Saturated Control of a Switched FES-Cycle with an Unknown Time-Varying Input Delay ⋆

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Abstract: A common rehabilitation for those with lower limb movement disorders is motorized functional electrical stimulation (FES) induced cycling. Motorized FES-cycling is a switched system with uncertain dynamics, unknown disturbances, and there exists an unknown time-varying input delay between the application/removal of stimulation and the onset/removal of muscle force. This is further complicated by the fact that each participant has varying levels of sensitivity to the FES input, and the stimulation must be bounded to ensure comfort and safety. In this paper, saturated FES and motor controllers are developed for an FES-cycle that ensure safety and comfort of the participant, while likewise being robust to uncertain parameters in the dynamics, unknown disturbances, and an unknown time-varying input delay. A Lyapunov-based stability analysis is performed to ensure uniformly ultimately bounded cadence tracking.

Keywords: Functional Electrical Stimulation (FES), Saturated Control, Input Delay, Switched System, Human-Robot Interaction

1. INTRODUCTION

Throughout the world, there are millions of people with a variety of neurological conditions (NCs) such as stroke, Parkinson's Disease (PD), and traumatic brain injury (TBI), among others (Cousin et al., 2019). Functional electrical stimulation (FES) induced cycling is a common treatment for those with NCs that result in lower limb movement disorders (Gföhler et al., 2001; Pons et al., 1989; Schutte et al., 1993; Bellman et al., 2017; Cousin et al., 2019). FES-cycling has been shown to improve physiological motor control (Ferrante et al., 2008), increase bone mineral density (Mohr et al., 1997), and provide numerous other benefits (Cousin et al., 2019). Although FES-cycling has a variety of benefits, implementing closed-loop control of a FES-cycle has numerous challenges (Bellman et al., 2017; Cousin et al., 2019; Downey et al., 2017; Allen et al., 2020c,b). For example, there exists an unknown and complex electrophysiological process between the application of FES and the associated muscle force, which results in a potentially destabilizing input delay.

In the author's prior works, closed-loop FES controllers have been developed to compensate for FES-induced input delays for FES-cycling (Allen et al., 2019a,b, 2020a). The FES-cycling controllers in Allen et al. (2019a,b, 2020a) assumed an unknown and constant delay in Allen et al. (2019b), and assumed an unknown and time-varying delay in Allen et al. (2019a, 2020a), but in Allen et al. (2019a) a constant estimate of the delay was implemented, whereas in Allen et al. (2020a) a time-varying estimate was utilized. However, the aforementioned controllers are all functions of the system's states and unmodeled disturbances or large initial conditions may result in large FES inputs, which may be unsafe or uncomfortable.

Similar to the authors' previous works (Allen et al., 2019b,a, 2020a), a cadence tracking control system is developed for a switched FES-cycle that is robust to uncertainties in the system, unknown disturbances, and a time-varying input delay. Additionally, a delay-dependent trigger condition is developed to schedule the activation/deactivation of FES to yield effective muscle contractions. However, in the author's prior works, FES and motor controllers are proven to be bounded, but the bound is unknown, which could result in high FES inputs that cause discomfort/pain or motor inputs that exceed motor capabilities. The focus of this work is to guarantee safety and comfort by developing saturated FES and motor controllers where the control bounds are known and can be adjusted a priori. Furthermore, a Lyapunov-like stability analysis is performed to guarantee uniformly ultimately bounded cadence tracking errors despite using saturated controllers.

2. DYNAMICS

In this paper, delayed functions are defined as

$$h_\tau \triangleq \begin{cases} h\left(t-\tau\left(t\right)\right) & t-\tau\left(t\right)\geq t_0 \\ 0 & t-\tau\left(t\right)< t_0 \end{cases},$$

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where $t \in \mathbb{R}_{\geq 0}$ denotes the time, $t_0 \in \mathbb{R}_{\geq 0}$ denotes the initial time, and $\tau : \mathbb{R}_{\geq 0} \to \mathbb{S}$ denotes the electromechanical delay (EMD), where $\mathbb{S} \subset \mathbb{R}$ denotes the set of possible delay values (Merad et al., 2016; Allen et al., 2020c). The EMD represents the time latency between the application or removal of the FES input and the corresponding onset or elimination of muscle force [1]. The cycle-rider dynamics can be modeled as (Bellman et al., 2017) [2]

$$\begin{aligned} M(q)\ddot{q} + V(q,\dot{q})\dot{q} + G(q) + P(q,\dot{q}) + b_c\dot{q} + d(t) \\ = B_M^\tau(q,\dot{q},\tau,t)\,u_\tau + B_E(q,\dot{q})\,u_e(t), \end{aligned} \tag{1}$$

where the measurable crank angle and velocity are denoted by $q : \mathbb{R}_{\geq 0} \to \mathcal{Q}$ and $\dot{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, respectively, the set of all possible crank angles is denoted by $\mathcal{Q} \subseteq \mathbb{R}$, and the unmeasured acceleration is denoted by $\ddot{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}$. The inertial, gravitational, and centripetal-Coriolis effects are respectively denoted as $M : \mathcal{Q} \to \mathbb{R}_{>0}$, $G : \mathcal{Q} \to \mathbb{R}$, and $V : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$. The passive viscoelastic tissue torques, system disturbances, and the cycle's viscous damping coefficient are denoted by $P : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$, $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $b_c \in \mathbb{R}_{>0}$, respectively. The implemented FES control input, which is subsequently designed, is denoted by $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and the unknown, lumped muscle control effectiveness, $B_M^\tau : \mathcal{Q} \times \mathbb{R} \times \mathbb{S} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, maps the delayed FES control input, u_τ, into an output torque and is defined as

$$B_M^\tau(q,\dot{q},\tau,t) \triangleq \sum_{m \in \mathcal{M}} B_m(q,\dot{q},t)\, k_m \sigma_{m,\tau}(q_\tau,\dot{q}_\tau), \tag{2}$$

where $m \in \mathcal{M} \triangleq \{RQ, RH, RG, LQ, LH, LG\}$ indicates the right (R) and left (L) quadriceps femoris (Q), hamstrings (H), and gluteal (G) muscle groups, $k_m \in \mathbb{R}_{>0}$, $\forall m \in \mathcal{M}$ are selectable constants, and $\sigma_{m,\tau} : \mathcal{Q} \times \mathbb{R} \to \{0, 1\}, \forall m \in \mathcal{M}$ are delayed switching signals. The unknown, time-varying, and nonlinear muscle control effectiveness for each stimulated muscle group is denoted by $B_m : \mathcal{Q} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $\forall m \in \mathcal{M}$.

The delayed switching signals, $\sigma_{m,\tau}(q_\tau,\dot{q}_\tau)$, $\forall m \in \mathcal{M}$, are included in (2) to indicate which muscle groups received the delayed FES control input, u_τ, at time $t-\tau$. The implemented piecewise right-continuous FES switching signals, denoted by $\sigma_m : \mathcal{Q} \times \mathbb{R} \to \{0, 1\}, \forall m \in \mathcal{M}$, are designed as

$$\sigma_m(q,\dot{q}) \triangleq \begin{cases} 1, & q_\alpha \in \mathcal{Q}_m \\ 0, & \text{otherwise} \end{cases}, \tag{3}$$

$\forall m \in \mathcal{M}$, where $q_\alpha : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ denotes a trigger condition. The function q_α uses an upper bound on the delay (e.g., see the experimental results in Merad et al. (2016)) to determine the appropriate time to activate/deactivate the stimulation of each muscle group such that muscle contractions occur within desired regions for each muscle, denoted by $\mathcal{Q}_m \subset \mathcal{Q}, \forall m \in \mathcal{M}$, and called the muscle's FES region. Each muscle's FES region is defined, using the method detailed in Bellman et al. (2017), as

$$\mathcal{Q}_m \triangleq \{q \in \mathcal{Q} \mid T_m(q) > \varepsilon_m\}, \tag{4}$$

$\forall m \in \mathcal{M}$, where $T_m : \mathcal{Q} \to \mathbb{R}$ denotes a torque transfer ratio and $\varepsilon_m \in \mathbb{R}_{>0}$ is a lower threshold designed to limit the FES regions to crank angles where it is kinematically efficient for a muscle contraction to contribute to forward pedaling (i.e., positive crank motion). The combined FES region of the cycle is defined as $\mathcal{Q}_{FES} \triangleq \bigcup_{m \in \mathcal{M}} \{\mathcal{Q}_m\}$ and the remainder is called a kinematic deadzone and is defined as $\mathcal{Q}_e \triangleq \mathcal{Q} \setminus \mathcal{Q}_{FES}$.

The torque contribution due to the motor in (1) is comprised of the subsequently designed motor control input, denoted by $u_e : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and the motor control effectiveness, denoted by $B_E : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$, which maps the motor control input, u_e, into an output torque and is defined as

$$B_E(q,\dot{q}) \triangleq B_e k_e \sigma_e(q,\dot{q}), \tag{5}$$

where $k_e \in \mathbb{R}_{>0}$ is a selectable constant, and $B_e \in \mathbb{R}_{>0}$ denotes the unknown motor effectiveness. The piecewise right-continuous motor switching signal, denoted by $\sigma_e : \mathcal{Q} \times \mathbb{R} \to \{0, 1\}$, is designed as

$$\sigma_e(q,\dot{q}) \triangleq \begin{cases} 1, \ q \in \mathcal{Q}_e \\ 1, \ q \in \mathcal{Q}_{FES}, \ \sum\limits_{m \in \mathcal{M}} \sigma_m = 0 \\ 0, \ \text{otherwise} \end{cases}. \tag{6}$$

The parameters of the combined cycle-rider system shown in (1) are unknown for the cycle and each rider, however the subsequent control development only requires for bounds to be established on each parameter. The switched system in (1) has the following properties (Bellman et al., 2017). **Property: 1** $c_m \leq M \leq c_M$, where c_m, $c_M \in \mathbb{R}_{>0}$ are known constants. **Property: 2** $|V| \leq c_V|\dot{q}|$, where $c_V \in \mathbb{R}_{>0}$ is a known constant and $|\cdot|$ denotes the absolute value. **Property: 3** $|G| \leq c_G$, where $c_G \in \mathbb{R}_{>0}$ is a known constant. **Property: 4** $|P| \leq c_{P1} + c_{P2}|\dot{q}|$, where c_{P1}, $c_{P2} \in \mathbb{R}_{>0}$ are known constants. **Property: 5** $b_c\dot{q} \leq c_c|\dot{q}|$, where $c_c \in \mathbb{R}_{>0}$ is a known constant. **Property: 6** $|d| \leq c_d$, where $c_d \in \mathbb{R}_{>0}$ is a known constant. **Property: 7** The muscle control effectiveness B_m is lower and upper bounded $\forall m \in \mathcal{M}$, and thus, when $\sum_{m \in \mathcal{M}} \sigma_{m,\tau} > 0$, $c_b \leq B_M^\tau \leq c_B$, where $c_b, c_B \in \mathbb{R}_{>0}$ are known constants. **Property: 8** The motor control effectiveness is bounded such that when $\sigma_e = 1$, $c_e \leq B_E \leq c_E$, where c_e, $c_E \in \mathbb{R}_{>0}$ are known constants. **Property: 9** The delay is bounded such that $\underline{\tau} \leq \tau \leq \bar{\tau}$, where $\underline{\tau}$, $\bar{\tau} \in \mathbb{R}_{>0}$ are known constants.

To aid the subsequent control design and analysis, the vector $\operatorname{Tanh}(\cdot) \in \mathbb{R}^n$ is defined as follows

$$\operatorname{Tanh}(\xi) \triangleq [\tanh(\xi_1), ..., \tanh(\xi_n)]^T, \tag{7}$$

where $\xi = [\xi_1, ..., \xi_n]^T \in \mathbb{R}^n$. Based on the definition in (7), the following inequalities hold $\forall \xi \in \mathbb{R}^n$ (Zhang et al., 2000):

$$\|\xi\|^2 \geq \sum_{i=1}^{n} \ln(\cosh(\xi_i)) \geq \ln(\cosh(\|\xi\|)) \geq \frac{1}{2}\tanh^2(\|\xi\|), \tag{8}$$

$$\|\xi\| > \|\operatorname{Tanh}(\xi)\|, \quad \|\operatorname{Tanh}(\xi)\|^2 \geq \tanh^2(\|\xi\|), \tag{9}$$

$$\frac{\|\xi\|}{\tanh(\|\xi\|)} \leq \|\xi\| + 1. \tag{10}$$

[1] In some literature the EMD corresponds to the time latency between the onset of EMG activity and muscle force Nordez et al. (2009).

[2] For notational brevity, all explicit dependence on time, t, within the terms $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$ is suppressed.

3. CONTROL DEVELOPMENT

One objective is for the bicycle crank to track a desired cadence, denoted by $\dot{q}_d : \mathbb{R}_{\geq 0} \to \mathbb{R}$, despite the dynamic model being uncertain and having an unknown time-varying input delay. To quantify the control objective a measurable cadence tracking error, $\dot{e} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, is defined as [3]

$$\dot{e} \triangleq \dot{q}_d - \dot{q}. \tag{11}$$

A measurable auxiliary tracking error, $r : \mathbb{R}_{\geq 0} \to \mathbb{R}$, is defined as

$$r \triangleq \dot{e} + \alpha e_u, \tag{12}$$

where $\alpha \in \mathbb{R}_{\geq 0}$ is a selectable constant. The auxiliary error signal, $e_u : \mathbb{R}_{\geq 0} \to \mathbb{R}$, is defined as

$$e_u \triangleq -\int_{t-\hat{\tau}}^{t} \sigma_s(\theta) u(\theta) d\theta, \tag{13}$$

and is used to inject a delay-free input into the closed-loop error system. The constant estimate of the delay is denoted by $\hat{\tau} \in \mathbb{R}_{>0}$ and by Property 9, the delay estimation error, $\tilde{\tau} = \tau - \hat{\tau}$, can be upper bounded such that $|\tilde{\tau}| \leq \bar{\tilde{\tau}}$, where $\bar{\tilde{\tau}} \in \mathbb{R}_{>0}$ is a known constant. The piecewise right-continuous switching signal denoted by $\sigma_s : \mathbb{R}_{\geq 0} \to \{0, 1\}$ is defined as

$$\sigma_s(t) \triangleq \begin{cases} 1, & \sum_{m \in \mathcal{M}} \sigma_m > 0 \\ 0, & \text{otherwise} \end{cases}, \tag{14}$$

and indicates when stimulation is applied. The open-loop error system is obtained by substituting (11) and (13) into (12) and taking the time derivative of (12), solving (1) for $\ddot{q}$ and substituting into the time derivative of (12), and adding and subtracting $\frac{B_M^\tau}{M} u_{\hat{\tau}}$ to yield

$$\begin{aligned} \dot{r} &= \chi + \frac{B_M^\tau}{M}(u_{\hat{\tau}} - u_\tau) - \frac{B_E}{M} u_e \\ &+ \left(\sigma_{s,\hat{\tau}}\alpha - \frac{B_M^\tau}{M}\right) u_{\hat{\tau}} - \sigma_s \alpha u, \end{aligned} \tag{15}$$

where $\chi : \mathcal{Q} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined as $\chi \triangleq \ddot{q}_d + \frac{1}{M}[V\dot{q} + G + P + b_c \dot{q} + d]$. By using Properties 1-6, (11), and (12), the auxiliary term χ can be bounded as

$$|\chi| \leq \Phi + \rho(\|z\|)\|z\|, \tag{16}$$

where $\Phi \in \mathbb{R}_{>0}$ is a known constant, $\rho(\cdot)$ is a positive, strictly increasing, radially unbounded, and globally invertible function, and $z \in \mathbb{R}^2$ is a composite error vector defined as

$$z \triangleq [\, r \;\; e_u \,]^T. \tag{17}$$

A secondary control objective is to design saturated FES and motor controllers that can meet the cadence tracking objective. Based on the open-loop error system in (15), the bound in (16), and the subsequent stability analysis, the FES and motor control inputs are designed as [4]

$$u \triangleq k_s \tanh(r), \tag{18}$$

and

$$u_e \triangleq k_1 \mathrm{sgn}(r) + k_2 \tanh(r), \tag{19}$$

respectively, where k_s, k_1, $k_2 \in \mathbb{R}_{>0}$ are selectable constants and $\mathrm{sgn}(\cdot)$ denotes the signum function. Notice that the FES and motor control inputs are bounded by selectable gain constants, since $|u| \leq k_s$ and $|u_e| \leq k_1 + k_2$. Note, the stimulation input (i.e. pulse width) to each of the rider's muscles is defined as $u_m \triangleq k_m \sigma_m u, \forall m \in \mathcal{M}$, and the stimulation input can be bounded by $|u_m| \leq k_m k_s$, $\forall m \in \mathcal{M}$. Likewise, the current input to the motor is defined as $u_E \triangleq k_e \sigma_e u_e$ and can be bounded by $|u_E| \leq k_e(k_1 + k_2)$. The bounds on the FES controller allow for the maximum stimulation input to be limited, resulting in a safer and more comfortable experience for the participant. Likewise, the current input into the motor is limited to ensure the input does not exceed the motor specifications. Substituting (18) and (19) into (15) yields the closed-error system

$$\begin{aligned} \dot{r} &= \chi + k_s \frac{B_M^\tau}{M}(\tanh(r_{\hat{\tau}}) - \tanh(r_\tau)) \\ &- \frac{B_E}{M}(k_1 \mathrm{sgn}(r) + k_2 \tanh(r)) \\ &+ \left(\sigma_{s,\hat{\tau}}\alpha - \frac{B_M^\tau}{M}\right) k_s \tanh(r_{\hat{\tau}}) \\ &- \sigma_s \alpha k_s \tanh(r). \end{aligned} \tag{20}$$

To facilitate the subsequent stability analysis, and based on the closed-loop error system in (20), the following Lyapunov-Krasovskii functionals Q_1, $Q_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ are defined as

$$Q_1 \triangleq \frac{1}{2}(\varepsilon_1\omega_1 + \varepsilon_2\omega_2) k_s \int_{t-\hat{\tau}}^{t} \tanh^2(r(\theta)) d\theta, \tag{21}$$

$$Q_2 \triangleq \frac{\omega_3 k_s}{\hat{\tau}} \int_{t-\hat{\tau}}^{t} \int_{s}^{t} \tanh^2(r(\theta)) d\theta ds, \tag{22}$$

where $\varepsilon_1, \varepsilon_2, \omega_1, \omega_2, \omega_3 \in \mathbb{R}_{>0}$ are selectable constants. Further, the auxiliary bounding constants $\beta, \beta_1, \beta_2, \delta \in \mathbb{R}_{>0}$ are defined as

$$\beta \triangleq \min(\beta_1, \beta_2), \tag{23}$$

$$\begin{aligned} \beta_1 \triangleq \min &\left(k_s \left(\frac{1}{2}\alpha - \varepsilon_1\omega_1 - \varepsilon_2\omega_2 - \omega_3 \right), \right. \\ &\left. \frac{\omega_3}{3k_s\hat{\tau}^2} - \frac{\omega_1 k_s}{\varepsilon_1}, \frac{\omega_3}{3\hat{\tau}\left(\frac{1}{2}(\varepsilon_1\omega_1 + \varepsilon_2\omega_2)\right)}, \frac{1}{3\hat{\tau}} \right), \end{aligned} \tag{24}$$

$$\begin{aligned} \beta_2 \triangleq \min &\left(\frac{c_e}{2c_M} k_{21} - k_s(\varepsilon_1\omega_1 + \varepsilon_2\omega_2 + \omega_3), \right. \\ &\left. \frac{\omega_3}{3k_s\hat{\tau}^2} - \frac{\omega_1 k_s}{\varepsilon_1}, \frac{\omega_3}{3\hat{\tau}\left(\frac{1}{2}(\varepsilon_1\omega_1 + \varepsilon_2\omega_2)\right)}, \frac{1}{3\hat{\tau}} \right), \end{aligned} \tag{25}$$

$$\delta \triangleq \max\left(\frac{1}{\alpha k_s}, \frac{c_M}{2c_e k_{21}} \right), \tag{26}$$

and the gain conditions are defined as

$$\alpha > 2(\varepsilon_1\omega_1 + \varepsilon_2\omega_2 + \omega_3), \quad \omega_3 > \frac{3k_s^2\hat{\tau}^2\omega_1}{\varepsilon_1}, \tag{27}$$

$$\varepsilon_2\omega_2 \geq \max\left(\left| \alpha - \frac{c_b}{c_M} \right|, \left| \alpha - \frac{c_B}{c_m} \right| \right), \tag{28}$$

$$k_1 \geq \frac{c_M}{c_e}\left(\Phi + k_s \bar{\tilde{\tau}} \Upsilon \frac{c_B}{c_m} + k_s \hat{\tau} \Upsilon \max\left(\frac{c_b}{c_M}, \alpha \right) \right), \tag{29}$$

$$k_2 \geq k_{21} + k_{22}, \tag{30}$$

$$k_{21} > \frac{2c_M k_s}{c_e}(\varepsilon_1\omega_1 + \varepsilon_2\omega_2 + \omega_3), \tag{31}$$

[3] For notational brevity, all functional dependencies are hereafter suppressed unless required for clarity of exposition.

[4] A sliding mode term is included in the motor controller to achieve asymptotic cadence tracking if the motor control input was always available. However, since the FES controller does not have a sliding mode term, the overall cadence tracking result yields an ultimate bound. Therefore, the sliding mode term could be removed from the motor controller if desired, and an overall maximum ultimate bound could be determined.

$$k_{22} \geq \frac{k_s c_M}{c_e} \max\left(\frac{c_b}{c_M}, \alpha\right), \tag{32}$$

where $k_{21}, k_{22} \in \mathbb{R}_{>0}$ are selectable constants, and $\Upsilon \in \mathbb{R}_{>0}$ is a subsequently defined known bounding constant.

4. STABILITY ANALYSIS

Let switching times be denoted by $\{t_n^i\}$, $i \in \{m, e\}$, $n \in \{0, 1, 2, ...\}$, which represent the time instances when B_E becomes zero ($i = m$), or the time instances when B_E becomes nonzero ($i = e$). Let $V_L : \mathcal{D} \to \mathbb{R}_{\geq 0}$ denote a positive definite, continuously differentiable, common Lyapunov function candidate on a domain $\mathcal{D} \subseteq \mathbb{R}^4$, that is defined as

$$V_L \triangleq \ln(\cosh(r)) + \frac{1}{2}\omega_1 e_u^2 + Q_1 + Q_2. \tag{33}$$

By using (8), (33) can be bounded as

$$\underbrace{\lambda_1 \ln(\cosh(\|y\|))}_{\phi_1(\|y\|)} \leq V_L \leq \underbrace{\lambda_2 \|y\|^2}_{\phi_2(\|y\|)}, \tag{34}$$

where $y \in \mathbb{R}^4$ is defined as

$$y \triangleq \left[\, z^T \ \sqrt{Q_1} \ \sqrt{Q_2} \,\right]^T, \tag{35}$$

and $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ are known constants defined as

$$\lambda_1 \triangleq \min\left(1, \frac{\omega_1}{2}\right), \quad \lambda_2 \triangleq \max\left(1, \frac{\omega_1}{2}\right).$$

Theorem 1. For the cycle-rider dynamics in (1), along with Properties 1-9, the FES and motor control inputs defined in (18) and (19) ensure uniformly ultimately bounded tracking in the sense that

$$\|y(t)\| < \overline{d}, \quad \forall t > T\left(\overline{d}, \|y(t_0)\|\right), \tag{36}$$

where $\overline{d}, T \in \mathbb{R}_{>0}$ denote the ultimate bound, and the ultimate time to reach the ultimate bound, respectively, provided the gain conditions in (27)-(32) are satisfied, along with the following sufficient gain condition

$$\frac{\beta}{2\delta} \geq \rho^2(\overline{\mu})(\overline{\mu} + 1)^2, \tag{37}$$

where $\overline{\mu} \in \mathbb{R}_{>0}$ is defined as $\overline{\mu} \triangleq \max\left(\overline{d}, \|y(t_0)\|\right)$.

Proof. Since the motor controller, B_E, and B_M^τ are discontinuous, a generalized solution to the time derivative of (33) exists almost everywhere (a.e.) within $t \in [t_0, \infty)$, and $\dot{V}_L(y) \overset{\text{a.e.}}{\in} \dot{\tilde{V}}_L(y)$, where $\dot{\tilde{V}}_L$ is the generalized time derivative of V_L. Using the calculus of $K[\cdot]$ from Paden and Sastry (1987), applying the Leibniz integral rule to (13), (21), (22), and using (20) yields the following generalized time derivative of (33)

$$\begin{aligned}
\dot{\tilde{V}}_L \subseteq \tanh(r) \Bigg(& \chi + k_s \frac{K[B_M^\tau]}{M}\left(\tanh(r_{\hat{\tau}}) - \tanh(r_\tau)\right) \\
& - \frac{K[B_E]}{M}\left(k_1 K[\operatorname{sgn}(r)] + k_2 \tanh(r)\right) \\
& - K[\sigma_s]\,\alpha k_s \tanh(r) \\
& + \left(K[\sigma_{s,\hat{\tau}}]\,\alpha - \frac{K[B_M^\tau]}{M}\right) k_s \tanh(r_{\hat{\tau}}) \Bigg) \\
& + \omega_1 k_s e_u \left(-K[\sigma_s] \tanh(r) + K[\sigma_{s,\hat{\tau}}] \tanh(r_{\hat{\tau}})\right) \\
& + \frac{1}{2}(\varepsilon_1\omega_1 + \varepsilon_2\omega_2) k_s \left(\tanh^2(r) - \tanh^2(r_{\hat{\tau}})\right) \\
& + \frac{\omega_3 k_s}{\hat{\tau}} \left(\hat{\tau} \tanh^2(r) - \int_{t-\hat{\tau}}^{t} \tanh^2(r(\theta))\, d\theta\right),
\end{aligned} \tag{38}$$

where, $K[\operatorname{sgn}(\cdot)] = SGN(\cdot)$ such that $SGN(\cdot) = \{1\}$ if $(\cdot) > 0$, $[-1, 1]$ if $(\cdot) = 0$, and $\{-1\}$ if $(\cdot) < 0$. To obtain a result for all time, the expression in (38) must be evaluated for each possible switching condition combination. From the switching conditions defined in (3), (6), and (14) and the expression in (38) it can be seen that there exists 9 unique cases. Case 1, which represents the case when the motor is not in use, will be examined first. Then Cases 2-9 will be examined using an overall upper bound.

Case 1 represents the only case when the motor is not in use, which occurs only when FES-induced muscle contractions are present in the system (i.e., $t \in \left[t_n^m, t_{n+1}^e\right)$). Therefore, setting $K[\sigma_s] = 1$, $K[\sigma_{s,\hat{\tau}}] = 1$, and $K[B_E] = 0$, using Properties 1 and 7, choosing ε_2 and ω_2 such that $\max\left(\left|\alpha - \frac{c_b}{c_M}\right|, \left|\alpha - \frac{c_B}{c_m}\right|\right) \leq \varepsilon_2\omega_2$, and using the fact that $\dot{V}_L(y) \overset{\text{a.e.}}{\in} \dot{\tilde{V}}_L(y)$ allows (38) to be evaluated during Case 1 and upper bounded as follows

$$\begin{aligned}
\dot{V}_L \overset{\text{a.e.}}{\leq} & |\tanh(r)|\,|\chi| - \alpha k_s \tanh^2(r) \\
& + k_s \frac{c_B}{c_m} |\tanh(r)|\,|(\tanh(r_{\hat{\tau}}) - \tanh(r_\tau))| \\
& + k_s \varepsilon_2\omega_2 |\tanh(r)\tanh(r_{\hat{\tau}})| + \omega_3 k_s \tanh^2(r) \\
& + \omega_1 k_s |e_u \tanh(r)| + \omega_1 k_s |e_u \tanh(r_{\hat{\tau}})| \\
& + \frac{1}{2}(\varepsilon_1\omega_1 + \varepsilon_2\omega_2) k_s \left(\tanh^2(r) - \tanh^2(r_{\hat{\tau}})\right) \\
& - \frac{\omega_3 k_s}{\hat{\tau}} \int_{t-\hat{\tau}}^{t} \tanh^2(r(\theta))\, d\theta.
\end{aligned} \tag{39}$$

Provided that $\|y(\cdot)\| < \gamma$, $\forall \cdot \in [t_0, t)$, it could be shown using Properties 1, 7, and 8, (16), (20), and the fact that $\|y\| \geq \|z\|$ that

$$\dot{r}(\cdot) \leq c_1 + c_2\gamma + c_3\gamma^2 \leq \Upsilon, \tag{40}$$

$\forall \cdot \in [t_0, t)$, where $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ are known constants. Now, using (40), the Mean Value Theorem can be used to further bound (39) as

$$\begin{aligned}
\dot{V}_L \overset{\text{a.e.}}{\leq} & \left(|\chi| + k_s \overline{\overline{\tau}}\Upsilon \frac{c_B}{c_m}\right) |\tanh(r)| - \alpha k_s \tanh^2(r) \\
& + k_s \varepsilon_2\omega_2 |\tanh(r)\tanh(r_{\hat{\tau}})| + \omega_3 k_s \tanh^2(r) \\
& + \omega_1 k_s |e_u \tanh(r)| + \omega_1 k_s |e_u \tanh(r_{\hat{\tau}})| \\
& + \frac{1}{2}(\varepsilon_1\omega_1 + \varepsilon_2\omega_2) k_s \left(\tanh^2(r) - \tanh^2(r_{\hat{\tau}})\right) \\
& - \frac{\omega_3 k_s}{\hat{\tau}} \int_{t-\hat{\tau}}^{t} \tanh^2(r(\theta))\, d\theta.
\end{aligned} \tag{41}$$

Using Young's Inequality to upper bound select terms in (41) yields the following upper bound

$$\begin{aligned}
\dot{V}_L \overset{\text{a.e.}}{\leq} & \left(|\chi| + k_s \overline{\overline{\tau}}\Upsilon \frac{c_B}{c_m}\right) |\tanh(r)| - \alpha k_s \tanh^2(r) \\
& + \left(\frac{\omega_1 k_s}{\varepsilon_1}\right) e_u^2 + k_s(\varepsilon_1\omega_1 + \varepsilon_2\omega_2 + \omega_3)\tanh^2(r) \\
& - \frac{\omega_3 k_s}{\hat{\tau}} \int_{t-\hat{\tau}}^{t} \tanh^2(r(\theta))\, d\theta.
\end{aligned} \tag{42}$$

Substituting (16) into (42) and completing the squares on the $|\tanh(r)|$ terms yield

$$\dot{V}_L \overset{\text{a.e.}}{\le} -k_s\left(\frac{1}{2}\alpha - \varepsilon_1\omega_1 - \varepsilon_2\omega_2 - \omega_3\right)\tanh^2(r)$$
$$\left(\frac{\omega_1 k_s}{\varepsilon_1}\right)e_u^2 - \frac{\omega_3 k_s}{\hat{\tau}}\int_{t-\hat{\tau}}^{t}\tanh^2(r(\theta))\,d\theta$$
$$+\frac{1}{\alpha k_s}\left(\rho^2(\|z\|)\|z\|^2 + \left(\Phi + k_s\bar{\bar{\tau}}\Upsilon\frac{c_B}{c_m}\right)^2\right). \tag{43}$$

To facilitate the analysis, the Cauchy-Schwarz inequality can be used with (13) and (18) to yield

$$e_u^2 \le \hat{\tau}k_s^2\int_{t-\hat{\tau}}^{t}\tanh^2(r(\theta))\,d\theta. \tag{44}$$

Furthermore, Q_2 can be upper bounded as

$$Q_2 \le \omega_3 k_s\int_{t-\hat{\tau}}^{t}\tanh^2(r(\theta))\,d\theta. \tag{45}$$

The following inequality is obtained by using (21), (44), and (45):

$$-\frac{\omega_3 k_s}{\hat{\tau}}\int_{t-\hat{\tau}}^{t}\tanh^2(r(\theta))\,d\theta \le -\frac{\omega_3}{3\hat{\tau}\left(\frac{1}{2}(\varepsilon_1\omega_1+\varepsilon_2\omega_2)\right)}Q_1 - \frac{\omega_3}{3k_s\hat{\tau}^2}e_u^2 - \frac{1}{3\hat{\tau}}Q_2. \tag{46}$$

Substituting (46) into (43) and using the definition of β_1 in (24) yields

$$\dot{V}_L \overset{\text{a.e.}}{\le} -\beta_1\|x\|^2 + \frac{1}{\alpha k_s}\rho^2(\|z\|)\|z\|^2 + v, \tag{47}$$

$\forall t \in \left[t_n^m, t_{n+1}^e\right)$, provided that $\|y(\cdot)\| < \gamma,\ \forall \cdot \in [t_0, t)$, where $x \in \mathbb{R}^4$ is defined as

$$x \triangleq \left[\tanh(r)\ e_u\ \sqrt{Q_1}\ \sqrt{Q_2}\right]^T, \tag{48}$$

and the auxiliary constant, $v \in \mathbb{R}_{>0}$, is defined as

$$v \triangleq \frac{1}{\alpha k_s}\left(\Phi + k_s\bar{\bar{\tau}}\Upsilon\frac{c_B}{c_m}\right)^2. \tag{49}$$

Now the Cases 2-9 will be considered, which represent the cases when the motor is active (i.e., $t \in \left[t_n^e, t_{n+1}^m\right)$). An overall upper bound for Cases 2-9 is determined to facilitate the analysis. Note, by individually considering each case, utilizing Properties 1 and 7, and selecting ε_2 and ω_2 such that $\alpha - \frac{c_b}{c_M} \le \left|\alpha - \frac{c_b}{c_M}\right| \le \varepsilon_2\omega_2$ and $\frac{c_B}{c_m} - \alpha \le \left|\alpha - \frac{c_B}{c_m}\right| \le \varepsilon_2\omega_2$, it could be proven that

$$\left|\left(K[\sigma_{s,\hat{\tau}}]\alpha - \frac{K[B_M^{\tau}]}{M}\right)\right| \le \varepsilon_2\omega_2 + \max\left(\frac{c_b}{c_M}, \alpha\right). \tag{50}$$

The inequality in (38) can be upper bounded for Cases 2-9 by considering each case individually, using Properties 7 and 8, and utilizing the inequality in (50) to yield

$$\dot{V}_L \overset{\text{a.e.}}{\le} |\chi||\tanh(r)| - \frac{c_e}{c_M}k_1|\tanh(r)|$$
$$+k_s\frac{c_B}{c_m}|\tanh(r)||(\tanh(r_{\hat{\tau}}) - \tanh(r_\tau))|$$
$$-\frac{c_e}{c_M}k_2\tanh^2(r) + k_s\varepsilon_2\omega_2|\tanh(r)\tanh(r_{\hat{\tau}})|$$
$$+k_s\max\left(\frac{c_b}{c_M}, \alpha\right)|\tanh(r)\tanh(r_{\hat{\tau}})|$$
$$+\omega_1 k_s|e_u\tanh(r)| + \omega_1 k_s|e_u\tanh(r_{\hat{\tau}})|$$
$$+\frac{1}{2}(\varepsilon_1\omega_1+\varepsilon_2\omega_2)k_s\left(\tanh^2(r) - \tanh^2(r_{\hat{\tau}})\right)$$
$$+\omega_3 k_s\tanh^2(r) - \frac{\omega_3 k_s}{\hat{\tau}}\int_{t-\hat{\tau}}^{t}\tanh^2(r(\theta))\,d\theta. \tag{51}$$

Now, by following a similar development as for Case 1 the following upper bound for (51) can be obtained

$$\dot{V}_L \overset{\text{a.e.}}{\le} -\beta_2\|x\|^2 + \frac{c_M}{2c_e k_{21}}\rho^2(\|z\|)\|z\|^2, \tag{52}$$

$\forall t \in \left[t_n^e, t_{n+1}^m\right)$, provided that $\|y(\cdot)\| < \gamma,\ \forall \cdot \in [t_0, t)$, where β_2 is defined in (25).

An upper bound for every case can be obtained by upper bounding both (47) and (52) by

$$\dot{V}_L \overset{\text{a.e.}}{\le} -\beta\|x\|^2 + \delta\rho^2(\|z\|)\|z\|^2 + v, \tag{53}$$

$\forall t \in [t_0, \infty)$, where

$$\delta \triangleq \max\left(\frac{1}{\alpha k_s}, \frac{c_M}{2c_e k_{21}}\right). \tag{54}$$

From (53), (33) can be considered a common Lyapunov-like function across the entire crank cycle (i.e., for every case (Roy et al., 2018)). Additionally, notice that $\beta > 0$ provided the gain conditions in (27)-(31) are satisfied. Using (9), (35), and (48) it can be proven that $\|x\|^2 \ge \tanh^2(\|y\|)$. Furthermore, (53) can be upper bounded by using the fact that $\|y\| \ge \|z\|$ and $\|x\|^2 \ge \tanh^2(\|y\|)$ to yield

$$\dot{V}_L \overset{\text{a.e.}}{\le} \underbrace{-\frac{\beta}{2}\tanh^2(\|y\|)}_{\phi_3(\|y\|)} + v, \tag{55}$$

$\forall t \in [t_0, \infty)$, provided that $\|y(\cdot)\| < \gamma,\ \forall \cdot \in [t_0, t)$ and that the following inequality is satisfied for all time

$$-\frac{\beta}{2}\tanh^2(\|y\|) + \delta\rho^2(\|y\|)\|y\|^2 \le 0. \tag{56}$$

The expression in (56) can be rewritten as follows

$$\rho^2(\|y\|)\left(\frac{\|y\|}{\tanh(\|y\|)}\right)^2 \le \frac{\beta}{2\delta}. \tag{57}$$

A sufficient condition for (57) is obtained by using the properties in (10) as

$$\rho^2(\|y\|)(\|y\|+1)^2 \le \frac{\beta}{2\delta}. \tag{58}$$

Notice that the left-hand side of (58) is strictly increasing with respect to $\|y\|$. Therefore, the condition in (58) implies that $\|y\|$ must be bounded. Let γ denote the maximum value of $\|y\|$ such that (58) holds. Thus, (58) is satisfied for all time if $\|y(t)\| \le \gamma, \forall t \in [t_0, \infty)$. Therefore, a sufficient condition for $\|y(\cdot)\| < \gamma,\ \forall \cdot \in [t_0, t)$ is that the condition in (58) is satisfied for all time. Given (34) and (55), $\|y(\cdot)\|$ is uniformly ultimately bounded (Corless and Leitmann, 1981) in the sense that

$$\|y(t)\| \le \bar{d}, \quad \forall t \ge T\left(\bar{d}, \|y(t_0)\|\right), \tag{59}$$

provided the gain conditions in (27)-(31) and the sufficient condition in (58) are satisfied for all time. In (59), $\bar{d}$ denotes the ultimate bound of $\|y(t)\|$ and is determined according to Corless and Leitmann as

$$\bar{d} > \left(\phi_1^{-1} \circ \phi_2\right) \left(\phi_3^{-1}(v)\right), \tag{60}$$

where ϕ_1 and ϕ_2 are defined in (34), ϕ_3 is defined in (55), v is defined in (49), and T denotes the ultimate time to reach the ultimate bound and is defined as Corless and Leitmann

$$T \triangleq \begin{cases} 0 & \|y(t_0)\| \leq \kappa \\ \dfrac{\phi_2\left(\|y(t_0)\|\right) - \phi_1\left(\left(\phi_2^{-1} \circ \phi_1\right)(\bar{d})\right)}{\phi_3\left(\phi_2^{-1} \circ \phi_1\right)(\bar{d}) - v} & \|y(t_0)\| > \kappa. \end{cases} \tag{61}$$

where $\kappa \triangleq \left(\phi_2^{-1} \circ \phi_1\right)(\bar{d})$. From (59), a sufficient condition for (58) to be satisfied for all time is provided in (37), which is expressed in terms of the initial condition and the ultimate bound of the composite error signal $\|y\|$.

5. CONCLUSION

A saturated control system is developed such that the upper bound of both the motor and FES controller is known a priori and can be adjusted by modifying the feedback control gains. A Lyapunov-like analysis was performed to ensure uniformly ultimately bounded cadence tracking for an uncertain nonlinear dynamic switched system, unknown bounded additive disturbances, and an unknown time-varying input delay. Additionally, switching conditions were developed to activate/deactivate stimulation to yield effective muscle contractions. In future efforts, the control system will be modified to ensure both position and cadence tracking. Additionally, experiments will be performed on participants with NCs to validate the current and modified control systems followed by an in-depth analysis of the data.

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