



Regularity for $C^{1,\alpha}$ Interface Transmission Problems

LUIS A. CAFFARELLI, MARÍA SORIA-CARRO & PABLO RAÚL STINGA

Communicated by A. FIGALLI

Abstract

We study the existence, uniqueness, and optimal regularity of solutions to transmission problems for harmonic functions with $C^{1,\alpha}$ interfaces. For this, we develop a novel geometric stability argument based on the mean value property.

1. Introduction

Transmission problems in classical elasticity theory were first introduced by PICONE in 1954; see [14]. In the following years, contributions were made by LIONS [10], STAMPACCHIA [16] and CAMPANATO [3]. In 1960, SCHECHTER generalized the theory to include smooth elliptic operators in nondivergence form in domains with smooth interfaces [15]. Since then, transmission problems have been of great interest due to their applications in different areas in science. For instance, LADYZHENSKAYA and URAL'TSEVA considered in [7] the so-called diffraction problem. For other recent developments, see [5, 8, 9, 12].

As a particular feature, and in contrast with free boundary problems, transmission problems deal with a fixed interface where solutions change abruptly and the primary focus is to study their behavior across this surface. Additionally, these problems cannot be treated separately as boundary value problems per se, as solutions interact with each other from each side of the interface through the transmission condition.

We study the existence, uniqueness and regularity of solutions to a transmission problem for harmonic functions. One of our main novelties is that the transmission interface has only $C^{1,\alpha}$ regularity. Furthermore, we build up a new fine geometric argument based on the mean value property to show that solutions are $C^{1,\alpha}$ up to each side of the interface.

The setting is the following: let Ω be a smooth, bounded domain of $\mathbb{R}^n$, $n \geq 2$. Let Ω_1 be a subdomain of Ω such that $\Omega_1 \subset\subset \Omega$ and set $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. Suppose

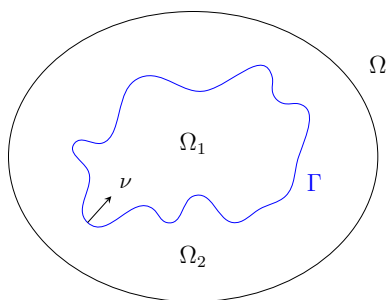


Fig. 1. Geometry for the transmission problem (TP)

that the interface Γ between Ω_1 and Ω_2 , namely, $\Gamma = \partial\Omega_1$, is a $C^{1,\alpha}$ manifold, for some $0 < \alpha < 1$. Then $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$. For a function $u : \overline{\Omega} \rightarrow \mathbb{R}$, we denote

$$u_1 = u|_{\overline{\Omega}_1} \quad \text{and} \quad u_2 = u|_{\overline{\Omega}_2}.$$

We consider the problem of finding a continuous function $u : \overline{\Omega} \rightarrow \mathbb{R}$ such that

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ (u_1)_\nu - (u_2)_\nu = g & \text{on } \Gamma. \end{cases} \quad (\text{TP})$$

Here $g \in C^{0,\alpha}(\Gamma)$ and ν is the unit normal vector on Γ that is interior to Ω_1 ; see Fig. 1. This is a transmission problem in the spirit of Schechter in [15], where Γ is the transmission interface. In contrast to our problem, [15] only deals with $\Gamma \in C^\infty$. The last two equations on (TP) are called the *transmission conditions*.

If in (TP) we set $g \equiv 0$ then u is a harmonic function in Ω . Therefore, in order to have a meaningful elliptic transmission condition, we assume that

$$g(x) \geq 0 \quad \text{for all } x \in \Gamma.$$

Hence, u will not be differentiable at those points on Γ where $g > 0$. In turn, we prove that u is $C^{1,\alpha}$ from each side up to Γ . In (TP) we have also imposed homogeneous Dirichlet boundary condition on $\partial\Omega$. This is not a restriction, since we can always add to u a harmonic function v in Ω such that $v = \phi$ on $\partial\Omega$, to make $u_2 = \phi$ on $\partial\Omega$. The one dimensional case is excluded because one can easily find explicit solutions.

Our main result is the following:

Theorem 1.1. *There exists a unique classical solution u to the transmission problem (TP). Moreover, $u_1 \in C^{1,\alpha}(\overline{\Omega}_1)$, $u_2 \in C^{1,\alpha}(\overline{\Omega}_2)$, and there exists $C = C(n, \alpha, \Gamma) > 0$ such that*

$$\|u_1\|_{C^{1,\alpha}(\overline{\Omega}_1)} + \|u_2\|_{C^{1,\alpha}(\overline{\Omega}_2)} \leq C \|g\|_{C^{0,\alpha}(\Gamma)}.$$

The appropriate notion of solution to (TP) comes from computing Δu in the sense of distributions. Indeed, if u and Γ were sufficiently smooth and $\varphi \in C_c^\infty(\Omega)$, then

$$(\Delta u)(\varphi) = \int_{\Omega} u \Delta \varphi \, dx = \int_{\Gamma} ((u_1)_\nu - (u_2)_\nu) \varphi \, dH^{n-1} = \int_{\Gamma} g \varphi \, dH^{n-1}.$$

Thus Δu is a singular measure concentrated on Γ with density g . In Section 2 we show that there exists a unique distributional solution $u \in C_0(\overline{\Omega})$ to (TP), where $C_0(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$. Moreover, we prove that u is Log-Lipschitz in $\overline{\Omega}$, see Theorem 2.2. The main issue is the optimal regularity of u up to Γ . Theorem 1.1 will be a consequence of our next result.

Theorem 1.2. (Pointwise $C^{1,\alpha}$ boundary regularity) *Let $\Gamma = \{(y', \psi(y')) : y' \in B'_1\}$, where ψ is a $C^{1,\alpha}$ function, for some $0 < \alpha < 1$. Assume that $0 \in \Gamma$. Let u be a distributional solution to the transmission problem*

$$\Delta u = g \, dH^{n-1} \Big|_{\Gamma},$$

where $g \in L^\infty(\Gamma)$, $g \geq 0$, and $g \in C^{0,\alpha}(0)$. Then there are linear polynomials $P(x) = A \cdot x + B$, and $Q(x) = C \cdot x + B$ such that

$$\begin{aligned} |u_1(x) - P(x)| &\leq D|x|^{1+\alpha} && \text{for all } x \in \Omega_1 \cap B_{1/2} \\ |u_2(x) - Q(x)| &\leq D|x|^{1+\alpha} && \text{for all } x \in \Omega_2 \cap B_{1/2}, \end{aligned}$$

with

$$|A| + |B| + |C| + D \leq C_0 \|\psi\|_{C^{1,\alpha}(B'_1)} ([g]_{C^\alpha(0)} + \|g\|_{L^\infty(\Gamma)})$$

and $C_0 = C_0(n, \alpha) > 0$.

The key tool to prove Theorem 1.2 is a *stability* result, obtained via a novel geometric approach we develop, which is based on the mean value property and the maximum principle, see Theorem 4.2. In fact, our idea is to explicitly construct classical solutions to problems with flat interfaces that are close to u . With this, we can transfer the regularity from classical solutions to u . Indeed, as shown in Section 3, solutions to flat problems have the expected optimal regularity up to the interface. More precisely, we show that if the flatness and oscillation of the interface Γ are controlled, then we can construct a solution for a flat interface problem, where the flat interface does not intersect Γ . We also quantify how close solutions must be, depending only on the geometric properties of Γ and the basic regularity of u . These ingredients are crucial for the first step in the proof of Theorem 1.2, see Lemma 5.1. To close the argument, one needs to use these approximations at each scale. Through these techniques, and similar to the case of elliptic equations [2], we are able to find that flat solutions are asymptotically close to non-flat solutions.

Our geometric techniques developed in Section 4 are constructive and quantitative, and provide a precise understanding of the underlying geometry of the

transmission problem. Furthermore, this paper is essentially self-contained. We believe that the tools presented here could be used in free boundary problems, an idea we will explore in the future. Finally, notice that our results are also useful in terms of numerical analysis, as our constructions give explicit rates of approximation.

The paper is organized as follows: in Section 2 we prove the existence, uniqueness and global Log-Lipschitz regularity of the solution u to (TP). Section 3 deals with the case when the transmission interface is flat. Our geometric stability result based on the mean value property is proved in Section 4. The proof of Theorems 1.2 and 1.1 are given in Sects. 5 and 6, respectively. The appendix contains some basic geometric considerations about integration on Lipschitz domains.

Notation. For a point $x \in \mathbb{R}^n$ we write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. The gradient in the variables x' is denoted by ∇' , dH^{n-1} is the $(n-1)$ -dimensional Hausdorff measure in $\mathbb{R}^n$ and $B'_r(x')$ denotes the ball in $\mathbb{R}^{n-1}$ of radius $r > 0$ centered at x' . When the ball is centered at the origin $x' = 0'$ or $x = 0 = (0', 0)$, we will just write B'_r or B_r .

2. Existence, Uniqueness and Global Log-Lipschitz Regularity

As we mentioned in the Introduction, the notion of solution to (TP) comes from computing Δu in the sense of distributions.

Definition 2.1. (Distributional solution) We say that $u \in C_0(\overline{\Omega})$ is a distributional solution to (TP) if for any $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_{\Omega} u \Delta \varphi \, dx = \int_{\Gamma} g \varphi \, dH^{n-1}.$$

In this case, we write

$$\Delta u = g \, dH^{n-1}|_{\Gamma}.$$

Even though the definition of distributional solution makes sense for $u \in L^1_{\text{loc}}(\Omega)$, we ask u to be continuous up to the boundary so that the boundary condition $u = 0$ is well-defined.

Recall that a bounded function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is in the space $\text{LogLip}(\overline{\Omega})$ if

$$[u]_{\text{LogLip}(\overline{\Omega})} = \sup_{\substack{x, y \in \overline{\Omega} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y| \log |x - y|} < \infty.$$

Theorem 2.2. (Existence, uniqueness, and Log-Lipschitz global regularity) *Let Γ be a Lipschitz interface, and $g \in L^\infty(\Gamma)$. Then the unique distributional solution $u \in C_0(\overline{\Omega})$ to (TP) is given by*

$$u(x) = \int_{\Gamma} G(x, y) g(y) \, dH^{n-1} \quad \text{for } x \in \Omega, \quad (2.1)$$

where $G(x, y)$ is the Green's function for the Laplacian in Ω . Furthermore, $u \in \text{LogLip}(\overline{\Omega})$ and there exists $C = C(n, \Gamma, \Omega) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + [u]_{\text{LogLip}(\overline{\Omega})} \leq C \|g\|_{L^\infty(\Gamma)}.$$

Proof. Let u be as in (2.1). By using a partition of unity on Γ , it is enough to assume that $\Gamma = \psi(\mathbb{R}^{n-1})$ where $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a Lipschitz function and that $g(y', \psi(y'))$ has compact support in B'_1 (see Appendix 7). Then, for any $x \in \Omega$,

$$\begin{aligned} |u(x)| &\leq \int_{\Gamma} |G(x, y)| g(y) dH_y^{n-1} \\ &= \int_{B'_1} |G(x, (y', \psi(y')))| g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} dy' \\ &\leq C(n, \Gamma) \|g\|_{L^\infty(\Gamma)} \int_{B'_1} \frac{1}{|(x' - y', x_n - \psi(y'))|^{n-2}} dy' \\ &\leq C(n, \Gamma) \|g\|_{L^\infty(\Gamma)} \int_{B'_1} \frac{1}{|x' - y'|^{n-2}} dy' \\ &\leq C(n, \Gamma) \|g\|_{L^\infty(\Gamma)}. \end{aligned}$$

Thus, the integral defining u in (2.1) is absolutely convergent and u is bounded.

Next, for any $\varphi \in C_c^\infty(\Omega)$, by Fubini's Theorem and the symmetry $G(x, y) = G(y, x)$,

$$\begin{aligned} \int_{\Omega} u(x) \Delta \varphi(x) dx &= \int_{\Omega} \left[\int_{\Gamma} G(x, y) g(y) dH^{n-1} \right] \Delta \varphi(x) dx \\ &= \int_{\Gamma} g(y) \int_{\Omega} G(y, x) \Delta_x \varphi(x) dx dH^{n-1} = \int_{\Gamma} g(y) \varphi(y) dH^{n-1}. \end{aligned}$$

Moreover, since $G(\bar{x}, y) = 0$ for $\bar{x} \in \partial\Omega$ and $y \in \Omega$, by dominated convergence we see that $u(x)$ converges to 0 as $x \in \Omega$ converges to $\bar{x}$.

Now we show that $u \in \text{LogLip}(\bar{\Omega})$. Since u is harmonic in $\Omega \setminus \Gamma$, we only need to prove the regularity of u near Γ . Suppose that $x_1, x_2 \in K$, where $K \subset \Omega$ is a compact set containing Γ . Let $0 < d < 1$. If $|x_1 - x_2| \geq d$, then

$$|u(x_1) - u(x_2)| \leq \frac{2\|u\|_{L^\infty(\Omega)}}{d} d \leq C|x_1 - x_2|.$$

Assume next that $|x_1 - x_2| = \delta < d$. If $n \geq 3$, then, since $B_{2\delta}(x_1) \subset B_{4\delta}(x_2)$, by classical estimates for the Green's function,

$$\begin{aligned} |u(x_1) - u(x_2)| &\leq \int_{\Gamma} |G(x_1, y) - G(x_2, y)| |g(y)| dH^{n-1} \\ &\leq C_{n,K} \|g\|_{L^\infty(\Gamma)} \left[\int_{B_{2\delta}(x_1) \cap \Gamma} \frac{1}{|x_1 - y|^{n-2}} dH^{n-1} + \int_{B_{4\delta}(x_2) \cap \Gamma} \frac{1}{|x_2 - y|^{n-2}} dH^{n-1} \right. \\ &\quad \left. + \int_{\Gamma \setminus (B_{2\delta}(x_1) \cup B_{4\delta}(x_2))} \frac{|x_1 - x_2|}{|x_1 - y|^{n-1}} dH^{n-1} \right] \\ &\leq C_{n,K,\Gamma} \|g\|_{L^\infty(\Gamma)} \left[\int_{B'_{2\delta}(x'_1)} \frac{1}{|x'_1 - y'|^{n-2}} dy' + \int_{B'_{4\delta}(x'_2)} \frac{1}{|x'_2 - y'|^{n-2}} dy' \right. \\ &\quad \left. + |x_1 - x_2| \int_{B'_1 \setminus B'_{2\delta}(x'_1)} \frac{1}{|x'_1 - y'|^{n-1}} dy' \right] \\ &\leq C_{n,K,\Gamma} \|g\|_{L^\infty(\Gamma)} (|x_1 - x_2| + |x_1 - x_2| |\log |x_1 - x_2||). \end{aligned}$$

The estimate in dimension $n = 2$ follows the same lines.

For uniqueness, if $u, v \in C_0(\overline{\Omega})$ are distributional solutions, then

$$\int_{\Omega} (u - v) \Delta \varphi \, dx = 0 \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

Hence, $u - v \in C_0(\overline{\Omega})$ is harmonic in Ω and, as a consequence, $u \equiv v$.

Remark 2.3. Note that if $u \in \text{LogLip}(\overline{\Omega})$ then $u \in C^{0,\gamma}(\overline{\Omega})$ for every $0 < \gamma < 1$ and there exists $C = C(\Omega, \gamma) > 0$ such that

$$[u]_{C^{0,\gamma}(\overline{\Omega})} \leq C[u]_{\text{LogLip}(\overline{\Omega})}.$$

3. Flat Problems

For the next results, we fix the following notation. For $a \in \mathbb{R}$ we denote

$$\begin{aligned} B_{r,a} &= B_r(0', a) \\ B_{r,a}^+ &= B_r(0', a) \cap \{x_n > a\} \\ B_{r,a}^- &= B_r(0', a) \cap \{x_n < a\} \\ T_{r,a} &= \{x \in B_r(0', a) : x_n = a\} \\ T_a &= B_1 \cap \{x_n = a\} \\ T_a^+ &= \{x_n \geq a\} \\ T_a^- &= \{x_n \leq a\}. \end{aligned}$$

When $a = 0$, we use the simplified notation $T = T_0$ and $B_r^\pm = B_{r,0}^\pm$.

Theorem 3.1. (Flat problem) *Let $r > 0$ and $a \in \mathbb{R}$. Given $0 < \alpha, \gamma < 1$, let $g \in C^{0,\alpha}(T_{r,a})$ and $f \in C^{0,\gamma}(\overline{B_{r,a}})$. Then there exists a unique solution $v \in C^\infty(B_{r,a} \setminus T_{r,a}) \cap C^{0,\gamma}(\overline{B_{r,a}})$ to the flat transmission problem*

$$\begin{cases} \Delta v = g \, dH^{n-1} \Big|_{T_{r,a}} & \text{in } B_{r,a} \\ v = f & \text{on } \partial B_{r,a}. \end{cases}$$

Moreover, if we let $v^\pm = v \chi_{\overline{B_{r,a}^\pm}}$, then $v^\pm \in C^{1,\alpha}(\overline{B_{r/2,a}^\pm})$ and

$$\|v^\pm\|_{C^{1,\alpha}(\overline{B_{r/2,a}^\pm})} \leq C(\|g\|_{C^{0,\alpha}(T_{r,a})} + \|f\|_{L^\infty(\partial B_{r,a})}),$$

where $C = C(n, \alpha, r) > 0$. If $g \in C^{k-1,\alpha}(T_{r,a})$, $k \geq 1$, then $v \in C^{k,\alpha}(\overline{B_{r/2,a}^\pm})$ and

$$\|v^\pm\|_{C^{k,\alpha}(\overline{B_{r/2,a}^\pm})} \leq C(\|g\|_{C^{k-1,\alpha}(T_{r,a})} + \|f\|_{L^\infty(\partial B_{r,a})}),$$

where $C = C(n, \alpha, r, k) > 0$.

Proof. By subtracting from v the harmonic function h in $B_{r,a}$ that coincides with f on $\partial B_{r,a}$, it is enough to assume that $f = 0$ on $\partial B_{r,a}$. We consider only the case $k = 1$, that is, $g \in C^{0,\alpha}(T_{r,a})$. When $k \geq 1$ the proof is completely analogous. Moreover, it is sufficient to prove the result for $a = 0$ and $r = 1$. Indeed suppose that g is as in the statement, and let $\tilde{g}$ be defined on T , so that

$$g(x', x_n) = r^{-1} \tilde{g}(r^{-1}x', r^{-1}(x_n - a)),$$

whenever $x \in T_{r,a}$. If $\tilde{v}$ is the corresponding solution in B_1 , then

$$v(x', x_n) = \tilde{v}(r^{-1}x', r^{-1}(x_n - a)) \quad \text{for } x \in \overline{B_{r,a}}$$

is the unique solution to $\Delta v = g \, dH^{n-1}|_{T_{r,a}}$ such that $v = 0$ on $\partial B_{r,a}$. Moreover, we have the following control of the norms:

$$\begin{aligned} \|v^\pm\|_{C^{1,\alpha}(\overline{B_{r/2,a}^\pm})} &= \|\tilde{v}^\pm\|_{L^\infty(\overline{B_{1/2}^\pm})} + r^{-1} \|\nabla \tilde{v}^\pm\|_{L^\infty(\overline{B_{1/2}^\pm})} + r^{-(1+\alpha)} [\nabla \tilde{v}^\pm]_{C^{0,\alpha}(\overline{B_{1/2}^\pm})} \\ &\leq \max\{1, r^{-1}, r^{-(1+\alpha)}\} \|\tilde{v}^\pm\|_{C^{1,\alpha}(\overline{B_{1/2}^\pm})} \\ &\leq C_n \max\{1, r^{-1}, r^{-(1+\alpha)}\} \|\tilde{g}\|_{C^{0,\alpha}(T)} \\ &\leq C_n \max\{1, r^{-1}, r^{-(1+\alpha)}\} (r \|g\|_{L^\infty(T_{r,a})} + r^{1+\alpha} [g]_{C^{0,\alpha}(T_{r,a})}) \\ &\leq C \|g\|_{C^{0,\alpha}(T_{r,a})}, \end{aligned}$$

where $C > 0$ is as in the statement.

Let v^+ be the solution to the mixed boundary value problem

$$\begin{cases} \Delta v^+ = 0 & \text{in } B_1^+ \\ v^+ = 0 & \text{on } \partial B_1^+ \setminus T \\ v_{x_n}^+ = g/2 & \text{on } T. \end{cases}$$

By classical elliptic regularity, $v^+ \in C^\infty(B_1^+) \cap C^{1,\alpha}(\overline{B_{1/2}^+})$ and

$$\|v^+\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C_0 \|g\|_{C^{0,\alpha}(T)}$$

for some $C_0 = C_0(n) > 0$. The reflection of v^+ onto B_1^- given by $v^-(x', x_n) = v^+(x', -x_n)$, whenever $x_n \leq 0$, solves

$$\begin{cases} \Delta v^- = 0 & \text{in } B_1^- \\ v^- = 0 & \text{on } \partial B_1^- \setminus T \\ v_{x_n}^- = -g/2 & \text{on } T. \end{cases}$$

It follows that $v = v^+ \chi_{\overline{B_1^+}} + v^- \chi_{\overline{B_1^-}}$ is the unique distributional solution to $\Delta v = g \, dH^{n-1}|_T$ such that $v = 0$ on ∂B_1 . By the same argument as in the proof of Theorem 2.2, it is clear that $v \in C^\infty(B_1 \setminus T) \cap \text{LogLip}(\overline{B_1})$. Moreover, $v^\pm \in C^{1,\alpha}(\overline{B_{1/2}^\pm})$ with

$$\|v^\pm\|_{C^{1,\alpha}(\overline{B_{1/2}^\pm})} \leq C_n \|g\|_{C^{0,\alpha}(T)}$$

for some $C = C(n) > 0$, as desired. $\square$

Corollary 3.2. *Given $|a| < 1/4$, $c_0 > 0$, and $f \in C^{0,\gamma}(\overline{B_1})$, with $0 < \gamma < 1$, there exists a unique solution $v \in C^\infty(B_1 \setminus T_a) \cap C^{0,\gamma}(\overline{B_1})$ to*

$$\begin{cases} \Delta v = c_0 \, dH^{n-1}|_{T_a} & \text{in } B_1 \\ v = f & \text{on } \partial B_1 \end{cases}$$

such that, for any $k \geq 1$,

$$\|v^\pm\|_{C^{k,\alpha}(\overline{B_{1/2}} \cap T_a^\pm)} \leq C(c_0 + \|f\|_{L^\infty(\partial B_1)}),$$

where $C = C(n, \alpha, k) > 0$.

Proof. Fix $k \geq 1$. By Theorem 3.1 with $r = 4$, there is a unique solution $w \in C^\infty(B_{4,a} \setminus T_{4,a}) \cap C^{0,\gamma}(\overline{B_{4,a}})$ to $\Delta w = c_0 \, dH^{n-1}|_{T_{4,a}}$ such that $w = 0$ on $\partial B_{4,a}$. Moreover, $\|w^\pm\|_{C^{k,\alpha}(\overline{B_{2,a}^\pm})} \leq Cc_0$, for some $C = C(n, \alpha, k) > 0$. Let h be the harmonic function in B_1 such that $h = w - f$ on ∂B_1 . Then $h \in C^\infty(B_1) \cap C^{0,\gamma}(\overline{B_1})$, and

$$\|h\|_{C^{k,\alpha}(\overline{B_{1/2}})} \leq C(\|w\|_{L^\infty(\partial B_1)} + \|f\|_{L^\infty(\partial B_1)}) \leq C(c_0 + \|f\|_{L^\infty(\partial B_1)}),$$

where $C = C(n, \alpha, k) > 0$. Define $v = w - h$ on $\overline{B_1}$. Then v is the unique solution to $\Delta v = g \, dH^{n-1}|_{T_a}$ with $v = f$ on ∂B_1 . Moreover, since $\overline{B_{1/2}} \cap T_a^\pm \subset \overline{B_{2,a}^\pm}$,

$$\|v^\pm\|_{C^{k,\alpha}(\overline{B_{1/2}} \cap T_a^\pm)} \leq \|w^\pm\|_{C^{k,\alpha}(\overline{B_{2,a}^\pm})} + \|h\|_{C^{k,\alpha}(\overline{B_{1/2}})} \leq C(c_0 + \|f\|_{L^\infty(\partial B_1)}).$$

□

4. The Stability Result

In this section we prove our stability result, Theorem 4.2. As we mentioned at the beginning, our argument is based on the mean value property and, therefore, it is self-contained.

Fix $\varepsilon > 0$, and let $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$ and $\Gamma_\varepsilon = \{x \in \Omega : \text{dist}(x, \Gamma) < \varepsilon\}$. Consider the average

$$u_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} u(y) \, dy \quad \text{for } x \in \Omega_\varepsilon.$$

Proposition 4.1. *(Properties of averages) Let u be the distributional solution given in Theorem 2.2. The following properties hold:*

- (i) *If $B_\varepsilon(x) \cap \Gamma = \emptyset$ then $u_\varepsilon(x) = u(x)$.*
- (ii) *$u_\varepsilon \rightarrow u$ uniformly in compact subsets of Ω , as $\varepsilon \rightarrow 0$.*

(iii) If $g \in L^\infty(\Gamma)$ then $g_\varepsilon \in C_c(\Gamma_\varepsilon)$, where

$$g_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{\Gamma \cap B_\varepsilon(x)} g(y) \, dH^{n-1} \quad \text{for } x \in \Gamma_\varepsilon.$$

Moreover, $\Delta u_\varepsilon(x) = g_\varepsilon(x)$ for any $x \in \Omega_\varepsilon$.

Proof. Since u is harmonic outside of Γ , (i) is immediate by the mean value property.

For (ii), recall by Remark 2.3 that $u \in C^{0,\gamma}(\overline{\Omega})$. Therefore,

$$|u_\varepsilon(x) - u(x)| \leq \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} |u(y) - u(x)| \, dy \leq C \|g\|_{L^\infty(\Gamma)} \varepsilon^\gamma \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

We now show (iii). If $g \in L^\infty(\Gamma)$, by dominated convergence, $g_\varepsilon \in C_c(\Gamma_\varepsilon)$. Moreover, for any $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} (\Delta u_\varepsilon)(\varphi) &= \int_{\Omega} u_\varepsilon(x) \Delta \varphi(x) \, dx \\ &= \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \int_{\Omega} u(x+y) \Delta \varphi(x) \, dx \, dy \\ &= \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \int_{\Omega} u(z) \Delta \varphi(z-y) \, dz \, dy \\ &= \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \int_{\Gamma} g(z) \varphi(z-y) \, dH_z^{n-1} \, dy \\ &= \frac{1}{|B_\varepsilon|} \int_{\Gamma} \left[\int_{B_\varepsilon} \varphi(z-y) \, dy \right] g(z) \, dH_z^{n-1} \\ &= \frac{1}{|B_\varepsilon|} \int_{\Gamma} \left[\int_{\Omega} \chi_{B_\varepsilon}(z-y) \varphi(y) \, dy \right] g(z) \, dH_z^{n-1} \\ &= \frac{1}{|B_\varepsilon|} \int_{\Omega} \int_{\Gamma} \chi_{B_\varepsilon}(z-y) g(z) \, dH_z^{n-1} \varphi(y) \, dy \\ &= \int_{\Omega} \left[\frac{1}{|B_\varepsilon|} \int_{\Gamma \cap B_\varepsilon(y)} g(z) \, dH_z^{n-1} \right] \varphi(y) \, dy = \int_{\Omega} g_\varepsilon(y) \varphi(y) \, dy. \end{aligned}$$

□

Theorem 4.2. (Stability) Let $0 < \varepsilon, \theta < 1/2$ and $0 < \delta, \gamma < 1$ be given, and let $\Gamma = \{(y', \psi(y')) : y' \in B'_1\}$, where ψ is a Lipschitz function. Assume that Γ is $\theta\varepsilon$ -flat in B_1 in the sense that

$$\Gamma \subset \{x \in B_1 : |x_n| < \theta\varepsilon\}$$

and that Γ is also ε -horizontal in B_1 ; that is,

$$1 - \varepsilon \leq v(x) \cdot (0', 1) = (1 + |\nabla' \psi(x')|^2)^{-1/2} \leq 1$$

for every $x \in \Gamma$, where $\nu(x)$ denotes the upward pointing normal on Γ . Then there exists $C = C(n) > 0$ such that for any $u \in C^{0,\gamma}(\overline{B_1})$ and $g \in L^\infty(\Gamma)$ satisfying

$$\begin{cases} \Delta u = g \, dH^{n-1}|_\Gamma & \text{in } B_1 \\ |g - 1| \leq \delta & \text{on } \Gamma, \end{cases}$$

the classical solution $v \in C^\infty(B_1 \setminus T_{-\theta\varepsilon}) \cap C^{0,\gamma}(\overline{B_1})$ to the flat problem

$$\begin{cases} \Delta v = dH^{n-1}|_{T_{-\theta\varepsilon}} & \text{in } B_1 \\ v = u & \text{on } \partial B_1 \end{cases}$$

satisfies

$$|u - v| \leq C(\theta + \delta + \varepsilon^\gamma) \quad \text{in } B_{1/2}. \quad (4.1)$$

Remark 4.3. The interface for the flat problem in Theorem 4.2 is $T_{-\theta\varepsilon} = B_1 \cap \{x_n = -\theta\varepsilon\}$, which lies below Γ in the x_n -direction. To approximate u with the solution to a flat problem where the interface lies above Γ in the x_n -direction, it is enough to consider the classical solution v to

$$\begin{cases} \Delta v = dH^{n-1}|_{T_{\theta\varepsilon}} & \text{in } B_1 \\ v = u & \text{on } \partial B_1. \end{cases}$$

In this case, the same conclusion as in Theorem 4.2 holds.

Before proceeding with the proof, we need the following geometric result (see Fig. 2):

Lemma 4.4. Let Γ be as in Theorem 4.2. Define $M = 1 + 2\theta$ and let $x \in B_{1-M\varepsilon}$ be such that $\text{dist}(x, \Gamma) < \varepsilon$. Then

$$\{y' : (y', \psi(y')) \in B_\varepsilon(x)\} \subset B'_{((M\varepsilon)^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x') = \{y' : (y', -\theta\varepsilon) \in B_{M\varepsilon}(x)\} \quad (4.2)$$

and

$$\{y' : (y', \psi(y')) \in B_{M\varepsilon}(x)\} \supset B'_{(\varepsilon^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x') = \{y' : (y', -\theta\varepsilon) \in B_\varepsilon(x)\}. \quad (4.3)$$

Proof. If x is as in the statement then, by the flatness condition on Γ , we have $|x_n| < (1 + \theta)\varepsilon$.

Let us prove (4.2). Suppose first that $-\theta\varepsilon < x_n < \theta\varepsilon$. Then

$$\{y' : (y', \psi(y')) \in B_\varepsilon(x)\} \subset \{y' : (y', x_n) \in B_\varepsilon(x)\} = B'_\varepsilon(x').$$

Since

$$\begin{aligned} (M\varepsilon)^2 - (x_n + \theta\varepsilon)^2 &= (1 + 2\theta)^2\varepsilon^2 - (x_n + \theta\varepsilon)^2 \\ &\geq (1 + 4\theta + 4\theta^2)\varepsilon^2 - (2\theta\varepsilon)^2 = \varepsilon^2 + 4\theta\varepsilon^2 > \varepsilon^2, \end{aligned}$$

we see that $B'_\varepsilon(x') \subset B'_{((M\varepsilon)^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x')$ and the conclusion follows. Assume now that $\theta\varepsilon \leq x_n < (1 + \theta)\varepsilon$. Notice that

$$\{y' : (y', \psi(y')) \in B_\varepsilon(x)\} \subset \{y' : (y', \theta\varepsilon) \in B_\varepsilon(x)\} = B'_{(\varepsilon^2 - (x_n - \theta\varepsilon)^2)^{1/2}}(x').$$

Since

$$\begin{aligned} & (M\varepsilon)^2 - (x_n + \theta\varepsilon)^2 - (\varepsilon^2 - (x_n - \theta\varepsilon)^2) \\ &= (1 + 2\theta)^2\varepsilon^2 - (x_n^2 + 2\theta\varepsilon x_n + (\theta\varepsilon)^2) - \varepsilon^2 + (x_n^2 - 2\theta\varepsilon x_n + (\theta\varepsilon)^2) \\ &= 4\theta\varepsilon^2 + 4\theta^2\varepsilon^2 - 4\theta\varepsilon x_n \geq 4\theta\varepsilon^2 \geq 0, \end{aligned}$$

we find that $B'_{(\varepsilon^2 - (x_n - \theta\varepsilon)^2)^{1/2}}(x') \subset B'_{((M\varepsilon)^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x')$, as desired. The last case is when $-(1 + \theta)\varepsilon < x_n \leq -\theta\varepsilon$. Here it is clear that, since $M > 1$,

$$\begin{aligned} \{y' : (y', \psi(y')) \in B_\varepsilon(x)\} &\subset \{y' : (y', -\theta\varepsilon) \in B_\varepsilon(x)\} \\ &= B'_{(\varepsilon^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x') \\ &\subset B'_{((M\varepsilon)^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x'). \end{aligned}$$

This concludes the proof of (4.2).

For (4.3), notice that if $x_n \geq (1 - \theta)\varepsilon$ then the inclusion follows as $\{y' : (y', -\theta\varepsilon) \in B_\varepsilon(x)\} = \emptyset$. We therefore assume that $-(1 + \theta)\varepsilon < x_n < (1 - \theta)\varepsilon$. If $x_n \geq -\theta\varepsilon$, then

$$\begin{aligned} \{y' : (y', \psi(y')) \in B_{M\varepsilon}(x)\} &\supset \{y' : (y', -\theta\varepsilon) \in B_{M\varepsilon}(x)\} \\ &= B'_{((M\varepsilon)^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x') \\ &\supset B'_{(\varepsilon^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x'), \end{aligned}$$

because $M > 1$. If $-(1 + \theta)\varepsilon < x_n < -\theta\varepsilon$, then

$$\{y' : (y', \psi(y')) \in B_{M\varepsilon}(x)\} \supset \{y' : (y', \theta\varepsilon) \in B_{M\varepsilon}(x)\} = B'_{((M\varepsilon)^2 - (x_n - \theta\varepsilon)^2)^{1/2}}(x')$$

and

$$\begin{aligned} & (M\varepsilon)^2 - (x_n - \theta\varepsilon)^2 - (\varepsilon^2 - (x_n + \theta\varepsilon)^2) \\ &= (1 + 2\theta)^2\varepsilon^2 - (x_n^2 - 2\theta\varepsilon x_n + (\theta\varepsilon)^2) \\ &\quad - \varepsilon^2 + (x_n^2 + 2\theta\varepsilon x_n + (\theta\varepsilon)^2) \\ &= 4\theta\varepsilon^2 + 4\theta^2\varepsilon^2 + 4\theta\varepsilon x_n \geq 0, \end{aligned}$$

so that $B'_{((M\varepsilon)^2 - (x_n - \theta\varepsilon)^2)^{1/2}}(x') \supset B'_{(\varepsilon^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x')$, as desired, whence, (4.3) holds. $\square$

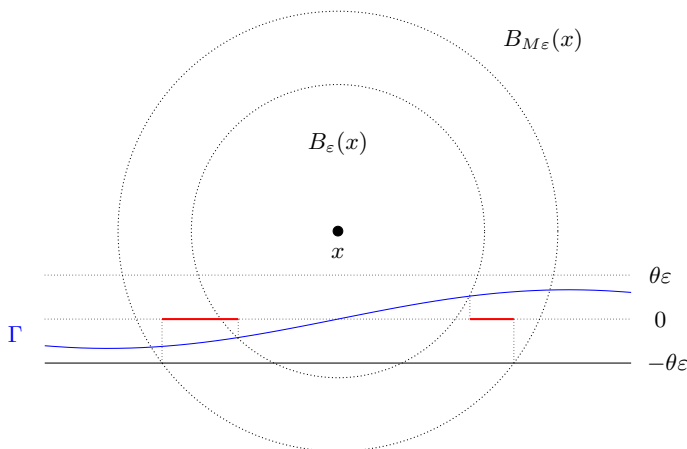


Fig. 2. The red set is $\{y' : (y', -\theta\epsilon) \in B_{M\epsilon}(x)\} \setminus \{y' : (y', \psi(y')) \in B_\epsilon(x)\}$

Proof of Theorem 4.2. Let $M = 1 + 2\theta > 1$. By Corollary 3.2 with $a = -\theta\epsilon$, and $c_0 = M^n(1 + \delta)(1 - \epsilon)^{-1}$ there is a unique classical solution $\underline{w}$ to the flat transmission problem

$$\begin{cases} \Delta \underline{w} = M^n(1 + \delta)(1 - \epsilon)^{-1} dH^{n-1} \big|_{T_{-\theta\epsilon}} & \text{in } B_1 \\ \underline{w} = u & \text{on } \partial B_1. \end{cases}$$

Moreover, since $u \in C^{0,\gamma}(\overline{B_1})$, then $\underline{w} \in C^\infty(B_1 \setminus T_{-\theta\epsilon}) \cap C^{0,\gamma}(\overline{B_1})$.

Define the averages

$$u_\epsilon(x) = \frac{1}{|B_\epsilon|} \int_{B_\epsilon(x)} u(y) dy \quad \text{for } x \in B_{1-\epsilon} \subset B_1$$

and

$$\underline{w}_{M\epsilon}(x) = \frac{1}{|B_{M\epsilon}|} \int_{B_{M\epsilon}(x)} \underline{w}(y) dy \quad \text{for } x \in B_{1-M\epsilon} \subset B_1.$$

By Proposition 4.1(iii), $\Delta u_\epsilon(x) = g_\epsilon(x)$ for every $x \in B_{1-\epsilon}$, and

$$\Delta \underline{w}_{M\epsilon}(x) = \frac{1}{|B_{M\epsilon}|} \int_{B_{M\epsilon}(x) \cap T_{-\theta\epsilon}} M^n(1 + \delta)(1 - \epsilon)^{-1} dH^{n-1} \quad \text{for } x \in B_{1-M\epsilon}.$$

In addition, notice that

$$\text{supp}(\Delta u_\epsilon) \subset \{x \in B_{1-\epsilon} : \text{dist}(x, \Gamma) < \epsilon\}$$

and

$$\text{supp}(\Delta \underline{w}_{M\epsilon}) \subset \{x \in B_{1-M\epsilon} : |x_n| < M\epsilon\}.$$

Since Γ is $\theta\epsilon$ -flat in B_1 and $M = 1 + 2\theta$, it follows that

$$\text{supp}(\Delta u_\epsilon) \subset \text{supp}(\Delta \underline{w}_{M\epsilon}).$$

Let us first show that

$$\Delta \underline{w}_{M\varepsilon} \geq \Delta u_\varepsilon \quad \text{in } B_{1-M\varepsilon}.$$

If $x \notin \text{supp}(g_\varepsilon)$ there is nothing to prove because $\Delta \underline{w}_{M\varepsilon} \geq 0$ in $B_{1-M\varepsilon}$. Let us then take $x \in B_{1-M\varepsilon}$ such that $\text{dist}(x, \Gamma) < \varepsilon$. Using that $0 < g \leq 1 + \delta$, Γ is ε -horizontal and (4.2) in Lemma 4.4, we get

$$\begin{aligned} \Delta \underline{w}_{M\varepsilon}(x) &= \frac{1}{M^n |B_\varepsilon|} \int_{B_{M\varepsilon}(x) \cap T_{-\theta\varepsilon}} M^n (1 + \delta) (1 - \varepsilon)^{-1} dH^{n-1} \\ &\geq \frac{1}{|B_\varepsilon|} \int_{\{y': (y', -\theta\varepsilon) \in B_{M\varepsilon}(x)\}} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} dy' \\ &\geq \frac{1}{|B_\varepsilon|} \int_{\{y': (y', \psi(y')) \in B_\varepsilon(x)\}} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} dy' \\ &= \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x) \cap \Gamma} g dH^{n-1} = \Delta u_\varepsilon(x). \end{aligned}$$

We also have

$$\underline{w}_{M\varepsilon} \leq u_\varepsilon + C\varepsilon^\gamma \quad \text{on } \partial B_{1-M\varepsilon}$$

for some $C = C(n, \Gamma) > 0$. Indeed, fix any $x \in \partial B_{1-M\varepsilon}$, and let $z \in \partial B_1$ be such that $\text{dist}(x, \partial B_1) = |x - z| = M\varepsilon$. By using that $\underline{w}, u \in C^{0,\gamma}(\overline{B_1})$, and $\underline{w} = u$ on ∂B_1 ,

$$\begin{aligned} \underline{w}_{M\varepsilon}(x) - u_\varepsilon(x) &= (\underline{w}_{M\varepsilon}(x) - \underline{w}(x)) + (\underline{w}(x) - \underline{w}(z)) \\ &\quad + (u(z) - u(x)) + (u(x) - u_\varepsilon(x)) \\ &\leq \frac{1}{|B_{M\varepsilon}|} \int_{B_{M\varepsilon}(x)} |\underline{w}(y) - \underline{w}(x)| dy + C \|g\|_{L^\infty(\Gamma)} |x - z|^\gamma \\ &\quad + \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} |u(y) - u(x)| dy \\ &\leq C\varepsilon^\gamma, \end{aligned} \tag{4.4}$$

where $C = C(n) > 0$, because Γ is ε -horizontal and $|g - 1| \leq \delta$ on Γ . Hence, by the maximum principle, $\underline{w}_{M\varepsilon} - u_\varepsilon \leq C\varepsilon^\gamma$ in $B_{1-M\varepsilon}$. Consequently, by arguing similarly as to (4.4), it follows that, for some $C = C(n) > 0$,

$$\underline{w} - u \leq C\varepsilon^\gamma \quad \text{in } B_{1-M\varepsilon}. \tag{4.5}$$

Secondly, consider the classical solution $\bar{w}$ to the flat transmission problem

$$\begin{cases} \Delta \bar{w} = M^{-n} (1 - \delta) dH^{n-1} \Big|_{T_{-\theta\varepsilon}} & \text{in } B_1 \\ \bar{w} = u & \text{on } \partial B_1 \end{cases}$$

and the corresponding averages $\bar{w}_\varepsilon$ and $u_{M\varepsilon}$ of $\bar{w}$ and u , respectively. Since $g \geq 1 - \delta$, by (4.3) in Lemma 4.4, we find that

$$\Delta \bar{w}_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x) \cap T_{-\theta\varepsilon}} M^{-n} (1 - \delta) dH^{n-1}$$

$$\begin{aligned}
&\leq \frac{1}{|B_{M\varepsilon}|} \int_{\{y': (y', -\theta\varepsilon) \in B_\varepsilon(x)\}} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} dy' \\
&\leq \frac{1}{|B_{M\varepsilon}|} \int_{\{y': (y', \psi(y')) \in B_{M\varepsilon}(x)\}} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} dy' \\
&= \frac{1}{|B_{M\varepsilon}|} \int_{B_{M\varepsilon}(x) \cap \Gamma} g dH^{n-1} = \Delta u_{M\varepsilon}(x).
\end{aligned}$$

By using parallel arguments to those in (4.4) we also get that, for some $C = C(n) > 0$,

$$u - \bar{w} \leq C\varepsilon^\gamma \quad \text{in } B_{1-M\varepsilon}. \quad (4.6)$$

Define $w = \frac{\underline{w} + \bar{w}}{2}$. By (4.5) and (4.6),

$$u - w \leq \bar{w} + C\varepsilon^\gamma - \frac{\underline{w} + \bar{w}}{2} = \frac{\bar{w} - \underline{w}}{2} + C\varepsilon^\gamma$$

and

$$u - w \leq \underline{w} - C\varepsilon^\gamma - \frac{\underline{w} + \bar{w}}{2} = \frac{\underline{w} - \bar{w}}{2} - C\varepsilon^\gamma.$$

Hence,

$$\|u - w\|_{L^\infty(B_{1/2})} \leq \frac{1}{2} \|\bar{w} - \underline{w}\|_{L^\infty(B_{1/2})} + C\varepsilon^\gamma.$$

Since

$$\begin{cases} \Delta(\bar{w} - \underline{w}) = [M^{-n}(1 - \delta) - M^n(1 + \delta)(1 - \varepsilon)^{-1}] dH^{n-1} \Big|_{T_{-\theta\varepsilon}} & \text{in } B_1 \\ \bar{w} - \underline{w} = 0 & \text{on } \partial B_1 \end{cases}$$

by Theorem 3.1,

$$\|\bar{w} - \underline{w}\|_{L^\infty(B_{1/2})} \leq C|M^n(1 + \delta)(1 - \varepsilon)^{-1} - M^{-n}(1 - \delta)| \leq C(\theta + \delta + \varepsilon)$$

for some $C = C(n) > 0$. Therefore,

$$\|u - w\|_{L^\infty(B_{1/2})} \leq C(\theta + \delta + \varepsilon^\gamma). \quad (4.7)$$

Also, $\Delta w = (1 + \eta) dH^{n-1} \Big|_{T_{-\theta\varepsilon}}$, where

$$1 + \eta = \frac{M^n(1 + \delta)(1 - \varepsilon)^{-1} + M^{-n}(1 - \delta)}{2}.$$

Observe that, since $0 < \theta, \varepsilon < 1/2, 0 < \delta < 1$, it follows that

$$\begin{aligned}
|\eta| &= \frac{|M^{2n}(1 + \delta) + (1 - \delta)(1 - \varepsilon) - 2(1 - \varepsilon)M^n|}{2(1 - \varepsilon)M^n} \\
&\leq C(|(1 + 2\theta)^{2n} + 1 - 2(1 + 2\theta)^n| + \delta + \varepsilon) \leq C(\theta + \delta + \varepsilon),
\end{aligned} \quad (4.8)$$

where $C = C(n) > 0$.

Let $v \in C^\infty(B_1 \setminus T_{-\theta\varepsilon}) \cap C^{0,\gamma}(\overline{B_1})$ be the solution to

$$\begin{cases} \Delta v = dH^{n-1}|_{T_{-\theta\varepsilon}} & \text{in } B_1 \\ v = u & \text{on } \partial B_1; \end{cases}$$

see Corollary 3.2. Then $v - w$ solves

$$\begin{cases} \Delta(v - w) = \eta dH^{n-1}|_{T_{-\theta\varepsilon}} & \text{in } B_1 \\ v - w = 0 & \text{on } \partial B_1. \end{cases}$$

Therefore, by (4.8),

$$\|v - w\|_{L^\infty(B_1)} \leq C|\eta| \leq C(\theta + \delta + \varepsilon), \quad (4.9)$$

where $C = C(n) > 0$. From (4.7) and (4.9) the estimate on the statement is proved. $\square$

Remark 4.5. (Divergence form equations) Recall that our proof of Theorem 4.2 is self-contained, based on the mean value property for harmonic functions and the maximum principle. In view of recently developed mean value formulas for solutions to divergence form elliptic equations by BLANK-HAO [1], the natural question of extending our geometric techniques to transmission problems for divergence form elliptic equations arise. For this case, our maximum principle techniques must be replaced by energy methods. More importantly, not much is known about the geometry of the mean value sets from [1], so it is not clear at all how to mimic geometric arguments such as those in Lemma 4.4.

Remark 4.6. (Nondivergence form equations) The second natural question would be to extend our methods to transmission problems with nondivergence form elliptic equations, where the maximum principle is a more adequate tool. In this situation, not only there are no useful mean value formulas available, but also the notion of distributional solution we consider in this paper does not apply anymore. The first step would be to prove existence, uniqueness, and some initial regularity of viscosity solutions. This is an open problem.

5. Proof of Theorem 1.2

Throughout this section, Γ is an interface in B_1 given by the graph of a function $x_n = \psi(x') : T \rightarrow \mathbb{R}$. Thus, we can write $B_1 = \Omega_1 \cup \Gamma \cup \Omega_2$, where $\Omega_1 = \{x = (x', x_n) \in B_1 : x_n > \psi(x')\}$. We also assume that $0 \in \Gamma$.

5.1. Preliminary Lemmas

Lemma 5.1. *Given $0 < \alpha, \gamma < 1$, there exist constants $C_0 > 0$, $0 < \lambda < 1/2$, $0 < \theta, \delta, \varepsilon < \lambda$ depending only on n, α and γ , such that for any $u \in C^{0,\gamma}(\overline{B_1})$ satisfying*

$$\begin{cases} \Delta u = g \, dH^{n-1}|_{\Gamma} & \text{in } B_1 \\ |u| \leq 1 & \text{in } B_1 \\ |g - 1| \leq \delta & \text{on } \Gamma \end{cases}$$

if Γ is $\theta\varepsilon$ -flat and ε -horizontal in B_1 , then there are linear polynomials $P_1(x) = A \cdot x + B$ and $Q_1(x) = C \cdot x + B$, with $A, C \in \mathbb{R}^n$, $B \in \mathbb{R}$, and $|A| + |B| + |C| \leq C_0$, such that

$$\begin{aligned} |u_1(x) - P_1(x)| &\leq \lambda^{1+\alpha} && \text{for all } x \in \Omega_1 \cap B_\lambda \\ |u_2(x) - Q_1(x)| &\leq \lambda^{1+\alpha} && \text{for all } x \in \Omega_2 \cap B_\lambda. \end{aligned}$$

Moreover, $\nabla' P_1 = \nabla' Q_1$ and $(P_1)_{x_n} - (Q_1)_{x_n} = 1$.

Proof. Fix $0 < \theta, \delta, \varepsilon < \lambda < 1/2$ to be chosen later. Consider the solutions

$$\begin{aligned} \underline{v} &= \underline{v}^+ \chi_{\overline{B_1} \cap T_{-\theta\varepsilon}^+} + \underline{v}^- \chi_{\overline{B_1} \cap T_{-\theta\varepsilon}^-} \\ \bar{v} &= \bar{v}^+ \chi_{\overline{B_1} \cap T_{\theta\varepsilon}^+} + \bar{v}^- \chi_{\overline{B_1} \cap T_{\theta\varepsilon}^-} \end{aligned}$$

to the flat transmission problems given in Theorem 4.2, and Remark 4.3, respectively. By Corollary 3.2 with $k = 2$,

$$\|\underline{v}^+\|_{C^{2,\alpha}(\overline{B_{1/2} \cap T_{-\theta\varepsilon}^+})} + \|\bar{v}^-\|_{C^{2,\alpha}(\overline{B_{1/2} \cap T_{\theta\varepsilon}^-})} \leq C(1 + \|u\|_{L^\infty(B_1)}) \leq C_0$$

for some $C_0 = C_0(n) > 0$. In particular,

$$|\underline{v}(0)| + |\nabla \underline{v}(0)| + |\bar{v}(0)| + |\nabla \bar{v}(0)| \leq C_0.$$

Let h be the harmonic function in B_1 such that $h = u$ on ∂B_1 . Define

$$\begin{aligned} P_1(x) &= \underline{v}(0) + \nabla \underline{v}(0) \cdot x + \left[\frac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0) \right] x_n \\ Q_1(x) &= \bar{v}(0) + \nabla \bar{v}(0) \cdot x + \left[-\frac{1}{2} - \bar{v}_{x_n}(0) + h_{x_n}(0) \right] x_n. \end{aligned}$$

Then P_1 and Q_1 are small perturbations of the linear parts of $\underline{v}$ and $\bar{v}$ at the origin, respectively. To see this, first note that the functions $\underline{v}(x', x_n) - h(x', x_n)$ and $\bar{v}(x', -x_n) - h(x', -x_n)$ satisfy the same transmission problem on $T_{-\theta\varepsilon}$ with zero data on ∂B_1 . By uniqueness,

$$\underline{v}(x', x_n) - h(x', x_n) = \bar{v}(x', -x_n) - h(x', -x_n) \quad \text{for all } x \in \overline{B_1}.$$

In particular, $\underline{v}(x', 0) = \bar{v}(x', 0)$, $\nabla' \underline{v}(x', 0) = \nabla' \bar{v}(x', 0)$, and thus, $P_1(0) = Q_1(0)$, and $\nabla' P_1 = \nabla' \underline{v}(0) = \nabla' \bar{v}(0) = \nabla' Q_1$. Clearly, $(P_1)_{x_n} - (Q_1)_{x_n} = 1$. Moreover,

$$\underline{v}_{x_n}(x', 0) - h_{x_n}(x', 0) = -\bar{v}_{x_n}(x', 0) + h_{x_n}(x', 0)$$

and thus, $|\frac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)| = |-\frac{1}{2} - \bar{v}_{x_n}(0) + h_{x_n}(0)|$. Let us show that

$$|\frac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)| \leq D(\theta\varepsilon)^\gamma \quad (5.1)$$

for some $D = D(n) > 0$. Recall that by the construction of $\underline{v}$ in Corollary 3.2, we can write $\underline{v} = w - H$, where $w \in C^\infty(B_{4,-\theta\varepsilon} \setminus T_{-\theta\varepsilon}) \cap \bar{C}^{0,\gamma}(\overline{B_{4,-\theta\varepsilon}})$ is the harmonic function in $B_{4,-\theta\varepsilon}$ such that $w = 0$ on $\partial B_{4,-\theta\varepsilon}$, and H is the harmonic function in B_1 , with $H = w - u$ on ∂B_1 . Then

$$|\frac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)| \leq |w_{x_n}(0) - \frac{1}{2}| + |(H + h)_{x_n}(0)|.$$

In particular, $w_{x_n}(0) = w_{x_n}^+(0)$, where w^+ is the harmonic function in $B_{4,-\theta\varepsilon}^+$ such that $w = 0$ on $\partial B_{4,-\theta\varepsilon}^+ \setminus T_{-\theta\varepsilon}$, and $w_{x_n}^+ = \frac{1}{2}$ on $T_{-\theta\varepsilon}$. By the mean value theorem,

$$w_{x_n}(0) - \frac{1}{2} = w_{x_n}^+(0', 0) - w_{x_n}^+(0', -\theta\varepsilon) = w_{x_n x_n}^+(0', \xi)\theta\varepsilon$$

for some $-\theta\varepsilon \leq \xi \leq 0$. Moreover, by Theorem 3.1, $\|w^+\|_{C^{2,\alpha}(\overline{B_{2,-\theta\varepsilon}^+})} \leq D_0$, for some constant $D_0 = D_0(n) > 0$. Hence,

$$|w_{x_n}(0) - \frac{1}{2}| \leq D_0\theta\varepsilon.$$

Next, note that $H + h$ is harmonic in B_1 , and $H + h = w$ on ∂B_1 . Consider the harmonic function ϕ in $B_{1-\theta\varepsilon, -\theta\varepsilon}$ such that $\phi = w$ on $B_{1-\theta\varepsilon, -\theta\varepsilon}$. Observe that $B_{1-\theta\varepsilon, -\theta\varepsilon} \subset B_1$. Since w is symmetric with respect to the plane $T_{-\theta\varepsilon}$, it follows that $\phi_{x_n}(x', -\theta\varepsilon) = 0$ for any $(x', -\theta\varepsilon) \in B_{1-\theta\varepsilon, -\theta\varepsilon}$. Therefore, $|\phi_{x_n}(0)| \leq D_0\theta\varepsilon$. By interior estimates, the maximum principle, and the facts that $w \in C^{0,\gamma}(\overline{B_1})$ and $\text{dist}(\partial B_1, \partial B_{1-\theta\varepsilon, -\theta\varepsilon}) \leq 2\theta\varepsilon$,

$$|(H + h)_{x_n}(0) - \phi_{x_n}(0)| \leq D_1\|(H + h) - w\|_{L^\infty(\partial B_{1-\theta\varepsilon, -\theta\varepsilon})} \leq D_1(\theta\varepsilon)^\gamma$$

for some $D_1 = D_1(n) > 0$, and thus,

$$|(H + h)_{x_n}(0)| \leq D_1(\theta\varepsilon)^\gamma + |\phi_{x_n}(0)| \leq D_1(\theta\varepsilon)^\gamma + D_0\theta\varepsilon \leq D(\theta\varepsilon)^\gamma$$

for some $D = D(n) > 0$. Therefore, (5.1) holds.

If $x \in \Omega_1 \cap B_{1/2}$, by Theorem 4.2 and (5.1), there are constants $C, D > 0$, depending only on n , such that

$$\begin{aligned} |u_1(x) - P_1(x)| &\leq |u(x) - \underline{v}(x)| + |\underline{v}(x) - P_1(x)| \\ &\leq |u(x) - \underline{v}(x)| + |\underline{v}(x) - \underline{v}(0) - \nabla \underline{v}(0)| \\ &\quad + |\frac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)| |x_n| \\ &\leq C(\theta + \delta + \varepsilon^\gamma) + \|D^2 \underline{v}\|_{L^\infty(\Omega_1 \cap B_{1/2})} |x|^2 + D(\theta\varepsilon)^\gamma |x_n| \\ &\leq C(\theta + \delta + \varepsilon^\gamma) + C_0 |x|^2 + D(\theta\varepsilon)^\gamma |x_n|. \end{aligned}$$

Similarly, if $x \in \Omega_2 \cap B_{1/2}$,

$$|u_2(x) - Q_1(x)| \leq C(\theta + \delta + \varepsilon^\gamma) + C_0 |x|^2 + D(\theta\varepsilon)^\gamma |x_n|.$$

First, choose $0 < \lambda < 1/2$ such that

$$C_0|x|^2 \leq \frac{\lambda^{1+\alpha}}{2} \quad \text{for all } x \in B_\lambda.$$

Then, choose $0 < \theta, \delta, \varepsilon < \lambda$ such that

$$C(\theta + \delta + \varepsilon^\gamma) + D(\theta\varepsilon)^\gamma \lambda \leq \frac{\lambda^{1+\alpha}}{2}.$$

□

Lemma 5.2. *Given $0 < \alpha < 1$, there exist $C_0 > 0$, $0 < \lambda < 1/2$, and $0 < \delta < 1$, depending only on n and α , such that for any distributional solution $u \in C(\overline{B_1})$ to*

$$\begin{cases} \Delta u = g \, dH^{n-1}|_\Gamma & \text{in } B_1 \\ |u| \leq 1 & \text{in } B_1 \\ |g| \leq \delta & \text{on } \Gamma, \end{cases}$$

there is a linear polynomial $P(x) = A \cdot x + B$, with $A \in \mathbb{R}^n$, $B \in \mathbb{R}$ and $|A| + |B| \leq C_0$, such that

$$|u(x) - P(x)| \leq \lambda^{1+\alpha} \quad \text{for all } x \in B_\lambda.$$

Proof. Fix $\lambda, \delta > 0$ to be determined. Let v be the harmonic function in B_1 such that $v = u$ on ∂B_1 . Then, the difference $w = u - v$ is the distributional solution to

$$\begin{cases} \Delta w = g \, dH^{n-1}|_\Gamma & \text{in } B_1 \\ w = 0 & \text{on } \partial B_1. \end{cases}$$

Moreover, $\|w\|_{L^\infty(B_1)} \leq C\|g\|_{L^\infty(\Gamma)} \leq C\delta$, where $C = C(n) > 0$. Define $P(x) = v(0) + \nabla v(0) \cdot x$. By interior estimates and the maximum principle, we have

$$\|D^j v\|_{L^\infty(B_{1/2})} \leq C_0 \|v\|_{L^\infty(B_{1/2})} \leq C_0 \quad \text{for all } j \geq 0,$$

where $C_0 = C_0(n, j) > 0$. Hence, for $x \in B_\lambda$, with $0 < \lambda < 1/2$, we get

$$\begin{aligned} |u(x) - P(x)| &\leq |u(x) - v(x)| + |v(x) - P(x)| \\ &\leq C\delta + \|D^2 v\|_{L^\infty(B_{1/2})} |x|^2 \\ &\leq C\delta + C_0 \lambda^2. \end{aligned}$$

First, choose $0 < \lambda < 1/2$, such that $C_0 \lambda^2 \leq \lambda^{1+\alpha}/2$. Then choose $0 < \delta < 1$ such that $C\delta \leq \lambda^{1+\alpha}/2$. □

5.2. Proof of Theorem 1.2

Fix $0 < \alpha, \gamma < 1$. Let $C_0, \lambda, \theta, \varepsilon, \delta > 0$ be the minimum of the corresponding constants given in Lemmas 5.1 and 5.2. Let $0 < \delta_0 < \min \left\{ \delta, \theta\varepsilon, \frac{\lambda^{1+\alpha}}{2} \right\}$. First, we normalize the problem. Recall that we are assuming that $0 \in \Gamma$, that is, that $\psi(0') = 0$.

- (i) By rotation, we can assume that $v(0) = e_n$. In particular, $\nabla' \psi(0') = 0'$.
- (ii) If $g(0) \neq 0$, we can suppose that $g(0) = 1$. Indeed, we consider $v = u/g(0)$. The case $g(0) = 0$ will be addressed at the end.
- (iii) Assume that $\|u\|_{L^\infty(B_1)} \leq 1$, and that

$$[g]_{C^{0,\alpha}(0)} = \sup_{x \in \Gamma \cap B_1, x \neq 0} \frac{|g(x) - g(0)|}{|x|^\alpha} \leq \delta_0.$$

Indeed, one can consider

$$v = \delta_0 \frac{u}{\|u\|_{L^\infty(B_1)} + [g]_{C^{0,\alpha}(0)}}.$$

- (iv) Also, we let $[\psi]_{C^{1,\alpha}(0)} \leq [\psi]_{C^{1,\alpha}(B'_1)} \leq \delta_0$. Recall that

$$[\psi]_{C^{1,\alpha}(0)} = \sup_{x' \in B'_1, x' \neq 0'} \frac{|\nabla' \psi(x') - \nabla' \psi(0')|}{|x'|^\alpha} = \sup_{x' \in B'_1, x' \neq 0'} \frac{|\nabla' \psi(x')|}{|x'|^\alpha}.$$

Then, for this normalization one can take

$$\phi = \delta_0 \frac{\psi}{[\psi]_{C^{1,\alpha}(B'_1)}}.$$

We make an abuse of notation and call the solution, the interface, the parametrization and the right hand side as in the statement, namely, u, Γ, ψ , and g , respectively.

It is enough to prove the following.

Claim. For all $k \geq 1$, there exist linear polynomials $P_k = A_k \cdot x + B_k$ and $Q_k = C_k \cdot x + B_k$ such that

$$\lambda^k |A_{k+1} - A_k| + \lambda^k |C_{k+1} - C_k| + |B_{k+1} - B_k| \leq C_0 \lambda^{k(1+\alpha)}$$

where $C_0 = C_0(n) > 0$, and such that

$$\begin{aligned} |u_1(x) - P_k(x)| &\leq \lambda^{k(1+\alpha)} && \text{for all } x \in \Omega_1 \cap B_{\lambda^k} \\ |u_2(x) - Q_k(x)| &\leq \lambda^{k(1+\alpha)} && \text{for all } x \in \Omega_2 \cap B_{\lambda^k}. \end{aligned}$$

Moreover, $\nabla' P_k = \nabla' Q_k$ and $(P_k)_{x_n} - (Q_k)_{x_n} = 1$.

We prove the claim by induction. Let us start with the case $k = 1$. By the normalization, u, Γ and g satisfy the assumptions on Lemma 5.1. Indeed, by (i) and (iv), for any $(x', x_n) \in \Gamma$,

$$|x_n| = |\psi(x')| = |\psi(x') - \psi(0') - \nabla' \psi(0') \cdot x'| \leq [\psi]_{C^{1,\alpha}(0)} \leq \delta_0 \leq \theta\varepsilon.$$

Also, $1 \leq (1 + |\nabla' \psi(x')|^2)^{1/2} \leq (1 + \delta_0^2)^{1/2} \leq (1 - \varepsilon)^{-1}$. Moreover, by (iii),

$$|g(x) - 1| = |g(x) - g(0)| \leq [g]_{C^{0,\alpha}(0)} |x|^\alpha \leq \delta_0 \leq \delta \quad \text{for all } x \in \Gamma.$$

Hence, by Lemma 5.1, there are linear polynomials $P_1(x) = A_1 \cdot x + B_1$, and $Q_1(x) = C_1 \cdot x + B_1$, with $A_1, C_1 \in \mathbb{R}^n$, $B_1 \in \mathbb{R}$, and $|A_1| + |B_1| + |C_1| \leq C_0$, such that

$$\begin{aligned} |u_1(x) - P_1(x)| &\leq \lambda^{1+\alpha} & \text{for all } x \in \Omega_1 \cap B_\lambda \\ |u_2(x) - Q_1(x)| &\leq \lambda^{1+\alpha} & \text{for all } x \in \Omega_2 \cap B_\lambda. \end{aligned}$$

Moreover, $\nabla' P_1 = \nabla' Q_1$, and $(P_1)_{x_n} - (Q_1)_{x_n} = 1$.

For the induction step, assume that the claim holds for some $k \geq 1$, and let P_k and Q_k be such polynomials. Denote by

$$\begin{aligned} \Omega_{i,\lambda^k} &= \{x \in B_1 : \lambda^k x \in \Omega_i\} \quad \text{for } i = 1, 2 \\ \Gamma_{\lambda^k} &= \{x \in B_1 : \lambda^k x \in \Gamma\}. \end{aligned}$$

Note that if ψ_{λ^k} is a parametrization of Γ_{λ^k} in B'_1 , then $\psi_{\lambda^k}(x') = \lambda^{-k} \psi(\lambda^k x')$. In particular, $\nabla' \psi_{\lambda^k}(x') = \nabla' \psi(\lambda^k x)$, and thus, for $x \in \Gamma_{\lambda^k}$, if $\nu_{\lambda^k}(x)$ is the normal vector on x pointing at $\Omega_{\lambda^k,1}$, then $\nu_{\lambda^k}(x) = \nu(\lambda^k x)$. Define $\mathcal{P}_k = P_k \chi_{\Omega_1} + Q_k \chi_{\Omega_2}$. Consider the rescaled function

$$w(x) = \frac{u(\lambda^k x) - \mathcal{P}_k(\lambda^k x)}{\lambda^{k(1+\alpha)}} \quad \text{for } x \in B_1. \quad (5.2)$$

By the induction hypothesis, $\|w\|_{L^\infty(B_1)} \leq 1$. Notice that w is a piecewise continuous function with a jump discontinuity on Γ_{λ^k} . In fact, if

$$w_1 = w|_{\overline{\Omega}_{1,\lambda^k}}, \quad w_2 = w|_{\overline{\Omega}_{2,\lambda^k}},$$

then for $x \in \Gamma_{\lambda^k}$, by the normalization (iv), and the induction hypothesis, we have

$$\begin{aligned} |(w_1 - w_2)(x)| &= \frac{|Q_k(\lambda^k x) - P_k(\lambda^k x)|}{\lambda^{k(1+\alpha)}} = \lambda^{-k\alpha} |x_n| \\ &\leq \lambda^{-k\alpha} \sup_{x \in \Gamma_{\lambda^k}} |x_n| \\ &\leq \sup_{x' \in B'_1} \frac{|\psi_{\lambda^k}(x')|}{\lambda^{k\alpha}} \leq [\psi]_{C^{1,\alpha}(0)} \leq \delta_0. \end{aligned} \quad (5.3)$$

Let $v = v_1 \chi_{\overline{\Omega}_{1,\lambda^k}} + v_2 \chi_{\overline{\Omega}_{2,\lambda^k}}$, where v_1 and v_2 are the solutions to

$$\begin{cases} \Delta v_i = 0 & \text{in } \Omega_{i,\lambda^k} \\ v_i = w_i & \text{on } \partial\Omega_{i,\lambda^k} \setminus \Gamma_{\lambda^k} \\ v_i = \frac{w_1 + w_2}{2} & \text{on } \Gamma_{\lambda^k} \end{cases}$$

for $i = 1, 2$. Then $v \in C^0(\overline{B_1})$. Moreover,

$$\begin{cases} \Delta(v_i - w_i) = 0 & \text{in } \Omega_{i,\lambda^k} \\ v_i - w_i = 0 & \text{on } \partial\Omega_{i,\lambda^k} \setminus \Gamma_{\lambda^k} \\ v_i - w_i = (-1)^i \frac{w_1 - w_2}{2} & \text{on } \Gamma_{\lambda^k}. \end{cases} \quad (5.4)$$

By the maximum principle and (5.3) it follows that

$$\begin{aligned} \|v - w\|_{L^\infty(B_1)} &\leq \|v_1 - w_1\|_{L^\infty(\Omega_{1,\lambda^k})} + \|v_2 - w_2\|_{L^\infty(\Omega_{2,\lambda^k})} \\ &= \|w_1 - w_2\|_{L^\infty(\Gamma_{\lambda^k})} \leq \delta_0. \end{aligned} \quad (5.5)$$

We compute the distributional Laplacian of v and estimate its size. For any $\varphi \in C_c^\infty(B_1)$,

$$\begin{aligned} \Delta v(\varphi) &= \int_{B_1} v(x) \Delta \varphi(x) \, dx \\ &= \int_{\Omega_{1,\lambda^k}} v_1(x) \Delta \varphi(x) \, dx + \int_{\Omega_{2,\lambda^k}} v_2(x) \Delta \varphi(x) \, dx \\ &= \int_{\Omega_{1,\lambda^k}} (v_1 - w_1)(x) \Delta \varphi(x) \, dx + \int_{\Omega_{2,\lambda^k}} (v_2 - w_2)(x) \Delta \varphi(x) \, dx \\ &\quad + \int_{B_1} w(x) \Delta \varphi(x) \, dx \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

For $i = 1, 2$, by Green's formula,

$$\begin{aligned} I_i &= \frac{1}{2} \int_{\Gamma_{\lambda^k}} (w_1 - w_2)(x) \varphi_{v_{\lambda^k}}(x) \, dH^{n-1} \\ &\quad + (-1)^{i+1} \int_{\Gamma_{\lambda^k}} (v_i - w_i)_{v_{\lambda^k}}(x) \varphi(x) \, dH^{n-1}, \end{aligned}$$

where we recall that v_{λ^k} is the unit normal vector on Γ_{λ^k} pointing at Ω_{1,λ^k} . Note that

$$I_3 = \Delta w(\varphi) = \Delta \left(\frac{u(\lambda^k x)}{\lambda^k(1+\alpha)} \right) (\varphi) - \Delta \left(\frac{\mathcal{P}_k(\lambda^k x)}{\lambda^k(1+\alpha)} \right) (\varphi).$$

Since u is a distributional solution, by doing a change of variables, we get

$$\begin{aligned} \Delta(u(\lambda^k x))(\varphi) &= \int_{B_1} u(\lambda^k x) \Delta \varphi(x) \, dx \\ &= \lambda^{k(2-n)} \int_{B_{\lambda^k}} u(y) \Delta_y \varphi(\lambda^{-k} y) \, dy \\ &= \lambda^{k(2-n)} \int_{\Gamma \cap B_{\lambda^k}} g(y) \varphi(\lambda^{-k} y) \, dH_y^{n-1} \end{aligned}$$

$$= \lambda^k \int_{\Gamma_{\lambda^k}} g(\lambda^k x) \varphi(x) \, dH^{n-1}.$$

Also, by Green's formula, the induction hypothesis and (5.3),

$$\begin{aligned} \Delta(\mathcal{P}_k(\lambda^k x))(\varphi) &= \lambda^k \int_{\Gamma_{\lambda^k}} [\nabla P_k(\lambda^k x) - \nabla Q_k(\lambda^k x)] \cdot \nu_{\lambda^k}(x) \varphi(x) \, dH^{n-1} \\ &\quad + \int_{\Gamma_{\lambda^k}} [Q_k(\lambda^k x) - P_k(\lambda^k x)] \varphi_{\nu_{\lambda^k}}(x) \, dH^{n-1} \\ &= \lambda^k \int_{\Gamma_{\lambda^k}} v_n(\lambda^k x) \varphi(x) \, dH^{n-1} \\ &\quad + \lambda^{k(1+\alpha)} \int_{\Gamma_{\lambda^k}} (w_1 - w_2)(x) \varphi_{\nu_{\lambda^k}}(x) \, dH^{n-1}. \end{aligned}$$

Then

$$I_3 = \int_{\Gamma_{\lambda^k}} \tilde{g}(x) \varphi(x) \, dH^{n-1} - \int_{\Gamma_{\lambda^k}} (w_1 - w_2)(x) \varphi_{\nu_{\lambda^k}}(x) \, dH^{n-1},$$

where

$$\tilde{g}(x) = \frac{g(\lambda^k x) - v_n(\lambda^k x)}{\lambda^{k\alpha}}.$$

Therefore,

$$\Delta v(\varphi) = \int_{\Gamma_{\lambda^k}} \left[(v_1 - w_1)_{\nu_{\lambda^k}}(x) - (v_2 - w_2)_{\nu_{\lambda^k}}(x) + \tilde{g}(x) \right] \varphi(x) \, dH^{n-1}.$$

By $C^{1,\alpha}$ boundary estimates for harmonic functions in (5.4), and by (5.3),

$$\|(v_i - w_i)_{\nu_{\lambda^k}}\|_{L^\infty(\Gamma_{\lambda^k} \cap B_{3/4})} \leq C \|w_1 - w_2\|_{L^\infty(\Gamma_{\lambda^k})} \leq C \delta_0$$

where $C = C(n, \alpha) > 0$. Moreover, for $x \in \Gamma_{\lambda^k}$, by the normalization,

$$\begin{aligned} |\tilde{g}(x)| &\leq \frac{|g(\lambda^k x) - 1|}{\lambda^{k\alpha}} + \frac{|1 - v_n(\lambda^k x)|}{\lambda^{k\alpha}} \\ &\leq [g]_{C^{0,\alpha}(0)} + [v_n]_{C^{0,\alpha}(0)} \leq \delta_0 + \delta_0 = 2\delta_0. \end{aligned}$$

By the maximum principle, $\|v\|_{L^\infty(B_1)} \leq \|w\|_{L^\infty(B_1)} \leq 1$. Therefore, we can apply Lemma 5.2 to v to find a linear polynomial $P(x) = A \cdot x + B$, with $A \in \mathbb{R}^n$, $B \in \mathbb{R}$ and $|A| + |B| \leq C_0$, such that

$$|v(x) - P(x)| \leq \frac{\lambda^{1+\alpha}}{2} \quad \text{for all } x \in B_\lambda.$$

Hence, for any $x \in B_\lambda$, by the estimate above and (5.5),

$$|w(x) - P(x)| \leq |w(x) - v(x)| + |v(x) - P(x)|$$

$$\begin{aligned} &\leq \delta_0 + \frac{\lambda^{1+\alpha}}{2} \\ &\leq \lambda^{1+\alpha}, \end{aligned}$$

since $\delta_0 \leq \lambda^{1+\alpha}/2$. According to (5.2),

$$\left| \frac{u(\lambda^k x) - \mathcal{P}_k(\lambda^k x)}{\lambda^{k(1+\alpha)}} - P(x) \right| \leq \lambda^{1+\alpha} \quad \text{for all } x \in B_\lambda,$$

or equivalently, for $y = \lambda^k x$,

$$|u(y) - \mathcal{P}_k(y) - \lambda^{k(1+\alpha)} P(y/\lambda^k)| \leq \lambda^{(k+1)(1+\alpha)} \quad \text{for all } y \in B_{\lambda^{k+1}}.$$

Define the polynomials P_{k+1} and Q_{k+1} as

$$P_{k+1}(y) = P_k(y) + \lambda^{k(1+\alpha)} P(y/\lambda^k), \quad Q_{k+1}(y) = Q_k(y) + \lambda^{k(1+\alpha)} P(y/\lambda^k).$$

From the previous estimate, it follows that

$$\begin{aligned} |u_1(y) - P_{k+1}(y)| &\leq \lambda^{(k+1)(1+\alpha)} & \text{for all } y \in \Omega_1 \cap B_{\lambda^{k+1}} \\ |u_2(y) - Q_{k+1}(y)| &\leq \lambda^{(k+1)(1+\alpha)} & \text{for all } y \in \Omega_2 \cap B_{\lambda^{k+1}}. \end{aligned}$$

Moreover, since $P_k(0) = Q_k(0)$, and $\nabla' P_k = \nabla' Q_k$, it is clear that $P_{k+1}(0) = Q_{k+1}(0)$, and $\nabla' P_{k+1} = \nabla' Q_{k+1}$. Also, $(P_{k+1})_{x_n} - (Q_{k+1})_{x_n} = (P_k)_{x_n} - (Q_k)_{x_n} = 1$. If $P_{k+1}(y) = A_{k+1} \cdot y + B_{k+1}$ and $Q_{k+1}(y) = C_{k+1} \cdot y + B_{k+1}$ then

$$A_{k+1} = A_k + \lambda^{k\alpha} A, \quad B_{k+1} = B_k + \lambda^{k(1+\alpha)} B, \quad C_{k+1} = C_k + \lambda^{k\alpha} A.$$

By the estimate $|A| + |B| \leq C_0$, we conclude

$$\lambda^k |A_{k+1} - A_k| + \lambda^k |C_{k+1} - C_k| + |B_{k+1} - B_k| \leq C_0 \lambda^{k(1+\alpha)}.$$

The proof of the claim is completed.

Finally, we consider the case $g(0) = 0$. As before, it is enough to prove the following.

Claim. For all $k \geq 1$, there exists a linear polynomial $P_k = A_k \cdot x + B_k$ such that

$$\lambda^k |A_{k+1} - A_k| + |B_{k+1} - B_k| \leq C_0 \lambda^{k(1+\alpha)}$$

where $C_0 = C_0(n) > 0$, and such that

$$|u(x) - P_k(x)| \leq \lambda^{k(1+\alpha)} \quad \text{for all } x \in \Omega \cap B_{\lambda^k}.$$

The proof is by induction. For $k = 1$, since $\|u\|_{L^\infty(B_1)} \leq 1$, and

$$\|g\|_{L^\infty(\Gamma)} = \sup_{x \in \Gamma} |g(x) - g(0)| \leq \delta_0$$

we can apply Lemma 5.2 to u . Then we find a linear polynomial $P_1(x) = A_1 \cdot x + B_1$, with $A_1 \in \mathbb{R}^n$, $B_1 \in \mathbb{R}$, and $|A_1| + |B_1| \leq C_0$, such that

$$|u(x) - P_1(x)| \leq \lambda^{1+\alpha} \quad \text{for all } x \in B_\lambda.$$

Assume the claim holds for $k \geq 1$. Define

$$w(x) = \frac{u(\lambda^k x) - P_k(\lambda^k x)}{\lambda^{k(1+\alpha)}} \quad \text{for } x \in B_1.$$

Then, for any $\varphi \in C_c^\infty(B_1)$,

$$\Delta w(\varphi) = \frac{\Delta(u(\lambda^k x))(\varphi)}{\lambda^{k(1+\alpha)}} = \int_{\Gamma_{\lambda^k}} \frac{g(\lambda^k x)}{\lambda^{k\alpha}} \varphi(x) \, dH^{n-1}.$$

Also, for any $x \in \Gamma_{\lambda^k}$,

$$\frac{|g(\lambda^k x)|}{\lambda^{k\alpha}} = \frac{|g(\lambda^k x) - g(0)|}{\lambda^{k\alpha}} \leq [g]_{C^{0,\alpha}(0)} \leq \delta_0.$$

Then the claim follows for $k+1$, by applying, once again, Lemma 5.2. □

6. Proof of Theorem 1.1

To prove Theorem 1.1 we need Campanato's characterization of $C^{1,\alpha}$ spaces [4] and a technical result that patches the interior and boundary estimates together. We believe that the latter belongs to the folklore (see, for example, [13]) but, for the sake of completeness, we will give a proof.

Theorem 6.1. (Campanato) *Let u be a measurable function defined on a bounded $C^{1,\alpha}$ domain Ω . Then $u \in C^{1,\alpha}(\overline{\Omega})$ if and only if there exists $C_0 > 0$ such that for any $x \in \overline{\Omega}$, there exists a linear polynomial $Q_x(z)$ such that*

$$|u(z) - Q_x(z)| \leq C_0 |x - z|^{1+\alpha}$$

for all $z \in B_1(x) \cap \Omega$. In this case, if C_ denotes the least constant $C_0 > 0$ for which the property above holds, then*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \sim C_* + \sup_{x \in \overline{\Omega}} |Q_x|,$$

where $|Q_x|$ denotes the sum of the coefficients of the polynomial $Q_x(z)$.

Proposition 6.2. *Let S be a collection of measurable functions defined on a bounded $C^{1,\alpha}$ domain Ω . For $x \in \Omega$, we let $d_x = \text{dist}(x, \partial\Omega)$. Fix $u \in S$, and suppose the following hold.*

- (i) (Interior estimates). *There exist $A, C, D > 0$ such that for any $x \in \Omega$ there exists a linear polynomial $P_x(z)$ such that*

$$\|P_x\|_{L^\infty(B)} + d_x \|\nabla P_x\|_{L^\infty(B)} \leq C \|u\|_{L^\infty(B)}$$

and

$$|u(z) - P_x(z)| \leq \left(A \frac{\|u\|_{L^\infty(B)}}{d_x^{1+\alpha}} + D \right) |z - x|^{1+\alpha}$$

for all $z \in B \equiv B_{d_x/2}(x)$.

(ii) (Boundary estimates). There exists $E > 0$ such that for any $y \in \partial\Omega$, there is a linear polynomial $P_y(z)$ such that

$$\|P_y\|_{L^\infty(\Omega)} + \|\nabla P_y\|_{L^\infty(\Omega)} \leq E$$

and

$$|u(z) - P_y(z)| \leq E|z - y|^{1+\alpha}$$

for all $z \in \overline{\Omega}$.

(iii) (Invariance property). For any $u \in S$, and any $y \in \partial\Omega$, with corresponding linear polynomial P_y as in (ii), the function $v = u - P_y$ also satisfies the estimates of (i).

Then $S \subset C^{1,\alpha}(\overline{\Omega})$, and there exists $M > 0$, depending only on A, C, D, E such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq M\|u\|_{L^\infty(\Omega)}.$$

Proof. We need to show that any $u \in S$ satisfies the Campanato characterization from Theorem 6.1. Let us pick any point $x \in \overline{\Omega}$. If $x \in \partial\Omega$ then the polynomial $Q_x(z) \equiv P_x(z)$, where $P_x(z)$ is as in assumption (ii), satisfies the Campanato condition with $C_0 = E$.

Suppose next that $x \in \Omega$. Let $y \in \partial\Omega$ be a boundary point that realizes the distance from x to the boundary, namely, $d_x = |x - y|$. Let $P_y(z)$ be the linear polynomial that satisfies (ii). Consider the function $v(z) = u(z) - P_y(z)$. By (iii), there is a linear polynomial $P_x(z)$ such that the conditions in (i) are met for v in place of u . We claim that the polynomial Q_x for the Campanato condition is

$$Q_x(z) \equiv P_y(z) + P_x(z).$$

To show this, we split the argument into two cases.

Case 1. Suppose that $|z - x| < d_x/2$. This is the case when we can apply (i) for $v - P_x$:

$$\begin{aligned} |u(z) - Q_x(z)| &= |u(z) - P_y(z) - P_x(z)| = |v(z) - P_x(z)| \\ &\leq \left(A \frac{\|v\|_{L^\infty(B_{d_x/2}(x))}}{d_x^{1+\alpha}} + D \right) |z - x|^{1+\alpha} \\ &= \left(A \frac{\|u - P_y\|_{L^\infty(B_{d_x/2}(x))}}{d_x^{1+\alpha}} + D \right) |z - x|^{1+\alpha}. \end{aligned}$$

Now, we notice that, by (ii), by the choice of y , and the fact that $|z - x| < d_x/2$,

$$|u(z) - P_y(z)| \leq E|z - y|^{1+\alpha} \leq E(3/2d_x)^{1+\alpha} \leq 2^{1+\alpha} E d_x^{1+\alpha}.$$

Hence,

$$|u(z) - Q_x(z)| \leq (2^{1+\alpha} A E + D) |z - x|^{1+\alpha}$$

and $C_0 = 2^{1+\alpha} A E + D$.

Case 2. Suppose that $|z - x| \geq d_x/2$. By the estimate in (i) for $P_x(z)$, we get

$$\begin{aligned} |P_x(z)| &= |P_x(x) + \nabla P_x(x) \cdot (z - x)| \\ &\leq C\|u - P_y\|_{L^\infty(B)} + Cd_x^{-1}\|u - P_y\|_{L^\infty(B)}|z - x|. \end{aligned}$$

Also, by the boundary estimate in (ii),

$$\|u - P_y\|_{L^\infty(B)} \leq (3/2)^{1+\alpha} Ed_x^{1+\alpha}.$$

Hence,

$$\begin{aligned} |u(z) - Q_x(z)| &\leq |u(z) - P_y(z)| + |P_x(z)| \\ &\leq E|z - y|^{1+\alpha} + C\|u - P_y\|_{L^\infty(B)} + Cd_x^{-1}\|u - P_y\|_{L^\infty(B)}|z - x| \\ &\leq 3^{1+\alpha}E|z - x|^{1+\alpha} + 3^{1+\alpha}CEd_x^{1+\alpha} + Cd_x^{-1}(3/2)^{1+\alpha}Ed_x^{1+\alpha}|z - x| \\ &\leq 3^{1+\alpha}E(1 + 2C)|z - x|^{1+\alpha}. \end{aligned}$$

Thus, in this case, the Campanato constant is $C_0 = 3^{1+\alpha}E(1 + 2C)$. $\square$

Proof of Theorem 1.1. Let $u \in \text{LogLip}(\overline{\Omega})$ be the solution given by Theorem 2.2. We will show the statement for the function $u_2 : \overline{\Omega}_2 \rightarrow \mathbb{R}$, and we can argue similarly for $u_1 : \overline{\Omega}_1 \rightarrow \mathbb{R}$.

The following holds:

- (i) (*Interior estimates*). For any $x \in \Omega_2$, there exists a linear polynomial $P_x(z)$ such that

$$\|P_x\|_{L^\infty(B)} + d_x\|\nabla P_x\|_{L^\infty(B)} \leq (1 + 2n)\|u_2\|_{L^\infty(B)}$$

and

$$|u_2(z) - P_x(z)| \leq 2^{\alpha-1}n \frac{\|u\|_{L^\infty(B)}}{d_x^{1+\alpha}}|z - x|^{1+\alpha}$$

for all $z \in B \equiv B_{d_x/2}(x)$.

Indeed, fix $x \in \Omega_2$. Since u_2 is harmonic, it is smooth in Ω_2 , so we can define

$$P_x(z) = u_2(x) + \nabla u_2(x) \cdot (z - x).$$

Then, by classical interior estimates for harmonic functions,

$$\begin{aligned} \|P_x\|_{L^\infty(B)} + d_x\|\nabla P_x\|_{L^\infty(B)} &\leq \|u_2\|_{L^\infty(B)} + d_x\|\nabla u_2\|_{L^\infty(B)} + d_x\|\nabla u_2\|_{L^\infty(B)} \\ &\leq \|u_2\|_{L^\infty(B)} + 2n\|u_2\|_{L^\infty(B)} \\ &\leq (1 + 2n)\|\nabla u_2\|_{L^\infty(B)}. \end{aligned}$$

Moreover,

$$\begin{aligned} |u_2(z) - P_x(z)| &\leq \|D^2 u_2\|_{L^\infty(B)}|z - x|^2 \\ &\leq n \frac{\|u_2\|_{L^\infty(B)}}{d_x^2}|z - x|^2 \leq 2^{\alpha-1}n \frac{\|u_2\|_{L^\infty(B)}}{d_x^{1+\alpha}}|z - x|^{1+\alpha}. \end{aligned}$$

(ii) (*Boundary estimates*). Consider $\partial\Omega_2 = \Gamma \cup \partial\Omega$.

If $y \in \Gamma$, by Theorem 1.2, there exists a linear polynomial $P_y(z)$ such that

$$\|P_y\|_{L^\infty(\Omega_2)} + \|\nabla P_y\|_{L^\infty(\Omega_2)} \leq E$$

and

$$|u_2(z) - P_y(z)| \leq E|z - y|^{1+\alpha}$$

for all $z \in \overline{\Omega}_2$, with $E \leq C_0\|\psi\|_{C^{1,\alpha}(B'_1)}\|g\|_{C^{0,\alpha}(\Gamma)}$, and $C_0 = C_0(n, \alpha) > 0$. If $y \in \partial\Omega \in C^\infty$, since $u_2 = 0$, then, by classical boundary regularity for harmonic functions, $u_2 \in C^{1,\alpha}(\overline{B} \cap \Omega)$, with $B \equiv B_r(y)$, for some $r > 0$ sufficiently small. By Theorem 6.1, there exists a linear polynomial $P_y(z)$ such that

$$|u_2(z) - P_y(z)| \leq C_0|z - y|^{1+\alpha}$$

for all $z \in \overline{\Omega}_2$, for some $C_0(n, \alpha) > 0$.

(iii) (*Invariance property*). Fix $y \in \partial\Omega_2$, and let P_y be the corresponding linear polynomial given in (ii). Clearly, the function $v = u_2 - P_y$ is harmonic in Ω_2 , so it satisfies the interior estimates in (i).

Therefore, by Theorem 6.2, we have $u_2 \in C^{1,\alpha}(\overline{\Omega}_2)$, and there exists a constant $C > 0$, depending only on n, α and Γ such that $\|u_2\|_{C^{1,\alpha}(\overline{\Omega}_2)} \leq C\|g\|_{C^{0,\alpha}(\Gamma)}$. $\square$

Acknowledgements. We would like to thank the referees for some very valuable remarks that helped us improve the presentation of this paper.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix

A special Lipschitz domain Ω in $\mathbb{R}^n$ is a set of the form

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > \psi(x')\},$$

where $\psi \in \text{Lip}(\mathbb{R}^{n-1})$, that is, there exists $M > 0$ such that

$$|\psi(x') - \psi(y')| \leq M|x' - y'| \quad \text{for all } x', y' \in \mathbb{R}^{n-1}.$$

In other words, Ω is the set of points lying above the graph of a Lipschitz function ψ . Then, by Rademacher's Theorem, ψ is Fréchet differentiable almost everywhere with $\|\nabla\psi\|_{L^\infty(\mathbb{R}^{n-1})} \leq M$. On $\partial\Omega$ we thus have

$$dH^{n-1}|_{\partial\Omega} = \sqrt{1 + |\nabla\psi(x')|^2} dx' \quad \text{and} \quad \nu(x', \psi(x')) = \frac{(\nabla\psi(x'), -1)}{\sqrt{1 + |\nabla\psi(x')|^2}},$$

where $x = (x', \psi(x')) \in \partial\Omega$. For a measurable function f on $\partial\Omega$, we have

$$\int_{\partial\Omega} f(x) \, dH^{n-1} = \int_{\mathbb{R}^{n-1}} f(x', \psi(x')) \sqrt{1 + |\nabla \psi(x')|^2} \, dx'.$$

For more details see [6, 11].

A bounded Lipschitz domain in $\mathbb{R}^n$ is a bounded domain Ω such that the boundary $\partial\Omega$ can be covered by finitely many open balls B_j in $\mathbb{R}^n$, $j = 1, \dots, J$, centered at $\partial\Omega$, such that

$$B_j \cap \Omega = B_j \cap \Omega_j, \quad j = 1, \dots, J,$$

where Ω_j are rotations of suitable special Lipschitz domains given by Lipschitz functions ψ_j . One may then assume that $\partial\Omega \cap B_j$ can be represented in local coordinates by $x_n = \psi_j(x')$, where ψ_j is a Lipschitz function on $\mathbb{R}^{n-1}$ with $\psi_j(0') = 0$. Recall also that if ψ is a Lipschitz function defined on an set $A \subset \mathbb{R}^{n-1}$, with Lipschitz constant M , then there exists an extension $\bar{\psi} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ of ψ such that $\bar{\psi} = \psi$ on A and the Lipschitz constant of $\bar{\psi}$ does not exceed M , see [6].

Let $\Omega_0 = \Omega \cap (\bigcup_{j=1}^J B_j)^c$. A partition of unity $\{\xi_j\}_{j=0}^J$ subordinated to $\{\Omega_0, B_1, \dots, B_J\}$ is a family of nonnegative smooth functions ξ_j on $\mathbb{R}^n$ such that

$$\xi_0 \in C_c^\infty(\Omega_0) \quad \xi_j \in C_c^\infty(B_j), \quad j = 1, \dots, J \quad \text{and} \quad \sum_{j=0}^J \xi_j(x) = 1 \quad \text{for all } x \in \bar{\Omega}.$$

It follows that $0 \leq \xi_j \leq 1$, $j = 0, 1, \dots, J$. Obviously the family $\{\xi_j\}_{j=1}^J$ is a partition of unity subordinated to the open cover $\{B_1, \dots, B_J\}$ of $\partial\Omega$ and $\sum_{j=1}^J \xi_j(x) = 1$ for every $x \in \partial\Omega$.

Let $f : \Gamma \rightarrow \mathbb{R}$ be a measurable function, where $\Gamma = \partial\Omega$ is the boundary of a bounded Lipschitz domain Ω . Consider the balls B_j , $j = 1, \dots, J$, that cover Γ as above, and the corresponding Lipschitz functions $\psi_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. Let $\{\xi_j\}_{j=1}^J$ be a smooth partition of unity subordinated to the open cover $\{B_j\}_{j=1}^J$ of Γ . Then

$$\int_{\Gamma} f \, dH^{n-1} = \sum_{j=1}^J \int_{\Gamma} \xi_j f \, dH^{n-1} = \sum_{j=1}^J \int_{B_j \cap \Gamma} \xi_j f \, dH^{n-1}.$$

Let us consider each one of the terms in the sum above separately. We study the following situation: let B be a ball and let $\bar{f} : B \cap \Gamma \rightarrow \mathbb{R}$ of compact support in $B \cap \Gamma$. Let $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ be a Lipschitz function such that $\psi(B'_1) = B \cap \Gamma$. Then, by extending trivially $\bar{f}$ to the rest of the graph of ψ and using the coarea formula [6, 11],

$$\begin{aligned} \int_{B \cap \Gamma} \bar{f} \, dH^{n-1} &= \int_{\psi(B'_1)} \bar{f} \, dH^{n-1} = \int_{\psi(\mathbb{R}^{n-1})} \bar{f} \, dH^{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \bar{f}(y', \psi(y')) \sqrt{1 + |\nabla \psi(y')|^2} \, dy' \\ &= \int_{B'_1} \bar{f}(y', \psi(y')) \sqrt{1 + |\nabla \psi(y')|^2} \, dy'. \end{aligned}$$

Remark. The contents of this work are part of the second author's PhD dissertation. She presented these results at the AMS Fall Central Sectional Meeting at University of Michigan, Ann Arbor (Oct. 2018), the Barcelona Analysis Conference at Universitat de Barcelona (Jun. 2019), and the Midwest Geometry Conference at Iowa State University (Sep. 2019).

References

1. BLANK, I., HAO, Z.: The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators. *Commun. Anal. Geom.* **23**, 129–158, 2015
2. CAFFARELLI, L.A.: Elliptic second order equations. *Rend. Sem. Mat. Fis. Milano* **58**, 253–284, 1988
3. CAMPANATO, S.: Sul problema di M. Picone relativo all'equilibrio di un corpo elastico incastrato. *Ricerche Mat.* **6**, 125–149, 1957
4. CAMPANATO, S.: Proprietà di hölderianità di alcune classi di funzioni. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **17**, 175–188, 1963
5. CITTI, G., FERRARI, F.: A sharp regularity result of solutions of a transmission problem. *Proc. Amer. Math. Soc.* **140**, 615–620, 2012
6. EVANS, L.C., GARIEPI, R.F.: *Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics*. CRC Press, Boca Raton 1992
7. LADYZHENSKAYA, O.A., URAL'TSEVA, N.N.: *Linear and Quasilinear Elliptic Equations*. Academic Press, New York 1968
8. LI, Y., NIRENBERG, L.: Estimates for elliptic systems from composite material. Dedicated to the memory of Jürgen K. Moser. *Commun. Pure Appl. Math.* **56**, 892–925, 2003
9. LI, Y., VOGELIUS, M.: Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. *Arch. Ration. Mech. Anal.* **153**, 91–151, 2000
10. LIONS, J.L.: Contributions à un problème de M. M. Picone. *Ann. Mat. Pura Appl.* **4**(41), 201–219, 1956
11. MAGGI, F.: *Sets of Finite Perimeter and Geometric Variational Problems. An Introduction to Geometric Measure Theory, Cambridge Studies in Advanced Mathematics*, vol. 135. Cambridge University Press, Cambridge 2012
12. MATEU, J., OROBITG, J., VERDERA, J.: Extra cancellation of even Calderón–Zygmund operators and quasiconformal mappings. *J. Math. Pures Appl. (9)* **91**, 402–431, 2009
13. MILAKIS, E., SILVESTRE, L.E.: Regularity for fully nonlinear elliptic equations with Neumann boundary data. *Commun. Partial Differ. Equ.* **31**, 1227–1252, 2006
14. PICONE, M.: *Sur un problème nouveau pour l'équation linéaire aux dérivées partielles de la théorie mathématique classique de l'élasticité*. Colloque sur les équations aux dérivées partielles, Bruxelles May 1954
15. SCHECHTER, M.: A generalization of the problem of transmission. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **14**, 207–236, 1960
16. STAMPACCHIA, G.: Su un problema relativo alle equazioni di tipo ellittico del secondo ordine. *Ricerche Mat.* **5**, 3–24, 1956

L.A. CAFFARELLI, M. SORIA-CARRO
Department of Mathematics,
The University of Texas at Austin,
Austin
TX
78712 USA.
e-mail: caffarel@math.utexas.edu
e-mail: maria.soriac@math.utexas.edu

and

P. R. STINGA
Department of Mathematics,
Iowa State University,
396 Carver Hall,
Ames
IA
50011 USA.
e-mail: stinga@iastate.edu

(Received April 15, 2020 / Accepted January 28, 2021)

Published online February 19, 2021

© The Author(s), under exclusive licence to Springer-Verlag GmbH, DE part of Springer
Nature (2021)