.

Now suppose that A is subhomogeneous. By Lemma 2.16 of [66], t_i' has the form $\mathrm{tr}_i \circ \pi_i$, where $\pi_1, \pi_2, ..., \pi_n$ are in $\mathrm{Irr}(A)$. It follows that (e 16.2) holds.

Note that $\hat{A}_l$ is a locally compact Hausdorff space (see Proposition 4.4.10 of [95], for example). Fix a metric on $\hat{A}_l$. For each $\pi_i \in \hat{A}_l$, there exists $\delta_i > 0$ such that for any irreducible representation $x \in \hat{A}_l$ with $\mathrm{dist}(x, \pi_i) < \delta_i$, we have

$$|\mathrm{tr}(x(f)) - \mathrm{tr}(\pi_i(f))| < \sigma/64k \quad \text{for all } \ f \in \mathcal{H},$$

where tr is the unique trace of M_l.

Suppose that, for each l, $\hat{A}_l$ has no isolated points. Fix j and, for each i with $\mathrm{tr}_i \circ \pi_i \in E$, choose $p_{i,j}$ distinct points in a neighborhood $O(\pi_i, \delta_i)$ of π_i (in $\hat{A}_l$) with diameter less than δ_i. Let $\{\pi_{1,j}, \pi_{2,j}, ..., \pi_{k,j}\}$ be the resulting set of k elements (see (e 16.7)). Then, one has (by (e 16.3) and (e 16.8))

(e 16.9)
$$|\tau(f) - \frac{1}{k}(\mathrm{tr}_{1,j}(f(\pi_{1,j})) + \mathrm{tr}_{2,j}(f(\pi_{2,j})) + \cdots + \mathrm{tr}_{k,j}(f(\pi_{k,j})))| < \sigma \ \text{ for all } \ f \in \mathcal{H}$$

(where $\mathrm{tr}_{i,j}$ is the tracial state of $\pi_{i,j}(A)$), as desired. □

LEMMA 16.2. *Let $\mathcal{H}$ be a finite subset of $C([0,1] \times \mathbb{T}) \otimes M_r$ (for some $r \ge 1$) and let $\sigma > 0$. Then there is an integer $N \ge 1$ such that for any finite Borel measure μ on $[a,b] \times \mathbb{T}$ with $\|\mu\| \le 1$ and any $k \ge N$, there are $x_1, x_2, ..., x_m$ in $(0,1) \times \mathbb{T}$ for some $m \le N$ such that*

$$|\int_{(a,b)\times\mathbb{T}} \mathrm{tr}(h) d\mu - \frac{1}{k}(\mathrm{tr}(h(x_1)) + \mathrm{tr}(h(x_2)) + \cdots + \mathrm{tr}(h(x_m)))| < \sigma \ \textit{for all} \ h \in \mathcal{H},$$

where $[a,b] \subset [0,1]$ and tr *is the tracial state of M_r.*

PROOF. Let $A = C([0,1] \times \mathbb{T}) \otimes M_r$. Note that, for each μ as specified, $\tau(f) = \int_{(a,b)\times\mathbb{T}} \mathrm{tr}(f) d\mu$ is a trace on A with $\|\tau\| \le 1$. Therefore, the conclusion follows immediately from Lemma 16.1 if one allows $x_1, x_2, ..., x_m$ to be in $[0,1] \times \mathbb{T}$. The equicontinuity of $\mathcal{H}$ then allows us to require these points to be in $(0,1) \times \mathbb{T}$. □

The following fact is known to experts.

LEMMA 16.3. *Let $C = \bigoplus_{i=1}^k C(X_i) \otimes M_{r(i)}$, where each X_i is a connected compact metric space. Let $\mathcal{H} \subset C$ be a finite subset and let $\sigma > 0$. Then there is an integer $N \geq 1$ satisfying the following condition: for any positive homomorphism $\kappa : K_0(C) \to K_0(M_s) = \mathbb{Z}$ with $\kappa([1_{C(X_i)\otimes M_{r(i)}}]) \geq N$ (for each i) and any $\tau \in T(C)$ such that*

$$\rho_C(x)(\tau) := (1/s)(\kappa(x)) \ \ \textit{for all } x \in K_0(C), \tag{e 16.10}$$

there is a homomorphism $\varphi : C \to M_s$ such that $\varphi_{*_0} = \kappa$ and

$$|\mathrm{tr} \circ \varphi(h) - \tau(h)| < \sigma \ \ \textit{for all } h \in \mathcal{H},$$

where tr *is the tracial state on M_s.*

PROOF. For convenience, we present a proof, using Lemma 16.1. It is clear that the general case can be reduced to the case that $C = C(X) \otimes M_r$ for some connected compact metric space X and $r \geq 1$. Let σ and $\mathcal{H}$ be given. Without loss of generality, we may assume that $\mathcal{H}$ is in the unit ball of C. Let N_1 be as in 16.1 for σ and $\mathcal{H}$. Let $N = N_1 r$.

Suppose now τ and κ are given such that $\kappa([1_C]) \geq N$ and (e 16.10) holds. Let $D : K_0(C) \to \mathbb{Z}$ be defined by the rank function and let $e \in C$ be a projection of (constant) rank one. Then $\kappa([e]) \geq N_1$. The assumption (e 16.10) means that $\rho_C(x)(\tau) = (1/s)(\kappa([e])D(x))$ for all $x \in K_0(C)$. In particular, $1 = (1/s)(\kappa([e])r) = (1/s)\kappa([1_C])$ and $s = \kappa([e])r$. Let $k = \kappa([e]) \geq N_1$ and $\pi_1, \pi_2, ..., \pi_k$ be given by (the second part of) Lemma 16.1. Define $\varphi(f) = \mathrm{diag}(\pi_1(f), \pi_2(f), ..., \pi_k(f))$. Then, by the choice of N_1 and by Lemma 16.1, one has

$$|\mathrm{tr}(\varphi(h)) - \tau(h)| < 1/\sigma \ \text{ for all } \ h \in \mathcal{H}.$$

Moreover, $\varphi_{0*} = \kappa$.

□

LEMMA 16.4. *Let $C = C(\mathbb{T}) \otimes F_1$, where $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$, or $C = F_1$. Let $\mathcal{H} \subset C$ be a finite subset, and let $\epsilon > 0$. There is $\delta > 0$ satisfying the following condition: For any M_s, any positive order-unit preserving map $\kappa : K_0(C) \to K_0(M_s)$, and any tracial state $\tau \in T(C)$ such that*

$$|\rho_{M_s}(\kappa(p))(\mathrm{tr}) - \tau(p)| < \delta$$

for all projections p in C, where tr *is the tracial state on M_s, there is a tracial state $\tilde{\tau} \in T(C)$ such that*

$$(1/s)(\kappa([p])) = \tilde{\tau}(p) \ \ \textit{for all } p \in C$$

and

$$|\tau(h) - \tilde{\tau}(h)| < \epsilon \ \text{ for all } \ h \in \mathcal{H}.$$

PROOF. We may assume that $\mathcal{H}$ is in the unit ball of C. Let $\delta = \varepsilon/l$. We may write $\tau = \sum_{j=1}^{l} \alpha_j \tau_j$, where τ_j is a tracial state on $C(\mathbb{T}) \otimes M_{r(j)}$, $\alpha_j \in \mathbb{R}_+$, and $\sum_{j=1}^{k} \alpha_j = 1$. Let $\beta_j = (1/s)(\kappa([1_{C(\mathbb{T})\otimes M_{R(j)}}])$, $j = 1, 2, ..., l$. Put $\tilde{\tau} = \sum_{j=1}^{l} \beta_j \tau_j$. Then $\mathrm{tr}(\kappa(p)) = \tilde{\tau}(p)$ for all projections $p \in C$; and for any $h \in \mathcal{H}$,

$$|\tilde{\tau}(h) - \tau(h)| \le \sum_{j=1}^{l} |\beta_j - \alpha_j| < \varepsilon,$$

as desired. □

REMARK 16.5. (1) Let C be a separable stably finite C^*-algebra and let $A = C \otimes C(\mathbb{T}) = C(\mathbb{T}, C)$. Recall that $K_0(A) = K_0(C) \oplus \boldsymbol{\beta}(K_1(C)) \cong K_0(C) \oplus K_1(C)$ (see 2.14). If $K_i(C)$ is finitely generated, then $K_i(A)$ is also finitely generated, $i = 0, 1$ (see 3.5). Fix a point $t_0 \in \mathbb{T}$. Let $\pi_{t_0} : A \to C$ be the point evaluation at t_0 defined by $\pi_{t_0}(f) = f(t_0)$ for all $f \in C(\mathbb{T}, C)$. Define $\iota : C \to C(\mathbb{T}, C) = A$ by $\iota(c)(t) = c$ for all $t \in \mathbb{T}$ and $c \in C$. Then $\pi_{t_0} \circ \iota = \mathrm{id}_C$. Thus, the homomorphism $(\pi_{t_0})_{*0} : K_0(A) \to K_0(C)$ induced by π_{t_0} maps $K_0(A)$ onto $K_0(C)$ and $((\pi_{t_0})_{*0})|_{K_0(C)}$ is an order isomorphism. In particular, we may write $\ker(\pi_{t_0})_{*0} = \boldsymbol{\beta}(K_1(C))$. In other words, if $p(t), q(t) \in M_N(C(\mathbb{T}, C))$ are projections and $[p] - [q] \in \boldsymbol{\beta}(K_1(C))$, then $[p(t)] = [q(t)]$ in $K_0(C)$ for all $t \in \mathbb{T}$. Note that, any tracial state $\tau \in T(A)$ may be written as $\tau(f) = \int_{\mathbb{T}} s(f)(t) d\mu(t)$, where $s \in T(C)$ and μ is a probability Borel measure on $\mathbb{T}$. It follows that $\tau(p) - \tau(q) = 0$ if $[p] - [q] \in \ker(\pi_{t_0})_{*0}$ for all $\tau \in T(A)$. This, in particular, implies that $\boldsymbol{\beta}(K_1(C)) \subset \ker\rho_A$.

(2) Let $C = A(F_1, F_2, \varphi_0, \varphi_1)$ and let $\pi_e : C \to F_1$ be as in 3.1 which gives an order embedding $(\pi_e)_{*0} : K_0(C) \to K_0(F_1)$ (see 3.5). Let $\varphi_i' : C(\mathbb{T}) \otimes F_1 \to C(\mathbb{T}) \otimes F_2$ be defined by $\varphi_i'(f \otimes d) = f \otimes \varphi_i(d)$ for all $f \in C(\mathbb{T})$ and $d \in F_1$, where $i = 0, 1$. Then $(A = C \otimes C(\mathbb{T}))$

$$\begin{aligned} A \quad = \quad & \{(f, g) \in C([0,1] \times \mathbb{T}, F_2) \oplus (C(\mathbb{T}) \otimes F_1) : \\ & \qquad\qquad f(0, t) = \varphi_0'(g(t)) \text{ and } f(1, t) = \varphi_1'(g(t))\}. \end{aligned}$$

Let $\pi_e' : A \to C(\mathbb{T}, F_1)$ be defined by $\pi_e'(c \otimes f) = \pi_e(c) \otimes f$ for all $c \in C$ and $f \in C(\mathbb{T})$. Note that, by 3.5, $\ker\rho_C = \{0\}$. Then, by (1) above, $\ker\rho_A = \boldsymbol{\beta}(K_1(A))$. One then verifies easily that the map $(\pi_e')_{*0} : K_0(A) = K_0(C) \otimes C(\mathbb{T}) = K_0(C) \oplus \boldsymbol{\beta}(K_1(C)) \to F_1 \otimes C(\mathbb{T}) = K_0(F_1)$ is given by $(\pi_e')_{*0}(x \oplus y) = (\pi_e)_{*0}(x)$ for all $x \in K_0(A)$ and $y \in \boldsymbol{\beta}(K_1(C))$. In other words, $K_0(A)/\ker\rho_A$ is embedded into $K_0(C(\mathbb{T}, F_1)$.

LEMMA 16.6. *Let $A = C$ for some $C \in \mathcal{C}$ or $A = C \otimes C(\mathbb{T})$ for some $C \in \mathcal{C}$. Let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. Let $\mathcal{H} \subset A$ be a finite subset, and let $\sigma > 0$. Then there are a finite subset $\mathcal{H}_1 \subset A_+$ and a positive integer K such that for any $\tau \in T(A)$ satisfying*

$$\tau(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \tag{e 16.11}$$

and any positive homomorphism $\kappa : K_0(A) \to K_0(M_s)$ *with* $s = \kappa([1_A]) \in \mathbb{N}$ *satisfying*

$$\tau(x) = (1/s)(\kappa(x)) \text{ for all } x \in K_0(A), \tag{e 16.12}$$

there is a unital homomorphism $\varphi : A \to M_{sK}$ *such that* $\varphi_{*0} = K\kappa$ *and*

$$|\mathrm{tr}' \circ \varphi(h) - \tau(h)| < \sigma \text{ for all } h \in \mathcal{H},$$

where tr' *is the tracial state on* M_{sK}.

PROOF. We will consider the case that $A = C(\mathbb{T}) \otimes C$. The case $A = C$ can be proved in the same manner but simpler.

Let $C = \{(f, g) \in C([0,1], F_2) \otimes F_1 : f(0) = \varphi_0(g) \text{ and } f(1) = \varphi_1(g)\}$. Write $F_1 = M_{R_1} \oplus M_{R_2} \oplus \cdots \oplus M_{R_l}$ and $F_2 = M_{r(1)} \oplus \cdots M_{r(k)}$ and $C([0,1] \times \mathbb{T}, F_2) = \bigoplus_{i=1}^k C([0,1] \times \mathbb{T}, M_{r(j)})$. Then, keeping the notation of part (2) of 16.5, we write

$$\begin{aligned} rA \;\; = \;\; & \{(f, g) \in C([0,1] \times \mathbb{T}, F_2) \oplus (C(\mathbb{T}) \otimes F_1) : \\ & \qquad f(0, t) = \varphi_0'(g(t)) \text{ and } f(1, t) = \varphi_1'(g(t))\}. \end{aligned}$$

In particular, $A \in \mathcal{D}_1$. Let $\pi_i : C(\mathbb{T}) \otimes F_1 \to C(\mathbb{T}) \otimes M_{R_i}$ be the projection to the ith summand and $\varepsilon_j : C([0,1] \times \mathbb{T}) \otimes F_2 \to C([0,1] \times \mathbb{T}) \otimes M_{r(j)}$ the projection to the j-th summand, $1 \le j \le k$. Set $\varphi_{0,j} = \pi_j \circ \varphi_0 : F_1 \to M_{r(j)}$ and $\varphi_{1,j} = \pi_j \circ \varphi_1 : F_1 \to M_{r(i)}$. Let $\varphi_{0,j}' : C(\mathbb{T}) \otimes F_1 \to C(\mathbb{T}) \otimes M_{r(j)}$ and $\varphi_{1,j}' : C(\mathbb{T}) \otimes F_1 \to C(\mathbb{T}) \otimes M_{r(j)}$ be the homomorphisms induced by $\varphi_{0,j}$ and $\varphi_{1,j}$, respectively. Let $\pi_e : C \to F_1$ and $\pi_e' : A \to C(\mathbb{T}) \otimes F_1$ be as in part (2) of 16.5.

Let $h = (h_f, h_g) \in A$. In what follows, for $\xi \in (0,1) \times \mathbb{T}$, we will use $\varepsilon_i(h)(\xi)$ for $\varepsilon_i(h_f)(\xi)$, and, we will use $\varepsilon_i(h)((0,t))$ for $\varphi_{0,i}'(\pi_e'(h))(t)$ (for $t \in \mathbb{T}$) and $\varepsilon_i(h)((1,t))$ for $\varphi_{1,i}'(\pi_e'(h))(t)$ (for $t \in \mathbb{T}$), whenever it is convenient.

By 3.15, $K_0(C)$ is finitely generated by minimal projections in $M_m(C)$. Replacing C by $M_m(C)$, without loss of generality, we may assume that $K_0(A)$ is generated by $\{p_1, p_2, ..., p_c\}$, where $p_i \in C$ are minimal projections, $i = 1, 2, ..., c$. In what follows we will identify p_i with $p_i \otimes 1_{C(\mathbb{T})}$ whenever it is convenient.

Note that $K_0(A) = K_0(C) \oplus \boldsymbol{\beta}(K_1(C)) \cong K_0(C) \oplus K_1(C)$, $\mathrm{ker}\rho_C = \{0\}$, and $\mathrm{ker}\rho_A = \boldsymbol{\beta}(K_1(C))$ (see 16.5). Therefore, $\kappa|_{\mathrm{ker}\rho_A} = \{0\}$. Let $\kappa_{00} : K_0(C) \to K_0(M_s)$ be the positive homomorphism induced by κ.

Note also $K_0(C(\mathbb{T}) \otimes F_1) \cong K_0(F_1) = \mathbb{Z}^l$. Let e_i be a minimal projection of M_{R_i}, $i = 1, 2, ..., l$. Let $I = \mathrm{ker}\pi_e'$. Since π_e is surjective (see 3.1), there are $h_i \in A_+$ such that $\|h_i\| \le 1$ and $\pi_e'(h_i) = e_i$, $i = 1, 2, ..., l$. We may assume that $\varepsilon_i(h_j)(r, t) = \varphi_{0,i}'(e_j)$ if $r \in [0, 1/4]$ and $\varepsilon_i(h_j)(r, t) = \varphi_{1,i}'(e_j)$ if $r \in [3/4, 1]$.

We may also assume that $\mathcal{H}$ is a subset of the unit ball of A which contains 1_A as well as $\{p_1, p_2, ..., p_c\}$. Let $\overline{\mathcal{H}} = \{\pi_e'(h) : h \in \mathcal{H}\}$.

Let N_0 (in place of N) be the integer for $\pi_e'(A)$ (in place of C), $\overline{\mathcal{H}}$ (in place of $\mathcal{H}$) and $\sigma/64$ (in place of σ) given by Lemma 16.3. Let us fix the metric

$d(r \times t, r' \times t') = \sqrt{|r-r'|^2+|t-t'|^2}$ for all $r \times t, r' \times t' \in [0,1] \times \mathbb{T}$. There exists $1/4 > \delta_0 > 0$ such that, if $\mathrm{dist}(\xi, \xi') < \delta_0$ ($\xi, \xi' \in [0,1] \times \mathbb{T}$), or $|t-t'| < \delta_0$ ($t, t' \in \mathbb{T}$), or $0 < r < \delta_0$, then, for any $h = (h_f, h_g) \in \mathcal{H}$, one has

$$\|h_f(\xi) - h_f(\xi')\| < \sigma/64kN_0l, \quad \|h_g(t') - h_g(t)\| < \sigma/64N_0kl, \tag{e 16.13}$$

and, for all $t \in \mathbb{T}$,

$$\|h_f(r,t) - \varphi_0'(h_g)(t)\| < \sigma/64kN_0l \text{ and } \|h_f(1-r,t) - \varphi_1'(h_g)(t)\| < \sigma/64kN_0l.$$

Choose $a_I \in I_+$ such that $\|a_I\| \le 1$, $a(r,t) = 1_{F_2}$ and $a(1-r,t) = 1_{F_2}$ if $1 > r > \delta_0/2$, and $a_I(r,t) = a_I(1-r,t) = 0$ if $0 < r < \delta_0/4$, for all $t \in \mathbb{T}$.

Now we choose $\mathcal{H}_1$. For each $1 \le j \le k$, find a $g_j \in (C_0(0,1) \otimes \mathbb{T} \otimes M_{r(j)})_+ \setminus \{0\}$ such that $g_j(r,t) = 0$ if $r \notin (\delta_0/2, 1-\delta_0/2)$ and $\|g_j\| \le 1$. Find $g_j', g_j'' \in (C_0(0,1) \otimes \mathbb{T} \otimes M_{r(j)})_+ \setminus \{0\}$ such that $\|g_j'\|, \|g_j''\| \le 1$, $g_j'(r,t) = 0$ if $r \notin (0, \delta_0/2)$ and $g_j''(r,t) = 0$ if $r \notin (1-\delta_0/4, 1)$, $j = 1, 2, ..., k$. We will also use g_j for the elements of $I \subset A$ such that $\varepsilon_j(g_j)(\xi) = g_j(\xi)$ for all $\xi \in (0,1) \times \mathbb{T}$, $\varepsilon_i(g_j) = 0$ for all $i \neq j$. We may also view g_j' and g_j'' as elements of I in exactly the same manner. Let $h_i' = (1-a_I)h_i(1-a_I)$, $i = 1, 2, ..., l$. Note that $h_i'(r,t) = h_i(r,t)$ if $r \in [0, \delta_0/4] \cup [1-\delta_0/4, 1]$ and $h_i'(r,t) = 0$ if $r \in [\delta_0/2, 1-\delta_0/2]$.

Recall that we identify p_i with $p_i \otimes 1_{C(\mathbb{T})}$, $i = 1, 2, ..., c$. Put

$$\mathcal{P}' = \{g_i'p_j : g_i'p_j \neq 0, 1 \le i \le k, 1 \le j \le c\} \cup \{g_i''p_j'' : g_i''p_j'' \neq 0, q \le i \le k, 1 \le j \le c\}$$

and put

$$\mathcal{H}_1 = \{1_A\} \cup \{h_i, h_i' : 1 \le i \le l\} \cup \{g_j, g_j', g_j'' : 1 \le j \le k\} \cup \mathcal{P}'.$$

Put

$$\sigma_1 = \min\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}/2 \quad \text{and} \quad \sigma_2 = \sigma_1 \cdot \sigma/64kl. \tag{e 16.14}$$

Let M be the integer in Lemma 15.1 associated with the pair $K_0(A)$ (as G) and $\mathbb{Z}^l$ (see 3.5). Let K_1 (in place of R) be the integer provided by 15.1 for $G = K_0(A)$, σ_1 and σ_2. Let N_j be the integer provided by Lemma 16.2 for $\sigma_2/8k\prod_{j=1}^k r(j)$ (in place of σ) for $r = r(j)$, $j = 1, 2, ..., l$. Let $\bar{N} = \max\{N_j : 1 \le j \le k\}$. Put $d_0 = \prod_{j=1}^k r(j)$. Let $K = K_1 \cdot N_0 \cdot \bar{N} \cdot M \cdot d_0$.

Now suppose that κ and $\tau \in T(A)$ are given satisfying (e 16.11) and (e 16.12). We may write (see 2.14 of [66])

$$\tau(a) = (\sum_{i=1}^k \int_{(0,1)\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(a)(\xi))d\mu_i(\xi)) + t_e \circ \pi_e'(a) \text{ for all } a \in A, \tag{e 16.15}$$

where μ_i is a Borel measure on $(0,1) \times \mathbb{T}$ with $\|\mu_i\| \le 1$, tr_i is the normalized trace on $M_{r(i)}$, and t_e is a trace (with $\|t\| \le 1$) on $C(\mathbb{T}) \otimes F_1$. Consider the finite set $\{\mu_1, \mu_2, ..., \mu_k\}$. By % applying Lemma 4.10, one can find two points $\delta_0'' < \delta_0'$ in $(15\delta_0/16\delta_0, \delta_0]$ such that

$$\int_{[\delta_0'',\delta_0']\times\mathbb{T}} d\mu_i < \sigma_2/8k, \ \ i = 1, 2, ..., k. \tag{e 16.16}$$

Note, for each j, $\varepsilon_i(p_j)(\xi)$ is constant on $(0,1)\times\mathbb{T}$ for each i. If $\varepsilon_i(p_j) \neq 0$, then $g_i'p_j \in \mathcal{P}'$. We have (as $0 < \delta_0/2 < \delta_0' \le \delta_0$ and $\varepsilon_s(g_j') = 0$, if $s \neq j$)

$$\begin{aligned}\int_{(0,\delta_0')\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(p_j))d\mu_i &\ge \int_{(0,1)\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(g_i'p_j))d\mu_i \\ &= \sum_{s=1}^{k} \int_{(0,1)\times\mathbb{T}} \mathrm{tr}_s(\varepsilon_s(g_i'p_j))d\mu_s = \tau(g_i'p_j) \ge 2\sigma_1.\end{aligned} \tag{e 16.17}$$

It follows from Lemma 16.2 that, for each i, there are $t_{i,j} \in (0,1)\times\mathbb{T}$, $j = 1, 2, ..., m(i) \le \bar{N}$, such that

$$|\int_{[\delta_0',1-\delta_0']\times\mathbb{T}} \mathrm{tr}_i(f)d\mu_i - (1/\bar{N})\sum_{j=1}^{m(i)} \mathrm{tr}_i(f(t_{i,j}))| < \sigma_2/8k \tag{e 16.18}$$

for all $f \in \mathcal{H}$. For each i, define $\rho_i, \rho_i' : A \to \mathbb{C}$ by

$$\rho_i(f) = \int_{(0,1)\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(f))d\mu_i - (1/\bar{N})\sum_{j=1}^{m(i)} \mathrm{tr}_i(\varepsilon_i(f(t_{i,j}))), \tag{e 16.19}$$

$$\rho_i'(f) = \int_{(0,1-\delta_0']\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(f))d\mu_i - (1/\bar{N})\sum_{j=1}^{m(i)} \mathrm{tr}_i(\varepsilon_i(f(t_{i,j}))) \tag{e 16.20}$$

for all $f \in A$. Then, since $\varepsilon_i(p_j)(\xi)$ is constant on $[0,1]\times\mathbb{T}$,

$$\rho_i(p_j) = \int_{(0,1)\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(p_j))d\mu_i - (1/\bar{N})\sum_{s=1}^{m(i)} \mathrm{tr}_i(\varepsilon_i(p_j(t_{i,s}))) \tag{e 16.21}$$

$$= \mathrm{tr}_i(\varepsilon_i(p_j))\rho_i(1_A). \tag{e 16.22}$$

If $\varepsilon_i(p_j) \neq 0$, by (e 16.18) and (e 16.17),

$$\begin{aligned}\rho_i'(p_j) &= \int_{(0,\delta_0')\times\mathbb{T}} \mathrm{tr}_i(p_j)d\mu_i + \int_{[\delta_0',1-\delta_0']\times\mathbb{T}} \mathrm{tr}_i(p_j)d\mu_i(t) - (1/\bar{N})\sum_{s=1}^{m(i)} \mathrm{tr}_i(p_j(t_{i,s})) \\ &> \int_{(0,\delta_0')\times\mathbb{T}} \mathrm{tr}_i(p_j)d\mu_i - \sigma_2/2k \ge 2\sigma_1 - \sigma_2/2k > 0.\end{aligned}$$

Put $\alpha_i' = \rho_i'(1_A)$, $i = 1, 2, ..., k$. Let $\nu_{0,i}$ and $\nu_{1,i}$ be the Borel measures on $\mathbb{T}$ given by

$$\int_{\mathbb{T}} \mathrm{tr}_i(f(t))d\nu_{0,i}(t) = \int_{(0,\delta_0')\times\mathbb{T}} \mathrm{tr}_i(1_C\otimes f)d\mu_i \ \text{ and} \tag{e 16.23}$$

$$\int_{\mathbb{T}} \mathrm{tr}_i(f(t))d\nu_{1,i} = \int_{(1-\delta_0',1)\times\mathbb{T}} \mathrm{tr}_i(1_C\otimes f)d\mu_i \tag{e 16.24}$$

for all $f \in C(\mathbb{T}, M_{r(i)})$, $1 \le i \le k$. Note that $\|v_{0,i}\| \ge 2\sigma_1$, and by (e 16.18),

$$\begin{aligned} & |\rho_1'(1_A) - \|v_{0,i}\|| \\ = \ & |\rho_1'(1_A) - \int_{\mathbb{T}} \mathrm{tr}_i(1_A)d\mu_i| < \sigma_2/8k, \ \ i = 1, 2, ..., k. \end{aligned} \tag{e 16.25}$$

Define $T_{0,i}, T_{1,i} : A \to \mathbb{C}$ by

$$T_{0,i}(a) = \frac{\alpha_i'}{\|\nu_{0,i}\|}\int_{\mathbb{T}} \mathrm{tr}_i\circ\varphi_{0,i}'\circ\pi_e'(a)d\nu_{0,i} \ \text{ and} \tag{e 16.26}$$

$$T_{1,i}(a) = \int_{\mathbb{T}} \mathrm{tr}_i\circ\varphi_{1,i}'\circ\pi_e'(a)d\nu_{1,i} \tag{e 16.27}$$

for all $a \in A$. Note, for any $h \in A$ and $t \in \mathbb{T}$,

$$\mathrm{tr}_i((\varphi_{0,i}'\circ\pi_e'(h))(t)) = \mathrm{tr}_i(\varepsilon_i(h)((0,t))) \tag{e 16.28}$$

and

$$\mathrm{tr}_i((\varphi_{1,i}\circ\pi_e'(h))(t)) = \mathrm{tr}_i(\varepsilon_i(h)((1,t))).$$

Therefore, for $h \in \mathcal{H}$, by (e 16.13), (e 16.23), (e 16.24), (e 16.14), and (e 16.25),

$$\begin{aligned} & |\int_{((0,\delta_0')\cup(1-\delta_0',1))\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(h))d\mu_i - (T_{0,i}(h) + T_{1,i}(h))| \\ \le \ & \frac{2\sigma}{64kl} + |\int_{(0,\delta_0')\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(h)(\delta_0'/2, t))d\mu_i - T_{0,i}(h)| \\ & +|\int_{(1-\delta_0',1)\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(h)(1-\delta_0'/2, t))d\mu_i - T_{1,i}(h)| \\ = \ & |\int_{\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(h)(\delta_0'/2, t))d\nu_{0,i} - T_{0,i}(h)| \\ & +|\int_{\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(h)(1-\delta_0'/2, t))d\nu_{1,i} - T_{1,i}(h)| + \frac{\sigma}{32kl} \\ \le \ & |\int_{\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(h)(0, t))d\nu_{0,i} - T_{0,i}(h)| + |\int_{\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(h)(1, t))d\nu_{1,i} - T_{1,i}(h)| \\ & +\frac{\sigma}{32kl} + \frac{2\sigma}{64kl} \\ < \ & |1 - \frac{\rho_i'(1_A)}{\|\nu_{0,i}\|}|\|\nu_{0,i}\| + 0 + \frac{\sigma}{16kl} < \sigma_2/8k + \frac{\sigma}{16kl}. \end{aligned} \tag{e 16.29}$$

Let $\bar{h}_j \in C([0,1]\times\mathbb{T})\otimes F_2$ be such that $\|\bar{h}_j\| \le 1$, $\bar{h}_j(r,t) = \varphi_0'(e_j)$ for $r \in [0,\delta_0'']$, $\bar{h}_j(r,t) = 0$ if $r \in [\delta_0', 1-\delta_0']$, and $\bar{h}_j(r,t) = \varphi_1'(e_j)$ if $r \in [1-\delta_0'', 1]$, $j = 1,2,...,l$. Then $\bar{h}_j \ge h_j'$, $j = 1,2,...,l$. Moreover (by (e 16.25) again),

$$\begin{aligned}
&|\int_{((0,\delta_0')\cup(1-\delta_0',1))\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(\bar{h}_j))d\mu_i - (T_{0,i}(\bar{h}_j) + T_{1,i}(\bar{h}_j))| \\
\le\ & \sigma_2/8k + |\int_{(0,\delta_0')\times\mathbb{T}} \mathrm{tr}_i(\varphi_{0,i}'(e_j))d\mu_i - T_{0,i}(\bar{h}_j)| \\
& +|\int_{(1-\delta_0',1)\times\mathbb{T}} \mathrm{tr}_i(\varphi_{1,i}'(e_j))d\mu_i - T_{1,i}(\bar{h}_j)| \\
\le\ & \sigma_2/8k + |1 - \frac{\rho_i'(1_A)}{\|\nu_{0,i}\|}|\|\nu_{0,i}\| + 0 < \sigma_2/8k + \sigma_2/8k = \sigma_2/4k.
\end{aligned} \tag{e 16.30}$$

Since $\mathrm{tr}_i(\varepsilon_i(p_j))$ is constant on each open set $(0,1)\times\mathbb{T}$, put $L_{i,j} = \mathrm{tr}_i(\varepsilon_i(p_j))$. Then one checks (using (e 16.22) among other items) that

$$\begin{aligned}
& T_{0,i}(p_j) + T_{1,i}(p_j) \\
=\ & \frac{\alpha_i'}{\|\nu_{0,i}\|}\int_{\mathbb{T}} \mathrm{tr}_i\circ\varphi_{0,i}\circ\pi_e'(p_j)d\nu_{0,i} + \int_{\mathbb{T}} \mathrm{tr}_i\circ\varphi_{1,i}\circ\pi_e'(p_j)d\nu_{1,i} \\
=\ & \alpha_i'\mathrm{tr}_i(\varepsilon_i(p_j)) + \|\nu_{1,i}\|\mathrm{tr}_i(\varepsilon_i(p_j)) = L_{i,j}(\rho_i'(1_A) + \int_{\mathbb{T}} d\nu_{1,i}) \\
=\ & L_{i,j}(\rho_i(1_A)) = \rho_i(p_j).
\end{aligned} \tag{e 16.31}$$

Let

$$\begin{aligned}
T_1(a) &= t_e\circ\pi_e'(a) + \sum_{i=1}^k (T_{0,i}(a) + T_{1,i}(a)), T_2(a) \\
&= \sum_{i=1}^k [(1/\bar{N})\sum_{j=1}^{m(i)} \mathrm{tr}_i(a(t_{i,j}) \\
T(a) &= T_1(a) + T_2(a)t_e\circ\pi_e'(a) + \sum_{i=1}^k (T_{0,i}(a) + T_{1,i}(a) \\
&\quad +(1/\bar{N})\sum_{j=1}^{m(i)} \mathrm{tr}_i(a(t_{i,j})))
\end{aligned} \tag{e 16.32}$$

for all $a \in A$. Thus, by (e 16.31) (see also (e 16.15)),

$$T(p_j) = t_e \circ \pi_e'(p_j) + \sum_{i=1}^{k} (\rho_i(p_j) + (1/\bar{N}) \times \sum_{j=1}^{m(i)} \mathrm{tr}_i(p_j(t_{i,j}))) = \tau(p_j) \tag{e 16.33}$$

for all j. Then T_1 and T_2 are traces on A and T is a tracial state on A. Define

$$T_1'(b) = t_e(b) + \sum_{i=1}^{k} (\frac{\alpha_i'}{\|\nu_{0,i}\|} \int_{\mathbb{T}} \mathrm{tr}_i \circ \varphi_{0,i}(b) d\nu_{0,i} + \int_{\mathbb{T}} \mathrm{tr}_i \circ \varphi_{1,i}(b) d\nu_{1,i}) \quad \text{for all } b \in C(\mathbb{T}) \otimes F_1.$$

In what follows we will also use T_1' for the extension on $A \otimes M_m$ defined by

$$\begin{aligned} T_1'(b \otimes x) &= t_e(b)\mathrm{Tr}_m(x) \\ &+ \sum_{i=1}^{k} (\frac{\alpha_i'}{\|\nu_{0,i}\|} \int_{\mathbb{T}} \mathrm{tr}_i \circ \varphi_{0,i}(b) \mathrm{Tr}_m(x) d\nu_{0,i} \\ &+ \int_{\mathbb{T}} \mathrm{tr}_i \circ \varphi_{1,i}(b) \mathrm{Tr}_m(x) d\nu_{1,i}) \end{aligned} \tag{e 16.34}$$

for all $b \in C(\mathbb{T}) \otimes F_1$ and $x \in M_m$, where $\mathrm{Tr}_m : M_m \to \mathbb{C}$ is the non-normalized trace.

By (e 16.33) and (e 16.12), and by (e 16.18) and (e 16.30), one has

$$(1/s) \circ \kappa(p) = T(p) \quad \text{for all } p \in K_0(A) \text{ and} \tag{e 16.35}$$

$$|\tau(h) - T(h)| < \sigma_2/8k + \sigma_2/8k + \sigma/16kl < \sigma/2 \quad \text{for all } h \in \mathcal{H}. \tag{e 16.36}$$

Put $d_j = \prod_{i \neq j} r(i)$ and $d = d_0(\sum_{i=1}^{k} m(i))$. Note $d_i r(i) = d_0$ for all $1 \leq i \leq k$. It follows that

$$d = \bar{N} d_0 T_2(1_A). \tag{e 16.37}$$

Define $\Psi : A \to M_d$ by $\Psi(a) = \bigoplus_{i=1}^{k} (\sum_{j=1}^{m(i)} \bar{\varepsilon}_i(a)(t_{i,j}))$ for all $a \in A$, where

$$\bar{\varepsilon}_i(a) = \mathrm{diag}(\overbrace{\varepsilon_i(a), \varepsilon_i(a), ..., \varepsilon_i(a)}^{d_i \bar{N}}) = \varepsilon_i(a) \otimes 1_{d_i \bar{N}} \quad \text{for all } a \in A. \tag{e 16.38}$$

Denote by t_d the tracial state of M_d. Then

$$T_2(1_A) t_d(\Psi(a)) = T_2(a) \quad \text{for all } a \in A. \tag{e 16.39}$$

Let $\kappa_0 : K_0(A) \to \mathbb{Z}$ be given by Ψ. By hypothesis, $s\tau(p_i) \in \mathbb{Z}$, $j = 1, 2, ..., c$. By (e 16.33),

$$-s\bar{N}d_0T_1(p_j) = s\bar{N}d_0(T(p_j) - T_2(p_j)) \tag{e 16.40}$$
$$= s\bar{N}d_0([\tau(p_j) - \sum_{i=1}^{k}((1/\bar{N})\sum_{j=1}^{m(i)} \mathrm{tr}_i(p(t_{i,j})))])$$
$$= s\bar{N}d_0\tau(p_j) - d_0\sum_{i=1}^{k}(\sum_{j=1}^{m(i)} \mathrm{tr}_i(p(t_{i,j}))) \in \mathbb{Z}$$

$(s\tau(p_j) \in \mathbb{Z})$. We have

(e 16.41) $T_1'(e_j)$

$$= t_e(e_j) + \sum_{i=1}^{k}(\frac{\alpha_i'}{\|\nu_{0,i}\|}\int_{\mathbb{T}} \mathrm{tr}_i \circ \varphi_{0,i}(e_j)d\nu_{0,i}$$
$$+ \int_{\mathbb{T}} \mathrm{tr}_i \circ \varphi_{1,i}(e_j)d\nu_{1,i})$$
$$= t_e \circ \pi_e'(\bar{h}_j') + \sum_{i=1}^{k}(T_{0,i}(\bar{h}_j') + T_{1,i}(\bar{h}_j'))$$
$$\geq t_e \circ \pi_e'(\bar{h}_j')$$
$$+ \sum_{i=1}^{k}(\int_{((0,\delta_0')\cup(1-\delta_0',1))\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(\bar{h}_j'))d\mu_i - \sigma_2/4k) \text{ (by (e 16.31))}$$
$$= t_e \circ \pi_e'(\bar{h}_j') + \sum_{i=1}^{k}\int_{(0,1)\times\mathbb{T}} \mathrm{tr}_i(\varepsilon_i(\bar{h}_j'))d\mu_i - \sigma_2/4$$
$$(\varepsilon_i(\bar{h}_j)(\xi) = 0, \text{ for } \xi \in [\delta_0', 1-\delta_0'])$$
$$= \tau(\bar{h}_j') - \sigma_2/4 \geq \tau(h_j') - \sigma_2/4$$
$$\geq \Delta(h_j') - \sigma_2/4 \geq 2\sigma_1 - \sigma_2/4 \geq \sigma_1 \quad \text{(by (e 16.11))}.$$

Let $\kappa_1 : K_0(A) \to \mathbb{Z}$ be defined by $\kappa_1|_{\ker\rho_A} = 0$ and by $\kappa_1([p_j]) = s\bar{N}d_0T_1(p_j)$, $j = 1, 2, ..., c$. Let $\kappa_1' : \mathbb{Z}^l \to \mathbb{R}$ be defined by $s\bar{N}d_0T_1'$ (see (e 16.35)). As in (2) of 16.5 (see also 3.5 and (1) of 16.5), we may view $K_0(A)/\ker\rho_A$ as $K_0(C) \subset \mathbb{Z}^l$. Then $\kappa_1' \circ (\pi_e')_{*0} = \kappa_1$. In particular, $\kappa_1'([\pi_e'(1_A)]) = \kappa_1([1_A]) = s\bar{N}d_0T_1([1_A])$. Note that, by (e 16.41),

$$\kappa_1'([e_j]) = s\bar{N}d_0T_1'(e_j) \geq s\bar{N}d_0\sigma_1 \geq \sigma_1, \;\; j = 1, 2, ..., l. \tag{e 16.42}$$

By the choice of K_1 and M, and by applying 15.1, there is an order homomorphism $\kappa_2 : K_0(C(\mathbb{T}) \otimes F_1) = \mathbb{Z}^l \to \mathbb{Z}$ such that

$$\kappa_2|_{K_0(A)} = K_1 M \kappa_1 \quad \text{and} \quad |\kappa_1'(e_j) - (1/K_1M)\kappa_2(e_j)| < \sigma_2. \tag{e 16.43}$$

Write $T_1' = \sum_{j=1}^l \alpha_j t_j \circ \pi_j'$, where each t_j is a tracial state on $C(\mathbb{T}) \otimes M_{R(j)}$, and $\alpha_j = \kappa_1'(e_j)/s\bar{N}d_0$, $j = 1, 2, ..., l$. Write $\beta_j = (1/K_1 M s\bar{N}d_0)\kappa_2(e_j)$, $j = 1, 2, ..., l$. Then, by (e 16.43),

$$|\alpha_j - \beta_j| < \sigma_2/s\bar{N}d_0, \;\; j = 1, 2, ..., l. \tag{e 16.44}$$

Put $T_1'' = \sum_{j=1}^l \beta_j \circ \pi_j'$ and $T_1''' = T_1''/\|T_1''\|$. Note $T_1''' \in T(C(\mathbb{T}) \otimes F_1)$ and

$$\begin{aligned} \|T_1''\| &= T_1''(\pi_e'(1_A)) = (1/K_1 M s\bar{N}d_0)\kappa_2([\pi_e'(1_A)]) \\ &= (1/s\bar{N}d_0)\kappa_1([1_A]) = T_1([1_A]). \end{aligned}$$

We also have $(1/K_1 M s\bar{N}d_0 T_1([1_A]))\kappa_2([p]) = T_1'''(p)$ for all projections p in $C(\mathbb{T}) \otimes F_1$. Put $K_2 = K_1 N_0 M \bar{N} d_0 T_1([1_A])$. It follows from Lemma 16.3 that there is a unital homomorphism $\Phi : C(\mathbb{T}) \otimes F_1 \to M_{sK_2}$ such that

$$\Phi_{*0} = N_0\kappa_2 \quad \text{and} \quad |\mathrm{tr} \circ \Phi(h) - T_1'''(h)| < \sigma/64 \tag{e 16.45}$$

for all $h \in \overline{\mathcal{H}}$, where tr is the tracial state on M_{sK_2}. Recall (see (e 16.12) and (e 16.33)) that

$$T_1(1_A) + T_2(1_A) = T(1_A) = (1/s)\kappa([1_A]) = 1.$$

Thus, $sK = sN_0\bar{N}K_1 M d_0(T_1(1_A) + T_2(1_A))$. Define $\varphi : A \to M_{sK}$ by

$$\varphi(a) = \Phi \circ \pi_e'(a) \oplus \tilde{\Psi}(a) \;\text{ for all }\; a \in A, \tag{e 16.46}$$

where $\tilde{\Psi}$ is the direct sum of sN_0K_1M copies of Ψ (recall (e 16.37)). By (e 16.45), (e 16.43), (e 16.39), (e 16.32), and (e 16.35), for any projection $p \in A$, one has

$$\frac{(\varphi)_{*0}(p)}{sK} \;\; = \;\; T_1(p) + \sum_{i=1}^{k}(1/\bar{N})\sum_{j=1}^{m(i)} \mathrm{tr}_j(p(t_{i,j})) = T(p) = (1/s)\kappa,$$

and hence

$$\varphi_{*0} = K\kappa. \tag{e 16.47}$$

By (e 16.45) and (e 16.44),

$$|(T_1(1_A)\mathrm{tr} \circ \Phi(\pi_e'(h)) - T_1(h)| < (T_1(1_A))(\sigma/64 + l\sigma_2/s\bar{N}d_0) < \sigma/2 \tag{e 16.48}$$

for all $h \in \mathcal{H}$. Note that (recall the definition of K and K_2, and (e 16.37))

$$\begin{aligned} sK_2/sK &= T_1(1_A)/(T_1(1_A)+T_2(1_A)) = T_1(1_A) \\ &\quad \text{and } sN_0K_1Md/sK = T_2(1_A). \end{aligned} \tag{e 16.49}$$

It follows from (e 16.46), the first part of (e 16.49), (e 16.36), (e 16.48), the second part of (e 16.49) and (e 16.39) that

$$\begin{aligned} &|\mathrm{tr}' \circ \varphi(h) - \tau(h)| \\ &\quad \le |\mathrm{tr}' \circ \varphi(h) - T(h)| + |T(h) - \tau(h)| \\ &\quad < |T_1(1_A)\mathrm{tr} \circ \Phi(\pi_e'(h)) - T_1(h)| + |\mathrm{tr}' \circ \tilde{\Psi}(h) - T_2(h)| + \sigma/2 \\ &\quad < \sigma/2 + |T_2(1_A)t_d(\Psi(h)) - T_2(h)| + \sigma/2 = \sigma \end{aligned}$$

for all $h \in \mathcal{H}$. (Recall tr' is the tracial state of M_{sK} and tr is the tracial state on M_{sK_2}.) ☐

LEMMA 16.7. *Let $A = C \otimes B$ be a unital C^*-algebra, where $C \in \mathcal{C}$ and $B = C(X)$, where X is one point, or $X = \mathbb{T}$. Let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\mathcal{H} \subset A$ be a finite subset and let $\epsilon > 0$. There exist a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$ and a finite subset of projections $\mathcal{P} \subset M_n(A)$ (for some $n \ge 1$), and there is $\delta > 0$ such that if a tracial state $\tau \in T(A)$ satisfies*

$$\tau(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \tag{e 16.50}$$

and $\kappa : K_0(A) \to K_0(M_s)$ is any order preserving homomorphism with $\kappa([1_A]) = [1_{M_s}]$ satisfying

$$|\rho_{M_s}(\kappa([p]))(\mathrm{tr}) - \tau(p)| < \delta \tag{e 16.51}$$

for all projections $p \in \mathcal{P}$, where tr is the tracial state on M_s, then there is a tracial state $\tilde{\tau} \in T(A)$ such that

$$\rho_{M_s}(\kappa(x))(\mathrm{tr}) = \tilde{\tau}(x) \text{ for all } x \in K_0(A) \text{ and } |\tau(h) - \tilde{\tau}(h)| < \epsilon \text{ for all } h \in \mathcal{H}.$$

PROOF. Let $C = A(F_1, F_2, \varphi_1, \varphi_2)$. Write $A = C \otimes C(X) = C(X, C)$, where X is a single point or $X = \mathbb{T}$. Note that $K_0(A) = K_0(C) \oplus \boldsymbol{\beta}(K_1(C)) \cong K_0(C) \oplus K_1(C)$, $\ker\rho_C = \{0\}$ and $\ker\rho_A = \boldsymbol{\beta}(K_1(C))$ (see 2.14). Without loss of generality, we may assume that the projections in A generate $K_0(A)$, by replacing A by $M_N(A)$ for some integer $N \ge 1$. Let $\mathcal{P}'$ be a finite subset of projections in C which generates $K_0(C)_+$ (see 3.15) which we also assume to contain $[1_A]$. Let $\iota : C \to C(X, C) = A$ be defined by $\iota(c)(t) = c$ for all $t \in \mathbb{T}$ and $c \in C$ and let $\mathcal{P} = \{\iota(p) : p \in \mathcal{P}'\}$. Then, by Remark 16.5, $\{\rho_A([p]) : p \in \mathcal{P}\}$ generates $\rho_A(K_0(A))_+$. Without loss of generality, we may assume that $\mathcal{H} \subset A_+^{\mathbf{1}} \setminus \{0\}$. Let

$\mathcal{H}_1 = \mathcal{H} \cup \mathcal{P}$ and let $\sigma_0 = \min\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}$. Without loss of generality, we may assume that $0 < \varepsilon < 1$. Let $\delta = \varepsilon \cdot \sigma_0/128$.

Now suppose that τ and κ satisfy the hypotheses for the above mentioned $\mathcal{H}_1$, $\mathcal{P}$, and δ. Note that, for any $s \in S_{[1_{M_s}]}(K_0(M_s))$, $s \circ \kappa$ is a state on $K_0(A)$. By Corollary 3.4 of [8], $s \circ \kappa$ is induced by a tracial state of A. It follows that $\kappa(\mathrm{ker}\rho_A) \subset \mathrm{ker}\rho_{M_s} = \{0\}$. In what follows, identifying $K_0(M_s)$ with $\mathbb{Z}$, we may view κ as an order preserving homomorphism from $K_0(A)$ to $\mathbb{Z} \subset \mathbb{R}$. Define $\eta = (1 - \varepsilon/3)(\kappa/s - \tau) : K_0(A) \to \mathbb{R}$, where $\tau(x) := \rho_A(x)(\tau)$ for $x \in K_0(A)$. Let $d_A : K_0(A) \to K_0(A)/\mathrm{ker}\rho_A$ be the quotient map. Then κ, τ, and η factor through $K_0(A)/\mathrm{ker}\rho_A$. Choose $\gamma : K_0(A)/\mathrm{ker}\rho_A \to \mathbb{R}$ such that $\gamma \circ d_A = (\varepsilon/3s)\kappa + \eta$.

Let $\pi_e' : A \to F_1 \otimes C(\mathbb{T})$ be the homomorphism defined in Remark 16.5. Let $\Psi_0 = \pi_{t_0} \circ \pi_e' : A \to F_1$, where $\pi_{t_0} : F_1 \otimes C(\mathbb{T}) \to F_1$ is the point evaluation at $t_0 \in \mathbb{T}$. Note that, as in 16.5 and 3.5,

$$K_0(A)/\mathrm{ker}\rho_A \cong K_0(C). \tag{e 16.52}$$

As computed in 3.5, $K_0(A)/\mathrm{ker}\rho_A \cong K_0(C) = (\Psi_0)_{*0}(K_0(A)) \subset K_0(F_1)$. For each $p \in \mathcal{P}$, from the assumption (e 16.51), one computes that

$$|\eta([p])| < (1 - \varepsilon/3)\delta. \tag{e 16.53}$$

Therefore, by (e 16.50), (e 16.51), (e 16.53), and the choice of δ and σ_0,

$$\begin{aligned}
\gamma(d_A([p])) &= (\varepsilon/3)(\kappa([p])/s) + \eta([p]) \\
&> (\varepsilon/3)(\Delta(\hat{p}) - \delta) - (1 - \varepsilon/3)\delta \\
&\geq (\varepsilon/3)(1 - 1/128)\sigma_0 - (1 - \varepsilon/3)\varepsilon\sigma_0/128 \\
&= \varepsilon\sigma_0[(\frac{1}{3} - \frac{1}{3 \cdot 128}) - (\frac{1}{128} - \frac{\varepsilon}{3 \cdot 128})] > 0
\end{aligned}$$

for all $p \in \mathcal{P}$. In other words, γ is positive. By 2.8 of [61], there is then a positive homomorphism $\gamma_1 : K_0(F_1) \to \mathbb{R}$ such that $\gamma_1 \circ (\Psi_0)_{*0} = \gamma \circ d_A$. It is well known that there is a (non-normalized) trace T_0 on F_1 such that $\gamma_1([q]) = T_0(q)$ for all projections $q \in F_1$.

Consider the trace $\tau' = (1 - \varepsilon/3)\tau + T_0 \circ \Psi_0$ on A. Then, for any projection $p \in A$,

$$\begin{aligned}
\tau'(p) &= (1 - \varepsilon/3)\tau(p) + T_0 \circ \Psi_0(p) \\
&= (1 - \varepsilon/3)\tau(p) + (\varepsilon/3s)\kappa([p]) + \eta([p]) \\
&= (1 - \varepsilon/3)\tau(p) + (\varepsilon/3s)\kappa([p]) + (1 - \varepsilon/3)(\kappa([p])/s - \tau(p)) \\
&= (1/s)\kappa([p]).
\end{aligned}$$

Since $(1/s)\kappa([1_A]) = 1$, $\tau' \in T(A)$. We also compute that, by (e 16.53),

$$|T_0 \circ \Psi_0(1_A)| = |\gamma \circ d_A([1_A])| < \varepsilon/2. \tag{e 16.54}$$

Therefore, we also have

$$|\tau'(h) - \tau(h)| < \varepsilon \text{ for all } h \in \mathcal{H}. \tag{e 16.55}$$

$\square$

LEMMA 16.8. *Let $C = C(X, C_0)$ for some $C_0 \in \mathcal{C}$, where X is a point or $X = \mathbb{T}$. Let $\Delta : A_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\mathcal{H} \subset A$ be a finite subset and let $\sigma > 0$. Then there are a finite subset $\mathcal{H}_1 \subset A_+^{\mathbf{1}} \setminus \{0\}$, $\delta > 0$, a finite subset $\mathcal{P} \subset K_0(A)$, and a positive integer K such that for any $\tau \in T(A)$ satisfying*

$$\tau(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1$$

and any positive homomorphism $\kappa : K_0(A) \to K_0(M_s) = \mathbb{Z}$ with $s = \kappa([1_A])$ such that

$$|\rho_A(x)(\tau) - (1/s)(\kappa(x))| < \delta$$

for all $x \in \mathcal{P}$, there is a unital homomorphism $\varphi : A \to M_{sK}$ such that $\varphi_{*0} = K\kappa$ and

$$|\mathrm{tr}' \circ \varphi(h) - \tau(h)| < \sigma \text{ for all } h \in \mathcal{H},$$

where tr' is the tracial state on M_{sK}.

PROOF. Note that there is an integer $n \geq 1$ such that the projections in $M_n(A)$ generate $K_0(A)$. Therefore this lemma is a corollary of Lemma 16.6 and Lemma 16.7. $\square$

LEMMA 16.9. *Let $C = C(X, C_0)$ for some $C_0 \in \mathcal{C}$, where X is a point or $X = \mathbb{T}$. Let $\Delta : C_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\mathcal{F}, \mathcal{H} \subset C$ be finite subsets, and let $\epsilon > 0$, $\sigma > 0$. Then there are a finite subset $\mathcal{H}_1 \subset C_+^{\mathbf{1}} \setminus \{0\}$, $\delta > 0$, a finite subset $\mathcal{P} \subset K_0(C)$, and a positive integer K such that for any continuous affine map $\gamma : T(C([0,1])) \to T(C)$ satisfying*

$$\gamma(\tau)(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \text{ and for all } \tau \in T(C([0,1])),$$

and any positive homomorphism $\kappa : K_0(C) \to K_0(M_s(C([0,1])))$ with $\kappa([1_C]) = s$ such that

$$|\rho_A(x)(\gamma(\tau)) - (1/s)\tau(\kappa(x))| < \delta \text{ for all } \tau \in T(C([0,1])),$$

for all $x \in \mathcal{P}$, there is an $\mathcal{F}$-ϵ-multiplicative completely positive linear map $\varphi : C \to M_{sK}(C([0,1]))$ such that $[\varphi]|_{K_0(C)} = K\kappa$ (note that $K_0(C)$ is finitely generated; see 16.5 *and the end of* 2.12*) and*

$$|\tau \circ \varphi(h) - \gamma'(\tau)(h)| < \sigma \text{ for all } h \in \mathcal{H},$$

where $\gamma' : T(M_{sK}(C([0,1]))) \to T(C)$ is induced by γ. Furthermore, the map φ_0 can be chosen such that $\varphi_0 = \pi_0 \circ \varphi$ and $\varphi_1 = \pi_1 \circ \varphi$ are homomorphisms, where $\pi_t : M_{sK}(C([0,1]) \to M_{sK}$ is the point evaluation at $t \in [0,1]$. In the case that $C \in \mathcal{C}$ (i.e., X is a point), the map φ can be chosen to be a homomorphism.

PROOF. Since any C^*-algebras in $\mathcal{C}$ are semiprojective (see the line above 3.2), the second part of the statement follows directly from the first part of the statement. Thus, we need only show the first part of the statement. Without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of A, $1_A \in \mathcal{F}$ and $\{ab : a, b \in \mathcal{F}\} \subset \mathcal{H}$. To simplify notation, without loss of generality, by replacing C by $M_r(C)$ for some $r \geq 1$, we may assume that the set of projections in C generates $\rho_C(K_0(C))$ (see 3.15 and 16.5).

Since the K-theory of C is finitely generated, there is $m' \in \mathbb{N}$ such that, for any $x \in \mathrm{Tor}(K_i(C)) = 0$, $i = 0, 1$,

$$mx = 0 \text{ for some integer } 0 < m \leq m'.$$

Put $m_1 = (m')!$. Let $\mathcal{H}_{1,1} \subset C_+^1 \setminus \{0\}$ (in place of $\mathcal{H}_1$) and $\sigma_1 > 0$ (in place of δ) be the finite subsets and the positive constant provided by Theorem 4.18 for C (in place of A), $\varepsilon_0 = \min\{\sigma/3, \epsilon/3\}$ (in place of ϵ), $\mathcal{H}$ (in place of $\mathcal{F}$), and $\Delta/2$. (We will not need the finite set $\mathcal{P}$ of Theorem 4.18, since $K_0(C)$ is finitely generated and when we apply Theorem 4.18, we will require that both maps induce the same KL map.)

Let $\mathcal{H}_{1,2} \subset C$ (in place of $\mathcal{H}_1$) be a finite subset, $\delta > 0$ be a positive number, $\mathcal{P} \subset K_0(C)$ be a finite subset, and K' be an integer as provided by Lemma 16.8 for C, $\Delta/2$ (in place of Δ), $\mathcal{H} \cup \mathcal{H}_{1,1}$ (in place of $\mathcal{H}$), and the positive number $\min\{\sigma/16, \sigma_1/8, \{\Delta(\hat{h})/4 : h \in \mathcal{H}_{1,1}\}\}$ (in place of σ). We may assume that $\mathcal{P}$ is represented by P, a finite set of projections in C (see Remark 16.6).

Put $\mathcal{H}_1 = \mathcal{H}_{1,1} \cup \mathcal{H}_{1,2}$ and $K = m_1 K'$. Then, let $\gamma : T(C([0,1])) \to T(C)$ be a continuous affine map with

$$\gamma(\tau)(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1,$$

and let $\kappa : K_0(C) \to K_0(M_s(C([0,1])))$ with $\kappa([1_C]) = s$ be such that

$$|\rho_\gamma(\tau)(x) - (1/s)\tau(\kappa(x))| < \delta \text{ for all } x \in \mathcal{P} \text{ for all } \tau \in T(C([0,1])).$$

Since γ is continuous, there is a partition

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

such that, for any $0 \leq i \leq n-1$ and any $x \in [x_i, x_{i+1}]$, one has

$$|\gamma(\tau_x)(h) - \gamma(\tau_{x_i})(h)| < \min\{\sigma/8, \sigma_1/4\} \text{ for all } h \in \mathcal{H}_1, \tag{e 16.56}$$

where $\tau_x \in T(M_s(C([0,1])))$ is the extremal trace concentrated at x.

For any $0 \leq i \leq n$, consider the trace $\tilde{\tau}_i = \gamma(\tau_{x_i}) \in T(C)$. It is clear that

$$|\tilde{\tau}_i(x) - \mathrm{tr}((\pi_{x_i})_{*0} \circ \kappa(x))| < \delta \text{ for all } x \in \mathcal{P} \text{ and } \tilde{\tau}_i(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_{1,2},$$

where tr is the tracial state of M_s (π_x was defined in the statement of the lemma). By Lemma 16.8, there exists a unital homomorphism $\varphi_i' : C \to M_{sK'}$ such that

$$(\varphi_i')_{*0} = K\kappa,$$

as we identify $K_0(C([0,1], M_s))$ with $\mathbb{Z}$ and
(e 16.57)
$$|\mathrm{tr} \circ \varphi_i'(h) - \tau_{x_i}(h)| < \min\{\sigma/16, \Delta(\hat{h})/4, \sigma_1/8,\ h \in \mathcal{H}_{1,1}\} \text{ for all } h \in \mathcal{H} \cup \mathcal{H}_{1,1}.$$

Note $(\varphi_i')_{*1} = 0$ since $K_1(M_{sK'}) = \{0\}$. In particular, by (e 16.57), one has that, for any $0 \le i \le n-1$,

$$|\mathrm{tr} \circ \varphi_i'(h) - \mathrm{tr} \circ \varphi_{i+1}'(h)| < \sigma_1 \text{ for all } h \in \mathcal{H}_{1,1}.$$

Note that $\gamma(\tau_{x_i})(h) > \Delta(\hat{h})$ for any $h \in \mathcal{H}_{1,1}$ by hypothesis. It then also follows from (e 16.57) that, for any $0 \le i \le n$,

$$\mathrm{tr} \circ \varphi_i'(h) > \Delta(\hat{h})/2 \text{ for all } h \in \mathcal{H}_{1,1}.$$

Define the amplification φ_i'' as

$$\varphi_i'' := \varphi_i' \otimes 1_{\mathrm{M}_{m_1}(\mathbb{C})} : C \to M_{sK} = M_{m_1(sK')}.$$

Then $(\varphi_i'')_{*j} = (\varphi_{i+1}'')_{*j}$, $j = 0, 1$. By the choice of m_1, $[\varphi_i'']|_{K_j(C,\mathbb{Z}/k\mathbb{Z})} = 0 = [\varphi_{i+1}'']|_{K_j(C,\mathbb{Z}/k\mathbb{Z})}$ on each non-zero $K_j(C, \mathbb{Z}/k\mathbb{Z})$, $j = 0, 1$, $k = 2, 3, ...$ Therefore

$$[\varphi_i''] = [\varphi_{i+1}''] \quad \text{in } KL(C, M_{sK}).$$

It then follows from Theorem 4.18 that there is a unitary $u_1 \in M_s$ such that

$$\|\varphi_0''(h) - \mathrm{Ad}u_1 \circ \varphi_1''(h)\| < \varepsilon_0 \text{ for all } h \in \mathcal{H}.$$

Consider the maps $\mathrm{Ad}u_1 \circ \varphi_1''$ and φ_2''. Applying Theorem 4.18 again, one obtains a unitary $u_2 \in M_{sK}$ such that

$$\|\mathrm{Ad}u_1 \circ \varphi_1''(h) - \mathrm{Ad}u_2 \circ \varphi_2''(h)\| < \varepsilon_0 \text{ for all } h \in \mathcal{H}.$$

Repeating this argument for all $i = 1, ..., n$, one obtains unitaries $u_i \in M_{sK}$ such that

$$\|\mathrm{Ad}u_i \circ \varphi_i''(h) - \mathrm{Ad}u_{i+1} \circ \varphi_{i+1}''(h)\| < \varepsilon_0 \text{ for all } h \in \mathcal{H}.$$

Then define $\varphi_0 = \varphi_0''$ and $\varphi_i = \mathrm{Ad}u_i \circ \varphi_i''$, and one has

(e 16.58)
$$\|\varphi_i(h) - \varphi_{i+1}(h)\| < \varepsilon_0 \text{ for all } h \in \mathcal{H}.$$

Define the linear map $\varphi : C \to M_{sK}([0,1])$ by

$$\varphi(f)(t) = \frac{t - x_i}{x_{i+1} - x_i}\varphi_i(f) + \frac{x_{i+1} - t}{x_{i+1} - x_i}\varphi_{i+1}(f), \quad \text{if } t \in [x_i, x_{i+1}].$$

Since each φ_i is a homomorphism, by (e 16.58), the map φ is $\mathcal{F}$-ϵ-multiplicative. It is clear that $[\varphi]|_{K_0(C)} = K\kappa$. On the other hand, for any $x \in [x_i, x_{i+1}]$ for some $i = 1, ..., n-1$, one has that for any $h \in \mathcal{H}$,

$$\begin{aligned}
&|\gamma(\tau_x)(h) - \tau_x \circ \varphi(h)| \\
&= |\gamma(\tau_x)(h) - (\frac{x - x_i}{x_{i+1} - x_i}\mathrm{tr}(\varphi_i(h)) + \frac{x_{i+1} - x}{x_{i+1} - x_i}\mathrm{tr}(\varphi_{i+1}(h)))| \\
&< |\gamma(\tau_x)(h) - (\frac{x - x_i}{x_{i+1} - x_i}\gamma(\tau_{x_i})(h) + \frac{x_{i+1} - x}{x_{i+1} - x_i}\gamma(\tau_{x_{i+1}})(h))| + \sigma/4 \\
&\qquad\qquad \text{(by (e 16.57))} \\
&< |\gamma(\tau_x)(h) - \gamma(\tau_{x_{i+1}})(h))| + 3\sigma/8 \qquad \text{(by (e 16.56))} \\
&< \sigma/2 \qquad \text{(by (e 16.56))}.
\end{aligned}$$

Hence for any $h \in \mathcal{H}$,

$$|\gamma(\tau)(h) - \tau \circ \varphi(h)| < \sigma$$

for any $\tau \in T(M_{sK}(C([0,1])))$.

Note that $\pi_0 \circ \varphi = \varphi_0$ and $\pi_1 \circ \varphi = \varphi_n$ which are homomorphisms. Thus the map φ satisfies the conclusion of the lemma.

□

THEOREM 16.10. *Let $C = C(X, C_0)$, where $C_0 \in \mathcal{C}$ and X is a point, or $X = \mathbb{T}$. Let $\Delta : C_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\mathcal{F}, \mathcal{H} \subset C$ be finite subsets, and let $1 > \sigma, \epsilon > 0$. There exist a finite subset $\mathcal{H}_1 \subset C_+^{\mathbf{1}} \setminus \{0\}$, $\delta > 0$, a finite subset $\mathcal{P} \subset K_0(C)$, and a positive integer K such that for any continuous affine map $\gamma : T(D) \to T(C)$ satisfying*

$$\gamma(\tau)(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \text{ for all } \tau \in T(D),$$

where D is a C^*-algebra in $\mathcal{C}$, any positive homomorphism $\kappa : K_0(C) \to K_0(D)$ with $\kappa([1_C]) = s[1_D]$ for some integer $s \geq 1$ satisfying

$$|\rho_C(x)(\gamma(\tau)) - (1/s)\tau(\kappa(x))| < \delta \text{ for all } \tau \in T(D)$$

and for all $x \in \mathcal{P}$, there is a $\mathcal{F}$-ϵ-multiplicative positive linear map $\varphi : C \to M_{sK}(D)$ such that

$$[\varphi]|_{K_0(A)} = K\kappa$$

(see 2.12) and

$$|(1/(sK))\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma \tag{e 16.59}$$

$$\text{for all } h \in \mathcal{H} \text{ and } \tau \in T(D).$$

In the case that $C \in \mathcal{C}$, the map φ can be chosen to be a homomorphism.

Proof. Recall that if $\Phi : B_1 \to B_2$ is a map from a C^*-algebra B_1 to a C^*-algebra B_2, then we will continue to use Φ for the amplification $\Phi \otimes \mathrm{id}_{M_n} : B_1 \otimes M_n \to B_2 \otimes M_n$. We will use this practice in this proof and in the rest of this section. As in the proof of Lemma 16.9, since C^*-algebras in $\mathcal{C}$ are semiprojective, we only need to prove the first part of the statement. Without loss of generality, one may assume that $\mathcal{F}$ is a subset of the unit ball of C, $1_C \in \mathcal{F}$, and $\{ab : a, b \in \mathcal{F}\} \subset \mathcal{H}$. Fix $1 > \varepsilon, \sigma > 0$. Replacing C by $M_m(C)$ for some integer $m \geq 1$, and applying 3.15, we may find a finite subset P of projections in C such that $\mathcal{P} = \{[p] : p \in P\}$ generates $K_0(C)$. We may also assume that $P \subset \mathcal{H}$.

Since the K-groups of C are finitely generated (as abelian groups), there is $m' \in \mathbb{N}$ such that, for any $x \in \mathrm{Tor}(K_i(C))$, $i = 0, 1$, $mx = 0$ for some $0 < m \leq m'$. Set $m_1 = m!$.

Let $\mathcal{H}_{1,1} \subset C_+^1 \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\mathcal{H}'_{1,1} \subset C_{s.a.}$ (in place of $\mathcal{H}_2$) be finite subsets and $\mathcal{Q} \subset \underline{K}(C)$ (in place of $\mathcal{P}$) be another finite subset, and $\sigma_1 > 0$ (in place of δ) be a positive number as provided by Theorem 4.18 with respect to C (in place of A), $\min\{\sigma/4, \epsilon/6\}$ (in place of ϵ), $\mathcal{H}$ (in place of $\mathcal{F}$), and Δ.

Set $\sigma_0 = \frac{1}{2}\min\{\sigma/16, \sigma_1/4, \min\{\Delta(\hat{h})/2 : \ h \in \mathcal{H}_{1,1}\}\}$. Let $\mathcal{H}_{1,2} \subset C_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, let σ_2 (in place of δ) be a positive number, and K_1 (in place of K) be an integer as provided by Lemma 16.8 with respect to $\mathcal{H} \cup \mathcal{H}_{1,1} \cup \mathcal{H}'_{1,1}$ and σ_0 (in place of σ) and Δ. Without loss of generality, we may assume that $\mathcal{H}_{1,2} \supset \mathcal{H}_{1,1}$.

Let $\mathcal{H}_{1,3} \subset C_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\sigma_3 > 0$ (in place of δ), and K_2 (in place of K) be finite subsets and a constant as provided by Lemma 16.9 with respect to C, $\mathcal{H} \cup \mathcal{H}'_{1,1} \cup \mathcal{H}_{1,2}$ (in place of $\mathcal{H}$), σ_0 (in place of σ), $\varepsilon/12$ (in place of ε), $\mathcal{H}$ (in place of $\mathcal{F}$), and Δ (with the same $\mathcal{P}$ as above).

Put $\mathcal{H}_1 = \mathcal{H}_{1,1} \cup \mathcal{H}_{1,2} \cup \mathcal{H}_{1,3} \cup P$, $\delta = \min\{\sigma_1/2, \sigma_2, \sigma_3, 1/4\}$, and $K = m_1 K_1 K_2$. Let

$$\begin{aligned} D = A(F_1, F_2, \psi_0, \psi_1) \quad &= \quad \{(f, a) \in C([0,1], F_2) \oplus F_1 : f(0) = \psi_0(a) \\ &\qquad \text{and } \ f(1) = \psi_1(a)\} \end{aligned}$$

be any C^*-algebra in $\mathcal{C}$, where $\psi_i : F_1 \to F_2$ is a unital homomorphism, $i = 0, 1$, and let $\gamma : T(D) \to T(C)$ be a given continuous affine map satisfying

$$\gamma(\tau)(h) > \Delta(\hat{h}) \ \text{ for all } \ h \in \mathcal{H}_1 \ \text{ for all } \ \tau \in T(D).$$

Write $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$, $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(k)}$ and $I_j = C([0,1], M_{r(j)})$, $j = 1, 2, ..., k$. Denote by $\pi_{e,j} : D \to M_{R(j)}$ the homomorphism which is the composition of $\pi_e : D \to F_1$ (defined in 3.1) and the projection from F_1 onto $M_{R(j)}$ $(1 \leq j \leq l)$. Denote by $\pi^{I_j} : D \to I_j$ the restriction map defined by $(f, a) \to f|_{[0,1]_j}$ (see 3.17).

Let $\kappa : K_0(C) \to K_0(M_s(D))$ be any positive map with $s[1_D] = \kappa([1_C])$ satisfying

$$|\rho_C(x)(\gamma(\tau)) - (1/s)\tau(\kappa(x))| < \delta \ \text{ for all } \ \tau \in T(D)$$

and for all $x \in \mathcal{P}$. Write $C([0,1], F_2) = I_1 \oplus I_2 \oplus \cdots \oplus I_k$ with $I_i = C([0,1], M_{r(i)})$, $i = 1, ..., k$. Note that γ induces a continuous affine map $\gamma_i : T(I_i) \to T(C)$ defined by $\gamma_i(\tau) = \gamma(\tau \circ \pi^{I_i})$ for each $1 \le i \le k$. It is clear that for any $1 \le i \le k$, one has that

$$\gamma_i(\tau)(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_{1,3} \text{ and } \text{ for all } \tau \in T(I_i), \tag{e 16.60}$$

and

$$|\rho_C(x)(\gamma_i(\tau)) - \tau((\pi^{I_i})_{*0} \circ \kappa(x))| < \delta \le \sigma_3 \quad \text{for all } \tau \in T(M_s(I_i)) \tag{e 16.61}$$

and for all $x \in \mathcal{P}$ and for any $1 \le i \le k$. Since

$$\gamma(\tau)(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_{1,2} \text{ and for all } \tau \in T(D), \text{ and} \tag{e 16.62}$$

$$|\rho_C(x)(\gamma(\tau)) - (1/s)\tau(\kappa(x))| < \delta \text{ for all } \tau \in T(D) \tag{e 16.63}$$

and for all $x \in \mathcal{P}$, one has that, for each j,

$$\gamma \circ (\pi_{e,j})^*(\mathrm{tr}'_j)(h) > \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_{1,2} \text{ and} \tag{e 16.64}$$

$$|\rho_C(x)(\gamma \circ (\pi_{e,j})^*(\mathrm{tr}'_j)) - \mathrm{tr}'_j((\pi_{e,j})_{*0} \circ \kappa(x))| < \delta \le \sigma_2, \tag{e 16.65}$$

where tr'_j is the tracial state on $M_{R(j)}$, for all $x \in K_0(C)$ and where $\gamma \circ (\pi_{e,j})^*(\mathrm{tr}'_j) = \gamma(\mathrm{tr}'_j \circ \pi_{e,j})$.

Using (e 16.64), (e 16.65) and applying 16.8 to $(\pi_{e,j})_{*0} \circ K_1K_2\kappa$, one obtains a homomorphism $\varphi'_j : C \to M_{R(j)} \otimes M_{sK_1K_2}$ such that

$$(\varphi'_j)_{*0} = (\pi_{e,j})_{*0} \circ K_1K_2\kappa \text{ and} \tag{e 16.66}$$

$$|\mathrm{tr}_j \circ \varphi'_j(h) - (\gamma \circ (\pi_{e,j})^*(\mathrm{tr}'_j))(h)| < \sigma_0 \quad \text{for all } h \in \mathcal{H} \cup \mathcal{H}_{1,1} \cup \mathcal{H}'_{1,1}, \tag{e 16.67}$$

where tr_j is the tracial state on $M_{R(j)} \otimes M_{sK_1K_2}$. By (e 16.64), (e 16.67) and the choice of σ_0,

$$\mathrm{tr}_j \circ \varphi'_j(h) \ge \Delta(\hat{h})/2 \text{ for all } h \in \mathcal{H}_{1,1}. \tag{e 16.68}$$

Set $\varphi' = \bigoplus_{j=1}^{l} \varphi'_j : C \to F_1 \otimes \mathrm{M}_{sK_1K_2}$. Then, for all $t \in T(F_1 \otimes \mathrm{M}_{sK_1K_2})$,

$$t \circ \varphi'(h) \ge \Delta(\hat{h})/2 \text{ for all } h \in \mathcal{H}_{1,1}. \tag{e 16.69}$$

By (e 16.67), for all $h \in \mathcal{H} \cup \mathcal{H}_{1,1}\mathcal{H}'_{1,1}$ and for all $t \in T(F_1)$,

$$|(t \otimes \mathrm{tr}_{sK_1K_2}) \circ \varphi'(h) - \gamma(t \circ \pi_e)(h)| < \sigma_0, \tag{e 16.70}$$

where $\mathrm{tr}_{sK_1K_2}$ is the tracial state of $M_{sK_1K_2}$. Applying Lemma 16.9, using (e 16.60) and (e 16.61), and tensoring the resulting map with $1_{M_{K_1}}$, for any $1 \le i \le k$, one obtains an $\mathcal{H}$-$\varepsilon/4$-multiplicative contractive completely positive linear map $\varphi_i : C \to I_i \otimes M_{sK_1K_2}$ such that $[\varphi_i]|_{K_0(C)} = (\pi^{I_i})_{*0} \circ K_1K_2\kappa$ and

$$|\tilde{\tau} \circ \varphi_i(h) - ((\gamma \circ (\pi^{I_i})^*(\tau))(h)| < \sigma_0 \tag{e 16.71}$$

for all $h \in \mathcal{H} \cup \mathcal{H}_{1,1} \cup \mathcal{H}'_{1,1}$, and for all $\tau \in T(I_i)$ and $\tilde{\tau} = \tau \otimes \mathrm{tr}_{sK_1K_2}$. Furthermore, as in the conclusion of Lemma 16.9, the restrictions of φ_i to both boundaries (end points of the interval) are homomorphisms.

For each $1 \le i \le k$, denote by $\pi_{i,t}$ the evaluations of $I_i \otimes M_{sK_1K_2}$ at the point $t \in [0,1]$. Also, for each $1 \le i \le k$, define $\psi_{0,i} = (q_i \circ \psi_0) \otimes M_{sK_1K_2}$ and $\psi_{1,i} = (q_i \circ \psi_1) \otimes \mathrm{id}_{M_{sK_1K_2}}$, where $q_i : F_2 \to M_{r(i)}$ is the projection map. Then one has

$$\psi_{0,i} \circ \pi_e = \pi_{i,0} \circ \pi^{I_i}. \tag{e 16.72}$$

(Recall that a map Φ is identified with $\Phi \otimes \mathrm{id}_{sK_1K_2}$). It follows that

$$\begin{aligned} &(\psi_{0,i} \circ \varphi')_{*0} \\ &= (\psi_{0,i})_{*0} \circ (\sum_{j=1}^{l} (\pi_{e,j})_{*0}) \circ K_1K_2\kappa \\ &= (\psi_{0,i})_{*0} \circ (\pi_e)_{*0} \circ K_1K_2\kappa = (\pi_{i,0} \circ \pi^{I_i})_{*0} \circ K_1K_2\kappa \\ &= (\pi_{i,0})_{*0} \circ [\varphi_i]|_{K_0(A)}. \end{aligned} \tag{e 16.73}$$

Note that $\psi_{0,i}$ is unital. Therefore, by (e 16.69), for any $\mathrm{tr}^{(i)} \in T(M_{r(i)} \otimes M_{sK_1K_2})$, one has

$$\mathrm{tr}^{(i)} \circ (\psi_{0,i} \circ \varphi')(h) \ge \Delta(\hat{h})/2 \text{ for all } h \in \mathcal{H}_{1,1}, \tag{e 16.74}$$

and, since $\pi_{i,0}$ is unital, by (e 16.71) and (e 16.62),

$$\mathrm{tr}^{(i)} \circ (\pi_{i,0} \circ \varphi_i)(h) \ge \Delta(\hat{h})/2 \text{ for all } h \in \mathcal{H}_{1,1}. \tag{e 16.75}$$

Let $t^{(i)}$ be the tracial state of $M_{r(i)}$. It follows from (e 16.71), (e 16.72) and (e 16.67) that

$$\begin{aligned} &\mathrm{tr}^{(i)} \circ (\pi_{i,0} \circ \varphi_i)(h) \\ &\approx_{\sigma_0} \gamma((\pi^{I_i})^*)(t^{(i)} \circ \pi_{i,0})(h) = \gamma(t^{(i)} \circ \pi_{i,0} \circ \pi^{I_i})(h) \\ &= \gamma(t^{(i)} \circ \psi_{i,0} \circ \pi_e)(h) \approx_{\sigma_0} \mathrm{tr}^{(i)} \circ \varphi'(h) \\ &\qquad \text{for all } h \in \mathcal{H}_{1,1} \cup \mathcal{H}'_{1,1}. \end{aligned} \tag{e 16.76}$$

Consider the amplifications

$$\text{(e 16.77)} \qquad \begin{aligned} \varphi_i^{\sim}: &= \varphi_i \otimes 1_{\mathrm{M}_{m_1}(\mathbb{C})} : C \to I_i \otimes M_{sK} \text{ and} \\ \varphi'': &= \varphi' \otimes 1_{\mathrm{M}_{m_1}(\mathbb{C})} : C \to F_1 \otimes M_{sK}. \end{aligned}$$

Since $K_1(M_{r(i)}) = \{0\}$, by (e 16.73), $(\psi_{0,i} \circ \varphi'')_{*j} = (\pi_{i,0} \circ \varphi_i^{\sim})_{*j}$, $j = 0, 1$. By the choice of m_1, $[\psi_{0,i} \circ \varphi'']|_{K_j(C,\mathbb{Z}/k\mathbb{Z})} = 0 = [\pi_{i,0} \circ \varphi_i^{\sim}]|_{K_j(C,\mathbb{Z}/k\mathbb{Z})}$, $j = 0, 1$, and $i = 0, 1$, for any k such that $K_i(C, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$. It follows that

$$[\psi_{0,i} \circ \varphi''] = [\pi_{i,0} \circ \varphi_i^{\sim}] \quad \text{in } KL(C, \mathrm{M}_{r(i)sK}).$$

Therefore, by Theorem 4.18 (by also (e 16.74), (e 16.75), and (e 16.76)), there is a unitary $u_{i,0} \in M_{r(i)} \otimes M_{sK}$ such that

$$\|\mathrm{Ad} u_{i,0} \circ \pi_{i,0} \circ \varphi_i^{\sim}(f) - \psi_{0,i} \circ \varphi''(f)\| < \min\{\sigma/4, \epsilon/6\} \text{ for all } f \in \mathcal{H}.$$

Exactly the same argument shows that there is a unitary $u_{i,1} \in M_{r(i)} \otimes M_{sK}$ such that

$$\|\mathrm{Ad} u_{i,1} \circ \pi_{i,1} \circ \varphi_i^{\sim}(f) - \psi_{1,i} \circ \varphi''(f)\| < \min\{\sigma/4, \epsilon/6\} \text{ for all } f \in \mathcal{H}.$$

Choose two paths of unitaries $\{u_{i,0}(t) : t \in [0, 1/2]\} \subset M_{r(i)} \otimes M_{sK}$ such that $u_{i,0}(0) = u_{i,0}$ and $u_{i,0}(1/2) = 1_{M_{r(i)} \otimes M_{sK}}$, and $\{u_{i,1}(t) : t \in [1/2, 1]\} \subset M_{r(i)} \otimes M_{sK}$, such that $u_{i,1}(1/2) = 1_{M_{r(i)} \otimes M_{sK}}$ and $u_{i,1}(1) = u_{i,1}$ Put $u_i(t) = u_{i,0}(t)$ if $t \in [0, 1/2)$ and $u_i(t) = u_{i,1}(t)$ if $t \in [1/2, 1]$. Define $\varphi_{i,I} : C \to I_i \otimes M_{sK}$ by

$$\pi_t \circ \varphi_{i,I} = \mathrm{Ad}\, u_i(t) \circ \pi_t \circ \varphi_i',$$

where $\pi_t : I_i \otimes M_{sK} \to M_{r(i)} \otimes M_{sK}$ is the point evaluation at $t \in [0, 1]$.

One has that, for each i,

$$\begin{aligned} &\|\pi_{i,0} \circ \varphi_{i,I}(f) - \psi_{0,i} \circ \varphi''(f)\| < \min\{\sigma/4, \epsilon/6\} \text{ and} \\ &\|\pi_{i,1} \circ \varphi_{i,I}(f) - \psi_{1,i} \circ \varphi''(f)\| < \min\{\sigma/4, \epsilon/6\} \text{ for all } f \in \mathcal{H}. \end{aligned}$$

For each $1 \le i \le k$, let $\epsilon_i < 1/2$ be a positive number such that

$$\text{(e 16.78)} \qquad \begin{aligned} &\|\varphi_{i,I}(f)(t) - \psi_{0,i} \circ \varphi''(f)\| < \min\{\sigma/4, \epsilon/6\} \\ &\quad \text{for all } f \in \mathcal{H} \text{ for all } t \in [0, \epsilon_i], \text{ and} \end{aligned}$$

$$\text{(e 16.79)} \qquad \begin{aligned} &\|\varphi_{i,I}(f)(t) - \psi_{1,i} \circ \varphi''(f)\| < \min\{\sigma/4, \epsilon/6\} \\ &\quad \text{for all } f \in \mathcal{H} \text{ for all } t \in [1 - \epsilon_i, 1]. \end{aligned}$$

Define $\Phi_i : C \to I_i \otimes M_{sK}$ to be

$$\Phi_i(f)(t) = \begin{cases} (\varepsilon_i - t/\varepsilon_i)(\psi_{0,i} \circ \varphi'') + (t/\epsilon_i)\varphi_{i.I}(f)(\epsilon_i), & \text{if } t \in [0, \epsilon_i], \\ \varphi_{i,I}(f)(t), & \text{if } t \in [\epsilon_i, 1-\epsilon_i] \\ ((t-1+\epsilon_i)/\epsilon_i)(\psi_{1,i} \circ \varphi'') + ((1-t)/\epsilon_i)\varphi_{i,I}(f)(\epsilon_i), & \text{if } t \in [1-\epsilon_i, 1]. \end{cases},$$

The map Φ_i is not necessarily a homomorphism, but it is $\mathcal{F}$-ϵ-multiplicative. Moreover, it satisfies the relations

$$\begin{aligned} \text{(e 16.80)} \qquad \pi_{i,0} \circ \Phi_i(f) &= \psi_{0,i} \circ \varphi''(f) \text{ and} \\ \pi_{i,1} \circ \Phi_i(f) &= \psi_{1,i} \circ \varphi''(f) \text{ for all } f \in \mathcal{H}, i = 1, ..., k. \end{aligned}$$

Define $\Phi' : C \to C([0,1], F_2) \otimes M_{sK}$ by $\pi_{i,t} \circ \Phi' = \pi_t \circ \Phi_i$, where $\pi_{i,t} : C([0,1], F_2) \otimes M_{sK} \to M_{r(i)} \otimes M_{sK}$ is defined by the point evaluation at $t \in [0,1]$ (on the i-th direct summand). Define

$$\varphi(f) = (\Phi'(f), \varphi''(f)).$$

It follows from (e 16.80) that φ is an $\mathcal{F}$-ϵ-multiplicative contractive completely positive linear map from C to $D \otimes M_{sK}$. It follows from (e 16.66) (see also (**??**)) that

$$\text{(e 16.81)} \quad [\pi_e \circ \varphi(p)] = [\varphi''(p)] = (\pi_e)_{*0} \circ K\kappa([p]) \text{ for all } p \in \mathcal{P}.$$

Since $(\pi_e)_{*0} : K_0(D) \to \mathbb{Z}^l$ is injective (see 3.5), one has

$$\text{(e 16.82)} \qquad \varphi_{*0} = K\kappa.$$

For any $\tau_0 \in T(F_1 \otimes M_{sK})$, let $\tau = \tau_0 \circ \pi_e$. Note also $\pi_e \circ \varphi = \varphi''$. By (e 16.70),

$$\text{(e 16.83)} \qquad |\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma/16 \text{ for all } h \in \mathcal{H}.$$

Let $\tau \in T(D)$ be defined by $\tau(f) = \sum_{j=1}^k \int_{(0,1)} \mathrm{tr}_{r(j)sK}(\pi^{I_j}(f))d\mu_j$ for all $f \in D \otimes M_{sK}$, where $\mathrm{tr}_{r(j)sK}$ is the tracial state of $M_{r(j)sK}$ and μ_j is a positive Borel measure on $(0,1)$ with $\sum_{j=1}^k \|\mu_j\| = 1$. Then, by the definition of φ, (e 16.78), (e 16.79), and (e 16.71),

$$\text{(e 16.84)} \qquad |\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma/4 + \sigma/16 = 5\sigma/16 \text{ for all } h \in \mathcal{H}.$$

In general, $\tau \in T(D \otimes M_{sK})$ has the form

$$\tau(f) = \tau_0 \circ \pi_e(f) + \sum_{j=1}^k \int_{(0,1)} \mathrm{tr}_{r(j)sK}(\pi^{I_j}(f))d\mu_j \text{ for all } f \in D \otimes M_{sK},$$

where $\|\tau_0\| + \sum_{j=1}^k \|\mu_j\| = 1$ (see the proof of 3.10). It follows from (e 16.83) and (e 16.84) that

$$|(1/sK)\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma \text{ for all } h \in \mathcal{H}$$

and for all $\tau \in T(D)$. □

LEMMA 16.11. *Let $C \in \mathcal{C}$. For any $\varepsilon > 0$ and any finite subset $\mathcal{H} \subset C_{s.a.}$, there exists a finite subset of extremal traces $E \subset T(C)$ and a continuous affine map $\lambda : T(C) \to \triangle_E$, where $\triangle_E$ is the convex hull of of E, such that*

$$|\lambda(\tau)(h) - \tau(h)| < \varepsilon, \quad h \in \mathcal{H},\ \tau \in T(C). \tag{e 16.85}$$

PROOF. We may assume that $\mathcal{H}$ is in the unit ball of C. Write

$$C = A(F_1, F_2, \psi_0, \psi_1),$$

where $F_1 = M_{R(1)} \oplus M_{R(2)} \oplus \cdots \oplus M_{R(l)}$ and $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(k)}$. Let $\pi_{e,i} : C \to M_{R(i)}$ be the surjective homomorphism defined by the composition of π_e and the projection from F_1 onto $M_{R(i)}$, and $\pi^{I_j} : C \to C([0,1], M_{r(j)})$ the restriction which may also be viewed as the restriction of the projection from $C([0,1], F_2)$ to $C([0,1], M_{r(i)})$. Denote by $\pi_t \circ \pi^{I_j}$ the composition of π^{I_j} and the point evaluation at $t \in [0,1]$. There is $\delta > 0$ such that, for any $h \in \mathcal{H}$,

$$\|\pi^{I_j}(h)(t) - \pi^{I_j}(h)(t')\| < \varepsilon/16 \text{ for all } h \in \mathcal{H} \tag{e 16.86}$$

and $|t - t'| < \delta$, $t, t' \in [0,1]$.

Let $g_1, g_2, ..., g_n$ be a partition of unity over the interval $[\delta, 1-\delta]$ subordinate to an open cover with order 2 such that each $\mathrm{supp}(g_i)$ has diameter $< \delta$ and $g_s g_{s'} \neq 0$ implies that $|s - s'| \leq 1$. Let $t_s \in \mathrm{supp}(g_s) \cap [\delta, 1-\delta]$ be a point. We may assume that $t_s < t_{s+1}$. We may further choose $t_1 = \delta$ and $t_n = 1 - \delta$ and assume that $g_1(\delta) = 1$ and $g_n(1-\delta) = 1$, choosing an appropriate open cover of order 2.

Extend g_s to $[0,1]$ by defining $g_s(t) = 0$ if $t \in [0,\delta) \cup (1-\delta, 1]$ for $s = 2, 3, ..., n-1$ and

$$\begin{aligned} g_1(t) &= g_1(\delta)(t/\delta) \text{ for } t \in [0,\delta) \text{ and} \\ g_n(t) &= g_n(1-\delta)(1-t)/\delta \text{ for } t \in (1-\delta, 1]. \end{aligned} \tag{e 16.87}$$

Define $g_0 = 1 - \sum_{s=1}^n g_s$. Then $g_0(t) = 0$ for all $t \in [\delta, 1-\delta]$. Put $\bar{g}_s = (g_s \cdot 1_{F_2}, 0) \in C$ for $1 \leq s \leq n$, and $\bar{g}_0 = (g_0 \cdot 1_{F_2}, 1_{F_1})$, so that $\psi_0 \circ \pi_e(\bar{g}_0) = \psi_0(1_{F_1})$ and $\psi_0(\pi_e(g_0)) = \psi_1(1_{F_1})$. Let $g_{s,j} = \pi^{I_j}(\bar{g}_s)$, $s = 1, 2, ..., n$, $j = 1, 2, ..., k$. Let $p_i \in F_1$ be the support projection corresponding to the summand $M_{R(i)}$. Choose $d_i \in C([0,1], F_2)$ such that $d_i(t) = \psi_0(p_i)$ for $t \in [0,\delta]$ and $d_i(t) = \psi_1(p_i)$ for $t \in [1-\delta, 1]$ and $0 \leq d_i(t) \leq 1$ for $t \in (\delta, 1-\delta)$. Note that $\bar{d}_i = (d_i \cdot 1_{F_2}, p_i) \in C$. We may assume that $\sum_{i=1}^l g_0 d_i = g_0$ and $\sum_{i=1}^l \bar{g}_0 \bar{d}_i = \bar{g}_0$. Denote by tr_i the tracial state on $M_{R(i)}$ and tr'_j the tracial state on $M_{r(j)}$, $i = 1, 2, ..., l$, and $j = 1, 2, ..., k$. Let

$$E = \{\mathrm{tr}_i \circ \pi_{e,i} : 1 \leq i \leq l\} \cup \bigcup_{s=1}^n \{\mathrm{tr}'_j \circ \pi_{t_s} \circ \pi^{I_j} : 1 \leq j \leq k\}.$$

Let $\triangle_E$ be the convex hull of E. Define $\lambda : T(C) \to \triangle_E$ by

$$\lambda(\tau)(f) = \sum_{j=1}^{k}\sum_{s=1}^{n} \tau(g_{s,j})\mathrm{tr}'_j \circ (\pi^{I_j}(f)(t_s)) + \sum_{i=1}^{l} \tau(\bar{g}_0\bar{d}_i)\mathrm{tr}_i \circ \pi_{e,i}(f),$$

where we view $g_{s,j} \in C_0((0,1), M_{r(j)}) \subset C$, for all $f \in C$. It is clear that λ is a continuous affine map. Note that if $h \in C$ ($\pi_{e,j}(\bar{d}_i) = 0$, if $i \neq j$),

$$\begin{aligned} \text{(e 16.88)} \qquad & \lambda(\mathrm{tr}_j \circ \pi_{e,j})(h) \\ &= \sum_{i=1}^{l} \mathrm{tr}_j \circ \pi_{e,j}(\bar{g}_0\bar{d}_i)\mathrm{tr}_i \circ \pi_{e,i}(h) \\ &= \mathrm{tr}_j \circ \pi_{e,j}(\bar{g}_0\bar{d}_j)\mathrm{tr}_j \circ \pi_{e,j}(h) = \mathrm{tr}_j \circ \pi_{e,j}(h). \end{aligned}$$

If $\tau(f) = \mathrm{tr}'_j \circ (\pi^{I_j}(f)(t))$ with $t \in (\delta, 1-\delta)$, then, for $h \in \mathcal{H}$,

$$\begin{aligned} \text{(e 16.89)} \qquad \tau(h) &= \mathrm{tr}'_j \circ (\pi^{I_j}(h)(t)) = (\sum_{s=1}^{n} \mathrm{tr}'_j \circ (\pi^{I_j}(h\bar{g}_s)(t))) \\ &\approx_{2\varepsilon/16} \sum_{s=1}^{n} \mathrm{tr}'_j(\pi^{I_j}(h)(t_s)(\pi^{I_j})(\bar{g}_s)(t)) \\ &= \sum_{s=1}^{n} g_{s,j}(t)\mathrm{tr}'_j \circ \pi^{I_j}(h)(t_s) \\ &= \sum_{s=1}^{n} \tau(\bar{g}_{s,j})\mathrm{tr}'_j(\pi^{I_j}(h))(t_s) \\ &= \sum_{i=1}^{k}\sum_{s=1}^{n} \tau(\bar{g}_{s,i})\mathrm{tr}'_i(\pi^{I_i}(h))(t_s) \\ &\quad + \sum_{i=1}^{l} 0 \cdot \mathrm{tr}_i \circ \pi_{e,i}(h) = \lambda(\tau)(h), \end{aligned}$$

where we note that $\tau(\bar{g}_{s,i}) = 0$ if $i \neq j$. If τ has the form $\tau(f) = \mathrm{tr}'_j(\pi^{I_j}(f)(t))$ for some fixed $t \in (0, \delta]$, then for $h \in \mathcal{H}$ with $h = (h_0, h_1)$, where $h_0 \in C([0,1], F_2)$ and $h_1 \in F_1$ are such that $\psi_0(h_1) = h_0(0) = h(0)$ and $\psi_1(h_1) = h_0(1) = h(1)$

(using $\sum_{i=1}^{l} p_i = 1_{F_1}$ and $p_i = 1_{M_{r(i)}}$),

(e 16.90)

$$\begin{aligned}
\tau(h) &= \mathrm{tr}'_j(\pi^{I_j}(h)(t)) \\
&= \mathrm{tr}'_j(\pi^{I_j}(h\bar{g}_1)(t)) + \mathrm{tr}'_j(\pi^{I_j}(h\bar{g}_0)(t))) \approx_{\varepsilon/8} \mathrm{tr}'_j(\pi^{I_j}(h)(\delta)\pi^{I_j}(\bar{g}_1)(t)) \\
&\quad + \mathrm{tr}'_j(\pi^{I_j}(h)(0)\pi^{I_j}(\bar{g}_0)(t)) \\
&= g_1(t)\mathrm{tr}'_j(\pi^{I_j}(h)(\delta)) + g_0(t)\mathrm{tr}'_j \circ (\psi_0(h_1)) \\
&= \tau(\bar{g}_{1,j})\mathrm{tr}'_j(\pi^{I_j}(h)(t_1)) + g_0(t)\sum_{i=1}^{l} \mathrm{tr}'_j(\psi_0(h_1 p_i)) \\
&= \tau(\bar{g}_{1,j})\mathrm{tr}'_j \circ (\pi^{I_j}(h)(t_1)) + g_0(t)\big(\sum_{i=1}^{l} \mathrm{tr}'_j(\psi_0(p_i))\mathrm{tr}_i \circ \pi_{e,i}(h)\big) \\
&= \tau(\bar{g}_{1,j})\mathrm{tr}'_j(\pi^{I_j}(h)(t_1)) + g_0(t)\sum_{i=1}^{l} \mathrm{tr}'_j \circ (\pi^{I_j}(\bar{d}_i)(t)))\mathrm{tr}_i(\pi_{e,i}(h)) \\
&= \tau(g_{1,j})\mathrm{tr}'_j(\pi^{I_j}(h)(t_1)) + \sum_{i=1}^{l} \mathrm{tr}'_j \circ (\pi^{I_j}(\bar{g}_0\bar{d}_i)(t))\mathrm{tr}_i(\pi_{e,i}(h)) \\
&= \tau(g_{1,j})\mathrm{tr}'_j(\pi^{I_j}(h)(t_1)) + \sum_{i=1}^{l} \tau(\bar{g}_0\bar{d}_i)\mathrm{tr}_i(\pi_{e,i}(h)) = \lambda(\tau)(h).
\end{aligned}$$

The same argument as above shows that, if

$$\tau(f) = \mathrm{tr}'_j \circ (\pi^{I_j}(f)(t)), \quad t \in [1-\delta, 1),$$

then

$$\tau(h) \approx_{\varepsilon/8} \lambda(\tau)(h) \text{ for all } h \in \mathcal{H}. \tag{e 16.91}$$

It follows from (e 16.88), (e 16.89), (e 16.90), and (e 16.91) that

$$|\tau(h) - \lambda(\tau)(h)| < \varepsilon/8 \text{ for all } h \in \mathcal{H}$$

and for all extreme points of $\tau \in T(C)$. By Choquet's Theorem, for each $\tau \in T(C)$, there exist a Borel probability measure μ_τ on the extreme points $\partial_e T(C)$ of $T(C)$ such that

$$\tau(f) = \int_{\partial_e T(C)} f(t) d\mu_\tau \text{ for all } f \in \mathrm{Aff}(T(C)).$$

Therefore, for each $h \in \mathcal{H}$,

$$\tau(h) = \int_{\partial_e(T(C)} \hat{h}(t)d\mu_\tau \approx_{\varepsilon/8} \int_{\partial_e T(C)} \hat{h}(\lambda(t))d\mu_\tau = \lambda(\tau)(h) \text{ for all } \tau \in T(C),$$

as desired. □

LEMMA 16.12. *Let C be a unital stably finite C^*-algebra, and let $A \in \mathcal{B}_1$ (or $\mathcal{B}_0$). Let $\alpha : T(A) \to T(C)$ be a continuous affine map.*

(1) For any finite subset $\mathcal{H} \subset C_{s.a.}$, and any $\sigma > 0$, there are a C^*-subalgebra $D \subset A$ and a continuous affine map $\gamma : T(D) \to T(C)$ such that $D \in \mathcal{C}$ (or $\mathcal{C}_0$), and

$$|\gamma(\imath(\tau))(h) - \alpha(\tau)(h)| < \sigma \text{ for all } \tau \in T(A) \text{ for all } h \in \mathcal{H},$$

where $\imath : T(A) \ni \tau \to \frac{1}{\tau(p)}\tau|_D \in T(D)$, $p = 1_D$ and $\tau(1-p) < \sigma$ for all $\tau \in T(A)$.

(2) If there is a non-increasing map: $\Delta_0 : C_+^{\mathbf{1}} \setminus \{0\} \to (0,1)$, and there are a finite subset $\mathcal{H}_1 \subset C_+$ and $\sigma_1 > 0$ such that

$$\alpha(\tau)(g) \geq \Delta_0(g) > \sigma_1 \text{ for all } g \in \mathcal{H}_1 \text{ and } \text{ for all } \tau \in T(A),$$

then the affine map γ can be chosen so that

$$\gamma(\tau)(g) \geq \Delta_0(g) > \sigma_1 \text{ for all } g \in \mathcal{H}_1 \text{ and } \text{ for all } \tau \in T(D).$$

(3) If the positive cone of $K_0(C)$ is generated by a finite subset $\mathcal{P}$ of projections and there is an order-unit map $\kappa : K_0(C) \to K_0(A)$ which is compatible with α and strictly positive, then, for any $\delta > 0$, the C^*-subalgebra D and γ above can be chosen so that there are also positive homomorphisms $\kappa_0 : K_0(C) \to K_0((1-p)A(1-p))$ and $\kappa_1 : K_0(C) \to K_0(D)$ such that κ_1 is strictly positive, $\kappa_1([1_C]) = [1_D]$, $\kappa = \kappa_0 + \imath_{*0} \circ \kappa_1$, where $\imath : D \to A$ is the embedding,, and

$$|\gamma(\tau)(q) - \rho_D(\kappa_1([q]))(\tau)| < \delta \text{ for all } q \in \mathcal{P} \text{ and } \tau \in T(D). \tag{e 16.92}$$

(4) Moreover, in addition to (3), if $A \cong A \otimes U$ for some infinite dimensional UHF-algebra, for any given positive integer K, the C^*-algebra D can be chosen so that $D = M_K(D_1)$ for some $D_1 \in \mathcal{C}$ (or $D_1 \in \mathcal{C}_0$) and $\kappa_1 = K\kappa_1'$, where $\kappa_1' : K_0(C) \to K_0(D_1)$ is a strictly positive homomorphism. Furthermore, κ_0 can also be chosen to be strictly positive.

PROOF. Write $\mathcal{H} = \{h_1, h_2, ..., h_m\}$. We may assume that $\|h_i\| \leq 1$, $i = 1, 2, ..., m$. Choose $f_1, f_2, ..., f_m \in A_{s.a.}$ such that that $\tau(f_i) = h_i(\alpha(\tau))$ for all $\tau \in T(A)$ and $\|f_i\| \leq 2$, $i = 1, 2, ..., m$ (see 9.2 of [71]). Put $\mathcal{F} = \{1_A, f_1, f_2, ..., f_m\}$.

Let $\delta > 0$ and let $\mathcal{G}_1$ (in place of $\mathcal{G}$) be a finite subset as given by Lemma 9.4 of [71] for A, $\sigma/16$ (in place of ε), and $\mathcal{F}$. Let $\sigma_1 = \min\{\sigma/16, \delta/16, 1/16\}$. We may assume that $\mathcal{G}_1 \supset \mathcal{F}$ and $\|g\| \leq 1$ for $g \in \mathcal{G}_1$. Put $\mathcal{G} = \{g, gh : g, h \in \mathcal{G}_1\}$. Since $A \in \mathcal{B}_1$ (or $\mathcal{B}_0$), by the definition of $\mathcal{B}_1$ (or $\mathcal{B}_0$), there is a $D \in \mathcal{C}$ (or $\mathcal{C}_0$) with $p = 1_D$ such that, for all $g \in \mathcal{G}$, $\|pg - gp\| < \sigma_1/32$, $pgp \in_{\sigma_1/34} D$, and

$\tau(1-p) < \sigma_1/4$ for all $\tau \in T(A)$. Define $L' : A \to pAp$ by $L'(a) = pap$ for all $a \in A$. Since D is amenable, there is a unital completely positive linear map $L'' : pAp \to D$ such that $\|L''(pgp) - pgp\| < \sigma_1/32$ for all $g \in \mathcal{G}$. Put $L = L'' \circ L'$. Then L is $\mathcal{G}$-$\sigma_1/4$-multiplicative. We estimate, for all $\tau \in T(A)$ and for all $g \in \mathcal{G}$ (since $\tau((1-p)gp) = \tau(pg(1-p)) = 0$),

$$\tau(L(g) - g) \approx_{\sigma_1/32} \tau((1-p)g(1-p)) \text{ and } \tau((1-p)g(1-p)) < \sigma_1/4 \qquad \text{(e 16.93)}$$

Furthermore, we may assume that $L_0 : A \to (1-p)A(1-p)$ defined by $L_0(a) = (1-p)a(1-p)$ is also $\mathcal{G}$-$\sigma_1/2$-multiplicative. By the choice of δ and $\mathcal{G}$, it follows from Lemma 9.4 of [71] that, for each $\tau \in T(D)$, there is $\gamma'(\tau) \in T(A)$ such that

$$|\tau(L(f)) - \gamma'(\tau)(f)| < \sigma/16 \text{ for all } f \in \mathcal{F}. \qquad \text{(e 16.94)}$$

Applying Lemma 16.11, one obtains $t_1, t_2, ..., t_n \in \partial_e T(D)$ and a continuous affine map $\lambda : T(D) \to \triangle$ such that

$$|\tau(d) - \lambda(\tau)(d)| < \sigma/16 \text{ for all } \tau \in T(D) \text{ and } d \in L(\mathcal{F}), \qquad \text{(e 16.95)}$$

where $\triangle$ is the convex hull of $\{t_1, t_2, ..., t_n\}$. Define an affine map $\lambda_1 : \triangle \to T(A)$ by

$$\lambda_1(t_i) = \gamma'(t_i), \ i = 1, 2, ..., m. \qquad \text{(e 16.96)}$$

Define $\gamma = \alpha \circ \lambda_1 \circ \lambda$. Then, for $1 \le j \le n$, by (e 16.94), (e 16.95) (recall $\tau(1-p) < \sigma_1/4$),

$$\begin{aligned} \gamma(\imath(\tau))(h_j) &= \alpha \circ \lambda_1 \circ \lambda(\imath(\tau))(h_j) = \lambda_1 \circ \lambda(\imath(\tau))(f_j) \\ &\approx_{\sigma/16} \lambda(\imath(\tau))(L(f_j)) \approx_{\sigma/16} \imath(\tau)(L(f_j)) \\ &= (1/\tau(p))\tau(L(f_j)) \\ &\approx_{(9\sigma_1/32)/(1-\sigma_1/4)} (1/\tau(p))\tau(f_j) \quad \text{(by (e 16.93))} \\ &\approx_{(\sigma_1/2)/(1-\sigma_1/4)} \tau(f_j) = \alpha(\tau)(h_j), \end{aligned}$$

and this proves (1). Note that it follows from the construction that $\gamma(\tau) \in \alpha(T(A))$, and hence (2) also holds.

To show that (3) holds, let $\mathcal{P} = \{p_1, p_2, .., p_N\}$ be a finite subset of non-zero projections such that $\{[p_1], [p_2], ..., [p_N]\}$ generates $K_0(C)_+$ as a positive cone. Let $F_0 = \kappa(K_0(C))$ and $(F_0)_+ = \kappa(K_0(C)_+)$. Replacing C by $M_N(C)$ and A by $M_N(A)$, we may assume that $\kappa(\mathcal{P})$ is represented by a finite subset $\mathcal{Q}$ of projections in A. Denote by q_p a projection in A such that $[q_p] = \kappa([p])$ for $p \in \mathcal{P}$. In the proof of (1) above, we may choose that $\mathcal{G}_1 \supset \mathcal{Q}$. It is clear, in proof of (1), that, with sufficiently large $\mathcal{G}_1$ and small $\sigma_1/2$, we may assume that

$[L_0]|_{\kappa(K_0(C))}$ and $[L]|_{\kappa(K_0(C))}$ are well defined, and $[L_0(q)] \geq 0$ and $[L(q)] \geq 0$ for all $q \in \mathcal{Q}$. In other words, $[L_0]|_{(F_0)_+}$ and $[L]|_{(F_0)_+}$ are a positive. Moreover, $[\mathrm{id}_A]|_{F_0} = [L_0]|_{F_0} + [\imath \circ L]|_{F_0}$. Define $\kappa_0 := [L_0] \circ \kappa$ and $\kappa_1 := [L] \circ \kappa$. Then $\kappa = \kappa_0 + \imath_{*0} \circ \kappa_1$. Since $\kappa([1_C]) = [1_A]$ and L is unital, $\kappa_1([1_C]) = [1_D]$. Thus, by case (1), it remains to show that κ_1 is strictly positive. Let

$$\sigma_0 = \min\{\alpha(\tau)(p) : p \in \mathcal{P} \text{ and } \tau \in T(A)\}.$$

Note that $\rho_A(\kappa([p]))(\tau) = \alpha(\tau)(p)$ for all $\tau \in T(A)$ and all $p \in \mathcal{P}$. Since κ is strictly positive, $\alpha(\tau)(p) > 0$ for all $\tau \in T(A)$. Since $T(A)$ is compact and $\mathcal{P}$ is finite, it follows that $\sigma_0 > 0$.

In the proof of (1) above, choose $\sigma < \sigma_0/4$ and $\mathcal{H} \supset \mathcal{P}$. Then, by the proof of (1) (see (e 16.93)), for any $p \in \mathcal{P} \setminus \{0\}$,

$$\tau(\imath \circ L(q_p)) > 3\sigma_0/4 \text{ for all } \tau \in T(A). \tag{e 16.97}$$

There is also a projection $q' \in A$ such that $\|\imath \circ L(q_p) - q'\| < \sigma_1$ and $[q'] = [q_p]$ as L is contractive and $\mathcal{G}_1$-$\sigma_1/2$-multiplicative. It follows that $\imath_{*0} \circ \kappa_1([p]) > 0$ for all $p \in \mathcal{P}$. Let $x \in K_0(C)_+ \setminus \{0\}$. Since the positive cone $K_0(C)_+$ is finitely generated by $\{[p] : p \in \mathcal{P}\}$, one may write $x = \sum_{i=1}^{l} m_i[p_i] \in K_0(C)_+$, where $m_i \in \mathbb{Z}_+$, and for some i, $m_i > 0$. Therefore $\imath_{*0} \circ \kappa_1(x) > 0$. It follows that κ_1 is strictly positive.

To see that (4) holds, we first assume that (3) holds. Note that we may choose $D \subset A \otimes 1_U$. Choose a projection $e \in U$ such that

$$0 < t_0(e) < \delta_0 < \delta - \max\{|\gamma(\tau)(p) - \tau(\kappa_1([p])| : p \in \mathcal{P} \text{ and } \tau \in T(D)\},$$

where t_0 is the unique tracial state of U and $[1-e]$ is divisible by K. We then replace κ_1 by $\kappa_2 : K_0(A) \to K_0(D_2)$, where $D_2 = D \otimes (1-e)$ and $\kappa_2([p]) = \kappa_1([p]) \otimes [1-e]$. Define $\kappa_3([p]) = \kappa_1([p]) \otimes [e]$. Then let $\kappa_4 : K_0(C) \to K_0((1-(p\otimes(1-e)))A(1-(p\otimes(1-e))))$ be defined by $\kappa_4 = \kappa_0 + \imath_{*0} \circ \kappa_3$, where $\imath : D \otimes e \to A \otimes U \cong A$ is the embedding. We then replace κ_0 by κ_4 and κ_1 by κ_2. Note that, now, κ_4 is strictly positive. □

17. Maps from Homogeneous C^*-algebras to C^*-algebras in $\mathcal{C}$

The proof of the following lemma is similar to that of Theorem 16.10.

Lemma 17.1. *Let X be a connected finite CW-complex. Let $\mathcal{H} \subset C(X)$ be a finite subset, and let $\sigma > 0$. There exists a finite subset $\mathcal{H}_{1,1} \subset C(X)_+$ with the following property: for any $\sigma_{1,1} > 0$, there is a finite subset $\mathcal{H}_{1,2} \subset C(X)_+$ satisfying the following condition: for any $\sigma_{1,2} > 0$, there is a positive integer M such that for any $D \in \mathcal{C}$ with the dimension of any irreducible representation of D at least M, for any continuous affine map $\gamma : T(D) \to T(C(X))$ satisfying*

$$\gamma(\tau)(h) > \sigma_{1,1} \textit{ for all } h \in \mathcal{H}_{1,1} \textit{ and } \tau \in T(D), \textit{ and}$$
$$\gamma(\tau)(h) > \sigma_{1,2} \textit{ for all } h \in \mathcal{H}_{1,2} \textit{ and } \tau \in T(D),$$

there is a homomorphism $\varphi : C(X) \to D$ *such that*

$$|\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma \text{ for all } h \in \mathcal{H} \text{ and } \tau \in T(D).$$

Moreover, if $D \in \mathcal{C}_0$*, then there is a point evaluation* $\Psi : C(X) \to D$ *such that* $[\varphi] = [\Psi]$.

PROOF. We may assume that every element of $\mathcal{H}$ has norm at most 1.

Let $\eta > 0$ be such that for any $f \in \mathcal{H}$ and any $x, x' \in X$ with $d(x, x') < \eta$, one has

$$|f(x) - f(x')| < \sigma/4.$$

Since X is compact, one can choose a finite subset $\mathcal{H}_{1,1} \subset C(X)_+$ such that for any open ball $O_{\eta/24} \subset X$, of radius $\eta/24$, there is a non-zero element $h \in \mathcal{H}_{1,1}$ with $\mathrm{supp}(h) \subset O_{\eta/24}$. We assume that $\|h\| \le 1$ for all $h \in \mathcal{H}_{1,1}$. Consequently, if there is $\sigma_{1,1} > 0$ such that

$$\tau(h) > \sigma_{1,1} \text{ for all } h \in \mathcal{H}_{1,1},$$

then

$$\mu_\tau(O_{\eta/24}) > \sigma_{1,1}$$

for any open ball $O_{\eta/24}$ with radius $\eta/24$, where μ_τ is the probability measure induced by τ.

Fix $\sigma_{1,1} > 0$. Let δ and $\mathcal{G} \subset C(X)$ (in place of $\mathcal{G}$) be the constant and finite subset provided by Lemma 6.2 of [78] for $\sigma/2$ (in place of ϵ), $\mathcal{H}$ (in place of $\mathcal{F}$), and $\sigma_{1,1}/\eta$ (in place of σ).

Let $\mathcal{H}_{1,2} \subset C(X)_+$ (in place of $\mathcal{H}_1$) be the finite subset provided by Theorem 4.3 for δ (in place of ϵ) and $\mathcal{G}$ (in place of $\mathcal{F}$). We may assume that $\mathcal{H}_{1,1} \subset \mathcal{H}_{1,2}$.

Let $\sigma_{1,2} > 0$. Then let $\mathcal{H}_2 \subset C(X)$ (in place of $\mathcal{H}_2$) and σ_2 be the finite subset, and positive constant provided by Theorem 4.3 for $\sigma_{1,2}$ and $\mathcal{H}_{1,2}$ (in place of σ_1 and $\mathcal{H}_1$).

Let M (in place of N) be the constant provided by Corollary 2.5 of [56] for $\mathcal{H}_2 \cup \mathcal{H}_{1,2}$ (in place of F) and $\min\{\sigma/4, \sigma_2/4, \sigma_{1,2}/2, \sigma_{1,1}/2\}$ (in place of ϵ).

Let $D = D(F_1, F_2, \psi_0, \psi_1)$ be a C^*-algebra in $\mathcal{C}$ with the dimensions of its irreducible representations at least M. Write $F_1 = M_{R(1)} \oplus \cdots \oplus M_{R(l)}$ and $F_2 = M_{r(1)} \oplus M_{r(2)} \oplus \cdots \oplus M_{r(k)}$. Denote by $q_i : F_2 \to M_{r(i)}$ the quotient map, and write $\psi_{0,i} = q_i \circ \psi_0$ and $\psi_{1,i} = q_i \circ \psi_1$.

Let $\gamma : T(D) \to T(C(X))$ be a map as in the lemma. Write $C([0,1], F_2) = I_1 \oplus \cdots \oplus I_k$ with $I_i = C([0,1], M_{r(i)})$, $i = 1, ..., k$. Then γ induces a continuous affine map $\gamma_i : T(I_i) \to T(C(X))$ by $\gamma_i(\tau) = \gamma(\tau \circ \pi^{I_i})$, where π^{I_i} is the restriction map $D \to I_i$. It is then clear that

$$\gamma_i(\tau)(h) > \sigma_{1,2} \text{ for all } \quad h \in \mathcal{H}_{1,2}, \quad \tau \in T(I_i).$$

Denote by $\pi_{e,j} : D \to M_{R(j)}$ the composition of π_e (see 3.1) with the projection from F_1 onto $M_{R(j)}$. By Corollary 2.5 of [56], for each $1 \le i \le k$, there is a homomorphism $\varphi_i : C(X) \to I_i$ such that

$$|\tau \circ \varphi_i(h) - \gamma_i(\tau)(h)| < \min\{\sigma/4, \sigma_2/4, \sigma_{1,2}/2, \sigma_{1,1}/2\} \tag{e 17.1}$$

for all $h \in \mathcal{H}_2 \cup \mathcal{H}_{1,2}$ and for all $\tau \in T(I_i)$, and for any j, there is also a homomorphism $\varphi'_j : C(X) \to M_{R(j)}$ such that

$$|\mathrm{tr}_{R(j)} \circ \varphi'_j(h) - \gamma \circ (\pi_{e,j})^*(\mathrm{tr}_{R(j)})(h)| < \min\{\sigma/4, \sigma_2/4, \sigma_{1,2}/2, \sigma_{1,1}/2\} \tag{e 17.2}$$

for all $h \in \mathcal{H}_2 \cup \mathcal{H}_{1,2} \cup \mathcal{H}_{1,1}$ (we use tr_m for the tracial state of M_m). Set $\varphi' = \bigoplus_j \varphi'_j$ and let $\pi_t : I_i \to M_{r(i)}$ be the point evaluation at $t \in [0,1]$. Then

$$|\mathrm{tr}_{r(i)} \circ (\psi_{0,i} \circ \varphi')(h) - \mathrm{tr}_{r(i)} \circ (\pi_0 \circ \varphi_i)(h)| \le \sigma_2/2 \text{ for all } h \in \mathcal{H}_2$$

and

$$\mathrm{tr}_{r(i)} \circ (\psi_{0,i} \circ \varphi')(h) \ge \sigma_{1,2}/2 \tag{e 17.3}$$

and

$$\mathrm{tr}_{r(i)} \circ (\pi_0 \circ \varphi_i)(h) \ge \sigma_{1,2}/2 \text{ for all } h \in \mathcal{H}_{1,2}.$$

By Theorem 4.3, there is a unitary $u_{i,0} \in M_{r(i)}$ such that

$$||\mathrm{Ad}u_{i,0} \circ \pi_0 \circ \varphi_i(f) - \psi_{0,i} \circ \varphi'(f)|| < \delta \text{ for all } f \in \mathcal{G}.$$

Exactly the same argument shows that there is a unitary $u_{i,1} \in M_{r(i)}$ such that

$$\|\mathrm{Ad}u_{i,1} \circ \pi_1 \circ \varphi_i(f) - \psi_{1,i} \circ \varphi'(f)\| < \delta \text{ for all } f \in \mathcal{G}.$$

Choose two paths of unitaries, $\{u_{i,0}(t) : t \in [0,1/2]\} \subset M_{r(i)}$ such that $u_{i,0}(0) = u_{i,0}$ and $u_{i,0}(1/2) = 1_{M_{r(i)}}$, and $\{u_{i,1}(t) : t \in [1/2,1]\} \subset M_{r(i)}$ such that $u_{i,1}(1/2) = 1_{M_{r(i)}}$ and $u_{i,1}(1) = u_{i,1}$. Put $u_i(t) = u_{i,0}(t)$ if $t \in [0,1/2)$ and $u_i(t) = u_{i,1}(t)$ if $t \in [1/2,1]$. Define $\tilde{\varphi}_i : C(X) \to I_i$ by

$$\pi_t \circ \tilde{\varphi}_i = \mathrm{Ad}\, u_i(t) \circ \pi_t \circ \varphi_i.$$

Then

$$\|\pi_0 \circ \tilde{\varphi}_i(f) - \psi_{0,i} \circ \varphi'(f)\| < \delta \text{ and } \|\pi_1 \circ \tilde{\varphi}_i(f) - \psi_{1,i} \circ \varphi'(f)\| < \delta \tag{e 17.4}$$

for all $f \in \mathcal{G}, i = 1, ..., k$.

Note that it also follows from (e 17.1) and (e 17.3) that

$$\mathrm{tr}_{r(i)} \circ (\psi_{0,i} \circ \varphi')(h) \ge \sigma_{1,1}/2 \quad \text{and} \quad \mathrm{tr}_{r(i)} \circ (\pi_0 \circ \tilde{\varphi}_i)(h) \ge \sigma_{1,1}/2 \text{ for all } h \in \mathcal{H}_{1,1}.$$

Hence,

$$\mu_{\mathrm{tr}_{r(i)} \circ (\psi_{0,i} \circ \varphi')}(O_{\eta/24}) \ge \sigma_{1,1} \quad \text{and} \quad \mu_{\mathrm{tr}_{r(i)} \circ (\pi_0 \circ \tilde{\varphi}_i)}(O_{\eta/24}) \ge \sigma_{1,1}.$$

Thus, by Lemma 6.2 of [78], for each $1 \le i \le k$, there are two unital homomorphisms

$$\Phi_{0,i}, \Phi'_{0,i} : C(X) \to C([0,1], M_{r(i)})$$

such that

$$\pi_0 \circ \Phi_{0,i} = \psi_{0,i} \circ \varphi', \quad \pi_0 \circ \Phi'_{0,i} = \pi_0 \circ \tilde{\varphi}_i,$$

$$||\pi_t \circ \Phi_{0,i}(f) - \psi_{0,i} \circ \varphi'(f)|| < \sigma/2, \quad ||\pi_t \circ \Phi'_{0,i}(f) - \pi_0 \circ \tilde{\varphi}_i(f)|| < \sigma/2$$

for all $f \in \mathcal{H}$ and $t \in [0,1]$, and there is a unitary $w_{i,0} \in M_{r(i)}$ (in place of u) such that

$$\pi_1 \circ \Phi_{0,i} = \mathrm{Ad} w_{i,0} \circ \pi_1 \circ \Phi'_{0,i}.$$

The same argument shows that, for each $1 \le i \le k$, there are two unital homomorphisms $\Phi_{1,i}, \Phi'_{1,i} : C(X) \to C([0,1], M_{r(i)})$ such that

$$\pi_1 \circ \Phi_{1,i} = \psi_{1,i} \circ \varphi', \quad \pi_1 \circ \Phi'_{1,i} = \pi_1 \circ \tilde{\varphi}_i,$$

$$||\pi_t \circ \Phi_{1,i}(f) - \psi_{1,i} \circ \varphi'(f)|| < \sigma/2, \quad ||\pi_t \circ \Phi'_{1,i}(f) - \pi_0 \circ \tilde{\varphi}_i(f)|| < \sigma/2$$

for all $f \in \mathcal{H}$ and $t \in [0,1]$, and there is a unitary $w_{i,1} \in M_{r(i)}$ (in place of u) such that

$$\pi_0 \circ \Phi_{1,i} = \mathrm{Ad} w_{i,1} \circ \pi_0 \circ \Phi'_{1,i}.$$

Choose two continuous paths $\{w_{i,0}(t) : t \in [0,1]\}$, $\{w_{i,1}(t) : t \in [0,1]\}$ in $M_{r(i)}$ such that $w_{i,0}(0) = w_{i,0}$, $w_{i,0}(1) = 1_{M_{r(i)}}$ and $w_{i,1}(1) = 1_{M_{r(i)}}$ and $w_{i,1}(0) = w_{i,1}$.

For each $1 \le i \le k$, by the continuity of γ_i, there is $1 > \epsilon_i > 0$ such that

$$|\gamma_i(\tau_x)(h) - \gamma_i(\tau_y)(h)| < \sigma/4 \text{ for all } h \in \mathcal{H},$$

provided that $|x-y| < \epsilon_i$, where τ_x and τ_y are the extremal traces of I_i concentrated on x and y.

Define the map $\tilde{\tilde{\varphi}}_i : C(X) \to I_i$ by

$$\pi_t \circ \tilde{\tilde{\varphi}}_i = \begin{cases} \pi_{\frac{3t}{\epsilon_i}} \circ \Phi_{0,i}, & t \in [0, \epsilon_i/3), \\ \mathrm{Ad}(w_{i,0}(\frac{3t}{\epsilon_i} - 1)) \circ \pi_1 \circ \Phi'_{0,i}, & t \in [\epsilon_i/3, 2\epsilon_i/3), \\ \pi_{3-\frac{3t}{\epsilon_i}} \circ \Phi'_{0,i}, & t \in [2\epsilon_i/3, \epsilon_i), \\ \pi_{\frac{t-\epsilon_i}{1-2\epsilon_i}} \circ \tilde{\varphi}_i, & t \in [\epsilon_i, 1-\varepsilon_i), \\ \pi_{\frac{1-2\varepsilon_i/3-t}{\varepsilon_i/3}} \circ \Phi'_{1,i}, & t \in [1-\varepsilon_i, 1-2\varepsilon_i/3], \\ \mathrm{Ad}(w_{i,1}(\dfrac{(1-\varepsilon_i/3)-t}{\epsilon_i/3})) \circ \pi_0 \circ \Phi'_{1,i}, & t \in [1-2\epsilon_i/3, 1-\epsilon_i/3], \\ \pi_{\frac{t-1+\varepsilon_i/3}{\epsilon_i/3}} \circ \Phi_{1,i}, & t \in [1-\epsilon_i/3, 1]. \end{cases}$$

Then,

$$\pi_0 \circ \tilde{\tilde{\varphi}}_i = \psi_{0,i} \circ \varphi' \quad \text{and} \quad \pi_1 \circ \tilde{\tilde{\varphi}}_i = \psi_{1,i} \circ \varphi'. \tag{e 17.5}$$

One can also estimate, by the choice of ε_i and the definition of $\tilde{\tilde{\varphi}}_i$, that

$$|\tau_t \circ \tilde{\tilde{\varphi}}_i(h) - \gamma_i(\tau_t)(h)| < \sigma \text{ for all } t \in [0,1] \text{ and } h \in \mathcal{H}, \tag{e 17.6}$$

where τ_t is the extremal tracial state of I_i concentrated on $t \in [0,1]$.

Define $\Phi : C(X) \to C([0,1], F_2)$ by $\Phi(f) = \bigoplus_{i=1}^k \tilde{\tilde{\varphi}}_i(f)$ for all $f \in C(X)$. Define $\varphi : C(X) \to C([0,1], F_2) \oplus F_1$ by $\varphi(f) = (\Phi(f), \varphi'(f))$ for $f \in C(X)$. By (e 17.5), φ is a homomorphism from $C(X)$ to D. By (e 17.6) and (e 17.2), one has that

$$|\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma \text{ for all } h \in \mathcal{H}$$

and for all $\tau \in T(D)$, as desired.

To see the last part of the lemma, one assumes that $D \in \mathcal{C}_0$. Consider $\pi_e \circ \varphi : C(X) \to F_1$ (where $D = D(F_1, F_2, \psi_0, \psi_1)$ as above). Since $\pi_e \circ \varphi$ has finite dimensional range, it is a point evaluation. We may write $\pi_e \circ \varphi(f) = \sum_{i=1}^m f(x_i)p_i$ for all $f \in C(X)$, where $\{x_1, x_2, ..., x_m\} \subset X$ and $\{p_1, p_2, ..., p_m\} \subset F_1$ is a set of mutually orthogonal projections. Let $I = \{f \in C(X) : f(x_1) = 0\}$ and let $\imath : I \to C(X)$ be the embedding. It follows that $[\pi_e \circ \varphi \circ \imath] = 0$. By 3.5, $[\pi_e]$ is injective on each $K_i(D)$ and on each $K_i(D, \mathbb{Z}/k\mathbb{Z})$ ($k \geq 2$, $i = 0, 1$). Hence $[\varphi \circ \imath] = 0$. Choose $\Psi(f) = f(x) \cdot 1_D$. Since X is connected, $[\varphi] = [\Psi]$ (see the end of Remark 4.4). □

COROLLARY 17.2. *Let X be a connected finite CW-complex. Let $\Delta : C(X)_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\mathcal{H} \subset C(X)$ be a finite subset and let $\sigma > 0$. Then there exist a finite subset $\mathcal{H}_1 \subset C(X)_+^{\mathbf{1}} \setminus \{0\}$ and a positive integer M such that for any $D \in \mathcal{C}(X)$ with the dimension of any irreducible representation of D at least M, and for any continuous affine map $\gamma : T(D) \to T(C(X))$ satisfying*

$$\gamma(\tau)(h) > \Delta(\hat{h}) \textit{ for all } h \in \mathcal{H}_1 \textit{ and } \textit{ for all } \tau \in T(D),$$

there is a homomorphism $\varphi : C(X) \to D$ such that

$$|\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma \textit{ for all } h \in \mathcal{H}.$$

PROOF. Let $\mathcal{H}_{1,1}$ be the subset of Lemma 17.1 with respect to $\mathcal{H}$ and σ. Then put

$$\sigma_{1,1} = \min\{\Delta(\hat{h}) : h \in \mathcal{H}_{1,1}\}.$$

Let $\mathcal{H}_{1,2}$ be the finite subset of Lemma 17.1 with respect to $\sigma_{1,1}$, and then put

$$\sigma_{1,2} = \min\{\Delta(\hat{h}) : h \in \mathcal{H}_{1,2}\}.$$

Let M be the positive integer of Lemma 17.1 with respect to $\sigma_{1,2}$. Then it follows from Lemma 17.1 that the finite subset

$$\mathcal{H}_1 := \mathcal{H}_{1,1} \cup \mathcal{H}_{1,2}$$

and the positive integer M are as desired in the corollary. □

THEOREM 17.3. *Let X be a connected finite CW complex, and let $A \in \mathcal{B}_0$ be a unital separable simple C^*-algebra. Suppose that $\gamma : T(A) \to T_f(C(X)))$ (see Definition 2.2) is a continuous affine map. Then, for any $\sigma > 0$, and any finite subset $\mathcal{H} \subset C(X)_{s.a.}$, there exists a unital homomorphism $h : C(X) \to A$ such that*

$$[h] = [\Psi] \in KL(C(X), A)$$

and

$$|\tau \circ h(f) - \gamma(\tau)(f)| < \sigma \text{ for all } f \in \mathcal{H} \text{ and } \tau \in T(A), \tag{e 17.7}$$

where Ψ is a homomorphism with a finite dimensional image.

PROOF. We may assume that every element of $\mathcal{H}$ has norm at most one. Let $\mathcal{H}_{1,1} \subset C(X)_+ \setminus \{0\}$ be the finite subset of Lemma 17.1 with respect to $\mathcal{H}$ (in place of $\mathcal{H}$), $\sigma/4$ (in place of σ), and $C(X)$. Since $\gamma(T(A)) \subset T_f(C(X))$, there is $\sigma_{1,1} > 0$ such that

$$\gamma(\tau)(h) > \sigma_{1,1}, \text{ for all } h \in \mathcal{H}_{1,1} \text{ and for all } \tau \in T(A).$$

Let $\mathcal{H}_{1,2} \subset C(X)_+ \setminus \{0\}$ be the finite subset of Lemma 17.1 with respect to $\sigma_{1,1}$. Again, since $\gamma(T(A)) \subset T_f(C(X))$, there is $\sigma_{1,2} > 0$ such that

$$\gamma(\tau)(h) > \sigma_{1,2}, \text{ for all } h \in \mathcal{H}_{1,2} \text{ and for all } \tau \in T(A).$$

Let M be the constant of Lemma 17.1 with respect to $\sigma_{1,2}$. Note that $A \in \mathcal{B}_0$. By (1) and (2) of Lemma 16.12, there are a C^*-subalgebra $D \subset A$ with $D \in \mathcal{C}_0$, and a continuous affine map $\gamma' : T(D) \to T(C(X))$ such that

$$|\gamma'(\frac{1}{\tau(p)}\tau|_D)(f) - \gamma(\tau)(f)| < \sigma/4, \text{ for all } \tau \in T(A), \text{ for all } f \in \mathcal{H}, \tag{e 17.8}$$

where $p = 1_D$, $\tau(1-p) < \sigma/(4+\sigma)$ for all $\tau \in T(A)$,

$$\gamma'(\tau)(h) > \sigma_{1,1}, \text{ for all } \tau \in T(D) \text{ for all } h \in \mathcal{H}_{1,1}, \text{ and} \tag{e 17.9}$$

$$\gamma'(\tau)(h) > \sigma_{1,2}, \text{ for all } \tau \in T(D) \text{ for all } h \in \mathcal{H}_{1,2}. \tag{e 17.10}$$

Moreover, since A is simple, one may assume that the dimension of any irreducible representation of D is at least M (see 10.1). Thus, by (e 17.9) and (e 17.10), one applies Lemma 17.1 to D, $C(X)$, and γ' (in place of γ) to obtain a homomorphism $\varphi : C(X) \to D$ such that

$$|\tau \circ \varphi(f) - \gamma'(\tau)(f)| < \sigma/4, \text{ for all } f \in \mathcal{H} \text{ and for all } \tau \in T(D). \tag{e 17.11}$$

Moreover, we may assume that $[\varphi] = [\Phi_0] \in KL(C(X), D)$ for a homomorphism $\Phi_0 : C(X) \to D$ with a finite dimensional image since we assume that $D \in \mathcal{C}_0$. Pick a point $x \in X$, and define $h : C(X) \to A$ by

$$f \mapsto f(x)(1-p) \oplus \varphi(f) \ \text{ for all } \ f \in C(X).$$

For any $f \in \mathcal{H}$ and any $\tau \in T(A)$, one has

$$\begin{aligned} |\tau \circ h(f) - \gamma(\tau)(f)\| &\le |\tau \circ \varphi(f) - \gamma(\tau)(f)| + \sigma/4 \\ &< |\tau \circ \varphi(f) - \gamma'(\frac{1}{\tau(p)}\tau|_D)(f)| + \sigma/2 \\ &< |\tau \circ \varphi(f) - \frac{1}{\tau(p)}\tau \circ \varphi(f)| + 3\sigma/4 < \sigma. \end{aligned}$$

Define $\Psi : C(X) \to A$ by $\Psi(f) = f(x)(1-p) \oplus \Phi_0(f)$ for all $f \in C(X)$. Then $[h] = [\Psi]$. □

18. KK-attainability of the Building Blocks

Definition 18.1 (9.1 of [71]). Let $\mathcal{D}$ be a class of unital C^*-algebras. A C^*-algebra C is said to be KK-attainable with respect to $\mathcal{D}$ if for any $A \in \mathcal{D}$ and any $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))^{++}$, there exists a sequence of contractive completely positive linear maps $L_n : C \to A \otimes \mathcal{K}$ such that

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \ \text{ for all } \ a,\, b \in C \ \text{ and} \tag{e 18.1}$$

$$[\{L_n\}] = \alpha. \tag{e 18.2}$$

(The latter means that, for any finite subset $\mathcal{P} \subset \underline{K}(C)$, $[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all large n.) If C satisfies the UCT, then $\mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))^{++}$ may be replaced by $KL(C, A)^{++}$. In what follows, we will use $\mathcal{B}_{ui}$ to denote the class of those separable C^*-algebras of the form $A \otimes U$, where $A \in \mathcal{B}_i$ and U is a UHF-algebra of infinite type, $i = 0, 1$.

Theorem 18.2. (Theorem 5.9 of [60]; see 6.1.11 of [63]) *Let A be a separable C^*-algebra satisfying the UCT and let B be a unital separable C^*-algebra. Assume that A is the closure of an increasing sequence $\{A_n\}$ of amenable residually finite dimensional C^*-subalgebras. Then for any $\alpha \in KL(A, B)$, there exist two sequences of completely positive contractions $\varphi_n^{(i)} : A \to B \otimes \mathcal{K}$ $(i = 1, 2)$ satisfying the following conditions:*

(1) $\|\varphi_n^{(i)}(ab) - \varphi_n^{(i)}(a)\varphi_n^{(i)}(b)\| \to 0$ *as* $n \to \infty$*;*

(2) *for each finite subset $\mathcal{P}$, $[\varphi_n^{(i)}]|_{\mathcal{P}}$ is well defined for sufficiently large n, $i = 1, 2$, and, for any n, the image of $\varphi_n^{(2)}$ is contained in a finite dimensional sub-C^*-algebra of $B \otimes \mathcal{K}$;*

(3) *for each finite subset* $\mathcal{P} \subset \underline{K}(A)$*, there exists* $m > 0$ *such that*

$$[\varphi_n^{(1)}]|_{\mathcal{P}} = (\alpha + [\varphi_n^{(2)}])|_{\mathcal{P}} \text{ for all } n \geq m;$$

(4) *for each* n*,* $\varphi_n^{(2)}$ *is a homomorphism on* A_n*.*

Moreover, the condition that A_n *is amenable could be replaced by the condition that* B *is amenable.*

Note that α does not need to be positive as stated in 6.1.11 of [63]. In fact, the proof of Theorem 5.9 of [60] does not require that α be positive, if one does not require the part (5) of Theorem 5.9 of [60] (there was also a typo which is corrected above).

LEMMA 18.3. *Let* $C = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ *and let* $N_1, N_2 \geq 1$ *be integers. There exists* $\sigma > 0$ *satisfying the following condition: Let* $A \in \mathcal{C}$ *be another* C^**-algebra and let* $\kappa : K_0(C) \to K_0(A)$ *be an order preserving homomorphism such that, for any non-zero element* $x \in K_0(C)_+$*,* $N_1\kappa(x) > [1_A]$ *and* $\kappa([1_C]) \leq N_2[1_A]$*. Then, for any* $\tau \in T(A)$*, there exists* $t \in T(C)$ *such that*

$$t(h) \geq \sigma \int_{[0,1]} \mathrm{T}(\lambda(h)(s))d\mu(s) \text{ for all } h \in C_+ \text{ and} \tag{e 18.3}$$

$$\frac{\rho_A(\kappa(x))(\tau)}{\rho_A(\kappa([1_C]))(\tau)} = \rho_C(x)(t) \text{ for all } x \in K_0(C)_+, \tag{e 18.4}$$

where $\lambda : C \to C([0,1], F_2)$ *is the homomorphism given by* (e 3.1) *and* $\mathrm{T}(b) = \sum_{i=1}^k \mathrm{tr}_i(\psi_i(b))$ *for all* $b \in F_2$*, where* tr_i *is the normalized tracial state on the* i*-th simple direct summand* $M_{r(i)}$ *of* F_2*,* $\psi_i : F_2 \to M_{r(i)}$ *is the projection map, and* μ *is Lebesgue measure on* $[0,1]$*.*

PROOF. In what follows, if B is a C^*-algebra, $\tau \in T(B)$ and $x \in K_0(B)$, we will use the following convention:

$$\tau(x) := \rho_B(x)(\tau).$$

We also write $F_1 = M_{R(1)} \bigoplus M_{R(2)} \bigoplus \cdots \bigoplus M_{R(l)}$. Denote by $q_j : F_1 \to M_{R(j)}$ the projection map and tr_j the tracial state of $M_{R(j)}$. Recall $\pi_e : C \to F_1$ is the quotient map defined in 3.1. Define $T : K_0(F_2) \to \mathbb{R}$ by

$$T(x) = \sum_{i=1}^k \rho_{M_{r(i)}}((\psi_i)_{*0}(x))(\mathrm{tr}_i)$$

for $x \in K_0(F_2)$, where $F_2 = M_{r(1)} \bigoplus M_{r(2)} \bigoplus \cdots \bigoplus M_{r(k)}$. Note $T([\varphi_0(\pi_e(1_C))]) = k$, where $\varphi_0 : F_1 \to F_2$ is given by C.

Let $p_1, p_2, ..., p_s$ be a set of minimal projections in $M_m(C)$ for some integer m such that $\{p_1, p_2, ..., p_s\}$ generates $K_0(C)_+$ (see 3.15). It follows that there is a $\sigma_{00} > 0$ such that

$$\sigma_{00}mk < 1/2. \tag{e 18.5}$$

Set $\sigma_0 = \sigma_{00}/2N_1$ and $\sigma = \sigma_0/N_2$. Write

$$\begin{aligned} A &= A(F_1', F_2', \varphi_0', \varphi_1') \\ &= \{(g, c) \in C([0,1], F_2') \oplus F_1' : g(0) = {\varphi_0}'(c) \text{ and } g(1) = {\varphi_1}'(c),\ c \in F_1'\}. \end{aligned}$$

Denote by $\pi_e' : A \to F_1'$ the quotient map (see 3.1). Write $F_1' = M_{R(1)'} \oplus M_{R(2)'} \oplus \cdots \oplus M_{R(l')'}$, and let $q_i' : F_1' \to M_{R(i)'}$ be the projection map and tr_i' the tracial state of $M_{R(i)'}$, $i = 1, 2, ..., l'$.

Note $K_0(F_1') = \mathbb{Z}^{l'}$. View $(q_i' \circ \pi_e')_{*0} \circ \kappa$ as a positive homomorphism from $K_0(C) \to \mathbb{R}$ (as well as a homomorphism from $K_0(C)$ to $K_0(M_{R(i)'}) = \mathbb{Z}$). Since

$$N_1\kappa(x) > [1_A] \text{ for all } x \in K_0(C)_+ \setminus \{0\},$$

one has that

$$N_1((q_i' \circ \pi_e')_{*0} \circ \kappa)(x) > R(i)' > 0 \text{ for all } x \in K_0(C)_+ \setminus \{0\}. \tag{e 18.6}$$

Then, for $p \in \{p_1, p_2, ..., p_s\}$,

$$\begin{aligned} & N_1((q_i' \circ \pi_e')_{*0} \circ \kappa)([p]) - \sigma_{00}R(i)'T \circ (\varphi_0 \circ \pi_e)(p) \\ &= N_1((q_i' \circ \pi_e')_{*0} \circ \kappa)([p]) - \sigma_{00}R(i)'(\sum_{i=1}^{k} \rho_{M_{r(i)}}(\psi_i(p(0)))(\mathrm{tr}_i)) \\ &\geq N_1((q_i' \circ \pi_e')_{*0} \circ \kappa)([p]) - \sigma_{00}R(i)'km > 0. \end{aligned}$$

Define $\Gamma_i : K_0(C) \to \mathbb{R}$ by

$$\Gamma_i(x) = ((q_i' \circ \pi_e')_{*0} \circ \kappa)(x) - \sigma_0 R(i)' \cdot T \circ (\varphi_0 \circ \pi)_{*0}(x) \text{ for all } x \in K_0(C).$$

Then Γ_i is positive (since $\sigma_0 < \sigma_{00}/N_1$). Note that $(\pi_e)_{*0} : K_0(C) \to K_0(F_1)$ is an order embedding (see 3.5). It follows from Theorem 3.2 of [48] (see also 2.8 of [61]) that there are positive homomorphisms $\tilde{\Gamma}_i : K_0(F_1) \to \mathbb{R}$ such that $\tilde{\Gamma}_i \circ (\pi_e)_{*0} = \Gamma_i$ and $\tilde{\Gamma}_i([e_j]) = \alpha_{i,j} \geq 0$, $j = 1, 2, ..., l$, $i = 1, 2, ..., l'$, where $e_j = q_j \circ \pi_e(1_C)$, $j = 1, 2, ..., l$. Using the fact that the homomorphisms from $K_0(F_1) = \mathbb{Z}^l$ is determined by their values on the canonical basis, one gets

$$\begin{aligned} & \sum_{j=1}^{l} \frac{\mathrm{rank} q_j(\pi_e(p))}{\mathrm{rank} q_j(\pi_e(1_C))} \alpha_{i,j} + \sum_{j=1}^{k} \frac{\mathrm{rank}\psi_j(\varphi_0 \circ \pi_e(p))}{\mathrm{rank}\psi_j(\varphi_0 \circ \pi_e(1_C))} \sigma_0 R(i)' \\ &= \Gamma_i([p]) + \sigma_0 R(i)' \cdot T(\varphi_0 \circ \pi_e)_{*0}([p]) = (q_i' \circ \pi_e')_{*0} \circ \kappa([p]) \end{aligned} \tag{e 18.7}$$

for any projection $p \in M_m(C)$, where $m \geq 1$ is an integer. In particular, $(q_i' \circ \pi_e')_{*0} \circ \kappa([1_C]) = \sum_{j=1}^{l} \alpha_{i,j} + k\sigma_0 R(i)'$. For any $\tau \in T(A)$, by 2.8 of [61], since $K_0(A)$ is order embedded into $K_0(F_1')$ (see 3.5), there is $\tau' \in T(F_1')$ (see also Corollary 3.4 of [8]) such that

$$\tau' \circ \pi_e'(x) = \tau(x) \text{ for all } x \in K_0(A). \tag{e 18.8}$$

Write $\tau' = \sum_{i=1}^{l'} \lambda_{i,\tau} \mathrm{tr}_i' \circ q_i'$, where $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{l'} \lambda_{i,\tau} = 1$. Put

$$\begin{aligned} \beta = \tau(\kappa([1_C])) &= \tau'((\pi_e')_{*0}(\kappa([1_C]))) \\ &= \sum_{i=1}^{l'} \lambda_{i,\tau}(1/R(i)')((q_i' \circ \pi_e')_{*0} \circ \kappa([1_C])) \end{aligned} \tag{e 18.9}$$

(with the identification $K_0(M_{R(i)'}) = \mathbb{Z}$). Then $\beta \leq N_2$. For each i, define, for $(s, b) \in C$,

$$t_i((s,b)) = \frac{1}{\beta R(i)'}\left(\sum_{j=1}^{k} \sigma_0 R(i)' \int_{[0,1]} \mathrm{tr}_j(\psi_j(s(t)))d\mu(t) + \sum_{j=1}^{l} \alpha_{i,j} \bar{\mathrm{tr}}_j(q_j(b))\right).$$

Then, if $h = (s, b) \in C_+$ (recall $\beta \leq N_2$),

$$\begin{aligned} t_i(h) &\geq \frac{\sigma_0 R(i)'}{\beta R(i)'}(\sum_{j=1}^{k} \int_{[0,1]} \mathrm{tr}_j(\psi_j(s(t)))d\mu(t)) \\ &= \frac{\sigma_0}{\beta} \int_{[0,1]} T(\lambda(h)(t))d\mu \geq \sigma \int_{[0,1]} T(\lambda(h)(t))d\mu. \end{aligned}$$

For $\tau \in T(A)$ mentioned above, define $t_\tau = \sum_{i=1}^{l'} \lambda_{i,\tau} t_i$. It is straightforward to verify

$$t_\tau(h) \geq \sigma \int_{[0,1]} \mathrm{T}(\lambda(h)(t))d\mu(t) \text{ for all } h \in C_+.$$

Moreover, for each i, by (e 18.7), for every projection $p = (p, \pi_e(p)) \in M_N(C)$ (for any $N \geq 1$),

$$\begin{aligned} t_i(p) &= \frac{1}{\beta R(i)'}\left(\sum_{j=1}^{k} \sigma_0 R(i)' \int_{[0,1]} \mathrm{tr}_j(\psi_i(p(t)))d\mu(t) + \sum_{j=1}^{l} \alpha_{i,j} \bar{\mathrm{tr}}_j(q_j(\pi_e(p)))\right) \\ &= \frac{1}{\beta R(i)'}\left(\sum_{j=1}^{k} \sigma_0 R(i)' \frac{\mathrm{rank}(\psi_j(\varphi_0 \circ \pi_e(p)))}{\mathrm{rank}\psi_j(\varphi_0 \circ \pi_e(1_C))} + \sum_{j=1}^{l} \alpha_{i,j} \frac{\mathrm{rank} q_j(\pi_e(p))}{\mathrm{rank} q_j(\pi_e(1_C))}\right) \\ &= \frac{1}{\beta R(i)'}\left(\sigma_0 R(i)' \cdot T \circ \varphi_0 \circ (\pi_e)_{*0}([p]) + \Gamma_i([p])\right) \\ &= \frac{1}{\beta}(1/R(i)')(q_i' \circ \pi_e')_{*0}(\kappa([p])) = \frac{1}{\beta}(\mathrm{tr}_i'(q_i' \circ \pi_e')_{*0}(\kappa([p])). \end{aligned} \tag{e 18.10}$$

In particular,

$$t_i(1_C) = \frac{1}{\beta}(\mathrm{tr}_{\mathrm{i}}'(q_i' \circ \pi_e')_{*0}(\kappa([1_C])) = \frac{1}{\beta}(1/R(i)')(q_i' \circ \pi_e')_{*0}(\kappa([1_C])).$$

It follows that $t_\tau(1_C) = \beta/\beta = 1$ (see (e 18.9)) and $t_\tau \in T(C)$. Finally, by (e 18.9), (e 18.10) and (e 18.8),

$$\begin{aligned} t_\tau(x) &= (1/\beta)\sum_{i=1}^{l'} \lambda_{i,\tau}\mathrm{tr}_i'((q_i' \circ \pi_e')_{*0}(\kappa(x))) \\ &= (1/\tau(\kappa([1_C])))(\tau'((\pi_e')_{*0}(\kappa(x)))) \\ &= \tau(\kappa(x))/\tau(\kappa([1_C])) \text{ for all } x \in K_0(C). \end{aligned}$$

□

PROPOSITION 18.4. *Let $S \in \mathcal{C}$ and $N \geq 1$. There exists an integer $K \geq 1$ satisfying the following condition: For any positive homomorphism $\kappa : K_0(S) \to K_0(A)$ which satisfies $\kappa([1_S]) \leq [1_A]$ and $N\kappa([p]) > [1_A]$ for any $[p] \in K_0(S)_+ \setminus \{0\}$, where $A \in \mathcal{C}$, there exists a homomorphism $\varphi : S \to M_K(A)$ such that $\varphi_{*0} = K\kappa$. If we further assume $\kappa([1_S]) = [1_A]$, then φ can be chosen to be unital.*

PROOF. Write

$$\begin{aligned} S &= A(F_1, F_2, \varphi_0, \varphi_1) \\ &= \{(f,g) : (f,g) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_0(g),\ \ f(1) = \varphi_1(g)\}. \end{aligned}$$

Denote by $\lambda : S \to C([0,1], F_2)$ the map given by (e 3.1). Since A has stable rank one (see 3.3), the proposition is known for the case that S is finite dimensional (see, for example, Lemma 7.3.2 (ii) of [108]). So we may assume S is not finite dimensional. Since C has stabel rank one, by considering each summand of S, we may reduce the general case to the case that S has only one direct summand (or is minimal—see 3.1). In particular, we may assume that $\ker\varphi_0 \cap \ker\varphi_1 = \{0\}$. It follows that that λ is injective.

To simplify the notation, without loss of generality, replacing S by $M_r(S)$ for some integer $r \geq 1$ (and replacing A by $M_r(A)$), let us assume that projections of S generate $K_0(S)_+$ (see 3.15). Note also, by 3.5, since there are only finitely many elements of $K_0(S)_+$ which are dominated by $[1_S]$, there are only finitely many elements of $K_0(S)_+$ which can be represented by projections in S.

Let $\sigma > 0$ be as given by 18.3 (associated with the integers $N_1 = N$ and $N_2 = 1$). Define $\Delta : S_+^{q,1} \setminus \{0\} \to (0,1)$ by

$$\Delta(\hat{h}) = \sigma/2 \int_{[0,1]} \mathrm{T}(\lambda(h)(t))d\mu(t)$$

for all $h = (\lambda(h), \pi_e(h)) \in S_+$, where $\mathrm{T}(c) = \sum_{i=1}^{k} \mathrm{tr}_i(\psi_i(c))$ for all $c \in F_2$, where $\psi_i : F_2 \to M_{r(i)}$ is the projection map onto the i-th simple direct summand of F_2 (so we assume that F_2 has k simple direct summands) and tr_i is the normalized tracial state on $M_{r(i)}$. Note that σ depends only on S and N_1. So Δ depends only on N and S.

Let $\mathcal{H}_1$, $\delta > 0$, $\mathcal{P}$, and K be the finite subsets and constants provided by Theorem 16.10 with respect to S, Δ, an arbitrarily chosen finite set $\mathcal{H}$ (containing 1_S), and an arbitrarily chosen $1 > \sigma_1 = 1/2 > 0$ (in place of σ). Note that, when the finite subset $\mathcal{P}$ is given, one can replace it by a finite subset $\mathcal{P}'$ such that the subgroup generated by $\mathcal{P}'$ contains $\mathcal{P}$, as long as δ is chosen to be sufficiently small. Therefore, since we have assumed that projections in S generate $K_0(S)_+$, choosing a sufficiently small δ, we may assume that $\mathcal{P}$ is represented by $P \subset S$, where P is a finite subset of projections such that every projection $q \in S$ is equivalent to one of the projections in $\mathcal{P}$. (We will only apply a part of Theorem 16.10 and will not use (e 16.59)).

Note that, by hypothesis, $\kappa([p])$ is the class of a full projection in $M_N(A)$. Without loss of generality, applying 3.19, we may assume that $[1_A] = \kappa([1_S])$.

Let $\mathcal{Q} \subset M_{r'}(A)$ for some $r' \geq 1$ be a finite set of projections such that $\kappa(\mathcal{P})$ can be represented by projections in $\mathcal{Q}$. It follows from Lemma 16.11 that there is a finite subset $\mathcal{T}$ of extreme points of $T(A)$ and there exists a continuous affine map $\gamma' : T(A) \to C_{\mathcal{T}}$ such that

$$|\gamma'(\tau)(p) - \tau(p)| < \delta/2 \ \text{ for all } \ p \in \mathcal{Q} \text{ and for } \ \text{all } \tau \in T(A), \tag{e 18.11}$$

where $C_{\mathcal{T}}$ is the convex hull of $\mathcal{T}$.

Note that any C^*-algebra in the class $\mathcal{C}$ is of type I, it is amenable and in particular it is exact. Therefore, by Corollary 3.4 of [8], for each $s \in C_{\mathcal{T}}$, there is a tracial state $t_s \in T(S)$ such that

$$r_S(t_s)(x) = r_A(s)(\kappa(x)) \ \text{ for all } \ x \in K_0(S), \tag{e 18.12}$$

where $r_S : T(S) \to S_{[1_S]}(K_0(S))$ and $r_A : T(A) \to S_{[1_A]}(K_0(A))$ are the induced maps from the tracial state spaces to the state spaces of the K_0-groups. It follows from Lemma 18.3 that we may choose t_s such that

$$t_s(h) > \Delta(\hat{h}) \ \text{ for all } \ h \in S_+^{\mathbf{1}}. \tag{e 18.13}$$

For each $s \in \mathcal{T}$, define $\lambda(s) = t_s$ (where t_s satisfies (e 18.12) and (e 18.13)). This extends to a continuous affine map $\lambda : C_{\mathcal{T}} \to T(S)$. Put $\gamma = \lambda \circ \gamma'$. Then, for any $\tau \in T(A)$,

$$\gamma(\tau)(h) > \Delta(\hat{h}) \ \text{ for all } \ h \in \mathcal{H}_1 \ \text{ and } \ (\text{by (e 18.12)})$$

$$\begin{aligned} & |\gamma(\tau)(q) - \tau(\kappa([q]))| \\ & \quad \leq \ |\lambda(\gamma'(\tau))(q) - \gamma'(\tau)(\kappa([q]))| + |\gamma'(\tau)(\kappa([q])) - \tau(\kappa([q]))| \\ & \quad = \ |\gamma'(\tau)(\kappa([q])) - \tau(\kappa([q]))| < \delta/2 \end{aligned}$$

for all projections $q \in S$. One then applies Theorem 16.10 to obtain a unital homomorphism $\varphi : S \to M_K(A)$ such that $[\varphi] = K\kappa$. ☐

LEMMA 18.5. *Let $C \in \mathcal{C}$. Then there is $M > 0$ satisfying the following condition: Let $A_1 \in \mathcal{B}_1$ and let $A = A_1 \otimes U$ for some UHF-algebra U of infinite type and let $\kappa : (K_0(C), {K_0}^+(C)) \to (K_0(A), {K_0}^+(A))$ be a strictly positive homomorphism with multiplicity M (see the lines above (15.5)). Then there exists a homomorphism $\varphi : C \to M_m(A)$ (for some integer $m \geq 1$) such that $\varphi_{*0} = \kappa$ and $\varphi_{*1} = 0$.*

PROOF. Write $C = A(F_1, F_2, \varphi_0, \varphi_1)$. Denote by M the constant of Corollary 15.3 for $G = K_0(C) \subset K_0(F_1) = \mathbb{Z}^l$. Let $\kappa : K_0(C) \to K_0(A)$ be a positive homomorphism satisfying the condition of the lemma. Let $e \in M_r(A)$ (for some integer $r \geq 1$) be a projection such that $\kappa([1_C]) = e$. Replacing A by $eM_r(A)e$, without loss of generality, we may assume that κ is unital. Since $K_0(C)_+$ is finitely generated, $K_0(A)_+$ is simple and κ is strictly positive, there is N such that for any non-zero element $x \in K_0(C)_+$, one has that $N\kappa(x) > 2[1_A]$. Let K be the natural number of Proposition 18.4 with respect to C and $2N$.

We may also assume that $M_r(C)$ contains a set of minimal projections such that every minimal element of $K_0(C)_+ \setminus \{0\}$ is represented by a minimal projection from the set (see 3.15).

By Lemma 15.6, for any positive map κ with multiplicity M, one has $\kappa = \kappa_1 + \kappa_2$ and there are positive homomorphisms $\lambda_1 : K_0(C) \to \mathbb{Z}^n$, $\gamma_1 : \mathbb{Z}^n \to K_0(A)$, and $\lambda_2 : K_0(C) \to K_0(C')$ such that λ_1 has multiplicity M, λ_2 has multiplicity MK, $\kappa_1 = \gamma_1 \circ \lambda_1$, $\kappa_2 = \imath_{*0} \circ \lambda_2$, and $C' \subset A$ is a C^*-subalgebra with $C' \in \mathcal{C}$, where $\imath : C' \to A$ is the embedding. Moreover,

$$\lambda_2([1_C]) = [1_{C'}] \quad \text{and} \quad 2N\lambda_2(x) > \lambda_2([1_C]) > 0 \tag{e 18.14}$$

for any $x \in K_0(C)_+ \setminus \{0\}$ (here λ_1 plays the role of φ_1 and λ_2 plays the role of φ_2 in Lemma 15.6). Let $\lambda_1([1_C]) = (r_1, r_2, ..., r_n)$, where $r_i \in \mathbb{Z}_+$, $i = 1, 2, ..., n$. Note that we may assume that $r_i \neq 0$ for all i; otherwise, we replace $\mathbb{Z}^n$ by $\mathbb{Z}^{n_1}$ for some $1 \leq n_1 < n$.

Let $\imath_A : A \to A \otimes U$ be defined by $\imath_A(a) = a \otimes 1_U$ and $j : A \otimes U \to A$ such that $j \circ \imath_A$ is approximately unitarily equivalent to id_A. In particular, $[j \circ \imath_A] = [\mathrm{id}_A]$.

Let us first assume that $\lambda_1 \neq 0$. Let R_0 be as in Corollary 15.3 associated with $K_0(C) = G \subset \mathbb{Z}^l$ and $\lambda_1 : K_0(C) \to \mathbb{Z}^n$, which has multiplicity M. Put $F_3 = M_{r_1} \oplus M_{r_2} \oplus \cdots \oplus M_{r_n}$ (recall $\lambda_1([1_C]) = (r_1, r_2, ..., r_n)$). Since A has stable rank one, there is a homomorphism $\psi_0 : F_3 \to A$ such that $(\psi_0)_{*0} = \gamma_1$. Write $U = \lim_{n\to\infty}(M_{R(n)}, h_n)$, where $h_n : M_{R(n)} \to M_{R(n+1)}$ is a unital embedding. Choose $R(n) \geq R_0$. Consider the unital homomorphism $j_{F_3} : F_3 \to F_3 \otimes M_{R(n)}$ defined by $j_{F_3}(a) = a \otimes 1_{M_{R(n)}}$ for all $a \in F_3$, and consider the unital homomorphism $\psi_0 \otimes h_{n,\infty} : F_3 \otimes M_{R(n)} \to A \otimes U$ defined by $(\psi_0 \otimes h_{n,\infty})(a \otimes b) = \psi_0(a) \otimes h_{n,\infty}(b)$ for all $a \in F_3$ and $b \in M_{R(n)}$. We have, for any projection $p \in F_3$ (recall $A = A_1 \otimes U$)

$$((\psi_0 \otimes h_{n,\infty}) \circ j_{F_3})_{*0}([p]) = [\psi_0(p) \otimes 1_U] = (\imath_A)_{*0} \circ (\psi_0)_{*0}([p]) \in K_0(A).$$

It follows that

$$(j_{*0}\circ(\psi_0 \otimes h_{n,\infty})_{*0} \circ (j_{F_3})_{*0}= (j \circ \imath_A)_{*0} \circ (\psi_0)_{*0} = (\psi_0)_{*0}.$$

Now the map $(j_{F_3})_{*0} \circ \lambda_1 : K_0(C) \to K_0(F_3) = \mathbb{Z}^n$ has multiplicity $MR(n)$. Applying Corollary 15.3, we obtain a positive homomorphism $\lambda_1' : K_0(F_1) = \mathbb{Z}^l \to K_0(F_3)$ such that $(\lambda_1')_{*0} \circ (\pi_e)_{*0} = (j_{F_3})_{*0} \circ \lambda_1$. The construction above can be summarized by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}^l \arrow[r, "\lambda_1'"] & K_0(F_3 \otimes M_{R(n)}) \arrow[r, "(\psi_0 \otimes h_{n,\infty})_{*0}"] & K_0(A \otimes U) \arrow[d, shift left, "j"] \\
K_0(C) \arrow[u, "(\pi_e)_{*0}"'] \arrow[r, "\lambda_1"] & K_0(F_3) \arrow[u, "(j_{F_3})_{*0}"'] \arrow[r, "\gamma_1=(\psi_0)_{*0}"] & K_0(A). \arrow[u, shift left, "\imath"]
\end{tikzcd}$$

We obtain a homomorphism $h_0 : F_1 \to F_3 \otimes M_{R(n)}$ such that $(h_0)_{*0} = \lambda_1'$. Define $H_1 = h_0 \circ \pi_e : C \to F_3 \otimes M_{R(n)}$ and $H_2 = (j \circ \psi_0 \otimes h_{n,\infty}) \circ H_1 : C \to A \otimes U$. Then, by the commutative diagram above,

$$(H_2)_{*0} = \kappa_1. \tag{e 18.15}$$

If $\lambda_1 = 0$, then $\kappa_1 = 0$, and we choose $H_2 = 0$.

Since λ_2 has multiplicity K, there exists $\lambda_2' : K_0(C) \to K_0(C')$ such that $K\lambda_2' = \lambda_2$. Since $K_0(C')$ is weakly unperforated, λ_2' is positive. Moreover, by (e 18.14),

$$2KN\lambda_2'(x) > K\lambda_2'([1_C]) = \lambda_2([1_C]) = [1_{C'}] > 0. \tag{e 18.16}$$

Since $K_0(C')$ is weakly unperforated, we have

$$2N\lambda_2'(x) > \lambda_2'([1_C]) > 0 \text{ for all } x \in K_0(C)_+ \setminus \{0\}. \tag{e 18.17}$$

There is a projection $e \in M_k(C')$ for some integer $k \geq 1$ such that $\lambda_2'([1_C]) = [e]$. Define $C'' = eM_k(C')e$. By (e 18.16), e is full in C'. In fact, $K[e] = [1_{C'}]$. In other words, $M_K(C'') \cong C'$. By 3.19, $C'' \in \mathcal{C}$. Applying Proposition 18.4, we obtain a unital homomorphism $H_3' : C \to C'$ such that $(H_3')_{*0} = K\lambda_2' = \lambda_2$. Put $H_3 = \imath \circ H_3'$. Note that $[1_A] = \kappa([1_C]) = \kappa_1([1_C]) + \kappa_2([1_C])$. Conjugating by a unitary, we may assume that $H_2(1_C) + H_3(1_C) = 1_A$. Then it is easy to check that the map $\varphi : C \to A \otimes U$ defined by $\varphi(c) = H_2(c) + H_3(c)$ for all $c \in C$ meets the requirements. □

LEMMA 18.6 (cf. Lemma 9.8 of [71]). *Let A be a unital C^*-algebra and let B_1 be a unital separable simple C^*-algebra in $\mathcal{B}_0$, and let $B = B_1 \otimes U$ for some UHF-algebra of infinite type and $C \in \mathcal{C}_0$ be a C^*-subalgebra of B. Let $G \subset \underline{K}(A)$ be a finitely generated subgroup. Suppose that there exists an $\mathcal{F}$-δ-multiplicative contractive completely positive linear map $\psi : A \to C \subset B$ such that $\psi(1_A) = p$*

is projection and $[\psi]|_G$ *is well defined. Then, for any* $\varepsilon > 0$, *there exist a* C^*-*subalgebra* $C_1 \cong C$ *of* B *and an* $\mathcal{F}$-δ-*multiplicative contractive completely positive linear map* $L : A \to C_1 \subset B$ *such that*

$$\text{(e 18.18)} \qquad [L]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})} = [\psi]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})} \ \ \text{and} \ \ \tau(1_{C_1}) < \varepsilon$$

for all $\tau \in T(B)$ *and for all* $k \geq 1$ *and such that* $G \cap K_0(A, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, *where* L *and* ψ *are viewed as maps to* B. *Furthermore, if* $[\psi]|_{G\cap K_0(A)}$ *is positive then so also is* $[L]|_{G\cap K_0(A)}$.

PROOF. Let $\imath : U \to U \otimes U$ be defined by $\imath(a) = a \otimes 1_U$ and let $j : U \otimes U \to U$ be an isomorphism such that $j \circ \imath$ is approximately unitarily equivalent to id_U. Put $j_B = \mathrm{id}_{B_1} \otimes j : B \otimes U \,(= B_1 \otimes U \otimes U) \to B \,(= B_1 \otimes U)$. Let $1 > \varepsilon > 0$. Suppose that

$$G \cap K_0(A, \mathbb{Z}/k\mathbb{Z}) = \{0\} \ \text{ for all } \ k \geq K$$

for some integer $K \geq 1$. Find a projection $e_0 \in U$ such that $\tau_0(e_0) < \varepsilon$ and $1_U = e_0 + \sum_{i=1}^m p_i$, where $m = 2lK!$ and $1/l < \varepsilon$ and $p_1, p_2, ..., p_m$ are mutually orthogonal and mutually equivalent projections in U. Choose $C_1' = C \otimes e_0 \subset B \otimes U$ and $C_1 = j_B(C_1')$. Then $C_1 \cong C$. Let $\varphi : C \to C_1$ be the isomorphism defined by $\varphi(c) = j_B(c \otimes e_0)$ for all $c \in C$. Put $L = \varphi \circ \psi$. Note that $K_1(C) = K_1(C_1) = \{0\}$. Both $[L]$ and $[\psi]$ map $\mathcal{G} \cap K_0(A, \mathbb{Z}/k\mathbb{Z})$ to $K_0(B)/kK_0(B) \subset K_0(B, \mathbb{Z}/k\mathbb{Z})$ and factor through $K_0(C, \mathbb{Z}/k\mathbb{Z})$. It follows that (recall every element in $K_0(\cdot, \mathbb{Z}/k\mathbb{Z})$ is k-torsion)

$$[L]|_{G\cap K_0(A.\mathbb{Z}/k\mathbb{Z})} = [\psi]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})}.$$

In the case that $[\psi]|_{G\cap K_0(A)}$ is positive, it follows from the definition of L that $[L]|_{G\cap K_0(A)}$ is also positive. ☐

THEOREM 18.7. *Let* C *and* A *be unital stably finite* C^*-*algebras and let* $\alpha \in KL_e(C, A)^{++}$.

(i) *If* $C \in \mathbf{H}$, *or* $C \in \mathcal{C}$, *and* A_1 *is a unital simple* C^*-*algebra in* $\mathcal{B}_0$ *and* $A = A_1 \otimes U$ *for some UHF-algebra* U *of infinite type, then there exists a sequence of completely positive linear maps* $L_n : C \to A$ *such that*

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \ \text{ for all } \ a, b \in C \ \text{ and } \ [\{L_n\}] = \alpha;$$

(ii) *if* $C \in \mathcal{C}_0$ *and* $A_1 \in \mathcal{B}_1$, *the conclusion above also holds;*

(iii) *if* $C = M_n(C(S^2))$ *for some integer* $n \geq 1$, $A = A_1 \otimes U$ *and* $A_1 \in \mathcal{B}_1$, *then there is a unital homomorphism* $h : C \to A$ *such that* $[h] = \alpha$;

(iv) *if* $C \in \mathbf{H}$ *with* $K_1(C)$ *torsion,* $M_n(C(S^2))$ *is not a direct summand of* C, *and* $A = A_1 \otimes Q$, *where* A_1 *is unital and* A *has stable rank one, then there exists a unital homomorphism* $h : C \to A$ *such that* $[h] = \alpha$;

(v) *if* $C = M_n(C(\mathbb{T}))$ *for some integer* $n \geq 1$, *then for any unital* C^*-*algebra* A *with stable rank one, there is a unital homomorphism* $h : C \to A$ *such that* $[h] = \alpha$.

PROOF. Let us first consider (iii). This is a special case of Lemma 2.19 of [92]. Let us provide a proof here. In this case one has that $K_0(C) = \mathbb{Z} \oplus \mathrm{ker}\rho_C \cong \mathbb{Z} \oplus \mathbb{Z}$ is free abelian and $K_1(C) = \{0\}$. Write $A = A_1 \otimes U$, where $K_0(U) = D \subset \mathbb{Q}$ is identified with a dense subgroup of $\mathbb{Q}$ and $1_U = 1$. Let $\alpha_0 = \alpha|_{K_0(C)}$. Then $\alpha_0([1_C]) = [1_A]$ and $\alpha_0(x) \in \mathrm{ker}\rho_A$ for all $x \in \mathrm{ker}\rho_C$ (since α_0 is order preserving). Let $\xi \in \mathrm{ker}\rho_C = \mathbb{Z}$ be a generator and $\alpha_0(\xi) = \zeta \in \mathrm{ker}\rho_A$. Let B be a the unital simple AF-algebra with

$$(K_0(B), K_0(B)_+, [1_{B_0}]) = (D \oplus \mathbb{Z}, (D \oplus \mathbb{Z})_+, (1, 0)),$$

where

$$(D \oplus \mathbb{Z})_+ = \{(d, m) : d > 0, m \in \mathbb{Z}\} \cup \{(0, 0)\}.$$

Note that $K_0(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$ which is order isomorphic to $K_0(C(S^2))$. There is a standard continuous map which maps $s : \mathbb{T}^2 \to S^2$ (with $\mathbb{T}^2$ viewed as a 2-cell attached to a figure 8, the map is defined by sending the figure 8 to a single point) such that $\gamma : C(S^2) \to C(\mathbb{T}^2)$ defined by $\gamma(f) = f \circ s$ has the property that $\gamma_{*0} = \mathrm{id}_{\mathbb{Z}\oplus\mathbb{Z}}$. It follows from [35] that there is a unital homomorphism $h_0' : C(\mathbb{T}^2) \to B$ such that $(h_0')_{*0}(\xi) = (0, 1)$. Define $h_0 = h_0' \circ \gamma$. There is a positive order-unit preserving homomorphism $\lambda : D \to K_0(A)$ (given by the embedding $a \to 1_A \otimes a$ from $U \to A_1 \otimes U$). Define a homomorphism $\kappa_0 : K_0(B) \to K_0(A)$ by $\kappa_0((r, 0)) = \lambda(r)$ for all $r \in D$ and $\kappa_0((0, 1)) = \zeta$. Since A has stable rank one and B is AF, it is known that there is a unital homomorphism $\varphi : B \to A$ such that

$$\varphi_{*0} = \kappa_0. \tag{e 18.19}$$

Define $L = \varphi \circ h_0$. Then, $[L] = \alpha$. This proves (iii).

For (iv), we note that $K_i(A)$ is torsion free and divisible. One may reduce the general case to the case that C has only one direct summand $C = PM_n(C(X))P$, where X is a connected finite CW complex and $P \in M_n(C(X))$ is a projection. Note that we assume here $K_1(C)$ is torsion and $X \neq S^2$. In this case (see 14.5 and 13.27), $K_0(C) = \mathbb{Z} \oplus \mathrm{Tor}(K_0(C))$, $K_1(C) = \{0\}$, or $K_0(C) = \mathbb{Z}$ and $K_1(C)$ is finite. Suppose that P has rank $r \geq 1$. Choose $x_0 \in X$. Let $\pi_{x_0} : C \to M_r$ be defined by $\pi_{x_0}(f) = f(x_0)$ for all $f \in C$. Suppose that $e = (1, 0) \in \mathbb{Z} \oplus \mathrm{Tor}(K_0(C))$ or $e = 1 \in \mathbb{Z}$. Choose a projection $p \in A$ such that $[p] = \alpha_0(e)$ (this is possible since A has stable rank one). Note that $[P] = (r, x)$, where $x \in \mathrm{Tor}(K_0(C))$, i.e., $[P] - re$ is a torsion element. Since $K_i(A)$ is torsion free, $\alpha_0(\mathrm{Tor}(K_0(C))) = 0$. Thus $r\alpha_0(e) = r[p] = \alpha_0([P]) = [1_A]$. Moreover, since $K_0(A)$ is torsion free, $[1_A]$ is a sum of r mutually equivalent projections which are all unitarily equivalent to p. Thus there is a unital homomorphism $h_0 : M_r \to A$ such that $h_0(e_{1,1}) = p$, where $e_{1,1} \in M_r$ is a rank one projection. Define $h : C \to A$ by $h = h_0 \circ \pi_{x_0}$. One verifies that $[h] = \alpha$.

Now we prove (i) and (ii). First, if $C \in \mathbf{H}$, as before, we may assume that C has a single direct summand. In this case, if $C = M_n(C(S^2))$, this follows from case (iii). If $C \in \mathbf{H}$ and $C \neq M_n(C(S^2))$, the statement follows from the same argument as that of Lemma 9.9 of [71], by replacing Lemma 9.8 of [71] by 18.6 above (and replacing F by C and F_1 by C_1) in the proof of Lemma 9.9 of [71].

Assume that $C \in \mathcal{C}$. Considering $\mathrm{Ad}\, w \circ L_n|_C$ for suitable unitary w (in $M_r(A)$), we may replace C by $M_r(C)$ for some $r \geq 1$ so that $K_0(C)_+$ is generated by minimal projections $\{p_1, p_2, ..., p_d\} \subset C$ (see 3.15). We may rearrange it so that $\{[p_1], [p_2], ..., [p_{d'}]\}$ $(0 < d' \leq d)$ forms a base for $K_0(C)$. Since A is simple and $\alpha(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$, there exists an integer $N \geq 1$ such that

$$N\alpha([p]) > 2[1_A] \text{ for all } [p] \in K_0(C)_+ \setminus \{0\}. \tag{e 18.20}$$

Let $M \geq 1$ be the integer given by Lemma 18.5 associated with C. Since C has a separating family of finite dimensional representations, by Theorem 18.2, there exist two sequences of completely positive contractions $\varphi_n^{(i)} : C \to A \otimes \mathcal{K}$ $(i = 0, 1)$ satisfying the following conditions:
(a) $\|\varphi_n^{(i)}(ab) - \varphi_n^{(i)}(a)\varphi_n^{(i)}(b)\| \to 0$, for all $a, b \in C$, as $n \to \infty$,
(b) for any n, $\varphi_n^{(1)}$ is a homomorphism with finite dimensional range and, consequently, for any finite subset $\mathcal{P} \subset \underline{K}(C)$, the map $[\varphi_n^{(i)}]|_{\mathcal{P}}$ is well defined for all sufficiently large n, and
(c) for each finite subset $\mathcal{P} \subset \underline{K}(C)$, there exists $m > 0$ such that

$$[\varphi_n^{(0)}]|_{\mathcal{P}} = \alpha + [\varphi_n^{(1)}]|_{\mathcal{P}} \text{ for all } n > m.$$

Since C is semiprojective and the positive cone of the K_0-group is finitely generated (see the end of 3.1), there are homomorphisms φ_0 and φ_1 from $C \to A \otimes \mathcal{K}$ such that

$$[\varphi_0] = \alpha + [\varphi_1].$$

Without loss of generality, we may assume that φ_0 and φ_1 are homomorphisms from C to $M_r(A)$ for some r. Note that $M_r(A) \in \mathcal{B}_0$ (or $\in \mathcal{B}_1$, when $C \in \mathcal{C}_0$).

Since $K_i(C)$ is finitely generated $(i = 0, 1)$, there exists $n_0 \geq 1$ such that every element $\kappa \in KL(C, A)$ is determined by κ on $K_i(C)$ and $K_i(C, \mathbb{Z}/n\mathbb{Z})$ for $2 \leq n \leq n_0$, $i = 0, 1$ (see Corollary 2.11 of [20]). Let $\mathcal{P} \subset \underline{K}(C)$ be a finite subset which generates the group

$$\bigoplus_{i=0,1} (K_i(C) \oplus \bigoplus_{2 \leq n \leq n_0} K_i(C, \mathbb{Z}/n\mathbb{Z})).$$

Let $K = n_0!$. Let $\mathcal{G}$ be a finite subset of $M_r(A)$ which contains $\{\varphi_0(p_i), \varphi_1(p_i); i = 1, ..., d\}$. We may assume that $\mathcal{P}_0 := \{[p_i], i = 1, 2, ...,d\} \subset \mathcal{P}$. Let $G(\mathcal{P})$ be the subgroup generated by $\mathcal{P}$. Note that $K_0(C) \subset G(\mathcal{P})$.

Let

$$T = \max\{\tau(\varphi_0(p_i)) + KM\tau(\varphi_1(p_i)) : 1 \leq i \leq d; \tau \in T(A)\}.$$

Choose $r_0 > 0$ such that

$$NTr_0 < 1/2. \tag{e 18.21}$$

Let $\mathcal{Q} = [\varphi_0](\mathcal{P}) \cup [\varphi_1](\mathcal{P}) \cup \alpha(\mathcal{P})$. Let $1 > \varepsilon > 0$. By Proposition 14.15, for ε and r_0 as above, there are a non-zero projection $e \in M_r(A)$, a C^*-subalgebra $B \in \mathcal{C}_0$ (or $B \in \mathcal{C}$ for case (ii)) with $e = 1_B$, and $\mathcal{G}$-ε-multiplicative contractive completely positive linear maps $L_1 : M_r(A) \to (1-e)M_r(A)(1-e)$ and $L_2 : M_r(A) \to B$ with the following properties:

(1) $\|L_1(a) + L_2(a) - a\| < \varepsilon/2$ for all $a \in \mathcal{G}$;
(2) $[L_i]|_{\mathcal{Q}}$ is well defined, $i = 1, 2$;
(3) $[L_1]|_{\mathcal{Q}} + [\imath \circ L_2]|_{\mathcal{Q}} = [\mathrm{id}]|_{\mathcal{Q}}$;
(4) $\tau \circ [L_1](g) \le r_0 \tau(g)$ for all $g \in \alpha(\mathcal{P}_0)$ and $\tau \in T(A)$;
(5) for any $x \in \mathcal{Q}$, there exists $y \in \underline{K}(B)$ such that $x - [L_1](x) = [\imath \circ L_2](x) = KM[\imath](y)$; and
(6) there exist positive elements $\{f_i\} \subset K_0(B)_+$ such that for $i = 1, ..., d$,

$$\alpha([p_i]) - [L_1](\alpha([p_i]) = [\imath \circ L_2](\alpha([p_i])) = KM\imath_{*0}(f_i),$$

where $\imath : B \to M_r(A)$ is the embedding. Here we also write $[L_1]$ as a homomorphism on the subgroup generated $\mathcal{Q}$. By (18), since $K = n_0!$,

$$[\imath \circ L_2 \circ \varphi_0]|_{K_i(C,\mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} = [\imath \circ L_2 \circ \varphi_1]|_{K_i(C,\mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} = 0, \ \ i = 0, 1,$$

and $n = 1, 2, ..., n_0$. It follows that

$$\begin{aligned}[L_1 \circ \varphi_0]|_{K_i(C,\mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} &= [\varphi_0]|_{K_i(C,\mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} \quad \text{and}\\ [L_1 \circ \varphi_1]|_{K_i(C,\mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} &= [\varphi_1]|_{K_i(C,\mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}},\end{aligned}$$

$i = 0, 1$ and $n = 1, 2, ..., n_0$. Furthermore, for the case $B \in \mathcal{C}_0$ we have $K_1(B) = 0$, and consequently

$$[\imath \circ L_2]|_{K_1(C)\cap\mathcal{P}} = 0.$$

It follows that

$$\begin{aligned}[L_1 \circ \varphi_0]|_{K_1(C)\cap\mathcal{P}} &= [\varphi_0]|_{K_1(C)\cap\mathcal{P}},\\ [L_1 \circ \varphi_1]|_{K_1(C)\cap\mathcal{P}} &= [\varphi_1]|_{K_1(C)\cap\mathcal{P}}.\end{aligned} \tag{e 18.22}$$

In the second case when we assume that $C \in \mathcal{C}_0$ and $A_1 \in \mathcal{B}_1$, then $K_1(C) = 0$. Therefore (e 18.22) above also holds.

Denote by $\Psi := \varphi_0 \oplus \bigoplus_{KM-1} \varphi_1$. One then has

$$\begin{aligned}[L_1 \circ \Psi]_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} &= [L_1 \circ \varphi_0]_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} + (KM-1)[L_1 \circ \varphi_1]_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}}\\ &= [L_1 \circ \varphi_0]_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} - [L_1 \circ \varphi_1]_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}}\\ &= [\varphi_0]_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}} - [\varphi_1]_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}}\\ &= \alpha|_{K_i(C,\ \mathbb{Z}/n\mathbb{Z})\cap\mathcal{P}},\end{aligned}$$

where $i = 0, 1,\ n = 1, 2, ..., n_0$. Note that we also assume $[L_1 \circ \Psi]|_{\mathcal{G}(\mathcal{P})}$ is a homomorphism. By (e 18.20), (18), and (e 18.21),

$$N(\tau(\alpha([p_i])) - [L_1 \circ \Psi]([p_i])) \geq 2 - Nr_0T \geq 3/2 \text{ for all } \tau \in T(A).$$

Since the strict order on $K_0(A)$ is determined by traces (see 9.11 and 9.10), one has that $N(\alpha([p_i]) - [L_1 \circ \Psi]([p_i])) > [1_A]$. Moreover, one also has

$$\begin{aligned} & \alpha([p_i]) - [L_1 \circ \Psi]([p_i]) \\ = \ & \alpha([p_i]) - ([L_1 \circ \alpha]([p_i]) + KM[L_1 \circ \varphi_1]([p_i])) \\ = \ & (\alpha([p_i]) - [L_1 \circ \alpha]([p_i])) - KM[L_1 \circ \varphi_1]([p_i]) \\ = \ & KM(\imath_{*0}(f_j) - [L_1] \circ [\varphi_1]([p_i])) \\ = \ & KMf_i', \qquad\qquad \text{where } f_i' = \imath_{*0}(f_i) - [L_1] \circ [\varphi_1]([p_i]). \end{aligned}$$

Note that $f_j' \in K_0(A)_+ \setminus \{0\}$, $j = 1, 2, ..., d$. Note also that $\alpha - [L_1 \circ \Psi]$ defines a homomorphism on $K_0(C)$. Since $Mf_i' \in K_0(A)$, $i = 1, 2, ..., d'$, the map $\beta : K_0(C) \to K_0(A)$ defined by $\beta = (1/K)(\alpha - [L_1 \circ \Psi])|_{K_0(C)}$ which maps $[p_i]$ to Mf_i' $(1 \leq i \leq d')$ is a homomorphism. In fact, one has $\beta([p_j]) = Mf_j' + z_j$, $j = 1, 2, ..., d$, where $Kz_j = 0$. Therefore $\beta([p_j]) \in K_0(A)_+ \setminus \{0\}$ (recall the order of $K_0(A)$ is determined by the traces), $j = 1, 2, ..., d$.

Since the semigroup $K_0(C)_+$ is generated by $[p_1], [p_2], ..., [p_d]$, we have $\beta(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$. Since β has multiplicity M, by the choice of M and by Lemma 18.5, there exists a homomorphism $H : C \to M_R(A)$ (it may not be unital) for some integer $R \geq 1$ such that

$$H_{*0} = \beta \quad \text{and} \quad H_{*1} = 0.$$

Consider the map $\varphi' := L_1 \circ \Psi \oplus (\bigoplus_{i=1}^{K} H) : C \to A \otimes \mathcal{K}$. One has that

$$[\varphi']|_{K_0(C) \cap \mathcal{P}} = [L_1 \circ \Psi]|_{K_0(C) \cap \mathcal{P}} + K\beta = \alpha|_{K_0(C) \cap \mathcal{P}}.$$

Since every element in $K_i(\cdot, \mathbb{Z}/n\mathbb{Z})$ is n-torsion,

$$K[H]|_{K_i(C,\ \mathbb{Z}/n\mathbb{Z}) \cap \mathcal{P}} = 0, \quad i = 0, 1,\ n = 1, 2, ..., n_0,$$

and therefore

$$[\varphi']_{K_i(C,\ \mathbb{Z}/n\mathbb{Z}) \cap \mathcal{P}} = [L_1 \circ \Psi]|_{K_i(C,\ \mathbb{Z}/n\mathbb{Z}) \cap \mathcal{P}} = \alpha|_{K_i(C,\ \mathbb{Z}/n\mathbb{Z}) \cap \mathcal{P}},$$

$i = 0, 1,\ n = 1, 2, ..., n_0$. We also have that

$$[\varphi']_{K_1(C) \cap \mathcal{P}} = [L_1 \circ \Psi]|_{K_1(C) \cap \mathcal{P}} = \alpha|_{K_1(C) \cap \mathcal{P}}.$$

Therefore,

$$[\varphi']|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Since $[\varphi'(1_C)] = [1_A]$ and A has stable rank one, there is a unitary u in a matrix algebra over A such that the map $\varphi = \mathrm{Ad}(u) \circ \varphi'$ satisfies $\varphi(1_C) = 1_A$, as desired.

Case (v) is standard and is well known. □

COROLLARY 18.8. *Any C^*-algebra A given by Theorem* 14.10 *is KK-attainable with respect to $\mathcal{B}_{u0}$.*

PROOF. Note that $A = \lim_{n\to\infty}(A_n, \varphi_n)$, where each A_n is a finite direct sum of C^*-algebras in $\mathbf{H}$ or $\mathcal{C}_0$ and φ_n is injective. Let $A_n' = \varphi_{n,\infty}(A_n)\,(\cong A_n)$ and $\imath_n : A_n' \to A$ be the embedding. By Lemma 2.2 of [20], for any C^*-algebra B, $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) = \lim_{n\to\infty} \mathrm{Hom}_\Lambda(\underline{K}(A_n'), \underline{K}(B))$. Let $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$. Then, for each n, there is $\alpha_n \in \mathrm{Hom}_\Lambda(\underline{K}(A_n'), \underline{K}(B))$ such that $\alpha|_{[\imath_n](\underline{K}(A_n))} = \alpha_n$ and $\alpha_{n+l}|_{[\varphi_{n,n+l}](\underline{K}(A_n'))} = \alpha_n$ for all $l \geq 1$. Noting that $\imath_n(p) \neq 0$ for any non-zero projection $p \in M_m(A_n')$ (for any $m > 0$), one has $\alpha_n \in \mathrm{Hom}_\Lambda(\underline{K}(A_n'), \underline{K}(B))^{++}$. Let $q \in M_m(B)$ be a projection for some integer $m \geq 1$ such that $[q] = \alpha([1_A])$. Since A is unital, we may assume that φ_n is unital (for large n). Then $[q] = \alpha_n([1_{A_n'}])$. Let $B_1 = qM_m(B)q$. Then $\alpha_n \in KL_e(A_n', B_1)^{++}$. Recall that B has stable rank one. By considering each summand of A_n', by Theorem 18.7 that, for each n, there exists a sequence of contractive completely positive linear maps $L_{k,n} : A_n' \to B_1$ such that $\lim_{k\to\infty} \|L_{k,n}(ab) - L_{k,n}(a)L_{k,n}(b)\| = 0$ for all $a, b \in A_k'$ and $[\{L_{k,n}\}] = \alpha_n$. Since each A_m' is separable and amenable, there exists a sequence of contractive completely positive linear maps $\Psi_m : A \to A_m'$ such that $\lim_{n\to\infty} \Psi_m(a) = a$ (see, for example, 2.3.13 of [63]). It is standard that, after passing to two suitable subsequences $\{m_n\}$ and $\{k_n\}$, the sequence $\{L_n = L_{k_n,m_n} \circ \Psi_{m_n}\}$ has the property that

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \text{ for all } a, b \in A \text{ and } [\{L_n\}] = \alpha.$$

Thus, A is KK-attainable with respect to $\mathcal{B}_{u0}$. □

COROLLARY 18.9. *Let $C \in \mathcal{C}$, $A \in \mathcal{B}_{u0}$, and $\alpha \in KK(C, A)^{++}$ be such that $\alpha([1_C]) = [p]$ for some projection $p \in A$. Then there exists a homomorphism $\varphi : C \to A$ such that, $\varphi_{*0} = \alpha$.*

PROOF. This is a special case of Theorem 18.7 since C is semiprojective. □

COROLLARY 18.10. *Let $A \in \mathcal{B}_{u0}$. Then there exist a unital simple C^*-algebra $B_1 = \varinjlim(C_n, \varphi_n)$, where each C_n is in $\mathcal{C}_0$, and a UHF algebra U of infinite type such that, for $B = B_1 \otimes U$, we have*

$$\begin{aligned} (\text{e}\,18.23) \qquad & (K_0(B), K_0(B)_+, [1_B], T(B), r_B) \\ &= (\rho_A(K_0(A)), \rho_A(K_0(A)_+), \rho_A([1_A]), T(A), r_A). \end{aligned}$$

Moreover, for each n, there is a homomorphism $h_n : C_n \otimes U \to M_2(A)$ such that

$$(\text{e}\,18.24) \qquad \rho_A \circ (h_n)_{*0} = (\varphi_{n,\infty} \otimes \mathrm{id}_U)_{*0}.$$

PROOF. Recall that $r_A : T(A) \to S_u(K_0(A))$ is defined by $r_A(\tau)(x) = \rho_A(x)(\tau)$ for all $x \in K_0(A)$. Therefore r_A also induces a continuous affine map from $T(A)$

to $S_u(\rho_A(K_0(A)))$ by defining $r_A(\tau)(\rho_A(x)) = \rho_A(x)(\tau)$ for all $x \in K_0(A)$. If $s \in S_u(\rho_A(K_0(A)))$, then $s \circ \rho_A \in S_u(K_0(A))$. By Corollary 3.4 of [8], $r_A : T(A) \to S_u(K_0(A))$ is surjective. It follows that r_A also maps $T(A)$ onto $S_u(\rho_A(K_0(A)))$. Consider the tuple

$$(\rho_A(K_0(A)), \rho_A(K_0(A)_+), \rho_A([1_A]), T(A), r_A).$$

Since $A \cong A \otimes U_1$, for a UHF algebra U_1 of infinite type, A has the property (SP) (see [7]), and therefore the ordered group $(K_0(A), K_0(A)_+, [1_A])$ has the property (SP) in the sense of Theorem 14.10; that is, for any positive real number $0 < s < 1$, there is $g \in K_0(A)_+$ such that $\tau(g) < s$ for any $\tau \in T(A)$. Then it is clear that the scaled ordered group $(\rho_A(K_0(A)), (\rho_A(K_0(A)_+), \rho_A(1_A))$ also has the property (SP) in the sense of Theorem 14.10. Therefore, by Theorem 14.10, there is a simple unital C^*-algebra $B_1 = \varinjlim(C_n, \varphi_n)$, where each $C_n \in \mathcal{C}_0$, such that

$$\begin{aligned}&(K_0(B_1), K_0(B_1)_+, [1_{B_1}], T(B_1), r_{B_1})\\ &\quad\cong \quad (\rho_A(K_0(A)), \rho_A(K_0(A)_+), \rho_A(1_A), T(A), r_A).\end{aligned}$$

Let $U = U_1$ and $B = B_1 \otimes U$. Then

$$\begin{aligned}&(K_0(B), K_0(B)_+, [1_B], T(B), r_B)\\ &\quad\cong \quad (\rho_A(K_0(A \otimes U)), \rho_A(K_0(A \otimes U)_+), \rho_A(1_A), T(A \otimes U), r_{A\otimes U}).\end{aligned}$$

Let $\imath : U \to U \otimes U$ be defined by $\imath(a) = a \otimes 1_U$ for all $a \in U$ and let $j : U \otimes U \to U$ be an isomorphism; recall that j^{-1} is approximately unitarily equivalent to $\imath$. Define $\imath_A : A \to A \otimes U$ by $\imath_A = \mathrm{id}_A \otimes \imath$ and define $j_A = \mathrm{id}_A \otimes j$. It follows that j_A induces an order isomorphism:

$$\begin{aligned}&(\rho_A(K_0(A \otimes U)), \rho_A(K_0(A \otimes U)_+), \rho_A(1_{A\otimes U}), T(A \otimes U), r_{A\otimes U})\\ &\qquad\qquad \cong (\rho_A(K_0(A)), \rho_A(K_0(A)_+), \rho_A(1_A), T(A), r_A).\end{aligned}$$

Therefore (e 18.23) holds.

Clearly, B has the inductive limit decomposition

$$B = \varinjlim(C_n \otimes U, \varphi_n \otimes \mathrm{id}_U).$$

For each n, consider the positive homomorphism $(\varphi_{n,\infty})_{*0} : K_0(C_n) \to K_0(B_1) \cong \rho_A(K_0(A))$. Since $\rho_A(K_0(A))$ is torsion free, $(\varphi_{n,\infty})_{*0}(K_0(C_n))$ is a free abelian group. There is a group homomorphism $\kappa_n : K_0(C_n) \to K_0(A)$ such that (recall that the order of $K_0(A)$ by traces) is determined by

$$\rho_A \circ \kappa_n = (\varphi_{n,\infty})_{*0} \text{ and } \kappa_n([1_{C_n}]) \leq 2[1_A].$$

Note that κ_n is positive since $\rho_A \circ \kappa_n$ is. By Corollary 18.9, there is a homomorphism $h_n' : C_n \to M_2(A)$ such that $(h_n')_{*0} = \kappa_n$. It is clear that $h_n := j_A \circ (h_n' \otimes \mathrm{id}_U)$ satisfies the desired condition. □

LEMMA 18.11. *Let $C \in \mathcal{C}$. Let $\sigma > 0$ and let $\mathcal{H} \subset C_{s.a.}$ be a finite subset. Let $A \in \mathcal{B}_{u0}$. Then for any $\kappa \in KL_e(C, A)^{++}$ and any continuous affine map $\gamma : T(A) \to T_f(C)$ (see 2.2) which is compatible to κ, there is a unital homomorphism $\varphi : C \to A$ such that*

$$[\varphi] = \kappa \quad \textit{and} \quad |\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma \ \textit{for all}\ h \in \mathcal{H}\ \textit{and for all}\ \tau \in T(A).$$

Moreover, the conclusion above also holds if $C \in \mathcal{C}_0$ and $A \in \mathcal{B}_{u1}$.

PROOF. We assume $C \in \mathcal{C}$ and $B \in \mathcal{B}_{u1}$ first. Let $\sigma > 0$. We may assume that every element of $\mathcal{H}$ has norm at most one. Let κ and γ be given. Define $\Delta : C_+^{q,1} \setminus \{0\} \to (0, 1)$ by

$$\Delta(\hat{h}) = \inf\{\gamma(\tau)(h)/2 : \ \tau \in T(A)\}.$$

Let $\mathcal{H}_1 \subset C_+ \setminus \{0\}$, δ, and K be the finite subset, positive constant, and the positive integer of Theorem 16.10 with respect to C, Δ, $\mathcal{H}$, and $\sigma/4$ (in place of σ). Let $\mathcal{P} \subset M_m(C)$ (for some $m \geq 1$) be a finite subset of projections which generates $K_0(C)_+$. We may assume that $\mathcal{H}_1$ is in the unit ball of C.

Set

$$\sigma_1 = \min\{\sigma/2, \min\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}\} > 0.$$

By Lemma 16.12 (apply to $\sigma_1/(8+\sigma_1)$), there are a C^*-subalgebra $D \subset A$ with $D \in \mathcal{C}$ and with $1_D = p \in A$, and a continuous affine map $\gamma' : T(D) \to T(C)$ such that

(e 18.25)
$$|\gamma'(\frac{1}{\tau(p)}\tau|_D)(f) - \gamma(\tau)(f)| < \sigma_1/(8 + \sigma_1) \ \text{for all}\ \tau \in T(A)\ \text{for all}\ f \in \mathcal{H} \cup \mathcal{H}_1,$$

where $\tau(1-p) < \sigma_1/(8+\sigma_1)$ for all $\tau \in T(A)$. Moreover (by (2) of Lemma 16.12)

$$\gamma'(\tau)(h) > \Delta(\hat{h}) \ \text{for all}\ \tau \in T(D)\ \text{for all}\ h \in \mathcal{H}_1. \tag{e 18.26}$$

Denote by $\imath : D \to pAp$ the embedding. Moreover, as by (3) and (4) of Lemma 16.12, there are also positive homomorphisms $\kappa_{0,0} : K_0(C) \to K_0((1-p)A(1-p))$ and $\kappa_{1,0} : K_0(C) \to K_0(D)$ such that $\kappa_{1,0}$ is strictly positive, $\kappa_{1,0}([1_C]) = [1_D]$, $\kappa_{1,0}$ has multiplicity K, and

$$\kappa|_{K_0(C)} = \kappa_{0,0} + \imath_{*0} \circ \kappa_{1,0} \ \text{and}\ |\gamma'(\tau)(q) - \rho_D(\kappa_{1,0}(q))(\tau)| < \delta, \quad q \in \mathcal{P},\ \tau \in T(D).$$

Suppose that $A = A_1 \otimes U$ for some $A_1 \in \mathcal{B}_1$ and a UHF-algebra U of infinite type. By the last part of 16.12, we may assume that $\kappa_{0,0}$ is also strictly positive. Let $x \in K_0(D)_+$ be such that $Kx = [1_D] = \kappa_{1,0}([1_C])$. Then there is a projection $e \in D$ such that $[e] = x$ as D has stable rank one. Note e is full and $D_1 := eDe$ belongs to $\mathcal{C}$ and in case $D \in \mathcal{C}_0$, $D_1 \in \mathcal{C}_0$ (see 3.19). We may write $D_1 \otimes M_K = D$. Define $\lambda : T(D_1) \to T(D)$ by $\lambda(t)(a \otimes b) = t(a)\mathrm{tr}(b)$, where tr is the tracial state of M_K. Define $\gamma'' : T(D_1) \to T(C)$ by $\gamma''(t) = \gamma'(\lambda(t))$. Note that $K_0(D_1) = K_0(D)$. Then, for any $q \in \mathcal{P}$ and any $t \in T(D_1)$,

$$|\gamma''(t)(q) - \rho_{D_1}((1/K)\kappa_{1,0})([q]))(t)| = |\gamma'(\lambda(t))(q) - \rho_D(\kappa_{1,0})([q])(\lambda(t))| < \delta$$

and

$$\gamma''(t)(h) = \gamma'(\lambda(t))(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1. \qquad \text{(see also (e 18.26))}.$$

Therefore, by Lemma 16.10, there is a homomorphism

$$\varphi_1 : C \to M_K(D_1) = D \subset A$$

such that

$$(\varphi_1)_{*0} = K(1/K)\kappa_{1,0} = \kappa_{1,0}$$

and

$$|(1/K)(t \otimes \mathrm{Tr})(\varphi_1(h)) - \gamma''(t)(h)| < \sigma/4 \quad \text{for all } h \in \mathcal{H} \text{ and } t \in T(D_1), \tag{e 18.27}$$

where Tr is the unnormalized trace of M_K. Note that $(1/K)(\tau \otimes \mathrm{Tr}) = \lambda(\tau)$ for all $\tau \in T(D_1)$. By (e 18.27),

$$|\tau \circ \varphi_1(h) - \gamma'(\tau)(h)| < \sigma/4 \text{ for all } h \in \mathcal{H} \text{ and } \tau \in T(D).$$

Since A is simple, $K_i((1-p)A(1-p)) = K_i(A)$, $i = 0, 1$. Let $\kappa_0 = \kappa - [\imath \circ \varphi_1] \in KL(C, A) = KL(C, (1-p)A(1-p))$. Then $\kappa_0|_{K_0(C)} = \kappa_{0,0}$. Since $\kappa_0|_{K_0(C)}$ is equal to $\kappa_{0,0}$, it is strictly positive and $\kappa_0([1_C]) = [1-p]$. Note that $(1-p)A(1-p) \otimes U \cong (1-p)A(1-p)$. Therefore, by Theorem 18.7, since C is semiprojective, there is a homomorphism $\varphi_0 : C \to (1-p)A(1-p)$ such that $[\varphi_0] = \kappa_0$. Note that this holds for both the case that $A \in \mathcal{B}_{u0}$ (by (i) of Theorem 18.7) and the case that $C \in \mathcal{C}_0$ and $A \in \mathcal{B}_{u1}$ (apply (ii) of of Theorem 18.7).

Consider the homomorphism

$$\varphi := \varphi_0 \oplus \imath \circ \varphi_1 : C \to (1-p)A(1-p) \oplus D \subset A.$$

One has that $[\varphi] = \kappa$ and, for all $h \in \mathcal{H}$ and $\tau \in T(A)$,

$$\begin{aligned} |\tau \circ \varphi(h) - \gamma(\tau)(h)| &\leq |\tau \circ \varphi_1(h) - \gamma(\tau)(h)| + \sigma/4 \\ &< |\tau \circ \varphi_1(h) - \gamma'(\frac{1}{\tau(p)}\tau|_D)(h)| + \sigma/2 \\ &< |\tau \circ \varphi_1(h) - \frac{1}{\tau(p)}\tau \circ \varphi_1(h)| + 3\sigma/4 < \sigma, \end{aligned}$$

as desired. □

It turns out that KK-attainability implies the following existence theorem.

Proposition 18.12. *Let $A \in \mathcal{B}_0$, and assume that A is KK-attainable with respect to $\mathcal{B}_{u0}$. Then for any $B \in \mathcal{B}_{u0}$, any $\alpha \in KL(A,B)^{++}$, and any continuous affine map $\gamma : T(B) \to T(A)$ which is compatible with α, there is a sequence of contractive completely positive linear maps $L_n : A \to B$ such that*

$$\lim_{n\to\infty} ||L_n(ab) - L_n(a)L_n(b)|| = 0 \text{ for all } a, b \in A,$$

$$[\{L_n\}] = \alpha, \text{ and}$$

$$\lim_{n\to\infty} \sup\{|\tau \circ L_n(f) - \gamma(\tau)(f)| : \tau \in T(A)\} = 0 \text{ for all } f \in A.$$

Proof. The proof is the same as that of Proposition 9.7 of [71]. Instead of using Corollary 9.6 of [71], one uses Lemma 18.11. ☐

19. The Class $\mathcal{N}_1$

Let A be a unital C^*-algebra such that $A \otimes Q \in \mathcal{B}_1$. In this section, we will show that $A \otimes Q \in \mathcal{B}_0$. Note that this is proved without assuming $A \otimes Q$ is nuclear. However, it implies that $\mathcal{N}_1 = \mathcal{N}_0$. If we assume that $A \otimes Q$ has finite nuclear dimension, then by using Lemma 18.11 and a characterization of the class TAS by Winter, a much shorter proof of Theorem 19.2 could be given here.

Lemma 19.1. *Let $A \in \mathcal{B}_1$ be such that $A \cong A \otimes Q$. Then the following property holds: For any $\varepsilon > 0$, any two non-zero mutually orthogonal elements $a_1, a_2 \in A_+$, and any finite subset $\mathcal{F} \subset A$, there exist a projection $q \in A$ and a C^*-subalgebra $C_1 \in \mathcal{C}$ with $1_{C_1} = q$ such that*

(1) $\|[x, q]\| < \varepsilon/16$, $x \in \mathcal{F}$,
(2) $qxq \in_{\varepsilon/16} C_1$, $x \in \mathcal{F}$, and
(3) $1 - q \lesssim a_1$.

Suppose that $\Delta : (C_1)_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ is an order preserving map such that

$$\tau(c) \geq \Delta(\hat{c}) \text{ for all } \tau \in T(A) \text{ and } c \in (C_1)_+^{\mathbf{1}} \setminus \{0\}.$$

(By 12.5 such a Δ always exists.) Suppose also that $\mathcal{H} \subset (C_1)_+ \setminus \{0\}$ and $\mathcal{F}_1 \subset C_1$ are finite subsets. Then, there exist another projection $p \in A$ with $p \leq q$, a C^*-subalgebra $C_2 \in \mathcal{C}$ of A with $1_{C_2} = p$, and a unital homomorphism $H : C_1 \to C_2$ such that

(4) $\|[x, p]\| < \varepsilon/16$, $x \in \mathcal{F}$,
(5) $\|H(y) - pyp\| < \varepsilon/16$, $y \in \mathcal{F}_1$, and
(6) $1 - p \lesssim a_1 + a_2$.

Moreover, C_1 *and* H *may be chosen in such a way that* $K_1(C_1)$ *may be written as* $\mathbb{Z}^m \oplus G_0$ *with* $H_{*1}(G_0) = \{0\}$, *and* $H_{*1}|_{\mathbb{Z}^m}$ *and* $(j \circ H)_{*1}|_{\mathbb{Z}^m}$ *both injective, where* $j : C_2 \to A$ *is the embedding (*m *could be zero, and in this case,* $G_0 = K_1(C_1)$*). Furthermore, we may choose* C_2 *and* H *such that*

$$\tau(j \circ H(c)) \geq 3\Delta(\hat{c})/4 \ \ \textit{for all} \ \ c \in \mathcal{H} \ \ \textit{and} \ \ \ \textit{for all} \ \ \tau \in T(A).$$

PROOF. Since $A \in \mathcal{B}_1$, there exist a projection $q \in A$ and a C^*-subalgebra $C_1 \in \mathcal{C}$ of A with $1_{C_1} = q$ such that

(a) $\|[x, q]\| < \varepsilon/16$, $x \in \mathcal{F}$,

(b) $qxq \in_{\varepsilon/16} C_1$, $x \in \mathcal{F}$, and

(c) $1 - q \lesssim a_1$.

There are two non-zero mutually orthogonal elements a_2' and a_3 in $\overline{a_2Aa_2}$. Since $A \cong A \otimes Q$, $K_1(A)$ is torsion free. Denote by $\iota : C_1 \to qAq$ the embedding. Since $K_1(C_1)$ is finitely generated, we may write $K_1(C_1) = G_1 \oplus G_0$, where $G_1 \cong \mathbb{Z}^{m_1}$, $\iota_{*1}|_{G_1}$ is injective, and $\iota_{*1}|_{G_0} = 0$. Define

$$\sigma = \min\{\Delta(\hat{h})/16 : h \in \mathcal{H}\} > 0.$$

Choose an element $a_2'' \in \overline{a_2'Aa_2'}$ such that $d_\tau(a_2'') < \sigma$ for all $\tau \in T(A)$ (recall $d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}})$).

Suppose that G_0 is generated by $[v_1], [v_2], ..., [v_l]$, where $v_i \in U(C_1)$ (note that, by 3.3, C_1 has stable rank one). Then, we may write $\iota(v_k) = \prod_{s=1}^{l(k)} \exp(ih_{s,k})$ (since $\imath_{*1}([v_k]) = 0$ in $K_1(A)$), where $h_{s,k} \in (qAq)_{s.a.}$, $s = 1, 2, ..., l(k)$, $k = 1, 2, ..., l$.

Let $\mathcal{F}_1$ be a finite subset of C_1 with the following property: if $x \in \mathcal{F}$, then there is $y \in \mathcal{F}_1$ such that $\|qxq - y\| < \varepsilon/16$.

Let $\mathcal{F}_2$ be a finite subset of qAq, to be determined, which at least contains $\mathcal{F} \cup \mathcal{F}_1 \cup \mathcal{H}$ and the elements $h_{s,k}$, $\exp(ih_{s,k})$, $s = 1, 2, ..., l(k)$, $k = 1, 2, ..., l$.

Let $0 < \delta < \min\{\varepsilon/64, \sigma/4\}$, to be determined. Since, by 9.9, $qAq \in \mathcal{B}_1$, one obtains a non-zero projection $q_1 \in qAq$ and a C^*-subalgebra $C_2 \subset qAq$ such that

(d) $\|[x, q_1]\| < \delta$, $x \in \mathcal{F}_2$,

(e) $q_1xq_1 \in_\delta C_2$, $x \in \mathcal{F}_2$, and

(f) $1 - q_1 \lesssim a_2{}''$.

With sufficiently small δ and large $\mathcal{F}_2$, using the semiprojectivity of C_1, we obtain unital homomorphisms $h_1 : C_1 \to C_2$ such that

$$\|h_1(a) - q_1aq_1\| < \min\{\varepsilon/16, \sigma\} \ \text{ for all } \ a \in \mathcal{F}_1 \cup \mathcal{H}. \tag{e 19.1}$$

By the choice of σ and a_2'', and by (e 19.1), one computes that, for all $\tau \in T(A)$,

$$\tau(j \circ h_1(c)) \geq 7\Delta(\hat{c})/8 \ \text{ for all } \ c \in \mathcal{H}, \tag{e 19.2}$$

where j is the embedding from C_2 into qAq. Note that, when $\mathcal{F}_2$ is large enough, $(h_1)_{*1}(G_0) = \{0\}$. Since $K_1(C) = \mathbb{Z}^{m_1} \oplus G_0$, G_0 is a direct summand of $\ker (h_1)_{*1}$.

We may write $K_1(C_1) = G_2 \oplus G_{2,0} \oplus G_{2,0,0}$, where $G_2 \cong \mathbb{Z}^{m_2}$ with $m_2 \le m_1$, G_2 is a subgroup of G_1, $G_{2,0,0} \supset G_0$, $(h_1)_{*1}(G_{2,0,0}) = \{0\}$, $(j \circ h_1)_{*1}|_{G_{2,0}} = 0$, and $(j \circ h_1)_{*1}|_{G_2}$ is injective. Here we use the fact that $K_1(A)$ is torsion free (and $G_{2,0,0}$ could be chosen to be just G_0 itself). If $G_{2,0} = \{0\}$, we are done. Otherwise, $m_2 < m_1$. We also note that $(j \circ h_1)_{*1}(G_{2,0} \oplus G_{2,0,0}) = \{0\}$.

We will repeat the process to construct h_2, and consider $h_2 \circ h_1$. Then we may write $K_1(C_1) = G_3 \oplus G_{3,0} \oplus G_{3,0,0}$ with $G_3 \cong \mathbb{Z}^{m_3}$ $(m_3 \le m_2)$, $G_3 \subset G_2$, $G_{3,0,0} \supset G_{2,0} \oplus G_{2,0,0}$, $(j \circ h_2 \circ h_1)_{*1}(G_{3,0}) = \{0\}$, and $(j \circ h_2 \circ h_1)_{*1}|_{G_3}$ injective. Again, if $G_{3,0} = \{0\}$, we are done (choose $H = h_2 \circ h_1$). Otherwise, $m_3 < m_2 < m_1$. We continue this process. This process stops at a finite stage since m_1 is finite. This proves the lemma. $\square$

THEOREM 19.2. *Let A_1 be a unital simple separable C^*-algebra such that $A_1 \otimes Q \in \mathcal{B}_1$. Then $A_1 \otimes Q \in \mathcal{B}_0$.*

PROOF. Let $A = A_1 \otimes Q$. Suppose that $A \in \mathcal{B}_1$. Let $\varepsilon > 0$, let $a \in A_+ \setminus \{0\}$, and let $\mathcal{F} \subset A$ be a finite subset. Since A has the property (SP) (see 9.16), we obtain three non-zero and mutually orthogonal projections $e_0, e_1, e_2 \in \overline{aAa}$. There exist a projection $q_1 \in A$ and a C^*-subalgebra $C_1 \in \mathcal{C}$ of A with $1_{C_1} = q_1$ such that

$$\|[x, q_1]\| < \varepsilon/16 \text{ and } q_1 x q_1 \in_{\varepsilon/16} C_1 \text{ for all } x \in \mathcal{F}, \text{ and} \tag{e 19.3}$$

$$1 - q_1 \lesssim e_0. \tag{e 19.4}$$

Let $\mathcal{F}_1 \subset C_1 \subset A$ be a finite subset such that, for any $x \in \mathcal{F}$, there is $y \in \mathcal{F}_1$ such that $\|q_1 x q_1 - y\| < \varepsilon/16$. For each $h \in (C_1)_+ \setminus \{0\}$, define

$$\Delta(\hat{h}) = (1/2) \inf\{\tau(h) : \tau \in T(A)\}.$$

Then the function $\Delta : (C_1)_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ preserves the order. Let $\mathcal{H}_1 \subset (C_1)_+^{\mathbf{1}} \setminus \{0\}$ be a finite subset, $\gamma_1, \gamma_2 > 0$, $\delta > 0$, $\mathcal{G} \subset C_1$ be a finite subset, $\mathcal{P} \subset \underline{K}(C_1)$ be a finite subset, $\mathcal{H}_2 \subset (C_1)_{s.a.}$ be a finite subset, and $\mathcal{U} \subset J_c(K_1(C_1) \subset U(C_1)/CU(C_1)$ (see 2.16) be a finite subset as provided by Theorem 12.7 (and Corollary 12.8) for $C = C_1$, $\varepsilon/16$ (in place of ε), $\mathcal{F}_1$ (in place of $\mathcal{F}$), and $\Delta/2$ (in place of Δ).

By Lemma 19.1, there exist another projection $q_2 \in A$ such that $q_2 \le q_1$, a C^*-subalgebra $C_2 \in \mathcal{C}$ of A with $q_2 = 1_{C_2}$, and a unital homomorphism $H : C_1 \to C_2$ such that

$$\|[x, q_2]\| < \varepsilon/16 \text{ for all } x \in \mathcal{F}, \tag{e 19.5}$$

$$\|j \circ H(y) - q_2 y q_2\| < \varepsilon/16 \text{ for all } y \in \mathcal{F}_1, \tag{e 19.6}$$

$$\tau(j \circ H(c)) \ge 3\Delta(\hat{c})/4 \text{ for all } c \in \mathcal{H} \text{ and } \text{ for all } \tau \in T(A), \text{ and} \tag{e 19.7}$$

$$1 - q_2 \lesssim e_0 + e_1. \tag{e 19.8}$$

Moreover, we may write $K_1(C_1) = \mathbb{Z}^m \oplus G_{00}$, where $H_{*1}(G_{00}) = \{0\}$, and $H_{*1}|_{\mathbb{Z}^m}$ and $(j \circ H)_{*1}|_{\mathbb{Z}^m}$ are injective, where $j : C_2 \to A$ is the embedding. Let $A_2 = q_2Aq_2$ and denote by $j_1 : C_2 \to A_2$ the embedding. Put

$$\sigma_0 = \inf\{\tau(e_2) : \tau \in T(A)\} > 0. \tag{e 19.9}$$

By Theorem 14.10, there exists a unital simple C^*-algebra $B \cong B \otimes Q$ with $B = \varinjlim(B_n, \imath_n)$ in such a way that each B_n is equal to $B_{n,0} \oplus B_{n,1}$ with $B_{n,0} \in \mathbf{H}$ (see 14.5) and $B_{n,1} \in \mathcal{C}_0$, each $\imath_n$ is injective, and

$$\lim_{n\to\infty} \max\{\tau(1_{B_{n,0}}) : \tau \in T(B)\} = 0, \tag{e 19.10}$$
$$\text{and such that } \mathrm{Ell}(B) = \mathrm{Ell}(A_2).$$

We may assume (choosing a smaller δ) that $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_0$, with $\pi(\mathcal{U}_1)$ generating $\mathbb{Z}^m \subset K_1(C_1)$ and $\pi(\mathcal{U}_0) \subset G_{00}$, where $\pi : U(C_1)/CU(C_1) \to K_1(C_1)$ is the quotient map. Recall that $J_c : K_1(C_1) \to U(C_1)/CU(C_1)$ is a fixed splitting map as defined in 2.16. Suppose that $\bar{v}_1, \bar{v}_2, ..., \bar{v}_m$ form a set of independent generators for $J_c^{(1)}(\mathbb{Z}^m) \cong \mathbb{Z}^m$. We may assume that $\mathcal{U}_1 = \{\bar{v}_1, \bar{v}_2, ..., \bar{v}_m\}$. Put

$$\gamma_3 = \min\{\Delta(\hat{h})/2 : h \in \mathcal{H}_1\}.$$

Note $H^{\ddagger}(\mathcal{U}_0) \subset U_0(C_2)/CU(C_2)$. By Lemma 2.18, we may choose a finite subset $\mathcal{H}_3 \subset (C_2)_{s.a.}$ and $\sigma > 0$ with the following property: for any two unital homomorphisms $h_1, h_2 : C_2 \to D$, for any unital C^*-algebra D of stable rank one, if, for all $\tau \in T(D)$,

$$|\tau \circ h_1(g) - \tau \circ h_2(g)| < \sigma \ \text{ for all } \ g \in \mathcal{H}_3, \tag{e 19.11}$$

then

$$\mathrm{dist}(h_1^{\ddagger}(\bar{v}), h_2^{\ddagger}(\bar{v})) < \gamma_2/8 \tag{e 19.12}$$

for all $\bar{v} \in H^{\ddagger}(\mathcal{U}_0) \subset U_0(C_2)/CU(C_2)$. Without loss of generality, we may assume that $\|h\| \le 1$ for all $h \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$.

Let $\Gamma = (\kappa_0, \kappa_1, \kappa_T) : \mathrm{Ell}(A_2) \to \mathrm{Ell}(B)$ be the above identification, where $\kappa_0 : K_0(A_2) \to K_0(B)$ is an order isomorphism with $\kappa_0([q_2]) = [1_B]$, $\kappa_1 : K_1(A_2) \to K_1(B)$ is an isomorphism, and $\kappa_T : T(A_2) \to T(B)$ is an affine homeomorphism such that $r_{A_2}(\kappa_T^{-1}(\tau))(x) = r_B(\tau)(\kappa_0(x))$ for all $x \in K_0(A_2)$ and for all $\tau \in T(B)$ (or, equivalently, $r_{A_2}(t)(\kappa_0^{-1}(y)) = r_B(\kappa_T(t))(y)$ for all $y \in K_0(B)$ and for all $t \in T(A_2)$). Since B satisfies the UCT, there is an element $\kappa^{-1} \in KK_e(B, A_2)^{++}$ (see 2.10) such that $\kappa^{-1}|_{K_i(B)} = \kappa_i^{-1}$, $i = 0, 1$. Note that since $A_2 \cong A_2 \otimes Q$, one has $\mathrm{Hom}_\Lambda(\underline{K}(B), \underline{K}(A_2)) = \mathrm{Hom}(K_*(B), K_*(A_2))$.

Let $\kappa^{(2)} \in KK_e(C_2, B)^{++}$ be such that $\kappa^{(2)}|_{K_i(C_2)} = \kappa_i \circ j_{*i}$, $i = 0, 1$. In fact, since $K_i(B) = K_i(B) \otimes Q$ $(i = 0, 1)$, by the UCT, $\kappa^{(2)}$ is uniquely determined by $\kappa_i \circ j_{*i}$.

It follows from Lemma 18.11 that there exists a unital homomorphism $\varphi : C_2 \to B$ such that

$$[\varphi] = \kappa^{(2)} \text{ and} \tag{e 19.13}$$

$$|\tau(\varphi(h)) - \gamma(\tau)(h)| < \min\{\gamma_1, \gamma_2, \gamma_3, \sigma\}/8 \tag{e 19.14}$$

for all $h \in H(\mathcal{H}_1) \cup H(\mathcal{H}_2) \cup \mathcal{H}_3$, where $\gamma : T(B) \to T_f(C_2)$ is induced by κ_T and the embedding j. In particular, φ_{*1} is injective on $H_{*1}(\mathbb{Z}^m)$ (since j_{*1} is injective on $H_{*1}(\mathbb{Z}^m)$ and κ_1 is an isomorphism).

Since C_2 is semiprojective, without loss of generality, we may assume $\varphi(C_2) \subset j_{1,\infty}(B_1) \subset B$. In what follows, we use the notation φ' for the homomorphism from C_2 to B_1 such that $\varphi = j_{1,\infty} \circ \varphi'$. We may also assume that, in the decomposition $B_1 = B_{1,0} \oplus B_{1,1}$,

$$\tau(1_{B_{1,0}}) < \min\{\sigma_0/4, \gamma_1/8, \gamma_2/8, \gamma_3/8, \sigma/8\} \quad \text{for all } \tau \in T(B). \tag{e 19.15}$$

We have $(\imath_{1,\infty} \circ \varphi')_{*1} = \varphi_{*1}$, and φ_{*1} is injective on $H_{*1}(\mathbb{Z}^m)$. Consequently $(\imath_{1,\infty})_{*1}$ is injective on $(\varphi')_{*1}(H_{*1}(\mathbb{Z}^m))$.

Let $G_1 = H^{\ddagger}(J_c^{(1)}(\mathbb{Z}^m)) \subset U(C_2)/CU(C_2)$ and $G_0 = H^{\ddagger}(J_c^{(1)}(G_{00}))$. Note, by the construction, $G_0 \subset U_0(C_2)/CU(C_2)$. Since $\kappa|_{K_1(B)}$ is an isomorphism and $(\imath_{1,\infty})_{*1}$ is injective on $(\varphi')_{*1}(H_{*1}(\mathbb{Z}^m))$, $\varphi^{\ddagger}|_{G_1}$ is injective. Let $\mathcal{H}_4 = P(\varphi(\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3))$, where $P : B_1 \to B_{1,1}$ is the projection.

Note that since $A \cong A \otimes Q$, $KL(B_1, A) = \mathrm{Hom}(K_*(B_1), K_*(A))$. Let $e_3 \in A_2$ be a projection such that $[e_3] = \kappa_0^{-1}([\imath_{1,\infty})(1_{B_{1,1}})])$, and let $A_3 = e_3Ae_3$. There is $\kappa'_{B_{1,1}} \in KK(B_{1,1}, A_2)^{++}$ such that $\kappa'_{B_{1,1}}$ induces $(\kappa_*^{-1} \circ (\imath_{1,\infty})_*)|_{K_*(B_{1,1})}$. As $K_i(B_{1,1})$ is finitely generated and $A \cong A \otimes Q$ (and $K_*(A_3) = K_*(A)$), by the UCT, the map $\kappa'_{B_{1,1}}$ is uniquely determined by $(\kappa_*^{-1} \circ (\imath_{1,\infty})_*)|_{K_*(B_{1,1})}$. It follows from the second part of Lemma 18.11 that there is a unital monomorphism $\psi_1 : B_{1,1} \to e_3Ae_3$ such that

$$[\psi_1] = \kappa'_{B_{1,1}} \tag{e 19.16}$$

and

$$\begin{aligned} &|\tau \circ \psi_1(g) - \gamma'(\tau)(\imath_{1,\infty}(g))| \\ &\quad < \min\{\tau(([e_3]))/4, \gamma_1/8, \gamma_2/8, \gamma_3/8, \sigma/8\} \text{ for all } g \in \mathcal{H}_4 \end{aligned} \tag{e 19.17}$$

for all $\tau \in T(A_3)$, where $\gamma' : T(A_3) \to T_f(B_{1,1})$ is the map induced by κ_T^{-1} and $\imath_{1,\infty}$.

Write $B_{1,0} = B_{1,0,1} \oplus B_{1,0,2}$, where $B_{1,0,1}$ is a finite direct sum of circle algebras and $K_1(B_{1,0,2})$ is finite (according to the definition of **H**—see 14.5). Since $B \cong B \otimes Q$, we may assume that $(\imath_{1,\infty})_{*1}|_{K_1(B_{1,0.2})} = \{0\}$.

Since $A_2 \cong A_2 \otimes Q$, by parts (iii) and (iv) of Theorem 18.7, there exists a unital homomorphism $\psi_2 : B_{1,0,2} \to e_4 A_2 e_4$ such that $[\psi_2]$ is equal to $\kappa'_{B_{1,0,2}}$, which induces $\kappa_*^{-1} \circ (\imath_{1,\infty})_*|_{K_*(B_{1,0,2})}$, where $e_4 \in A_2$ is a projection orthogonal to e_3 and $[e_4] = \kappa_0^{-1} \circ (\imath_{1,\infty})_{*0}([1_{B_{1,0,2}}])$. We have $(\psi_2)_{*1} = 0$, since $K_i(A)$ is torsion free. Let $\psi_3 : B_{1,1} \oplus B_{1,0,2} \to (e_3+e_4)A_2(e_3+e_4)$ be defined by $\psi_3(a,b) = \psi_1(a) \oplus \psi_2(b)$ for $a \in B_{1,1}$ and $b \in B_{1,0,2}$.

Let $P_1 : B_1 \to B_{1,0,1}$. Then, since $(\imath_{1,\infty})_{*1}|_{K_1(B_{1,0,2})} = \{0\}$, and $K_1(B_{1,1}) = \{0\}$, the restriction $(P_1)_{*1}|_{\varphi_{*1}(H_{*1}(\mathbb{Z}^m))}$ is injective. Also, $P_1^{\ddagger}$ is injective on $\varphi^{\ddagger} \circ H^{\ddagger}(J_c^{(1)}(\mathbb{Z}^m))$. Put $G_1' = P_1^{\ddagger} \circ \varphi^{\ddagger} \circ H^{\ddagger}(J_c^{(1)}(\mathbb{Z}^m)) \subset U(B_{1,0,1})/CU(B_{1,0,1})$. Then $G_1' \cong \mathbb{Z}^m$.

It follows from Theorem 18.7 that there is a unital homomorphism $\psi_4' : B_{1,0,1} \to (q_2 - e_3 - e_4)A_2(q_2 - e_3 - e_4)$ such that $[\psi_4'] = \kappa'_{B_{1,0,1}}$ which induces $(\kappa_*^{-1} \circ (\imath_{1,\infty})_*)|_{K_*(B_{1,0,1})}$. Let

$$z_i = P_1^{\ddagger} \circ \varphi'^{\ddagger} \circ H^{\ddagger}(\bar{v}_i) \ \text{ and } \ \xi_i = \psi_3^{\ddagger} \circ (\mathrm{id}_{B_1} - P_1)^{\ddagger} \circ \varphi'^{\ddagger} \circ H^{\ddagger}(\bar{v}_i),$$

$i = 1, 2, ..., m$. It should be noted that, since $(\psi_1)_{*1} = (\psi_2)_{*1} = 0$, $\xi_i \in U_0(A_2)/CU(A_2)$, $i = 1, 2, ..., m$. Moreover, since

$$(\psi_4')_{*1} \circ (P_1)_{*1} \circ \varphi'_{*1}(x) = j_{*1}(x) \ \text{ for all } \ x \in H_{*1}(\mathbb{Z}^m) \subset K_1(C_2),$$

$$\pi((j \circ H)^{\ddagger}(\bar{v}_i))\pi((\psi_4')^{\ddagger}(z_i))^{-1} = 0 \ \text{ in } \ K_1(A).$$

Define a homomorphism $\lambda : G_1' \to U(A_2)/CU(A_2)$ by

$$\lambda(z_i) = (j \circ H)^{\ddagger}(\bar{v}_i)((\psi_4')^{\ddagger}(z_i)\xi_i)^{-1}, \ \ i = 1, 2, ..., m. \tag{e 19.18}$$

Note that $\lambda(z_i) \in U_0(A)/CU(A)$, $i = 1, 2, ..., m$. By Lemma 11.5, the abelian group $U_0(A)/CU(A)$ is divisible. There exists a homomorphism

$$\bar{\lambda} : U(B_{1,0,1})/CU(B_{1,0,1}) \to U(A_2)/CU(A_2)$$

such that $\bar{\lambda}|_{G_1'} = \lambda$. Define a homomorphism $\lambda_1 : U(B_{1,0,1})/CU(B_{1,0,1}) \to U(A_2)/CU(A_2)$ by $\lambda_1(x) = (\psi_4')^{\ddagger}(x)\bar{\lambda}(x)$ for all $x \in U(B_{1,0,1})/CU(B_{1,0,1})$. By Theorem 11.10, the homomorphism

$$U((q_2 - e_3 - e_4)A_2(q_2 - e_3 - e_4))/CU((q_2 - e_3 - e_4)A_2(q_2 - e_3 - e_4)) \to U(A_2)/CU(A_2)$$

given by sending u to $\mathrm{diag}(u, e_3 + e_4)$ is an isomorphism. Since $B_{1,0,1}$ is a circle algebra, one easily obtains a unital homomorphism

$$\psi_4 : B_{1,0,1} \to (q_2 - e_3 - e_4)A_2(q_2 - e_3 - e_4)$$

such that

$$(\psi_4)_{*i} = (\psi_4')_{*i} = \kappa_i^{-1} \circ (\imath_{1,\infty})_{*i}|_{B_{1,0,1}} \ (i = 0, 1) \ \text{ and } \ \psi_4^{\ddagger} = \lambda_1.$$

Define $\psi : B_1 \to A_2$ by $\psi(a, b) = \psi_3(a) \oplus \psi_4(b)$ for $a \in B_{1,1} \oplus B_{1,0,2}$ and $b \in B_{1,0,1}$. Then, we have

$$\psi_{*i} = \kappa_i^{-1} \circ (\imath_{1,\infty})_{*i}, \ \ i = 0, 1. \tag{e 19.19}$$

Since $(\imath_{1,\infty} \circ \varphi')_{*i} = \kappa_i \circ j_{*i}$, $i = 0, 1$, we compute that

$$(\psi \circ \varphi' \circ H)_{*i} = \kappa_i^{-1} \circ (\imath_{1,\infty})_{*i} \circ (\varphi' \circ H)_{*i} = (j \circ H)_{*i}, \ \ i = 0, 1.$$

Thus, since $A \cong A \otimes Q$ and C_1 satisfies the UCT,

$$[\psi \circ \varphi' \circ H] = [j \circ H] \ \text{ in } \ KK(C_1, A). \tag{e 19.20}$$

On the other hand,

$$\begin{aligned} &\tau(\psi(1_{B_{1,0}})) \\ &= \ r_{A_2}(\tau)(\kappa_{*0}^{-1}((\imath_{1,\infty})_{*0}([1_{B_{1,0}}])) \\ &= \ r_B(\kappa_T(\tau))((\imath_{1,\infty})_{*0}([1_{B_{1,0}}])) < \sigma_0 \ \text{ for all } \ \tau \in T(A). \end{aligned}$$

By (e 19.9) and (e 19.15), and by 9.11,

$$[\psi(1_{B_{1,0}})] \le [e_2]. \tag{e 19.21}$$

By (e 19.14), (e 19.15), and (e 19.17),

$$\begin{aligned} &|\tau \circ \psi \circ \varphi'(g) - \tau(g)| \\ &< \ \min\{\sigma, \gamma_1, \gamma_2, \gamma_3, \sigma\} \ \text{ for all } \ g \in H(\mathcal{H}_1) \cup H(\mathcal{H}_2) \cup \mathcal{H}_3. \end{aligned}$$

Denote by ψ' the composition $\psi \circ \varphi'$. We have, for all $\tau \in T(A_2)$,

$$\begin{aligned} &|\tau \circ \psi'(g) - \tau(j(g))| < \min\{\gamma_1, \gamma_2, \gamma_4\} \\ &\text{for all } \ g \in H(\mathcal{H}_1) \cup H(\mathcal{H}_2). \end{aligned} \tag{e 19.22}$$

We then compute that

$$\tau \circ \psi'(H(h)) \ge \Delta(\hat{h})/2 \ \text{ for all } \ h \in \mathcal{H}_1, \ \text{ and } \ \tau \in T(A_2). \tag{e 19.23}$$

By (e 19.18) and the definition of λ_1, we have

$$\begin{aligned} &(\psi' \circ H)^{\ddagger}(\bar{v}_i) \\ &= \ \psi^{\ddagger} \circ (\varphi' \circ H)^{\ddagger}(\bar{v}_i) \\ &= \ \left(\psi_3^{\ddagger} \circ (\mathrm{id}_{B_1} - P_1)^{\ddagger} \circ (\varphi'^{\ddagger} \circ H^{\ddagger})(\bar{v}_i)\right) \left(\psi_4^{\ddagger}(P_1^{\ddagger}(\varphi'^{\ddagger} \circ H^{\ddagger}(\bar{v}_i))\right) \\ &= \ \xi_i \lambda_1(z_i) = (j \circ H)^{\ddagger}(\bar{v}_i), \ \ i = 1, 2, ..., m. \end{aligned}$$

By the choice of $\mathcal{H}_3$ and σ, we also have

$$\text{(e 19.24)} \qquad \mathrm{dist}((\psi')^{\ddagger}(H(\bar{v})), (j \circ H)^{\ddagger}(\bar{v})) < \gamma_2 \text{ for all } \bar{v} \in \mathcal{U}_0.$$

It follows from Theorem 12.7 and Corollary 12.8 (on using (e 19.20), (e 19.22), (e 19.23), and (e 19.24)) that there is a unitary $U \in q_2Aq_2$ such that

$$\text{(e 19.25)} \quad \|\mathrm{Ad}\, U \circ \psi'(H(y)) - j \circ H(y)\| < \varepsilon/16 \text{ for all } y \in \mathcal{F}_1.$$

Let $C_3 = \mathrm{Ad}\, U \circ \psi(B_{1,1})$ and $p = 1_{C_3} = \mathrm{Ad}\, U \circ \psi(1_{B_{1,1}})$. Since $1_{B_{1,1}}$ is in the center of B_1, for any $y \in \mathcal{F}_1$, $\varphi'(y)1_{B_{1,1}} = 1_{B_{1,1},}\varphi'(y)$. Therefore, $\mathrm{Ad}\, U \circ \psi'(y)p = p\mathrm{Ad}\, U \circ \psi \circ \varphi'(y) = p\mathrm{Ad}\, U \circ \psi'(y)$. Note that, for any $y \in F_1$, we have

$$\|y - j \circ H(y) \oplus (1 - q_2)y(1 - q_2)\| < \varepsilon/8,$$

and $p < q_2$. Hence,

$$\|py - yp\| \le \|py - p\mathrm{Ad}\, U \circ \psi'(y)\| + \|p\mathrm{Ad}\, U \circ \psi'(y) - yp\| < \varepsilon/8 + \varepsilon/8 = \varepsilon/4$$

for all $y \in \mathcal{F}_1$. Combining this with (e 19.5), we have

$$\|px - xp\| = \|pq_2x - xq_2p\| < 2\varepsilon/16 + \|pq_2xq_2 - q_2xq_2p\| < \varepsilon \text{ for all } x \in \mathcal{F}.$$

Let $x \in \mathcal{F}$. Choose $y \in \mathcal{F}_1$ such that $\|q_2xq_2 - q_2yq_2\| = \|q_2(q_1xq_1 - y)q_2\| < \varepsilon/16$. Then, by (e 19.6) and (e 19.25),

$$\begin{aligned} &\|pxp - p(\mathrm{Ad}\, U \circ \psi'(H(y)))p\| \\ &\le \quad \|pxp - pq_2xq_2p\| + \|pq_2xq_2p - pq_2yq_2p\| \\ &\quad + \|pq_2yq_2p - p(j \circ H(y))p\| \\ &\quad + \|p(j \circ H(y))p - p(\mathrm{Ad}\, U \circ \psi'(H(y)))p\| \\ &< \quad 0 + \varepsilon/16 + \varepsilon/16 + \varepsilon/16 = 3\varepsilon/8. \end{aligned}$$

However, $p\mathrm{Ad}\, U \circ \psi'(y)p = \mathrm{Ad}\, U \circ (\psi(1_{B_{1,1}}\varphi'(y)1_{B_{1,1}})) \in C_3$ for all $y \in \mathcal{F}_1$. Therefore,

$$pxp \in_{\varepsilon} C_3.$$

We then estimate that, by (e 19.8) and by (e 19.21),

$$[1 - p] \le [1 - q_2] + [\psi'(1_{B_{1,0}})] \le [e_0 \oplus e_1 \oplus e_2] \le [a].$$

Since $B_{1,1} \in \mathcal{C}_0$, by applying 3.20, C_3 can be approximated by C^*-subalgebras in $\mathcal{C}_0$. It follows that $A \in \mathcal{B}_0$. □

COROLLARY 19.3. *If A is a unital separable amenable simple C^*-algebra such that $A \otimes Q \in \mathcal{B}_1$, then, for any infinite dimensional UHF-algebra U, $A \otimes U \in \mathcal{B}_0$.*

PROOF. It follows from Theorem 19.2 that $A \otimes Q \in \mathcal{B}_0$. Then, by Lemma 3.20 and by Theorem 3.4 of [87], $A \otimes U \in \mathcal{B}_0$ for every infinite dimensional UHF-algebra. □

20. **KK-attainability of the C*-algebras in $\mathcal{B}_0$** The main purpose of this section is to establish Theorem 20.16. It is an existence theorem for maps from an algebra in the sub-class $\mathcal{B}_0$ to a C^*-algebra as in Theorem 14.10 (and elsewhere). The construction is similar to that of Section 2 of [67], and, roughly, we will construct a map factoring through a C^*-subalgebra (in $\mathcal{C}_0$) of the given C^*-algebra in $\mathcal{B}_0$, and also require this map to carry the given KL-element. But since the ordered K_0-group of a C^*-algebra in $\mathcal{C}_0$ in general is not a Riesz group, extra work has to be done to take care of this difficulty (similar work also appears in [91], [92], and [93]).

20.1. Let us proceed as in Section 2 of [67]. Let $A \in \mathcal{B}_0$ be a separable C^*-algebra and assume that A has the property (SP). By Lemma 9.8, the C^*-algebra A can be embedded as a C*-subalgebra of $\prod M_{n_k}/\bigoplus M_{n_k}$ for some sequence $\{n_k\}$; in other words, A is MF in the sense of Blackadar and Kirchberg (Theorem 3.2.2 of [6]). Since A is assumed to be amenable, by Theorem 5.2.2 of [6], the C^*-algebra A is strong NF, and hence, by Proposition 6.1.6 of [6], there is an increasing family of residually finite dimensional sub-C^*-algebras $\{A_n\}$ with union dense in A. Let us assume that $1 \in A_1$.

Let us set up the initial stage: Choose a dense sequence $\{x_0, x_1, x_2, ..., x_n, ...\}$ of elements in the unit ball of A. Let $\mathcal{P}_0 \subset M_\infty(A)$ be a finite subset of projections. We assume that $x_0 \in A_1$ and $\mathcal{P}_0 \subset M_\infty(A_1)$. Choose a finite subset $\mathcal{F}_0$ in the unit ball of $A_1 \subset A$ with $\{1, x_0\} \subset \mathcal{F}_0$, and $\eta_0 > 0$ such that $[L']|_{\mathcal{P}_0}$ is well defined for any $\mathcal{F}_0$-$2\eta_0$-multiplicative contractive completely positive linear map L' from A. Moreover, by choosing even larger $\mathcal{F}_0$ and smaller η_0, we may assume that

$$[L']|_{\mathcal{P}_0} = [L'']|_{\mathcal{P}_0}, \tag{e 20.1}$$

if $\|L'(x) - L''(x)\| < 2\eta_0$ for all $x \in \mathcal{F}_0$ and L'' is also an $\mathcal{F}_0$-$2\eta_0$-multiplicative contractive completely positive linear map. Since A_1 is a RFD algebra, one can choose a homomorphism h_0 from A_1 to a finite-dimensional C^*-algebra F_0 which is non-zero on $\mathcal{F}_0$.

Let $\delta_0 < \min\{\eta_0/2, 1/2\}$.

Since A is assumed to have the property (SP), by Lemma 2.1 of [90], there is a non-zero homomorphism $h' : F_0 \to A$ with $h'(1_{F_0}) = e_0 \in A$ such that

(1) $||e_0 x - x e_0|| < \delta_0/256$ and
(2) $||h' \circ h_0(x) - e_0 x e_0|| < \delta_0/256$ for all $x \in \mathcal{F}_0$.

Since F_0 has finite dimension, it follows from Arveson's Extension Theorem that the homomorphism $h_0 : A_1 \to F_0$ can be extended to a contractive completely positive linear map from A to F_0; let us still denote this by h_0.

Put $H = h' \circ h_0 : A \to A$. Note that $e_0 = H(1)$. Since the hereditary C^*-subalgebra $(1 - e_0)A(1 - e_0)$ is in the class $\mathcal{B}_0$ again (see Theorem 9.9), there is a projection $q_1' \leq 1 - e_0$ and a C^*-subalgebra $S_1' \in \mathcal{C}_0$ (of $(1 - e_0)A(1 - e_0)$) with $1_{S_1'} = q_1'$ such that

(3) $||q_1'x - xq_1'|| < \delta_0/256$ for any $x \in (1-e_0)\mathcal{F}_0(1-e_0)$,
(4) $\mathrm{dist}(q_1'xq_1', S_1') < \delta_0/256$ for any $x \in (1-e_0)\mathcal{F}_0(1-e_0)$, and
(5) $\tau(1-e_0-q_1') < 1/16$ for any tracial state τ on A.

Put $q_1 = q_1' + e_0$ and $S_1 = S_1' \oplus h'(F_0)$. One has

(6) $||q_1x - xq_1|| < \delta_0/64$ for any $x \in \mathcal{F}_0$,
(7) $\mathrm{dist}(q_1xq_1, S_1) < \delta_0/64$ for any $x \in \mathcal{F}_0$, and
(8) $\tau(1-q_1) = \tau(1-q_1'-e_0) < 1/16$ for any tracial state τ on A.

Let $\bar{\mathcal{F}}_0 \subset S_1$ be a finite subset such that $\mathrm{dist}(q_1yq_1, \bar{\mathcal{F}}_0) < \delta_0/64$ for all $y \in \mathcal{F}_0$. Let $\mathcal{G}_1$ be a finite generating subset of S_1 (see 3.1) which is in the unit ball.

Since S_1 is amenable, by Theorem 2.3.13 of [63], there is a contractive completely positive linear map $L_0' : q_1Aq_1 \to S_1$ such that

$$\|L_0'(s) - s\| < \delta_0/256 \ \text{ for all } \ s \in \mathcal{G}_1 \cup \bar{\mathcal{F}}_0. \tag{e 20.2}$$

Set $L_0(a) = L_0'(q_1aq_1)$ for any $a \in A$. Then L_0 is a completely positive contraction from A to S_1 such that

$$\|L_0(s) - s\| < \delta_0/128 \ \text{ for all } \ s \in \mathcal{G}_1 \cup \bar{\mathcal{F}}_0. \tag{e 20.3}$$

Consequently, by the definition of $\bar{\mathcal{F}}_0$ we have

$$\|L_0(s) - s\| < \frac{\delta_0}{64} + \frac{\delta_0}{128} + \frac{\delta_0}{64} < \frac{\delta_0}{16} \quad \text{for all } \ s \in \mathcal{G}_1 \cup (q_1\mathcal{F}_0q_1). \tag{e 20.4}$$

It follows that L_0 is $\mathcal{F}_0 \cup \mathcal{G}_1$-$\delta_0/16$-multiplicative. This ends the initial stage of the proof. Let us now prove the following claim.

Claim: There exist an increasing sequence of finite subsets $\{\mathcal{F}_n\}_{n=0}^{\infty}$ of the unit ball of A, starting with $\mathcal{F}_0$ above, with $\mathcal{F}_n \supset \{x_1, x_2, \cdots, x_n\}$ (consequently, $\bigcup_{k=0}^{\infty} \mathcal{F}_k$ is dense in the unit ball of A), two decreasing sequences of positive numbers $\{\eta_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ starting with η_0, δ_0 above, finite generating subsets $\mathcal{G}_n \subset S_n$ (in the unit ball of S_n), starting with $\mathcal{G}_1$ (see the end of 3.1), a sequence of homomorphisms $h_{n-1} : S_{n-1} \to S_n$ (for $n \geq 2$), and a sequence of $\mathcal{F}_{n-1} \cup \mathcal{G}_n$-$\delta_{n-1}/2$-multiplicative contractive completely positive linear maps $L_{n-1} : A \to S_n$ (for $n \geq 1$) such that:

$$||q_nx - xq_n|| < \delta_{n-1}/64 \ \text{ for all } \ x \in \mathcal{F}_{n-1}; \tag{e 20.5}$$

$$\mathrm{dist}(q_nxq_n, S_n) < \delta_{n-1}/64, \quad \text{for all } \ x \in \mathcal{F}_{n-1}; \tag{e 20.6}$$

$$\tau(1-q_n) < 1/2^{n+1} \ \text{ for all } \ \tau \in T(A); \tag{e 20.7}$$

$$\mathcal{G}_n \subset \mathcal{F}_n, \ \text{ for } n \geq 1 \text{ (recall } \mathcal{G}_n \text{ generates } S_n\text{)}; \tag{e 20.8}$$

$$||L_{n-1}(a) - h_{n-1}(a)|| < \delta_{n-2}/64 \ \text{ for all } \ n \geq 2, \quad \text{for all } a \in L_{n-2}(\mathcal{F}_{n-2}) \cup \mathcal{G}_{n-1}; \tag{e 20.9}$$

$$L_{n-1}(a) = L_{n-1}(q_n a q_n) \text{ for all } a \in A; \tag{e 20.10}$$

$$\|L_{n-1}(a) - a\| < \delta_{n-1}/16 \text{ for all } a \in \mathcal{G}_n \cup q_n \mathcal{F}_{n-1} q_n; \tag{e 20.11}$$

$$\delta_n < \eta_n/2, \quad \delta_n < \delta_{n-1}/2; \tag{e 20.12}$$

$$\|\big((1-q_n)x(1-q_n) + L_{n-1}(x)\big) - x\| < \delta_{n-1}/8, \quad \text{for all } x \in \mathcal{F}_{n-1}; \tag{e 20.13}$$

$$[L_{n-1}]|_{(\iota_{n-1})_{*0}(K_0(S_{n-1}))} \text{ is well defined;} \tag{e 20.14}$$

if $\tilde{L} : A \to A$ is an $\mathcal{F}_{n-1}$-$2\eta_{n-1}$ multiplicative contractive completely positive linear map with $\|\tilde{L}(x) - L_{n-1}(x)\| < 2\eta_{n-1}$ for all $x \in \mathcal{F}_{n-1}$, then

$$[\iota_n \circ L_{n-1}]|_{(\iota_{n-1})_{*0}(K_0(S_{n-1}))} = [\tilde{L}]|_{(\iota_{n-1})_{*0}(K_0(S_{n-1}))}; \tag{e 20.15}$$

if $L : S_n \to B$ ($n \geq 1$, B is any C^*-algebra,) is a $\mathcal{G}_n$-$2\eta_n$ -multiplicative contractive completely positive linear map, there exists a homomorphism $h : S_n \to B$ such that

$$\|L(a) - h(a)\| < \delta_{n-1}/64 \text{ for all } a \in \mathcal{G}_n \cup L_{n-1}(\mathcal{F}_{n-1}); \tag{e 20.16}$$

if $L' : A \to B$ is an $\mathcal{F}_n$-$2\eta_n$ -multiplicative contractive completely positive linear map, then

$$[L']|_{(\iota_n)_{*0}(K_0(S_n))} \text{ is well defined;} \tag{e 20.17}$$

and, if $L'' : A \to B$ is another $\mathcal{F}_n$-$2\eta_n$ multiplicative contractive completely positive linear map with $\|L'(x) - L''(x)\| < 2\eta_n$ for all $x \in \mathcal{F}_n$, then

$$[L']|_{(\iota_n)_{*0}(K_0(S_n))} = [L'']|_{(\iota_n)_{*0}(K_0(S_n))}. \tag{e 20.18}$$

We have constructed $\mathcal{F}_0$ η_0, δ_0, q_1, S_1 (with $1_{S_1} = q_1$), a finite generating set $\mathcal{G}_1 \subset S_1$, and $L_0 : A \to S_1$ in the initial stage. Note that (e 20.5)–(e 20.7), (e 20.10), (e 20.11), and (e 20.14) for $n = 1$ follow from our construction (see (6), (7), (8), and (e 20.4)). To begin the induction, let us first define η_1, δ_1, and the subset $\mathcal{F}_1$ (L_1 and h_1 should be defined after we define q_2 and S_2, so it is not necessary to define them here). Since S_1 is semiprojective (see the end of 3.1), there exists a positive number $\eta_1 < \eta_0/2$ satisfying the following condition: for any $\mathcal{G}_1$-$2\eta_1$-multiplicative contractive completely positive linear map L from S_1 to another C^*-algebra B (recall that $\mathcal{G}_1$ is a generating subset of S_1), there exists a homomorphism $h : S_1 \to B$ such that

$$\|L(a) - h(a)\| < \delta_0/64 \text{ for all } a \in \mathcal{G}_1 \cup L_0(\mathcal{F}_0). \tag{e 20.19}$$

Moreover, since $K_0(S_1)$ is finitely generated, we may assume that there is a finite subset $\mathcal{F}_1'$ such that $[L']|_{(\iota_1)_{*0}(K_0(S_1))}$ is well defined, if L' is an $\mathcal{F}_1'$-$2\eta_1$-multiplicative contractive completely positive linear map from A. Furthermore,

$$[L']|_{(\iota_1)_{*0}(K_0(S_1))} = [L'']|_{(\iota_1)_{*0}(K_0(S_1))} \tag{e 20.20}$$

if $\|L'(x) - L''(x)\| < 2\eta_1$ for all $x \in \mathcal{F}_1'$ and L'' is also an $\mathcal{F}_1'$-$2\eta_1$-multiplicative contractive completely positive linear map from A. Let $\delta_1 < \min\{\eta_1/2, \delta_0/2\}$, and $\mathcal{F}_1 = \mathcal{F}_0 \cup \mathcal{F}_1' \cup \mathcal{G}_1 \cup L_0(\mathcal{F}_0) \cup \{x_0, x_1\}$. Then (e 20.8), (e 20.12), (e 20.16), (e 20.17), and (e 20.18) hold for $n = 1$. (Note that (e 20.10), (e 20.14), and (e 20.15) only make sense for $n \geq 2$.)

For $k \geq 2$, suppose that we have already constructed $\mathcal{F}_{k-1} \subset A$ (and all $\mathcal{F}_i$ with $1 \leq i < k-1$), $S_{k-1} \in \mathcal{C}_0$ (and all S_i with $1 \leq i < k-1$), a finite generating subset $\mathcal{G}_{k-1}$ of S_{k-1} (and all $\mathcal{G}_i$ with $1 \leq i < k-1$), η_{k-1}, δ_{k-1} (and all η_i, δ_i with $1 \leq i < k-1$), and a homomorphism $h_{k-2} : S_{k-2} \to S_{k-1}$ if $k \geq 3$ (and all h_i with $1 \leq i < k-2$), a completely positive linear contraction $L_{k-2} : A \to S_{k-1}$ (and all L_i with $0 \leq i < k-2$) as described in (e 20.5)–(e 20.18) with $k-1$ in place of n.

Since $A \in \mathcal{B}_0$, for the k above, there exist a projection $q_k \in A$ and a C^*-subalgebra $S_k \in \mathcal{C}_0$ with $1_{S_k} = q_k$ such that

$$\|q_k x - x q_k\| < \delta_{k-1}/64 \text{ for all } x \in \mathcal{F}_{k-1}, \tag{e 20.21}$$

$$\operatorname{dist}(q_k x q_k, S_k) < \delta_{k-1}/64 \text{ for all } x \in \mathcal{F}_{k-1}, \text{ and} \tag{e 20.22}$$

$$\tau(1 - q_k) < 1/2^{k+1} \text{ for all } \tau \in T(A). \tag{e 20.23}$$

Hence (e 20.5),(e 20.6), and (e 20.7) hold for $n = k$.

Let $\mathcal{G}_k \subset S_k$ be a finite generating subset in the unit ball of $\mathcal{G}_k$.

Define $\Lambda_k : A \to q_k A q_k$ by $\Lambda_k(a) = q_k a q_k$ for all $a \in A$. Then, by (e 20.21), Λ_k is $\mathcal{F}_{k-1}$-$\delta_{k-1}/64$-multiplicative. Let $\bar{\mathcal{F}}_{k-1} \subset S_k$ be a finite subset such that $\operatorname{dist}(q_k x q_k, \bar{\mathcal{F}}_{k-1}) < \delta_k/64$ for all $x \in \mathcal{F}_{k-1}$.

Since S_k is amenable, by Theorem 2.3.12 of [63], there exists a unital completely positive linear map $L_k' : q_k A q_k \to S_k$ such that

$$\|L_k'(a) - a\| < \delta_{k-1}/64 \text{ for all } a \in \mathcal{G}_k \cup \bar{\mathcal{F}}_{k-1}. \tag{e 20.24}$$

Let $L_{k-1} : A \to S_k$ be defined by $L_{k-1} = L_k' \circ \Lambda_k$. Evidently, (e 20.10) holds for k in place of n. Also,

$$\|L_{k-1}(a) - a\| < \delta_{k-1}/64 \text{ for all } a \in \mathcal{G}_k \cup \bar{\mathcal{F}}_{k-1}. \tag{e 20.25}$$

Since $\operatorname{dist}(a, \bar{\mathcal{F}}_{k-1}) < \delta_{k-1}/64$ for each $a \in q_k \mathcal{F}_{k-1} q_k$, we have

$$\|L_{k-1}(a) - a\| < \delta_{k-1}/16 \text{ for all } a \in \mathcal{G}_k \cup q_k \mathcal{F}_{k-1} q_k, \tag{e 20.26}$$

which is (e 20.11) for k in place of n. Combining (e 20.21) and (e 20.26), we conclude that

$$\|\big((1-q_k)x(1-q_k)+L_{k-1}(x)\big)-x\|<\delta_{k-1}/8, \text{ for all } x\in\mathcal{F}_{k-1},$$

which is (e 20.14) for $n=k$, and, further, L_{k-1} is $\mathcal{F}_{k-1}$-δ_{k-1}-multiplicative, and hence $\mathcal{F}_{k-1}$-η_{k-1}-multiplicative as $\delta_{k-1}<\eta_{k-1}$. By (e 20.17), and (e 20.18) for $n=k-1$, (e 20.14) and (e 20.15) hold for $n=k$. By (e 20.8) for $n=k-1$, we have $\mathcal{G}_{k-1}\subset\mathcal{F}_{k-1}$. Then by (e 20.16) for $n=k-1$, there is a homomorphism $h_{k-1}:S_{k-1}\to S_k$ such that

$$||L_{k-1}(a)-h_{k-1}(a)||<\delta_{k-2}/64 \tag{e 20.27}$$

for all $a\in L_{k-2}(\mathcal{F}_{k-2})\cup\mathcal{G}_{k-1}$, which is (e 20.10) for $n=k$.

Since S_k is semiprojective (see the end of 3.1), there exists a positive number $\eta_k<\eta_{k-1}/2<\eta_0/2$ satisfying the following condition: for any $\mathcal{G}_k$-$2\eta_k$-multiplicative contractive completely positive linear map L from S_k to another C^*-algebra B (recall that $\mathcal{G}_k$ is a generating subset of S_k), there exists a homomorphism $h:S_k\to B$ such that

$$\|L(a)-h(a)\|<\delta_{k-1}/64 \text{ for all } a\in\mathcal{G}_k\cup L_{k-1}(\mathcal{F}_{k-1}), \tag{e 20.28}$$

which is (e 20.16) for $n=k$.

Moreover, since $K_0(S_k)$ is finitely generated, we may assume that there is a finite subset $\mathcal{F}'_k$ such that $[L']|_{(\iota_k)_{*0}(K_0(S_k))}$ is well defined, if L' is an $\mathcal{F}'_k$-$2\eta_k$-multiplicative contractive completely positive linear map from A. That is, (e 20.17) holds for k in place of n, provided $\mathcal{F}'_k\subset\mathcal{F}_k$. Furthermore,

$$[L']|_{(\iota_k)_{*0}(K_0(S_k))}=[L'']|_{(\iota_k)_{*0}(K_0(S_k))} \tag{e 20.29}$$

if $\|L'(x)-L''(x)\|<2\eta_k$ for all $x\in\mathcal{F}'_k$ and L'' is also an $\mathcal{F}'_k$-$2\eta_k$-multiplicative contractive completely positive linear map from A. In other words, (e 20.18) holds for k in place of n, provided $\mathcal{F}'_k\subset\mathcal{F}_k$. Let $\mathcal{F}_k=\mathcal{F}_{k-1}\cup\mathcal{F}'_k\cup\mathcal{G}_k\cup L_{k-1}(\mathcal{F}_{k-1})\cup\{x_0,x_1,\cdots,x_k\}$. Then (e 20.8), (e 20.17), and (e 20.18) hold for $n=k$. Let $\delta_k<\min\{\eta_k/2,\delta_{k-1}/2\}$. Then (e 20.12) holds for $n=k$. This completes the proof of the claim.

20.2. Let $\Psi_n:A\to(1-q_{n+1})A(1-q_{n+1})$ denote the cutting-down map sending a to $(1-q_{n+1})a(1-q_{n+1})$, and let $J_n:A\to A$ denote the map sending a to $\Psi_n(a)\oplus L_n(a)$. By (e 20.14), the maps Ψ_n and J_n are $\mathcal{F}_n$-$\delta_n/2$-multiplicative. Set $J_{m,n}=J_{n-1}\circ\cdots\circ J_m$ and $h_{m,n}=h_{n-1}\circ\cdots\circ h_m:S_m\to S_n$. Again by (e 20.14), $J_{m,n}$ is $\mathcal{F}_m$-δ_m-multiplicative and $\|J_{m,n}(x)-x\|<\delta_m$ for all $x\in\mathcal{F}_m$. We shall also use $L_n,\Psi_n,J_n,J_{m,n},h_m$, and $h_{m,n}$ for the extensions of these maps to a matrix algebra over A.

By (e 20.12), (e 20.17), and (e 20.18), $J_{m,n}$ induces a well-defined map $[J_{m,n}|_{\mathcal{P}_0\cup(\iota_k)_{*0}(K_0(S_m))}]$, which agrees with the identity map.

Using the same argument as that of Lemma 2.7 of [67], one has the following lemma.

LEMMA 20.3 (Lemma 2.7 of [67]). *Let $\mathcal{P} \subset M_m(A)$ be a finite set of projections. Assume that $\mathcal{F}_0$ (in 20.1) is sufficiently large and δ_0 (in 20.1) is sufficiently small that $[L_n \circ J_{1,n}]|_{\mathcal{P}}$ and $[L_n \circ J_{1,n}]|_{G_0}$ are well defined, where G_0 is the subgroup of $K_0(A)$ generated by $\mathcal{P}$. Then*

$$\lim_{n\to\infty} \sup_{\tau\in T(A)} |\tau([\iota_{n+1} \circ L_n \circ J_{1,n}]([p])) - \tau([p])| = 0$$

for any $p \in \mathcal{P}$.

Furthermore, we have

$$|\tau(h_{k,k+n+1} \circ [L_{k-1}]([p])) - \tau(h_{k,k+n} \circ [L_{k-1}]([p])| < (1/2)^{n+k} \text{ and} \tag{e 20.30}$$

$$\tau(h_{k,k+n} \circ [L_{k-1}]([p])) \geq (1 - \sum_{i=1}^{n} 1/2^{i+k})\tau([L_{k-1}]([p])) > 0 \tag{e 20.31}$$

for all $p \in \mathcal{P}$ and for all $\tau \in T(A)$, and

$$\tau(h_{k,k+n}([q])) \geq (1 - \sum_{i=1}^{n} 1/2^{i+k})\tau([q]) > 0 \text{ for all } \tau \in T(A) \tag{e 20.32}$$

and for all $q \in K_0(S_k)_+ \setminus \{0\}$.

REMARK 20.4. Since A is stably finite and assumed to be amenable, and therefore exact, any state of $K_0(A)$ is the restriction of a tracial state of A ([8] and [51]). Thus, the lemma above still holds if one replaces the trace τ by any state τ_0 on $K_0(A)$.

20.5. Fix a finite subset $\mathcal{P}$ of projections of $M_r(A)$ (for some $r \geq 1$) and an integer $N \geq 1$ such that $[L_{N+i}]|_{\mathcal{P}}$, $[J_{N+i}]|_{\mathcal{P}}$ and $[\Psi_{N+i}]|_{\mathcal{P}}$ are all well defined. Keep the notation of 20.2. Then, on $\mathcal{P}$,

$$\begin{aligned} [L_{N+1} \circ J_N] &= [L_{N+1} \circ L_N] \oplus [L_{N+1} \circ \Psi_N] \\ &= [h_{N+1} \circ L_N] \oplus [L_{N+1} \circ \Psi_N], \text{ and} \end{aligned}$$

$$\begin{aligned} &[L_{N+2} \circ J_{N,N+2}] \\ &= [L_{N+2} \circ L_{N+1} \circ J_N] \oplus [L_{N+2} \circ \Psi_{N+1} \circ J_N] \\ &= [L_{N+2} \circ L_{N+1} \circ L_N] \oplus [L_{N+2} \circ L_{N+1} \circ \Psi_N] \\ &\quad \oplus [L_{N+2} \circ \Psi_{N+1} \circ J_N] \\ &= [h_{N+1,N+3}] \circ [L_N] \oplus [L_{N+2} \circ L_{N+1} \circ \Psi_N] \\ &\quad \oplus [L_{N+2} \circ \Psi_{N+1} \circ J_N]. \end{aligned}$$

Moreover, on $\mathcal{P}$,

$$\begin{aligned}
[L_{N+n}\circ J_{N,N+n}] \quad = \quad & [h_{N+1,N+n+1}]\circ[L_N]\oplus[L_{N+n}\circ\Psi_{N+n-1}\circ J_{N,N+n-1}]\\
&\oplus[L_{N+n}\circ L_{N+n-1}\circ\Psi_{N+n-2}\circ J_{N,N+n-2}]\\
&\oplus[L_{N+n}\circ L_{N+n-1}\circ L_{N+n-2}\circ\Psi_{N+n-3}\circ J_{N,N+n-3}]\\
&\oplus\cdots\oplus[L_{N+n}\circ L_{N+n-1}\circ\cdots\circ L_{N+2}\circ\Psi_{N+1}\circ J_N]\\
&\oplus[L_{N+n}\circ L_{N+n-1}\circ\cdots\circ L_{N+1}\circ\Psi_N].
\end{aligned}$$

Set $\psi_N^N = L_N$, $\psi_{N+1}^N = L_{N+1}\circ\Psi_N$, $\psi_{N+2}^N = L_{N+2}\circ\Psi_{N+1}, ...,$ and $\psi_{N+n}^N = L_{N+n}\circ\Psi_{N+n-1}$, $n=1,2,....$

(Note that $\psi_{N+i}^N=\psi_{N+i}^{N+1}=\cdots=\psi_{N+i}^{N+i-1}$. We insist on the notation ψ_{N+i}^N in order to emphasize that our estimation starts with a fixed index N, and goes on to $N+1$, $N+2$, and so on, and we only change the subscripts and keep the superscript the same to emphasize our beginning index N (see Corollary 20.8 below).)

20.6. (a) For each S_n, since the abelian group $K_0(S_n)$ is finitely generated and torsion free, there is a set of free generators $\{e_1^n, e_2^n, ..., e_{l_n}^n\}\subset K_0(S_n)$. By Theorem 3.15, the positive cone of the ordered group $K_0(S_n)$ is finitely generated; denote the (semigroup) generators (exactly the the minimal non-zero positive elements) by $\{s_1^n, s_2^n, ..., s_{r_n}^n\}\subset K_0(S_n)_+\backslash\{0\}$. Then there is an $r_n\times l_n$ integer-valued matrix R_n' such that

$$\vec{s}_n = R_n'\vec{e}_n,$$

where $\vec{s}_n=(s_1^n, s_2^n, ..., s_{r_n}^n)^{\mathrm{T}}$ and $\vec{e}_n=(e_1^n, e_2^n, ..., e_{l_n}^n)^{\mathrm{T}}$. In particular, for any ordered group H, and any elements $h_1, h_2, ..., h_{l_n}\in H$, the map $e_i^n\mapsto h_i$, $i=1, ..., l_n$, induces an abelian-group homomorphism $\varphi: K_0(S_n)\to H$, and the map φ is positive (or strictly positive) if and only if

$$R_n'\vec{h}\in H_+^{r_n}\quad(\text{or } R_n'\vec{h}\in(H_+\setminus\{0\})^{r_n}),$$

where $\vec{h}=(h_1, h_2, ..., h_{l_n})^T\in H^{l_n}$. Moreover, for each e_i^n, write it as $e_i^n=(e_i^n)_+-(e_i^n)_-$ with $(e_i^n)_+, (e_i^n)_-\in K_0(S_n)_+$ and fix this decomposition. Define a $r_n\times 2l_n$ matrix

$$R_n = R_n'\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & -1\end{pmatrix}_{l_n\times l_{2n}}.$$

Then one has

$$\vec{s}_n = R_n\vec{e}_{n,\pm},$$

where $\vec{e}_{n,\pm} = ((e_1^n)_+, (e_1^n)_-, ..., (e_{l_n}^n)_+, (e_{l_n}^n)_-)^{\mathrm{T}}$. Thus, for any ordered group H, and any elements $h_{1,+}, h_{1,-}, ..., h_{l_n,+}, h_{l_n,-} \in H$, the map $e_i^n \mapsto (h_{i,+} - h_{i,-})$, $i = 1, ..., l_n$, induces a positive (or strictly positive) homomorphism if and only if

$$R_n \vec{h}_\pm \in H_+^{r_n} \quad (\text{or } R_n \vec{h}_\pm \in (H_+ \setminus \{0\})^{r_n}) \ ,$$

where $\vec{h}_\pm = (h_{1,+}, h_{1,-}, ..., h_{l_n,+}, h_{l_n,-})^T \in H^{l_n}$.

(b) Let A be as above. Let B be an inductive limit C^*-algebra as in Theorem 14.10 such that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Let $\alpha \in KL(A, B)$ be an element which implements the isomorphism above.

Let $e_i^n, e_{i,\pm}^n \in K_0(S_n)$ be as above, $i = 1, 2, ..., l_n$. Let $s(0) = 0, s(n) = \sum_{i=1}^n 2l_i$. Put

$$\alpha(\imath_n \circ h_{j,n}(e_{i,+}^j)) = g_{s(j-1)+2i-1}^{(n)}, \quad \alpha(\imath_n \circ h_{j,n}(e_{i,-}^j)) = g_{s(j-1)+2i}^{(n)},$$

and

$$g_l^{(n)} = 0 \text{ if } l > s(n).$$

Let $a_j^{(n)} = \rho_B(g_j^{(n)}) \in \mathrm{Aff}(T(B))_+$ for $j = 1, 2, ...$ Then, by Lemma 20.3, $\lim_{n\to\infty} a_j^n = a_j > 0$ uniformly on $T(B)$.

For $j \in \{1, 2, ..., n\}$, let $(s_1^j, s_2^j, ..., s_{r_j}^j) \in K_0(S_j)_+ \setminus \{0\}$ be the generators of the positive cone $K_0(S_j)_+$, and let R_j be the $r_j \times 2l_j$ matrix as in part (a). Then

$$\begin{aligned} & R_j(g_{s(j-1)+1}^{(n)}, g_{s(j-1)+2}^{(n)}, ..., g_{s(j)}^{(n)})^T \\ & \quad = \quad (\alpha(\imath_n \circ h_{j,n}(s_1^j)), \alpha(\imath_n \circ h_{j,n}(s_2^j)), ..., \alpha(\imath_n \circ h_{j,n}(s_{r_j}^j)))^T \end{aligned}$$

and

$$\begin{aligned} & R_j(a_{s(j-1)+1}, a_{s(j-1)+2}, ..., a_{s(j)})^T \\ & \quad = \quad \lim_{n\to\infty} (\rho_B(\alpha(\imath_n \circ h_{j,n}(s_1^j))), \rho_B(\alpha(\imath_n \circ h_{j,n}(s_2^j))), ..., \rho_B(\alpha(\imath_n \circ h_{j,n}(s_{r_j}^j))))^T. \end{aligned}$$

Note that for each $\tau \in T(B)$, $\tau \circ \rho_B \circ \alpha$ is a state on $K_0(A)$, which (by [8] and [51]) can be extended to a trace on A. From Lemma 20.3 ((e 20.32)), for all $1 \le j \le n$, and for all $\tau \in T(B)$,

$$\tau(\rho_B(\alpha(\imath_n \circ h_{j,n}(s_i^j)))) > (1 - \sum_{k=1}^{\infty} 1/2^{j+k})\tau(\rho_B(\alpha(\imath_j(s_i^j)))) > 0,$$
$$\text{for all } i \in \{1, 2, ..., r_j\}.$$

Hence each entry of $R_j(a_{s(j-1)+1}, a_{s(j-1)+2}, ..., a_{s(j)})^T$ is a strictly positive element of $\mathrm{Aff}(T(B))^{++}$. Let $\bar{R}_n = \mathrm{diag}(R_1, R_2, ..., R_n)$. Then

$$0 \ll \bar{R}_n(a_1, a_2, ..., a_{s(n)})^T \in (\mathrm{Aff}(T(B)))^{\sum_{i=1}^n r_k},$$

i.e., each coordinate is strictly positive on $T(B)$.

Furthermore, $\bar{R}_n(g_1^{(n)}, g_2^{(n)}, ..., g_{s(n)}^{(n)}) \in (K_0(B)_+ \setminus \{0\})^{\sum_{i=1}^n r_k}$. In particular, for each positive integer $N_0 < n$, we also have

$$\mathrm{diag}(R_{N_0+1}, R_{N_0+1}, ..., R_n)(g_{s(N_0)+1}^{(n)}, g_{s(N_0)+2}^{(n)}, ..., g_{s(n)}^{(n)}) \in (K_0(B)_+ \backslash \{0\})^{\sum_{i=N_0+1}^n r_k}.$$

(c) Since $\{e_1^n, e_2^n, ..., e_{l_n}^n\}$ is a set of $\mathbb{Z}$-linearly independent generators of the free abelian group $K_0(S_n)$, for any projection p in a matrix algebra over S_n, there is a unique l_n-tuple of *integers* $m_1^n([p]), ..., m_{l_n}^n([p])$ such that $[p] = \sum_{j=1}^{l_n} m_j^n([p])e_j^n$ and, for any homomorphism $\tau : K_0(S_n) \to \mathbb{R}$, one has

$$\begin{aligned} \tau([p]) = \langle \vec{m}_n([p]), \tau(\vec{e}_n) \rangle &= \sum_{i=1}^{l_n} m_i^n([p])\tau(e_i^n) \\ &= \sum_{i=1}^{l_n} m_i^n([p])\tau((e_i^n)_+) - m_i^n([p])\tau((e_i^n)_-), \end{aligned}$$

where $\vec{m}_n([p]) = (m_1^n([p]), ..., m_{l_n}^n([p]))^T$ and $\vec{e_n} = (e_1^n, e_2^n, ..., e_{l_n}^n)$.

For each $p \in M_m(A)$, for some integer $m \geq 1$, denote by $[\psi_{k+j}^k(p)]$ the element of $K_0(S_{k+j+1})$ associated with $\psi_{k+j}^k(p)$. Let $\imath_n : S_n \to A$ be the embedding. Consider the map

$$\overline{(\imath_n)_{*0}} : \vec{e_n} \mapsto (((\imath_n)_{*0}(e_1^n), (\imath_n)_{*0}(e_2^n), ..., (\imath_n)_{*0}(e_{l_n}^n)).$$

Then, by Lemma 20.3 and Remark 20.4, one has the following lemma.

LEMMA 20.7. *With notation as above, for any $p \in \mathcal{P}$, for each fixed k, one has that*

$$\begin{aligned} \tau(p) = \quad & \lim_{n\to\infty} \sum_{j=1}^{n} (\sum_{i=1}^{l_{k+j}} m_i^{k+j}([\psi_{k+j-1}^k(p)])\tau((\imath_{k+n} \circ h_{k+j,k+n})_{*0}(e_i^{k+j})_+) \\ & -m_i^{k+j}(\psi_{k+j-1}^k(p)])\tau((\imath_{k+n} \circ h_{k+j,k+n})_{*0}(e_i^{k+j})_-)) \end{aligned}$$

uniformly on $\mathrm{S}(K_0(A))$. *Moreover,* $\rho_A \circ (\imath_{n+k})_{*0} \circ h_{k+j,k+n}(e_{i,\pm}^{k+j})$ *converges uniformly to a strictly positive element of* $\mathrm{Aff}(\mathrm{S}(K_0(A)))$ *as* $n \to \infty$.

PROOF. We first compute that, if $j > 1$, then

$$\begin{aligned}&\sum_{i=1}^{l_{k+j}} m_i^{k+j}([\psi^k_{k+j-1}]([p]))\tau((\imath_{k+n}\circ h_{k+j,k+n})_{*0}(e_i^{k+j}))\\&=\tau([L_{k+n-1}\circ\cdots\circ L_{k+j+1}\circ L_{k+j}\circ L_{k+j-1}\circ\Psi_{k+j-2}]([p])),\end{aligned}$$

and, if $j = 1$, then

$$\begin{aligned}&\sum_{i=1}^{l_{k+1}} m_i^{k+1}([\psi^k_k]([p]))\tau((\imath_{k+n}\circ h_{k+1,k+n})_{*0}(e_i^{k+1}))\\&=\tau([L_{k+n-1}\circ\cdots\circ L_{k+1}\circ L_k]([p])).\end{aligned}$$

Thus (see 20.5),

$$\begin{aligned}&\sum_{j=1}^{n}(\sum_{i=1}^{l_{k+j}} m_i^{k+j}([\psi^k_{k+j-1}(p)])\tau((\imath_n\circ h_{k+j,k+n})_{*0}(e_i^{k+j})))\\&=\tau([L_{k+n-1}\circ\cdots\circ L_{k+1}\circ L_k]([p]))\\&\quad+\sum_{j=2}^{n}\tau([L_{k+n-1}\circ\cdots\circ L_{k+j+1}\circ L_{k+j}\circ L_{k+j-1}\circ\Psi_{k+j-1}]([p]))\\&=\tau([L_{k+n-1}\circ\cdots\circ L_{k+1}\circ L_k]([p]))\\&\quad+\sum_{j=2}^{n}\tau([L_{k+n-1}\circ\cdots\circ L_{k+j+1}\circ L_{k+j}\circ L_{k+j-1}\circ\Psi_{k+j-2}\circ J_{k,k+j-2}]([p]))\\&=\tau([L_{k+n-1}\circ J_{k,n+k-1}]([p])).\end{aligned}$$

The first part of the conclusion then follows from Lemma 20.3. The second part also follows. ☐

One then has the following

COROLLARY 20.8. *With notation as above, in particular with $\mathcal{P}$ be a finite subset of projections in a matrix algebra over A, let G_0 be the subgroup of $K_0(A)$ generated by $\mathcal{P}$, and let $k \geq 1$ be an integer. Denote by $\tilde{\rho}: G_0 \to \prod\mathbb{Z}$ the map defined (see* 20.6 *(c)) by*

$[p] \mapsto$

$(m_1^{k+1}(g_0), -m_1^{k+1}(g_0), m_2^{k+1}(g_0), -m_2^{k+1}(g_0), \cdots m_{l_{k+1}}^{k+1}(g_0), -m_{l_{k+1}}^{k+1}(g_0),$

$m_1^{k+2}(g_1), -m_1^{k+2}(g_1), m_2^{k+2}(g_1), -m_2^{k+2}(g_1), ..., m_{l_{k+2}}^{k+2}(g_1), -m_{l_{k+2}}^{k+2}(g_1),$

....

$m_1^{k+i+1}(g_2), -m_1^{k+i+1}(g_i), m_2^{k+i+1}(g_i), -m_2^{k+i+1}(g_i), ..., m_{l_{k+i+1}}^{k+i+1}(g_i), -m_{l_{k+i+1}}^{k+i+1}(g_i), \cdots),$

where $g_i = [\psi^k_{k+i}(p)]$, $i = 0, 1, 2, ...$ *If* $g \in \mathcal{G}_0$ *and* $\tilde{\rho}(g) = 0$*, then* $\tau(g) = 0$ *for* $\tau \in T(A)$.

By the definition of the maps $\tilde{\rho}$ and $H = h' \circ h_0 : A \to F_0 \to A$, where h_0, h' are as in 20.1, using the same argument as that of Lemma 2.12 of [67], one has the following lemma.

LEMMA 20.9. *Let* $\mathcal{P}$ *be a finite subset of projections in* $M_k(A_1) \subset M_k(A)$*. Then there are a finite subset* $\mathcal{F}_1 \subset A_1$ *and* $\delta_0 > 0$ *such that if the above construction starts with* $\mathcal{F}_1$ *and* δ_0*, then*

$$\ker \tilde{\rho} \subset \ker[h_0] \quad \textit{and} \quad \ker \tilde{\rho} \subset \ker[H].$$

The K_0-part of the existence theorem will deal with maps which almost factor through the map $\tilde{\rho}$, and this lemma will help us to handle the elements of $K_0(A)$ which vanish under $\tilde{\rho}$. Moreover, to get such a K_0-homomorphism, one also needs to find a copy of the generating subset of the positive cone of $K_0(S)$ inside the image of $\tilde{\rho}$ as an ordered group for a certain algebra $S \in \mathcal{C}_0$. In order to do so, one needs the following technical lemma, which is essentially Lemma 3.4 of [67].

LEMMA 20.10 (Lemma 3.4 of [67]). *Part (a): Let* S *be a compact convex set, and let* $\mathrm{Aff}(S)$ *be the space of real continuous affine functions on* S*. Let* $\mathbb{D}$ *be a dense ordered subgroup of* $\mathrm{Aff}(S)$ *(with the strict pointwise order), and let* G *be an ordered group with the strict order determined by a surjective homomorphism* $\rho : G \to \mathbb{D}$ *(i.e.,* $g \geq 0$*, if and only if* $g = 0$*, or* $\rho(g) > 0$*). Let* $\{x_{ij}\}_{1 \leq i \leq r, 1 \leq j < \infty}$ *be an* $r \times \infty$ *matrix having rank* r *and with* $x_{ij} \in \mathbb{Z}$ *for each* i, j*. Let* $g^{(n)}_j \in G$ *be such that* $\rho(g^{(n)}_j) = a^{(n)}_j$*, where* $\{a^{(n)}_j\}$ *is a sequence of positive elements of* $\mathbb{D}$ *such that* $a^{(n)}_j \to a_j\,(> 0)$ *uniformly on* S *as* $n \to \infty$.

Suppose that there is a sequence of integers $s(n)$ *satisfying the following condition:*

Let $\widetilde{v_n} = (g^{(n)}_j)_{s(n)\times 1}$ *be the initial part of* $(g^{(n)}_j)_{1 \leq j < \infty}$ *and let*

$$\widetilde{y_n} = (x_{ij})_{r \times s(n)} \widetilde{v_n}.$$

Set $y_n = \rho^{(r)}(\widetilde{y_n})$*, where* $\rho^{(r)} = \mathrm{diag}(\underbrace{\rho, \rho, \cdots, \rho}_{r})$*. Suppose that there exists* $z = (z_j)_{r\times 1}$ *such that* $y_n \to z$ *on* S *uniformly.*

With the condition above, there exist $\delta > 0$*, a positive integer* $K > 0$*, and a positive integer* N *with the following property:*

If $n \geq N$*,* M *is a positive integer, and if* $\tilde{z}' \in (K^3G)^r$ *(i.e., there is* $\tilde{z}'' \in G^r$ *such that* $K^3\tilde{z}'' = \tilde{z}'$*) satisfies* $||z - Mz'|| < \delta$*, where* $z' = (z'_1, z'_2, \cdots, z'_r)^T$ *with* $z'_j = \rho(\tilde{z}'_j)$ *, then there is a* $\tilde{u} = (\tilde{c}_j)_{s(n)\times 1} \in G^{s(n)}$ *such that*

$$(x_{ij})_{r\times s(n)}\tilde{u} = \tilde{z}'.$$

Part (b): With the condition above if we further assume that each $s(n)$ can be written as $s(n)=\sum_{k=1}^{n} l_k$, where the l_k are positive integers, and for each k, there is an $r_k \times l_k$ matrix R_k with entries in $\mathbb{Z}$ (for some $r_k \in \mathbb{N}$) such that the block diagonal matrix

$$\bar{R}_n = \mathrm{diag}(R_1, R_2, ..., R_n)$$

satisfies

$$\text{(e 20.33)} \qquad \bar{R}_n\bar{g}_n \in (G_+ \setminus \{0\})^{\sum_{k=1}^{n} r_k} \text{ and } \bar{R}_n\tilde{a} \in (\mathrm{Aff}(S)^{++})^{\sum_{k=1}^{n} r_k},$$

where $\bar{g}_n = \left(g_1^{(n)}, g_2^{(n)}, ..., g_{s(n)}^{(n)}\right)^T$ and $\tilde{a}=(a_1,a_2,...,a_{s(n)})^T$ (recall the definition of $\mathrm{Aff}(S)^{++}$ –see 2.2), $n=1,2,...,$ then there exist δ, K, N as described above but with $\tilde{u}$ satisfying the extra condition that

$$\text{(e 20.34)} \qquad \bar{R}_n\tilde{u} > 0.$$

PROOF. The part (a) is essentially Lemma 3.4 of [67]. To prove part (b), we will repeat the argument of [67] and show that if (e 20.33) holds, then $\tilde{u}=(\tilde{c}_j)_{s(n)\times 1}$ can be chosen to make (e 20.34) hold.

Without loss of generality, we may suppose that $(x_{ij})_{r\times r}$ has rank r. Choose N_0 such that $s(N_0) \geq r$. Write

$$\bar{R}_{N_0}(a_1,a_2,...,a_{s(N_0)})^T = (b_1,b_2,...,b_l)^T \in (\mathrm{Aff}(S)^{++})^l,$$

where $l=\sum_{i=1}^{N_0} r_i$. Let $\varepsilon_0 = \frac{1}{4}\min_{1\le j\le l}\inf_{s\in S}\{b_j(s)\} > 0$. There is a positive number $\delta_0 < \varepsilon_0$ such that if

$$\|(a_1', a_2', ..., a_{s(N_0)}')^T - (a_1,a_2,...,a_{s(N_0)})^T\| < \delta_0,$$

then

$$\|\bar{R}_{N_0}(a_1', a_2', ..., a_{s(N_0)}')^T - (b_1,b_2,...,b_l)^T\| < \varepsilon_0.$$

We further assume that $\delta_0 < \frac{1}{4}\min_{1\le j\le s(N_0)}\inf_{s\in S}\{a_j(s)\}$. Consequently, if $(h_1,h_2,...,,h_{s(N_0)}) \in G^{s(N_0)}$ satisfies

$$\|(\rho(h_1),\rho(h_2),...,,\rho(h_{s(N_0)}))^T - (a_1,a_2,...,a_{s(N_0)})^T\| < \delta_0,$$

then

$$\text{(e 20.35)} \qquad \begin{aligned} &(h_1,h_2,...,h_{s(N_0)}) \in (G_+\setminus\{0\})^{s(N_0)}, \text{ and} \\ &\bar{R}_{N_0}(h_1,h_2,...,h_{s(N_0)}) \in (G_+\setminus\{0\})^l. \end{aligned}$$

Since $A:=(x_{ij})_{r\times r}$ has rank r, there is an invertible matrix $B \in M_r(\mathbb{Q})$ with $BA = I_r$. There is an integer $K>0$ such that all all entries of KB and $K(B)^{-1}$

are integers. Choose a positive number $\delta < \delta_0$ such that if $\|Mz' - z\| < \delta$, then $\|B(Mz') - B(z)\| < \frac{1}{8}\delta_0$. (Note δ does not depend on M.)

Let $N \geq N_0$ be such that if $n \geq N$, then

$$\|y_n - z\| < \delta, \qquad \text{and} \quad \|a_j^{(n)} - a_j\| < \frac{1}{4}\delta_0, \ \ \forall \ \ j \in \{1, 2, ..., s(N_0)\},$$

where we recall that $y_n = (x_{ij})_{r \times s(n)}(a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})^T \to z$ as $n \to \infty$. Hence

$$\|B(y_n) - B(z)\| < \frac{1}{8}\delta_0. \tag{e 20.36}$$

Let us show that K, N, and δ as defined above are as desired.

Suppose that $n \geq N \geq N_0$. Set $A_n = (x_{ij})_{r \times s(n)}$. Then $BA_n = C_n$, where $C_n = (I_r, D_n')$ for some $r \times (s(n) - r)$ matrix D_n'. Since all entries of A_n and KB are integers, KD_n' is also a matrix with integer entries.

Recall that $\rho(g_j^{(n)}) = a_j^{(n)}$, and so from the first part of (e 20.33), we have

$$\bar{R}_n(a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})^T \in (\mathrm{Aff}(S)^{++})^{\sum_{k=1}^n r_k}.$$

For each $n \geq N$ (by the continuity of the linear maps), there is $0 < \delta_1 < \delta/4$ such that if $(x_1, x_2, ..., x_{s(n)}) \in \mathrm{Aff}(S)^{s(n)}$ satisfies

$$\|(x_1, x_2, ..., x_{s(n)}) - (a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})\| < \delta_1,$$

then

$$\begin{aligned} \|(0_{r \times r}, D_n')(x_1, x_2, ..., x_{s(n)})^T \\ -(0_{r \times r}, D_n')(a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})^T\| < \frac{\delta_0}{4}, \end{aligned} \tag{e 20.37}$$

and $\bar{R}_n(x_1, x_2, ..., x_{s(n)})^T \in (\mathrm{Aff}(S)^{++})^{\sum_{k=1}^n r_k}$. In particular, we have

$$\begin{aligned} \mathrm{diag}(R_{N_0+1}, R_{N_0+2}, ..., R_n)(x_{s(N_0)+1}, x_{s(N_0)+2}, ..., x_{s(n)})^T \\ \in (\mathrm{Aff}(S)^{++})^{\sum_{k=N_0+1}^n r_k}. \end{aligned} \tag{e 20.38}$$

Since $\mathbb{D}$ is dense in $\mathrm{Aff}(S)$, there is $\xi \in G_+^{s(n)}$ such that $\xi = (\tilde{d}_j)_{s(n) \times 1}$ and

$$||MK^3\rho(\tilde{d}_j) - a_j^{(n)}|| < \delta_1, \quad \text{for all} \quad j = 1, 2, ..., s(n). \tag{e 20.39}$$

Suppose that $\tilde{z}'$ and $\tilde{z}''$ are as described in the lemma. That is, $\tilde{z}'' \in G^r$, $\tilde{z}' = K^3\tilde{z}''$, and $\|M\rho^r(\tilde{z}') - z\| < \delta$. Consequently, $\|MB\rho^r(\tilde{z}') - By_n\| < \frac{1}{4}\delta_0$ and $\|M\rho^r(\tilde{z}') - y_n\| < 2\delta$.

Let $D_n = (0_{r\times r}, D'_n)$. Then

$$C_n - D_n = (I_r, 0_{r\times s(n)-r}). \tag{e 20.40}$$

Since both KB and KD_n are matrices over $\mathbb{Z}$, we can define

$$u' = (KB)\tilde{z}'' - (KD_n)\xi \in G^r. \tag{e 20.41}$$

(Warning: G^r is a $\mathbb{Z}$-module rather than a $\mathbb{Q}$-vector space, so only an integer matrix can act on it. That is, $B\tilde{z}''$ does not make sense, but $(KB)\tilde{z}''$ makes sense.)

Let $\xi' = (\tilde{d}_{r+1}, \tilde{d}_{r+1}, ..., \tilde{d}_{s(n)})^T \in G^{s(n)-r}$ consist of the last $s(n)-r$ coordinates of ξ. Then $D_n\xi = D'_n\xi'$. Put $u' = (\tilde{c_1}, \tilde{c_2}, ..., \tilde{c_r})^T \in G^r_+$, and

$$u = \begin{pmatrix} u' \\ K\xi' \end{pmatrix} = (\tilde{c_1}, \tilde{c_2}, ..., \tilde{c_r}, K\tilde{d}_{r+1}, K\tilde{d}_{r+2}, ..., K\tilde{d}_{s(n)})^T \in G^{s(n)}.$$

Then

$$\begin{aligned} A_nK^2u &= (KB^{-1})(KB)A_nu \\ &= (KB^{-1}(KI_r, KD'_n)u \\ &= (KB^{-1})(Ku' + (K^2D'_n)\xi') \\ &= (KB^{-1})(K((KB)\tilde{z}'' - (KD_n)\xi) + (K^2D_n)\xi) \\ &= (KB^{-1})(KB)(K\tilde{z}'') \\ &= K^3\tilde{z}'' = \tilde{z}'. \end{aligned}$$

Let $a' = (a_1^{(n)}, a_2^{(n)}, ..., a_r^{(n)})^T \in \mathrm{Aff}(S)^r$. Then we have (using (e 20.41), (e 20.37), and (e 20.40)),

$$\begin{aligned} &||MK^2\rho^r(u') - a'|| \\ &= ||MB\rho^r(\tilde{z}') - MK^3D_n\rho^{s(n)}(\xi) - a'|| \\ &\le ||By_n - MK^3D_n\rho^{s(n)}(\xi) - a'\| + \frac{\delta_0}{4} \\ &\le ||By_n - D_n(a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})^T - a'|| + \frac{\delta_0}{4} + \frac{\delta_0}{4} \\ &= ||BA_n(a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})^T - D_n(a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})^T - a'|| + \frac{\delta_0}{2} \\ &= ||(C_n - D_n)(a_1^{(n)}, a_2^{(n)}, ..., a_{s(n)}^{(n)})^T - a'|| + \frac{\delta_0}{2} = ||0|| + \frac{\delta_0}{2}. \end{aligned}$$

Let $\tilde{u} = K^2 u = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}$ with

$$\tilde{u}_1 = (K^2\tilde{c}_1, K^2\tilde{c}_2, ..., K^2\tilde{c}_r, K^3\tilde{d}_{r+1}, K^3\tilde{d}_{r+2}, ...,K^3\tilde{d}_{s(N_0)})^T$$

and $\tilde{u}_2 = (K^3\tilde{d}_{s(N_0)+1}, K^3\tilde{d}_{s(N_0)+2}, ..., K^3\tilde{d}_{s(n)})^T$. Then we have

$$A_n\tilde{u} = \tilde{z}'.$$

Combining (e 20.38) and (e 20.39), we have

$$\mathrm{diag}(R_{N_0+1}, R_{N_0+2}, ..., R_n)\tilde{u}_2 \in (G_+ \setminus \{0\})^{\sum_{k=N_0+1}^n r_k}.$$

Furthermore, combining (e 20.39) with the estimate above, we have

$$\|M\rho^{s(N_0)}(\tilde{u}_1) - (a_1^{(n)}, a_2^{(n)}, ...,a_{s(N_0)}^{(n)})^T\| < \frac{\delta_0}{2}.$$

Hence,

$$\|M\rho^{s(N_0)}(\tilde{u}_1) - (a_1, a_2, ...,a_{s(N_0)})^T\| < \frac{\delta_0}{2} + \frac{\delta_0}{4} < \delta_0.$$

By the choice of δ_0 (see (e 20.35)), we have $M\bar{R}_{N_0}\tilde{u}_1 \in (G_+ \setminus \{0\})^{\sum_{k=1}^{N_0} r_k}$. In other words, $\bar{R}_{N_0}\tilde{u}_1 \in (G_+ \setminus \{0\})^{\sum_{k=1}^{N_0} r_k}$. Hence $\bar{R}_n\tilde{u} \in (G_+ \setminus \{0\})^{\sum_{k=1}^{n} r_k}$. This ends the proof. ☐

DEFINITION 20.11. A unital stably finite C^*-algebra A is said to have the property of K_0-density, if $\rho_A(K_0(A))$ is dense in $\mathrm{Aff}(\mathrm{S}_{[1]}(K_0(A))$, where $\mathrm{S}_{[1]}(K_0(A))$ is the convex set of states of $K_0(A)$ (i.e., the convex set of all positive homomorphisms $r : K_0(A) \to \mathbb{R}$ satisfying $r([1]) = 1$).

REMARK 20.12. By Corollary 7.9 of [47], the linear space spanned by $\rho(K_0(A))$ is always dense in $\mathrm{Aff}(\mathrm{S}_{[1]}(K_0(A))$. Therefore, the unital stably finite C^*-algebra of the form of $A \otimes U$ for a UHF-algebra U always has the K_0-density property. Moreover, any unital stably finite C^*-algebra A which is tracially approximately divisible has the K_0-density property.

REMARK 20.13. Not all C^*-algebras in $\mathcal{B}_1$ with the (SP) property satisfy the K_0-density property. The following example illustrates this: Consider

$$G = \{(a, b) \in \mathbb{Q} \oplus \mathbb{Q} : 2a - b \in \mathbb{Z}\} \text{ and}$$
$$G_+ \setminus \{0\} = (\mathbb{Q}_+ \setminus \{0\} \oplus \mathbb{Q}_+ \setminus \{0\}) \cap G,$$

and $1 = (1, 1) \in G_+$ as unit. Then $(G, G_+, 1)$ is a weakly unperforated rationally Riesz simple ordered group (but not a Riesz group—see [84]). Evidently G has the property (SP); but the image of G is not dense in $\mathrm{Aff}(\mathrm{S}_{[1]}(G) = \mathbb{R} \oplus \mathbb{R}$, as $(1/2, 1/2) \in \mathbb{R} \oplus \mathbb{R}$ is not in the closure of the image of G. We leave the details to the readers.

PROPOSITION 20.14. *Let $A \in \mathcal{B}_0$ be an amenable C^*-algebra which has the K_0-density property and satisfies the UCT, and let B_1 be an inductive limit C^*-algebra as in Theorem* 14.10 *such that*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)),$$

and such that B satisfies the condition of Corollary 14.14 *for B there, where $B = B_1 \otimes U$ for a UHF-algebra U of infinite type. Let $\alpha \in KL(A, B)$ be an element which implements the isomorphism above. Then, there is a sequence of completely positive linear maps $L_n : A \to B$ such that*

$$\lim_{n\to\infty} ||L_n(ab) - L_n(a)L_n(b)|| = 0 \text{ for all } a, b \in A \text{ and } [\{L_n\}] = \alpha.$$

(By $[\{L_n\}] = \alpha$, we mean for any finite set $\mathcal{P} \subset \underline{K}(A)$, there is a positive integer N such that if $n \geq N$, then $[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$.)

PROOF. By Lemma 9.8, A is the closure of an increasing union of RFD C^*-subalgebras $\{A_n\}$. Fix a finite subset $\mathcal{P} \subset \underline{K}(A)$. Let $\mathcal{F}_1$ be a finite subset of A and let $\delta_0 > 0$ be such that for any $\mathcal{F}_1$-δ_0-multiplicative contractive linear map L from A, the map $[L]|_{\mathcal{P}}$ is well defined. We may assume that $\mathcal{F}_1 \subset A_1$ and there exists $\mathcal{P}' \subset \underline{K}(A_1)$ such that $[\iota](\mathcal{P}') = \mathcal{P}$, where $\iota : A_1 \to A$ is the embedding. Let $G = G(\mathcal{P})$ be the subgroup generated by $\mathcal{P}$ and let $\mathcal{P}_0{}' \subset \mathcal{P}'$ be such that $\mathcal{P}_0 := \iota_{*0}(\mathcal{P}'_0)$ generates $G \cap K_0(A)$. We may suppose that $\mathcal{P}_0 = \{[p_1], ..., [p_l]\}$, where $p_1, ..., p_l$ are projections in a matrix algebra over A_1, where we identify $\iota(p_i)$ with p_i. Let G_0 be the group generated by $\mathcal{P}_0$. Moreover, we may assume that $\mathcal{F}_1$ and δ_0 satisfy the conclusion of Lemma 20.9. Let k_0 be an integer such that $G(\mathcal{P}) \cap \mathrm{K}_i(A, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ for any $k \geq k_0$, $i = 0, 1$.

By (the “Moreover” part of) Theorem 18.2, there are two $\mathcal{F}_1$-$\delta_0/2$ multiplicative contractive completely positive linear maps Φ_0, Φ_1 from A to $B \otimes \mathcal{K}$ (B is amenable) such that

$$[\Phi_0]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} + [\Phi_1]|_{\mathcal{P}}$$

and the image of Φ_1 is contained in a finite dimensional C^*-subalgebra. Moreover, we may also assume that Φ_1 is a homomorphism when it is restricted to A_1, and the image is a finite dimensional C^*-algebra. With Φ_1 in the role of h_0, we can proceed with the construction as described at the beginning of this section. We will keep the same notation.

Consider the map $\tilde{\rho} : G(\mathcal{P}) \cap \mathrm{K}_0(A) \to l^\infty(\mathbb{Z})$ defined in Corollary 20.8. The linear span of the subset $\{\tilde{\rho}(p_1), ..., \tilde{\rho}(p_l)\}$ over $\mathbb{Q}$ will have finite rank, say r. So, we may assume that $\{\tilde{\rho}(p_1), ..., \tilde{\rho}(p_r)\}$ is linearly independent and its $\mathbb{Q}$-linear span is the whole subspace. Therefore, there is an integer M such that for any $g \in \tilde{\rho}(G_0)$, the element Mg is in the subgroup generated by $\{\tilde{\rho}(p_1), ..., \tilde{\rho}(p_r)\}$. Let $x_{ij} = (\tilde{\rho}(p_i))_j$, and $z_i = \rho_B(\alpha([p_i]) \in \mathbb{D}$, where $\mathbb{D} := \rho_B(K_0(B)) \subset \mathrm{Aff}(S_{[1]}(K_0(B))$. Since A is assumed to have the K_0-density property, so also is B. Therefore the image $\mathbb{D}$ is a dense subgroup of $\mathrm{Aff}(\mathrm{S}_{[1]}(K_0(B)))$.

Let $\{S_j\}$ be the sequence of C^*-subalgebras in $\mathcal{C}_0$ in the construction at the beginning of this section. Fix $k \geq 1$. Let $e_i^{k+j}, e_{i,\pm}^{k+j} \in K_0(S_{k+j})$, $i = 1, 2, ..., l_{k+j}$, and the $r_{k+j} \times 2l_{k+j}$ matrix R_{k+j} be as described in 20.6. Let $s(j) = \sum_{i=1}^{j} 2l_{k+i}$, $j = 1, 2, ...$ Put

$$\alpha([\imath_n \circ h_{k+j,n}(e_{i,+}^{k+j})]) = g_{s(j-1)+2i-1}^{(n)} \in K_0(B)_+, \tag{e 20.42}$$

$$\alpha([\imath_n \circ h_{k+j,n}(e_{i,-}^{k+j})]) = g_{s(j-1)+2i}^{(n)} \in K_0(B)_+, \tag{e 20.43}$$

$i = 1, 2, ..., l_{k+j}$, and $a_j^{(n)} = \rho_B(g_j^{(n)})$, $j = 1, 2, ..., s(n) = \sum_{j=1}^{n} l_{k+j}$, $n = 1, 2, ...$ Note that $a_j^{(n)} \in \mathbb{D}^+ \backslash \{0\}$. It follows from Lemma 20.3 that $\lim_{n\to\infty} a_{s(j-1)+2i}^{(n)} = a_{s(j-1)+2i} = \lim_{n\to\infty} \rho_B(\alpha(g_{s(j-1)+2i}^{(n)}))$
> 0 and $\lim_{n\to\infty} a_{s(j-1)+2i-1}^{(n)} = a_{s(j-1)+2i-1} = \lim_{n\to\infty} \rho_B(\alpha(g_{s(j-1)+2i-1}^{(n)})) > 0$ uniformly. Moreover, by Lemma 20.7, $\sum_{j=1}^{n} x_{ij} a_j^{(n)} \to z_i$ uniformly. Furthermore, by 20.6 (b), we know that $\{x_{ij}\}$, g_j^n, and R_n satisfy the conditions of Lemma 20.10, with $K_0(B)$ in place of G and $T(B)$ in place of S. So Lemma 20.10 applies. Fix K and δ obtained from Lemma 20.10.

Let $\Psi := \Phi_0 \oplus (\overbrace{\Phi_1 \oplus \cdots \oplus \Phi_1}^{MK^3(k_0+1)!-1})$. Since Φ_1 factors through a finite dimensional C^*-algebra, it is zero when restricted to $K_1(A) \cap G$ and $K_1(A, \mathbb{Z}/k\mathbb{Z}) \cap G$ for $2 \leq k \leq k_0$. Moreover, the map $(\overbrace{\Phi_1 \oplus \cdots \oplus \Phi_1}^{MK^3(k_0+1)!})$ vanishes on $K_0(A, \mathbb{Z}/k\mathbb{Z})$ for $2 \leq k \leq k_0$. Therefore we have

$$[\Psi]|_{K_1(A)\cap G} = \alpha|_{K_1(A)\cap G}, \quad [\Psi]|_{K_1(A,\mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_1(A,\mathbb{Z}/k\mathbb{Z})\cap G}$$

and $[\Psi]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G}$. We may assume that $\Psi(1_A)$ is a projection in $\mathrm{M}_m(B)$ for some integer m.

We may also assume that there exist projections $\{p_1', ..., p_l'\}$ in $B \otimes \mathcal{K}$ which are sufficiently close to $\{\Psi(p_1), ..., \Psi(p_l)\}$, respectively, that $[p_i'] = [\Psi(p_i)]$. Note that $B \in \mathcal{B}_0$, and hence the strict order on the projections of B is determined by traces. Thus there is a projection $q_i' \leq p_i'$ such that $[q_i'] = MK^3(k_0+1)![\Phi_1(p_i)]$. Set $e_i' = p_i' - q_i'$, and let $\mathcal{P}_1 = \Psi(\mathcal{P}) \cup \Phi_1(\mathcal{P}) \cup \{p_i', q_i', e_i'; i = 1, ..., l\}$. Denote by G_1 the group generated by $\mathcal{P}_1$. Recall that $G_0 = G(\mathcal{P}) \cap K_0(A)$ and $\rho_A(G_0)$ is a free abelian group, and decompose it as $G_{00} \oplus G_{01}$, where G_{00} is the infinitesimal part of G_0. Fix this decomposition and denote by $\{d_1, ..., d_t\}$ the positive elements which generate G_{01}.

Note that B belongs to the class of Corollary 14.14, and so $M_r(B)$ does also. Applying Corollary 14.14 to $M_r(B)$ with any finite subset $\mathcal{G}$, any $\varepsilon > 0$ and any $0 < r_0 < \delta < 1$, one has a $\mathcal{G}$-ϵ-multiplicative map $L : M_m(B) \to M_m(B)$ with the following properties:

(1) $[L]|_{\mathcal{P}_1}$ and $[L]|_{G_1}$ are well defined;

(2) $[L]$ induces the identity maps on the infinitesimal part of $G_1 \cap \mathrm{K}_0(B)$, $G_1 \cap \mathrm{K}_1(B)$, $G_1 \cap \mathrm{K}_0(B, \mathbb{Z}/k\mathbb{Z})$, and $G_1 \cap \mathrm{K}_1(B, \mathbb{Z}/k\mathbb{Z})$ for the k with $G_1 \cap \mathrm{K}_i(B, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, $i = 0, 1$;

(3) $\tau \circ [L](g) \le r_0 \tau(g)$ for all $g \in G_1 \cap \mathrm{K}_0(B)$ and $\tau \in T(B)$; and

(4) there exist positive elements $\{f_i\} \subset K_0(B)_+$ such that for $i = 1, ..., t$,

$$\alpha(d_i) - [L](\alpha(d_i)) = MK^3(k_0+1)! f_i.$$

Using the compactness of $T(B)$ and strict comparison for positive elements of B, the positive number r_0 can be chosen sufficiently small that $\tau \circ [L] \circ [\Psi]([p_i]) < \delta/2$ for all $\tau \in T(B)$, and $\alpha([p_i]) - [L \circ \Psi]([p_i]) > 0$, $i = 1, 2, ..., l$.

Let $[p_i] = \sum_{j=1}^t m_j d_j + s_i$, where $m_j \in \mathbb{Z}$ and $s_i \in G_{00}$. Note that, by (2) above, $(\alpha - [L] \circ \alpha)(s_i) = 0$. Then we have

$$\begin{array}{lll} \text{(e 20.44)} & \alpha([p_i]) - [L \circ \Psi]([p_i]) & \\ & = & \alpha([p_i]) - ([L \circ \alpha]([p_i]) + MK^3(k_0+1)![L \circ \Phi_1]([p_i])) \\ & = & (\alpha(\sum m_j d_j) - [L \circ \alpha](\sum m_j d_j)) \\ & & -MK^3(k_0+1)![L \circ \Phi_1]([p_i]) \\ & = & MK^3(k_0+1)!(\sum m_j f_j - [L] \circ [\Phi_1]([p_i])) \\ & = & MK^3(k_0+1)! f_i', \end{array}$$

where $f_i' = \sum m_j f_j - [L] \circ [\Phi_1]([p_i])$, for $i = 1, 2, ..., l$. Set $\beta([p_i]) = K^3(k_0+1)! f_i'$, $i = 1, 2, ..., l$.

Let us now construct a map $h' : A \to B$. It will be constructed by factoring through the K_0-group of some C^*-algebra in the class $\mathcal{C}_0$ in the construction given at the beginning of this section. Let $\tilde{z}_i{}' = \beta([p_i])$, and $z_i' = \rho_B(\tilde{z}_i{}') \in \mathrm{Aff}(S_{[1]}(K_0(B)))$. Then we have

$$\begin{array}{lll} ||Mz' - z||_\infty & = & \max_i\{||\rho_B(\alpha([p_i]) - [L \circ \Psi]([p_i])) - \rho(\alpha([p_i]))||\} \\ & = & \max_i\{\sup_{\tau \in T(B)}\{\tau \circ [L] \circ [\Psi]([p_i])\}\} \le \delta/2, \end{array}$$

where $z = (z_1, z_2, ..., z_r)$ and $z' = (z_1', z_2', ..., z_r')$. By Lemma 20.10, for sufficiently large n, one obtains $\tilde{u} = (u_1, u_2, ..., u_{s(n)}) \in K_0(B)^{s(n)}$ such that

$$\text{(e 20.45)} \qquad \sum_{j=1}^{s(n)} x_{ij} u_j = \tilde{z}_i'.$$

More importantly,

$$\text{(e 20.46)} \qquad \bar{R}_n \tilde{u} > 0.$$

It follows from 20.6(a) (recall that k is fixed) that, for each $1 \leq j \leq n$, the map

$$e_i^{k+j} \mapsto (u_{s(j-1)+2i-1} - u_{s(j-1)+2i}),\ 1 \leq i \leq l_{k+j},$$

defines a strictly positive homomorphism $\kappa_0^{(k+j)}$ from $K_0(S_{k+j})$ to $K_0(B)$. Since $B \in \mathcal{B}_{u0}$, by Corollary 18.9, there is a homomorphism $h' : D \to \mathrm{M}_m(B)$ for some large m such that $h'_{*0}|_{S_k} = \kappa_0^{(k)}$, where $D = S_{k+1} \oplus \cdots \oplus S_{k+n}$. By (e 20.45), one has, keeping the notation of the construction at the beginning of this section, for $[\psi^k_{k+j}(p_i)] = \sum_{l=1}^{l_{k+j}} m_i^{k+j}([\psi^k_{k+j}(p_i)])e_l^{k+j}$,

$$\begin{aligned}
\kappa_0^{k+j}([\psi^k_{k+j}(p_i)]) &= \sum_{l=1}^{l_{k+j}} m_i^{k+j}([\psi^k_{k+j}(p_i)])\kappa_0^{k+j}(e_l^{k+j}) \\
&= \sum_{l=1}^{l_{k+j}} m_i^{k+j}([\psi^k_{k+j}(p_i)])(u_{s(j-1)+2l-1} - u_{s(j-1)+2l}) \\
&= \sum_{l=1}^{l_{k+j}} x_{i,j+l}u_{s(j-1)+l}.
\end{aligned}$$

Hence,

$$h'_{*0}([\psi^k_k(p_i)],[\psi^k_{k+1}(p_i)], [\psi^k_{k+2}(p_i)], ..., [\psi^k_{n+k-1}(p_i)])= (x_{i,j})_{s(n)\times 1}\tilde{u} = \beta([p_i]),$$

$i = 1, ..., r$. Now, define $h'' : A \to D \to \mathrm{M}_m(B)$ by

$$h'' = h' \circ (\psi^k_k \oplus \psi^k_{k+1} \oplus \psi^k_{k+2} \oplus \cdots \oplus \psi^k_{k+n-1}).$$

Then h'' is $\mathcal{F}$-δ-multiplicative.

For any $x \in \ker\tilde{\rho}$, by Lemma 20.9, $x \in \ker\,(\rho_B \circ \alpha) \cap \ker[H]$ and $x \in \ker[h_0] = \ker[\Phi_1]$. Therefore, we have $[\Phi_1](x) = 0$ and $[\Psi](x) = \alpha(x)$. Note that $\alpha(x)$ also vanishes under any state of $(K_0(B), {K_0}^+(B))$, and we have $[L] \circ \alpha(x) = \alpha(x)$. So, we get

$$\alpha(x) - [L \circ \Psi](x) = 0.$$

Thus, $(\alpha - [L \circ \Psi])|_{\ker\tilde{\rho}} = 0$. Hence we may view $\alpha - [L \circ \Psi]$ as a homomorphism from $\tilde{\rho}(G_0)$. Recall that

$$(\alpha - [L \circ \Psi])([p_i]) = M\beta([p_i]),\ \ i = 1, 2, ..., r. \tag{e 20.47}$$

Set h to be the direct sum of M copies of h''. The map h is $\mathcal{F}$-δ-multiplicative, and

$$[h]([p_i]) = \alpha([p_i]) - [L] \circ [\Psi]([p_i]) \quad i = 1, ..., r.$$

Note that $[h]$ has multiplicity $MK^3(k_0 + 1)!$ (see (e 20.44)), and $D \in \mathcal{C}_0$ (the algebras in $\mathcal{C}$ with trivial K_1 group). One concludes that h induces the zero map on $G \cap K_1(A)$, $G \cap K_1(A, \mathbb{Z}/m\mathbb{Z})$, and $G \cap K_1(A, \mathbb{Z}/m\mathbb{Z})$ for $m \le k_0$. Therefore, we have

$$[h]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} - [L] \circ [\Psi]|_{\mathcal{P}}.$$

Set $L_1 = (L \circ \Psi) \oplus h$. L_1 is $\mathcal{F}$-δ multiplicative and

$$[L_1]|_{\mathcal{P}} = [h]|_{\mathcal{P}} + [L] \circ [\Psi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

We may assume $L_1(1_A) = 1_B$ by conjugating with a unitary in $M_m(B)$. Then L_1 is an $\mathcal{F}$-δ-multiplicative map from A to B, and $[L_1]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$.

Since A is separable, $\underline{K}(A)$ is countable. It follows that one obtains a sequence of contractive completely positive linear maps from A to B such that

$$\lim_{n\to\infty} ||L_n(ab) - L_n(a)L_n(b)|| = 0 \text{ for all } a, b \in A \quad \text{and} \quad [\{L_n\}] = \alpha.$$

□

COROLLARY 20.15. *Let A be a separable amenable C^*-algebra in the class $\mathcal{B}_0$ satisfying the UCT. Then A is KK-attainable with respect to $\mathcal{B}_{u0}$ (see* 18.1*).*

PROOF. Let C be any C^*-algebra in $\mathcal{B}_{u0}$, and let $\alpha \in KL(A, C)^{++}$. We may write $C = C_1 \otimes U$ for some $C_1 \in \mathcal{B}_0$ and for some UHF-algebra of infinite type.

For any C^*-algebra D, let us denote by $j_D : D \to D \otimes U$ the map defined by $j_D(d) = d \otimes 1_U$ for all $d \in D$.

Let $\alpha \in KL(A, C_1 \otimes U)^{++} = \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(C_1 \otimes U))^{++}$. Let us point out an easy fact that there is $\bar{\alpha} \in KL(A \otimes U, C_1 \otimes U)^{++}$ such that $\alpha = \bar{\alpha} \circ [j_A]$. To see this, recall that, since U is a UHF-algebra of infinite type, $[j_U] = [\mathrm{id}_U]$. Thus $[j_C] \circ \alpha = \alpha$ (as we identify C with $C \otimes U$).

For each $k \ge 1$, since $K_i(U)$ is torsion free, by Künneth formula, for $i = 0, 1$,

$$\begin{aligned} K_i(D \otimes U) &= K_i(D) \otimes K_0(U) \text{ and} \\ K_i(D \otimes U, \mathbb{Z}/k\mathbb{Z}) &= K_i(D, \mathbb{Z}/k\mathbb{Z}) \otimes K_0(U), \end{aligned}$$

where $D = A$, or $D = C$, $k = 2, 3, ...$ Define, for each $k \ge 1$, if $y = x \otimes r$, where $x \in K_i(A, \mathbb{Z}/k\mathbb{Z})$ and $r \in K_0(U)$, $\bar{\alpha}(y) = \alpha(x) \otimes r$. One checks that

$$\bar{\alpha} \circ [j_A](x) = \alpha(x \otimes 1) = [j_C] \circ \alpha(x) = \alpha(x) \text{ for all } x \in \underline{K}(A).$$

Therefore

$$\alpha = \bar{\alpha} \circ [j_A]. \tag{e 20.48}$$

By Theorem 14.10, there is a C^*-algebra B which is an inductive limit of C^*-algebras in the class $\mathcal{C}_0$ together with homogeneous C^*-algebras in the class $\mathbf{H}$ (see 14.5) such that

$$\begin{aligned} &(K_0(A_1), K_0(A_1)_+, [1_{A_1}]_0, K_1(A_1)) \\ &\cong (K_0(B_1), K_0(B_1)_+, [1_{B_1}]_0, K_1(B_1)), \end{aligned} \tag{e 20.49}$$

where $A_1 = A \otimes U$, and such that $B_1 := B \otimes U$ satisfies the condition of Corollary 14.14 for B there. Since A_1 satisfies the UCT, there is an invertible $\beta \in KL(A_1, B_1)^{++}$ such that β carries the isomorphism in (e 20.49). Applying Corollary 18.8, one obtains a sequence of unital completely positive linear maps $\{\Psi_n : B_1 \to C\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \|\Psi_n(ab) - \Psi_n(a)\Psi_n(b)\| = 0 \text{ for all } a, b \in B_1 \text{ and } [\{\Psi_n\}] = \bar{\alpha} \circ \beta^{-1}.$$

Note that $B \otimes U$ can be chosen as in Proposition 20.14. Applying Proposition 20.14 to $\beta \in KL(A_1, B \otimes U)$, one obtains a sequence of unital completely positive linear maps $\{\Phi_n\}$ from A_1 to $B \otimes U$ such that $[\{\Phi_n\}] = \beta$ and $\lim_{n\to\infty} \|\Phi_n(ab) - \Phi_n(a)\Phi_n(b)\| = 0$ for all a, $b \in A_1$. Choosing a subsequence $\{\Psi_{k(n)}\}$ and defining $L_n = \Psi_{k(n)} \circ \Phi_n \circ j_A$, one checks that (see (e 20.48))

$$[\{L_n\}] = [\{\Psi_{k(n)}\}] \circ [\{\Phi_n\}] \circ [j_A] = \bar{\alpha} \circ \beta^{-1} \circ \beta \circ [j_A] = \bar{\alpha} \circ [j_A] = \alpha \in KL(A, B)$$

and $\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0$ for all a, $b \in A$, as desired. □

THEOREM 20.16. *Let $A \in \mathcal{B}_0$ be a amenable C^*-algebra satisfying the UCT, and let $B \in \mathcal{B}_{u0}$. Then for any $\alpha \in KL(A, B)^{++}$, and any affine continuous map $\gamma : T(B) \to T(A)$ which is compatible with α, there is a sequence of completely positive linear maps $L_n : A \to B$ such that*

$$\lim_{n\to\infty} ||L_n(ab) - L_n(a)L_n(b)|| = 0 \text{ for all } a, b \in A,$$

$$[\{L_n\}] = \alpha, \quad \text{and} \quad \lim_{n\to\infty} \sup_{\tau \in T(B)} |\tau \circ L_n(f) - \gamma(\tau)(f)| = 0 \text{ for all } f \in A.$$

PROOF. This follows from Corollary 20.15 and Proposition 18.12 directly. □

21. The Isomorphism Theorem

DEFINITION 21.1. Let n and s be integers. Let Y be a connected finite CW complex with dimension no more than 3 such that $K_1(C(Y))$ is a finite group (no restriction on $K_0(C(Y))$), and let P be a projection in $\mathrm{M}_n(\mathrm{C}(Y))$ with rank $r \geq 6$. Note that Y could be a point. Let $D' = \bigoplus_{i=1}^{s} E_i$, where $E_i = M_{r_i}(C(\mathbb{T}))$ and where r_i is an integer, $i = 1, 2, ..., s$. Hence $K_1(D') \cong \mathbb{Z}^s$. Put $C' = D' \oplus PM_n(C(Y))P$. Then $K_1(C') = \mathrm{Tor}(K_1(C')) \oplus \mathbb{Z}^s$.

Let C be a finite direct sum of C^*-algebras of the form C' above and C^*-algebras in the class $\mathcal{C}_0$ (with trivial K_1-group). Write $C = D \oplus C_0 \oplus C_1$, where D is a finite direct sum of C^*-algebras of the form $M_{r_i}(C(\mathbb{T}))$, C_0 is a direct sum of C^*-algebras in the class $\mathcal{C}_0$, and C_1 is a finite direct sum of C^*-algebras of the form $PM_n(C(Y))P$, where Y is a connected finite CW complex with dimension no more than 3 such that $K_1(C(Y))$ is a finite abelian group and P has rank at least 6. Then, one has (recall that P has rank at least 6 on Y) $U(C)/U_0(C) = K_1(C) = K_1(D) \oplus \mathrm{Tor}(K_1(C))$ and

$$U(C)/CU(C) \cong U_0(C)/CU(C) \oplus K_1(D) \oplus \mathrm{Tor}(K_1(C)). \tag{e 21.1}$$

Here we identify $K_1(D)\oplus \mathrm{Tor}(K_1(C))$ with a subgroup of $U(C)/CU(C)$. Denote by π_0, π_1, π_2 the projection maps from $U(C)/CU(C)$ to each direct summand according to the decomposition above (so $\pi_0(U(C)/CU(C)) = U_0(C)/CU(C)$, $\pi_1(U(C)/CU(C)) = K_1(D)$ and $\pi_2(U(C)/CU(C)) = \mathrm{Tor}(K_1(C))$). We note that $\pi_i(U(C)/CU(C))$ is a subgroup of $U(C)/CU(C)$, $i = 0, 1, 2$. We may also write $U(C)/CU(C) = U(D)/CU(D) \oplus U_0(C_0)/CU(C_0) \oplus U(C_1)/CU(C_1)$. Denote by $\Pi : C \to D$ the projection map. Viewing D as a C^*-subalgebra of C, one has $\Pi|_D = \mathrm{id}_D$. Since $\pi_1(U(C)/CU(C))$ is a subgroup of $U(C)/CU(C)$, $\Pi^\ddagger \circ \pi_1$ is a homomorphism from $U(C)/CU(C)$ into $U(D)/CU(D)$ and $\Pi^\ddagger|_{\pi_1(U(C)/CU(C))}$ is injective (see 2.17 for the definition of $\Pi^\ddagger$).

In what follows in this section, if A is a unital C^*-algebra and $F \subset U(A)$ is a subset, then $\overline{F}$ is the image of F in $U(A)/CU(A)$ (see 2.16). We will frequently refer to the above notation later in this section.

Definition 21.1 plays a role similar to that which Definition 7.1 of [71] plays in [71]. The only difference is the appearance of $C_0 \in \mathcal{C}_0$. Since $K_1(C_0) = \{0\}$, as one will see in this section, this will not cause a new problem. However, since C^*-algebras in $\mathcal{C}_0$ have a different form, we will repeat many of the same arguments for the sake of completeness.

As in [71], we have the following lemmas to control the maps from $U(C)/CU(C)$ in the approximate intertwining argument in the proof of 21.9. The proofs are repetitions of the corresponding arguments in [71].

LEMMA 21.2 (see Lemma 7.2 of [71]). *Let C be the C^*-algebra defined in 21.1, let D, a finite direct sum of circle algebras, be the direct summand of C specified in 21.1, let $\mathcal{U} \subset U(C)$ be a finite subset, and denote by F the subgroup generated by $\mathcal{U}$. Let G be a subgroup of $U(C)/CU(C)$ which contains $\overline{F}$, the image of F in $U(C)/CU(C)$, and also contains $\pi_1(U(C)/CU(C))$ and $\pi_2(U(C)/CU(C))$. Suppose that the composed map $\gamma : \overline{F} \to U(D)/CU(D) \to U(D)/U_0(D)$ is injective—i.e., if $x, y \in \overline{F}$ and $x \neq y$, then $[x] \neq [y]$ in $U(D)/U_0(D)$. Let B be a unital C^*-algebra and $\Lambda : G \to U(B)/CU(B)$ be a homomorphism such that $\Lambda(G\cap(U_0(C)/CU(C))) \subset U_0(B)/CU(B)$. Let $\theta : \pi_2(U(C)/CU(C)) \to U(B)/CU(B)$ be defined by $\theta(g) = \Lambda|_{\pi_2(U(C)/CU(C))}(g^{-1})$ for any $g \in \pi_2(U(C)/CU(C))$. Then there is a homomorphism $\beta : U(D)/CU(D) \to U(B)/CU(B)$ with*

$$\beta(U_0(D)/CU(D)) \subset U_0(B)/CU(B)$$

such that

$$\beta \circ \Pi^\ddagger \circ \pi_1(\bar{w}) = \Lambda(\bar{w})(\theta \circ \pi_2(\bar{w})) \ \ for\ all\ \ w \in F,$$

where $\Pi : C \to D$ is defined in Definition 21.1. If, furthermore, $B \cong B_1 \otimes V$ for a unital C^*-algebra $B_1 \in \mathcal{B}_0$ and a UHF-algebra V of infinite type, and $\Lambda(G) \subset U_0(B)/CU(B)$, then $\beta \circ \Pi^\ddagger \circ (\pi_1)|_{\bar{F}} = \Lambda|_{\bar{F}}$.

The statement above is summarized in the following commutative diagram:

$$\begin{tikzcd}
\bar{F} \arrow[r, "\text{inclusion}"] \arrow[d, "\pi_1"] & G \arrow[d, "\Lambda+\theta\circ\pi_2"] \\
\pi_1(\bar{F}) \arrow[d, "\Pi^{\ddagger}"] & U(B)/CU(B) \\
U(D)/CU(D)\ . \arrow[ur, "\beta"'] &
\end{tikzcd}$$

PROOF. Note that $U(D)/U_0(D) \cong K_1(D)$, and this is a free abelian group, as D is a finite direct sum of C^*-algebras of the form $M_n(C(\mathbb{T}))$. Therefore $\gamma(\overline{F})$ as a subgroup of a free abelian group is free abelian. The proof is exactly the same as that of Lemma 7.2 of [71] (since it has nothing to do with the direct summands of C_0, and, $K_1(C_0) = \{0\}$ and $U(C_0)/CU(C_0) = U_0(C_0)/CU(C_0)$). However, we will repeat the proof.

Let $\kappa_1 : U(D)/CU(D) \to K_1(D) \subset U(C)/CU(C)$ be the quotient map and let $\eta : \pi_1(U(C)/CU(C)) \to K_1(D)$ be the map defined by $\eta = \kappa_1 \circ \Pi^{\ddagger}|_{\pi_1(U(C)/CU(C))}$. Note that $\pi_1(U(C)/CU(C))$ is identified with $K_1(D)$ and $\Pi^{\ddagger}|_{\pi(U(C)/CU(C)}$ is injective(see 21.1). Therefore η is an isomorphism. Since (the composed map) γ is injective and $\gamma(\overline{F})$ is free abelian, we conclude that $\kappa_1 \circ \Pi^{\ddagger} \circ \pi_1$ is also injective on $\overline{F}$. Since $U_0(C)/CU(C)$ is divisible (see, for example, Lemma 11.5), there is a homomorphism $\lambda : K_1(D) \to U_0(C)/CU(C)$ such that

$$\lambda|_{\kappa_1 \circ \Pi^{\ddagger} \circ \pi_1(\overline{F})} = \pi_0 \circ ((\kappa_1 \circ \Pi^{\ddagger} \circ \pi_1)|_{\overline{F}})^{-1},$$

where $((\kappa_1 \circ \Pi^{\ddagger} \circ \pi_1)|_{\overline{F}})^{-1} : \quad \eta \circ \pi_1(\overline{F}) \to \overline{F}$ is the inverse of the injective map $(\kappa_1 \circ \Pi^{\ddagger} \circ \pi_1)|_{\overline{F}}$. This could be viewed as the following commutative diagram:

$$\begin{tikzcd}
& K_1(D) \arrow[dr, "\lambda"] & \\
(\eta \circ \pi_1)(\bar{F}) \arrow[ur] \arrow[rr, "\pi_0\circ(\eta\circ\pi_1)^{-1}"] & & U_0(C)/CU(C).
\end{tikzcd}$$

Define

$$\beta = \Lambda((\eta^{-1} \circ \kappa_1) \oplus (\lambda \circ \kappa_1)),$$

a homomorphism from $U(D)/CU(D)$ to $U(B)/CU(B)$. Then, for any $\overline{w} \in \overline{F}$,

$$\beta(\Pi^{\ddagger} \circ \pi_1(\overline{w})) = \Lambda(\eta^{-1}(\kappa_1 \circ \Pi^{\ddagger}(\pi_1(\overline{w}))) \oplus \lambda \circ \kappa_1(\Pi^{\ddagger}(\pi_1(\overline{w})))) = \Lambda(\pi_1(\overline{w}) \oplus \pi_0(\overline{w})).$$

Recall that $\theta : \pi_2(U(C)/CU(C)) \to U(B)/CU(B)$ is defined by

$$\theta(x) = \Lambda(x^{-1}) \text{ for all } x \in \pi_2(U(C)/CU(C)).$$

Then

$$\beta(\Pi^{\ddagger}(\pi_1(\overline{w}))) = \Lambda(\overline{w})\theta(\pi_2(\overline{w})) \text{ for all } w \in F. \tag{e 21.2}$$

For the second part of the statement, one assumes that $\Lambda(G) \subset U_0(B)/CU(B)$. Then $\Lambda(\pi_2(U(C)/CU(C)))$ is a torsion subgroup of $U_0(B)/CU(B)$. But $U_0(B)/CU(B)$ is torsion free, as $B = B_1 \otimes V$, by Lemma 11.5, and hence $\theta = 0$. Thus, in the first diagram, with five groups, above, $(\Lambda + \theta \circ \pi_2)|_{\overline{F}} = \Lambda|_{\overline{F}}$, or with multiplicative notation, $\theta(\pi_2(\overline{w})) = \bar{1}$. The second part of the lemma follows from (e 21.2).

☐

Lemma 21.3 (see Lemma 7.3 of [71]). *Let $B \in \mathcal{B}_1$ be a separable simple C^*-algebra, and let C be as defined in 21.1. Let $\mathcal{U} \subset U(B)$ be a finite subset, and suppose that, with F the subgroup generated by $\mathcal{U}$, $\kappa_1^B(\bar{F})$ is free abelian, where $\kappa_1^B : U(B)/CU(B) \to K_1(B)$ is the quotient map. Suppose that $\alpha : K_1(C) \to K_1(B)$ is an injective homomorphism and $L : \bar{F} \to U(C)/CU(C)$ is an injective homomorphism with $L(\bar{F} \cap U_0(B)/CU(B)) \subset U_0(C)/CU(C)$ such that $\pi_1 \circ L$ is injective and*

$$\alpha \circ \kappa_1^C \circ L(g) = \kappa_1^B(g) \quad \textit{for all} \quad g \in \bar{F},$$

where $\kappa_1^C : U(C)/CU(C) \to K_1(C)$ is the quotient map. Then there exists a homomorphism $\beta : U(C)/CU(C) \to U(B)/CU(B)$ with $\beta(U_0(C)/CU(C)) \subset U_0(B)/CU(B)$ such that

$$\beta \circ L(f) = f \; \textit{for all} \; f \in \bar{F}.$$

Part of the statement can be summarized by the following commutative diagram:

$$\begin{tikzcd}
& \bar{F} \arrow[dl, "inclusion"'] \arrow[dr, "L"] & \\
U(B)/CU(B) \arrow[d, "\kappa_1^B"] & & U(C)/CU(C) \arrow[ll, dashed, "\beta"] \arrow[d, "\kappa_1^C"] \\
K_1(B) & & K_1(C). \arrow[ll, "\alpha"]
\end{tikzcd}$$

PROOF. We will repeat the proof of Lemma 7.3 of [71].

Let G be the preimage of the subgroup $\alpha\circ\kappa_1^C(U(C)/CU(C))$ of $U(B)/CU(B)$ under κ_1^B. So we have the short exact sequence

$$0\to U_0(B)/CU(B)\to G\to \alpha\circ\kappa_1^C(U(C)/CU(C))\to 0.$$

Since $U_0(B)/CU(B)$ is divisible (see 11.5), there is an injective homomorphism

$$\lambda:\alpha\circ\kappa_1^C(U(C)/CU(C))\to G$$

such that $\kappa_1^B\circ\lambda(g)=g$ for any $g\in\alpha\circ\kappa_1^C(U(C)/CU(C))$. Since $\alpha\circ\kappa_1^C\circ L(f)=\kappa_1^B(f)$ for any $f\in\overline{F}$, we have $\overline{F}\subset G$. Moreover, note that

$$(\lambda\circ\alpha\circ\kappa_1^C\circ L(f))^{-1}f\in U_0(B)/CU(B)\ \text{ for all }\ f\in\overline{F}.$$

Define $\psi:L(\overline{F})\to U_0(B)/CU(B)$ by

$$\psi(x)=\lambda\circ\alpha\circ\kappa_1^C(x^{-1})L^{-1}(x)$$

for $x\in L(\overline{F})$. Since $U_0(B)/CU(B)$ is divisible, there is a homomorphism $\tilde{\psi}:U(C)/CU(C)\to U_0(B)/CU(B)$ such that $\tilde{\psi}|_{L(\overline{F})}=\psi$. Now define

$$\beta(x)=\lambda\circ\alpha\circ\kappa_1^C(x)\tilde{\psi}(x)\ \text{ for all }\ x\in U(C)/CU(C).$$

Then $\beta(L(f))=f$ for $f\in\overline{F}$. □

LEMMA 21.4. *Let A be a unital separable C^*-algebra such that the subgroup generated by $\{\rho_A([p]):p\in A\text{ aprojection}\}$ is dense in its real linear span. Then, for any finite dimensional C^*-subalgebra $B\subset A$ (with $1_B=1_A$), $U(B)\subset CU(A)$.*

PROOF. Let $u\in U(B)$. Since B is finite dimensional, $u=\exp(i2\pi h)$ for some $h\in B_{s.a.}$. We may write $h=\sum_{i=1}^n\lambda_ip_i$, where $\lambda_i\in(0,1]$ and $\{p_1,p_2,...,p_n\}$ is a set of mutually orthogonal projections. Let $h(t)=\sum_{j=1}^n t\lambda_ip_i$ and $u(t)=\exp(ih(t))$ ($t\in[0,1]$). Then $\hat{h}$ is in the real linear span of $\{\rho_A([p]):p\in A\text{ a projection}\}$, where $\hat{h}(\tau)=\tau(h(t))$ for all $\tau\in T(A)$ and $t\in[0,1]$. By the assumption, this implies that $\Delta^1(u(t))=\hat{h}\in\overline{\rho_A^1(K_0(A))}$ (notation as in 2.8 and 2.10 of [46]). Then, applying Proposition 3.6 (2) of [46] with $k=1$, one has $u\in CU(A)$. □

LEMMA 21.5 (see Lemma 7.4 of [71]). *Let $B\cong A\otimes V$, where $A\in\mathcal{B}_0$ and V is an infinite dimensional UHF-algebra. Let $C=D\oplus C_0\oplus C_1$ be as defined in* 21.1. *Let F be a group generated by a finite subset $\mathcal{U}\subset U(C)$ such that $(\pi_1)|_{\overline{F}}$ is injective, where $\overline{F}$ is the image of F in $U(C)/CU(C)$. Suppose that $\alpha:U(C)/CU(C)\to U(B)/CU(B)$ is a homomorphism such that $\alpha(U_0(C)/CU(C))\subset U_0(B)/CU(B)$. Then, for any $\epsilon>0$, there are $\sigma>0$, $\delta>0$, and a finite subset $\mathcal{G}\subset C$ satisfying the following condition: if $\varphi=\varphi_0\oplus\varphi_1:C\to B$ (by such a decomposition, we mean there is a projection e_0 such that $\varphi_0:C\to e_0Be_0$ and $\varphi_1:C\to(1_B-e_0)B(1_B-e_0)$) are unital $\mathcal{G}$-δ-multiplicative completely positive linear maps such that*

(1) *φ_0 sends the identity of each direct summand of C to a projection and is non-zero on each (non-zero) direct summand of D,*

(2) *$\mathcal{G}$ is sufficiently large and δ is sufficiently small, depending only on F and C, that $\varphi^{\ddagger}$, $\varphi_0^{\ddagger}$, and $\varphi_1^{\ddagger}$ (associated with a finite subset containing $\mathcal{U}$ and $\varepsilon/4$—see 2.17) are well defined on a subgroup of $U(C)/CU(C)$ containing all of $\bar{F}$, $\pi_0(\bar{F})$, $\pi_1(U(C)/CU(C))$, and $\pi_2(U(C)/CU(C))$,*

(3) *$[\varphi]|_{K_1(C)} = \alpha_*$ where $\alpha_* : K_1(C) \to K_1(B)$ is the map induced by α, and $[\varphi_0] = [\varphi_{00}]$ in $KK(C, B)$ (note that $K_i(C)$ is finitely generated—see the end of 2.12), where φ_{00} is a homomorphism from C to B which has a finite dimensional image, and*

(4) *$\tau(\varphi_0(1_C)) < \sigma$ for all $\tau \in T(B)$ (assume $e_0 = \varphi_0(1_C)$),*

then there is a homomorphism $\Phi : C \to e_0Be_0$ such that

(i) *$[\Phi] = [\varphi_{00}]$ in $KK(C, B)$ and*

(ii) *$\alpha(\bar{w})^{-1}(\Phi \oplus \varphi_1)^{\ddagger}(\bar{w}) = \bar{g_w}$ where $g_w \in U_0(B)$ and $\mathrm{cel}(g_w) < \epsilon$ for any $w \in \mathcal{U}$ (see 2.17).*

Proof. The argument is almost exactly the same as that of Lemma 7.4 of [71] (see also [33] and [89]). Since the source algebra and target algebra in this lemma are different from the ones in 7.4 of [71], we will repeat most of the argument here. We would like to point out that, since $K_1(D)$ is free abelian, so also is $\pi_1(\overline{F})$. Also $\gamma(\overline{F})$ is free abelian, as $U(D)/U_0(D)$ is free abelian, where $\gamma : \overline{F} \to U(D)/CU(D) \to U(D)/U_0(D)$ is the composed map. Since π_1 is injective on $\overline{F}$, $\overline{F} \cap (U(C_0)/CU(C_0)) = \{\bar{1}\}$. We will retain the notation of 21.1.

We rewrite $D = \bigoplus_{i=1}^{s} E_i$, where $E_i = M_{r_i}(C(\mathbb{T}))$. Since E_i are semiprojective, we may assume $\varphi_0|_D$ and $\varphi_1|_D$ are homomorphisms, choosing $\mathcal{G}$ sufficiently large and δ sufficiently small. Denote by $\Pi_i : D \to E_i$ the quotient map. Let $z_i' \in M_{r_i}(C(\mathbb{T}))$ be given by $z_i'(z) = \mathrm{diag}(z, 1, \cdots, 1)$ for any $z \in \mathbb{T}$, and $z_i \in D$ be defined by $z_i = (1_{r_1}, \cdots, 1_{r_{i-1}}, z_i', 1_{r_{i+1}}, \cdots, 1_{r_s})$. Let S_1 be the subgroup of $\pi_1(U(C)/CU(C)) \cong K_1(D)$ (recall that we may view $\pi_1(U(C)/CU(C))$ as a subgroup of $U(C)/CU(C)$; see (e 21.1)) generated by $\{\overline{z_1'}, \overline{z_2'}, ..., \overline{z_s'}\}$. Then $S_1 = \pi_1(U(C)/CU(C))$ and $\pi_1(\overline{F}) \subset S_1$. Thus one obtains a finitely generated subgroup F_1 of $U(C)$ such that $F \subset F_1$ and $\overline{F_1} \supset S_1$. Moreover, $\pi_1|_{\overline{F_1}}$ is injective and $\pi_1(\overline{F_1}) = S_1 \supset \pi_1(\overline{F})$. There is a finite subset $\mathcal{U}_1 \subset U(C)$ which generates the subgroup F_1.

Therefore, to simplify matters, from now on, we may assume that $\mathcal{U} = \mathcal{U}_1$. and $\overline{F} = \overline{F_1}$. In particular, $\kappa_1^C(\overline{F_1}) = K_1(D)$, where $\kappa_1^C : U(C)/CU(C) \to K_1(C)$ is the quotient map. Let $G = \overline{F_1} \oplus \pi_2(U(C)/CU(C)) = \overline{F_1} \oplus \mathrm{Tor}(K_1(C))$. G is a finitely generated subgroup of $U(C)/CU(C)$ which contains $\overline{F}$ and

$$\pi_1(U(C)/CU(C)) \oplus \pi_2(U(C)/CU(C)).$$

Let $w \in U(C)$. We write $w = (w_1, w_2, w_3)$, where $w_1 \in D$ and $w_2 \in C_0$ and $w_3 \in C_1$ and $\pi_1(\bar{w}) = \pi_1(\bar{w}_1) = (\overline{z_1}^{k(1,w)}, \overline{z_2}^{k(2,w)}, ..., \overline{z_s}^{k(s,w)})$, where $k(i, w)$

is an integer, $1 \leq i \leq s$. Then $\Pi_i^{\ddagger}(\pi_1(\overline{w_1})) = \overline{z_i}^{k(i,w)}$ and $\Pi_i^{\ddagger}(\overline{w_1}) = \overline{z_i^{k(i,w)} g_{i,w}}$, where $g_{i,w} \in U_0(E_i)$, $i = 1, 2, ..., s$. Note that one can choose a function $g \in C(\mathbb{T})$ of the form $g(e^{2\pi i t}) = e^{2\pi i f(t)}$ such that $\det(g_{i,w}(z)) = \det(g(z) \cdot 1_{E_i})$. That is, $g_{i,w} = g \cdot 1_{E_i}$ in $U_0(E_i)/CU(E_i)$. Therefore, we can write

$$\Pi_i^{\ddagger}(\overline{w_1}) = \overline{z_i^{k(i,w)} g_{i,w} \cdot 1_{E_i}}. \tag{e 21.3}$$

Later, in this proof, we will view $g_{i,w}$ as a complex-valued continuous function on $\mathbb{T}$.

Let $l = \max\{\mathrm{cel}(g_{i,w}) : 1 \leq i \leq s, w \in \mathcal{U}\}$. Choose an integer $n_0 \geq 1$ such that $(2 + l)/n_0 < \varepsilon/4\pi$. Choose $0 < \sigma < 1/(n_0 + 1)$.

Since $K_i(C)$ is finitely generated ($i = 0, 1$), we may assume, with sufficiently small δ and large $\mathcal{G}$, that $[\psi]$ gives an element of $KK(C, B)$ (see 2.12) and homomorphisms $\psi^{\ddagger}$ can be defined on G (defined above) for any contractive completely positive linear map $\psi : C \to A'$ (for any unital C^*-algebra A') which maps $G \cap (U_0(C)/CU(C))$ into $U_0(A')/CU(A')$, and $\mathrm{dist}(\psi^{\ddagger}(\overline{u}), \overline{\langle \psi(u) \rangle}) < \varepsilon/16$ for all $u \in \mathcal{U}$ (see 2.17). Fix such δ and $\mathcal{G}$. Let φ, φ_0, and φ_1 be as given. Define $\varphi_0^{\ddagger}$ and $\varphi_1^{\ddagger}$ as in 2.17. Let $\varphi^{\ddagger}$ be chosen to be $\varphi_0^{\ddagger} + \varphi_1^{\ddagger}$. We note that $\varphi^{\ddagger}, \varphi_0^{\ddagger}$, and $\varphi_1^{\ddagger}$ are defined on G and map $G \cap (U_0(C)/CU(C))$ into $U_0(B)/CU(B)$. Define $L' : G \to U(B)/CU(B)$ by $L'(g) = \varphi_1^{\ddagger}(g^{-1})$ for all $g \in G$. Applying Lemma 21.2 twice (once for the case that α plays the role of L and then again for the case that L' plays the role of L), one obtains homomorphisms $\beta_1, \beta_2 : U(D)/CU(D) \to U(B)/CU(B)$ with $\beta_i(U_0(D)/CU(D)) \subset U_0(B)/CU(B)$ ($i = 1, 2$) such that

$$\beta_1 \circ \Pi^{\ddagger}(\pi_1(\bar{w})) = \alpha(\bar{w})\theta_1(\pi_2(\bar{w})) \ \text{ and } \ \beta_2 \circ \Pi^{\ddagger}(\pi_1(\bar{w})) = \varphi_1^{\ddagger}(\bar{w}^*)\theta_2(\pi_2(\bar{w})) \tag{e 21.4}$$

for all $\bar{w} \in \bar{F}$, where $\theta_1, \theta_2 : \pi_2(U(C)/CU(C)) \to U(B)/CU(B)$ are defined by $\theta_1(g) = \alpha(g^{-1})$ and $\theta_2(g) = \varphi_1{}^{\ddagger}(g) = L'(g^{-1})$ for all $g \in \pi_2(U(C)/CU(C))$. Denote by $\kappa_1 : U(B)/CU(B) \to K_1(B)$ the quotient map. Then, by (3) (as $[\varphi_{00}]|_{K_1(C)} = 0$) for any $g \in \pi_2(\overline{F})$,

$$(\kappa_1 \circ \theta_1)(g) = \kappa_1(\alpha(g^{-1})) = (\kappa_1 \circ \varphi^{\ddagger}(g))^{-1} = (\kappa_1 \circ \theta_2(g))^{-1}. \tag{e 21.5}$$

Thus, by (e 21.5),

$$\theta_1(g)\theta_2(g) \in \ker\kappa_1 = U_0(B)/CU(B) \ \text{ for all } \ g \in \pi_2(\bar{F}). \tag{e 21.6}$$

Since $\pi_2(U(C)/CU(C))$ is torsion and $U_0(B)/CU(B)$ is torsion free (see Lemma 11.5), we have

$$\theta_1(g)\theta_2(g) = \bar{1} \ \text{ for all } \ g \in \pi_2(\bar{F}). \tag{e 21.7}$$

Let $e_0 = \varphi_0(1_C)$. Write $e_0 = e_0^d \oplus e_0^0 \oplus e_0^1$, where $e_0^d = \varphi_0(1_D)$, $e_0^0 = \varphi_0(1_{C_0})$, and $e_0^1 = \varphi_0(1_{C_1})$. Let $e_{i,1}$ be a rank one projection in $E_i = M_{r_i}(C(\mathbb{T}))$, $i =$

$1, 2, ..., s$. Let $\{q_{i,j,1} : 1 \le j \le r_i\}$ be a set of mutually orthogonal and mutually equivalent projections in B such that $[q_{i,j,1}] = [\varphi_0(e_{j,1})] = [\varphi_{00}(e_{j,1})]$, $1 \le i \le s$, $1 \le j \le r_i$, and such that $\sum_{j=1}^{r_i} q_{i,j,1} = \varphi_0(1_{E_i})$. Let $q_{i,1,1,x}, q_{i,1,1,y}$ be two non-zero mutually orthogonal projections in B such that $q_{i,1,1,x} + q_{i,1,1,y} = q_{i,1,1}$.

Choose $x_i' \in U(q_{i,1,1,x}Bq_{i,1,1,x})$ and $y_i' \in U(q_{i,1,1,y}Bq_{i,1,1,y})$ such that $\overline{x_i'} = \beta_1(\overline{z_i})$ and $\overline{y_i'} = \beta_2(\overline{z_i})$ and $\overline{y_i}' = \beta_2(\overline{z_i})$, $i = 1, 2, ..., s$. This is possible because of Theorem 11.10. Here we use the simplified notation $\overline{u}$ to denote $\overline{u \oplus (1-p)}$ for any unitary u in the cut-down algebra pBp of B. Put $x_i = x_i' \oplus q_{i,1,1,y}$ and $y_i = y_i' \oplus q_{i,1,1,x}$, $i = 1, 2, ..., s$. Note that $x_iy_i = y_ix_i$.

Let $\tilde{\varphi}^i : C(\mathbb{T}) \to q_{i,1,1}Bq_{i,1,1}$ be defined by $\tilde{\varphi}^i(f) = f(x_iy_i)$ for any $f \in C(\mathbb{T})$ (defined as the continuous function f acting on the unitary element x_iy_i by functional calculus). Identifying $\varphi_0(1_{E_i})B\varphi_0(1_{E_i})$ with $M_{r_i}(q_{i,1,1}Bq_{i,1,1})$, we can define $\Phi_1 : D = \bigoplus_{i=1}^s E_i = \bigoplus_{i=1}^s M_{r_i}(C(\mathbb{T})) \to \bigoplus_{i=1}^s M_{r_i}(q_{i,1,1}Bq_{i,1,1})$ by $\Phi_1 = \bigoplus_{i=1}^s \tilde{\varphi}^i \otimes \mathrm{id}_{r_i}$. It is clear that for any $f \in (C(\mathbb{T})$,

$$(\Phi_1)^\ddagger(f \cdot 1_{E_i}) = \overline{f(x_iy_i)\varphi_0(1_{E_i})}.$$

Then $(\Phi_1)_{*0} = ((\varphi_{00})_{*0})|_{K_0(D)}$. Define $h : C \to B$ by $h(c) = \Phi_1(\Pi(c)) \oplus \varphi_1(c)$ for all $c \in C$ (recall that $\Pi : C \to D$ is the quotient map). Note that $[h]|_{K_1(C)} = [\Phi_1 \circ \Pi]|_{K_1(C)} + [\varphi_1]|_{K_1(C)}$. Define $h^\ddagger = (\Phi_1 \circ \Pi)^\ddagger + \varphi_1^\ddagger$. Recall that any $w \in U(C)$ can be written as $w = (w_1, w_2, w_3) \in D \oplus C_0 \oplus C_1$ such that $\Pi_i^\ddagger(\overline{w_1})$ is as described in (e 21.3). Then

$$(\Phi_1)^\ddagger(\overline{w_1}) = \prod_{i=1}^s \overline{x_i^{k(i,w)}\big(g_{i,w} \cdot 1_{E_i}\big)(x_iy_i)y_i^{k(i,w)}}.$$

We compute (viewing $g_{i,w}$ as a continuous function on $\mathbb{T}$), for all $w \in \mathcal{U}$ (also using (e 21.4) and (e 21.7)),

$$\begin{aligned}
\text{(e 21.8)}\quad h^\ddagger(\overline{w}) &= (\prod_{i=1}^s \overline{x_i^{k(i,w)}\big(g_{i,w} \cdot 1_{E_i}\big)y_i^{k(i,w)}})\varphi_1^\ddagger(\overline{w}) \\
&= \beta_1((\Pi)^\ddagger(\pi_1(\overline{w})))\Phi_1^\ddagger(\bigoplus_{i=1}^s \overline{g_{i,w} \cdot 1_{E_i}})\beta_2((\Pi)^\ddagger(\pi_1(\overline{w})))\varphi_1^\ddagger(\overline{w}) \\
&= \alpha(\overline{w})\theta_1(\pi_2(\overline{w}))\theta_2(\pi_2(\overline{w}))\Phi_1^\ddagger(\bigoplus_{i=1}^s \overline{g_{i,w} \cdot 1_{E_i}}) \\
&= \alpha(\overline{w})\Phi_1^\ddagger(\bigoplus_{i=1}^s \overline{g_{i,w} \cdot 1_{E_i}}).
\end{aligned}$$

Put $g_w' = \Phi_1(\bigoplus_{i=1}^s g_{i,w} \cdot 1_{E_i}) \oplus (1_B - \varphi_0(1_C))$. Recalling $B = A \otimes V$, choose mutually orthogonal and mutually equivalent projections $\{p_1, p_2, ..., p_{n_0}\}$ in $1_A \otimes$

$V \subset B$ such that $\tau(p_i) \geq 1/(n_0+1) > \sigma$ and $\sum_{i=1}^{n_0} \tau(p_i) < 1-\sigma$ for all $\tau \in T(A)$. Since $\tau(\varphi_0(1_C)) < \sigma$ for all $\tau \in T(B)$, and B has strict comparison (see 9.11), $(1-\varphi_0(1_C))B(1-\varphi_0(C))$ contains n_0 mutually orthogonal and mutually equivalent projections which are equivalent to $\varphi_0(1_C)$. By Lemma 6.4 of [71], there exists $g_{w,-} \in CU(B)$ such that $\mathrm{cel}(g_w' g_{w,-}) < (l/n_0)\pi < \varepsilon/2$.

Since $[\varphi_0] = [\varphi_{00}]$ in $KK(C,B)$, by (3), $[\varphi_1]|_{K_1(C)} = \alpha_*$. It follows from (e 21.8) that $h_{*1} = \alpha_*$. Therefore $(\Phi_1)_{*1} = 0$. Thus $[\Phi_1] = [\varphi_{00}|_D]$ in $KK(D,B)$. Define $\Phi_2 : C_0 \oplus C_1 \to (e_0^0 \oplus e_0^1)B(e_0^0 \oplus e_0^1)$ by $\Phi_2 = (\varphi_{00})|_{C_0 \oplus C_1}$. Define $\Phi : C \to e_0Be_0$ by $\Phi((f,g)) = \Phi_1(f) \oplus \Phi_2(g)$ for $f \in D$ and $g \in C_0 \oplus C_1$. Thus (i) holds.

For $w \in \mathcal{U}$, put $w'' = \Phi_2(\Pi^c(w)) \oplus (1_B - e_0^d)$, where $\Pi^c : C \to C_0 \oplus C_1$ is the quotient map. Note, since $B = A \otimes V$, by 2.25, the subgroup generated $\{\rho_B(p) : p \in B\}$ is dense in the real linear span of $\{\rho_B(p) : p \in B\}$. By 21.4, $\overline{w''} \in CU(B)$, as Φ_2 has finite dimensional range. Thus, for $w \in \mathcal{U}$, let $g_w = g_w' g_{w-}$, Then, $\mathrm{cel}(g_w) < \varepsilon$. Moreover, by (e 21.8),

$$\alpha(\overline{w})^{-1}(\Phi \oplus \varphi_1)^{\ddagger}(\bar{w}) = \alpha(\overline{w})^{-1}\overline{h(w)}\overline{w''} = \overline{g_w'} = \overline{g_w' g_{w-}} = \overline{g_w}. \tag{e 21.9}$$

Thus (ii) holds. □

LEMMA 21.6 (see Lemma 7.5 of [71]). *Let $B \cong A \otimes V$, where $A \in \mathcal{B}_1$ and V is an infinite dimensional UHF-algebra. Let $\mathcal{U} \subset U(B)$ be a finite subset such that $\kappa_1^B(\bar{F})$ is free abelian, where F is the subgroup generated by $\mathcal{U}$, and $\kappa_1^B : U(B)/CU(B) \to K_1(B)$ is the quotient map. Let $C = D \oplus C_0 \oplus C_1$ be as defined in 21.1 and let $\gamma : U(C)/CU(C) \to U(B)/CU(B)$ be a continuous homomorphism such that $\gamma_* := \gamma|_{K_1(C)}$ is injective (viewing $K_1(C) \subset U(C)/CU(C)$, see* (e 21.1)*). Suppose that $j, L : \overline{F} \to U(C)/CU(C)$ are two injective homomorphisms with $j(\overline{F \cap U_0(B)})$, $L(\overline{F \cap U_0(B)}) \subset U_0(C)/CU(C)$ such that $\kappa_1^B \circ \gamma \circ L = \kappa_1^B \circ \gamma \circ j = \kappa_1^B|_{\bar{F}}$.*

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following condition: Suppose that there is a homomorphism $\varphi : C \to B$ such that $\varphi^{\ddagger} = \gamma$, $\varphi_{*1} = \gamma|_{K_1(C)}$, and $\varphi = \varphi_0 \oplus \varphi_1 : C \to B$, where φ_0 and φ_1 are homomorphisms such that

1. *$\tau(\varphi_0(1_C)) < \delta$ for all $\tau \in T(B)$ and*
2. *$[\varphi_0]$= $[\varphi_{00}]$, where $\varphi_{00} : C \to B$ is a homomorphism with finite dimensional image and φ_{00} is not zero on each summand of D,*

then there is a homomorphism $\psi : C \to e_0Be_0$ ($e_0 = \varphi_0(1_C)$) such that

3. *$[\psi] = [\varphi_0]$ in $KL(C,B)$ and*
4. *$(\varphi^{\ddagger} \circ j(\bar{w}))^{-1}(\psi \oplus \varphi_1)^{\ddagger}(L(\bar{w})) = \overline{g_w}$ where $g_w \in U_0(B)$ and $\mathrm{cel}(g_w) < \epsilon$ for any $w \in \mathcal{U}$.*

PROOF. First, if $z \in \overline{F}$ and $\pi_1(L(z)) = \bar{1}$, then $L(z) = \pi_0(L(z)) \oplus \pi_2(L(z))$. Therefore, $\kappa_1^B \circ \varphi^{\ddagger}(L(z)) = \kappa_1^B \circ \varphi^{\ddagger}(\pi_2(L(z))$ is a torsion element as $\pi_2(U(C_1)/CU(C_1))$ is torsion. Since $\kappa_1^B(\overline{F})$ is free abelian and $\kappa_1^B \circ \gamma \circ L = \kappa_1^B|_{\bar{F}}$, we have $\kappa_1^B \circ \varphi^{\ddagger}(L(z)) = \bar{1}$. Since $\kappa_1^B \circ \varphi^{\ddagger} \circ L$ is injective, $L(z) = \bar{1}$. This implies that $\pi_1|_{(L(\overline{F}))}$ is injective. Note also $\pi_1(L(\overline{F}))$ is free . Exactly the same reason implies that $\pi_1|_{j(\overline{F})}$ is injective and $\pi_1(j(\overline{F}))$ is free abelian.

Now we repeat the proof of Lemma 7.5 of [71]. Let $\kappa_1^C : U(C)/CU(C) \to K_1(C)$ be the quotient map and $G = (\kappa_1^B)^{-1}(\gamma_* \circ \kappa_1^C(U(C)/CU(C)))$. Consider the short exact sequence

$$0 \to U_0(B)/CU(B) \to G \xrightarrow{\kappa_1^B} \gamma_* \circ \kappa_1^C(U(C)/CU(C)) \to 0.$$

Since $U_0(B)/CU(B)$ is divisible, there exists an injective homomorphism $\lambda : \gamma_* \circ \kappa_1^C(U(C)/CU(C)) \to G$ such that $\kappa_1^B \circ \lambda(g) = g$ for all $g \in \gamma_* \circ \kappa_1^C(U(C)/CU(C))$. Since $\kappa_1^B \circ \gamma \circ L(f) = \kappa_1^B(f) = \kappa_1^B \circ \gamma \circ j(f) \subset \varphi_{*1}(K_1(C)) = \varphi_{*1}(\kappa_1^C(C(U(C)/CU(C)))$ for all $f \in \overline{F}$, one obtains $\overline{F} \subset G$. Note that $\gamma_* \circ \kappa_1^C = \kappa_1^B \circ \gamma$. Thus, for any $f \in \overline{F}$,

$$\begin{aligned}\kappa_1^B((\lambda &\circ \gamma_* \circ \kappa_1^C \circ L(f))^{-1}(\gamma \circ j(f))) \\ &= (\gamma_* \circ \kappa_1^C \circ L(f))^{-1}\kappa_1^B(\gamma \circ j(f)) \\ &= (\kappa_1^B \circ \gamma \circ L(f))^{-1}(\kappa_1^B \circ \gamma \circ j(f)) = [1_B].\end{aligned}$$

It follows that

$$(\lambda \circ \gamma_* \circ \kappa_1^C \circ L(f))^{-1}(\varphi^\ddagger \circ j(f)) \in U_0(B)/CU(B) \text{ for all } f \in \overline{F}.$$

Define $\zeta : L(\overline{F}) \to U_0(B)/CU(B)$ by

$$\zeta(x) = \big((\lambda \circ \gamma_* \circ \kappa_1^C(x)\big)^{-1}(\gamma \circ j \circ L^{-1}(x)) \text{ for all } x \in L(\overline{F}).$$

Since $U_0(B)/CU(B)$ is divisible, there exists $\tilde{\zeta} : U(C)/UC(C) \to U_0(B)/CU(B)$ such that $\tilde{\zeta}|_{L(\overline{F})} = \zeta$. Define $\alpha : U(C)/CU(C) \to U(B)/CU(B)$ by $\alpha(x) = (\lambda \circ \gamma_* \circ \kappa_1^C(x))\tilde{\zeta}(x)$ for all $x \in U(C)/CU(C)$. Note that, for all $x \in U(C)/CU(C)$,

$$\alpha(\kappa_1^C(x)) = (\lambda \circ \gamma_* \circ \kappa_1^C(x))\tilde{\zeta}(\kappa_1^C(x)) = \lambda \circ \gamma_* \circ \kappa_1^C(x) = \gamma_* \circ \kappa_1^C(x).$$

In other words, $\alpha_* := \alpha|_{K_1(C)} = \gamma_*$. Note also that

$$\alpha(L(f)) = \gamma \circ j(f) \text{ for all } f \in \overline{F}.$$

Let $\mathcal{U}' \subset U(C)$ be a finite subset such that the subgroup $F' \subset U(C)$ generated by $\mathcal{U}'$ satisfies $\overline{F'} = L(\overline{F})$. Let $\varepsilon > 0$. Choose $\delta = \sigma$ as in 21.5 associated $\mathcal{U}'$ (in place of $\mathcal{U}$), F' (in place of F), α as mentioned above, and $L(\overline{F})$ (in place of $\overline{F}$). Suppose that $\varphi = \varphi_0 \oplus \varphi_1$ as described. Applying 21.5, one obtains a homomorphism $\psi : C \to e_0Be_0$ such that $[\psi] = [\varphi_{00}]$ in $KK(C, B)$ and

$$(\alpha(z))^{-1}(\psi \oplus \varphi)(z) = \overline{f_z} \text{ and } \mathrm{cel}(f_z) < \varepsilon,$$

where $f_z \in U(B)$, for all $z \in L(\overline{F})$. Then, for any $w \in \mathcal{U}$, let $g_w = f_{L(\overline{w})}$, so that

$$(\varphi^\ddagger \circ j(\overline{w}))^{-1}(\psi \oplus \varphi)(\overline{w}) = \alpha(L(\overline{w})^{-1}(\psi \oplus \varphi)(L(\overline{w}) = \overline{g_w} \text{ and } \mathrm{cel}(g_w) < \varepsilon.$$

This shows that ψ has the desired properties. □

REMARK 21.7. The roles that Lemma 21.5 and Lemma 21.6 play in the proof of the isomorphism theorem, Theorem 21.9 below, are the same as those played by Lemma 7.4 and 7.5 of [71] in the proof of Theorem 10.4 of [71].

The following statement is well known. For the reader's convenience, we include a proof.

LEMMA 21.8. *Let $(A_n, \varphi_{n,n+1})$ be a unital inductive sequence of separable C^*-algebras, and consider the inductive limit $A = \varinjlim A_n$. Assume that A is amenable. Let $\mathcal{F} \subset A$ be a finite subset, and let $\varepsilon > 0$. Then there is an integer $m \geq 1$ and a unital completely positive linear map $\Psi : A \to A_m$ such that*

$$\|\varphi_{m,\infty} \circ \Psi(f) - f\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

PROOF. Regard A as the C^*-subalgebra of $\prod A_n / \bigoplus A_n$ consisting of the equivalence classes of the sequences $(x_1, x_2, ..., x_n, ...)$ such that that there is N with $x_{n+1} = \varphi_n(x_n)$, $n = N, N+1, ...$. Since A is amenable, by the Choi-Effros lifting theorem, there is a unital completely positive linear map $\Phi : A \to \prod A_n$ such that $\pi \circ \Phi = \mathrm{id}_A$, where $\pi : \prod A_n \to \prod A_n / \bigoplus A_n$ is the quotient map. In particular, this implies that

$$\lim_{k\to\infty} \|\pi_k \circ \Phi(a) - a_k\| = 0, \tag{e 21.10}$$

if $a = \pi((a_1, a_2, ..., a_k, ...)) \in A$.

Write $\mathcal{F} = \{f_1, f_2, ..., f_l\}$, and for each f_i, fix a representative

$$f_i = \pi((f_{i,1}, f_{i,2}, ..., f_{i,k}, ...)).$$

In particular

$$\lim_{k\to\infty} \varphi_{k,\infty}(f_{i,k}) = f_i. \tag{e 21.11}$$

Then, for each f_i, one has

$$\begin{aligned} & \limsup_{k\to\infty} \|\varphi_{k,\infty} \circ \pi_k \circ \Phi(f_i) - f_i\| \\ = \ & \limsup_{k\to\infty} \|\varphi_{k,\infty} \circ \pi_k \circ \Phi(f_i) - \varphi_{k,\infty}(f_{i,k})\| \quad \text{(by (e 21.11))} \\ \leq \ & \limsup_{k\to\infty} \|\pi_k \circ \Phi(f_i) - f_{i,k}\| = 0 \quad \text{(by (e 21.10))}. \end{aligned}$$

There then exists $m \in \mathbb{N}$ such that

$$\|\varphi_{m,\infty} \circ \pi_m \circ \Phi(f_i) - f_i\| < \epsilon, \quad 1 \leq i \leq l.$$

This shows that the unital completely positive linear map $\Psi := \pi_m \circ \Phi$ satisfies the condition of the lemma. $\square$

THEOREM 21.9. *Let $A_1 \in \mathcal{B}_0$ be a unital separable simple amenable C^*-algebra satisfying the UCT, and let $A = A_1 \otimes V$ for some UHF-algebra U of infinite type. Let C be a C^*-algebra constructed as in Theorem* 14.10 *(denoted by A there) such that* $\mathrm{Ell}(C) \cong \mathrm{Ell}(A)$. *Then there is an isomorphism $\varphi : C \to A$ which carries the identification of* $\mathrm{Ell}(C) \cong \mathrm{Ell}(A)$.

PROOF. Let $C = \varinjlim(C_n, \iota_{n,n+1})$ be as constructed in 14.10 (as

$$A = \varinjlim(A_n, \psi_{n,n+1})$$

with $\iota_{n,n+1}$ in place of $\psi_{n,n+1}$ there), where $\iota_{n,n+1}$ is injective, unital, and has the decomposition $\iota_{n,n+1} = \iota^{(0)}_{n,n+1} \oplus \iota^{(1)}_{n,n+1}$. Put $\iota_n = \iota_{n,\infty}$, $n = 1, 2, ...$ By 14.13, we may assume that $C \in B_{u0}$. Note also $A \in \mathcal{B}_{u0}$. The proof will use the fact that both A and C have stable rank 1 (see Theorem 9.7) without further notice. We will also use $\kappa_1^B : U(B)/CU(B) \to K_1(B)$ for the quotient map for any unital C^*-algebra B with the property $U(B)/U_0(B) = K_1(B)$. We will fix splitting maps: $J_c^C : K_1(C) \to U(C)/CU(C)$ and $J_c^A : K_1(A) \to U(A)/CU(A)$ such that $\kappa_1^C \circ J_c^C = \mathrm{id}_{K_1(C)}$ and $\kappa_1^A \circ J_c^A = \mathrm{id}_{K_1(A)}$ (see 2.16).

Let $\gamma : T(A) \to T(C)$ be as given by the isomorphism $\mathrm{Ell}(C) \cong \mathrm{Ell}(A)$, and choose $\alpha \in KL(C, A)$ with $\alpha^{-1} \in KL(A, C)$ lifting the K_i-group isomorphisms from the isomorphism $\mathrm{Ell}(C) \cong \mathrm{Ell}(A)$ (see 2.4).

Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset C$ and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset A$ be increasing sequences of finite subsets with dense union. Let $1/2 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$ be a decreasing sequence of positive numbers with finite sum. Let $\mathcal{P}_{c,n} \subset \underline{K}(C)$ be finite subsets such that $\mathcal{P}_{c,n} \subset \mathcal{P}_{c,n+1}$ $(n \geq 1)$ and $\bigcup_{n=1}^{\infty} \mathcal{P}_{c,n} = \underline{K}(C)$, let $\mathcal{Q}_{c,n} \subset \underline{K}(A)$ be finite subsets such that $\mathcal{Q}_{c,n} \subset \mathcal{Q}_{c,n+1}$ $(n \geq 1)$ and $\bigcup_{n=1}^{\infty} \mathcal{Q}_{c,n} = \underline{K}(A)$, and let $\mathcal{H}(c, n) \subset C_{s.a.}$ and $\mathcal{H}(a, n) \subset A_{s.a.}$ be finite subsets such that $\mathcal{H}(c, n) \subset \mathcal{H}(c, n+1)$ $(n \geq 1)$ and $\bigcup_{n=1}^{\infty} \mathcal{H}(c, n)$ is dense in $C_{s.a.}$, and $\mathcal{H}(a, n) \subset \mathcal{H}(a, n+1)$ $(n \geq 1)$ and $\bigcup_{n=1}^{\infty} \mathcal{H}(a, n)$ is dense in $A_{s.a.}$. We may assume that $\mathcal{P}_{c,n}$ is in the image of $\underline{K}(C_{m(n)})$ under $\iota_{m(n)*}$ for some $m(n) \geq 1$.

We will repeatedly apply the part (a) of Theorem 12.11. Let $\delta_c^{(1)} > 0$ (in place of δ), $\mathcal{G}_c^{(1)} \subset C$ ($\mathcal{G}_c^{(1)}$ in place of $\mathcal{G}$ and C in place of A), $\sigma_{c,1}^{(1)}, \sigma_{c,2}^{(1)} > 0$ (in place of σ_1 and σ_2), $\mathcal{P}_c^{(1)} \subset \underline{K}(C)$ (in place of $\mathcal{P}$), $\overline{\mathcal{U}_c^{(1)}} \subset U(C)/CU(C))$ (in place of $\mathcal{U}$), and $\mathcal{H}_c^{(1)} \subset C_{s.a}$ (in place of $\mathcal{H}$) be as provided by the part(a) of 12.11 for C (in place of A), ε_1 (in place of ε), and $\mathcal{G}_1$ (in place of $\mathcal{F}$). Here, $\mathcal{U}_c^{(1)}$ is a finite subset of $U(C)$. As in Remark 12.12, we may assume that $\overline{\mathcal{U}_c^{(1)}} \subset J_c^C(K_1(C))$. We may also assume that the image of $\mathcal{U}_c^{(1)}$ in $K_1(C)$ is contained in $\mathcal{P}_c^{(1)}$. As in Remark 12.12, we may also assume that $\mathcal{U}_c^{(1)}$ is in the image of $U(C_n)$ for all $n \geq n_0$ for some large $n_0 \geq m(1)$ under the map ι_n. We may further assume that $\mathcal{P}_{c,1} \subset \mathcal{P}_c^{(1)}$ and $\mathcal{H}(c, 1) \subset \mathcal{H}_c^{(1)}$.

Denote by $F_c^{(1)} \subset U(C)$ the subgroup generated by $\mathcal{U}_c^{(1)}$. We may write $\overline{F_c}^{(1)} = (\overline{F_c^{(1)}})_0 \oplus \mathrm{Tor}(\overline{F_c^{(1)}})$ where $(\overline{F_c^{(1)}})_0$ is torsion free, as $F_c^{(1)}$ is finitely generated. Note $\overline{F_c}^{(1)}$, $(\overline{F_c^{(1)}})_0$, and $\mathrm{Tor}(\overline{F_c^{(1)}})$ are in $J_c(K_1(C))$. Choosing a smaller $\sigma_{c,2}^{(1)}$, we may assume that

$$\mathcal{U}_c^{(1)} = \mathcal{U}_{c,0}^{(1)} \sqcup \mathcal{U}_{c,1}^{(1)},$$

where $\overline{\mathcal{U}_{c,0}^{(1)}}$ generates $(\overline{F_c^{(1)}})_0$ and $\overline{\mathcal{U}_{c,1}^{(1)}}$ generates $\mathrm{Tor}(\overline{F_c^{(1)}})$ –namely, we can choose $\mathcal{U}_{c,0}^{(1)}$ and $\mathcal{U}_{c,1}^{(1)}$ so that $\mathcal{U}_c^{(1)} \subset \mathcal{U}_{c,0}^{(1)} \cdot \mathcal{U}_{c,1}^{(1)}$; then, choosing smaller $\sigma_{c,2}^{(1)}$, one can replace $\mathcal{U}_c^{(1)}$ by $\mathcal{U}_{c,0}^{(1)} \sqcup \mathcal{U}_{c,1}^{(1)}$. Note that for each $u \in \mathcal{U}_{c,1}^{(1)}$, one has $u^k \in CU(C)$, where k is the order of $\overline{u}$.

Let a finite subset $\mathcal{G}_{uc}^{(1)} \subset C$ and $\delta_{uc}^{(1)} > 0$ satisfy the following condition: for any $\mathcal{G}_{uc}^{(1)}$-$\delta_{c,u}^{(1)}$-multiplicative unital completely positive linear map $L' : C \to A'$ (for any unital C^*-algebra A' with $K_1(A') = U(A')/U_0(A')$), $(L')^\ddagger$ can be defined as a homomorphism on $\overline{F_c^{(1)}}$, $\mathrm{dist}((L')^\ddagger(\overline{u}), \overline{\langle L'(u)\rangle}) < \sigma_{c,2}^{(1)}/8$ for all $u \in \mathcal{U}_c^{(1)}$, and $\kappa_1^{A'} \circ (L')^\ddagger(\overline{u}) = [L'] \circ \kappa_1^C([u])$ for all $u \in \mathcal{U}_c^{(1)}$ (see 2.17). Since $\overline{F_c^{(1)}} \subset J_c(K_1(C))$, and $J_c(K_1(C)) \cap U_0(C)/CU(C)$ only contains the unit $\overline{1_C}$, and since $(L')^\ddagger$ is a homomorphism on $\overline{F_c^{(1)}}$, we have $(L')^\ddagger(\overline{F_c^{(1)}} \cap U_0(C)/CU(C)) = \overline{1_{A'}} \in U_0(A')/CU(A')$. Moreover, for any $u \in \mathcal{U}_{c,1}^{(1)}$ with $\overline{u}$ of order k, as $u^k \in CU(C)$, we may assume (see 2.17) that

$$\mathrm{dist}(\langle L'(u^k)\rangle, CU(A')) < \sigma_{c,2}^{(1)}/8. \tag{e 21.12}$$

Put $\mathcal{G}_c^{(1)+} = \mathcal{G}_c^{(1)} \cup \mathcal{G}_{uc}^{(1)}$ and $\delta_c^{(1)+} = \frac{1}{2}\min\{\delta_c^{(1)}, \delta_{uc}^{(1)}\}$.

Recall that both A and C are in $\mathcal{B}_{u0}$. By Theorem 20.16, there is a $\mathcal{G}_c^{(1)+}$-$\delta_c^{(1)+}$ -multiplicative contractive completely positive linear map $L_1 : C \to A$ such that

$$[L_1]|_{\mathcal{P}_c^{(1)}} = \alpha|_{\mathcal{P}_c^{(1)}} \quad \text{and} \tag{e 21.13}$$

$$|\tau \circ L_1(f) - \gamma(\tau)(f)| < \sigma_{c,1}^{(1)}/8 \ \text{ for all } \ f \in \mathcal{H}_c^{(1)} \ \text{ for all } \ \tau \in T(A). \tag{e 21.14}$$

By choosing $\mathcal{G}_c^{(1)+}$ large enough and $\delta_c^{(1)+}$ small enough, one may assume that $L_1^\ddagger$ is a homomorphism defined on $(\overline{F_c^{(1)}})$ (see 2.17). We may further assume that, $\mathrm{dist}(L_1^\ddagger(\overline{u}), \overline{\langle L_1(u)\rangle}) < \sigma_{c,2}^{(1)}/8$ for all $u \in \mathcal{U}_c^{(1)}$ and $\kappa_1^A \circ L_1^\ddagger = [L_1] \circ \kappa_1^C$ on $(\overline{F_c^{(1)}})$. Since $\alpha|_{K_1(C)}$ is an isomorphism, we may assume that $L_1^\ddagger|_{\overline{F_c^{(1)}}}$ is injective as κ_1^C is injective on $J_c(K_1(C))$. In particular, $L_1^\ddagger|_{(\overline{F_c^{(1)}})_0}$ is injective. Moreover, if $u \in \mathcal{U}_{c,1}^{(1)}$ and $\overline{u}$ is of order k, one may also assume (see (e 21.12)) that

$$\mathrm{dist}(L_1(u^k), CU(A)) < \sigma_{c,2}^{(1)}/8.$$

Applying the part (a) of Theorem 12.11 a second time, let $\delta_a^{(1)} > 0$ (in place of δ), $\mathcal{G}_a^{(1)} \subset A$ (in place of $\mathcal{G}$), $\sigma_{a,1}^{(1)}, \sigma_{a,2}^{(1)} > 0$ (in place of σ_1 and σ_2), $\mathcal{P}_a^{(1)} \subset \underline{K}(A)$

(in place of $\mathcal{P}$), $\overline{\mathcal{U}_a^{(1)}} \subset U(A)/CU(A)$ (in place of $\mathcal{U}$), and $\mathcal{H}_a^{(1)} \subset A_{s.a}$ (in place of $\mathcal{H}$) be as provided by the part (a) of Theorem 12.11 for A (in place of A), ε_2 (in place of ε), and $\mathcal{F}_1$ (in place of $\mathcal{F}$). We may assume that $\delta_a^{(1)} < \delta_c^{(1)+}/2$, $L_1(\mathcal{G}_c^{(1)+}) \subset \mathcal{G}_a^{(1)}$,

$$J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{\mathcal{U}_c^{(1)}})) \subset \overline{\mathcal{U}_a^{(1)}}, \tag{e 21.15}$$

and $\kappa_1^A(\overline{\mathcal{U}_a^{(1)}}) \cup [L_1](\mathcal{P}_c^{(1)}) \subset \mathcal{P}_a^{(1)}$. Here we also assume that $\mathcal{U}_a^{(1)} \subset U(A)$ is a finite subset. As in Remark 12.12, we may further assume that $\overline{\mathcal{U}_a^{(1)}} \subset J_c^A(K_1(A))$. Moreover, we may assume that $\mathcal{P}_{a,1} \subset \mathcal{P}_c^{(2)}$ and $\mathcal{H}(a,1) \subset \mathcal{H}_a^{(1)}$.

Denote by $\overline{F_a^{(1)}} \subset \overline{U(A)}$ the subgroup generated by $\overline{\mathcal{U}_a^{(1)}}$. Since $\mathcal{U}_a^{(1)}$ is finite, we can write $\overline{F_a^{(1)}} = (\overline{F_a^{(1)}})_0 \oplus \mathrm{Tor}(\overline{F_a^{(1)}})$, where $(\overline{F_a})_0$ is torsion free. Fix this decomposition. Without loss of generality (choosing a smaller $\sigma_{a,2}^{(1)}$), one may assume that

$$\mathcal{U}_a^{(1)} = \mathcal{U}_{a,0}^{(1)} \sqcup \mathcal{U}_{a,1}^{(1)},$$

where $\mathcal{U}_{a,0}^{(1)}$ generates $(\overline{F_a^{(1)}})_0$ and $\mathcal{U}_{a,1}^{(1)}$ generates $\mathrm{Tor}(\overline{F_a^{(1)}})$ (then the condition (e 21.15) should be changed to the condition that $J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{\mathcal{U}_c^{(1)}}))$ is in the subgroup generated by $\overline{\mathcal{U}_a^{(1)}}$). Enlarging $\mathcal{P}_a^{(1)}$, we may assume $\mathcal{P}_a^{(1)} \supset \kappa_1^A((\overline{F_a^{(1)}})_0)$ (in $K_1(A)$). Note that for each $u \in \mathcal{U}_{a,1}^{(1)}$, $u^k \in CU(A)$, where k is the order of $\overline{u}$.

Let $\mathcal{U}_{c,a}^{(1)'} \subset U(A)$ be a finite subset such that $\overline{\mathcal{U}_{c,a}^{(1)'}} = L_1^{\ddagger}(\overline{\mathcal{U}_c^{(1)}})$ and $\mathcal{U}_a^{(1)'} = \mathcal{U}_a^{(1)} \cup \mathcal{U}_{c,a}^{(1)'} \cup \{\langle L_1(u)\rangle : u \in \mathcal{U}_c^{(1)}\}$. Let $F_{a,1}$ be the subgroup of $U(A)$ generated by $\mathcal{U}_a^{(1)'}$. Since $\kappa_1^A(L_1^{\ddagger}(\overline{\mathcal{U}_c^{(1)}})) = \{\kappa_1^A(\overline{\langle L_1(u)\rangle}) : u \in \mathcal{U}_c^{(1)}\}$ and since $J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{\mathcal{U}_c^{(1)}}))$ is in the subgroup generated by $\overline{\mathcal{U}_a^{(1)}}$, $\overline{F_{a,1}} = \overline{F_a^{(1)}} + (U_0(A)/CU(A)) \cap \overline{F_{a,1}}$.

Let a finite subset $\mathcal{G}_{ua}^{(1)} \subset A$ and $\delta_{ua}^{(1)} > 0$ satisfy the following condition: for any $\mathcal{G}_{ua}^{(1)}$-$\delta_{c,a}^{(1)}$-multiplicative contractive completely positive linear map $L' : A \to B'$ (for any unital C^*-algebra B' with $K_1(B') = U(B')/U_0(B')$), $(L')^{\ddagger}$ can be chosen to be a homomorphism on $\overline{F_{a,1}}$, $\mathrm{dist}((L')^{\ddagger}(\overline{u}), \overline{\langle L'(u)\rangle}) < \sigma_{a,2}^{(1)}/8$ for all $u \in \mathcal{U}_a^{(1)'}$, $(L')^{\ddagger}(\overline{F_{a,1}} \cap U_0(A)/CU(A)) \subset U_0(B')/CU(B')$, and $\kappa_1^{B'} \circ (L')^{\ddagger}(\overline{u}) = [L'] \circ \kappa_1^C([u])$ for all $u \in \mathcal{U}_a^{(1)'}$ (see 2.17). We assume that, for $u \in \mathcal{U}_{a,1}^{(1)}$,

$$\mathrm{dist}(\langle L'(u^k)\rangle, CU(B')) < \sigma_{a,2}^{(1)}/8, \tag{e 21.16}$$

where k is the order of $\overline{u}$ (see 2.17). Put $\mathcal{G}_a^{(1)+} = \mathcal{G}_a^{(1)} \cup \mathcal{G}_{ua}^{(1)}$ and $\delta_a^{(1)+} = \min\{\delta_a^{(1)}, \delta_{ua}^{(1)}\}$.

By Theorem 20.16 and the amenability of C, there are a finite subset $\mathcal{G}'_a \supset \mathcal{G}_a^{(1)+}$, a positive number $\delta'_a < \delta_a^{(1)+}$, a sufficiently large integer $n \geq n_0$, and a $\mathcal{G}'_a$-δ'_a-multiplicative map $\Phi'_1 : A \to C_n$ such that, with $\sigma_{a,c,1}^{(1)} = \min\{\sigma_{a,1}^{(1)}, \sigma_{c,1}^{(1)}\}/6$,

$$[\iota_n \circ \Phi'_1]_{\mathcal{P}_a^{(1)} \cup [L_1](\mathcal{P}_c^{(1)})} = \alpha^{-1}|_{\mathcal{P}_a^{(1)} \cup [L_1](\mathcal{P}_c^{(1)})} \tag{e 21.17}$$

and for all $f \in \mathcal{H}_a^{(1)} \cup L_1(\mathcal{H}_c^{(1)})$ and $\tau \in T(C)$.

$$|\tau \circ \iota_n \circ \Phi_1'(f) - \gamma^{-1}(\tau)(f)| < \sigma_{a,c,1}^{(1)}/8. \tag{e 21.18}$$

Moreover, one may assume that $\Phi_1' \circ L_1$ is $\mathcal{G}_c^{(1)}$-$\delta_c^{(1)}$-multiplicative, $(\Phi_1')^{\ddagger}$ is a homomorphism defined on $\overline{F_{a,1}}$, $\mathrm{dist}((\Phi_1')^{\ddagger}(\overline{u}), \overline{\langle \Phi_1'(u) \rangle}) < \sigma_{a,c,1}^{(1)}/8$, and $\kappa_1^{C_n} \circ (\Phi_1')^{\ddagger}(\overline{u}) = [\Phi_1'](\kappa_1^A(\overline{u}))$ for all $u \in \mathcal{U}_a^{(1)'}$. Then, by (e 21.17), since $\alpha^{-1}|_{K_1(A)}$ is injective, $(\iota_n)_{*1} \circ [\Phi_1']$ is injective on $\kappa_1^A(\overline{F_{a,1}})$, which implies that $[\Phi_1']$ is injective on $\kappa_1^A(\overline{F_{a,1}})$. Since κ_1^A is injective on $J_c^A(K_1(A))$, we conclude that $\kappa_1^{C_n} \circ (\Phi_1')^{\ddagger}$ is injective on $\overline{F_a^{(1)}}$, which implies that $(\Phi_1')^{\ddagger}$ is injective on $\overline{F_a^{(1)}}$. Write $(\Phi_1' \circ L_1)^{\ddagger} := (\Phi_1')^{\ddagger} \circ L_1^{\ddagger}$. Then $(\Phi_1' \circ L_1)^{\ddagger}$ is defined on $\overline{F_c^{(1)}}$, since $L_1^{\ddagger}(\overline{F_c^{(1)}}) \subset \overline{F_{a,1}}$.

If $u \in F_c^{(1)}$, then $\overline{u} \in \overline{F_c^{(1)}} \subset J_c^C(K_1(C))$. Suppose that $\overline{u} \neq 0$. Then $[u] \neq 0$, since $\kappa_1^C \circ J_c^C = \mathrm{id}_{K_1(C)}$. Then $\kappa_1^A(L_1^{\ddagger}(\overline{u})) = [L_1] \circ \kappa_1^C(\overline{u}) = \alpha([u]) \neq 0$. It follows that $J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{u})) \neq 0$. Then $(\Phi_1')^{\ddagger}(J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{u}))) \neq 0$ since $J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{u})) \subset \overline{F_a^{(1)}}$ and $(\Phi_1')^{\ddagger}$ is injective on $\overline{F_a^{(1)}}$. Moreover, since $\kappa_1^C(\overline{\mathcal{U}_c^{(1)}}) \subset \mathcal{P}_c^{(1)}$, by (e 21.17), for any $u \in \mathcal{U}_c^{(1)}$,

$$\begin{aligned} \kappa_1^{C_n}((\Phi_1')^{\ddagger}(J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{u})))) &= [(\Phi_1')](\kappa_1^A(J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{u})))) \\ &= [(\Phi_1')](\kappa_1^A(L_1^{\ddagger}(\overline{u}))) = [\Phi_1']([L_1](\kappa_1^C(\overline{u}))) \neq 0. \end{aligned} \tag{e 21.19}$$

Put $z = L_1^{\ddagger}(\overline{u}) - J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{u})$. Then $z \in U_0(A)/CU(A)$. It follows that $(\Phi_1')^{\ddagger}(z) \in U_0(C_n)/CU(C_n)$. If $(\Phi_1')^{\ddagger}(L_1^{\ddagger}(\overline{u})) = 0$, then

$$\kappa_1^{C_n}((\Phi_1')^{\ddagger}(J_c^A \circ \kappa_1^A(L_1^{\ddagger}(\overline{u})))) = \kappa_1^{C_n}((\Phi_1')^{\ddagger}(L_1^{\ddagger}(\overline{u}))) - \kappa_1^{C_n}((\Phi_1')^{\ddagger}(z)) = 0,$$

which contradicts (e 21.19). This implies that $(\Phi_1')^{\ddagger}(L_1^{\ddagger}(\overline{u})) \neq 0$. In other words, $(\Phi_1' \circ L_1)^{\ddagger}$ is injective on $\overline{F_c^{(1)}}$.

Furthermore, one may also assume (see (e 21.16)) that, for $u \in \mathcal{U}_{a,1}^{(1)}$,

$$\mathrm{dist}(\iota_n \circ \Phi_1'(u^k), CU(C)) < \sigma_{a,2}^{(1)}/8, \tag{e 21.20}$$

where k is the order of $\overline{u}$, and (by (e 21.12), as $L_1(\mathcal{G}_c^{(1)+}) \subset \mathcal{G}_a^{(1)}$ and $\delta_a^{(1)+} < \delta_c^{(1)+}/2$)

$$\mathrm{dist}((\iota_n \circ \Phi_1' \circ L_1)(v^{k'}), CU(A)) < \sigma_{c,2}^{(1)}/8, \tag{e 21.21}$$

if $v \in \mathcal{U}_{c,1}^{(1)}$ and k' is the order of $\overline{v}$. It then follows from (e 21.13) and (e 21.17) that

$$[\iota_n \circ \Phi_1' \circ L_1]|_{\mathcal{P}_c^{(1)}} = [\mathrm{id}]|_{\mathcal{P}_c^{(1)}}; \tag{e 21.22}$$

and it follows from (e 21.14) and (e 21.18) that

$$|\tau\circ\iota_n\circ\Phi_1'\circ L_1(f)-\tau(f)| < 2\sigma_{c,1}^{(1)}/3 \text{ for all } f\in\mathcal{H}_c^{(1)} \text{ for all } \tau\in T(C). \tag{e 21.23}$$

Recall that $\overline{(F_c^{(1)})_0} \subset U(C)/CU(C)$ is a free abelian subgroup, generated by $\overline{\mathcal{U}_{c,0}^{(1)}}$. Since we have assumed that $\overline{\mathcal{U}_{c,0}^{(1)}}$ is in the image of $U(C_n)/CU(C_n)$, there is an injective homomorphism $j : \overline{(F_c^{(1)})_0} \to U(C_n)/CU(C_n)$ such that

$$\iota_n^{\ddagger}\circ j = \mathrm{id}\,|_{\overline{(F_c^{(1)})_0}}. \tag{e 21.24}$$

Moreover, by (e 21.22),

$$\kappa_1^C\circ\iota_n^{\ddagger}\circ(\Phi_1'\circ L_1)^{\ddagger}|_{\overline{(F_c^{(1)})_0}} = (\iota_n)_{*1}\circ[\Phi_1'\circ L_1]\circ\kappa_1^C|_{\overline{(F_c^{(1)})_0}} = \kappa_1^C|_{\overline{(F_c^{(1)})_0}} = \kappa_1^C\circ\iota_n^{\ddagger}\circ j.$$

Let δ be the constant of Lemma 21.6 with respect to C_n (in place of C), C (in place of B), $\sigma_{c,2}^{(1)}/2$ (in place of ε), $\iota_n^{\ddagger}$ (in place of γ), j, and $(\Phi_1'\circ L_1)^{\ddagger}|_{\overline{(F_c^{(1)})_0}}$ (in place of L). For any $n'>n$, we have

$$\iota_n = (\iota_{n',\infty}\circ\iota_{n',n'+1}^{(0)}\circ\iota_{n,n'})\oplus(\iota_{n',\infty}\circ\iota_{n',n'+1}^{(1)}\circ\iota_{n,n'}). \tag{e 21.25}$$

Denote $\iota_{n',\infty}\circ\iota_{n',n'+1}^{(0)}\circ\iota_{n,n'}$ by $\imath_n^{(0)}$ and $\iota_{n',\infty}\circ\iota_{n',n'+1}^{(1)}\circ\iota_{n,n'}$ by $\imath_n^{(1)}$. By (e 14.13), if n' is large enough (in particular, depending on δ above), the decomposition $\iota_n = \imath_n^{(0)}\oplus\imath_n^{(1)}$ satisfies

(1) $\tau(\imath_n^{(0)}(1_{C_n})) < \min\{\delta, \sigma_{c,1}^{(1)}/6, \sigma_{a,1}^{(1)}/6\}$ for all $\tau\in T(C)$, and
(2) $\imath_n^{(0)}$ has finite dimensional range, and is non-zero on each direct summand of C_n.

Then, by Lemma 21.6, there is a homomorphism $h : C_n\to e_0Ce_0$, where $e_0 = \imath_n^{(0)}(1_{C_n})$, such that

(3) $[h] = [\imath_n^{(0)}]$ in $KL(C_n, C)$, and
(4) for each $u\in\mathcal{U}_{c,0}^{(1)}$, one has that

$$(\iota_n^{\ddagger}\circ j(\overline{u}))^{-1}(h\oplus\imath_n^{(1)})^{\ddagger}((\Phi_1'\circ L_1)^{\ddagger}(\overline{u})) = \overline{g_u} \tag{e 21.26}$$

for some $g_u\in U_0(C)$ with $\mathrm{cel}(g_u) < \sigma_{c,2}^{(1)}/2$.

Define $\Phi_1 = (h\oplus\imath_n^{(1)})\circ\Phi_1'$. By (e 21.17) and (3), and, by (e 21.18) and (1),

$$[\Phi_1]_{\mathcal{P}_a^{(1)}\cup[L_1](\mathcal{P}_c^{(1)})} = \alpha^{-1}|_{\mathcal{P}_a^{(1)}\cup[L_1](\mathcal{P}_c^{(1)})} \text{ and} \tag{e 21.27}$$

$$|\tau(\Phi_1(f)) - \gamma(\tau)(f)| < \sigma_{c,1}^{(2)}/3 \text{ for all } f\in\mathcal{H}_c^{(2)}. \tag{e 21.28}$$

Note that Φ_1 is still $\mathcal{G}_a^{(1)+}$-$\delta_a^{(1)+}$-multiplicative, and hence (e 21.20) and (e 21.21) still hold with Φ_1' replaced by Φ_1. That is, for $u \in \mathcal{U}_{a,1}^{(1)}$,

$$\mathrm{dist}(\langle \Phi_1(u)\rangle^k, CU(C)) < \sigma_{a,2}^{(1)}/2, \tag{e 21.29}$$

where k is the order of $\overline{u}$, and

$$\mathrm{dist}(\langle (\Phi_1 \circ L_1)(v)\rangle^{k'}, CU(A)) < \sigma_{c,2}^{(1)}, \tag{e 21.30}$$

if $v \in \mathcal{U}_{c,1}^{(1)}$ and if k' is the order of $\overline{v}$. By (e 21.22), (3), and, (e 21.23), and (4), one has

$$[\Phi_1 \circ L_1]|_{\mathcal{P}_c^{(1)}} = [\mathrm{id}]|_{\mathcal{P}_c^{(1)}} \text{ and} \tag{e 21.31}$$

$$|\tau \circ \Phi_1 \circ L_1(f) - \tau(f)| < \sigma_{c,1}^{(1)} \text{ for all } f \in \mathcal{H}_c^{(1)} \text{ for all } \tau \in T(C). \tag{e 21.32}$$

Moreover, for any $u \in \mathcal{U}_{c,0}^{(1)}$, one has (by (e 21.24) and (e 21.26))

$$(\Phi_1 \circ L_1)^{\ddagger}(\overline{u}) \quad = \quad (\iota_n^{\ddagger} \circ j(\overline{u})) \cdot \overline{g_u} = \overline{u} \cdot \overline{g_u} \approx_{\sigma_{c,2}^{(1)}} \overline{u}. \tag{e 21.33}$$

Let $\overline{u} \in \overline{\mathcal{U}_{c,1}^{(1)}}$, with order k. By (e 21.30), there is a self-adjoint element $b \in C$ with $||b|| < \sigma_{c,2}^{(1)}$ such that

$$(u^*)^k(\langle (\Phi_1 \circ L_1)(u)\rangle)^k \exp(2\pi i b) \in CU(C)$$

(where we notice that $(u^*)^k \in CU(C)$), and hence

$$((u^*)(\langle \Phi_1 \circ L_1(u)\rangle)\exp(2\pi i b/k))^k \in CU(C).$$

Note that

$$(u^*)(\langle \Phi_1 \circ L_1(u)\rangle)\exp(2\pi i b/k) \in U_0(C)$$

and $U_0(C)/CU(C)$ is torsion free (Corollary 11.7). One has

$$(u^*)(\langle \Phi_1 \circ L_1(u)\rangle)\exp(2\pi i b/k) \in CU(C).$$

In particular, this implies that

$$\mathrm{dist}((\Phi_1 \circ L_1)^{\ddagger}(\bar{u}), \bar{u}) < \sigma_{c,2}^{(1)}/k \text{ for all } \bar{u} \in \mathcal{U}_{c,1}^{(1)}. \tag{e 21.34}$$

Combining this with (e 21.33), we have

$$\mathrm{dist}((\Phi_1 \circ L_1)^{\ddagger}(\bar{u}), \bar{u}) < \sigma_{c,2}^{(1)} \text{ for all } u \in \mathcal{U}_c^{(1)}. \tag{e 21.35}$$

Therefore, by (e 21.31), (e 21.32), and (e 21.35), applying part (a) of Theorem 12.11, we obtain a unitary U_1 such that

$$||U_1^*(\Phi_1 \circ L_1(f))U_1 - f|| < \varepsilon_1 \text{ for all } f \in \mathcal{G}_1.$$

Replacing Φ_1 by $\mathrm{Ad}(U_1) \circ \Phi_1$, we may assume that

$$||\Phi_1 \circ L_1(f) - f|| < \varepsilon_1 \text{ for all } f \in \mathcal{G}_1.$$

In other words, one has the diagram

$$\begin{tikzcd} C \arrow[r, "\mathrm{id}"] \arrow[d, "L_1"'] & C \\ A, \arrow[ur, "\Phi_1"'] & \end{tikzcd}$$

which is approximately commutative on the subset $\mathcal{G}_1$ to within ε_1. We also notice that, by (e 21.27), $[\Phi_1]$ is injective on $\kappa_1^A(\overline{F_a^{(1)}})$, which implies that $\kappa_1^C \circ \Phi_1^{\ddagger}$ is injective on $\overline{F_a^{(1)}}$, since $\overline{F_a^{(1)}} \subset J_c^A(K_1(A))$. It follows that $\Phi_1^{\ddagger}$ is injective on $\overline{F_a^{(1)}}$.

We will continue to apply the part (a) of Theorem 12.11 Let $\delta_c^{(2)} > 0$ (in place of δ), $\mathcal{G}_c^{(2)} \subset C$ (in place of $\mathcal{G}$), $\sigma_{c,1}^{(2)}, \sigma_{c,2}^{(2)} > 0$ (in place of σ_1 and σ_2), $\mathcal{P}_c^{(2)} \subset \underline{K}(C)$ (in place of $\mathcal{P}$), $\overline{\mathcal{U}_c^{(2)}} \subset U(C)/CU(C)$ (in place of $\mathcal{U}$), and $\mathcal{H}_c^{(2)} \subset C_{s.a}$ (in place of $\mathcal{H}$) be as provided by the part (a) of Theorem 12.11 for C (in place of A), ε_3 (in place of ε), and $\mathcal{G}_2$ (in place of $\mathcal{F}$). We may assume that $\mathcal{U}_c^{(2)} \subset U(C)$ is a finite subset. We may also assume, without loss of generality, that $\delta_c^{(2)} < \delta_a^{(1)+}/2$, $\Phi_1(\mathcal{G}_a^{(1)+}) \subset \mathcal{G}_c^{(2)}$,

$$J_c^C \circ \kappa_1^C(\Phi_1^{\ddagger}(\overline{\mathcal{U}_a^{(1)}})) \subset \overline{\mathcal{U}_c^{(2)}}. \tag{e 21.36}$$

By 12.12, we may assume that $\overline{\mathcal{U}_c^{(2)}} \subset J_c^C(K_1(C))$ and $\mathcal{U}_c^{(2)}$ is in the image of $U(C_m)$ under ι_m for all $m \geq n_1 > n_0$. We may also assume, without loss of generality, that $\kappa_1^C(\overline{\mathcal{U}_c^{(2)}}) \cup [\Phi_1](\mathcal{P}_a^{(1)}) \subset \mathcal{P}_c^{(2)}$. We may further assume that $\mathcal{P}_{c,2} \subset \mathcal{P}_c^{(2)}$ and $\mathcal{H}(c,2) \subset \mathcal{H}_c^{(2)}$.

Denote by $F_c^{(2)} \subset U(C)$ the subgroup generated by $\mathcal{U}_c^{(2)}$. Since $\mathcal{U}_c^{(2)}$ is finite, we may write $\overline{F_c^{(2)}} = (\overline{F_c^{(2)}})_0 \oplus \mathrm{Tor}(\overline{F_c^{(2)}})$, where $(\overline{F_c^{(2)}})_0$ is torsion free. Fix this decomposition. Choosing a smaller $\sigma_{c,2}^{(2)}$, we may assume that

$$\mathcal{U}_c^{(2)} = \mathcal{U}_{c,0}^{(2)} \sqcup \mathcal{U}_{c,1}^{(2)},$$

where $\overline{\mathcal{U}_{c,0}^{(2)}}$ generates $(\overline{F_c^{(2)}})_0$ and $\overline{\mathcal{U}_{c,1}^{(2)}}$ generates $\mathrm{Tor}(\overline{F_c^{(2)}})$ (then the condition (e 21.36) should be changed to the condition that $J_c^C \circ \kappa_1^C(\Phi_1^{\ddagger}(\overline{\mathcal{U}_a^{(1)}}))$ is in the subgroup generated by $\overline{\mathcal{U}_c^{(2)}}$). Note that, for each $u \in \mathcal{U}_{c,1}^{(2)}$, one has $u^k \in CU(C)$, where k is the order of $\overline{u}$.

Let $\mathcal{U}_{a,c}^{(2)'} \subset U(C)$ be a finite subset such that $\overline{\mathcal{U}_{a,c}^{(2)'}} = \Phi_1^{\ddagger}(\overline{\mathcal{U}_a^{(2)}})$ and $\mathcal{U}_c^{(2)'} = \mathcal{U}_c^{(2)} \cup \mathcal{U}_{a,c}^{(2)'} \cup \{\langle \Phi_1(u) \rangle : u \in \mathcal{U}_a^{(1)}\}$. Let $F_{c,2}$ be the subgroup of $U(C)$ generated by $\mathcal{U}_c^{(2)'}$. Since $\kappa_1^A(\Phi_1^{\ddagger}(\overline{\mathcal{U}_a^{(1)}})) = \{\kappa_1^A(\overline{\langle \Phi_1(u) \rangle}) : u \in \mathcal{U}_a^{(1)}\}$ and since $J_c^C \circ \kappa_1^C(\Phi_1^{\ddagger}(\overline{\mathcal{U}_a^{(1)}}))$ is in the subgroup generated by $\overline{\mathcal{U}_c^{(2)}}$, $\overline{F_{c,2}} = \overline{F_c^{(2)}} + (U_0(C)/CU(C)) \cap \overline{F_{c,2}}$.

Let a finite subset $\mathcal{G}_{uc}^{(2)} \subset C$ and $\delta_{uc}^{(2)} > 0$ satisfy the following condition: for any $\mathcal{G}_{uc}^{(2)}$-$\delta_{c,u}^{(2)}$-multiplicative unital completely positive linear map $L' : C \to A'$ (for any unital C^*-algebra A' with $K_1(A') = U(A')/U_0(A')$), $(L')^{\ddagger}$ can be defined as a homomorphism on $\overline{F_{c,2}}$, $\mathrm{dist}((L')^{\ddagger}(\overline{u}), \overline{\langle L'(u) \rangle}) < \min\{\sigma_{a,1}^{(1)}, \sigma_{c,2}^{(2)}\}/12$ for all $u \in \mathcal{U}_c^{(2)'}$, $(L')^{\ddagger}(\overline{F_{c,2}} \cap U_0(C)/CU(C)) \subset U_0(A')/CU(A')$, and $\kappa_1^{A'} \circ (L')^{\ddagger}(\overline{u}) = [L'] \circ \kappa_1^C([u])$ for all $u \in \mathcal{U}_c^{(2)}$ (see 2.17). Moreover, we may also assume that

$$\mathrm{dist}(\langle L'(u^k) \rangle, CU(A')) < \sigma_{c,2}^{(2)}/8, \tag{e 21.37}$$

if $u \in \mathcal{U}_{c,1}^{(2)}$ and if k is the order of $\overline{u}$ (see 2.17).

There are a finite subset $\mathcal{G}_0 \subset C$ and a positive number $\delta_0 > 0$ such that, for any two $\mathcal{G}_0$-δ_0-multiplicative contractive completely positive linear maps $L_1'', L_2'' : C \to A$, if

$$\|L_1''(c) - L_2''(c)\| < \delta_0 \text{ for all } c \in \mathcal{G}_0,$$

then

$$[L_1'']|_{\mathcal{P}_c^{(2)}} = [L_2'']|_{\mathcal{P}_c^{(2)}}$$

and

$$|\tau \circ L_1''(h) - \tau \circ L_2''(h)| < \min\{\sigma_{c,1}^{(2)}, \sigma_{a,1}^{(1)}\}/12$$

for all $h \in \mathcal{H}_c^{(2)} \cup \Phi_1(\mathcal{H}_a^{(1)})$ and for all $\tau \in T(A)$. Put $\mathcal{G}_c^{(2)+} = \mathcal{G}_c^{(2)} \cup \mathcal{G}_{uc}^{(2)} \cup \mathcal{G}_0$ and $\delta_c^{(2)+} = \min\{\delta_c^{(2)}, \delta_{uc}^{(2)}, \delta_0\}/4$.

Let $M_c = \max\{\|g\| : g \in \mathcal{G}_c^{(2)+}\}$. Note that, by Lemma 21.8, there exist a large $m \geq n_1$ and a unital contractive completely positive linear map $L_{0,2} : C \to C_m$ such that

$$\|\iota_m \circ L_{0,2}(g) - g\| < \delta_c^{(2)+}/8(M_c + 1) \text{ for all } g \in \mathcal{G}_c^{(2)+}. \tag{e 21.38}$$

Then $L_{0,2}$ is $\mathcal{G}_c^{(2)+}$-$\delta_c^{(2)+}/4$-multiplicative. Let us now fix such an $m \geq n_1$.

Let $\kappa_1^{C_m} : U(C_m)/CU(C_m) \to K_1(C_m)$ be the quotient map. We may then assume that $L_{0,2}^{\ddagger}$ is defined on $\overline{F_{c,2}}$ and injective on $\overline{F_c^{(2)}}$ (see the discussion regarding injectivity of $(\Phi_1')^{\ddagger}$—with α^{-1} replaced by $[\mathrm{id}_C]$) and that $\mathrm{dist}(L_{0,2}^{\ddagger}(\overline{u}), \overline{u}) < \min\{\sigma_{c,1}^{(2)}, \sigma_{a,1}^{(1)}\}/12$ for all $u \in \mathcal{U}_c^{(2)}$. It follows that $(L_{0,2} \circ \Phi_1)^{\ddagger} := L_{0,2}^{\ddagger} \circ \Phi_1^{\ddagger}$ is defined and injective on $\overline{(F_a^{(1)})}$ (see the discussion of the injectivity of $(\Phi_1' \circ L_1^{\ddagger})$). Moreover,

$$\text{(e 21.39)}\quad \kappa_1^{C_m} \circ (L_{0,2} \circ \Phi_1)^{\ddagger}(g) = [L_{0,2} \circ \Phi_1](\kappa_1^A(g)), \quad g \in \overline{(F_a^{(1)})_0},$$

and by (e 21.27), for any $g \in \overline{(F_a^{(1)})_0}$ (note that $\mathcal{P}_a^{(1)} \supset \kappa_{1,A}((\overline{F_a})_0)$(in $K_1(A)$)),

$$\begin{aligned} \text{(e 21.40)}\quad \alpha \circ [\iota_m] \circ [L_{0,2} \circ \Phi_1](\kappa_1^A(g)) &= \alpha \circ [\iota_m \circ L_{0,2}] \circ [\Phi_1](\kappa_1^A(g)) \\ \text{(e 21.41)}\quad &= \alpha \circ [\Phi_1](\kappa_1^A(g)) = \kappa_1^A(g). \end{aligned}$$

Hence,

$$\alpha \circ [\iota_m] \circ \kappa_1^{C_m} \circ (L_{0,2} \circ \Phi_1)^{\ddagger}(g) = \alpha \circ [\iota_m] \circ [L_{0,2} \circ \Phi_1](\kappa_1^A(g)) = \kappa_1^A(g)$$

for all $g \in \overline{(F_a^{(1)})_0}$, which also implies that $\alpha \circ [\iota_m]$ is injective on $[L_{0,2} \circ \Phi_1](\kappa_1^A(\overline{(F_a^{(1)})_0}))$ and $[L_{0,2} \circ \Phi_1]$ is injective on $\kappa_1^A(\overline{(F_a^{(1)})_0})$. By (e 21.39), $\kappa_1^{C_m} \circ (L_{0,2} \circ \Phi_1)^{\ddagger}$ is injective on $\overline{(F_a^{(1)})_0}$. Note that $\kappa_1^A(\overline{(F_a^{(1)})_0})$ is free abelian. Therefore $\pi_1 \circ (L_{0,2} \circ \Phi_1)^{\ddagger}$ is injective on $\overline{(F_a^{(1)})_0}$ (recall that π_1 is defined in 21.1). It follows from Lemma 21.3 (where B is replaced by A, C is replaced by C_m, F is replaced by $(F_a^{(1)})_0$, α is replaced by $\alpha \circ (\iota_m)_{*0}$, and L is replaced by $(L_{0,2} \circ \Phi_1)^{\ddagger}$) that there is a homomorphism $\beta : U(C_m)/CU(C_m) \to U(A)/CU(A)$ with $\beta(U_0(C_m)/CU(C_m)) \subset U_0(A)/CU(A)$ such that

$$\text{(e 21.42)}\qquad \beta \circ (L_{0,2} \circ \Phi_1)^{\ddagger}(f) = f \ \text{ for all } \ f \in \overline{(F_a^{(1)})_0}.$$

Put $\overline{F_{a,c,m}^{(1)}} = (L_{0,2} \circ \Phi_1)^{\ddagger}(\overline{(F_a^{(1)})_0})$. Since $\overline{(F_a^{(1)})_0} \subset J_c^A(K_1(A))$, the equations (e 21.39) and (e 21.41) also imply that $\kappa_1^{C_m} \circ (L_{0,2} \circ \Phi_1)^{\ddagger}$ is injective on $\overline{(F_a^{(1)})_0}$. In particular, $\overline{F_{a,c,m}^{(1)}}$ is free abelian and $\kappa_1^{C_m}$ is injective on $\overline{F_{a,c,m}^{(1)}}$. It follows that $\pi_1|_{\overline{F_{a,c,m}^{(1)}}}$ is injective (see 21.1 for the definition of π_1). Let $\mathcal{U}_{a,c,m}^{(1)} \subset U(C_m)$ be a finite subset whose image in $U(C_m)/CU(C_m)$ generates $\overline{F_{a,c,m}^{(1)}}$. We may assume that,

$$\text{(e 21.43)}\qquad (L_{0,2} \circ \Phi_1)^{\ddagger}(\overline{\mathcal{U}_{a,0}^{(1)}}) \subset \overline{\mathcal{U}_{a,c,m}^{(1)}}.$$

Let $\sigma > 0$, $\delta' > 0$ (in place of δ), and the finite subset $\mathcal{G}_c \subset C_m$ (in place of $\mathcal{G}$) be as provided by Lemma 21.5 with respect to $\sigma_{a,2}^{(2)}/4$ (in place of ε) (and C_m in place of C, A in place of B, $\mathcal{U}_{a,c,m}^{(1)}$ in place of $\mathcal{U}$, $\overline{F_{a,c,m}^{(1)}}$ in place of $\overline{F}$, and β in place of α). By Theorem 14.10 and Remark 14.11, just as in the decomposition of ι_n in (e 21.25), one may write $\iota_m = \imath_m^{(0)} \oplus \imath_m^{(1)}$, where $\imath_m^{(i)}$ is a homomorphism $(i = 0, 1)$, $\imath_m^{(0)}$ has finite dimensional range, and $\imath_m^{(0)}$ is non-zero on each direct summand of C_m. Moreover,

$$\tau(\imath_m^{(0)}(1_{C_m})) < \min\{\sigma, \sigma_{c,1}^{(2)}, \sigma_{a,1}^{(1)}\}/12 \text{ for all } \tau \in T(C). \tag{e 21.44}$$

Let E' be a finite set of generators (in the unit ball) of the finite dimensional C^*-subalgebra $\imath_m^{(0)}(C_m)$ containing $\imath_m^{(0)}(1_{C_m})$. Since C is simple,

$$\sigma_{00} = \inf\{\tau(\iota_m(a^*a) : a \in E', \, \tau \in T(C)\} > 0. \tag{e 21.45}$$

Put $\mathcal{H}_c^{(2)+} = \mathcal{H}_c^{(2)} \cup \Phi_1(\mathcal{H}_a^{(1)}) \cup \{a^*a : a \in E'\}$.

By Theorem 20.16, there is a $\mathcal{G}''$-δ''-multiplicative map $L_2' : C \to A$ such that

$$[L_2']|_{\mathcal{P}_c^{(2)}} = \alpha|_{\mathcal{P}_c^{(2)}} \tag{e 21.46}$$

and

$$|\tau \circ L_2'(f) - \gamma(\tau)(f)| < \min\{\sigma, \sigma_{00}, \sigma_{c,1}^{(2)}, \sigma_{a,1}^{(1)}\}/12 \tag{e 21.47}$$

for all $f \in \mathcal{H}_c^{(2)+}$ and for all $\tau \in T(A)$, where $\mathcal{G}'' \subset C$ is a finite subset and $\delta'' > 0$. We may assume that

$$\mathcal{G}'' \supset \mathcal{G}_0 \cup \mathcal{G}_c^{(2)+} \cup \iota_m(\mathcal{G}_c) \text{ and } \delta'' < \min\{\delta_0, \delta_c^{(2)+}, \delta'\}/2.$$

Fix a finite subset $\mathcal{G}_{c,m}^{(2)} \subset C_m$ and $0 < \delta_0^{(2)} < \min\{\sigma_{00}, \delta_0\}/2$. We may assume that $E' \subset \imath_m^{(0)}(\mathcal{G}_{c,m}^{(2)})$. Since every finite dimensional C^*-algebra is semiprojective, since $\imath_m^{(0)}(C_m)$ is finite dimensional, and since L_2' is chosen after C_m is chosen, with sufficiently large $\mathcal{G}''$ and small δ'', we may assume, without loss of generality, that there exists a homomorphism $h_0 : C_m \to A$ with finite dimensional range such that

$$\|h_0(g) - L_2' \circ \imath_m^{(0)}(g)\| < \min\{\delta_0^{(2)}, \sigma/2, \sigma_{a,1}^{(1)}/6, \sigma_{c,1}^{(2)}/6\} \tag{e 21.48}$$

for all $g \in \mathcal{G}_{c,m}^{(2)}$,

$$\|(1 - h_0(1_{C_m})L_2' \circ \imath_m^{(1)}(g)(1 - h_0(1_{C_m})) - L_2' \circ \imath_m^{(1)}(g)\| < \delta_0^{(2)} \tag{e 21.49}$$

for all $g \in \mathcal{G}_{c,m}^{(2)}$ and

$$\tau(h_0(1_{C_m})) < \min\{\sigma/2, \sigma_{a,1}^{(1)}/6, \sigma_{c,1}^{(2)}/6\} \text{ for all } \tau \in T(A). \tag{e 21.50}$$

Let $l_m : C_m \to (1-h_0(1_{C_m}))A(1-h_0(1_{C_m}))$ be defined by

$$l_m(c) = (1-h_0(1_{C_m})L_2'\circ \imath_m^{(1)}(g)(1-h_0(1_{C_m})) \text{ for all } c\in C_m.$$

Since m is now fixed and γ is a homeomorphism, by (e 21.47), (e 21.48) and (e 21.45), we may assume that L_2' is injective on $\imath_m^{(0)}(C_m)$. Since $\imath_m^{(0)}$ is non-zero on each summand of C_m, by (e 21.48), (e 21.47), and (e 21.45), we may also assume that h_0 is non-zero on each direct summand of C_m. Note that $L_2'\circ\imath_m = h_0'\oplus l_m^{(1)}$, where $h_0' = L_2'\circ\imath_m^{(0)}$ and $l_m^{(1)} = L_2'\circ\imath_m^{(1)}$. (Note that h_0' is close to h_0 by (e 21.48).)

Choosing a sufficiently large $\mathcal{G}''$ and small δ'', we may assume that $(L_2'\circ\iota_m)^\ddagger$ and $(l_m^{(1)})^\ddagger$ are defined on a subgroup of $U(C_m)/CU(C_m)$ containing $(L_{0,2}\circ\Phi_1)^\ddagger(\overline{(F_a^{(1)})_0})$, $\pi_0((L_{0,2}\circ\Phi_1)^\ddagger(\overline{(F_a^{(1)})_0}))$, $\pi_1(U(C_m)/CU(C_m))$, and $\pi_2(U(C_m)/CU(C_m))$. Moreover, for all $u\in\mathcal{U}_{a,c,m}^{(1)}$,

$$\mathrm{dist}((L_2'\circ\iota_m)^\ddagger(\overline{u}), \overline{\langle L_2'\circ\iota_m(u)\rangle}) < \sigma_{a,2}^{(1)}/4 \text{ and} \tag{e 21.51}$$

$$\mathrm{dist}(l_m^\ddagger(\overline{u\rangle}), \overline{\langle l_m(u)\rangle}) < \sigma_{a,2}^{(1)}/4. \tag{e 21.52}$$

Then, by Lemma 21.5 (with $h_0\oplus l_m^{(1)}$ in place of φ, h_0 in place of φ_0, and $l_m^{(1)} =L_2'\circ\iota_m^{(1)}$ in place of φ_1), there is a homomorphism $\psi_0 : C_m\to e_0'Ae_0'$, where $e_0' = h_0(1_{C_m})$, such that

(i) $[\psi_0]=[h_0]$ in $KK(C_m,A)$, and

(ii) for any $u\in\mathcal{U}_{a,0}^{(1)}$, one has

$$\beta((L_{0,2})^\ddagger\circ\Phi_1^\ddagger(\overline{u}))^{-1}(\psi_0\oplus l_m^{(1)})^\ddagger((L_{0,2})^\ddagger\circ\Phi_1^\ddagger(\overline{u})) = \overline{g_u} \tag{e 21.53}$$

for some $g_u\in U_0(A)$ with $\mathrm{cel}(g_u)<\sigma_{a,2}^{(1)}$.

Define $L_2 = (\psi_0\oplus l_m^1)\circ L_{0,2} : C_m\to A$ and $L_2^\ddagger = (\psi_0\oplus l_m^{(1)})^\ddagger\circ(L_{0,2})^\ddagger$. Then $[L_2]|_{\mathcal{P}_a^{(2)}} = [L_2']|_{\mathcal{P}_a^{(2)}}$ and, by (e 21.47) and (e 21.50),for all $f\in\mathcal{H}_c^{(2)}\cup\Phi_1(\mathcal{H}_a^{(1)})$

$$|\tau(L_2(f))-\gamma(\tau)(f)| < \min\{\sigma_{c,1}^{(2)},\sigma_{c,2}^{(2)}\}/6. \tag{e 21.54}$$

Also, for any $u\in\mathcal{U}_{a,0}^{(1)}$, by (e 21.53) and (e 21.42), one then has

$$(L_2\circ\Phi_1)^\ddagger(\overline{u}) = \beta(L_{2,0}^\ddagger\circ\Phi_1^\ddagger(\overline{u}))\cdot\overline{g_u} = \overline{u}\cdot\overline{g_u}\approx_{\sigma_{a,2}^{(1)}}\overline{u} \text{ for all } u\in\mathcal{U}_{a,0}^{(1)}. \tag{e 21.55}$$

Moreover, since $[\Phi_1](\mathcal{P}_a^{(1)})\subset P_c^{(2)}$, by (e 21.46), (i), (e 21.27), and by (e 21.47), and by (e 21.50),

$$[L_2\circ\Phi_1]|_{\mathcal{P}_a^{(1)}} = [\mathrm{id}]|_{\mathcal{P}_a^{(1)}}, \text{ and} \tag{e 21.56}$$

$$|\tau\circ L_2\circ\Phi_1(f)-\tau(f)|<\sigma_{a,1}^{(1)} \text{ for all } f\in\mathcal{H}_a^{(1)} \text{ and } \text{ for all } \tau\in T(A). \tag{e 21.57}$$

Note that L_2 is still $\mathcal{G}_c^{(2)+}$-$\delta_c^{(2)+}$-multiplicative and $L_2\circ\Phi_1$ is $\mathcal{G}_a^{(1)+}$-$\delta_a^{(1)+}$-multiplicative. One then has that for any $u\in\mathcal{U}_{a,1}^{(1)}$ (with k the order of $\overline{u}$),

$$\mathrm{dist}(\overline{\langle(L_2\circ\Phi_1)(u)\rangle}^k, CU(A))<\sigma_{a,2}^{(1)}$$

(see (e 21.16)). Therefore, there is a self-adjoint element $h\in A$ with $\|h\|<\sigma_{a,2}^{(1)}$ such that

$$(u^*)^k(\langle L_2\circ\Phi_1(u)\rangle)^k\exp(2\pi ih)\in CU(A),$$

and hence

$$((u^*)(\langle L_2\circ\Phi_1(u)\rangle)\exp(2\pi ih/k))^k\in CU(A).$$

Note that

$$(u^*)(\langle L_2\circ\Phi_1(u)\rangle)\exp(2\pi ih/k)\in U_0(A)$$

and $U_0(A)/CU(A)$ is torsion free (Corollary 11.7). One has that

$$(u^*)(\langle L_2\circ\Phi_1(u)\rangle)\exp(2\pi ih/k)\in CU(A).$$

In particular, this implies that

$$\mathrm{dist}((L_2\circ\Phi_1)^{\ddagger}(u),u)<\sigma_{a,2}^{(1)}.$$

Combining this with (e 21.55), one has

$$\mathrm{dist}((L_2\circ\Phi_1)^{\ddagger}(u),u)<\sigma_{a,2}^{(1)} \text{ for all } u\in\mathcal{U}_a^{(1)}. \tag{e 21.58}$$

Then, with (e 21.56), (e 21.57), and (e 21.58), applying Theorem 12.11, one obtains a unitary $W\in A$ such that

$$||W^*(L_2\circ\Phi_1(f))W-f||<\varepsilon_2 \text{ for all } f\in\mathcal{F}_1.$$

Replacing L_2 by $\mathrm{Ad}(W)\circ L_2$, one then has

$$||L_2\circ\Phi_1(f)-f||<\varepsilon_2 \text{ for all } f\in\mathcal{F}_1.$$

That is, one has the following diagram

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}"] \arrow[d, "L_1"'] & C \arrow[d, "L_2"] \\
A \arrow[ur, "\Phi_1"'] \arrow[r, "\mathrm{id}"'] & A,
\end{tikzcd}$$

with the upper triangle approximately commuting on $\mathcal{G}_1$ to within ε_1 and the lower triangle approximately commuting on $\mathcal{F}_1$ to within ε_2. Recall that L_2 is $\mathcal{G}_a^{(2)+}$-$\delta_a^{(2)+}$-multiplicative,

$$[L_2]|_{\mathcal{P}_a^{(2)}} = \alpha|_{\mathcal{P}_a^{(2)}}, \text{ and} \tag{e 21.59}$$

$$|\tau(L_2(f)) - \gamma(\tau)(f)| < \sigma_{a,1}^{(2)}/6 \text{ for all } f \in \mathcal{H}_a^{(2)} \qquad \text{(see (e 21.54)).} \tag{e 21.60}$$

Note, by the choice of $\mathcal{G}_a^{(2)+}$ and $\delta_a^{(2)+}$ and by (e 21.46),

$$\kappa_1^C \circ L_2^{\ddagger}|_{\overline{(F_a^{(2)})_0}} = [L_2] \circ \kappa_1^A|_{\overline{(F_a^{(2)})_0}} = \alpha \circ \kappa_1^A|_{\overline{(F_a^{(2)})_0}}. \tag{e 21.61}$$

This implies that $L_2^{\ddagger}$ is injective on $\overline{(F_a^{(2)})_0}$.

Just as $\delta_c^{(2)+}$, $\mathcal{G}_c^{(2)+}$, $\sigma_{c,1}^{(2)}$, $\sigma_{c,2}^{(2)}$, $\mathcal{P}_c^{(2)}$, and $\mathcal{H}_c^{(2)}$ were chosen during the construction of L_2, the construction can continue. By repeating this argument, one obtains the following approximate intertwining diagram

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}"] \arrow[d, "L_1"'] & C \arrow[r, "\mathrm{id}"] \arrow[d, "L_2"] & C \arrow[r, "\mathrm{id"] \arrow[d, "L_3"] & C \arrow[r] \arrow[d, "L_4"] & \cdots \\
A \arrow[r, "\mathrm{id}"'] \arrow[ur, "\Phi_1"'] & A \arrow[r, "\mathrm{id}"'] \arrow[ur, "\Phi_2"'] & A \arrow[r, "\mathrm{id}"'] \arrow[ur, "\Phi_3"'] & A \arrow[r] \arrow[ur] & \cdots ,
\end{tikzcd}$$

where

$$||\Phi_n \circ L_n(g) - g|| < \varepsilon_{2n-1} \text{ for all } g \in \mathcal{G}_n,$$

$$||L_{n+1} \circ \Phi_n(f) - f|| < \varepsilon_{2n} \text{ for all } f \in \mathcal{F}_n, \ n = 1, 2, ...,$$

$$[L_n]|_{\mathcal{P}_c^{(n)}} = \alpha|_{\mathcal{P}_c^{(n)}}, \ [\Phi_n]|_{\mathcal{P}_a^{(n)}} = \alpha^{-1}|_{\mathcal{P}_c^{(n)}},$$

$$|\tau(L_n(f)) - \gamma(\tau)(f)| < \sigma_{c,1}^{(n)}/3 \text{ for all } f \in \mathcal{H}_c^{(n)}, \text{ for all } \tau \in T(A), \text{ and}$$

$$|t(\Phi_n)(g) - \gamma^{-1}(t)(g)| < \sigma_{a,1}^{(n)}/3 \text{ for all } g \in \mathcal{H}_a^{(n)}, \text{ for all } t \in T(C).$$

By the choices of $\mathcal{G}_n$ and $\mathcal{F}_n$ and the fact that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, the standard Elliott approximate intertwining argument (Theorem 2.1 of [30]) applies, and shows that there is an isomorphism $L : A \cong B$ with inverse Φ such that $[L] = \alpha$, and L induces γ as desired. □

THEOREM 21.10. *Let $A_1, B_1 \in \mathcal{B}_0$ be two unital separable amenable simple C^*-algebras satisfying the UCT. Let $A = A_1 \otimes U_1$ and $B = B_1 \otimes U_2$, where U_1 and U_2 are two UHF-algebras of infinite type. Suppose that* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. *Then there exists an isomorphism $\varphi : A \to B$ which carries the isomorphism* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

PROOF. By Theorem 14.10, there is a C^*-algebra C, constructed as in Theorem 14.10, such that $\mathrm{Ell}(C) \cong \mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. By Theorem 21.9, one has that $C \cong A$ and $C \cong B$. In particular, $A \cong B$. ☐

COROLLARY 21.11. *Let A and B be as in* 21.10. *If there is a homomorphism $\Gamma : \mathrm{Ell}(B) \to \mathrm{Ell}(A)$ (see* 2.4*), in particular, $\Gamma(K_0(B)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$, and $\Gamma([1_B]) = [1_A]$, then there is a unital homomorphism $\varphi : B \to A$ such that φ induces Γ.*

PROOF. By Theorem 21.10, we may assume that $B \cong C$ for some C as constructed in Theorem 14.10. So, without loss of generality, we may assume that $B = C$.

The proof is basically the same as that of Theorem 21.9 but simpler since we only need to have a one-sided approximate intertwining. In particular, we do not need to construct Φ_1. Thus, once L_1 is constructed, we can go on to construct L_2.

Nevertheless, we will repeat the argument here. First we keep the first paragraph at the beginning of the proof of Theorem 21.9.

Now, since C satisfies the UCT, by hypothesis, there exist an element $\alpha \in KL(C,A)^{++}$ such that $\alpha|_{K_i(C)} = \Gamma|_{K_i(C)}$, $i = 0,1$, and a continuous affine map $\gamma : T(A) \to T(C)$ such that

$$r_A(\gamma(t))(x) = r_B(t)(\alpha(x)) \text{ for all } x \in K_0(C) \text{ and } \text{ for all } t \in T(A).$$

Note that $\Gamma([1_C]) = [1_A]$.

Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset C$ be an increasing sequence of finite subsets with dense union. Let $1/2 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$ be a decreasing sequence of positive numbers with finite sum. Let $\mathcal{P}_{c,n} \subset \underline{K}(C)$ be finite subsets such that $\mathcal{P}_{c,n} \subset \mathcal{P}_{c,n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{P}_{c,n} = \underline{K}(C)$, and $\mathcal{H}(c,n) \subset C_{s.a.}$ be finite subsets such that $\mathcal{H}(c,n) \subset \mathcal{H}(c,n+1)$ and $\bigcup_{n=1}^{\infty} \mathcal{H}(c,n)$ is dense in $C_{s.a.}$.

We will repeatedly apply the part (a) of Theorem 12.11. Let $\delta_c^{(1)} > 0$ (in place of δ), $\mathcal{G}_c^{(1)} \subset C$ (in place of $\mathcal{G}$ and in place of A), $\sigma_{c,1}^{(1)}, \sigma_{c,2}^{(1)} > 0$ (in place of σ_1 and σ_2), $\mathcal{P}_c^{(1)} \subset \underline{K}(C)$ (in place of $\mathcal{P}$), $\overline{\mathcal{U}_c^{(1)}} \subset U(C)/CU(C))$ (in place of $\mathcal{U}$), and $\mathcal{H}_c^{(1)} \subset C_{s.a}$ (in place of $\mathcal{H}$) be as provided by the part (a) of 12.11 for C (in place of A), ε_1 (in place of ε), and $\mathcal{G}_1$ (in place of $\mathcal{F}$). Here $\mathcal{U}_c^{(1)}$ is a finite subset of $U(C)$. As in Remark 12.12, we may assume that $\overline{\mathcal{U}_c^{(1)}} \subset J_c^C(K_1(C))$. Without loss of generality, we may assume that the image of $\mathcal{U}_c^{(1)}$ in $K_1(C)$ is contained in $\mathcal{P}_c^{(1)}$. As in Remark 12.12, we may also assume that, for all $n \geq n_0$ for some large $n_0 \geq 1$, there is a finite subset $\mathcal{V}_{n,c}^{(1)} \subset U(C_n)$ such that $\mathcal{U}_c^{(1)} = \iota_n(\mathcal{V}_{n,c}^{(1)})$ under the map ι_n. We may further assume that $\mathcal{P}_{c,1} \subset \mathcal{P}_c^{(1)}$ and $\mathcal{H}(c,1) \subset \mathcal{H}_c^{(1)}$.

Let $F_c^{(1)}$ be the subgroup generated by $\mathcal{U}_c^{(1)}$. To simplify notation, let us assume that $\kappa_1^C(\overline{F_c^{(1)}})$ is equal to the subgroup generated by $\mathcal{P}_c^{(1)} \cap K_1(C)$. Put $K_{1,c}^{(1)} = \kappa_1^C(\overline{F_c^{(1)}})$. Write $\alpha(K_{1,c}^{(1)}) = K_{1,c,a,f} \oplus \mathrm{Tor}(\alpha(K_{1,c}^{(1)})) \subset K_1(A)$, where

$K_{1,c,a,f}$ is free abelian. There is an injective homomorphism $j_\alpha : K_{1,c,a,f} \to K_{1,c}^{(1)}$ such that $\alpha \circ j_\alpha = \mathrm{id}_{K_{1,c,a,f}}$. We may write $K_{1,c}^{(1)} = j_\alpha(K_{1,c,a,f}) \oplus K_{2,c}^{(1)}$, where $K_{2,c}^{(1)}$ is the preimage of $\mathrm{Tor}(\alpha(K_{1,c}^{(1)}))$ under α. Recall that we assumed that $\overline{F_c^{(1)}} \subset J_c^C(K_1(C))$. Therefore, $\overline{F_c^{(1)}} = J_c^C(K_{1,c}^{(1)}) = J_c^C(j_\alpha(K_{1,c,a,f})) \oplus J_c^C(K_{2,c}^{(1)})$. Put $\overline{(F_c^{(1)})_0} = J_c^C(j_\alpha(K_{1,c,a,f}))$ (this is one of the differences from the proof of 21.9), which is free abelian. Note that $\alpha|_{\kappa_1^C(\overline{(F_c^{(1)})_0}}$ is injective. Without loss of generality (choosing a smaller $\delta_c^{(1)}$), we may assume that

$$\mathcal{U}_c^{(1)} = \mathcal{U}_{c,0}^{(1)} \sqcup \mathcal{U}_{c,1}^{(1)}, \tag{e 21.62}$$

where $\overline{\mathcal{U}_{c,0}^{(1)}}$ generates $\overline{(F_c^{(1)})_0}$ and $\mathcal{U}_{c,1}^{(1)}$ generates $J_c^C(K_{2,c}^{(1)})$ (another difference from the proof of 21.9). Let k_0 be an integer such that $x^{k_0} = 0$ for all $x \in \mathrm{Tor}(\alpha(K_{1,c}^{(1)}))$. Let $\mathcal{V}_{c,0}^{(1)} \subset U(C_m)$ be a finite subset such that $\iota_n(\mathcal{V}_{c,0}^{(1)}) = \mathcal{U}_{c,0}^{(1)}$.

Since $\overline{(F_c^{(1)})_0}$ is free abelian, there exists an injective homomorphism $j_0 : \overline{(F_c^{(1)})_0} \to J_c^{C_n}(K_1(C_n))$ such that $J_c^C \circ \iota_n^\ddagger \circ j_0 = \mathrm{id}_{\overline{(F_c^{(1)})_0}}$. In particular, $J_c^C \circ \iota_n^\ddagger$ is injective on $j_0(\overline{(F_c^{(1)})_0})$. It follows that $\pi_1|_{j_0(\overline{(F_c^{(1)})_0})}$ is injective.

Let a finite subset $\mathcal{G}_{uc}^{(1)} \subset C$ and $\delta_{uc}^{(1)} > 0$ satisfy the following condition: for any $\mathcal{G}_{uc}^{(1)}$-$\delta_{c,u}^{(1)}$-multiplicative unital completely positive linear map $L' : C \to A'$ (for any unital C^*-algebra A' with $K_1(A') = U(A')/U_0(A')$), $(L')^\ddagger$ can be defined as a homomorphism on $\overline{F_c^{(1)}}$, $\mathrm{dist}((L')^\ddagger(\overline{u}), \overline{\langle L'(u)\rangle}) < \sigma_{c,2}^{(1)}/4$ for all $u \in \mathcal{U}_c^{(1)}$, and $\kappa_1^{A'} \circ (L')^\ddagger(\overline{u}) = [L'] \circ \kappa_1^C([u])$ for all $u \in \mathcal{U}_c^{(1)}$ (see 2.17). Since $\overline{F_c^{(1)}} \subset J_c(K_1(C))$, and $J_c(K_1(C)) \cap U_0(C)/CU(C)$ only contains the unit $\overline{1_C}$ and since $(L')^\ddagger$ is a homomorphism on $\overline{F_c^{(1)}}$, we have $(L')^\ddagger(\overline{F_c^{(1)}} \cap U_0(C)/CU(C)) = \overline{1_{A'}} \in U_0(A')/CU(A')$. Moreover, we may also assume that, if $[L']|_{\kappa_1^C(\overline{F_c}^{(1)})} = [\alpha]|_{\kappa_1^C(\overline{F_c}^{(1)})}$, and if $u \in \mathcal{U}_{c,1}^{(1)}$, then

$$\mathrm{dist}(\langle L'(u)\rangle^k, CU(A)) < \sigma_{c,2}^{(1)}/4, \tag{e 21.63}$$

where $1 \le k \le k_0$ is the order of $\alpha([u])$ (this is another difference from the the proof of 21.9). Put $\mathcal{G}_c^{(1)+} = \mathcal{G}_c^{(1)} \cup \mathcal{G}_{uc}^{(1)}$ and $\delta_c^{(1)+} = \min\{\delta_c^{(1)}, \delta_{uc}^{(1)}\}$.

Recall that both A and C are in $\mathcal{B}_{u0}$. By Theorem 20.16, there is a $\mathcal{G}_c^{(1)+}$-$\delta_c^{(1)+}$ -multiplicative contractive completely positive linear map $L_1 : C \to A$ such that

(e 21.64) $[L_1]|_{\mathcal{P}_c^{(1)}} = \alpha|_{\mathcal{P}_c^{(1)}}$ and

(e 21.65) $|\tau \circ L_1(f) - \gamma(\tau)(f)| < \sigma_{c,1}^{(1)}/4$ for all $f \in \mathcal{H}_c^{(1)}$ for all $\tau \in T(A)$.

Choosing $\mathcal{G}_c^{(1)+}$ large enough and $\delta_c^{(1)+}$ small enough one may assume that $L_1^\ddagger$ is a homomorphism defined on $(\overline{F_c^{(1)}})$ (see 2.17). We may further assume that

$\mathrm{dist}(L_1^{\ddagger}(\overline{u}), \overline{\langle L_1(u)\rangle}) < \sigma_{c,2}^{(1)}/8$ for all $u \in \mathcal{U}_c^{(1)}$ and $\kappa_1^A \circ L_1^{\ddagger} = [L_1] \circ \kappa_1^C$ on $(\overline{F_c^{(1)}})$. Since $\alpha|_{\kappa_1^C(\overline{(F_c^{(1)})_0})}$ is injective, we may assume that $L_1^{\ddagger}|_{\overline{(F_c^{(1)})_0}}$ is injective as κ_1^C is injective on $J_c(K_1(C))$.

Moreover, if $1 \le k(\le k_0)$ is the order of $\alpha([u])$, we may also assume (see (e 21.12)) that

$$\mathrm{dist}(\langle L_1(u)\rangle^k, CU(A)) < \sigma_{c,2}^{(1)}/4 \ \text{for all such}\ u \in \mathcal{U}_{c,1}^{(1)}. \tag{e 21.66}$$

Furthermore, we may assume that there exists a finite subset $\mathcal{G}_{c,c}^{(1)} \subset C_n$ such that $\iota_n(\mathcal{G}_{c,c,}^{(1)}) = G_c^{(1)+}$.

We now construct L_2. We will continue to apply the part (a) of Theorem 12.11. Let $\delta_c^{(2)} > 0$ (in place of δ), $\mathcal{G}_c^{(2)} \subset C$ (in place of $\mathcal{G}$), $\sigma_{c,1}^{(2)}, \sigma_{c,2}^{(2)} > 0$ (in place of σ_1 and σ_2), $\mathcal{P}_c^{(2)} \subset \underline{K}(C)$ (in place of $\mathcal{P}$), $\overline{\mathcal{U}_c^{(2)}} \subset U(C)/CU(C)$ (in place of $\mathcal{U}$), and $\mathcal{H}_c^{(2)} \subset C_{s.a}$ (in place of $\mathcal{H}$) be as provided by the part (a) of Theorem 12.11 for C (in place of A), ε_2 (in place of ε), and $\mathcal{G}_2$ (in place of $\mathcal{F}$). We may assume that $\mathcal{U}_c^{(2)} \subset U(C)$ is a finite subset. We may also assume, without loss of generality, that $\delta_c^{(2)} < \delta_c^{(1)+}/2$, $\mathcal{G}_c^{(1)+} \subset \mathcal{G}_c^{(2)}$ and

$$\overline{\mathcal{U}_c^{(1)}} \subset \overline{\mathcal{U}_c^{(2)}}. \tag{e 21.67}$$

By Remark 12.12, we may assume that $\overline{\mathcal{U}_c^{(2)}} \subset J_c^C(K_1(C))$ and there exists an integer $n_1 > n_0$ such that $\mathcal{U}_c^{(2)} \subset \iota_m(U(C_m))$ for all $m \ge \max\{n_1, n\} \ge n_0$. We may also assume, without loss of generality, that $\mathcal{G}_c^{(1)} \subset \mathcal{G}_c^{(2)}$, $\mathcal{P}_c^{(1)} \cup \mathcal{P}_{c,2} \subset \mathcal{P}_c^{(2)}$, $\mathcal{U}_c^{(1)} \subset \mathcal{U}_c^{(2)}$, $\mathcal{H}_c^{(1)} \cup \mathcal{H}(c,2) \subset \mathcal{H}_c^{(2)}$, and $\delta_c^{(2)} < \delta_c^{(1)}$. We may further assume that $\mathcal{P}_c^{(2)} \subset \iota_m(\mathcal{P}_{c,c}^{(2)})$ for some finite subset $\mathcal{P}_{c,c}^{(2)}$ of $\underline{K}(C_m)$, and $\mathcal{H}_c^{(2)} \subset \iota_m(\mathcal{H}_{c,c}^{(2)})$ for some finite subset $\mathcal{H}_{c,c}^{(2)}$ of C_m (for all $m \ge \max\{n_1, n\}$).

Let $F_c^{(2)}$ be the subgroup generated by $\mathcal{U}_c^{(2)}$. Without loss of generality, to simplify notation, we may assume that $\kappa_1^C(\overline{F_c^{(2)}})$ is the same as the subgroup generated by $\mathcal{P}_c^{(2)} \cap K_1(C)$. Put $K_{1,c}^{(2)} = \kappa_1^C(\overline{F_c^{(2)}})$. Write $\alpha(K_{1,c}^{(2)}) = K_{2,c,a,f} \oplus \mathrm{Tor}(\alpha(K_{1,c}^{(2)}))$, where $K_{2,c,a,f}$ is free abelian. There is an injective homomorphism $j_\alpha^{(2)} : K_{2,c,a,f} \to K_{1,c}^{(2)}$ such that $\alpha \circ j_\alpha^{(2)} = \mathrm{id}_{K_{2,c,a,f}}$. We may write $K_{1,c}^{(2)} = j_\alpha^{(2)}(K_{2,c,a,f}) \oplus K_{2,c}^{(2)}$, where $K_{2,c}^{(2)}$ is the preimage of $\mathrm{Tor}(\alpha(K_{1,c}^{(2)}))$ under α. Recall that we assumed that $\overline{F_c^{(2)}} \subset J_c^C(K_1(C))$. Therefore $\overline{F_c^{(2)}} = J_c^C(K_{1,c}^{(2)}) = J_c^C(j_\alpha(K_{2,c,a,f})) \oplus J_c^C(K_{2,c}^{(2)})$. Put $\overline{(F_c^{(2)})_0} = J_c^C(j_\alpha(K_{2,c,a,f}))$; this group is free abelian. Without loss of generality (choosing a smaller $\delta_c^{(2)}$), we may assume that

$$\mathcal{U}_c^{(2)} = \mathcal{U}_{c,0}^{(2)} \sqcup \mathcal{U}_{c,1}^{(2)}, \tag{e 21.68}$$

where $\overline{\mathcal{U}_{c,0}^{(2)}}$ generates $\overline{(F_c^{(2)})_0}$ and $\mathcal{U}_{c,1}^{(2)}$ generates $J_c^C(K_{2,c}^{(2)})$ (then the condition (e 21.67) should be changed to the condition that $\overline{\mathcal{U}_c^{(1)}}$ be contained in the subgroup generated by $\overline{\mathcal{U}_c^{(2)}}$, which also implies $F_c^{(1)} \subset F_c^{(2)}$). Let k_0 be an integer such that $x^{k_0} = 0$ for all $x \in \mathrm{Tor}(\alpha(K_{1,c}^{(1)}))$.

Let a finite subset $\mathcal{G}_{uc}^{(2)} \subset C$ and $\delta_{uc}^{(2)} > 0$ satisfy the following condition: for any $\mathcal{G}_{uc}^{(2)}$-$\delta_{c,u}^{(2)}$-multiplicative unital completely positive linear map $L' : C \to A'$ (for any unital C^*-algebra A' with $K_1(A') = U(A')/U_0(A')$), $(L')^{\ddagger}$ can be defined as a homomorphism on $\overline{F_c^{(2)}}$, $\mathrm{dist}((L')^{\ddagger}(\overline{u}), \overline{\langle L'(u)\rangle}) < \min\{\sigma_{a,1}^{(1)}, \sigma_{c,2}^{(2)}\}/12$ for all $u \in {\mathcal{U}_c^{(2)}}'$, $(L')^{\ddagger}(\overline{F_c^{(2)}} \cap U_0(C)/CU(C))= \overline{1_{A'}} \in U_0(A')/CU(A')$, and $\kappa_1^{A'} \circ (L')^{\ddagger}(\overline{u}) = [L'] \circ \kappa_1^C([u])$ for all $u \in \mathcal{U}_c^{(2)}$ (see 2.17). Moreover, we may also assume that, if $[L']|_{\kappa_1^C(\overline{F_c}^{(2)})} = [\alpha]|_{\kappa_1^C(\overline{F_c}^{(2)})}$,

$$\text{(e 21.69)} \qquad \mathrm{dist}(\langle L'(u)\rangle^k, CU(A')) < \sigma_{c,2}^{(2)}/4 \ \text{ for all } \ u \in \mathcal{U}_{c,1}^{(2)},$$

where k is the order of $\alpha([u])$ (see Definition 2.17).

There are a finite subset $\mathcal{G}_0 \subset C$ and a positive number $\delta_0 > 0$ such that, for any two $\mathcal{G}_0$-δ_0-multiplicative unital completely positive linear maps $L_1'', L_2'' : C \to A$, if

$$\|L_1''(c) - L_2''(c)\| < \delta_0 \ \text{ for all } \ c \in \mathcal{G}_0,$$

then

$$[L_1'']|_{\mathcal{P}_c^{(2)}} = [L_2'']|_{\mathcal{P}_c^{(2)}} \ \text{ and}$$

$$|\tau \circ L_1''(h) - \tau \circ L_2''(h)| < \min\{\sigma_{c,1}^{(2)}, \sigma_{a,1}^{(1)}\}/12 \ \text{ for all } \ h \in \mathcal{H}_c^{(2)} \ \text{ for all } \ \tau \in T(A).$$

Put $\mathcal{G}_c^{(2)+} = \mathcal{G}_c^{(2)} \cup \mathcal{G}_{uc}^{(2)} \cup \mathcal{G}_0$ and $\delta_c^{(2)+} = \min\{\delta_c^{(2)}, \delta_{uc}^{(2)}, \delta_0, \sigma_{c,1}^{(1)}/8\pi\}/8$.

Since $\overline{(F_c^{(2)})_0}$ is free abelian, there exists an injective homomorphism $j_1 : \overline{(F_c^{(2)})_0} \to J_c^{C_m}(K_1(C_m))$ such that $J_c^C \circ \iota_m^{\ddagger} \circ j_1 = \mathrm{id}_{\overline{(F_c^{(2)})_0}}$. In particular, $J_c^C \circ \iota_m^{\ddagger}$ is injective on $j_1(\overline{(F_c^{(2)})_0})$. It follows that $\pi_1|_{j_1(\overline{(F_c^{(2)})_0})}$ is injective. Let $\mathcal{V}_c^{(2)} \subset U(C_m)$ be a finite subset such that $\iota_m(V_c^{(2)}) = \mathcal{U}_{c,0}^{(2)}$ and $\overline{V_c^{(2)}} \subset j_1(\overline{(F_c^{(1)})_0})$.

To simplify notation, without loss of generality, let us assume that there exists a finite subset $\mathcal{G}_{c,c}^{(2)} \subset C_m$ for some $m \geq n_1$ such that $\iota_m(\mathcal{G}_{c,c}^{(2)}) = \mathcal{G}_c^{(2)+} \cup \mathcal{H}_c^{(2)}$.

Let $M_c = \max\{\|g\| : g \in \mathcal{G}_c^{(2)+}, \text{ or } g \in \mathcal{G}_{c,c}^{(2)}\}$. Note, since C_m and C are both amenable, there exists a unital completely positive linear map $L_{0,2} : C \to C_m$ such that for all $g \in \mathcal{G}_c^{(2)+}$

$$\text{(e 21.70)} \qquad \|\iota_m \circ L_{0,2}(g) - g\| < \delta_c^{(2)+}/8(M_c + 1)$$

and for all $c \in \mathcal{G}_{c,c}^{(2)} \cup \mathcal{V}_c^{(2)} \cup \iota_{n,m}(\mathcal{V}_c^{(1)})$

$$\text{(e 21.71)} \qquad \|L_{0,2} \circ \iota_m(c) - c\| < \delta_c^{(2)+}/8(M_c + 1.$$

Then $L_{0,2}$ is $\mathcal{G}_c^{(2)+}$-$\delta_c^{(2)+}/4$-multiplicative. Let $\kappa_1^{C_m} : U(C_m)/CU(C_m) \to K_1(C_m)$ be the quotient map. We may then assume that $L_{0,2}^{\ddagger}$ is defined on $\overline{F_c^{(2)}}$ and injective on $\overline{F_c^{(2)}}$, and that $\mathrm{dist}(L_{0,2}^{\ddagger}(\overline{u}), \overline{u}) < \min\{\sigma_{c,1}^{(2)}, \sigma_{c,1}^{(1)}\}/12$ for all $u \in \mathcal{U}_c^{(2)}$.

Note that $J_c^C \circ \iota_m^{\ddagger} \circ \iota_{n,m}^{\ddagger} = J_c^C \circ \iota_n^{\ddagger}$ is injective on $j_0(\overline{(F_c^{(1)})_0})$. It follows that $\iota_{n,m}^{\ddagger}$ is injective on $j_0(\overline{(F_c^{(1)})_0})$. Also, $(\pi_1)|_{\iota_{n,m}^{\ddagger}(j_0(\overline{(F_c^{(1)})_0}))}$ is injective. Let $\beta' := (\iota_{n,m}^{\ddagger})^{-1}$ be defined on $\iota_{n,m}^{\ddagger}(j_0(\overline{(F_c^{(1)})_0}))$.

Let $\sigma > 0$, $\delta_c' > 0$ (in place of δ) and the finite subset $\mathcal{G}_c' \subset C_m$ (in place of $\mathcal{G}$) be as provided by Lemma 21.5, with respect to $\sigma_{a,2}^{(2)}/16\pi$ (in place of ε), C_m (in place of C), A (in place of B), $V_c^{(2)}$ (in place of $\mathcal{U}$), $\iota_{n,m}^{\ddagger}(j_0(\overline{(F_c^{(1)})_0})$ (in place of $\overline{F}$), and $L_1^{\ddagger} \circ \iota_n^{\ddagger} \circ \beta'$ (in place of α). By Theorem 14.10 and Remark 14.11, as in the decomposition in (e 21.25) of ι_n in the proof of Theorem 21.9, one may write $\iota_m = \imath_m^{(0)} \oplus \imath_m^{(1)}$, where $\imath_m^{(i)}$ is a homomorphism $(i = 0, 1)$, $\imath_m^{(0)}$ has finite dimensional range, and $\imath_m^{(0)}$ is non-zero on each non-zero direct summand of C_m. Moreover,

$$\tau(\imath_m^{(0)}(1_{C_m})) < \min\{\sigma, \sigma_{c,1}^{(2)}, \sigma_{c,1}^{(1)}\}/12 \ \text{ for all } \ \tau \in T(C). \tag{e 21.72}$$

Let E' be a finite set of generators (in the unit ball) of the finite dimensional C^*-subalgebra $\imath_m^{(0)}(C_m)$. Then

$$\sigma_{00} := \inf\{\tau((e')^* e') : e' \in E',\ \tau \in T(C)\} > 0. \tag{e 21.73}$$

Put $\mathcal{H}_c^{(2)+} = \mathcal{H}_c^{(2)} \cup \Phi_1(\mathcal{H}_a^{(1)}) \cup \{a^*a : a \in E'\}$.

Recall that both A and C are in class $\mathcal{B}_{u0}$. By Theorem 20.16, for any finite subset $\mathcal{G}'' \supset \mathcal{G}_c^{(2)+}$ and $0 < \delta'' < \delta_c^{(2)+}$, there is a $\mathcal{G}''$-δ'' -multiplicative map $L_2' : C \to A$ such that

$$[L_2']|_{\mathcal{P}_c^{(1)}} = \alpha|_{\mathcal{P}_c^{(2)}} \tag{e 21.74}$$

and

$$|\tau \circ L_2'(f) - \gamma(\tau)(f)| < \min\{\sigma, \sigma_{00}, \sigma_{c,1}^{(2)}, \sigma_{a,1}^{(1)}\}/12 \quad \text{for all } \ f \in \mathcal{H}_c^{(2)+} \tag{e 21.75}$$

and for all $\tau \in T(A)$, where $\mathcal{G}'' \subset C$ is a finite subset and $\delta'' > 0$. We may choose that

$$\mathcal{G}'' \supset \mathcal{G}_0 \cup \mathcal{G}_c^{(2)+} \cup \iota_m(\mathcal{G}_c) \ \text{ and } \ \delta'' < \min\{\delta_0, \delta_c^{(2)+}, \delta'\}/2.$$

Fix any finite subset $\mathcal{G}_{c,m}^{(2)} \subset C_m$ and any $0 < \delta_0^{(2)} < \min\{\sigma_{00}, \delta_0\}/2$. We may assume that $E' \subset \imath_m^{(0)}(\mathcal{G}_{c,m}^{(2)})$. Since every finite dimensional C^*-algebra is semiprojective, $\imath_m^{(0)}(C_m)$ is finite dimensional, and since L_2' is chosen after C_m is chosen, with sufficiently large $\mathcal{G}''$ and small δ'', we may assume, without loss of generality, that there exists a homomorphism $h_0 : C_m \to A$ with finite dimensional range such that

$$\|h_0(g) - L_2' \circ \imath_m^{(0)}(g)\| < \delta_0^{(2)} \text{ for all } g \in \mathcal{G}_{c,m}^{(2)}, \tag{e 21.76}$$

$$\|(1 - h_0(1_{C_m}))L_2' \circ \imath_m^{(1)}(g)(1 - h_0(1_{C_m})) - L_2' \circ \imath_m^{(1)}(g)\| < \delta_0^{(2)} \tag{e 21.77}$$

for all $g \in \mathcal{G}_{c,m}^{(2)}$ and

$$\tau(h_0(1_{C_m})) < \min\{\sigma/2, \sigma_{a,1}^{(1)}/3, \sigma_{c,1}^{(2)}/3\} \text{ for all } \tau \in T(A). \tag{e 21.78}$$

Let $l_m : C_m \to (1 - h_0(1_{C_m}))A(1 - h_0(1_{C_m}))$ be defined by

$$l_m(c) = (1 - h_0(1_{C_m})L_2' \circ \imath_m^{(1)}(g)(1 - h_0(1_{C_m})) \text{ for all } c \in C_m.$$

One may assume that $(L_2')^{\ddagger}$ is a homomorphism defined on $(\overline{F_c^{(2)}})$ (see 2.17). We may also assume that $\mathrm{dist}((L_2')^{\ddagger}(\overline{u}), \overline{\langle (L_2')(u)\rangle}) < \sigma_{c,2}^{(2)}/8$ for all $u \in \mathcal{U}_c^{(2)}$, and $\kappa_1^A \circ (L_2')^{\ddagger} = [L_2'] \circ \kappa_1^C$ on $(\overline{F_c^{(2)}})$. Since $\alpha|_{\kappa_1^C(\overline{(F_c^{(2)})_0})}$ is injective, we may assume that $(L_2')^{\ddagger}|_{\overline{(F_c^{(2)})_0}}$ is injective as κ_1^C is injective on $J_c(K_1(C))$. Moreover, if $1 \le k(\le k_0)$ is the order of $\alpha([u])$, we may also assume (see (e 21.12)) that

$$\mathrm{dist}(\langle L_2'(u)\rangle^k, CU(A)) < \sigma_{c,2}^{(2)}/4 \text{ for all } u \in \mathcal{U}_{c,1}^{(2)}. \tag{e 21.79}$$

Since m is now fixed and γ is a homeomorphism, by (e 21.74), (e 21.76), and (e 21.73), we may assume that L_2' is injective on $\imath_m^{(0)}(C_m)$. Since $\imath_m^{(0)}$ is non-zero on each direct summand of C_m, by (e 21.76), (e 21.76), and (e 21.73), we may also assume that h_0 is non-zero on each direct summand of C_m.

Choosing sufficiently large $\mathcal{G}''$ and small δ'', we may assume that $(L_2' \circ \iota_m)^{\ddagger}$ and $(l_m^{(1)})^{\ddagger}$ are defined on a subgroup of $U(C_m)/CU(C_m)$ containing $\iota_{n,m}^{\ddagger}(j_0(\overline{(F_c^{(2)})_0}))$, $\pi_0(\iota_{n,m}^{\ddagger}(j_0(\overline{(F_c^{(2)})_0})))$, $\pi_1(U(C_m)/CU(C_m))$, and $\pi_2(U(C_m)/CU(C_m))$. Moreover, for all $u \in \iota_{n,m}(\mathcal{V}_{c,0}^{(1)})$,

$$\mathrm{dist}((L_2' \circ \iota_m)^{\ddagger}(\overline{u}), \overline{\langle L_2' \circ \iota_m(u)\rangle}) < \sigma_{c,2}^{(1)}/4 \text{ and} \tag{e 21.80}$$

$$\mathrm{dist}(l_m^{\ddagger}(\overline{u\rangle}), \overline{\langle l_m(u)\rangle}) < \sigma_{c,2}^{(1)}/4. \tag{e 21.81}$$

Then, by Lemma 21.5 (with $L_2' \circ \imath_m$ in place of φ, and $L_2' \circ \iota_m^{(1)}$ in place of φ_1), there is a homomorphism $\psi_0 : C_m \to e_0'Ae_0'$, where $e_0' = h_0(1_{C_m})$, such that

(i) $[\psi_0] = [\varphi_0]$ in $KK(C_m, A)$, and

(ii) for any $u \in \iota_{n,m}^{\ddagger}(\mathcal{V}_{c,0}^{(1)})$, one has

$$(L_1^{\ddagger} \circ \iota_n^{\ddagger} \circ \beta'(\overline{u}))^{-1}(\psi_0 \oplus l_m^{(1)})^{\ddagger}(\overline{u})) = \overline{g_u} \tag{e 21.82}$$

for some $g_u \in U_0(C)$ with $\mathrm{cel}(g_u) < \sigma_{a,2}^{(1)}/64\pi$.

Define $L_2'' = (\psi_0 \oplus l_m^1) \circ L_{2,0} : C \to A$ and $(L_2'')^{\ddagger} = (\psi_0 \oplus l_m^{(1)})^{\ddagger} \circ L_{2,0}^{\ddagger}$. Then $[L_2'']|_{\mathcal{P}_a^{(2)}} = [L_2']|_{\mathcal{P}_a^{(2)}}$. Therefore

$$[L_2'']|_{\mathcal{P}_c^{(2)}} = \alpha|_{\mathcal{P}_c^{(2)}}. \tag{e 21.83}$$

We also have from (e 21.74) and (e 21.77)

$$|\tau(L_2''(f)) - \gamma(\tau)(f)| < \min\{\sigma_{c,1}^{(1)}, \sigma_{c,1}^{(2)}\}/3 \ \text{ for all } \ f \in \mathcal{H}_c^{(2)}. \tag{e 21.84}$$

For $u \in \mathcal{U}_{c,0}^{(1)}$, write $u = \iota_n(v)$ for some $v \in \mathcal{V}_c^{(1)}$, Then, by (e 21.71),

$$L_1^{\ddagger}(\overline{u}) = L_1^{\ddagger}(\iota_n(v)) = L_1^{\ddagger}(\iota_n^{\ddagger} \circ \beta'(\overline{\iota_{n,m}(v)}) \ \text{ and} \tag{e 21.85}$$

$$\begin{aligned} L_2''(u) &= (\psi_0 \oplus l_m^{(1)})(L_{2,0}(u)) = (\psi_0 \oplus l_m^{(1)})(L_{2,0}(\iota_m \circ \iota_{n,m}(v))) \\ &\approx_{\sigma_{c,2}^{(1)}/16\pi} (\psi_0 \oplus l_m^1)(\iota_{n,m}(v)). \end{aligned} \tag{e 21.86}$$

It follows from (e 21.82) that, on the one hand, for $u \in \mathcal{U}_{c,0}^{(1)}$,

$$\mathrm{dist}(L_1^{\ddagger}(\overline{u}), (L_2'')^{\ddagger}(\overline{u})) < \sigma_{c,2}^{(1)}. \tag{e 21.87}$$

On the other hand, for $u \in \mathcal{U}_{c,1}^{(1)}$, by (e 21.66) and (e 21.69), there exists $h \in A_{s.a.}$ with $\|h\| \le \sigma_{c,2}^{(1)}/2$ such that

$$((\langle L_1(u)\rangle^k)^* \langle L_2''(u)\rangle^k (\exp(2\pi h/k))^k \in CU(A).$$

Note that $[L_1(u)] = [L_2''(u)]$ for $u \in \mathcal{U}_c^{(1)}$. Then, since $U_0(A)/CU(A)$ is torsion free (see Corollary 11.7),

$$((\langle L_1(u)\rangle)^* \langle L_2''(u)\rangle (\exp(2\pi h/k)) \in CU(A).$$

Then, for $u \in \mathcal{U}_{c,1}^{(1)}$,

$$\mathrm{dist}(L_1^{\ddagger}(\overline{u}), (L_2'')^{\ddagger}(\overline{u})) < \sigma_{c,2}^{(1)}. \tag{e 21.88}$$

Now, combining (e 21.64), (e 21.65), (e 21.83), (e 21.84), (e 21.87), and (e 21.88), applying Theorem 12.11, we have a unitary $u \in A$ such that

$$\|\mathrm{Ad}\, u \circ L_2''(c) - L_1(c)\| < \varepsilon_1 \ \text{ for all } \ c \in \mathcal{F}_1. \tag{e 21.89}$$

Put $L_2 = \mathrm{Ad}\, u \circ L_2''$. We obtain the diagram

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}"] \arrow[d, "L_1"'] & C \arrow[d, "L_2"] \\
A \arrow[r, "\mathrm{id}"'] & A,
\end{tikzcd}$$

which is approximately commutative on $\mathcal{G}_1$ to within ε_1. Note that, by (e 21.83), we have

$$[L_2]|_{\mathcal{P}_c^{(2)}} = \alpha|_{\mathcal{P}_c^{(2)}}, \tag{e 21.90}$$

$$|\tau(L_2(f)) - \gamma(\tau)(f)| < \sigma_{c,1}^{(2)}/3 \ \text{ for all } \ f \in \mathcal{H}_c^{(2)}. \tag{e 21.91}$$

We also have that the restriction $\pi_1|_{\overline{j_1((F_c^{(2)})_0)}}$ is injective. By the choice of $\mathcal{G}_c^{(2)+}$, $\delta_c^{(2)+}$, $\mathcal{U}_{c,0}^{(2)}$, and $\mathcal{U}_{c,1}^{(2)}$, this process can continue.

Repeating this argument, one obtains the following approximate intertwining diagram:

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}"] \arrow[d, "L_1"'] & C \arrow[r, "\mathrm{id}"] \arrow[d, "L_2"] & C \arrow[r, "\mathrm{id}"] \arrow[d, "L_3"] & C \arrow[r] \arrow[d, "L_4"] & \cdots \\
A \arrow[r, "\mathrm{id}"'] & A \arrow[r, "\mathrm{id}"'] & A \arrow[r, "\mathrm{id}"'] & A \arrow[r] & \cdots,
\end{tikzcd}$$

where

$$||\mathrm{id}_A \circ L_n(g) - L_{n+1} \circ \mathrm{id}_C(g)|| < \varepsilon_{2n-1} \ \text{ for all } \ g \in \mathcal{G}_n, \ n = 1, 2, ...$$

Moreover, $[L_n]|_{\mathcal{P}_c^{(n)}} = \alpha|_{\mathcal{P}_c^{(n)}}$ and $|\tau(L_n(f)) - \gamma(\tau)(f)| < \sigma_{c,1}^{(n)}/3$ for all $f \in \mathcal{H}_c^{(n)}$. By the choices of $\mathcal{G}_n$ and $\mathcal{F}_n$ and the fact that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, the standard Elliott approximate intertwining argument (Theorem 2.1 of [30]) applies, and shows that there is a homomorphism $\varphi : C \to A$ such that $[\varphi] = \alpha$ and $\tau(\varphi(f)) = \gamma(\tau)(f)$ for all $f \in A_{s.a.}$ and $\tau \in T(C)$. □

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