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We dedicate this article to Nina Ural′tseva,
a great mathematician and wonderful person

THE LEADING EDGE OF A FREE BOUNDARY INTERACTING WITH A LINE OF FAST DIFFUSION

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The goal of this work is to explain an unexpected feature of the expanding level sets of the solutions of a system where a half-plane in which reaction-diffusion phenomena take place exchanges mass with a line having a large diffusion of its own. The system was proposed by H. Berestycki, L. Rossi and the second author as a model of enhancement of biological invasions by a line of fast diffusion. It was observed numerically by A.-C. Coulon that the leading edge of the front, rather than being located on the line, was in the lower half-plane.

We explain this behavior for a closely related free boundary problem. We construct travelling waves for this problem, and the analysis of their free boundary near the line confirms the predictions of the numerical simulations.

§1. Introduction

1.1. Model and question. Consider the cylinder

$$\Sigma = \{(x, y) \in \mathbb{R} \times (-L, 0)\}.$$

We look for a real $c > 0$, a function $u(x)$ defined for $x \in \mathbb{R}$, a function $v(x, y)$ defined in Σ, and a curve $\Gamma \subset \Sigma$ such that

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Ключевые слова: expanding level sets, reaction-diffusion phenomena, line of fast diffusion.

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$$
\begin{cases}
-d\Delta v + c\partial_x v = 0 & (x,y) \in \{v > 0\}, \\
|\nabla v| = 1 & ((x,y) \in \Gamma := \Sigma \cap \partial\{v > 0\}, \\
-Du_{xx} + c\partial_x u + 1/\mu u - v = 0 & \text{for } x \in \mathbb{R},\ y = 0, \\
v_y = \mu u - v & \text{for } x \in \mathbb{R},\ y = 0 \text{ and } v(x,0) > 0, \\
u_y(x,-L) = 0, & \\
u(-\infty) = 1/\mu,\ u(+\infty) = 0, & v(-\infty, y) = 1,\ v(+\infty, y) = 0.
\end{cases}
\tag{1.1}
$$

In (1.1), the real numbers μ, d, D are fixed positive constants, and the problem inside Σ is a well-known free boundary problem. We will also consider then the following more compact problem, with unknowns (c, Γ, u), the function u being this time defined in $\overline{\Sigma}$, solving

$$
\begin{cases}
-d\Delta u + c\partial_x u = & 0 \quad (x,y) \in \{u > 0\}, \\
|\nabla u| = & 1 \quad ((x,y) \in \Gamma := \Sigma \cap \partial\{u > 0\}, \\
-Du_{xx} + c\partial_x u + 1/\mu u_y = & 0 \quad \text{for } x \in \mathbb{R},\ y = 0, \\
u_y(x,-L) = & 0, \\
u(-\infty, y) = & 1, \quad u(+\infty, y) = 0.
\end{cases}
\tag{1.2}
$$

In both problems (1.1) and (1.2), we will see that $v_x \leqslant 0$ (respectively, $u_x \leqslant 0$) inside Σ, and that the free boundary Γ inside Σ will be an analytic curve. Assume that Γ intersects the line $\{y = 0\}$, say at $(x,y) = (0,0)$. We ask for the behavior of φ near $y = 0$.

1.2. Motivation. Our starting point is the system proposed by H. Berestycki, L. Rossi, and the second author to model the speed-up of biological invasions by lines of fast diffusion in [4]. In this model, the two-dimensional lower half-plane ("the field") in which reaction-diffusion phenomena occur interacts with the x axis ("the road"), which has a much faster diffusion D of its own. It will sometimes be useful to assume that $D \geqslant d$, but not always. Call $u(t,x)$ the density of individuals on the road, and $v(t,x,y)$ the density of individuals in the field. The road yields the fraction μu to the field, and retrieves the fraction νv in exchange; the converse process occurs for the field. The system for u and v is

$$
\begin{aligned}
\partial_t u - D\partial_{xx} u = &\ \nu v(t,x,0) - \mu u, && x \in \mathbb{R}, \\
\partial_t v - d\Delta v = &\ f(v), && (x,y) \in \mathbb{R} \times \mathbb{R}_-, \\
\partial_y v(t,x,0) = &\ \mu u(t,x,t) - \nu v(t,x,0), && x \in \mathbb{R}.
\end{aligned}
\tag{1.3}
$$

Here f is the usual logistic term, $f(v) = v - v^2$. A model involving only the unknown u can be obtained by forcing the (biologically reasonable) formula $\phi(x) = \psi(x,0)$; in other words, we take the (formal) limit $\delta \to 0$ of $\nu = \mu = \frac{1}{\delta}$.

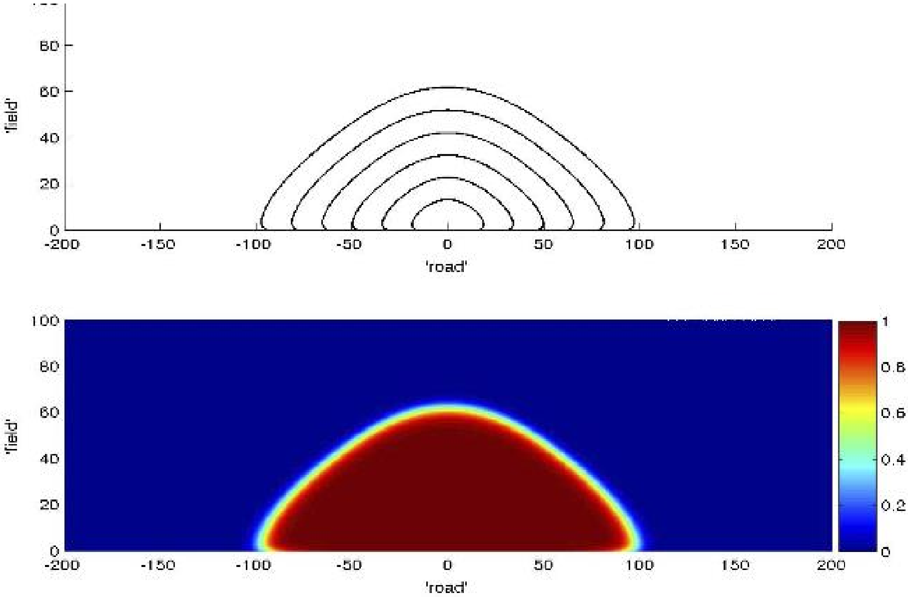
Still arguing in a formal way, we obtain $u = v$ on the road, and the exchange term is simply v_y. Thus, we obtain

$$\begin{aligned} \partial_t v - D\partial_{xx} v + v_y(t,x,0) = &\quad 0, &\qquad x \in \mathbb{R}, \\ \partial_t v - d\Delta v = &\quad f(v), &\qquad (x,y) \in \mathbb{R} \times \mathbb{R}_-. \end{aligned} \tag{1.4}$$

From now on, as is rather intuitively clear from the biological modelling, we will call (1.1) the two-species model, whereas problem (1.2) will be the single-species model. For the time being, let us only argue on system (1.3). The first question is how the stable state $(\nu/\mu, 1)$ invades the unstable state $(0,0)$. In [5] it was computed with $o_{t\to+\infty}(1)$ precision: for each direction e in the field, the level sets of v move with a velocity $w_*(e)$ which, quite surprisingly, does not obey the Huygens principle. The next step is to describe the asymptotic level sets with $O_{t\to+\infty}(1)$ precision; for this purpose numerical simulations were carried out by A.-C. Coulon, the simulations, which are part of a larger program of her thesis [8]. The above figures account for some of her results; the parameters are

$$f(v) = v - v^2, \quad D = 10, \quad u(0,x) = \mathbf{1}_{[-1,1]}(x), \quad v(0,x,y) \equiv 0.$$

The top figure represents the level set 0.5 of v at times 10, 20, 30, 40; the bottom figure represents the shape of $v(40,x,y)$. Notice that the simulations are carried out with and upward propagation instead of a downward propagation in our equations.



We have found these figures surprising, all the more as they are quite robust with respect to all the parameters. Indeed, a naive intuition would suggest that the leading edge of the invasion is located on the road, especially for large D. Such is manifestly not the case, the leading edge appears to be located in the field, at a distance to the road that seems to remain more or less constant in time.

A heuristic explanation is the following: the term $v(t,x,0)-\mu u(t,x)$ acts as an effective reaction term for u; given that, everything suggests that the invasion is driven by the road, especially for the large values of D. The immediate consequence is that $v_y(t,x,0)<0$, so the function v increases in the vicinity of the road, hence the observed behavior. The goal of this paper is to give a mathematically rigorous account of that fact.

1.3. From reaction-diffusion to free boundaries. Working directly on (1.3) or (1.4) to explain the above simulations did not lead to a conclusive answer. So, in order to understand the model a little better, we have chosen here to work on a limiting model, which will keep the main features of (1.3) or (1.4), this is why we have come up with (1.1) and (1.2). This system is indeed a limiting model of (1.3), let us explain why. Before that, we consider a permanent regime $(\partial_t u, \partial_t v) = (0,0)$ of (1.3), it is logical to think that it will take the form of a travelling wave

$$(u(t,x), v(t,x,y)) = (\phi(x+ct), \psi(x+ct,y)), \quad c>0. \tag{1.5}$$

It was proved in [10] that travelling waves are indeed attracting for (1.2) (and, more interestingly, the paper describes how the convergence occurs when D is very large). We have considered a propagation from right to left, thus the couple (ϕ,ψ) is an orbit of the following system

$$\begin{aligned} -D\partial_{xx}\phi + c\phi_x =&\ \nu\psi(x,0) - \mu\phi, && x\in\mathbb{R},\\ -d\Delta\psi + c\psi_x =&\ f(\psi), && (x,y)\in\Sigma,\\ \partial_y\psi(x,0) =&\ \mu\phi(x) - \nu\psi(x,0), && x\in\mathbb{R}, \end{aligned} \tag{1.6}$$

where we have omitted the Neumann condition for short. As for the model (1.4), the corresponding system is (we look for $v(t,x,y)$ under the form $\psi(x+ct,y)$):

$$\begin{aligned} -D\partial_{xx}\psi + c\phi_x + \psi_y =&\ 0, && x\in\mathbb{R},\\ -d\Delta\psi + c\psi_x =&\ f(\psi), && (x,y)\in\Sigma. \end{aligned} \tag{1.7}$$

A free boundary problem is obtained (once again in a formal way) in the limit as $\varepsilon\to 0$ of a sequence of solutions $(u_\varepsilon, v_\varepsilon)$ to (1.6) with $f=f_\varepsilon$, an approximation of the Dirac mass $\delta_{\psi=1}$. The ultimate result, which we will explain in this section, is the following. First, denote Γ the unknown boundary

$$\Gamma = \partial\big(\{\psi = 1\}\big).$$

To retrieve (1.1), we set (the functions u and v are not the same as in the time-dependent problem):

$$u(x) := 1/\mu - \phi(x),\ v(x,y) := 1 - \psi(x,y).$$

To retrieve (1.2), it suffices to set $u(x,y) := 1 - \psi(x,y)$ in (1.7).

Let us explain further, in an informal way, why a free boundary will develop in the plane. This is by no means a new idea, the introduction of reaction terms leading to free boundaries dates back to Zeldovich [14]. They have turned out to be very important to study asymptotic models in flame propagation, because they introduce different length scales that have the ability to explain subtle effects. Let us for instance quote the pioneering work of Sivashinsky [13] on the destabilisation of planar solutions under the effect of different diffusions. Actually, as these terms have become a paradigm in the theoretical study of flame models, they have given raise to a large amount of works that we will not be able to quote in detail here.

The first rigorous passage from the reaction-diffusion equation to the free boundary problem was done in the context of 1D travelling waves, by Berestycki, Nicolaenko, and Scheurer [3]. In several space dimensions, it was done by Berestycki, Nirenberg, and the first author (see [2]), a work that we will much use here. As far as evolution problems are concerned, the first rigorous passage to the limit was performed in a work of Vazquez and the first author [7], which raises difficulties due to the fact that the boundary condition is not of kinetic charcter, contrary to what happens in Stefan-type problems.

Now, we recall why a limiting solution (c,u,v) to (2.3) will develop a free boundary. For this, we consider the simple one-dimensional problem

$$\begin{array}{rl} -du'' + cu' + f_\varepsilon(u) = & 0 \text{ on } \mathbb{R}, \\ u(-\infty) = 0,\ u(+\infty) & = 1. \end{array} \tag{1.8}$$

Standard arguments show that, for a solution $(c_\varepsilon, u_\varepsilon)$ to (1.8), we have $c_\varepsilon > 0$ and $u'_\varepsilon < 0$. We may always assume that $u_\varepsilon(0) = \varepsilon$, so, for $x < 0$ we have $u_\varepsilon(x) = 1 - (1-\varepsilon)e^{cx}$. As for $x > 0$, we set

$$\xi = \frac{x}{\varepsilon}, \quad u_\varepsilon(x) = \varepsilon p_\varepsilon\Big(\frac{x}{\varepsilon}\Big).$$

The function p_ε solves

$$-dp'' + \varepsilon c_\varepsilon p' + p\varphi(p) = 0 \text{ on } \mathbb{R}_+, \quad p(0) = 1,\ p(+\infty) = 0.$$

Once again, standard arguments show that the term $\varepsilon c_\varepsilon p'$ may be neglected, so that multiplication by p' and integration over $\mathbb{R}_+$ yields

$$d\frac{(p'_\varepsilon)^2(0)}{2} = \int_0^{+\infty} p\varphi(p)dp = \frac{1}{2d}.$$

Scaling back, we obtain $u_\varepsilon(0) \sim 1$, matching derivatives yields $c = 1$. The limit (c, u) of $(u_\varepsilon, c_\varepsilon)$ satisfies therefore

$$-du'' + cu' = 0 \text{ on } \mathbb{R}_-, \quad d[u'](0) = 1.$$

This is the one-dimensional version of the problem inside Σ; here, Γ is the point $x = 0$.

1.4. Results. The question is therefore whether, and how, the free boundary Γ meets the fixed boundary $\{y = 0\}$. For the two unknowns problem (1.1), the scaling — both inside the domain and at the vicinity of the hitting point — is Lipschitz, which allows the use of a large body of existing ideas. On the other hand, Model (1.2) is not of the standard type, because the characteristic scales around the free boundary are different inside Σ and on the fixed boundary: in the latter we have $y \sim x^2$. However, before studying the free boundary Γ of Problem (1.2) near the origin, we first should make sure that the problem has a solution. Clearly, the issue is what happens in the vicinity of the axis $\{y = 0\}$, therefore we start with a situation for which travelling waves are known to exist.

Theorem 1.1. *System* (1.2) *has a solution* (c, Γ, u, v). *We have* $c > 0$ *and* $\partial_x u \leqslant 0$ (*and* < 0 *to the left of* Γ). *The function* v *is globally Lipschitz*: $|\nabla v| \leqslant C$ *for some universal* C (*therefore, u is* $C^{1,1}$ *on the line*). *The free boundary* Γ *is a graph* $(\varphi(y), y)$, *and also an analytic curve in* $\{y < 0\}$. *Moreover, it intersects the x-axis.*

Remark 1. A word will have to be said about the analyticity of the free boundary. Even though we are working in two space dimensions, singularities may occur, see [11] for a classification of them. That we are able to eliminate them from the start simplifies our work very much.

Remark 2. The uniqueness (up to translations) of (c, Γ, u, v) is probably true. The only issue is to examine the behavior of two solutions whose contact point lies at the intersection of the x axis and their respective free boundaries. This will not be pursued here.

Next, we study the free boundary in a neighborhood of the origin. By translation invariace, we may choose $\varphi(0) = 0$.

Theorem 1.2. *The free boundary* Γ *hits the line* $\{y = 0\}$ *at a point where* $u > 0$, *in other words we have* $u(\varphi(0), 0) > 0$. *In a neighborhood of* $(0, 0)$, Γ *is a graph in the x variable* $y = \psi(x)$, $x < 0$, *and there is* $\gamma > 0$ *such that*:

$$\psi(x) = \gamma x + o_{x \to 0^-}(x). \tag{1.9}$$

Let us turn to the model with one species.

Theorem 1.3. *Assume that $D \geqslant d$. System* (1.2) *has a solution* (c, Γ, u). *We have $c > 0$ and $\partial_x u \leqslant 0$ (and < 0 to the left of Γ). The function u is globally Lipschitz*: $|\nabla u| \leqslant C$ *for some universal C. The free boundary Γ is a graph $(\varphi(y), y)$, and also an analytic curve in $\{y < 0\}$. Moreover, it intersects the x-axis, so we may choose $\varphi(0) = 0$.*

Remark 3. We do not know whether the assumption $D \geqslant d$ is indispensable, or merely technical. In any case, it is consistent with our goal to study phenomena driven by a large diffusion on the road. In [4], the significant threshold is $D = 2d$, this is where the velocity of the wave for (1.2) exceeds that of the plane Fisher-KPP wave.

Theorem 1.4. *Assume that $D \geqslant d$. In a neighborhood of* $(0, 0)$ Γ *is a graph in the x variable*:

$$y = \psi(x), \quad x < 0, \quad \text{with } \psi(x) = -\frac{x^2}{2D} + o_{x \to 0^-}(x^2). \tag{1.10}$$

1.5. Discussion, organisation of the paper. We first notice, as far as the two species are concerned, an interesting loss of boundary condition between the reaction-diffusion system (1.6) and the free boundary problem (1.2), to the right of the free boundary. In other words, in this area, the road does not exchange individuals anymore with the field. Thus there is an asymptotic decoupling between the value of u and the value of v at the free boundary, and this accounts quite well for the numerical simulations. Let us revert to the old unknowns ϕ and ψ, which denote respectively the density of individuals on the road and in the field. At the intersection between the invasion front and the road, the density of individuals is at its maximum, whereas the road keeps feeding the field with individuals, as is stated by the exchange condition

$$\psi_y \sim \mu\phi - \psi \sim \mu u(0, 0) > 0.$$

Therefore, the invasion front in the field can only go further, which explains that its leading edge is not located on the road. This explains the simulations in [8].

Next, we observe a different behavior for the one species model. A heuristic reason is that the model is trying to accomodate both the exchange condition and the free boundary condition, and this is only done at the expense of a breakdown of the homogeneity near the road. The resulting sitation is the interaction of an obstacle problem on the road and a solution of the one phase problem in the field. This observation will be investigated further in a subsequent work.

The last remark concerns the derivation of the one species model from the two species model, in the limit of infinite exchange terms $\mu = \nu = \frac{1}{\delta}$. This passage was done in a rigorous way in [9] in the framework of a reaction-diffusion system; it would probably be possible to do it, such a task would probably be not entirely trivial. We postpone this matter to a future work, as it would not really add more to the understanding of the question at stake here.

The paper is organised as follows. In §2, we construct a solution to system (1.1). This is done by a classical approximation by a family of semilinear equations where the nonlinearity $f_\varepsilon(u)$ converges, in the measure sense, to $\delta_{\partial_{u>0}}$. The idea, as well as the inspiration for the proof, is taken from [2]. In §3, we study the free boundary of the two species model, the main argument will be a Liouville type theorem for a special class of global solutions. In §4, we construct the travelling wave, and the main part of the analysis is the gradient bound for u. Finally, in §5, we prove Theorem 1.4. The chief argument in all these (at times technical) considerations is that the free boundary condition creates such a rigidity that, unless the solutions behave as they are expected to behave, basic properties such as, for instance, the positivity of u, will not hold true.

§2. The travelling wave in the two species model

Let us consider a smooth function $\varphi(u)$, defined on $\mathbb{R}_+$, positive on $[0,1)$, zero outside, and such that

$$\int_0^{+\infty} u\varphi(u)du = \frac{1}{2d}. \tag{2.1}$$

Consider the sequence of reaction terms

$$f_\varepsilon(u) = \frac{u}{\varepsilon^2}\varphi\Big(\frac{u}{\varepsilon}\Big). \tag{2.2}$$

We will obtain a solution to (1.1) as the limit, as $\varepsilon \to 0$, of a sequence $(c_e, u_\varepsilon, v_\varepsilon)_\varepsilon$ of solutions of

$$\begin{cases} -d\Delta v + c\partial_x v + f_\varepsilon(v) = & 0, \quad (x,y)\in\Sigma, \\ \\ -Du_{xx} + c\partial_x u + 1/\mu u - v = & 0, \quad \text{for } x\in\mathbb{R},\ y=0, \\ v_y = & \mu u - v \quad \text{for } x\in\mathbb{R},\ y=0, \\ v_y(x,-L) = & 0, \\ \\ v(-\infty,y)=1,\ v(+\infty,y) = & 0, \quad u(-\infty)=1/\mu,\ u(+\infty)=0. \end{cases} \tag{2.3}$$

Let us come back to (2.3). For every $\varepsilon > 0$, (2.3) has (Dietrich [9, Theorem 1]) a unique solution $(c_\varepsilon, u_\varepsilon)$ such that $c_\varepsilon > 0$, $0 < u_\varepsilon < 1$, and $\partial_x u_\varepsilon, \partial_x v_\varepsilon < 0$; we will show that, up to a subsequence, $(c_\varepsilon, u_\varepsilon)$ converges, as $\varepsilon \to 0$, to a solution of (1.2).

2.1. Basic bounds. The first task is to show that the travelling wave velocity is uniformly bounded from above, that is, to prove the following.

Proposition 2.1. *There is $K > 0$ independent of ε such that $c_\varepsilon \leqslant K$.*

Proof. We may, even if it means translating u_ε, assume the normalisation condition

$$\min_{[-L,0]} v_\varepsilon(0, y) = \varepsilon. \tag{2.4}$$

Therefore, $f_\varepsilon(u) \equiv 0$ on $\mathbb{R}_- \times [-L, 0]$, and u_ε solves a purely linear equation. If

$$\rho \leqslant \min\Big(\frac{1}{D}, \frac{1}{d}\Big)c,$$

then

$$(\underline{u}(x), \underline{v}(x, y)) := \big(1 - (1-\varepsilon)e^{\rho x}\big)\Big(\frac{1}{\mu}, 1\Big)$$

is a subsolution to (1.2) on $\mathbb{R}_-$, thus $u(x, y) \geqslant \underline{u}(x)$ on $\mathbb{R}_-$. Choosing ρ to be the above minimum, we obtain

$$\partial_x v(0, y_e) \leqslant -(1-\varepsilon)\min\Big(\frac{1}{D}, \frac{1}{d}\Big)c,$$

where y_ε is a point in $[-L, 0]$ where the minimum in (2.4) is attained.

On the right half of Σ, another simple subsolution to (1.2) is

$$(\underline{u}_\varepsilon(x), \underline{v}_\varepsilon(x, y)) = \varepsilon p\Big(\frac{x}{\varepsilon}\Big)\Big(\frac{1}{\mu}, 1\Big),$$

with

$$-p'' + p\varphi(p) = 0 \text{ for } \xi > 0, \quad p(0) = 1,\ p(+\infty) = 0.$$

Since it is convex, it is a subsolution to the equation for v, and since it is monotone decreasing, it is also a subsolution of the equation inside the right half of Σ. The exchange condition for v is automatically satisfied. The classical sliding argument (slide $(u_\varepsilon, v_\varepsilon)$ until the two components exceed $(\underline{u}_\varepsilon, v_\varepsilon)$, then slide back until one of the two components reaches a contact point) yields $(u_\varepsilon, v_\varepsilon) \geqslant (\underline{u}_\varepsilon, v_\varepsilon)$. Thus

$$\partial_x v_\varepsilon(0, y_\varepsilon) \geqslant \underline{v}_\varepsilon'(0) = p'(0) = -\sqrt{2}.$$

This implies

$$(1-\varepsilon)\min\Big(\frac{1}{D}, \frac{1}{d}\Big)c \leqslant \sqrt{2},$$

the required bound. □

The next step is a uniform gradient bound on v_ε. Notice that u_ε satisfies a linear ODE with bounded right-hand side, thus is uniformly $C^{1,1}$ at this stage.

Proposition 2.2. *There is a universal $M > 0$ such that $|\nabla v_\varepsilon| \leqslant M$ in $\overline{\Sigma}$.*

For convenience, in the sequel we will drop the subscript $_\varepsilon$ for c_ε, u_ε and v_ε. The first ingredient is a gradient bound in $\{v \leqslant \varepsilon\}$, away from the road.

Lemma 2.3. *Consider $\lambda \in (0,1)$ and a point $(x_0, x_0) \in \overline{\Sigma}$ such that $u(x_0, y_0) = \lambda\varepsilon$. Assume that $y_0 \leqslant -2\varepsilon$. Then we have*

$$0 \leqslant u(x,y) \leqslant C\varepsilon, \quad (x,y) \in B_\varepsilon(x_0, y_0),$$

and

$$|\nabla u(x_0, y_0)| \leqslant C,$$

for some universal $C > 0$.

Proof. Notice, following [2], that the case of y_0 close to $-L$ is not really an issue, because one may, thanks to the Neumann condition at $y = -L$, extend the function u evenly in y to $\mathbb{R} \times (-2L, 0)$. This said, we do the classical Lipschitz scaling

$$v(x_0 + \varepsilon\xi, \varepsilon\zeta) = \varepsilon p(\xi, \zeta), \quad (\xi, \zeta) \in B_2(0), \tag{2.5}$$

and p solves

$$-d\Delta p + \varepsilon c \partial_\xi p + p\varphi(p) = 0 \quad (\xi, \zeta) \in B_2(0),\ p(0,0) = \lambda.$$

Then (see [2] again), from the Harnack inequality, p is universally controlled in $B_1(0)$, hence $\nabla u(x_0, y_0) = \nabla p(0,0)$ is universally controlled. ☐

Lemma 2.4. *Consider $\lambda \in (0,1)$, $y_0 \in [-\varepsilon, 0]$, and $x_0 \in \mathbb{R}$ such that*

$$u(x_0, y_0) = \lambda\varepsilon.$$

Then we have

$$0 \leqslant u(x,y) \leqslant C\varepsilon\lambda, \quad (x,y) \in B_\varepsilon(x_0, y_0),\ y \leqslant 0,$$

for some $C > 0$ that depends neither on ε, nor on λ.

Proof. Recall that u and v are bounded independently of ε. We redo the scaling (2.5), leaving u untouched. The only thing that has to be examined is the Neumann condition for p, which reads

$$p_\zeta(\xi, 0) + \varepsilon p(\xi, \zeta) = u(x_0 + \varepsilon\xi, 0). \tag{2.6}$$

We make the slight abuse of notations consisting in denoting by $u(\xi, \zeta)$ the function $u(x_0 + \varepsilon\xi, y_0 + \varepsilon\zeta)$, this function is clearly $C^{1,\alpha}$ uniformly in ε. We may indeed subtract from p any suitable harmonic function V satisfying the Neumann condition (2.6), thus V is uniformly $C^{2,\alpha}$ in $B_{3/4}$. We then apply the argument of Lemma 2.4 to $v - V$. ☐

These two lemmas lead to an effortless

Proof of Proposition 2.2. Let Γ_ε be the curve $\{v = \varepsilon\}$. The function v_y is bounded on Γ_ε, as well as on $\partial\Sigma\backslash\Gamma_\varepsilon$, moreover it satisfies $-\Delta y_y + c\partial_x v_y = 0$ to the left of Γ_ε, thus it is bounded everywhere. The same argument applies to v_x, but for this quantity we use the Robin condition $d\partial_y v_x + v_x = u_x$ (recall that u_x is uniformly bounded) and the Neumann conditon for v_x at the bottom of Σ. □

2.2. Convergence of the approximating sequence. The last ingredient that we need is a uniform lower bound on the travelling vave velocity. In this proposition we put the subscripts $_\varepsilon$ back in.

Proposition 2.5. *There is $c_0 > 0$ such that $c_\varepsilon \geqslant c_0$.*

Proof. We start with the following identity

$$c_\varepsilon = \frac{1}{L + d\mu} \int\limits_\Sigma f_\varepsilon(v_\varepsilon)\, dxdy, \tag{2.7}$$

obtained by integrating the system for $(u_\varepsilon, v_\varepsilon)$ over Σ. We may normalise v_ε so that

$$v_\varepsilon\Big(0, -\frac{L}{4}\Big) = \varepsilon.$$

From [2], the family of measures

$$\sigma_\varepsilon = \mathbf{1}_{B_{L/4}}\Big(0, -\frac{L}{4}\Big) f_\varepsilon(v_\varepsilon)\, dxdy$$

converges (possibly up to a subsequence), in the measure sense, to the image of the Lebesgue measure on a locally BV graph $\{h(y), y)\}$. Thus the sequence $\big((L+d\mu)c_\varepsilon\big)_\varepsilon$ converges, still up to a subsequence, to a limit that is larger than the length of Γ inside $B_{L/4}(0, -\frac{L}{4})$, a positive number. □

Putting everything together, we may prove the Theorem 1.1.

Proof of Theorem 1.1. Possibly up to a subsequence, the sequence $(u_\varepsilon, v_\varepsilon)_\varepsilon$ converges uniformly on $\overline{\Sigma}$, and in $H^1_{\rm loc}(\Sigma)$ weakly, to a function $(u(x), v(x, y))$ that is both Lipschitz and in $H^1_{\rm loc}(\Sigma)$. Notice that u is much smoother, it is $C^{2,1}$. Let us repeat here that the family of measures $(f_\varepsilon(v_\varepsilon)\, dxdy)_\varepsilon$ converges (see [2] again), in every set of the form $\mathbb{R} \times K$ (K is a compact subset of $[-L, 0)$), to the length measure of the graph

$$\Gamma = \{(h(y), y) \quad - L \leqslant y < 0\}.$$

For every $\delta \in (0, \frac{L}{4})$, identity (2.7) implies

$$\int_{\mathbb{R}\times[-L,-\delta]} f_\varepsilon(u)\,dxdy \leqslant (L+d\mu)c_\varepsilon. \tag{2.8}$$

Passing to the limit $\varepsilon \to 0$ yields

$$\int_{-L}^{-\delta} \sqrt{1+(h'(y))^2}dy \leqslant (L+d\mu)c. \tag{2.9}$$

Thus, the function h belongs to $BV([-L,0))$, thus can be extended by continuity to $y=0$. We may always assume that $h(0)=0$. It remains to prove that $\partial\{u>0\}$ is analytic. Let us set

$$F_\varepsilon(v) = \int_0^v f_\varepsilon(v)dv,$$

we claim that, for every $\varepsilon>0$, v_ε is a local minimiser of the energy

$$\int e^{-c_\varepsilon x}\Big(\frac{1}{2}|\nabla v|^2 + F_\varepsilon(v)\Big)\,dxdy.$$

More precisely, for every ball B whose closure is included in Σ, then v_ε minimises the energy

$$J_\varepsilon(\phi,B) = \int_B e^{-c_\varepsilon x}\Big(\frac{1}{2}|\nabla\phi|^2 + F_\varepsilon(\phi)\Big), \tag{2.10}$$

over all functions $\phi\in H^1(B)$ whose trace on ∂B is v_ε. This is easily seen from the monotonicity of v_ε in x, and a sliding argument. On the other hand, we use the two following facts, taken from [2]: first, the family $(v_\varepsilon)_\varepsilon$ is compact in $H^1_{\rm loc}(\Sigma)$, (ii). there is a uniform nondegeneracy property:

$$v_\varepsilon(x,y) \geqslant kd\big((x,y),\{v_\varepsilon\in(a\varepsilon,b\varepsilon)\}\big).$$

This implies that, for every ball B inside Σ we have:

$$\lim_{\varepsilon\to0} J_\varepsilon(v_\varepsilon,B) = \int_B \frac{e^{-cx}}{2}|\nabla v|^2\,dxdy + \int_{\{v>0\}\cap B} e^{-cx}\,dxdy := J(v,B).$$

On the other hand, for every $\phi\in H^1(B)$ whose trace on B is v, consider $\phi_\varepsilon\in H^1(B)$ whose trace on B is v_ε and such that the family $(\phi_\varepsilon)_\varepsilon$ converges to ϕ in $H^1(B)$. We have $J_\varepsilon(v_\varepsilon,B)\leqslant J_\varepsilon(\phi,B)$, which implies, sending $\varepsilon\to0$: $J(v,B)\leqslant J(\phi,B)$. Thus, v is a local minimizer of J. However, J is of the type of functionals treated in the paper [1]: its results are applicable, which imples the analyticity of $\partial\{v > 0\}$ inside Σ.

Finally, since v is Lipschitz, and because of the uniform convergence of $(v_\varepsilon)_\varepsilon$ to v, we have $v(x, 0) = 0$ if $x > 0$. □

Remark. The above argument also explains the loss of the exchange condition for v to the right of the free boundary, simply because the free boundary is forced to hit the road. There is a boundary layer in which the condition $v = 0$, located on a curve very close to the road, eventually overcomes the exchange condition in the limit $\varepsilon \to 0$.

§3. The two species model: the free boundary near (0,0)

The main feature of Model (1.1) is that the equation inside Σ, together with the free boundary conditions, can be studied in a neighborhood of a free boundary point up to the top of Σ *via* Lipschitz rescalings: $(x, y) = \delta(\xi, \zeta)$. This will enable us to show, in a relatively easy way, the linear behavior of the free boundary in the vicinity of the road. Let us first state a rigidity result in 2 space dimensions.

Theorem 3.1. *Pick* $\lambda \in [0, 1)$. *Let* $u(x, y)$ *solve*

$$\begin{array}{rll} \Delta u = & 0, & (x,y) \in \mathbb{R} \times \mathbb{R}_- \cap \{u > 0\}, \\ |\nabla u| = & 1, & (x,y) \in \partial\{u > 0\}, \\ u_y(x,0) = & \lambda & \text{if } x < 0, \\ u(x,0) = & 0 & \text{if } x > 0. \end{array} \tag{3.1}$$

Also assume that $\partial_x u \leqslant 0$. *Then we have*:

$$\partial\{u > 0\} = \{y = -\frac{\sqrt{1-\lambda^2}}{\lambda} x, \ x < 0\},$$

and

$$u(x, y) = \left(\sqrt{1-\lambda^2} x + \lambda y\right)^+ .$$

Proof. Let Ω be the positivity set of u, identify $\mathbb{R}^2$ with the complex plane, and set

$$f(z) = u_y(x, y) + i u_x(x, y), \quad z = x + iy.$$

Then f is analytic in Ω, let us assume that it is a nonconstant function. It is open from Ω onto its image, and maps Γ onto (a portion of) the unit circle, which we call γ_1, whereas it maps the negative x axis onto (a portion of) the vertical line $\{\mathrm{Im} Z = \lambda\}$, which we call γ_2. From its connectedness, $\gamma_1 \cup \gamma_2$ (but not only) enclose $f(\Omega)$. And, since $\partial_x u \leqslant 0$, $f(\Omega)$ is bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$, where γ_3 is a nonvoid, possibly very irregular curve. One of its end-points is on γ_1 — call it z_1; thus γ_1 is the segment $[z_1, \sqrt{1-\lambda^2} + i\lambda]$ — and the other on γ_2

— call it z_2. It stays in the upper half of the complex plane, and is constructed as follows: for each ray D starting from $\sqrt{1-\lambda^2}+i\lambda$ and not meeting $\gamma_1 \cup \gamma_2$, let Z_D be the furthest point from $\sqrt{1-\lambda^2}+i\lambda$ in $\overline{f(\Omega)}$: the set of all such Z_D's determines γ_3. Two cases are to be considered.

Case 1. *γ_3 has points strictly above the horizontal axis.* Let D be a ray starting from $\sqrt{1-\lambda^2}+i\lambda$, and let Z_0 be the furthest point in $D \cap \gamma_3$. If Z_0 has a preimage by f, we have a contradiction because f is open. If Z_0 has no preimage, this means the existence of a sequence $(z_n)_n$ such that

$$\lim_{n\to+\infty} |z_n| = 0 \text{ or } \lim_{n\to+\infty} |z_n| = +\infty,$$

and such that $\lim_{n\to+\infty} f(z_n) = Z_0$. The sequence of locally uniformly bounded functions

$$f_n(z) = f(\frac{z}{|z_n|}) \quad \text{if } |z_n| \to 0, \tag{3.2}$$

or

$$f_n(z) = f(z|z_n|) \quad \text{if } |z_n| \to +\infty, \tag{3.3}$$

converges, up to a subsequence, uniformly locally to an analytic function $f_\infty(z)$, by Montel's theorem. The function f_∞ takes the value Z_0, but it is also non-constant analytic, in a possibly different positivity set Ω_∞. However, by definition, Z_0 is still the furthest point between D and $f_\infty(\Omega_\infty)$, and the openness of f_∞ yields a contradiction. Thus f is constant, and so is ∇u.

Case 2. *γ_3 meets the horizontal axis.* Let $\mu \in [0,1]$ belong to γ_3. As above, there is a sequence $(z_n)_n$ whose modulus goes to 0 or $+\infty$, such that

$$\lim_{n\to+\infty} f(z_n) = \mu.$$

Let f_n be defined as in (3.2) or (3.3), and let f_∞ be one of its limits. By nondegeneracy (i.e., linear growth from each point of the free boundary, which is, in this case, inherited from its analyticity), there is a nontrivial solution u_∞ of (3.1), and a point (x_0, y_0) in the positivity set of u_∞ such that

$$\partial_x u_\infty(x_0, y_0) = 0.$$

Since $\partial_x u_\infty \leqslant 0$, it is zero everywhere. The only possibility for Γ_∞ is that it is horizontal, a contradiction with the fact that it has to meet the x axis. □

In particular, for $\lambda = 0$ we have:

$$\partial\{u > 0\} = \{y = 0\}, \quad u(x,y) = y^-.$$

The same argument as in Theorem 3.1 yields the following statement.

Corollary 3.2. *Pick $\lambda \in [0,1]$, and consider a solution $(\varphi(y), u(x,y))$, with $\varphi(y) < 0$ and $\partial_x u \leqslant 0$, of*

$$\begin{aligned} \Delta u = &\ 0,\ u > 0 \quad (x < \varphi(y)),\\ |\nabla u| = &\ 1 \quad (x = \varphi(y)),\\ u_y(x,0) = &\ \lambda \quad (x \in \mathbb{R}). \end{aligned} \tag{3.4}$$

Then φ is constant negative, and $\lambda = 1$.

As a consequence of these rigidity properties for 2D global solutions, we prove Theorem 1.2.

Proof of Theorem 1.2. Let (c, u, v, Γ) be the solution of (1.1). We claim that Γ hits the road at a point that we may assume, by translation invariance, to be $(0,0)$. Let us prove that $u(x_0) > 0$. Indeed, suppose the contrary and suppose that $u(x_0) = 0$. Consider a Lipschitz blow-up of v around $(x_0, 0)$:

$$v_\delta(\xi, \zeta) = \frac{v(\delta\xi, \delta\zeta)}{\delta}. \tag{3.5}$$

Call Γ_δ its free boundary, let us first show that, as $\delta \to 0$, it does not collapse on the x axis. In other words, we want to show the existence of $\rho_0 > 0$ such that, if Γ_δ is the positivity set of u_δ within $B_1(0)$, then

$$B_{\rho_0}(-1, -2\rho_0) \subset \Omega_\delta.$$

Assume that it is not the case, then we have $\lim_{\delta\to 0} v_\delta(\xi, 0) = 0$ uniformly on every compact in ξ, simply because v is Lipschitz. And so, by the exchange condition, we have $\lim_{\delta\to 0} \partial_y v_\delta(-1, 0) = 0$. On the other hand, let ζ_δ be the smallest $\zeta < 0$ such that $(-1, \zeta) \in \Gamma_\delta$. Clearly, ζ_δ tends to 0 as $\delta \to 0$. Rescale v_δ with $|\zeta_\delta|$, taking $(-1, \zeta_\delta)$ as the origin. That is, we set

$$v_\delta(\xi, \zeta) = |\zeta_\delta| V_\delta\left(\frac{\xi + 1}{|\zeta_\delta|}, \frac{\zeta - \zeta_\delta}{|\zeta_\delta|}\right).$$

(A subsequence of) the sequence $(V_\delta)_\delta$ will converge, as $\delta \to 0$, to a solution V_∞ of (3.4), with $\lambda = 0$, something that Corollary 3.2 excludes. As $\delta \to 0$, (a subsequence of) $(v_\delta)_\delta$ converges locally uniformly to a global solution V_∞ of (3.1), with $\lambda = 0$. By Theorem 3.1, we have $V_\infty(\xi, \zeta) = \xi^-$. Thus, the equation for u in a neighborhood of $x = x_0$ is

$$\begin{aligned} -Du'' + cu' + \mu u \sim x_0 - x, \quad x < 0,\\ u(0) = u'(0) = 0. \end{aligned}$$

This implies $u(x) \sim \frac{x^3}{6D}$ for $x < 0$ close to 0. Thus $u(x) < 0$ in a neighborhood of 0, a contradiction with the positivity of u. So, $u(0) > 0$, let us set

$$\lambda = \mu u(0) > 0.$$

We come back to the blow-up (3.5). Each converging subsequence has a solution of (3.1) as a limit, which is unique by virtue of Theorem 3.1. So, the whole blow-up converges to

$$V_\infty(x) = \big(\lambda y - \sqrt{1-\lambda^2}x\big)^+,$$

which implies, we come back to the solution (c, u, v, Γ) of (1.1), that Γ has slope $\gamma = \frac{\sqrt{1-\lambda^2}}{\lambda}$ at $(0,0)$. This proves Theorem 1.2. □

§4. The travelling wave of the single species model

As in §2, we will construct a solution to (1.2) as the limit, as $\varepsilon \to 0$, of a sequence $(c_e, u_\varepsilon)_\varepsilon$ of solutions of

$$\begin{cases} -d\Delta u + c\partial_x u + f_\varepsilon(u) = & 0, \quad (x,y) \in \Sigma, \\ -Du_{xx} + c\partial_x u + 1/\mu u_y = & 0 \quad \text{for } x \in \mathbb{R},\ y = 0, \\ u_y(x, -L) = & 0, \\ u(-\infty, y) = & 1, \quad u(+\infty, y) = 0, \end{cases} \tag{4.1}$$

the function f_ε being as in (2.2). For every $\varepsilon > 0$, (4.1) has (Dietrich [9] once again) a unique solution $(c_\varepsilon, u_\varepsilon)$ such that $c_\varepsilon > 0$, $0 < u_\varepsilon < 1$, and $\partial_x u_\varepsilon < 0$; we will show that, up to a subsequence, $(c_\varepsilon, u_\varepsilon)$ converges, as $\varepsilon \to 0$, to a solution of (1.2). Our first task is to show that c_ε is uniformly bounded from above, that is, to veryfy the following.

Proposition 4.1. *There is $K > 0$ independent of ε such that $c_\varepsilon \leqslant K$.*

Proof. The proof is similar to that of Proposition 2.1. In $\mathbb{R}_- \times [-L, 0]$, we work with the subsolution

$$\underline{u}(x) := 1 - (1-\varepsilon)e^{\rho x}$$

with

$$\rho \leqslant \min\Big(\frac{1}{D}, \frac{1}{d}\Big)c,$$

whereas, to the right of Σ, we work with

$$\underline{u}_\varepsilon(x) = \varepsilon p\Big(\frac{x}{\varepsilon}\Big),$$

with

$$-p'' + p\varphi(p) = 0 \text{ for } \xi > 0, \quad p(0) = 1,\ p(+\infty) = 0.$$

Since it is convex, it is a subsolution to the Wentzell boundary condition on $\mathbb{R}_+$, and since it is monotone decreasing, it is also a subsolution of the equation inside the right half of Σ. □

The next step is a uniform gradient bound on u_ε.

Theorem 4.2. *Assume that* $D \geqslant d$. *There is a universal* $M > 0$ *such that* $\|\nabla u_\varepsilon\|_\infty \leqslant M$.

This does not result from the application of known theorems, and will be rather involved. So, we first assume that it holds, and finish the construction of the travelling wave of (1.2).

4.1. Construction of a solution to the one species model, given Theorem 4.2. Let us first state an equivalent of Proposition 2.5

Proposition 4.3. *There is* $c_0 > 0$ *such that* $c_\varepsilon \geqslant c_0$.

Proof of Theorem 1.3. Arguing as in the proof of Theorem 1.1, we obtain a solution u to (1.2), at least for the free boundary problem inside, with a free boundary $(h(y), y)$ that meets the top of Σ at a point that we may assume to be 0. So, we have $u(x, 0) = 0$ if $x > 0$.

To the left of $\Gamma \cap \Sigma$ we have $u > 0$. We infer that $u(x, 0) > 0$ if $x < 0$. Indeed, assume that $u(x, 0) = 0$ for $x < \bar{x} < 0$, and that $u(x, 0) > 0$ if $x > \bar{x}$. As a consequence of the definition of Γ, there is $r \in (0, |\bar{x}|)$ such that

$$u > 0 \text{ in } B_r(\bar{x} - r, 0) \cap \{y < 0\}.$$

Wet come back to u_ε, the situation is a follows:

$$\begin{aligned} -d\Delta u_\varepsilon + c_\varepsilon \partial_x u_\varepsilon &= 0, \quad \bar{x} - r \leqslant x \leqslant \bar{x},\ -r < y < 0;\\ -D\partial_{xx} u_\varepsilon + c_\varepsilon \partial_x u_\varepsilon + \partial_y u_\varepsilon &= 0, \quad \bar{x} - r \leqslant x \leqslant \bar{x};\\ u_\varepsilon(x, 0) &= o_{\varepsilon\to 0}(1), \quad \bar{x} - r \leqslant x \leqslant \bar{x};\\ u_\varepsilon(x, y) \geqslant \lambda_r |y| &\text{ for some } \lambda_r > 0, \end{aligned} \tag{4.2}$$

the last inequality being valid because of the Hopf lemma for u applied at every point of the road between $(\bar{x} - r, 0)$ and $(\bar{x}, 0)$ on the one hand, and the fact, on the other hand, that the convergence of u_ε to u is better than uniform in $\{\bar{x} - r \leqslant x \leqslant \bar{x},\ -r \leqslant y \leqslant 0\}$ — the Schauder estimates being valid in this area. As a consequence, we have

$$\partial_y u_\varepsilon(x, 0) \leqslant -\lambda_r \text{ for } \bar{x} - r \leqslant x \leqslant \bar{x},\ -r \leqslant y < 0. \tag{4.3}$$

Since $u_x(\bar{x} - r, 0) = 0$, we have $u_\varepsilon(\bar{x} - r, 0) = o_{\varepsilon\to 0}(1)$. Integrating the Wentzell condition for u_ε between $\bar{x} - r$ and $\bar{x}$ and taking (4.3) into account, we obtain, when $\varepsilon > 0$ is sufficiently small:

$$u_\varepsilon(\bar{x}, 0) < 0,$$

a contradiction.

Let us finally show that u solves the full free boundary problem (1.2). From identity (2.7) and the boundedness of the sequence $(c_\varepsilon)_\varepsilon$, we deduce that $(f_\varepsilon(u_\varepsilon)\, dxdy)_\varepsilon$ converges (still up to a subsequence) to the length measure on Γ, plus a possible finite measure on the boundary $\mathbb{R}\times\{y=0\}$. We note that $u(x,0)$ is at least $C^{1,1}$: indeed, $\partial_y u_\varepsilon$ is uniformly bounded, due to Theorem 4.2. Thus $\partial_{xx}u_\varepsilon(\,\cdot\,,0)$ is also uniformly bounded, a property that passes on to $\partial_{xx}u$. So, the contribution of the limiting measure on $\mathbb{R}\times\{y=0\}$ is nonexistent, which proves that the Wentzell condition is satisfied both a.e. and in the distributional sense. The rest of the system is proved to be solved precisely as in Theorem 1.1. □

4.2. Gradient bound in the region $u\sim\varepsilon$. The outline of the proof is still, roughly, that of the Lipschitz bound in [2]. However, here, the vicinity of the road requires a special treatment and this is where we will, eventually, need $D\geqslant d$. In the sequel we will, for convenience, drop the subscript $_\varepsilon$ for c_ε and u_ε. First, recall the gradient bound in $\{u\leqslant\varepsilon\}$, away from the road.

Lemma 4.4. *Consider $\lambda\in(0,1)$ and a point $(x_0,y_0)\in\overline{\Sigma}$ such that*

$$u(x_0,y_0)=\lambda\varepsilon.$$

Assume that $y_0\leqslant-2\varepsilon$. Then we have

$$0\leqslant u(x,y)\leqslant C\varepsilon,\quad (x,y)\in B_\varepsilon(x_0,y_0),$$

and $|\nabla u(x_0,y_0)|\leqslant C$, for some universal $C>0$.

Proof. Set

$$p(\xi,\zeta)=\varepsilon u(x_0+\varepsilon\xi,y_0+\varepsilon\zeta),\tag{4.4}$$

and apply [2]. □

The main part of the task is therefore to bound ∇u at distance less than ε from the road. We start with the most extreme case, i.e., a point on the road.

Lemma 4.5. *Consider $\lambda\in(0,1)$ and $x_0\in\mathbb{R}$ such that*

$$u(x_0,0)=\lambda\varepsilon.$$

Then we have

$$0\leqslant u(x,y)\leqslant C\varepsilon\lambda,\quad (x,y)\in B_\varepsilon(x_0,y_0),\ y\leqslant 0,$$

for some $C>0$ that depends neither on ε nor on λ.

Proof. This looks, at first sight, like an innocent repetition of the previous lemma. However one quickly realises that the Lipschitz scaling yields the boundary condition

$$-Dp_{\xi\xi}+\varepsilon cp_\xi+\varepsilon/\mu p_\zeta=0,$$

so that half of the Wentzell condition is lost when one sets $\varepsilon = 0$. Of course this implies that p is linear on the road, hence constant in order to keep its positivity. However it does not prevent large gradients in y: assume for definiteness that $\lambda = 1$; if $p(-1,0) = M >> 1$, then

$$q(\xi,\zeta) = \frac{p(\xi,\zeta)}{M},$$

and q solves the equation

$$-\Delta q + q\varphi(Mq) = 0$$

inside the cylinder, while keeping the condition $q_{\xi\xi} = 0$. So, it is an asymptotic global solution for infinite M, which implies that one cannot hope for a uniform bound for p in this setting.

So, this time we use the mixed Lipschitz-quadratic scaling

$$u(x_0 + \sqrt{\varepsilon}\xi, \varepsilon\zeta) = \varepsilon\lambda p(\xi,\zeta), \quad (\xi,\zeta) \in B_2(0),\ \zeta < 0 \tag{4.5}$$

and p solves the system

$$\begin{cases} -d\varepsilon p_{\xi\xi} - dp_{\zeta\zeta} + \sqrt{\varepsilon}c\partial_\xi p + p\varphi(\frac{p}{\lambda}) = & 0, \quad (\xi,\zeta) \in \cap\{\zeta < 0\}, \\ -Dp_{\xi\xi} + \sqrt{\varepsilon}c\partial_\xi p + 1/\mu p_\zeta = & 0 \quad \text{for } -2 < \xi < 2,\ \zeta = 0, \\ p(0,0) = & \lambda. \end{cases} \tag{4.6}$$

We claim that $p(0,-1)$ is universally bounded, both with respect to λ and ε. Call $M_{\lambda,\varepsilon}$ this quantity, we have

$$p(\xi,-1) \geqslant M_{\lambda,\varepsilon} \text{ for } \xi \leqslant 0.$$

Assume that a subsequence of $(M_{\lambda,\varepsilon})_{\lambda,\varepsilon}$ (that we still relabel $(M_{\lambda,\varepsilon})_{\lambda,\varepsilon}$) with $\varepsilon \to 0$ grows to infinity. We claim the existence of $\xi_{\lambda,\varepsilon} < 0$ going to 0 as $\varepsilon \to 0$ and such that

$$p(\xi_{\lambda,\varepsilon},) \geqslant kM_{\lambda,\varepsilon} \tag{4.7}$$

for some universal $k > 0$. If this is not the case, then there is a universal $\xi_0 < 0$ such that

$$\lim_{\varepsilon\to 0} \frac{p(\xi,0)}{M_{\lambda,\varepsilon}} = 0 \text{ uniformly in } \xi \in [\xi_0, 0].$$

This entails the existence of a universal constant $\gamma_0 > 0$ such that

$$p_\zeta(\xi,0) \leqslant -\gamma_0 M_{\lambda,\varepsilon}, \quad \xi \in [\xi_0 + \sqrt{\varepsilon}, -\sqrt{\varepsilon}]. \tag{4.8}$$

Indeed, we have $p(\xi,\zeta) \geqslant \underline{p}(\xi,\zeta)$, where

$$\begin{aligned} -d\varepsilon \underline{p}_{\xi\xi} - d\underline{p}_{\zeta\zeta} + \sqrt{\varepsilon}^{3/2} c\underline{p}_\xi + \underline{p} = & \ 0 \quad (\xi \in (\xi_0, 0) \times (-1,0)), \\ \underline{p}(\xi,0) = p(\xi,0),\ \underline{p}(\xi,-1) = & \ M_{\lambda_\varepsilon} \quad (\xi \in (\xi_0,0)), \\ \underline{p}(0,\zeta) = p(\xi_0,\zeta) = & \ 0 \quad (\zeta \in (-1,0)). \end{aligned}$$

Rescaling in ξ — so as to recover a fully elliptic equation for p — shows that $\underline{p}_\zeta(\xi,0)$ satisfies an estimate of the type (4.8).

Now, from Rolle's theorem, there is $\xi'_{\lambda,\varepsilon} \in (\frac{\xi_0}{2},\frac{\xi_0}{4})$ such that $p_\xi(\xi'_{\lambda,\varepsilon},0) = o(M_{\lambda,\varepsilon})$ (recall that p is $o(M_{\lambda,\varepsilon})$ on $(\xi_0+\varepsilon,-\varepsilon)$). So, on $(\frac{3\xi_0}{4},\xi'_{\lambda,\varepsilon})$ we have:

$$Dp_{\xi\xi}(\xi,0) - \varepsilon c p_\xi(\xi,0) \leqslant -\frac{\gamma_0 M_{\lambda,\varepsilon}}{\mu},$$
$$p_\xi(\xi'_{\varepsilon,\lambda},0) = o(M_{\lambda,\varepsilon}), \quad p(\xi'_{\varepsilon,\lambda}(\xi'_{\varepsilon,\lambda},0) = o(M_{\lambda,\varepsilon}).$$

Integration of this very simple differential inequality between $\xi'_{\varepsilon,\lambda}$ and $\frac{3\xi_0}{4}$ yields

$$p\Big(\frac{3\xi_0}{4},0\Big) = o(M_{\lambda,\varepsilon}) - \frac{1}{2}\gamma_0 M_{\varepsilon,\lambda}\Big(\frac{3\xi_0}{4}-\xi'_{\lambda,\varepsilon}\Big)^2 \leqslant O(1) - \frac{9\xi_0^2}{16} < 0,$$

a contradiction.

So, we have found $\xi_{\varepsilon,\lambda}$, going to 0 as $\varepsilon \to 0$ such that (4.7) holds true. Thus, there exists $\xi''_{\varepsilon,\lambda} \in (\xi'_{\varepsilon,\lambda},0)$ such that

$$p(\xi''_{\varepsilon,\lambda},0) \in (1,2), \quad \lim_{\varepsilon\to 0} p_\xi(\xi''_{\varepsilon,\lambda}0) := a_{\lambda,\varepsilon} = -\infty.$$

Now we notice that $p_\zeta(\xi,0)$ is less than some universal $C>0$, for $\xi \in (\xi''_{\varepsilon,\lambda},+\infty)$: indeed, set

$$\widetilde{p}(\xi) = \begin{cases} p(\xi''_{\varepsilon,\lambda},0) & \text{if } \xi \leqslant \xi''_{\varepsilon,\lambda}, \\ p(\xi,0) & \text{if } \xi \geqslant \xi''_{\varepsilon,\lambda}, \end{cases}$$

so that $\widetilde{p}(\xi) \leqslant C$ (universal) on $(\xi''_{\varepsilon,\lambda},1)$. We have $p(\xi,\zeta) \leqslant \underline{q}(\xi,\zeta)$, where

$$\begin{aligned} -d\varepsilon \underline{q}_{\xi\xi} - d\underline{q}_{\zeta\zeta} + \sqrt{\varepsilon}^{3/2} c\underline{q}_\xi + \underline{q} = {}& 0 \quad (\xi \in (-1,1)\times(-1,0)), \\ \underline{q}(\xi,0) = \widetilde{p}(\xi),\ \underline{p}(\xi,-1) = {}& 0 \quad (\xi \in (-1,1)), \\ \underline{p}(\pm 1,\zeta) = {}& 0 \quad (\zeta \in (-1,0)), \end{aligned}$$

which satisfies

$$\underline{q}_\zeta(\xi,0) \leqslant C,\ C>0 \text{ (universal)}.$$

So this time we have

$$Dp_{\xi\xi}(\xi,0) - \varepsilon c p_\xi(\xi,0) \leqslant -C \text{ on } (\xi''_{\varepsilon,\lambda},1),$$

while

$$p_\xi(\xi''_{\varepsilon,\lambda},0) \in (0,2], \quad p(\xi'_{\varepsilon,\lambda}(\xi'_{\varepsilon,\lambda},0) = a_{\lambda,\varepsilon} \to -\infty.$$

Again, integration of the differential inequality on $(\xi''_{\varepsilon,\lambda},1)$ yields the existence of $\xi'''_{\varepsilon,\lambda} \geqslant \xi''_{\varepsilon,\lambda}$ such that $p(\xi'''_{\varepsilon,\lambda},) < 0$, a contradiction. Thus our claim that $p(0,-1)$ is universally bounded is proved.

It remains to see that $p(0,\zeta)$ is universally bounded for $\zeta \in [-1,0]$. First, notice that we have in fact proved that

$$p(\xi,-1) \text{ is universally bounded for } -1 \leqslant \xi \leqslant 1.$$

We have also proved that

$$p_\zeta(\xi,-1) \leqslant C, \quad C>0 \text{ is universal.}$$

Having this remark at hand, we may repeat the above argument, replacing -1 by ζ. □

Instructed by Lemma 4.5, we may now bound u in an ε-neighborhood of the road.

Lemma 4.6. *Consider $\lambda \in (0,1)$ and $(x_0,y_0) \in \mathbb{R}\times(-\varepsilon,0)$ such that*

$$u(x_0,y_0) = \lambda\varepsilon.$$

Then we have

$$0 \leqslant u(x,y) \leqslant C\varepsilon\lambda, \quad (x,y) \in B_{2\varepsilon}(x_0,y_0),\ y \leqslant 0,$$

for some $C>0$ that depends neither on ε, nor on λ.

Proof. We start again the mixed scaling (4.5), so that p solves (4.6). Set

$$y_0 = \varepsilon\zeta_0, \text{ thus } u(x_0,y_0) = \varepsilon p(0,\zeta_0) = \varepsilon\lambda.$$

The only thing that we have to prove is that $p(0,0)$ is universally bounded from above and below. The case where

$$\lim_{\varepsilon\to 0} \frac{p(0,0)}{p(0,\zeta_0)} = 0$$

amounts to the previous lemma, so it only remains to exclude the case where

$$\lim_{\varepsilon\to 0} \frac{p(0,0)}{p(0,\zeta_0)} = +\infty. \tag{4.9}$$

This time it will be more useful to work in the Lipschitz scaling (4.4). Set

$$M_{\varepsilon,\lambda} = \frac{p(0,0)}{p(0,\zeta_0)}, \quad q(\xi,\zeta) = \frac{p(\xi,\zeta)}{M_{\varepsilon,\lambda}},$$

so that q solves the equation

$$-d\Delta q + \varepsilon c\partial_\xi q + q\varphi(M_{\varepsilon,\lambda} q) = 0 \quad (\xi,\zeta) \in B_2(0),\ q(0,0)=1.$$

Now, notice a very simple bound for ∇q: the supremum of f_ε being of order ε, the elliptic estimates [12] for the original Wentzell problem (4.1) yield

$$\|D^2 u\|_{W^{2,p}(B\cap\bar\Sigma)} \leqslant \frac{C_p}{\varepsilon},$$

where B is any closed ball of radius 1, and C_p depends on p but not on ε. Hence we have

$$\|\nabla u\|_\infty \leqslant \frac{C}{\varepsilon}, \tag{4.10}$$

$C > 0$ is universal. Hence the condition for q on the road becomes

$$Dq_{\xi\xi} = \varepsilon c q_\xi + \varepsilon/\mu q_\zeta = \frac{O(1)}{M_{\varepsilon,\lambda}} = o(1).$$

This implies, in order to keep the positivity of q, that $q_\xi(0,0)$ is universally bounded; as a result, there is $\gamma_0 > 0$ universal, and $\xi_0 > 0$ universal as well, such that

$$\gamma_0 \leqslant q(\xi, 0) \leqslant \frac{1}{\gamma_0}, \quad \xi \in (-\xi_0, \xi_0).$$

Thus $q(\xi, \zeta \geqslant \underline{q}(\xi,\zeta)$ with

$$\begin{aligned} -d\Delta \underline{q} + \varepsilon c \underline{q}_\xi + \underline{q} = &\ 0 \quad ((\xi,\zeta) \in (-\xi_0, \xi_0) \times (-1, 0)), \\ \underline{q}(\xi, 0) = \gamma_0,\ \underline{q}(\xi, -1) = &\ 0 \quad (\xi \in (-\xi_0, \xi_0)), \\ \underline{q}(\pm\xi_0, \zeta) = &\ 0 \quad (\zeta \in (-1, 0)) \end{aligned}$$

and the strong maximum principle implies the existence of a universal $\gamma_0' > 0$ such that $\underline{q}(0, \zeta_0) \geqslant \gamma_0'$, a contradiction. □

Putting Lemmas 4.5 and 4.6 together, we obtain the gradient bound in the region where $f_\varepsilon(u)$ is active.

Corollary 4.7. *There is $M > 0$ (universal) such that $|\nabla u(x, y)| \leqslant M$ if $u(x, y) \leqslant \varepsilon$.*

Proof. Consider $\lambda \in (0, 1)$ and a point $(x_0, x_0) \in \overline{\Sigma}$ such that $u(x_0, y_0) = \lambda\varepsilon$. By Lemma 4.4, it is sufficient to assume that $y_0 \geqslant -2\varepsilon$. From Lemmas 4.5 and 4.6 we have, in the Lipschitz scaling (4.4):

$$0 \leqslant p(\xi, \zeta) \leqslant C, \quad (\xi, \zeta) \in B_2(0) \cap \zeta < 0.$$

The Wentzell condition reads once again as

$$p_{\xi\xi}(\xi, 0) = o(1),\ -2 \leqslant \xi \leqslant 2,$$

so that $p_\xi(\xi, 0) = o(1)$, for $-2 \leqslant \xi \leqslant 0$, in order to keep the positivity of p. Extend $p(\xi, 0)$ in a $C^{1,1}$ fashion inside $[-2, 2] \times [-2, 0]$, call $\widetilde{p}(\xi, \zeta)$ this extension. Then $p - \widetilde{p}$ solves an elliptic equation with bounded right-hand side inside $[-2, 2] \times [-2, 0]$, while satisfying a Dirichlet condition on the road segment $(-2, 2)$. Hence, the classical local elliptic estimates yield

$$\|p\|_{L^r(B_1(0) \cap \{\zeta < 0\})} \leqslant C_r,\ C_p > 0 \text{ depending only on } r.$$

This implies, in turn, the uniform boundedness of $|\nabla p(0, \zeta_0)|$, hence the corollary. □

4.3. The gradient bound away from the region $u \sim \varepsilon$. Let us mention from the outright that it is the sole place where we will need $D \geqslant d$. Once again, we do not know if this is essential.

Proof Theorem 4.2. We set

$$\Omega_\varepsilon = \Sigma \backslash \{u \leqslant \varepsilon\}.$$

Since $\partial_x u < 0$, the set $\{u \leqslant \varepsilon\}$ sits on the right of the smooth graph

$$\Gamma_\varepsilon = \big\{(h_\varepsilon(y), y),\ y \in [-L, 0]\big\},$$

where h_ε is a smooth function (whose derivatives may nonetheless blow-up as $\varepsilon \to 0$). We may assume that $h_\varepsilon(0) = 0$, and we set $x_\varepsilon = h_\varepsilon(-L)$; thus, we have

$$\partial\Omega_\varepsilon = (-\infty, 0) \times \{0\} \cup \Gamma_\varepsilon \cup (-\infty, x_\varepsilon) \times \{-L\}.$$

First, we bound u_y from above. On Γ_ε we have $u_y \leqslant C$. On $(-\infty, x_\varepsilon) \times \{-L\}$ we have $u_y = 0$. So, let us find a differential inequality for u_y on $(-\infty, 0) \times \{0\}$. In Ω_ε we have

$$-Du_{xx} + cu_x = D\Big(-u_{xx} + \frac{c}{d}u_x\Big) + \Big(1 - \frac{D}{d}\Big)cu_x = Du_{yy} + \Big(1 - \frac{D}{d}\Big)cu_x \geqslant Du_{yy},$$

simply because $c > 0$, $u_x < 0$, and $D \geqslant d$. This inequality carries over to the boundary $(-\infty, x_\varepsilon) \times \{0\}$ to yield $D\partial_y u_y + u_y \leqslant 0$. We also have $-D\Delta u_y + c\partial_x u_y = 0$ in Ω_ε, therefore u_y can only assume a positive maximum on Γ_ε, where it is uniformly bounded from above. If not, it is bounded from above by 0.

Now, we bound u_x from below. We now know that $u_y \leqslant C$, so, using $u_x < 0$ we write

$$\begin{aligned} u_{xx}(x,0) - cu_x(x,0) &\leqslant \quad C \quad (x < 0) \\ u_x(0,0) &= \quad O(1) \quad \text{because of Corollary 4.7} \end{aligned}$$

Integrating this inequality backward, and using Proposition 4.3, we obtain $u_x(x, 0) \geqslant -C$, $C > 0$ is universal. On $(-\infty, x_\varepsilon) \times \{0\}$ we have $\partial_y u_x = 0$, and u_x bounded on Γ_ε. Thus, u_x is bounded from below in Ω_ε.

Finally, we may bound u_y from below: now we know the boundedness of u_x, so the Wentzell condition becomes

$$D\partial_y u_y(x, 0) + u_y(x, 0) = O(1).$$

On Γ_ε, u_y is still bounded, while it still satisfies the Dirichlet condition on $(-\infty, x_\varepsilon)$. The lower bound follows from the maximum principle. $\square$

§5. The free boundary near $(0,0)$ in the single species model

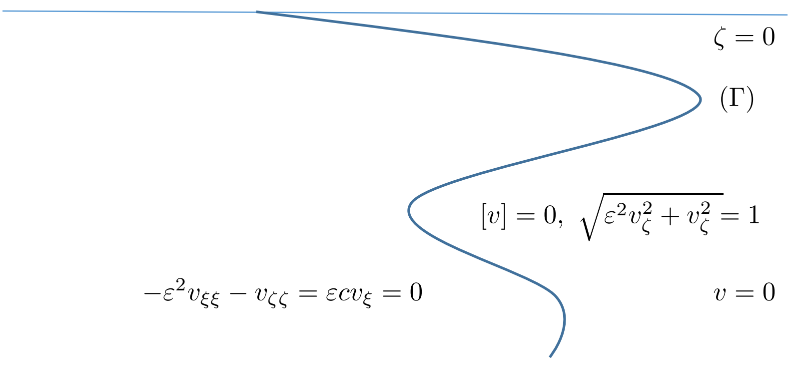
For the reader's convenience, we recall the system solved by (c,Γ,u):

$$\begin{cases} -d\Delta u + c\partial_x u = 0, & (x,y)\in\{u>0\},\\ u_\nu = 0, & (x,y)\in\Gamma,\\ -Du_{xx}+c\partial_x u + 1/\mu u_y = 0 & \text{for } x\in\mathbb{R},\ y=0,\\ u_y(x,-L) = 0, & \\ u(-\infty,y) = 1, & u(+\infty,y)=0. \end{cases} \tag{5.1}$$

Let us first explain the heuristics of Theorem 1.4. The starting point is a very simple global asymptotic solution, in the limit as $\varepsilon\to 0$ of the parabolic scaling:

$$u(x,y) = \varepsilon v\Big(\frac{x}{\sqrt{\varepsilon}},\frac{y}{\varepsilon}\Big). \tag{5.2}$$

In other words, we have the following situation:



Setting $\varepsilon = 0$ yields $v_{\zeta\zeta}\equiv 0$, hence $v(\xi,0)\equiv 1$ for $\xi<0$. So, we have $v(\xi,0) = \frac{\xi^2}{2D}$ and

$$v(\xi,\zeta) = \Big(\zeta + \frac{\xi^2}{2D}\Big)_+,$$

hence the free boundary has the form (1.10). So, we are going to make this rigorous in the remaining part of the section.

We may assume that Γ meets the axis $\{y=0\}$ at the point (0,0), and start the proof of Theorem 1.2. If $\bar X = (\bar x,\bar y)\in\Sigma$ is a free boundary point, and $\delta>0$ is so small that $B_{2\delta}(\bar X)\subset\Sigma$, rescale u at scale δ:

$$x-\bar x = \delta\xi,\ y-\bar y = \delta\zeta,\quad u(x,y) = \delta u_\delta(\xi,\zeta). \tag{5.3}$$

Let Γ_δ be the rescaled free boundary. As is well known, we have

$$\begin{aligned} -d\Delta u_\delta + \delta c\partial_\xi u_\delta &= 0 \quad (B_2(0)), \\ |\nabla u_\delta| &= 1 \quad (\Gamma_\delta), \\ u(0) &= 0. \end{aligned} \tag{5.4}$$

The uniform gradient bound comes from Theorem 1.1. We may therefore safely forget about u_ε, and concentrate on the solution $u(x,y)$ of (1.2). And so, in the sequel, we will use the letter ε for any small parameter, without any further reference to the approximate solution constructed in the preceding section.

This section more or less follows the organisation of §3: first, we will state a (more standard) rigidity property adapted to the needs of the asymptotic situations that we will encounter. We will then prove Theorem 1.4.

5.1. Another 2D rigidity result. The following proposition is a consequence of the 2D monotonicity formula.

Proposition 5.1. *Consider a solution $u(x,y)$ of*

$$\begin{aligned} \Delta u &= 0 \quad (\{u>0\}), \\ |\nabla u| &= 1 \quad (\partial\{u>0\}), \\ u(x,0) &= 0 \quad (x \in \mathbb{R}). \end{aligned} \tag{5.5}$$

Assume that $\partial\{u>0\}$ meets the x-axis at the point $(0,0)$, which is the only point where it may fail to be analytic. Then $u(x,y) = y^-$.

Proof. It is based on [6], Theorem 12.1. Set

$$u_1(x,y) = u(x,y), \quad u_2(x,y) = u(x,-y).$$

The functions u_1 and u_2 are harmonic in disjoint domains, and have a common zero line: the axis $\{y=0\}$. Thus ([6, Theorem 12.1]) the quantity

$$\frac{1}{R^4}\int_{B_R(0)} |\nabla u_1|^2\,dxdy \int_{B_R(0)} |\nabla u_2|^2\,dxdy$$

is monotone increasing in R. As a consequence, the quantity

$$J_u(R) = \frac{1}{R^2}\int_{B_R(0)} |\nabla u|^2\,dxdy$$

is also monotone increasing in R. Because of the gradient bound and nondegeneracy, the quantities $\lim_{R\to 0} J_u(R)$ and $\lim_{R\to+\infty} J_u(R)$ exist and are nonzero. Notice that J_u is invariant under Lipschitz scaling, that is,

$$J_u(R) = \int_{B_1(0)} |\nabla u_R|^2\,dxdy = J_{u_R}(1), \quad u_R(x,y) = \frac{1}{R}u(Rx,Ry).$$

The analyticity of the free boundary entails sufficient compactness to infer that the family $(u_R)_R$ converges in $H^1(B_1(0))$ as $R \to 0$ (respectively, $R \to +\infty$), possibly up to a subsequence, to a limit u^- (respectively, u^+). Another application of Theorem 12.1 of [6] to $u^\pm$ yields

$$J_{u^\pm}(r) = \text{Constant}, \quad \text{hence } J'_{u^\pm}(r) = 0.$$

Hence we have

$$\lim_{R\to 0} J'_u(R) = 0, \qquad \lim_{R\to+\infty} J'_u(R) = 0.$$

Following the proof of the theorem, we see that $\lim_{R\to 0} J'_u(R)$ (respectively, $\lim_{R\to+\infty} J'_u(R)$) is proportional to

$$\frac{\displaystyle\int_{\partial B_1(0)} |\nabla u^-|^2\, dxdy}{\displaystyle\int_{B_1(0)} |\nabla u^-|^2\, dxdy} - 2 \quad \left(\text{respectively, } \quad \frac{\displaystyle\int_{\partial B_1(0)} |\nabla u^+|^2\, dxdy}{\displaystyle\int_{B_1(0)} |\nabla u^+|^2\, dxdy} - 2\right).$$

However, still following the proof of the theorem, we find out that the above quantities are zero if and only if

$$u^\pm(y) = y^-;$$

actually the $u^\pm$ have to be proportional to y^-, but the free boundary relation imposes the proportionality coefficient. Thus J has the same limits at 0 and $+\infty$, thus J is constant. Thus $J'(R) \equiv 0$, and $u(x,y) = y^-$. This proves the proposition. □

5.2. Analysis of the free boundary. The first step is to show that the Lipschitz scale inside breaks down as the free boundary approaches the horizontal axis, and becomes flatter and flatter.

Lemma 5.2. *There is a universal $C > 0$ such that, for $\varepsilon > 0$ sufficiently small, we have*

$$\Gamma \cap B_\varepsilon(0) \subset (-\varepsilon, 0] \times (-C\varepsilon^2, 0].$$

Proof. Assume the lemma to be false, that is, there is a sequence $(x_\varepsilon, \delta_\varepsilon)$ with

$$\Gamma \cap \partial B_\varepsilon(0) = \{(x_\varepsilon, \delta_\varepsilon)\}$$

and

$$\lim_{\varepsilon\to 0} x_\varepsilon \sqrt{\delta_\varepsilon} = 0. \tag{5.6}$$

Drop the subscript $_\varepsilon$ and scale with δ as in (5.3), with $\bar{X} = (0,0)$. The Wentzell condition implies that

$$u_\delta(\xi, 0) = O(\delta\xi^2) \xrightarrow[\delta\to 0]{} 0.$$

The situation implies the existence of $\bar X_\delta$ in $B_1(0)\cap\{u_\delta>0\}$ whose distance to Γ_δ, as well as to $\{y=0,\}$, is universally controlled from below. From nondegeneracy, $u_\delta(\bar X_\delta)$ is universally controlled from below. We send δ to 0 and use the compactness of $(u_\delta)_\delta$ provided by the uniform gradient bound: we recover a function $u_\infty(\xi,\zeta)$, as well as an asymptotic smooth (possibly outside the origin) free boundary Γ_∞. Call Ω_∞ the area limited by $\{\zeta=0\}$ and Γ_∞, we have

$$-\Delta u_\infty = 0 \text{ in } \Omega_\infty, \quad u_\infty = 0 \text{ on } \partial\Omega_\infty. \tag{5.7}$$

Nondegeneracy implies that $u_\infty(\xi,-1)$ is uniformly controlled from below, which implies in turn

$$u_\zeta(\xi,0)\leqslant -2q$$

for some universal q. Returning to u_δ, we obtain the same kind of bound: for every $\rho>0$, there is $q_\rho>0$ such that

$$\partial_\zeta u_\delta(\xi,\zeta)\leqslant -q_\rho \text{ for small } \delta,\ -\xi_\delta<\xi\leqslant-\rho.$$

We have, because of (5.6):

$$\lim_{\delta\to0}\xi_\delta=+\infty.$$

The Wentzell condition at $\{\zeta=0\}$ implies

$$-D\partial_{\xi\xi}u_\delta+\delta c\partial_\xi u_\delta\geqslant q\delta(\mathbf{1}_{(-\xi_\delta,-\rho)}-C\mathbf{1}_{(-\rho,0)}),\ u_\delta(0,0)=\partial_\xi u_\delta(0,0)=0.$$

Integration of this inequality (this is by now a routine argument) yields

$$u_\delta(-\xi_\delta,0)\leqslant-\frac{q_\rho}{3}\delta<0,$$

the required contradiction. This proves the lemma. □

The next lemma shows that, if the free boundary wiggles before reaching the point (0,0), it does so in a controlled fashion.

Lemma 5.3. *For every $\varepsilon>0$, let $\zeta_\varepsilon^\pm$ be defined as*

$$\zeta_\varepsilon^-=\inf\{\zeta\in\mathbb{R}_+:(-\varepsilon,\zeta)\in\Gamma\},\quad \zeta_\varepsilon^+=\sup\{\zeta\in\mathbb{R}_+:(-\varepsilon,\zeta)\in\Gamma\}.$$

There is a universal $q\in(0,1)$ such that

$$|\zeta_\varepsilon^-|\geqslant q|\zeta_\varepsilon^+|.$$

Proof. Do the Lipschitz scaling (5.3) with $\bar X=(-\varepsilon,0)$ and $\delta=\zeta_\varepsilon^+$. Assume that, for a sequence $\delta_n\to0$ we have:

$$\lim_{n\to+\infty}\inf\{\zeta>0:\ (0,-\zeta)\in\Gamma_{\delta_n}\}=0.$$

This implies immediately

$$\lim_{n\to+\infty}u_{\delta_n}(\xi,0)=0,\quad \xi\geqslant0,\quad \lim_{n\to+\infty}\partial_\xi u_{\delta_n}(0,0)=0.$$

By compactness, we obtain a nontrivial limiting couple $(\Gamma_\infty, u_\infty)$ that solves the free boundary problem in $\mathbb{R} \times \mathbb{R}_-$ and is such that $(0,0) \in \Gamma_\infty$. For $\xi < 0$ we have $\partial_{\xi\xi} u_\infty(\xi, 0) = 0$, therefore $u_\infty(\xi, 0) \equiv 0$ for $\xi \in \mathbb{R}$. This is against Proposition 5.1. □

Lemma 5.3 will trigger an obvious analogy between the equation for y and the obstacle problem, in the sense that u behaves in a quadratic fashion in the vicinity of the point where the inside free boundary hits the axis $\{y = 0\}$. This is expressed in the next proposition.

Proposition 5.4. *For some $C > 0$ and for all $x \in [-1, 0]$, we have*

$$\frac{x^2}{C} \leqslant u(x, 0) \leqslant Cx^2.$$

Proof. Of course, it suffices to prove the result for small x. Consider any sequence $(\varepsilon_n)_n$ going to 0, without loss of generality we may shift the origin from $-\varepsilon_n$ to 0. Let us set

$$\delta_n := \zeta^+_{\varepsilon_n}.$$

We wish to prove that $u(-\varepsilon_n, 0)$ is of the order δ_n; because of the gradient bound, it is certainly $O(\delta_n)$. Let us prove the converse, i.e., that

$$\delta_n = O\big(u(-\varepsilon_n, 0)\big).$$

Assume this is not true. Do, as in the preceding situation, the Lipschitz scaling (5.3) with $\delta = \delta_n$. Let (Γ_n, u_n) be the scaled versions of u and Γ. Send $n \to +\infty$, we recover an unbounded analytic curve $\Gamma_\infty = \{h_\infty(\zeta), \zeta\}$ lying strictly below $\{\zeta = 0\}$, and a positive harmonic function u_∞ defined above Γ_∞ and satisfying the free boundary relation on Γ. This is impossible, due once again to Proposition 5.1. The same proposition will imply

$$\partial_\zeta u_\infty(0, 0) = 1,$$

so that, for the unscaled function $u(x, y)$, there is $q > 0$, universal, such that $u_y(x, 0) \geqslant q$ for $x <\in (-\varepsilon_n, 0)$ as soon as n is sufficiently large, except perhaps in an $o(\varepsilon_n)$-neighborhood of 0. So, by scaling, we have $u_y(x, 0) \geqslant q$ if $x < 0$ is sufficiently small. Integration of the Wentzell boundary condition yields the quadratic nondegeneracy of u on the road, in a small neighborhood of 0. □

The last ingredient that we need is that, at each point of the free boundary near $(0, 0)$, the normal is almost vertical.

Lemma 5.5. *There is $\delta_0 > 0$ such that, if $x \in (-\delta_0, 0)$, there is a unique $y = k(x)$ such that $(x, y) \in \Gamma$. For $y \in (-\delta_0, 0)$, let $\nu(x) = (\nu_1(x), \nu_2(x))$ be the* *outward normal to* Ω *at the point* $(x, k(x))$. *Then for some* $C > 0$ *we have*:

$$-Cx^2 \leqslant k(x) \leqslant -\frac{x^2}{C}. \tag{5.8}$$

Moreover, we have:

$$\lim_{x\to 0} \nu_1(x) = 0, \quad \lim_{x\to 0} \nu_2(x) = -1.$$

Proof. For every small ε, let δ_ε be the smallest δ such that

$$(-\varepsilon, -\delta) \in \Gamma,$$

from Lemma 5.3 the largest δ such that this property holds is also of order δ_ε. We do the Lipschitz scaling with

$$\bar{X} = (-\varepsilon, 0), \quad \delta = \delta_\varepsilon.$$

Let u_ε be the rescaled function, and send ε to 0. From the above lemmas, we obtain a couple $(\Gamma_\infty, u_\infty)$ where Γ_∞ is a union of analytic curves, one above the other, trapped in a bounded strip of the form $\mathbb{R} \times (-M, 0)$, and u_∞ solves the free boundary problem in the set Ω_∞ between Γ_∞ and $\{\zeta = 0\}$. Moreover, the top part of $\partial\Omega_\infty$ is a straight line, and there is $q > 0$ such that

$$u_\infty(\xi, 0) = q.$$

We also know that $\partial_\xi u_\infty \leqslant 0$, therefore it has two limits $u_\infty^\pm(\zeta)$ as $\zeta \to \pm\infty$. Hence these limits are nontrivial, and they also solve the free boundary problem. So, an easy computation yield

$$u_\infty^\pm(\zeta) = (q + \zeta)^+, \quad \zeta < 0.$$

Thus, we have

$$u_\infty(\xi, \zeta) = (q + \zeta)^+, \quad \Gamma_\infty = \{\zeta = -q\}.$$

Thus, the whole family $(u_\varepsilon)_\varepsilon$ converges to $(q + \zeta)^+$. For $\xi \in (-1, 1)$, this also implies the uniqueness of ζ such that $(\xi, \zeta) \in \Gamma$. Hence Γ_ε is an analytic graph $(\xi, k_\varepsilon(\xi))$ with $(k'_\varepsilon)_\varepsilon$ bounded by Lemma 5.3. Hence we have

$$\lim_{\varepsilon\to 0} k_\varepsilon(x) = -q, \quad \lim_{\varepsilon\to 0} k'_\varepsilon(x) = 0 \quad \text{uniformly in } \xi \in (-1, 1).$$

This implies the convergence of the normals, as stated in the lemma. □

Proof of Theorem 1.2. We finally revert to the Lipschitz-parabolic scaling, and set:

$$x = \sqrt{\varepsilon}\xi, \quad y = \varepsilon\zeta, \quad u_\varepsilon(\xi, \zeta) = \frac{1}{\varepsilon} u(\sqrt{\varepsilon}\xi, \varepsilon\zeta). \tag{5.9}$$

Let Γ_ε be, once again, the free boundary of u_ε. It suffices to prove that, for small $\varepsilon > 0$, the only $\zeta < 0$ such that $(-1, \zeta) \in \Gamma_\varepsilon$ satisfies

$$\zeta = -\frac{1}{2D}.$$

Pick any small $\delta \in (0,2)$. From Lemma 5.5, for $-2 \leqslant \xi \leqslant -\delta$, there is a unique $\zeta_\varepsilon(\xi) \leqslant 0$ such that

$$(\xi, \zeta_\varepsilon(\xi)) \in \Gamma_\varepsilon.$$

From Lemma 5.5 we have

$$\lim_{\varepsilon\to 0} \partial_\zeta u_\varepsilon(\xi, \zeta_\varepsilon(\xi)) = \lim_{\varepsilon\to 0} \partial_y u(\sqrt{\varepsilon}\xi, \varepsilon\zeta_\varepsilon(\xi)) = 1.$$

From the proof of the same lemma we have

$$\lim_{\varepsilon\to 0} \partial_\zeta u_\varepsilon(\xi, 0) = 1.$$

The Wentzell condition for u_ε is

$$\begin{aligned} D\partial_{\xi\xi}u_\varepsilon - \varepsilon c\partial_\xi u_\varepsilon = & \ \ \partial_\zeta u_\varepsilon \sim_{\varepsilon\to 0} 1 \quad \text{for } \xi \leqslant -\delta, \\ u_\varepsilon(0,0) = & \ \ \partial_\xi u_\varepsilon(0,0) = 0. \end{aligned}$$

Integrating this relation and taking the boundedness of $\partial_\zeta u_\varepsilon$ into account yields

$$\lim_{\varepsilon\to 0} u_\varepsilon(-1, 0) = -\frac{1}{2D}.$$

And, from the proof of Lemma 5.5, we have

$$\lim_{\varepsilon\to 0} u_\varepsilon(-1, \zeta) = (\zeta + \frac{1}{2D})^+.$$

This implies the result. □

Remark 4. The gradient bounds, as well as the upper and lower bounds on c, do not depend on L. In the proof of Theorem 1.2, the convergence of u_ε to $(\zeta + \frac{1}{2D})^+$ is uniform in $L \geqslant 1$.

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