SYMBOLS AND EQUIVARIANT BIRATIONAL GEOMETRY IN SMALL DIMENSIONS

BRENDAN HASSETT, ANDREW KRESCH, AND YURI TSCHINKEL

Abstract. We discuss the equivariant Burnside group and related new invariants in equivariant birational geometry, with a special emphasis on applications in low dimensions.

Let G be a finite group. Suppose that G acts birationally and generically freely on a variety over k, an algebraically closed field of characteristic zero. After resolving indeterminacy and singularities we may assume that G acts regularly on a smooth projective variety X. Classifying such actions

$$X \circlearrowleft G,$$

up to birational conjugation, especially in cases where $X \dashrightarrow \mathbb{P}^n$, is a long-standing problem. This entails understanding realizations of G in the *Cremona group* $\mathrm{BirAut}(\mathbb{P}^n)$. There is an enormous literature on this subject; we will summarize some key results in Section 1.

The focus of this survey is a new approach to the analysis of actions via new invariants extracted from the fixed points and stabilizer loci of X. This new approach has its origin in the study of specializations of birational type. Suppose a smooth projective variety specializes to a reduced normal crossing divisor – we seek to gain information about the general fiber from combinatorial structures associated with the special fiber [30, 21]. That circle of ideas, in turn, was inspired by motivic integration – another instance of this philosophy [26]. Actions of finite groups yield formally similar stratifications: we have the open subset, on which the group acts freely, and the locus with nontrivial stabilizers. The combinatorics of the resulting stratification – along with the representations of the stabilizers of the strata on the normal directions – sheds light on the original group action. In many cases, we can extract additional information on the birational type of these strata and the action of the normalizer of the stabilizer. We explain these constructions in detail, with many examples of small dimensions, referring to the original papers [20, 23]. Our goal is to illustrate how the layers of this new formalism reveal various levels of structure among

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G-birational types. In particular, we apply these new obstructions to cyclic actions on cubic fourfolds, including rational ones, and produce examples of nonlinearizable actions.

Here is the roadmap of the paper: After briefly recalling several classical results, we discuss the case of finite *abelian* groups G. We introduce the group of *equivariant birational types* $\mathcal{B}_n(G)$, as a quotient of a $\mathbb{Z}$-module of certain symbols by explicit defining relations and find a simpler presentation of these relations (Proposition 2.1). In Section 3 we explain, in numerous examples, how to compute the invariants on surfaces. In Section 4 we show that *all* symbols in $\mathcal{B}_n(G)$ are represented by smooth projective varieties with G-action. In Section 5 we introduce and study refined invariants of *abelian* actions, taking into account not only the representation on the tangent bundle to the fixed point strata, but also birational types of these strata. In Section 6 we exhibit cyclic actions on cubic fourfolds that are not equivariantly birational to linear actions; our main goal is to highlight the applications of the different invariants in representative examples. Finally, in Section 7 we consider *nonabelian* groups, define the *equivariant Burnside group*, which encodes new obstructions to equivariant rationality, and show how these obstructions work in a striking example, due to Iskovskikh [19]: distinguishing two birational actions of $\mathfrak{S}_2 \times \mathfrak{S}_3$ on rational surfaces.

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1. Brief history of previous work

These questions were considered by Bertini, Geiser, De Jonquières, Kantor, etc. over a century ago but continue to inspire new work:

- Manin [27], [28] studied G-surfaces both in the arithmetic and geometric context, focusing on the induced G-action on the geometric Picard group, and on cohomological invariants of that lattice;
- Iskovskikh [18] laid the groundwork for the G-birational classification of surfaces and their linkings;
- Bayle, Beauville, Blanc, and de Fernex [4, 5, 13, 9, 10] classified actions of finite abelian G on surfaces;
- Dolgachev and Iskovskikh [14] largely completed the surface case;

- Bogomolov and Prokhorov [11, 34] considered the stable conjugacy problem for the surface case using cohomological tools introduced by Manin;
- Prokhorov, Cheltsov, Shramov, and collaborators proved numerous theorems for threefolds – both concerning specific groups, such as simple groups [33, 12], as well as general structural properties [35].

Much of this work fits into the Minimal Model Program, using distinguished models to reduce the classification problem to an analysis of automorphisms of a restricted class of objects, e.g., del Pezzo surfaces. With a few exceptions – the application of the cohomology on the Néron-Severi group, by Manin and Bogomolov–Prokhorov, and the 'normalized fixed curve with action (NFCA)' invariant of Blanc [10] – invariants play a limited role.

A fundamental observation, recorded in [36, App. A], is that the presence of a point fixed by a given abelian subgroup H of G is a birational invariant of a smooth projective variety X with generically free G-action. Furthermore, Reichstein and Youssin showed that the *determinant* of the action of abelian stabilizers on the normal bundle to the locus with the given stabilizer, up to sign, is also a birational invariant [37]. However, for finite groups this is only useful when the minimal number of generators of the abelian group equals the dimension of the variety [36, Th. 1.1]. For cyclic groups, it is applicable only for curves.

The invariants defined in [20, 22, 23] record all eigenvalues for the action of abelian stabilizers, as well as additional information about the action on the components of the fixed locus, and on their function fields. These collections of data are turned into a G-birational invariant, via explicit *blowup relations*. The groups receiving these invariants, the *equivariant Burnside groups*, have an elaborate algebraic structure. And they led to new results in birational geometry, some of which will be discussed below.

2. Equivariant birational types

Here we restrict to the situation where G is *abelian* and consider only fixed points of $X \circlearrowleft G$. In general, there are no such fixed points and we obtain no information. However, large classes of actions do have fixed points, e.g., if G is cyclic and $h^i(X, \mathcal{O}_X) = 0$, for each $i > 0$, then the Atiyah-Bott holomorphic Lefschetz formula [2, Cor. 4.13] yields a fixed point. The vanishing assumption holds for rational and rationally connected X. If G is an abelian p-group (p prime) acting on

X without fixed points then every Chern number of X is divisible by p [17, Cor. 1.1.2].

To define an invariant of $X \circlearrowleft G$, we consider collections of weights for the action of G in the tangent bundle at G-fixed points in X. To formalize this, let

$$A = G^\vee = \mathrm{Hom}(G, \mathbb{G}_m)$$

be the character group of G, and $n = \dim X$. Let

$$\mathcal{S}_n(G)$$

be the free abelian group on symbols

$$[a_1, \dots, a_n], \quad a_j \in A, \quad \forall j,$$

subject to the conditions:

(**G**) **Generation:** $\{a_1, \dots, a_n\}$ generate A, i.e.,

$$\sum_{i=1}^{n} \mathbb{Z} a_i = A,$$

thus, n is at least the minimal number of generators of G;

(**S**) **Symmetry:** for each permutation $\sigma \in \mathfrak{S}_n$ we have

$$[a_{\sigma(1)}, \dots, a_{\sigma(n)}] = [a_1, \dots, a_n].$$

Let

$$\mathcal{S}_n(G) \to \mathcal{B}_n(G) \tag{2.1}$$

be the quotient, by relations, for all $2 \le r \le n$:

(**B**$_r$) **Blow-up:** for all $[a_1, \dots, a_r, b_1, \dots, b_{n-r}] \in \mathcal{S}_n(G)$ one has

$$[a_1, \dots, a_r, b_1, \dots, b_{n-r}] =$$

$$\sum_{1 \le i \le r,\, a_i \ne a_{i'} \text{ for } i' < i} [a_1 - a_i, \dots, a_i, \dots, a_r - a_i, b_1, \dots, b_{n-r}]. \tag{2.2}$$

These relations reflect the transformations of weights in tangent spaces to components of the fixed locus upon blowing up along a G-stable stratum.

From the definition, we have

$$\mathcal{B}_1(G) = \begin{cases} \mathbb{Z}^{\phi(N)} & \text{if } G \text{ is cyclic of order } N, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1. [23, Prop. 8.1] *For $n \geq 2$, all relations $(\mathbf{B}_r)$ are implied by relation $(\mathbf{B}_2)$.*

Thus, $\mathcal{B}_n(G)$ is obtained by imposing the relation:

(**B**) **Blow-up:** for all $[a_1, a_2, b_1, \dots, b_{n-2}] \in \mathcal{S}_n(G)$ one has

$$[a_1, a_2, b_1, \dots, b_{n-2}] =$$

$$\begin{cases} [a_1, a_2 - a_1, b_1, \dots, b_{n-2}] + [a_1 - a_2, a_2, b_1, \dots, b_{n-2}], & a_1 \neq a_2, \\ [0, a_1, b_1, \dots, b_{n-2}], & a_1 = a_2. \end{cases} \quad (2.3)$$

Proof of Proposition 2.1. We prove the result by induction on r. We first treat the case that $a_1, a_2, \dots, a_r$ are pairwise distinct; we drop the entries $b_1, \dots, b_{n-r}$ from the notation, as they do not take part in relations. Suppose $r \geq 3$. Then:

$$\begin{aligned} [a_1, \dots, a_r] &= (\text{by } (\mathbf{B}_{r-1}))\, [a_1, a_2 - a_1, \dots, a_{r-1} - a_1, a_r] + \cdots \\ &\qquad + [a_1 - a_{r-1}, \dots, a_{r-2} - a_{r-1}, a_{r-1}, a_r] \\ &= (\text{by } (\mathbf{S})) \qquad [a_1, a_r, a_2 - a_1, \dots, a_{r-1} - a_1] + \cdots \\ &\qquad + [a_{r-1}, a_r, a_1 - a_{r-1}, \dots, a_{r-2} - a_{r-1}] \\ &= (\text{by } (\mathbf{B}_2) + (\mathbf{S}), \text{ applied to each term}) \\ &[a_1, a_2 - a_1, \dots, a_r - a_1] + [a_1 - a_r, a_2 - a_1, \dots, a_{r-1} - a_1, a_r] \\ &\qquad + \cdots + \\ &[a_1 - a_{r-1}, \dots, a_{r-2} - a_{r-1}, a_{r-1}, a_r - a_{r-1}] \\ &\qquad + [a_1 - a_{r-1}, \dots, a_{r-2} - a_{r-1}, a_{r-1} - a_r, a_r]. \end{aligned}$$

The right-hand terms, taken together, are equal by $(\mathbf{B}_{r-1})$ to

$$[a_1 - a_r, \dots, a_{r-1} - a_r, a_r],$$

which together with the left-hand terms gives us what we want.

Next we treat the more general case $a_r \notin \{a_1, \dots, a_{r-1}\}$. In that case, for every i with $a_i \in \{a_1, \dots, a_{i-1}\}$ we omit the ith term on the right-hand side in the initial application of $(\mathbf{B}_{r-1})$ and $(\mathbf{S})$ and omit the ith line after the applications of $(\mathbf{B}_2)$ and $(\mathbf{S})$. We conclude as before.

Finally we treat the case $a_r \in \{a_1, \dots, a_{r-1}\}$. We start in the same way, by applying $(\mathbf{B}_{r-1})$ and $(\mathbf{S})$ as above. Now, when we apply $(\mathbf{B}_2)$, we have to pay special attention to terms with $a_i = a_r$: in the corresponding line we should leave the left-hand term but omit the right-hand term. Each of the remaining right-hand terms vanishes by an

application of $(\mathbf{B}_2)$ to a symbol of the form $[0,\dots]$, i.e., the vanishing of any symbol with two nonzero weights summing to 0. The left-hand terms give us directly what we want. □

2.1. **Antisymmetry.** Let

$$\mathcal{B}_n(G) \to \mathcal{B}_n^-(G)$$

be the projection to the quotient by the additional relation

$$[-a_1,\dots,a_n] = -[a_1,\dots,a_n]. \tag{2.4}$$

In particular, symbols of the type $[0,a_2,\dots,a_n]$ are in the kernel of the projection. We write

$$[a_1,\dots,a_n]^-$$

for the image of a standard generator $[a_1,\dots,a_n]$.

2.2. **Multiplication and co-multiplication.** Consider a short exact sequence

$$0 \to G' \to G \to G'' \to 0$$

and its dual

$$0 \to A'' \to A \to A' \to 0.$$

The *multiplication*

$$\nabla : \mathcal{B}_{n'}(G') \otimes \mathcal{B}_{n''}(G'') \to \mathcal{B}_n(G), \quad n'+n''=n,$$

is defined by

$$[a'_1,\dots,a'_{n'}] \otimes [a'_1,\dots,a'_{n'}] \mapsto \sum [a_1,\dots,a_{n'},a''_1,\dots a''_{n''}],$$

summing over all lifts $a_i \in A$ of $a'_i \in A'$. It descends to a similar map on quotients by the relation (2.4). .

The *co-multiplication* is defined only on $\mathcal{B}_n^-(G)$:

$$\Delta^- : \mathcal{B}_n^-(G) \to \mathcal{B}_{n'}^-(G') \otimes \mathcal{B}_{n''}^-(G''), \quad n'+n''=n.$$

On generators is takes the form

$$[a_1,\dots,a_n]^- \mapsto \sum [a_{I'} \bmod A'']^- \otimes [a_{I''}]^-,$$

where the sum is over subdivisions $\{1,\dots,n\} = I' \sqcup I''$ of cardinality n', respectively n'', such that

- $a_j \in A''$, for all $j \in I''$
- $a_j, j \in I''$, generate A''.

The correctness of this definition is proved as in [20, Prop. 11]. Here are the main steps: By [23, Prop. 8.1] (= Proposition 2.1), it suffices to check 2-term relations ($\mathbf{B}_2$), i.e., the image of the relation

$$[a_1, a_2, \ldots]^- = [a_1 - a_2, a_2, \ldots]^- + [a_1, a_2 - a_1, \ldots]^-$$

after applying co-multiplication. The only interesting part is when the first two arguments are distributed over different factors in the definition of co-multiplication.

The relation is the same relation as that for the $\mathcal{M}_n(G)$-groups, introduced and studied in [20], unless, $a_1 = a_2$ – recall that

$$[a, a, \ldots] = [a, 0, \ldots,] \in \mathcal{B}_n(G).$$

Since

$$0 = [a, -a, \ldots]^- = -[a, a, \ldots]^-$$

we have

$$[a, a, \ldots]^- = [a, 0, \ldots]^- = 0.$$

Now it suffices to repeat the argument in [20, Prop. 11]. Using the terminology of that paper, there are four cases, of which only (1) and (4) are relevant. In both cases, all terms are zero.

We return to the definition of an invariant for $X \circlearrowleft G$, $\dim(X) = n$, and G abelian. Consider irreducible components of the fixed locus

$$X^G = \coprod_\alpha F_\alpha,$$

and write

$$\beta_\alpha = [a_{1,\alpha}, \ldots, a_{n,\alpha}]$$

for the unordered n-tuple of weights for the G-action on the tangent space $\mathcal{T}_{x_\alpha} X$, for some $x_\alpha \in F_\alpha$ – this does not depend on the choice of x_α. The number of zero weights is $\dim(F_\alpha)$. We express

$$\beta(X \circlearrowleft G) = \sum_\alpha \beta_\alpha \in \mathcal{B}_n(G), \tag{2.5}$$

and write

$$\beta^-(X \circlearrowleft G) \in \mathcal{B}_n^-(G),$$

for the image under the projection.

Theorem 2.2. [20, Th. 3] *The class*

$$\beta(X \circlearrowleft G) \in \mathcal{B}_n(G)$$

is a G-birational invariant.

The proof relies on G-equivariant Weak Factorization, connecting G-birational varieties via blow-ups and blow-downs of smooth G-stable subvarieties.

Proposition 2.3. *Consider a linear, generically free, action of a cyclic group C_N, of order N, on $\mathbb{P}^n$, for $n \geq 2$. Then*

$$\beta^-(\mathbb{P}^n \circlearrowleft C_N) = 0 \in \mathcal{B}_n^-(C_N).$$

Proof. We know that all such actions are equivariantly birational, see, e.g., [37, Th. 7.1]. Thus it suffices to consider one such action. Take an action with weights $(1, 0, \dots, 0)$. It fixes a hyperplane, and a point, the corresponding class is

$$[1, 0, \dots, 0] + [-1, -1, \dots, -1] = [1, 0, \dots, 0] + [-1, 0, \dots, 0],$$

here, we repeatedly used relation (2.3) to transform the second term. □

Remark 2.4. We shall show in Section 3.3 that $[1, 0] + [-1, 0]$ is torsion in $\mathcal{B}_2(C_N)$. See [20, Prop. 7 and Lem. 32] for the case of prime N. The element is nontrivial when

$$N = 7, 9, 10, 11, 13, 14, 15, 17, \dots.$$

3. Computation of invariants on surfaces

3.1. **Sample computations of $\mathcal{B}_2(C_p)$.** This group is generated by symbols $[a_1, a_2]$, where

$$a_1, a_2 \in \mathrm{Hom}(C_p, \mathbb{G}_m) \cong \mathbb{Z}/p\mathbb{Z}$$

are not both trivial. To simplify notation, we write $a_i = 0, 1, \dots, p-1$. Note that

$$[a, a] = [0, a] \quad \text{and} \quad [a_1, a_2] = [a_2, a_1]$$

so that the symbols

$$[a_1, a_2]: \qquad 0 < a_1 \leq a_2 < p$$

generate $\mathcal{B}_2(C_p)$. The other relation – for $a_1 < a_2$ – takes the form

$$[a_1, a_2] = [a_1, a_2 - a_1] + [p + a_1 - a_2, a_2].$$

($p = 2$) We obtain relations

$$[1, 1] = [0, 1], \quad [0, 1] = [0, 1] + [1, 1]$$

forcing $\mathcal{B}_2(C_2) = 0$.

($p = 3$) We obtain relations

$$\begin{aligned}
[0,1] &= [0,1] + [2,1] = [0,1] + [1,2] \\
[0,2] &= [0,2] + [1,2] \\
[1,1] &= [0,1] \\
[1,2] &= [1,1] + [2,2] \\
[2,2] &= [0,2]
\end{aligned}$$

forcing

$$[1,2] = 0 \quad \text{and} \quad [0,1] = [1,1] = -[2,2] = -[0,2].$$

We conclude that $\mathcal{B}_2(C_3) \cong \mathbb{Z}$.

For instance, consider the standard diagonal action on $\mathbb{P}^2$

$$(x : y : z) \mapsto (x : \zeta_3 y : \zeta_3^2 z)$$

with the convention that ζ_N is a primitive Nth root of unity. This has invariant

$$\beta(\mathbb{P}^2 \circlearrowleft C_3) = [1,2] + [1,2] + [1,2] = 0.$$

On the other hand, the action of C_3 on the cubic surface

$$X = \{w^3 = f_3(x,y,z)\}, \quad (w : x : y : z) \mapsto (\zeta_3 w : x : y : z)$$

fixes the cubic curve $\{w = 0\}$ and we find that

$$\beta(X \circlearrowleft C_3) = [2,0] \neq 0,$$

thus X is not G-equivariantly birational to $\mathbb{P}^2$, with linear action. Note that β does not allow to distinguish among these cubic surfaces. Nor does it distinguish the cubic surfaces from the degree one del Pezzo surfaces with C_3-action

$$Y = \{w^2 = z^3 + f_6(x,y)\} \subset \mathbb{P}(3,1,1,2),$$

given by

$$(w : x : y : z) \mapsto (w : x : y : \zeta_3 z).$$

We shall see in Section 5 that taking into account the fixed locus $[F_\alpha]$ gives a complete invariant.

($p = 5$) We have relations

$$\begin{aligned}
[0,1] &= [0,1] + [4,1] \\
[0,2] &= [0,2] + [3,2] \\
[0,3] &= [0,3] + [2,3] \\
[0,4] &= [0,4] + [1,4]
\end{aligned}$$

forcing $[1,4]=[2,3]=0$. We also have

$$\begin{aligned}[1,2]&=[1,1]+[4,2]\\ [1,3]&=[1,2]+[3,3]\\ [1,4]&=[1,3]+[2,4]\\ [2,3]&=[2,1]+[4,3]\\ [2,4]&=[2,2]+[3,4]\\ [3,4]&=[3,1]+[4,4]\end{aligned}$$

which shows that $\mathcal{B}_2(C_5)$ is freely generated by

$$\beta_1=[1,1] \quad \text{and} \quad \beta_2=[1,2]$$

with

$$\begin{aligned}[1,3]=\beta_1-\beta_2,\quad [2,2]=2\beta_2-\beta_1,\quad [2,4]=\beta_2-\beta_1,\\ [3,3]=\beta_1-2\beta_2,\quad [3,4]=-\beta_2,\quad [4,4]=-\beta_1.\end{aligned}$$

For example, $\overline{M}_{0,5}$, a del Pezzo surface of degree 5, has a natural action of C_5 by permuting the coordinates with fixed points given by the roots of z^5-1 and z^5+1. We compute $\beta(\overline{M}_{0,5}\circlearrowleft C_5)$:

$$\beta_1+\beta_2+(\beta_1-\beta_2)+(2\beta_2-\beta_1)+(\beta_2-\beta_1)+(\beta_1-2\beta_2)+(-\beta_2)+(-\beta_1)=0.$$

Indeed, this action is in fact conjugate [5] to a linear action on $\mathbb{P}^2$

$$(x:y:z)\mapsto(x:\zeta_5 y:\zeta_5^4 z).$$

However, there is also a nontrivial action of C_5 on a del Pezzo surface of degree 1:

$$X=\{w^2=z^3+\lambda x^4 z+x(\mu x^5+y^5)\}\subset\mathbb{P}(3,1,1,2),$$

$$(w:x:y:z)\mapsto(w:x:\zeta_5 y:z),$$

with fixed locus an elliptic curve and invariant $\beta(X\circlearrowleft C_5)=[4,0]$.

Let us compute an example of nonprime order. The group $\mathcal{B}_2(C_4)$ has generators

$$[0,1],[0,3],[1,1],[1,2],[1,3],[2,3],[3,3]$$

and relations

$$\begin{aligned}
[0,1] &= [0,1] + [3,1] \\
[0,3] &= [0,3] + [1,3] \\
[1,1] &= [0,1] \\
[1,2] &= [1,1] + [3,2] \\
[1,3] &= [1,2] + [2,3] \\
[2,3] &= [2,1] + [3,3] \\
[3,3] &= [0,3]
\end{aligned}$$

whence

$$[1,3] = 0, \quad \beta_1 := [1,2] = -[2,3], \quad [0,3] = [3,3] = 2[2,3] = -2\beta_1,$$
$$[0,1] = [1,1] = 2[1,2] = 2\beta_1$$

and $\mathcal{B}_2(C_4) \cong \mathbb{Z}$.

Consider the del Pezzo surface of degree 1, given by

$$X = \{w^2 = z^3 + zL_2(x^2, y^2) + xyM_2(x^2, y^2)\} \subset \mathbb{P}(3,1,1,2),$$

where L_2 and M_2 are homogeneous of degree two. It admits a C_4-action

$$(w : x : y : z) \mapsto (iw : x : -y : -z)$$

with a unique fixed point $(0 : 1 : 0 : 0)$. The weights on the tangent bundle are $[2,3]$ whence

$$\beta(X \circlearrowleft C_4) \neq 0.$$

Observe that X^{C_2} is a curve of genus four.

See [8, §10.1] for a classification of automorphisms of large finite order N on del Pezzo surfaces:

(1) The surface

$$X = \{w^2 = z^3 + x(x^5 + y^5)\} \subset \mathbb{P}(3,1,1,2)$$

admits an automorphism of order 30

$$(w : x : y : z) \mapsto (-w : x : \zeta_5 y : \zeta_3 z)$$

with fixed point $(0 : 0 : 1 : 0)$ and with weights $[3,2]$, thus

$$\beta(X \circlearrowleft C_{30}) = [3,2] \neq 0 \in \mathcal{B}_2(C_{30}) \otimes \mathbb{Q},$$

by a computation in `Sage` [38]. This implies (see Remark 2.4) that this action is not conjugate to a linear action. Note that $\dim \mathcal{B}_2(C_{30}) \otimes \mathbb{Q} = 33$.

(2) The surface

$$X = \{w^2 = z^3 + xy(x^4 + y^4)\} \subset \mathbb{P}(3, 1, 1, 2)$$

admits an automorphism of order 24

$$(w : x : y : z) \mapsto (\zeta_8 w : x : iy : -i\zeta_3 z).$$

The fixed point is $(0 : 1 : 0 : 0)$ with symbol $[21, 22]$. Computing via `Sage` we find

$$\beta(X \circlearrowleft C_{24}) \neq 0 \in \mathcal{B}_2(C_{24}) \otimes \mathbb{Q}.$$

Here $\dim \mathcal{B}_2(C_{24}) \otimes \mathbb{Q} = 23$.

There are good reasons why we obtain nonvanishing invariants only when a curve is fixed: If $G = C_N$ is a cyclic group acting generically freely on a complex projective smooth rational surface $X \circlearrowleft G$ then the following are equivalent [8, Th. 4]:

- No $g \neq 1 \in G$ fixes a curve in X of positive genus.
- The subgroup G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$.

Even more is true: if G contains an element fixing a curve of positive genus then X is not even stably G-birational to projective space with a linear G-action, indeed, in this case $\mathrm{H}^1(G, \mathrm{Pic}(X)) \neq 0$ [11].

3.2. **Examples for noncyclic groups.** If G is a noncyclic abelian group then $\mathcal{B}_1(G) = 0$ by definition but there are actions on curves:

Example 3.1. Consider the action of $C_2 \times C_2$ on $\mathbb{P}^1$ by

$$g_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the elements

$$g_1^2 = g_1 g_2 g_1^{-1} g_2^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

acting trivially. Thus we obtain

$$\mathbb{P}^1 \circlearrowleft (C_2 \times C_2).$$

The group has no fixed points whence $\beta(\mathbb{P}^1 \circlearrowleft (C_2 \times C_2)) = 0$. The cyclic subgroups do have fixed points

$$(\mathbb{P}^1)^{\langle g_1 \rangle} = \{(1 : \pm i)\}, (\mathbb{P}^1)^{\langle g_2 \rangle} = \{(1 : 0), (0 : 1)\}, (\mathbb{P}^1)^{\langle g_1 g_2 \rangle} = \{(1 : \pm 1)\}.$$

We return to this in Example 5.2. In Section 5.2, we will discuss how to incorporate information from *all* points with nontrivial stabilizer.

We compute $\mathcal{B}_2(C_2 \times C_2)$. Writing

$$(C_2 \times C_2)^\vee = \{0, \chi_1, \chi_2, \chi_1 + \chi_2\},$$

the only admissible symbols are

$$[\chi_1, \chi_2], [\chi_1, \chi_1 + \chi_2], [\chi_1 + \chi_2, \chi_2]$$

with relations:

$$\begin{aligned} [\chi_1, \chi_2] &= [\chi_1, \chi_1 + \chi_2] + [\chi_1 + \chi_2, \chi_2] \\ [\chi_1, \chi_1 + \chi_2] &= [\chi_1, \chi_2] + [\chi_1 + \chi_2, \chi_2] \\ [\chi_1 + \chi_2, \chi_2] &= [\chi_1, \chi_1 + \chi_2] + [\chi_1, \chi_2]. \end{aligned}$$

Thus we obtain the Klein four group again

$$\mathcal{B}_2(C_2 \times C_2) = C_2 \times C_2.$$

The classification of finite abelian noncyclic actions on rational surfaces may be found in [8, §10.2]. Examples of actions of $C_2 \times C_2$ on rational surfaces include:

On $\mathbb{P}^1 \times \mathbb{P}^1$:

(1) $(x, y) \mapsto (\pm x^{\pm 1}, y)$, without fixed points;
(2) $(x, y) \mapsto (\pm x, \pm y)$, with fixed points $(0, 0), (0, \infty), (\infty, 0), (\infty, \infty)$, thus
$$\beta(\mathbb{P}^1 \times \mathbb{P}^1 \circlearrowleft C_2 \times C_2) = 4[\chi_1, \chi_2] = 0;$$
(3) the diagonal action
$$(x, y) \mapsto (-x, -y), (x, y) \mapsto (x^{-1}, y^{-1})$$
has no fixed points so the symbol sum is empty;

On conic fibrations over $\mathbb{P}^1$:

(4) $(x_1 : x_2) \times (y_1 : y_2) \mapsto$
$(x_1 : -x_2) \times (y_1 : y_2)$, respectively,
$(x_1 : x_2) \times (y_2(x_1 - bx_2)(x_1 + bx_2) : y_1(x_1 - ax_2)(x_1 + ax_2)$,
which also has four fixed points with the same symbol;

On a degree two del Pezzo surface:

(5) $(w : x : y : z) \mapsto (\pm w : x : y : \pm z)$ on
$$\{w^2 = L_4(x, y) + z^2 L_2(x, y) + z^4\},$$
with the involutions fixing a genus three curve and an elliptic curve meeting in four points whence $\beta(X \circlearrowleft C_2 \times C_2) = 0$;

On a degree one del Pezzo surface:

(6) $(w : x : y : z) \mapsto (\pm w : x : \pm y : z)$ on
$$\{w^2 = z^3 + zL_2(x^2, y^2) + L_3(x^2, y^2)\},$$
with the involutions fixing a genus four curve and a genus two curve meeting in six points whence $\beta(X \circlearrowleft C_2 \times C_2) = 0$.

None of these actions are distinguished by $\mathcal{B}_2(C_2 \times C_2)$. Case (1) is stably equivalent to the action on $\mathbb{P}^1$ described above. The second action is equivalent to a linear action on $\mathbb{P}^2$ – project from one of the fixed points. We return to these examples in Section 5.4.

3.3. **Linear actions yield torsion classes.** Let C_N act linearly and generically freely on $\mathbb{P}^n$. We saw in Proposition 2.3 that
$$\beta(\mathbb{P}^n \circlearrowleft C_N) = [a, 0, \dots, 0] + [-a, 0, \dots, 0]$$
for some a relatively prime to N. Remark 2.4 pointed out this is torsion for $n \geq 2$; we offer a proof now:

Proposition 3.2. *For $N \in \mathbb{N}$, an a with $\gcd(a, N) = 1$, and $n \geq 2$ the element*
$$[a, 0, \dots, 0] + [-a, 0, \dots, 0] \in \mathcal{B}_n(C_N)$$
is torsion.

Proof. It suffices to consider $n = 2$; the argument we present goes through without changes for $n > 2$.

We will work in $\mathcal{B}_2(C_N) \otimes \mathbb{Q}$. We use generators for this space arising from the alternative symbol formalism $\mathcal{M}_2(C_N)$ introduced in [20] with the property that [20, Prop. 7]
$$\mathcal{B}_2(C_N) \otimes \mathbb{Q} = \mathcal{M}_2(C_N) \otimes \mathbb{Q}.$$

For $a, b \in \mathrm{Hom}(C_N, \mathbb{G}_m)$ generating the group we set
$$\langle a, b\rangle = \begin{cases} [a, b] & \text{if } a, b \neq 0 \\ \frac{1}{2}[a, 0] & \text{if } a \neq 0, b = 0. \end{cases}$$
The advantage of these generators is that the relations are uniformly
$$\langle a, b\rangle = \langle a, b - a\rangle + \langle a - b, b\rangle \quad (\mathbf{B}),$$
even when $a = b$.

We follow the proof of [20, Th. 14]. For all a, b with $\gcd(a, b, N) = 1$ we write
$$\delta(a, b) := \langle a, b\rangle + \langle -a, b\rangle + \langle a, -b\rangle + \langle -a, -b\rangle .$$
We claim this is zero in $\mathcal{B}_2(C_N) \otimes \mathbb{Q}$.

First, we check that $\delta(a,b)$ is invariant under $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. This has generators

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

and $\delta(a,b) = \delta(-b,a)$ by the symmetry of the underlying symbols. We also have

$$\begin{aligned}
\delta(a,b-a) =& \langle a,b-a\rangle + \langle -a,b-a\rangle + \langle a,a-b\rangle + \langle -a,a-b\rangle \\
& \text{applying } \mathbf{B} \text{ to second and third terms above} \\
=& \langle a,b-a\rangle + \langle -a,b\rangle + \langle -b,b-a\rangle \\
& + \langle a,-b\rangle + \langle b,a-b\rangle + \langle -a,a-b\rangle \\
& \text{applying } \mathbf{B} \text{ to get first and four terms below} \\
=& \langle a,b\rangle + \langle -a,b\rangle + \langle a,-b\rangle + \langle -a,-b\rangle \\
=& \delta(a,b).
\end{aligned}$$

Average $\delta(a,b)$ over all pairs a,b with $\gcd(a,b,N)=1$ to obtain

$$S := \sum_{a,b} \delta(a,b) = 4 \sum_{a,b} \langle a,b\rangle .$$

Applying the blowup relation ($\mathbf{B}$) to all terms one finds

$$S = 2S,$$

which implies that $S = 0 \in \mathcal{B}_2(C_N) \otimes \mathbb{Q}$.

We may regard $\delta(a,b)$ and S as elements of $\mathcal{B}_2(C_N)$. It follows that $\delta(a,b)$ is torsion in $\mathcal{B}_2(C_N)$, annihilated by the number of summands in S. Substituting $b=0$, we obtain that

$$\delta(a,0) = [a,0] + [-a,0] = 0 \in \mathcal{B}_2(C_N) \otimes \mathbb{Q}.$$

□

The invariance of $\delta(a,0)$ shows that $[a,0]+[-a,0]$ is independent of the choice of a relatively prime to N.

3.4. **Algebraic structure in dimension 2.** For reference, we tabulate

$$\dim \mathcal{B}_2(G) \otimes \mathbb{Q},$$

for $G = C_N$ and small values of N:

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	0	1	1	2	2	3	3	5	4	6	7	8	7	13	10

For primes $p \geq 5$ there is a closed formula [20, §11]:

$$\dim \mathcal{B}_2(C_p) \otimes \mathbb{Q} = \frac{p^2-1}{24} + 1 = \frac{(p-5)(p-7)}{24} + \frac{p-1}{2}, \qquad (3.1)$$

which strongly suggested a connection with the modular curve $X_1(p)$! We also have

$$\dim \mathcal{B}_2^-(C_p) \otimes \mathbb{Q} = \frac{(p-5)(p-7)}{24}, \tag{3.2}$$

and, by [20, Prop. 30],

$$\mathcal{B}_1^-(C_p) \otimes \mathbb{Q} = \mathrm{Ker}(\mathcal{B}_2(C_p) \to \mathcal{B}_2^-(C_p)) \otimes \mathbb{Q}.$$

Computations in noncyclic cases have been performed by Zhijia Zhang[1]; we summarize the results: for primes $p \geq 5$ one has

$$\dim \mathcal{B}_2(C_p \times C_p) \otimes \mathbb{Q} = \frac{(p-1)(p^3+6p^2-p+6)}{24},$$

$$\dim \mathcal{B}_2^-(C_p \times C_p) \otimes \mathbb{Q} = \frac{(p-1)(p^3-p+12)}{24}$$

For $G = C_{N_1} \times C_{N_2}$ and small values of N_1, N_2, we have:

N_1	2	2	2	2	2	2	3	3	3	3	4	4	4	5	6
N_2	2	4	6	8	10	16	6	3	9	27	8	16	32	25	36
$d_{\mathbb{Q}}$	0	2	3	6	7	21	15	7	37	235	33	105	353	702	577
$d_{\mathbb{Q}}^-$	0	0	0	1	1	9	7	3	19	163	17	65	257	502	433
d_2	2	5	8	13	18	36	15	7	37	235	34	106	354	702	578
d_2^-	2	3	5	8	12	24	7	3	19	163	17	65	257	502	433

4. Reconstruction theorem

The examples offered so far might suggest that very few invariants in $\mathcal{B}_n(G)$ are actually realized geometrically by smooth projective varieties $X \circlearrowleft G$. If one allows nonrational examples far more invariants arise:

Proposition 4.1. *Let p be a prime. Then $\mathcal{B}_n(C_p)$ is generated as an abelian group by $\beta(X \circlearrowleft C_p)$, where X is smooth and projective.*

Proof. We proceed by induction on n. The case of $n = 1$ follows from the Riemann existence theorem applied to cyclic branched covers of degree p with the prescribed ramification data. (See also Lemma 4.2 below.)

For the symbols $[a_1, \ldots, a_{n-1}, 0]$ we construct $(n-1)$-dimensional varieties D with the prescribed invariants and $D \times \mathbb{P}^1$ with trivial action on the second factor. Since $[a, a, a_3, \ldots, a_n] = [0, a, a_3, \ldots, a_n]$ we may focus on symbols $[a_1, a_2, \ldots, a_n], 0 < a_1 < a_2 < \ldots < a_n < p$. We are reduced to the following lemma:

[1] see `https://cims.nyu.edu/~zz1753/ebgms/ebgms.html`

Lemma 4.2. *Any sum*

$$\sum m_{[a_1,a_2,\ldots,a_n]}[a_1,a_2,\ldots,a_n] \tag{4.1}$$

of symbols

$$[a_1,a_2,\ldots,a_n], \quad 0<a_1<a_2<\ldots<a_n<p,$$

with nonnegative coefficients, is realized as $\beta(X \circlearrowleft C_p)$, where X is smooth, projective, and irreducible.

For each symbol $[a_1,a_2,\ldots,a_n]$ appearing in the sum, take an n-dimensional representation $V_{[a_1,a_2,\ldots,a_n]}$ with the prescribed weights and the direct sum

$$W_{[a_1,a_2,\ldots,a_n]} = (V_{[a_1,a_2,\ldots,a_n]} \oplus \mathbb{C})^{m_{[a_1,a_2,\ldots,a_n]}}$$

where $\mathbb{C}$ is the trivial representation of C_p. Write

$$W = \oplus W_{[a_1,a_2,\ldots,a_n]},$$

where the index is over the terms appearing in (4.1), and consider the projectivization $\mathbb{P}(W)$ and the n-planes

$$P_{[a_1,a_2,\ldots,a_n],j} \subset \mathbb{P}(W_{[a_1,a_2,\ldots,a_n]}), \quad j=1,\ldots,m_{[a_1,a_2,\ldots,a_n]}$$

associated with the summands of $W_{[a_1,a_2,\ldots,a_n]}$, each with distinguished fixed point

$$p_{[a_1,a_2,\ldots,a_n],j} = (0:0:\cdots:0:1).$$

The action on

$$\mathcal{T}_{p_{[a_1,a_2,\ldots,a_n]}} P_{[a_1,a_2,\ldots,a_n],j}$$

coincides with the action on $V_{[a_1,a_2,\ldots,a_n]}$. The fixed points of $\mathbb{P}(W)$ correspond to the eigenspaces for the C_p action. Write $M = \sum m_{[a_1,a_2,\ldots,a_n]}$ for the number of summands; each weight occurs at most M times. Thus the fixed point loci are projective spaces of dimension $\leq M-1$.

Choose a high-degree smooth complete intersection X of dimension n, invariant under the action of C_p, containing the $p_{[a_1,a_2,\ldots,a_n],j}$ and tangent to $P_{[a_1,a_2,\ldots,a_n],j}$ for each $[a_1,a_2,\ldots,a_n]$. This complete intersection exists by polynomial interpolation applied to the quotient $\mathbb{P}(W)/C_p$; smoothness follows from Bertini's Theorem. Since complete intersections of positive dimension are connected, the resulting X is irreducible.

It only remains to show that such a complete intersection need not have fixed points beyond those specified. Now X has codimension $(M-1)(n+1)$, so we may assume it avoids the fixed point loci – away from the stipulated points $p_{[a_1,s_2,\ldots,a_n],j}$ – provided $(M-1)(n+1) \geq M$. It only remains to consider the special case $M=1$. Here we take

$$W = (V_{[a_1,a_2,\ldots,a_n]} \oplus \mathbb{C})^2,$$

imposing conditions at just one point $(0:0:\cdots:0:1)$. Here $X \subset \mathbb{P}(W)$ has codimension $n+1$ and the fixed point loci are $\mathbb{P}^1$'s, so we may avoid extraneous points of intersection. □

5. Refined invariants

5.1. **Encoding fixed points.** Since $\mathcal{B}_2(C_2) = 0$ this invariant says **nothing** about involutions of surfaces! Bertini, Geiser, and De Jonquières involutions are perhaps the most intricate parts of the classification, which relies on the study of fixed curves. This leads to a natural refinement of the invariants: For the symbols of type $[a, 0]$, corresponding to curves $F_\alpha \subset X$ fixed by C_N, we keep track of the (stable) birational equivalence class of F_α and the element of $\mathcal{B}_1(C_N)$ associated with $[a]$.

In general, [20] introduced a group combining the purely number-theoretic information encoded in $\mathcal{B}_n(G)$ with geometric information encoded in the Burnside group of fields from [21]. Let

$$\mathrm{Bir}_{n-1,m}(k), \quad 0 \le m \le n-1,$$

be the set of k-birational equivalence classes of $(n-1)$-dimensional irreducible varieties over k, which are k-birational to products $W \times \mathbb{A}^m$, but not to $W' \times \mathbb{A}^{m+1}$, for any W', and put

$$\mathcal{B}_n(G,k) := \oplus_{m=0}^{n-1} \oplus_{[Y]\in \mathrm{Bir}_{n-1,m}(k)} \mathcal{B}_{m+1}(G). \tag{5.1}$$

Let X be a smooth projective variety of dimension n with a regular, generically free, action of an abelian group G. Put

$$\beta_k(X \circlearrowleft G) := \sum_\alpha \beta_{k,\alpha},$$

where, as before, the sum is over components $F_\alpha \subset X^G$ of the G-fixed point locus, but in addition to the eigenvalues $a_1, \dots, a_{n-\dim(F_\alpha)} \in A$ in the tangent space $\mathcal{T}_{x_\alpha} X$ one keeps information about the function field of the component F_α. Choosing m_α so that

$$[F_\alpha \times \mathbb{A}^{n-1-\dim(F_\alpha)}] \in \mathrm{Bir}_{n-1,m_\alpha}(k)$$

we set

$$\beta_{k,\alpha} := [a_1, \dots, a_{n-\dim(F_\alpha)}, \underbrace{0, \dots, 0}_{m_\alpha+1-n+\dim(F_\alpha)}] \in \mathcal{B}_{m_\alpha+1}(G),$$

identified with the summand of (5.1) indexed by $[F_\alpha \times \mathbb{A}^{n-1-\dim(F_\alpha)}]$. When F_α is not uniruled we get a symbol in $\mathcal{B}_{\mathrm{codim}(F_\alpha)}(G)$.

Theorem 5.1. [20, Remark 5] *The class*

$$\beta_k(X \circlearrowleft G) \in \mathcal{B}_n(G,k)$$

is a G-birational invariant.

5.2. **Encoding points with nontrivial stabilizer.** We continue to assume that G is a finite abelian group, acting regularly and generically freely on a smooth variety X. Let $H \subset G$ arise as the stabilizer of some point of X, $F \subset X^H$ an irreducible component of the fixed point locus with generic stabilizer H, and Y the minimal G-invariant subvariety containing G. In Section 7.2, we will explain how to blow up X so that Y is always a disjoint union of translates of F.

Additional information about the action of G on X is contained in the action of the quotient G/H, which could act on the function field of F, or by translating F in X. The paper [23] introduced the group

$$\mathrm{Burn}_n(G)$$

as the quotient by certain relations of the free abelian group generated by symbols

$$(H, G/H \circlearrowright K, \beta), \tag{5.2}$$

where K is a G/H-Galois algebra over a field of transcendence degree $d \leq n$ over k, up to isomorphism, and β is a faithful $(n-d)$-dimensional representation of H (see [23, Def. 4.2] for a precise formulation of conditions on K and relations).

Passing to a suitable G-equivariant smooth projective model X – as discussed in Section 7.2 – its class is defined by

$$[X \circlearrowleft G] := \sum_{H \subseteq G} \sum_{Y} (H, G/H \circlearrowright K(Y), \beta_Y(X)) \in \mathrm{Burn}_n(G), \tag{5.3}$$

where the sum is over all G-invariant strata $Y \subset X$ with generic stabilizer H as above, the symbol records the eigenvalues of H in the tangent bundle to $x \in Y$ as well as the G/H-action on the total ring of fractions $K(Y)$.

Example 5.2. We revisit Example 3.1 using the notational conventions $\chi_1(g_2) = \chi_2(g_1) = 1$ and $G = C_2 \times C_2$. Here we have

$$\begin{aligned}[\mathbb{P}^1 \circlearrowleft G] =& (\langle 1 \rangle, G \circlearrowright k(t)) + (\langle g_1 \rangle, G/\langle g_1 \rangle \circlearrowright \{(1 : \pm i)\}, \chi_1) \\ &+ (\langle g_2 \rangle, G/\langle g_2 \rangle \circlearrowright \{(1:0), (0:1)\}, \chi_2) \\ &+ (\langle g_1 g_2 \rangle, G/\langle g_1 g_2 \rangle \circlearrowright \{(1 : \pm 1)\}, \chi_1 + \chi_2).\end{aligned}$$

The action on $K(\mathbb{P}^1) = k(t)$ is by $g_1(t) = -t$ and $g_2(t) = -1/t$.

Blowup relations ensure that $[X \circlearrowleft G]$ is a well-defined G-birational invariant – see Section 5.3 for more details.

There is a distinguished subgroup

$$\mathrm{Burn}_n^{\mathrm{triv}}(G) \subset \mathrm{Burn}_n(G)$$

generated by symbols $(1, G \subset K = K(X))$. For 'bootstrapping' purposes – where we seek invariants of n-folds in terms of lower-dimensional strata with nontrivial stabilizers – we may suppress these tautological symbols to get a quotient

$$\mathrm{Burn}_n(G) \to \mathrm{Burn}_n^{\mathrm{nontriv}}(G).$$

And there is also a natural quotient group

$$\mathrm{Burn}_n(G) \to \mathrm{Burn}_n^G(G)$$

obtained by suppressing all symbols $(H, G/H \subset K, \beta)$ where H is a proper subgroup of G. By [23, Prop. 8.1 and Prop. 8.2], there are natural surjective homomorphisms

$$\mathrm{Burn}_n^G(G) \to \mathcal{B}_n(G, k) \to \mathcal{B}_n(G).$$

Example 5.3. The group $\mathrm{Burn}_1^{\mathrm{nontriv}}(G)$ is freely generated by nontrivial subgroups $H \subset G$ and injective characters $a : H \hookrightarrow \mathbb{G}_m$, e.g. $\mathrm{Burn}_1^{\mathrm{nontriv}}(C_N) \cong \mathbb{Z}^{N-1}$ and $\mathrm{Burn}_1^{\mathrm{nontriv}}(C_2 \times C_2) = \mathbb{Z}^3$.

5.3. **Examples of blowup relations.** We illustrate the relations for fixed points of cyclic actions on surfaces. More computations are presented in Section 5.4; the reader may refer to Section 4 of [23] for the general formalism. All the key ideas are manifest in the surface case because the full set of blowup relations follows from those in codimension two – see [23, Prop. 8.1] and the special case Prop. 2.1 above.

Suppose that $G = C_N$ acts on the surface X with fixed point $\mathfrak{p}$ and weights a_1 and a_2 that generate $A = \mathrm{Hom}(C_N, \mathbb{G}_m)$. Let $\widetilde{X}$ denote the blowup of X at $\mathfrak{p}$ and $E \simeq \mathbb{P}^1$ the exceptional divisor. Let $\overline{H} = \ker(a_1 - a_2) \subset G$ denote the generic stabilizer of E which acts faithfully on the normal bundle $\mathcal{N}_{E/\widetilde{X}}$ via $\bar{a}_1 = \bar{a}_2 \in \overline{A} := \mathrm{Hom}(\overline{H}, \mathbb{G}_m)$.

Assume first that a_1 and a_2 are both prime to N, so $\mathfrak{p}$ is an isolated fixed point of X. The quotient $G/\overline{H}$ (when nontrivial) acts faithfully on $E = \mathbb{P}^1$ with fixed points $\mathfrak{p}_1$ and $\mathfrak{p}_2$. If $\overline{H} = G$ then $a_1 = a_2$ and we get the relation

$$(G, \mathrm{triv} \subset k = K(\mathfrak{p}), (a_1, a_1)) = (G, \mathrm{triv} \subset k(t) = K(E), (a_1)).$$

If $\mathrm{triv} \subsetneq H' \subsetneq G$ then

$$\begin{aligned}(G, \mathrm{triv} \subset k = K(\mathfrak{p}), (a_1, a_2)) &= (\overline{H}, G/\overline{H} \subset k(t) = K(E), \bar{a}_1 = \bar{a}_2) \\ &+ (G, \mathrm{triv} \subset k = K(\mathfrak{p}_2), (a_1, a_2 - a_1)) \\ &+ (G, \mathrm{triv} \subset k = K(\mathfrak{p}_1), (a_2, a_1 - a_2)) \end{aligned} \tag{5.4}$$

where $G/\overline{H}$ acts on t by a primitive dth root of unity with $d = |G/\overline{H}|$. If $\overline{H}$ is trivial then

$$\begin{aligned}(G, \mathrm{triv} \subset k = K(\mathfrak{p}),(a_1, a_2)) &= (G, \mathrm{triv} \subset k = K(\mathfrak{p}_2), (a_1, a_2 - a_1)) \\ &+ (G, \mathrm{triv} \subset k = K(\mathfrak{p}_1), (a_2, a_1 - a_2)). \end{aligned} \tag{5.5}$$

Assume now that $a_1 = m_1 n_1$ and $a_2 = m_2 n_2$ where $n_1, n_2 | N$ (and are relatively prime modulo N) and m_1 and m_2 are prime to N and each other. Then we have

$$\mathfrak{p} \in F_1 \cap F_2$$

for irreducible curves F_1 and F_2 with generic stabilizers C_{n_1} and C_{n_2} respectively. Let $\widetilde{F_1}, \widetilde{F_2} \subset \widetilde{X}$ denote the proper transforms of F_1 and F_2 and $\mathfrak{p}_1, \mathfrak{p}_2 \in E$ their intersections with the exceptional divisors. Thus the contribution to the strata containing $\mathfrak{p}$ is

$$\begin{aligned}&(G, \mathrm{triv} \subset K(\mathfrak{p}),(a_1, a_2)) \\ +(C_{n_1}, C_N/C_{n_1} \subset K(F_1), a_2) &+ (C_{n_2}, C_N/C_{n_2} \subset K(F_2), a_1)\end{aligned}$$

with the latter two terms appearing in the symbol sum on $\widetilde{X}$, with the F_i replaced by the $\widetilde{F_i}$. Note that $\overline{H} = \ker(a_1 - a_2) \subsetneq G$ because $a_1 \not\equiv a_2$. Here the blowup formula takes the form (5.4) or (5.5) depending on whether H' is trivial or not.

Now suppose that $a_2 = 0$. Let $F \subset X$ denote the irreducible component of the fixed locus containing $\mathfrak{p}$, so that a_1 is the character by which G acts on $\mathcal{N}_{F/X}$. Write $\widetilde{F} \subset \widetilde{X}$ for the proper transform of F, $\mathfrak{p}_1 = \widetilde{F} \cap E$, and $\mathfrak{p}_2 \in E$ for the other fixed point. Here we get the relation

$$(G, G \subset K(F), a_1) = (G, G \subset K(F), a_1) + (G, G \subset K(\mathfrak{p}_2), (a_1, -a_1)),$$

whence the latter term vanishes.

5.4. **Examples.** We now complement the computations in Section 3, for $G = C_N$, and small N. As before, we work over an algebraically-closed based field k of characteristic zero.

($N = 2$)

- $\mathcal{B}_2(C_2) = 0$.

- $\mathcal{B}_2(C_2, k) \cong \mathrm{Burn}_2^{C_2}(C_2)$; has a copy of $\mathcal{B}_1(C_2) = \mathbb{Z}$, for every isomorphism class of curves of positive genus.
- $\mathrm{Burn}_2(C_2) = \mathrm{Burn}_2^{\mathrm{triv}}(C_2) \oplus \mathrm{Burn}_2^{C_2}(C_2)$.

($N = 3$)

- $\mathcal{B}_2(C_3) \cong \mathbb{Z}$, generated by $[1, 1]$,

$$[1, 2] = 0, \qquad [2, 2] = -[1, 1].$$

- $\mathcal{B}_2(C_3, k) \cong \mathrm{Burn}_2^{C_3}(C_3)$, is a direct sum of $\mathbb{Z}$, corresponding to points and rational curves, and a copy of $\mathcal{B}_1(C_3) = \mathbb{Z}^2$ for every isomorphism class of curves of positive genus.
- $\mathrm{Burn}_2(C_3) = \mathrm{Burn}_2^{\mathrm{triv}}(C_3) \oplus \mathrm{Burn}_2^{C_3}(C_3)$.

($N = 4$)

- $\mathcal{B}_2(C_4) \cong \mathbb{Z}$, generated by $[1, 2]$,

$$[1, 1] = 2[1, 2], \quad [1, 3] = 0, \quad [2, 3] = -[1, 2], \quad [3, 3] = -2[1, 2].$$

- $\mathcal{B}_2(C_4, k)$ is a direct sum of $\mathbb{Z}$, corresponding to points and rational curves, and a copy of $\mathcal{B}_1(C_4) = \mathbb{Z}^2$ for every isomorphism class of curves of positive genus.
- $\mathrm{Burn}_2(C_4) = \mathrm{Burn}_2^{\mathrm{triv}}(C_4) \oplus \mathrm{Burn}_2^{\mathrm{nontriv}}(C_4)$: $\mathrm{Burn}_2^{\mathrm{nontriv}}(C_4)$ has, for every isomorphism class of curves of positive genus, a copy of $\mathcal{B}_1(C_4)$ and a copy of $\mathcal{B}_1(C_2)$, with an additional copy of $\mathcal{B}_1(C_2)$ for every isomorphism class of curves of positive genus with involution.

  We claim points and rational curves contribute

$$\mathbb{Z}^2 \subset \mathrm{Burn}_2^{\mathrm{nontriv}}(C_4),$$

  generated by $[1, 2]$ and $[2, 3]$ where

$$[i, j] = (C_4, k, (i, j)), \quad (i, j) = (1, 1), (1, 2), (1, 3), (2, 3), (3, 3).$$

  Abusing notation, write

$$[1, 0] = (C_4, k(t), 1) \quad [3, 0] = (C_4, k(t), 1).$$

  We write down the blowup relations, both orbits of points with special stabilizers and orbits on one-dimensional strata with

nontrivial stabilizer:

$$\begin{aligned}
[0,1] &= [0,1]+[1,3]\\
[0,3] &= [0,3]+[1,3]\\
[1,1] &= [1,0]\\
[1,2] &= [1,1]+[2,3]\\
[1,3] &= [1,2]+[2,3]+(C_2, C_2 \circlearrowright k(t), 1)\\
[2,3] &= [1,2]+[3,3]\\
[3,3] &= [3,0]\\
(C_2, C_2 \circlearrowright k^2, (1,1)) &= (C_2, C_2 \circlearrowright k(t)^2, 1)\\
(C_2, C_2 \circlearrowright k(t), 1) &= (C_2, C_2 \circlearrowright k(t), 1)+(C_2, C_2 \circlearrowright k(t)^2, 1)\\
(C_2, C_2 \circlearrowright k(t)^2, 1) &= (C_2, C_2 \circlearrowright k(t)^2, 1)+(C_2, C_2 \circlearrowright k(t)^2, 1)
\end{aligned}$$

Thus we find

$$\begin{gathered}
[1,3]=0,\\
[0,3]=[3,3]=-[1,2]+[2,3],\\
[0,1]=[1,1]=[1,2]-[2,3],\\
(C_2, C_2 \circlearrowright k(t), 1) = -[1,2]-[2,3],\\
(C_2, C_2 \circlearrowright k^2(t), 1) = 0,\\
(C_2, C_2 \circlearrowright k^2, (1,1)) = 0.
\end{gathered}$$

Here k^n denotes the total ring of fractions for an orbit of length n and $k^n(t)$ the total ring of fractions of the exceptional locus of the blowup of such an orbit. Furthermore, $C_2 \circlearrowright k(t)$ is via $t \mapsto -t$.

For example, the linear action on $\mathbb{P}^2$

$$(x : y : z) \mapsto (x : iy : -iz)$$

has invariant

$$[1,3]+[1,2]+[2,3]+(C_2, C_2 \circlearrowright k(y/z), 1) = 0 \in \mathrm{Burn}_2^{\mathrm{nontriv}}(C_4).$$

We close this section with a noncyclic example:

$(C_2 \times C_2)$
Write

$$G = C_2 \times C_2 = \{1, g_1, g_2, g_3 = g_1 g_2\}$$

and

$G^\vee = \{0, \chi_1, \chi_2, \chi_3 = \chi_1+\chi_2\}$, $\chi_1(g_1) = \chi_2(g_2) = -1, \chi_1(g_2) = \chi_2(g_1) = 1$

as before.

- $\mathcal{B}_2(C_2 \times C_2) = \{0, [\chi_1, \chi_2], [\chi_1, \chi_3], [\chi_2, \chi_3]\}$ with the structure of the Klein four group presented in Section 3.
- $\mathcal{B}_2(C_2 \times C_2, k) \cong \mathcal{B}_2(C_2 \times C_2)$ as $\mathcal{B}_1(C_2 \times C_2) = 0$.
- $\mathrm{Burn}_2(C_2 \times C_2) = \mathrm{Burn}^{\mathrm{triv}}(C_2 \times C_2) \oplus \mathrm{Burn}_2^{\mathrm{nontriv}}(C_2 \times C_2)$ where the second term is a direct sum of the subgroup R generated by points and rational curves, a copy of

  $$\mathrm{Burn}_1^{\mathrm{nontriv}}(C_2 \times C_2) \cong \mathbb{Z}^3$$

  for each curve of positive genus, and another copy for each curve of positive genus equipped with an involution.

The group R fits into an exact sequence

$$0 \to \mathrm{Burn}_1^{\mathrm{nontriv}}(C_2 \times C_2) \to R \to \mathcal{B}_2(C_2 \times C_2) \to 0$$

obtained by computing generators and relations.

Generators: Here we take $1 \le i < j \le 3$:

$$\begin{aligned} [\chi_i, \chi_j] &:= (G, \mathrm{triv} \circlearrowright k, (\chi_i, \chi_j)) \\ e_i &:= (\langle g_i \rangle, G/\langle g_i \rangle \circlearrowright k(t), \bar{\chi}_i = 1) \\ q_i &:= (\langle g_i \rangle, G/\langle g_i \rangle \circlearrowright k^2, (\bar{\chi}_i, \bar{\chi}_i) = (1,1)) \\ f_i &:= (\langle g_i \rangle, G/\langle g_i \rangle \circlearrowright k^2(t), \bar{\chi}_i = 1) \end{aligned}$$

Relations: Here we choose h so that $\{h, i, j\} = \{1, 2, 3\}$:

$$\begin{aligned} [\chi_i, \chi_j] &= e_h + [\chi_h, \chi_i] + [\chi_h, \chi_j] \quad \text{(blow up fixed point)} \\ q_i &= f_i \quad \text{(blow up orbit } q_i\text{)} \\ e_i &= e_i + f_i \quad \text{(blow up general orbit of } e_i\text{)} \end{aligned}$$

Thus the q_i and f_i are zero and we have

$$R/\langle e_1, e_2, e_3 \rangle = \mathcal{B}_2(C_2 \times C_2).$$

We revisit the actions of $C_2 \times C_2$ on rational surfaces in Section 3 using these new techniques:

(1) the action $(x, y) \mapsto (\pm x^{\pm 1}, y)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has invariant

$$f_1 + f_2 + f_3 = 0;$$

(2) the action $(x, y) \mapsto (\pm x, \pm y)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has invariant

$$2e_1 + 2e_2 + 4[\chi_1, \chi_2] = 0;$$

(3) the diagonal action on $\mathbb{P}^1 \times \mathbb{P}^1$ has invariant

$$2(q_1 + q_2 + q_3) = 0;$$

(4) the action on the conic fibration admits an elliptic curve

$$F = \{y_1^2(x_1 - ax_2)(x_1 + ax_2) = y_2^2(x_1 - bx_2)(x_1 + bx_2)\}$$

that is fixed by g_2 and fibers over $(x_1 : x_2) = (0 : 1), (1 : 0)$ fixed by g_1 and thus has invariant

$$4[\chi_1, \chi_2] + 2e_1 + (\langle g_2 \rangle, \langle g_1 \rangle \circlearrowleft K(F), 1) \neq 0$$

where g_1 acts on $K(F)$ by $x_1/x_2 \mapsto -x_1/x_2$;

(5) the action on the degree two del Pezzo surface has nontrivial invariant arising from the positive genus curves fixed by g_1 and g_2;

(6) the action on the degree one del Pezzo surface has nontrivial invariant arising from the positive genus curves fixed by the involution.

5.5. **Limitation of the birational invariant.** It is an elementary observation, recorded in [36, App. A], that the presence of a point fixed by a given abelian subgroup H of G is a birational invariant of a smooth projective variety X with generically free G-action. Two smooth n-dimensional projective varieties with generically free G-action might be distinguished in this way but nevertheless have the same class in $\mathrm{Burn}_n^{\mathrm{nontriv}}(G)$.

Indeed, letting C_2 act on $\mathbb{P}^1$, we consider the corresponding product action of $C_2 \times C_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$. As well, the action of $C_2 \times C_2$ on $\mathbb{P}^1$ gives rise to a diagonal action on $\mathbb{P}^1 \times \mathbb{P}^1$. The former, but not the latter, has a point fixed by $C_2 \times C_2$, so the actions belong to two distinct birational classes. However, both actions give rise to a vanishing class in $\mathrm{Burn}_2^{\mathrm{nontriv}}(C_2 \times C_2)$.

5.6. **Reprise: Cyclic groups on rational surfaces.** As already discussed in Section 3, the presence of higher genus curves in the fixed locus of the action of a cyclic group of prime order on a rational surface is an important invariant in the study of the plane Cremona group. For example, for $G = C_2$, these curves make up entirely the group $\mathrm{Burn}_2^G(G)$ and famously characterize birational involutions of the plane up to conjugation [4].

For more general cyclic groups acting on rational surfaces, we recover the NFCA invariant of Blanc [10], which governs his classification.

We recall the relevant definitions: Let $g \in \mathrm{Cr}_2$ be a nontrivial element of the plane Cremona group, of finite order m.

- *Normalized fixed curve* [13]:

$$\mathrm{NFC}(g) := \begin{cases} \emptyset \text{ if no curve of positive genus is fixed by } g, \\ \text{isomorphism class of the normalization of fixed curve.} \end{cases}$$

- *Normalized fixed curve with action*:

$$\mathrm{NFCA}(g) := \left((\mathrm{NFC}(g^r), g\mid_{\mathrm{NFC}(g^r)}\right)_{r=1}^{m-1},$$

where $g\mid_{\mathrm{NFC}(g^r)}$ is the automorphism induced by g on $\mathrm{NFC}(g^r)$.

One of the main results in this context is the following characterization:

Theorem 5.4. [10] *Two cyclic subgroups G and H of order m of Cr_2 are conjugate if and only if*

$$\mathrm{NFCA}(g) = \mathrm{NFCA}(h),$$

for some generators g of G and h of H.

It follows immediately from the definition that the information encoded in $\mathrm{Burn}_2(G)$, for $G = C_m$, is *equivalent* to $\mathrm{NFCA}(g)$.

Remark 5.5. It would be interesting to use symbol invariants to organize the classification of cyclic group actions on rational threefolds.

6. Cubic fourfolds

In this section we discuss several illustrative examples, showing various aspects of the new invariants introduced above. Equivariant geometry in dimension ≤ 3 is, in principle, accessible via the Minimal Model Program, and there is an extensive literature on factorizations of equivariant birational maps. We focus on dimension four, and in particular, on cubic fourfolds.

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. No examples are known to be irrational! Here we show that there are actions $X \circlearrowleft G$ where G-equivariant rationality fails, including actions on rational cubic fourfolds.

We found it useful to consult the classification of possible abelian automorphism groups of cubic fourfolds in [29]. Here is a list of $N > 1$, such that the cyclic group C_N acts on a smooth cubic fourfold:

$$N = 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 18, 21, 24, 30, 32, 33, 36, 48.$$

Note that

$$\mathcal{B}_4(C_N) \otimes \mathbb{Q} = 0, \quad \text{for all } N < 27,\ N = 30, 32.$$

We record their dimensions $d_{\mathbb{Q}}$ in the remaining cases:

N	33	36	48
$d_{\mathbb{Q}}$	2	3	7

One can also work with finite coefficients: let

$$d_p = d_p(N) := \dim \mathcal{B}_4(C_N) \otimes \mathbb{F}_p,$$

we have $d_2, d_3, d_5 = 0$, for all $N \leq 15$, and $N = 18, 21$. In the other cases, we find:

N	16	24	30	32	33	36	48
d_2	1	5	10	12	3	19	50
d_3	0	0	0	0	2	3	7
d_5	0	0	0	0	2	3	7
d_7	0	0	0	0	2	3	7

Thus, to exhibit applications of $\mathcal{B}_4(C_N)$ we have to look at large N.

Using $\mathcal{B}_n(G)$: Consider $X \subset \mathbb{P}^5$ given by

$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_4^2x_5 + x_5^2x_0 + x_0^3 = 0. \tag{6.1}$$

It carries the action of $G = C_{36}$, with weights

$$(0, 4, 28, 16, 9, 18)$$

on the ambient projective space, and isolated G-fixed points

$$P_1 = (0 : 1 : 0 : 0 : 0 : 0), \ldots, P_5 := (0 : 0 : 0 : 0 : 0 : 1).$$

Computing the weights in the corresponding tangent spaces, we find that $\beta(X)$ equals

$$[4, 24, 31, 22] + [28, 24, 19, 10] + [24, 12, 7, 34] + [9, 5, 17, 29] + [14, 26, 2, 9].$$

Solving a system of 443557 linear equations in 82251 variables, we find, by computer, that

$$\beta(X) \neq \beta(\mathbb{P}^4 \circlearrowleft C_{36}) = 0 \in \mathcal{B}_4(C_{36}) \otimes \mathbb{F}_2 = \mathbb{F}_2^{19}.$$

This implies that X is not G-equivariantly birational to $\mathbb{P}^4$.

Using co-multiplication: The fourfold $X \subset \mathbb{P}^5$ given by

$$x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_0 + x_0^3 + x_1^3 = 0 \tag{6.2}$$

carries an action of C_{48} with weights $(0, 16, 3, -6, 12, -24)$, and isolated fixed points. We find that $\beta(X) \in \mathcal{B}_4(C_{48})$ equals

$$[-3, 13, 9, -27] + [6, 22, 9, -18] + [-12, 4, -9, -18] + [40, 27, 18, 36].$$

Here, we apply the co-multiplication formula from Section 2.2 to the image $\beta^-(X)$ of $\beta(X)$ under the projection

$$\mathcal{B}_4(C_{48}) \to \mathcal{B}_4^-(C_{48}).$$

Let

$$G' := \mathbb{Z}/3\mathbb{Z}, \quad G'' := \mathbb{Z}/16\mathbb{Z}, \quad n' = 1 \quad \text{and} \quad n'' = 3.$$

We have a homomorphism

$$\nabla^- : \mathcal{B}_4^-(C_{48}) \to \mathcal{B}_1^-(C_3) \otimes \mathcal{B}_3^-(C_{16})$$

and we find that $\nabla^-(\beta^-(X))$ equals

$$[1]^- \otimes ([-1,3,-9]^- + [2,3,-6]^- + [-4,-3,-6]^- + [9,6,12]^-).$$

Now we are computing in $\mathcal{B}_3^-(C_{16})$, a much smaller group. We have

$$\dim \mathcal{B}_3(C_{16}) \otimes \mathbb{Q} = 3, \text{ but } \dim \mathcal{B}_3^-(C_{16}) \otimes \mathbb{Q} = 0,$$

however

$$\dim \mathcal{B}_3(C_{16}) \otimes \mathbb{F}_2 = 8 \text{ and } \dim \mathcal{B}_3^-(C_{16}) \otimes \mathbb{F}_2 = 7.$$

We find, by computer, that

$$[-4,-3,-6] = [9,6,12] \in \mathcal{B}_3(C_{16}) \otimes \mathbb{F}_2,$$

so that the sum of these terms does not contribute to $\nabla^-(\beta^-(X))$, and check that

$$[-1,3,-9]^- + [2,3,-6]^- = [1,2,10]^- \neq 0 \in \mathcal{B}_3^-(C_{16}).$$

It follows that

$$\nabla^-(\beta^-(X)) \neq 0, \quad \text{thus} \quad \beta^-(X) \neq 0 \in \mathcal{B}_4^-(C_{48}),$$

and this action of C_{48} on the cubic fourfold is not equivariantly birational to a linear action on $\mathbb{P}^4$.

Using $\mathcal{B}_n(G,k)$: There is also another way to analyze the fourfold in (6.2): observe that the divisor $Y \subset X$, a smooth cubic threefold given by $x_1 = 0$, is fixed by $C_3 \subset C_{48}$. This divisor is irrational, and we get a nontrivial contribution to $\beta_k(X) \in \mathcal{B}_4(C_3,k)$, in the summand labeled by $Y \in \mathrm{Bir}_{1,0}$; thus X is not even C_3-equivariantly birational to $\mathbb{P}^4$.

The fourfold $X \subset \mathbb{P}^5$ given by

$$f_3(x_0,x_1,x_2) + x_3^2x_4 + x_4^2x_5 + x_5^2f_1(x_0,x_1,x_2) = 0$$

carries the action of $G = C_8$, with weights

$$(0,0,0,1,6,4).$$

The fourfold is smooth, e.g., for $f_3 = x_0^3 + x_1^3 + x_2^3$ and $f_1 = x_0$.

Here, there are no isolated fixed points, but we find information from fixed point loci in higher dimension. The G-fixed locus contains the degree 3 curve Y given by

$$x_3 = x_4 = x_5 = f_3(x_0, x_1, x_2) = 0,$$

which is smooth for appropriate f_3. Thus we get a contribution to $\beta_k(X) \in \mathcal{B}_4(G, k)$, in the summand labeled by $Y \in \mathrm{Bir}_{3,2}$:

$$[7, 2, 4] \neq 0 \in \mathcal{B}_3(C_8) \cong \mathbb{F}_2.$$

Here, we solve 289 linear equations in 120 variables. This implies that X is not G-equivariantly birational to $\mathbb{P}^4$. Of course, this also follows by observing that the fourth power of the generator fixes a cubic threefold.

Using $\mathrm{Burn}_n(G)$**:** Consider $X \subset \mathbb{P}^5$ given by

$$x_0x_1^2 + x_0^2x_2 - x_0x_2^2 - 4x_0x_4^2 + x_1^2x_2 + x_3^2x_5 - x_2x_4^2 - x_5^3 = 0. \quad (6.3)$$

It carries the action of $G = C_6$, which acts with weights

$$(0, 0, 0, 1, 3, 4).$$

The cubic fourfold X is rational, since it contains the disjoint planes

$$x_0 = x_1 - x_4 = x_3 - x_5 = 0 \quad \text{and} \quad x_2 = x_1 - 2x_4 = x_3 + x_5 = 0.$$

Noticing a cubic surface $S \subset X$, with C_3-stabilizer and scalar action on the normal bundle, the fact that its C_2-action fixes an elliptic curve lets us conclude, by [11], that the cubic surface is not stably C_2-equivariantly rational; the corresponding symbol

$$[C_3, C_2 \circlearrowright K(S), \beta] \neq 0 \in \mathrm{Burn}_4(C_6),$$

moreover, it does not interact with any other symbols in $[X \circlearrowleft G]$, which implies that X is not G-birational to $\mathbb{P}^4$ with linear action. In this case, no subgroup of C_6 fixes a hyperplane section.

We discuss obstructions of such type in Section 7.4 below – formally, $\mathrm{Burn}_4(C_6)$ admits a projection to $\mathbb{Z}$ that distinguishes the equivariant birational class of X from that of $\mathbb{P}^4$ with linear action.

Kuznetsov [24] conjectures which cubic fourfolds X are rational, in terms of their derived categories $\mathsf{D}^b(X)$. Consider the line bundles $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ and the right orthogonal complement

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp}. \quad (6.4)$$

Conjecturally, X is rational if and only if $\mathcal{A}_X \cong \mathsf{D}^b(S)$, the bounded derived category of a K3 surface S. However, the K3 surface need

not be canonically determined as there are many examples of derived-equivalent but nonisomorphic K3 surfaces. Indeed the following conditions on complex projective K3 surfaces S_1 and S_2 are equivalent [31]

- $\mathsf{D}^b(S_1) \cong \mathsf{D}^b(S_2)$;
- the Mukai lattices
$$\widetilde{H}(S_1,\mathbb{Z}) \cong \widetilde{H}(S_2,\mathbb{Z})$$
as Hodge structures;
- the transcendental cohomology lattices
$$H^2_{\mathrm{tran}}(S_1,\mathbb{Z}) \cong H^2_{\mathrm{tran}}(S_2,\mathbb{Z})$$
as Hodge structures.

There is an alternative Hodge-theoretic version of the conjecture: A smooth cubic fourfold X is rational if and only if there exists a K3 surface S and an isomorphism of integral Hodge structures

$$H^4(X,\mathbb{Z})_{\mathrm{tran}} \cong H^2(S,\mathbb{Z})_{\mathrm{tran}}(-1), \tag{6.5}$$

where tran denotes the orthogonal complement of the Hodge classes and (-1) designates the Tate twist. Work of Addington and Thomas [1], and recent extensions [3, Cor. 1.7], show that the conditions (6.4) and (6.5) are equivalent. In particular, both are stable under specialization, consisting of an explicit countable collection of divisors in moduli [15]. The main theorem of [21] – that rationality is stable under specializations of smooth projective varieties – gives the equivalence of Kuznetsov's conjecture with the Hodge-theoretic statement.

Suppose then that X admits an action of a finite group G. If X is rational – and the conjectures are true – then G naturally acts on $\mathcal{A}_X$ and $\mathsf{D}^b(S)$, for each surface S arising in (6.4). There is an induced action on $\widetilde{H}^*(S,\mathbb{Z})$ as well. It is natural to speculate that a G-equivariant birational map $\mathbb{P}^4 \dashrightarrow X$ should imply that we may choose S in its derived equivalence class so that the G-action on the Mukai lattice is induced by a G-action on S.

There are several possible obstructions to finding such an S:

- if S exists then there exists a sublattice of algebraic classes
$$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H^2(S,\mathbb{Z})^\perp$$
in the G-invariant part of the abstract Mukai lattice arising from $\mathcal{A}_X$;
- the action of G on $\mathrm{Pic}(S)$ preserves the ample cone of S.

The first condition fails when G permutes various derived equivalent K3 surfaces. The second condition fails if G includes a Picard-Lefschetz transformation associated with a smooth rational curve $\mathbb{P}^1 \subset S$. Derived equivalent K3 surfaces might have very different automorphism groups [16, Ex. 23]; this paper discusses descent of derived equivalence in the presence of Galois actions.

We mention some results on when the group action can be lifted to the associated K3 surface [32, §8]:

- if $G \neq \{1\}, G$ acts on X symplectically, i.e., acts trivially on $\mathrm{H}^1(\Omega^3_X)$, then S is unique;
- if X is the Klein cubic fourfold

$$x_1^3 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_2 = 0$$

  then X admits a symplectic automorphism of order 11 and $\mathcal{A}_X \cong \mathsf{D}^b(S)$ for a unique K3 surface, which has no automorphism of order 11.

We speculate that the Klein example should not be C_{11}-equivariantly rational, even though

$$\beta(X \circlearrowleft C_{11}) = 0 \in \mathcal{B}_4(C_{11}),$$

as the C_{11}-action has isolated fixed points and the target group is trivial [20, §8].

Question 6.1. Let X be a smooth cubic fourfold with the action of a (finite) group G. Suppose that $\mathcal{A}_X \cong \mathsf{D}^b(S)$ for a K3 surface S with G-action, compatible with the isomorphism. Does it follow that

$$[X \circlearrowleft G] = [\mathbb{P}^4 \circlearrowleft G] \in \mathrm{Burn}_4(G),$$

for some action of G on $\mathbb{P}^4$?

It is mysterious how the invariants in the Burnside groups interact with the actions on the Hodge structures on the middle cohomology of X. Obstructions to G-equivariant rationality arise from fixed loci in various dimensions but the Hodge theory encodes codimension-two cycles only. The example (6.3), which is rational but not C_6-rational, is particularly striking to us: How is the cubic surface in the fixed locus coupled with the associated K3 surfaces?

7. Nonabelian invariants

In this section, G is a finite group, not necessarily abelian.

7.1. **The equivariant Burnside group.** As in Section 5.2, it is defined as the quotient of the $\mathbb{Z}$-module generated by symbols

$$(H, N_G(H)/H \circlearrowright K, \beta),$$

similar to those in (5.2), by **blow-up relations**. The required relations are a bit complicated but similar in spirit to what was written above; precise definitions are in [23, Section 4]. The resulting group

$$\mathrm{Burn}_n(G)$$

carries a rich combinatorial structure, that remains largely unexplored.

7.2. **Resolution of singularities.** The class of $X \circlearrowleft G$, a projective variety with a generically free G-action, is computed on a suitable model of the function field $K(X)$. We explain how such a model may be found in practice. While this is a corollary of Bergh's 'destackification' procedure [6], the approach here can be helpful in specific examples.

We first review the resolution process of [36, §3]. A variety with group action as above is *in standard form with respect to a G-invariant divisor Y* if

- X is smooth and Y is a normal crossings divisor;
- the G action on $X \setminus Y$ is free;
- for every $g \in G$ and irreducible component $Z \subset Y$ either $g(Z) = Z$ or $g(Z) \cap Z = \emptyset$.

We recall several fundamental results. First, we can always put actions in standard form:

> If X is smooth and Y is a G-invariant closed subset such that G acts freely on $X \setminus Y$ then there exists a resolution
>
> $$\pi : \widetilde{X} \to X$$
>
> obtained as a sequence of blowups along smooth G-invariant centers, such that $\widetilde{X}$ is in standard form with respect to $\mathrm{Exc}(\pi) \cup \pi^{-1}(Y)$ [36, Th. 3.2].

Using a canonical resolution procedure as in [7] – take an embedded resolution of $Y \subset X$ and then blow up the proper transform of Y – may assume that

$$\pi : \widetilde{X} \setminus \pi^{-1}(Y) \xrightarrow{\sim} X \setminus Y.$$

Even if we start with Y as the complement of the open set where G acts freely, the resulting strict normal crossings divisor may properly contain the locus with nontrivial stabilizers.

An action in standard form has stabilizers of special type:

Assume that X is in standard form with respect to Y and $x \in X$ lies on m irreducible components of Y then the stabilizer H of x is abelian with $\leq m$ generators [36, Th. 4.1].

The proof of [36, Th. 4.1] (see Remark 4.4) yields étale-local coordinates on X about x

$$x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_m$$

such that

- H acts diagonally on all the coordinates;
- $y_1 = \cdots = y_l = z_1 = \cdots = z_m = 0$ coincides with X^H, i.e., these are the coordinates on which H acts nontrivially;
- $y_1 \cdots y_m = 0$ coincides with Y and the associated characters of $\chi_i : H \to \mathbb{G}_m$ generate $H' = \mathrm{Hom}(H, \mathbb{G}_m)$ so the induced representation

$$(\chi_1, \ldots, \chi_m) : H \hookrightarrow \mathbb{G}_m^m \tag{7.1}$$

is injective.

Suppose that X is in standard form with respect to Y with irreducible components $Y_1, \ldots, Y_s$. For each orbit of these under the action of G, consider the reduced divisor obtained by summing over the orbit. The resulting divisors $D(1), \ldots, D(r)$ have smooth support – by the definition of standard form – and

$$D(1) \cup \cdots \cup D(r) = Y_1 \cup \cdots \cup Y_s.$$

The line bundles $\mathcal{O}_X(D(i))$ are naturally G-linearized and thus descend to line bundles on the quotient stack $[X/G]$ and we obtain

$$\varphi : [X/G] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m \quad (r \text{ factors}).$$

We claim φ is representable. It suffices to check this by showing that the induced homomorphism of stabilizers is injective at each point [39, Tag 04YY]. For $x \in X$, fix the indices $i_1, \ldots, i_m$ so that $D(i_1), \ldots, D(i_m)$ are the components of Y containing x, and consider the induced

$$\varphi_x : [X/G] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m \quad (m \text{ factors}).$$

The homomomorphism on stabilizers is given by (7.1) which is injective.

Thus we have established:

Proposition 7.1. *Let X be a smooth projective variety with a generically free action by a finite group G, in standard form with respect to a divisor Y. Then the Assumptions of* [23] *hold and the invariants constructed there may be evaluated on X.*

We are specifically using the formulation of Assumption 2 in [23, Rem. 3.2iii]. This is an alternate perspective on [23, Prop. 3.3], which guarantees these Assumptions can be satisfied on a suitable blowup.

7.3. **The class of $X \circlearrowleft G$.** On a suitable model X, we consider each stratum $F \subset X$ with nontrivial (abelian) stabilizer $H \subset G$, the action of the normalizer $N_G(H)/H$ on (the orbit of) the stratum, and the induced action of H on its normal bundle, and record these in a symbol. Then we define

$$[X \circlearrowleft G] := \sum_{H \subseteq G} \sum_{F} (H, N_G(H)/H \curvearrowright K(F), \beta) \in \mathrm{Burn}_n(G), \quad (7.2)$$

as (5.3). The proof that this is a G-birational invariant relies on G-equivariant Weak Factorization and combinatorics [23, Section 5].

7.4. **Elementary observations.** As we discussed, the presence of higher genus curves in the fixed locus of the action of a cyclic group of prime order on a rational surface is an important invariant in the study of the plane Cremona group; see, e.g., [10]. These make up entirely the group $\mathrm{Burn}_2^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ and entirely characterize birational involutions of the plane up to conjugation [4].

More generally, for any nontrivial cyclic subgroup H of G and birational class of an $(n-1)$-dimensional variety Y, not birational to $Z \times \mathbb{P}^1$ for any variety Z of dimension $n-2$, we have a projection from $\mathrm{Burn}_n(G)$ onto the free abelian group on the $N_G(H)$-conjugacy classes of pairs (H', a), where H' is a subgroup, $H \subset H' \subset N_G(H)$, and $a \in H^\vee$ is a primitive character. This sends

$$(H, \mathrm{Ind}_{H'/H}^{N_G(H)/H}(K(Y)), a),$$

for any $H'/H \curvearrowright K(Y)$, to the generator indexed by the conjugacy class $[(H', a)]$ of the pair.

A more refined version of this observation might also relax the restriction on Y but take into account the action $H'/H \curvearrowright K(Y)$. We do not go into details, but only point out, for instance, that for $n = 2$ and $Y = \mathbb{P}^1$ there is a projection

$$\mathrm{Burn}_2(G) \to \bigoplus_{\substack{[(H',a)] \\ H'/H \text{ not cyclic}}} \mathbb{Z}.$$

Taking G to be the dihedral group of order 12 and H the center of G, we may distinguish between the two inclusions of G into the plane Cremona group considered in [19], see Section 7.6 below.

7.5. **Dihedral group of order 12.** We now compute $\mathrm{Burn}_2(G)$, for $G = D_6$, the dihedral group with generators

$$\rho, \quad \sigma, \quad \text{with} \quad \rho^6 = \sigma^2 = \rho\sigma\rho\sigma = e_{D_6}.$$

We list abelian subgroups, up to conjugacy:

- order 6: $C_6 = \langle\rho\rangle$
- order 4: $D_2 = \langle\rho^3, \sigma\rangle$
- order 3: $C_3 = \langle\rho^2\rangle$
- order 2: central $C_2 = \langle\rho^3\rangle$, noncentral $S := \langle\sigma\rangle$, $S' := \langle\rho^3\sigma\rangle$
- order 1: triv

The subgroup of order 4 and two noncentral subgroups of order 2 have normalizer D_2, the others are normal.

As before, we use $K(X)$ to denote the function field of the underlying surface X, K to denote the algebra of functions of a one-dimensional stratum (perhaps reducible) with nontrivial stabilizer, and k^n to denote the algebra of functions of a zero-dimensional orbit of length n. When we blow up such an orbit, we use $k^n(t)$ to denote the total ring of fractions of the exceptional locus.

Generators:

$(C_6, \langle\bar\sigma\rangle \circlearrowright K, (1))$
$(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, j)), \quad j = 1, \dots, 4,$
$(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (2, 3))$
$(D_2, \mathrm{triv} \circlearrowright k, (a_1, a_2)), \quad a_1, a_2 \in \mathbb{F}_2^2$, generating $\mathbb{F}_2^2$
$(C_3, \langle\bar\rho, \bar\sigma\rangle \circlearrowright K, (1))$
$(C_3, \langle\bar\rho, \bar\sigma\rangle \circlearrowright k^4, (1, 1))$
$(C_2, \langle\bar\rho, \bar\sigma\rangle \circlearrowright K, (1))$
$(S, C_2 \circlearrowright K, (1))$
$(S', C_2 \circlearrowright K, (1))$
$(\mathrm{triv}, D_6 \circlearrowright K(X), \mathrm{triv})$

Relations:

$$(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 1)) = (C_6, \langle\bar\sigma\rangle \circlearrowright k^2(t), (1))$$
$$(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 2)) = (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 1)) + (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 4))$$
$$(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 3)) = (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 2)) + (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (2, 3)) + (C_2, \langle\bar\rho, \bar\sigma\rangle \circlearrowright k^2(t), (1)),$$

where $\bar\rho$ acts by cube roots of unity on t

$$(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 4)) = (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1, 3)) + (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (2, 3)) + (C_3, \langle\bar\rho, \bar\sigma\rangle \circlearrowright k^2(t), (1)),$$

where $\bar\rho$ acts by -1 on t

$0 = 2(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1,4)) + (C_2, \langle\bar\rho,\bar\sigma\rangle \circlearrowright k^2(t), (1)),$
where $\bar\rho$ acts by cube roots of unity on t

$$(C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (2,3)) = (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1,2)) + (C_6, \langle\bar\sigma\rangle \circlearrowright k^2, (1,3))$$

$$(D_2, \mathrm{triv} \circlearrowright k, ((1,0),(0,1))) = (D_2, \mathrm{triv} \circlearrowright k, ((1,0),(1,1)))+ (D_2, \mathrm{triv} \circlearrowright k, ((0,1),(1,1))) + (S', C_2 \circlearrowright k(t), (1))$$

$$(D_2, \mathrm{triv} \circlearrowright k, ((1,0),(1,1))) = (D_2, \mathrm{triv} \circlearrowright k, ((1,0),(0,1)))+ (D_2, \mathrm{triv} \circlearrowright k, ((0,1),(1,1))) + (C_2, \langle\bar\rho,\bar\sigma\rangle \circlearrowright k^3(t), (1)),$$

permutation action on k^3 with $\bar\sigma$ acting by -1 on t

$$(D_2, \mathrm{triv} \circlearrowright k, ((0,1),(1,1))) = (D_2, \mathrm{triv} \circlearrowright k, ((1,0),(0,1)))+ (D_2, \mathrm{triv} \circlearrowright k, ((1,0),(1,1))) + (S, C_2 \circlearrowright k(t), (1))$$

$$(C_3, \langle\bar\rho,\bar\sigma\rangle \circlearrowright k^4, (1,1)) = (C_3, \langle\bar\rho,\bar\sigma\rangle \circlearrowright k^4(t), (1))$$

$0 = 2(C_3, \langle\bar\rho,\bar\sigma\rangle \circlearrowright k^4, (1,1))$
$0 = (C_2, \langle\bar\rho,\bar\sigma\rangle \circlearrowright k^6(t), (1))$

$0 = (S, C_2 \circlearrowright k^2(t), (1))$
$0 = (S', C_2 \circlearrowright k^2(t), (1))$

7.6. **Embeddings of $\mathfrak{S}_3 \times C_2$ into the Cremona group.** Iskovskikh [19] exhibited two nonconjugate copies of $G = \mathfrak{S}_3 \times C_2 = D_6$ in $\mathrm{BirAut}(\mathbb{P}^2)$:

- the action on $x_1 + x_2 + x_3 = 0$ by permutation and reversing signs, with model $\mathbb{P}^2$;
- the action on $y_1y_2y_3 = 1$ by permutation and taking inverses, with model a sextic del Pezzo surface.

To justify the interest in these particular actions we observe that G is the Weyl group of the exceptional Lie group G_2, which acts on the Lie algebra of the torus, respectively on the torus itself, and it is natural to ask whether or not these actions are equivariantly birational. It turns out that they are stably G-birational [25, Proposition 9.11], but not G-birational. The proof of failure of G-birationality in [19] relies on the classification of links, via the G-equivariant Sarkisov program.

Here we explain how to apply $\mathrm{Burn}_2(G)$ to this problem. Note that neither model above satisfies the stabilizer condition required in the Definition (7.2)! We need to replace the surfaces by appropriate models X and Y, in particular, to blow up points:

- $(x_1, x_2, x_3) = (0, 0, 0)$, with G as stabilizer;
- $(y_1, y_2, y_3) = (1, 1, 1)$, with G as stabilizer, and
$$(\omega, \omega, \omega), \quad (\omega^2, \omega^2, \omega^2), \quad \omega = e^{2\pi i/3},$$
with $\mathfrak{S}_3$ as stabilizer.

We describe these actions in more detail, following closely [19]. The action on $\mathbb{P}^2$, with coordinates $(u_0 : u_1 : u_2)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad \rho^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

There is one fixed point, $(1 : 0 : 0)$; after blowing up this point, the exceptional curve is stabilized by the central involution ρ^3, and comes with a nontrivial $\mathfrak{S}_3$-action, contributing the symbol

$$(C_2, \mathfrak{S}_3 \circlearrowright K(\mathbb{P}^1), (1)) \tag{7.3}$$

to $[X \hookleftarrow G]$. Additionally, the line $\ell_0 := \{u_0 = 0\}$ has as stabilizer the central C_2, contributing the same symbol. There are also other contributing terms, of the shape:

- $(C_6, C_2 \circlearrowright k^2, \beta)$
- $(D_2, \mathrm{triv} \circlearrowright k, \beta')$

for some weights β, β'.

A better model for the second action is the quadric

$$v_0v_1 + v_1v_2 + v_2v_0 = 3w^2,$$

where $\mathfrak{S}_3$ permutes the coordinates $(v_0 : v_1 : v_2)$ and the central involution exchanges the sign on w. There are no G-fixed points, but a conic $R_0 := \{w = 0\}$ with stabilizer the central C_2 and a nontrivial action of $\mathfrak{S}_3$. There are also:

- a G-orbit of length 2:
$$\{P_1 := (1 : 1 : 1 : 1), P_2 := (1 : 1 : 1 : -1)\},$$
exchanged by the central involution, each point has stabilizer $\mathfrak{S}_3$ – these points have to be blown up, yielding a pair of conjugated $\mathbb{P}^1$, with a nontrivial $\mathfrak{S}_3$-action;
- another curve $R_1 := \{v_0 + v_1 + v_2 = 0\}$ with effective G-action;
- additional points with stabilizers C_6 and D_2 in R_0 and R_1.

The essential difference is that the symbol (7.3) appears *twice* for the action on $\mathbb{P}^2$, and only *once* for the action on the quadric: the pair of conjugated $\mathbb{P}^1$ with $\mathfrak{S}_3$-action has trivial stabilizer and does not contribute. Further blow-ups will not introduce new curves of this type. Formally, examining the relations in Section 7.5, we see that the symbol (7.3) is not equivalent to any combination of other symbols, i.e., it is independent of all other symbols. This implies that

$$[X \circlearrowleft G] \neq [Y \circlearrowleft G] \in \mathrm{Burn}_2(G),$$

thus X and Y are not G-equivariantly birational. Note, that X and Y are equivariantly birational for any proper subgroup of G.

Remark 7.2. One can view the symbol (7.3) as the analog of a curve of higher genus in the fixed locus of an element in the classification of abelian actions on surfaces, as discussed in Section 3.

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Brown University, Box 1917, 151 Thayer Street, Providence, RI 02912, USA
E-mail address: `bhassett@math.brown.edu`

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
E-mail address: `andrew.kresch@math.uzh.ch`

Courant Institute, 251 Mercer Street, New York, NY 10012, USA
E-mail address: `tschinkel@cims.nyu.edu`

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA