INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 4, Pages 500–523

A MODIFIED PRIMAL-DUAL WEAK GALERKIN FINITE ELEMENT METHOD FOR SECOND ORDER ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

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Abstract. A modified primal-dual weak Galerkin (M-PDWG) finite element method is designed for the second order elliptic equation in non-divergence form. Compared with the existing PDWG methods proposed in [6], the system of equations resulting from the M-PDWG scheme could be equivalently simplified into one equation involving only the primal variable by eliminating the dual variable (Lagrange multiplier). The resulting simplified system thus has significantly fewer degrees of freedom than the one resulting from existing PDWG scheme. Optimal order error estimates are derived for the numerical approximations in the discrete H^2-norm, H^1-norm and L^2-norm respectively. Extensive numerical results are demonstrated for both the smooth and non-smooth coefficients on convex and non-convex domains to verify the accuracy of the theory developed in this paper.

Key words. Primal-dual, weak Galerkin, finite element methods, non-divergence form, Cordès condition, polyhedral meshes.

1. Introduction

In this paper, we consider the second order elliptic equation in non-divergence form which seeks an unknown function $u = u(x)$ such that

$$
\begin{aligned}
\sum_{i,j=1}^{d} a_{ij}\partial^2_{ij}u &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1}
$$

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is an open bounded domain with Lipschitz continuous boundary $\partial\Omega$, the load function $f \in L^2(\Omega)$, and the coefficient tensor $a = (a_{ij})_{d\times d} \in [L^\infty(\Omega)]^{d\times d}$ is symmetric, uniformly bounded and positive definite in the sense that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$
C_1\xi^T\xi \le \xi^T a\xi \le C_2\xi^T\xi, \qquad \forall \xi \in \mathbb{R}^d,\ x \in \Omega. \tag{2}
$$

For the simplicity of notation, denote by $\mathcal{L} := \sum_{i,j=1}^{d} a_{ij}\partial^2_{ij}$ the second order partial differential operator.

The second order elliptic problem in non-divergence form arises in various applications such as probability and stochastic processes [2]. This type of problem also plays an important role in the research of fully nonlinear partial differential equations in conjunction with linearization techniques (e.g., the Newton's iterative method) [1, 3]. In such applications, the coefficient tensor $a(x)$ is often hardly smooth. Therefore, it is crucial to develop effective numerical methods for the model problem (1) with nonsmooth coefficient tensor. Readers are referred to [6] for more details of recent work developed for the model problem (1) . f The goal of

Received by the editors February 14, 2021 and, in revised form, April 5, 2021.
2000 *Mathematics Subject Classification.* 65N30, 65N12, 35J15, 35D35.

this paper is to develop a modified primal-dual weak Galerkin (M-PDWG) scheme for the second order elliptic problem in nondivergence form (1), which is different from and advantageous over the one proposed in [6]. The system of equations arising from the M-PDWG scheme could be equivalently simplified into one equation by eliminating its dual variable (Lagrange multiplier). The simplified system involves only the primal variable and thus has significantly fewer degrees of freedom compared to the PDWG scheme proposed in [6]. The main contribution of the present paper is that the numerical scheme admists a simplified form with reduced computational complexity. Our theory for the M-PDWG method is based on two assumptions: (1) the H^2-regularity of the exact solution of the model problem (1); and (2) the coefficient tensor $a(x)$ is piecewise continuous and satisfies the uniform ellipticity condition (2). Optimal order error estimates are established for the primal variable in a discrete H^2-norm and for the dual variable in the L^2-norm. Moreover, the convergence theory is derived for the primal variable in the H^1 norm and L^2 norm under some smoothness assumptions for the coefficient tensor $a(x)$. Numerical examples are presented to illustrate the accuracy of the theory developed for the M-PDWG method.

The paper is organized as follows. In Section 2, we present the weak formulation for the model problem (1). Section 3 is devoted to a review of weak second order differential operator and its discretization. In Section 4, we describe the M-PDWG finite element method for the model problem (1). Section 5 presents a simplified system resulting form the M-PDWG method proposed in Section 4. Section 6 is devoted to a stability analysis for the M-PDWG scheme. Section 7 presents the error equations for the numerical scheme. In Section 8, we derive an optimal order error estimate for the M-PDWG method in a discrete H^2 norm. Section 9 establishes some error estimates in the usual H^1 norm and L^2 norm for the primal variable. In Section 10, the numerical experiments are presented for the M-PDWG scheme for smooth and non-smooth coefficient tensor $a(x)$ on convex and non-convex domains.

2. Variational Formulations

We shall briefly review the weak formulation of the second order elliptic model problem (1) in non-divergence form [6].

Theorem 2.1. [4] *Assume (1) $\Omega \subset \mathbb{R}^d$ is a bounded convex domain; (2) the coefficient tensor $a = (a_{ij}) \in [L^\infty(\Omega)]^{d\times d}$ satisfies the ellipticity condition (2); and (3) the Cordès condition holds true; i.e., there exists an $\varepsilon \in (0, 1]$ such that*

$$\frac{\sum_{i,j=1}^d a_{ij}^2}{(\sum_{i=1}^d a_{ii})^2} \leq \frac{1}{d-1+\varepsilon} \qquad \text{in } \Omega. \tag{3}$$

There exists a unique strong solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of the model problem (1) satisfying

$$\|u\|_2 \leq C\|f\|_0, \tag{4}$$

for any given $f \in L^2(\Omega)$, where C is a constant depending on d, the diameter of Ω, C_1, C_2 and ε.

Throughout this paper, we assume the model problem (1) has a unique strong solution in $H^2(\Omega) \cap H_0^1(\Omega)$ with a priori estimate (4).

The variational formulation of the model problem (1) seeks $u \in X = H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$b(u,\sigma) = (f,\sigma) \qquad \forall \sigma \in Y = L^2(\Omega),$$

where

$$b(u,\sigma) = (\mathcal{L}u,\sigma). \tag{5}$$

The regularity assumption (4) implies that the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition

$$\sup_{v\in X, v\neq 0} \frac{b(v,\sigma)}{\|v\|_X} \geq \alpha \|\sigma\|_Y,$$

for all $\sigma \in Y$, where α is a generic constant related to the constant C in the H^2 regularity estimate (4), $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the H^2 norm and the L^2 norm, respectively.

3. Discrete Weak Second Order Partial Derivative

This section will briefly review the weak second order partial derivative and its discrete version [5, 6].

Let T be a polygonal or polyhedral domain with boundary ∂T. Denote by $v = \{v_0, v_b, \mathbf{v}_g\}$ the weak function on the element T, where $v_0 \in L^2(T)$ and $v_b \in L^2(\partial T)$ are the values of v in the interior and on the boundary of T; and $\mathbf{v}_g = (v_{g1}, \ldots, v_{gd}) \in [L^2(\partial T)]^d$ is the value of ∇v on the boundary of T. Note that v_b and $\mathbf{v}_g$ may not necessarily be related to the traces of v_0 and ∇v_0 on ∂T. It is feasible to take v_b as the trace of v_0 and leave $\mathbf{v}_g$ completely free or vice versa.

Let $W(T)$ be the local space of the weak functions on T; i.e.,

$$W(T) = \{v = \{v_0, v_b, \mathbf{v}_g\} : v_0 \in L^2(T), v_b \in L^2(\partial T), \mathbf{v}_g \in [L^2(\partial T)]^d\}. \tag{6}$$

The weak second order partial derivative of the weak function $v \in W(T)$, denoted by $\partial^2_{ij,w} v$, is defined as a bounded linear functional on the Sobolev space $H^2(T)$ satisfying

$$(\partial^2_{ij,w} v, \varphi)_T = (v_0, \partial^2_{ji}\varphi)_T - \langle v_b n_i, \partial_j \varphi\rangle_{\partial T} + \langle v_{gi}, \varphi n_j\rangle_{\partial T}, \tag{7}$$

for any $\varphi \in H^2(T)$, where $\mathbf{n} = (n_1, \cdots, n_d)$ is the unit outward normal direction on ∂T.

Denote by $P_r(T)$ the space of polynomials with degree no more than $r \geq 0$ on T. A discrete version of $\partial^2_{ij,w} v$, denoted by $\partial^2_{ij,w,r,T} v$, is defined as the unique polynomial in $P_r(T)$ such that

$$(\partial^2_{ij,w,r,T} v, \varphi)_T = (v_0, \partial^2_{ji}\varphi)_T - \langle v_b n_i, \partial_j \varphi\rangle_{\partial T} + \langle v_{gi}, \varphi n_j\rangle_{\partial T}, \quad \forall \varphi \in P_r(T). \tag{8}$$

Applying the usual integration by parts to the first term on the right-hand side of (8) yields

$$(\partial^2_{ij,w,r,T} v, \varphi)_T = (\partial^2_{ij} v_0, \varphi)_T - \langle (v_b - v_0) n_i, \partial_j \varphi\rangle_{\partial T} + \langle v_{gi} - \partial_i v_0, \varphi n_j\rangle_{\partial T}, \tag{9}$$

for all $\varphi \in P_r(T)$, provided that $v_0 \in H^2(T)$.

4. Primal-Dual Weak Galerkin

Denote by $\mathcal{T}_h$ a finite element partition of the domain Ω into polygons in 2D or polyhedra in 3D which is shape regular as described in [7]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges or flat faces. Denote by h_T the diameter of the element $T \in \mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h} h_T$ the meshsize of the partition $\mathcal{T}_h$.

Let $k \geq 2$. Denote by $W_k(T)$ the local space of discrete weak functions; i.e.,

$$W_k(T) := \{v = \{v_0, v_b, \mathbf{v}_g\} \in P_k(T) \times P_k(e) \times [P_{k-1}(e)]^d,\ e \in \partial T \cap \mathcal{E}_h\}. \tag{10}$$

Patching $W_k(T)$ over all the elements $T \in \mathcal{T}_h$ through common value for v_b on the interior interface $\mathcal{E}_h^0$ gives the weak finite element space; i.e.,

$$W_{h,k} := \big\{\{v_0, v_b, \mathbf{v}_g\} :\ \{v_0, v_b, \mathbf{v}_g\}|_T \in W_k(T),\ T \in \mathcal{T}_h\big\}.$$

Let $W_{h,k}^0$ be the subspace of $W_{h,k}$ with vanishing boundary value for v_b on $\partial\Omega$; i.e.,

$$W_{h,k}^0 = \{\{v_0, v_b, \mathbf{v}_g\} \in W_{h,k},\ v_b|_e = 0, e \subset \partial\Omega\}.$$

We further introduce the finite element space

$$V_{h,k} = \Big\{\sigma :\ \sigma|_T \in V_k(T),\ T \in \mathcal{T}_h\Big\},$$

where $V_k(T)$ is chosen as either $P_{k-2}(T)$ or $P_{k-1}(T)$, as appropriate. The choice of $V_k(T) = P_{k-2}(T)$ has fewer degrees of freedom, while the choice $V_k(T) = P_{k-1}(T)$ results in more accurate M-PDWG solution.

For simplicity of notation, denote by $\partial^2_{ij,w} v$ the discrete weak second order partial differential operator defined by (8) with $V_r(T) = V_k(T)$ on each element T; i.e.,

$$(\partial^2_{ij,w} v)|_T = \partial^2_{ij,w,r,T}(v|_T), \qquad v \in W_{h,k}.$$

We introduce the bilinear forms

$$b_h(v,\sigma) = \sum_{T\in\mathcal{T}_h} b_T(v,\sigma), \quad v \in W_{h,k},\ \sigma \in V_{h,k}, \tag{11}$$

$$s_h(u,v) = \sum_{T\in\mathcal{T}_h} s_T(u,v), \quad u,v \in W_{h,k}, \tag{12}$$

where

$$b_T(v,\sigma) = \sum_{i,j=1}^{d} (a_{ij}\partial^2_{ij,w} v, \sigma)_T, \tag{13}$$

$$s_T(u,v) = h_T^{-3}\langle u_0 - u_b, v_0 - v_b\rangle_{\partial T} + h_T^{-1}\langle \nabla u_0 - \mathbf{u}_g, \nabla v_0 - \mathbf{v}_g\rangle_{\partial T}. \tag{14}$$

We further introduce a symmetric and nonnegative continuous bilinear form

$$c_h(\cdot,\cdot) : V_{h,k} \times V_{h,k} \to \mathbb{R},$$

satisfying the continuity property; i.e., there exists a constant C such that

$$c_h(\lambda,\mu) \leq Ch^4 \|\lambda\|_{2,h}\|\mu\|_{2,h} \tag{15}$$

for any $\lambda, \mu \in V_{h,k}$, where $\|\cdot\|_{s,h}$ is a discrete H^2 norm with partial derivatives taken locally on each element. From the usual inverse inequality, we have

$$c_h(\lambda,\mu) \leq C\|\lambda\|_0\|\mu\|_0, \tag{16}$$

for any $\lambda, \mu \in V_{h,k}$, where $\|\cdot\|$ is the L_2 norm.

Algorithm 4.1. (M-PDWG Finite Element Method) *A modified primal-dual weak Galerkin scheme for solving the second order elliptic problem (1) in non-divergence form seeks* $(u_h; \lambda_h) \in W_{h,k}^0 \times V_{h,k}$ *satisfying*

$$s_h(u_h, v) + b_h(v, \lambda_h) = 0, \qquad \forall v \in W_{h,k}^0, \tag{17}$$

$$-c_h(\lambda_h, \sigma) + b_h(u_h, \sigma) = (f, \sigma), \qquad \forall \sigma \in V_{h,k}. \tag{18}$$

Here u_h *is the primal variable and* λ_h *is the dual variable or Lagrange multiplier.*

5. M-PDWG Finite Element Methods

In order to greatly reduce the degrees of freedom and the computational complexity of the M-PDWG method (17)-(18), we shall eliminate the dual variable λ_h from the M-PDWG system resulting in a simplified system involving the primal variable u_h only.

Deonte by $\langle \cdot, \cdot \rangle$ the duality pairing between the two spaces. For the bilinear forms $s_h(\cdot, \cdot)$, $b_h(\cdot, \cdot)$ and $c_h(\cdot, \cdot)$, we associate the operators $S \in \mathcal{L}(W_{h,k}^0; (W_{h,k}^0)')$, $B \in \mathcal{L}(W_{h,k}^0; V_{h,k}')$ and $C \in \mathcal{L}(V_{h,k}; V_{h,k}')$ defined by

$$\langle Su, v \rangle = s_h(u, v), \qquad \forall u, v \in W_{h,k}^0,$$
$$\langle Bu, \mu \rangle = b_h(u, \mu), \qquad \forall u \in W_{h,k}^0, \mu \in V_{h,k},$$
$$\langle C\lambda, \mu \rangle = c_h(\lambda, \mu), \qquad \forall \lambda, \mu \in V_{h,k},$$

where we assume $c_h(\cdot, \cdot)$ is suitably constructed so that (15) is satisfied and the matrix C is invertible. As a specific example, for any $\rho, \sigma \in V_{h,k}$, we may use

$$c_h(\rho, \sigma) = \sum_{T \in \mathcal{T}_h} h_T^4 \left\{ (\rho, \sigma)_T + (\nabla \rho, \nabla \sigma)_T + \sum_{i,j=1}^{d} (\partial_{ij}^2 \rho, \partial_{ij}^2 \sigma)_T \right\}. \tag{19}$$

Let $B' \in \mathcal{L}(V_{h,k}; (W_{h,k}^0)')$ be the dual operator of B; i.e.,

$$\langle B'\mu, u \rangle = \langle Bu, \mu \rangle = b_h(u, \mu), \qquad \forall u \in W_{h,k}^0, \mu \in V_{h,k}.$$

The M-PDWG scheme (17)-(18) can be equivalently rewritten as follows: Find $(u_h; \lambda_h) \in W_{h,k}^0 \times V_{h,k}$ satisfying

$$Su_h + B'\lambda_h = 0, \quad \text{in} \quad (W_{h,k}^0)', \tag{20}$$

$$-C\lambda_h + Bu_h = f, \quad \text{in} \quad (V_{h,k})', \tag{21}$$

where $(W_{h,k}^0)'$ and $(V_{h,k})'$ are the dual spaces of $W_{h,k}^0$ and $V_{h,k}$, respectively. Note that C is invertible. Using (21), we have

$$\lambda_h = -C^{-1}(f - Bu_h),$$

which, combined with (20), leads to a simplified system as follows: Find $u_h \in W_{h,k}^0$, such that

$$(S + B'C^{-1}B)u_h = B'C^{-1}f. \tag{22}$$

Compared with the PDWG scheme for the second order elliptic problem in nondivergence form proposed in [6], the M-PDWG scheme admits a simplified form (22) involving only the primal variable u_h. The idea of M-PDWG method can be generalized to PDWG methods for other model PDEs by adding appropriately chosen $c_h(\cdot, \cdot)$ terms.

6. Stability and Solvability

We shall demonstrate the existence and uniqueness for the M-PDWG solution arising from Algorithm 4.1 through an inf-sup condition for the bilinear form $b_h(\cdot,\cdot)$.

Let $k \geq 2$. On each element T, denote by Q_0 the L^2 projection onto $P_k(T)$. On each edge or face $e \subset \partial T$, denote by Q_b and $\mathbf{Q}_g = (Q_{g1}, \ldots, Q_{gd})$ the L^2 projections onto $P_k(e)$ and $[P_{k-1}(e)]^d$, respectively. For any function $w \in H^2(\Omega)$, denote by $Q_h w$ the L^2 projection onto the weak finite element space $W_{h,k}$ such that on each element T, we have

$$Q_h w = \{Q_0 w, Q_b w, \mathbf{Q}_g(\nabla w)\}. \tag{23}$$

Denote by $\mathcal{Q}_h$ the L^2 projection onto the space $V_{h,k}$.

Lemma 6.1. [5] *For any $w \in H^2(T)$, the commutative property holds true*

$$\partial^2_{ij,w}(Q_h w) = \mathcal{Q}_h(\partial^2_{ij} w), \qquad i, j = 1, \ldots, d. \tag{24}$$

We introduce the semi-norm for the weak finite element space $W_{h,k}$; i.e.,

$$\|v\|^2_{2,h} = \sum_{T\in\mathcal{T}_h} \|\sum_{i,j=1}^{d} \mathcal{Q}_h(a_{ij}\partial^2_{ij} v_0)\|^2_T + s_h(v,v), \qquad \forall v \in W_{h,k}. \tag{25}$$

Lemma 6.2. [6] *Assume that the coefficient matrix $a = (a_{ij})$ is uniformly piecewise continuous in Ω with respect to the finite element partition $\mathcal{T}_h$. There exists a fixed $h_0 > 0$ such that if $v = \{v_0, v_b, \mathbf{v}_g\} \in W^0_{h,k}$ satisfies $\|v\|_{2,h} = 0$, then we have $v \equiv 0$ for $h \leq h_0$.*

We further introduce another semi-norm for the weak finite element space $W_{h,k}$; i.e., for any $v \in W_{h,k}$,

$$|\!|\!|v|\!|\!|^2_2 = \sum_{T\in\mathcal{T}_h} \|\sum_{i,j=1}^{d} \mathcal{Q}_h(a_{ij}\partial^2_{ij,w} v)\|^2_T + s_h(v,v). \tag{26}$$

The two semi-norms defined in (25) and (26) are equivalent, which is stated in the following lemma.

Lemma 6.3. [6] *Assume that the coefficient tensor $a = (a_{ij})$ is uniformly piecewise continuous in Ω with respect to the finite element partition $\mathcal{T}_h$. For any $v \in W_{h,k}$, there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that*

$$\alpha_1 \|v\|_{2,h} \leq |\!|\!|v|\!|\!|_2 \leq \alpha_2 \|v\|_{2,h}.$$

Lemma 6.4. [6] *(inf-sup condition) Assume that the coefficient tensor $a = (a_{ij})$ is uniformly piecewise continuous in Ω with respect to the finite element partition $\mathcal{T}_h$. For any $\sigma \in V_{h,k}$, there exists $v_\sigma \in W^0_{h,k}$ satisfying*

$$b_h(v_\sigma, \sigma) \geq \frac{1}{2}\|\sigma\|^2_0, \tag{27}$$

$$\|v_\sigma\|^2_{2,h} \leq C\|\sigma\|^2_0, \tag{28}$$

provided that the meshsize $h < h_0$ for a sufficiently small, but fixed parameter $h_0 > 0$.

Theorem 6.5. *Assume that the coefficient matrix $a = (a_{ij})$ is uniformly piecewise smooth in Ω with respect to the finite element partition $\mathcal{T}_h$. The M-PDWG finite element scheme (17)-(18) has a unique solution $(u_h; \lambda_h) \in W_{h,k}^0 \times V_{h,k}$, provided that the meshsize $h < h_0$ holds true for a sufficiently small, but fixed parameter $h_0 > 0$.*

Proof. It suffices to show that the homogeneous problem of (17)-(18) has only the trivial solution. To this end, assume $f = 0$. By choosing $v = u_h$ and $\sigma = \lambda_h$ in (17)-(18) we arrive at

$$s_h(u_h, u_h) + c_h(\lambda_h, \lambda_h) = 0,$$

which implies $s_h(u_h, u_h) = 0$ and $c_h(\lambda_h, \lambda_h) = 0$. From $s_h(u_h, u_h) = 0$, we have $u_0 = u_b$ and $\nabla u_0 = \mathbf{u}_g$ on each ∂T, which gives $u_h \in C^1(\Omega)$. Therefore, from (17), we have

$$b_h(v, \lambda_h) = 0, \qquad \forall v \in W_{h,k}^0.$$

From Lemma 6.4, for $\lambda_h \in V_{h,k}$, there exists $v_{\lambda_h} \in W_{h,k}^0$ satisfying

$$0 = b_h(v_{\lambda_h}, \lambda_h) \geq \frac{1}{2}\|\lambda_h\|_0^2,$$

which gives $\lambda_h = 0$ on each $T \in \mathcal{T}_h$ and further $\lambda_h \equiv 0$ in Ω. Substituting $\lambda_h \equiv 0$ in Ω into (18) yields

$$\begin{aligned}
0 &= b_h(u_h, \sigma) \\
&= \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (a_{ij}\partial_{ij,w}^2 u_h, \sigma)_T \\
&= \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij,w}^2 u_h, \mathcal{Q}_h(a_{ij}\sigma))_T \\
&= \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij}^2 u_0, \mathcal{Q}_h(a_{ij}\sigma))_T - \langle (u_b - u_0)n_i, \partial_j \mathcal{Q}_h(a_{ij}\sigma)\rangle_{\partial T} \\
&\quad + \langle u_{gi} - \partial_i u_0, \mathcal{Q}_h(a_{ij}\sigma)n_j\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij}^2 u_0, \mathcal{Q}_h(a_{ij}\sigma))_T \\
&= \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij}^2 u_0, a_{ij}\sigma)_T \\
&= \sum_{T\in\mathcal{T}_h} (\sum_{i,j=1}^{d} a_{ij}\partial_{ij}^2 u_0, \sigma)_T \\
&= \sum_{T\in\mathcal{T}_h} (\mathcal{Q}_h(\sum_{i,j=1}^{d} a_{ij}\partial_{ij}^2 u_0), \sigma)_T
\end{aligned} \tag{29}$$

for any $\sigma \in V_{h,k}$, where we used (9) together with $u_0 = u_b$ and $\nabla u_0 = \mathbf{u}_g$ on each ∂T. Letting $\sigma = \mathcal{Q}_h(\sum_{i,j=1}^d a_{ij}\partial_{ij}^2 u_0)$ in (29) gives $\mathcal{Q}_h(\sum_{i,j=1}^d a_{ij}\partial_{ij}^2 u_0) = 0$ on each element $T \in \mathcal{T}_h$. This implies that $\sum_{i,j=1}^d a_{ij}\partial_{ij}^2 u_0 = 0$ on each element $T \in \mathcal{T}_h$. Note that $u_0 \in C^1(\Omega)$. Thus, we have $\sum_{i,j=1}^d a_{ij}\partial_{ij}^2 u_0 = 0$ in Ω. Since $u_h \in W_{h,k}^0$, we have $u_0 = u_b = 0$ on $\partial\Omega$. Therefore, $u_0 \equiv 0$ in Ω and further $u_h \equiv 0$ in Ω.

This completes the proof of the theorem. ☐

7. Error Equations

Let $(u_h;\lambda_h)\in W_{h,k}^0\times V_{h,k}$ be the M-PDWG solution arising from the numerical scheme (17)-(18). Note that the dual problem $b(v,\lambda)=0$ has a trivial solution $\lambda=0$ for any $v\in H^2(\Omega)\cap H_0^1(\Omega)$. The error functions are respectively defined as follows

$$e_h=u_h-Q_hu,\quad \gamma_h=\lambda_h-\mathcal{Q}_h\lambda=\lambda_h.$$

Lemma 7.1. *The following error equations for the M-PDWG scheme (17)-(18) hold true; i.e.,*

$$\begin{aligned}
&(30)\qquad & s_h(e_h,v)+b_h(v,\gamma_h) &= -s_h(Q_hu,v), &&\forall v\in W_{h,k}^0,\\
&(31)\qquad & -c_h(\gamma_h,\sigma)+b_h(e_h,\sigma) &= \ell_u(\sigma), &&\forall \sigma\in V_{h,k},
\end{aligned}$$

where

$$\ell_u(\sigma)=\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d((I-\mathcal{Q}_h)\partial_{ij}^2u,a_{ij}\sigma)_T. \tag{32}$$

Proof. First, by subtracting $s_h(Q_hu,v)$ from both sides of (17) we obtain

$$s_h(u_h-Q_hu,v)+b_h(v,\lambda_h)=-s_h(Q_hu,v),\qquad \forall v\in W_{h,k}^0,$$

which implies

$$s_h(e_h,v)+b_h(v,\gamma_h)=-s_h(Q_hu,v),\qquad \forall v\in W_{h,k}^0.$$

This completes the proof of the first error equation (30).

To derive (31), we use (1) and Lemma 6.1 to obtain

$$\begin{aligned}
b_h(Q_hu,\sigma)&=\sum_{T\in\mathcal{T}_h}(\sum_{i,j=1}^d a_{ij}\partial_{ij,w}^2Q_hu,\sigma)_T\\
&=\sum_{T\in\mathcal{T}_h}(\sum_{i,j=1}^d a_{ij}\mathcal{Q}_h\partial_{ij}^2u,\sigma)_T\\
&=\sum_{T\in\mathcal{T}_h}(\sum_{i,j=1}^d a_{ij}\partial_{ij}^2u,\sigma)_T+\sum_{T\in\mathcal{T}_h}(\sum_{i,j=1}^d a_{ij}(\mathcal{Q}_h-I)\partial_{ij}^2u,\sigma)_T\\
&=(f,\sigma)+\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d((\mathcal{Q}_h-I)\partial_{ij}^2u,a_{ij}\sigma)_T,
\end{aligned}$$

for all $\sigma\in V_{h,k}$. Now subtracting the above equation from (18) yields the error equation (31).

This completes the proof of the lemma. ☐

8. Error Estimates

Let $\mathcal{T}_h$ be a shape-regular finite element partition of the domain Ω. For any $T\in\mathcal{T}_h$, the following trace inequality holds true [7]:

$$\|\varphi\|_{\partial T}^2\le C(h_T^{-1}\|\varphi\|_T^2+h_T\|\nabla\varphi\|_T^2),\qquad \forall\varphi\in H^1(T). \tag{33}$$

Furthermore, assume φ is a polynomial on the element $T \in \mathcal{T}_h$. Applying the inverse inequality to (33) gives [7]

$$\|\varphi\|_{\partial T}^2 \leq C h_T^{-1} \|\varphi\|_T^2. \tag{34}$$

Lemma 8.1. [7] *Assume that $\mathcal{T}_h$ is a shape regular finite element partition of the domain Ω as specified in* [7]*. For any $0 \leq s \leq 2$ and $1 \leq m \leq k$, there holds*

$$\sum_{T\in\mathcal{T}_h} h_T^{2s} \|u - Q_0 u\|_{s,T}^2 \leq C h^{2(m+1)} \|u\|_{m+1}^2, \tag{35}$$

$$\sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} h_T^{2s} \|u - \mathcal{Q}_h u\|_{s,T}^2 \leq C h^{2(m-1)} \|u\|_{m-1}^2, \tag{36}$$

$$\sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} h_T^{2s} \|\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u\|_{s,T}^2 \leq C h^{2(m-1)} \|u\|_{m+1}^2. \tag{37}$$

We are ready to present the critical error estimates for the M-PDWG scheme (17)-(18), which is the main contribution of this paper.

Theorem 8.2. *Assume that the coefficient tensor $a = (a_{ij})$ is uniformly piecewise continuous in Ω with respect to the finite element partition $\mathcal{T}_h$. Let u be the exact solution of (1) and $(u_h; \lambda_h) \in W_{h,k}^0 \times V_{h,k}$ be the M-PDWG solution of (17)-(18), respectively. Assume that the exact solution u of (1) is sufficiently regular such that $u \in H^{k+1}(\Omega)$. There exists a constant C such that*

$$\|u_h - Q_h u\|_{2,h} + \|\lambda_h\|_0 + c_h(\lambda_h, \lambda_h)^{\frac{1}{2}} \leq C h^{k-1} \|u\|_{k+1}, \tag{38}$$

provided that the meshsize $h < h_0$ holds true for a sufficiently small, but fixed $h_0 > 0$.

Proof. From (30), we have

$$b_h(v, \gamma_h) = -s_h(Q_h u, v) - s_h(e_h, v). \tag{39}$$

Recall that

$$\begin{aligned} s_h(Q_h u, v) = & \sum_{T\in\mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, v_0 - v_b \rangle_{\partial T} \\ & + \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle \nabla Q_0 u - \mathbf{Q}_g(\nabla u), \nabla v_0 - \mathbf{v}_g \rangle_{\partial T}. \end{aligned} \tag{40}$$

The first term on the right-hand side of (40) can be estimated by using the Cauchy-Schwarz inequality, the trace inequality (33), and the estimate (35) with $m = k$ as follows

$$
\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h} h_T^{-3}\langle Q_0u-Q_bu, v_0-v_b\rangle_{\partial T}\Big|\\
=&\Big|\sum_{T\in\mathcal{T}_h} h_T^{-3}\langle Q_0u-u, v_0-v_b\rangle_{\partial T}\Big|\\
\le&\Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|u-Q_0u\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v_0-v_b\|_{\partial T}^2\Big)^{\frac12}\\
\le&C\Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\big(\|u-Q_0u\|_T^2+h_T^2\|u-Q_0u\|_{1,T}^2\big)\Big)^{\frac12}(s_h(v,v))^{\frac12}\\
\le&Ch^{k-1}\|u\|_{k+1}(s_h(v,v))^{\frac12}.
\end{aligned}
\tag{41}
$$

Similarly, the second term on the right-hand side of (40) has the following estimate

$$
\Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle \nabla Q_0u-\mathbf{Q}_g(\nabla u), \nabla v_0-\mathbf{v}_g\rangle_{\partial T}\Big|\le Ch^{k-1}\|u\|_{k+1}(s_h(v,v))^{\frac12}. \tag{42}
$$

Combining (40) - (42) gives

$$
|s_h(Q_hu,v)|\le Ch^{k-1}\|u\|_{k+1}(s_h(v,v))^{\frac12}. \tag{43}
$$

Using Cauchy-Schwarz inequality, it is easy to obtain

$$
|s_h(e_h,v)|\le \big(s_h(e_h,e_h)\big)^{\frac12}\big(s_h(v,v)\big)^{\frac12}. \tag{44}
$$

Substituting (43)-(44) into (39) gives

$$
|b_h(v,\gamma_h)|\le (Ch^{k-1}\|u\|_{k+1}+(s_h(e_h,e_h))^{\frac12})(s_h(v,v))^{\frac12},
$$

which from Lemma 6.4, for $\gamma_h\in V_{h,k}$, there exists $v_{\gamma_h}\in W_{h,k}^0$ such that

$$
\begin{aligned}
\frac12\|\gamma_h\|_0^2\le&|b_h(v_{\gamma_h},\gamma_h)|\\
\le&(Ch^{k-1}\|u\|_{k+1}+(s_h(e_h,e_h))^{\frac12})\|v_{\gamma_h}\|_{2,h}\\
\le&(Ch^{k-1}\|u\|_{k+1}+(s_h(e_h,e_h))^{\frac12})\|\gamma_h\|_0.
\end{aligned}
$$

Therefore, we have

$$
\|\gamma_h\|_0\le Ch^{k-1}\|u\|_{k+1}+(s_h(e_h,e_h))^{\frac12}. \tag{45}
$$

From (31), we have

$$
b_h(e_h,\sigma)=\ell_u(\sigma)+c_h(\gamma_h,\sigma). \tag{46}
$$

Using (32) and the estimate (37) with $m=k$ we have

$$
\begin{aligned}
|\ell_u(\sigma)|=&\Big|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d((I-\mathcal{Q}_h)\partial_{ij}^2u, a_{ij}\sigma)_T\Big|\\
\le&\sum_{i,j=1}^d\|a_{ij}\|_{L^\infty}\ \|(I-\mathcal{Q}_h)\partial_{ij}^2u\|_0\ \|\sigma\|_0\\
\le&Ch^{k-1}\|u\|_{k+1}\|\sigma\|_0.
\end{aligned}
\tag{47}
$$

Substituting (47) into (46), we have

$$|b_h(e_h,\sigma)| \le C(h^{k-1}\|u\|_{k+1} + \|\gamma_h\|_0)\|\sigma\|_0,$$

where we used (16). Taking $\sigma = \sum_{i,j=1}^d \mathcal{Q}_h(a_{ij}\partial^2_{ij,w}e_h)$ in the above equation gives

$$\Big(\sum_{T\in\mathcal{T}_h} \|\sum_{i,j=1}^d \mathcal{Q}_h(a_{ij}\partial^2_{ij,w}e_h)\|_T^2\Big)^{\frac{1}{2}} \le C(h^{k-1}\|u\|_{k+1} + \|\gamma_h\|_0). \tag{48}$$

Letting $v = e_h$ in (30) and $\sigma = \gamma_h$ in (31) gives

$$s_h(e_h,e_h) + c_h(\gamma_h,\gamma_h) = -s_h(Q_hu,e_h) - \ell_u(\gamma_h). \tag{49}$$

Substituting (43), (45) and (47) into (49) yields

$$\begin{aligned}
& s_h(e_h,e_h) + c_h(\gamma_h,\gamma_h)\\
\le & Ch^{k-1}\|u\|_{k+1}((s_h(e_h,e_h))^{\frac{1}{2}} + \|\gamma_h\|_0)\\
\le & Ch^{k-1}\|u\|_{k+1}((s_h(e_h,e_h))^{\frac{1}{2}} + Ch^{k-1}\|u\|_{k+1})\\
\le & Ch^{2k-2}\|u\|_{k+1}^2 + C\frac{1}{\epsilon}h^{2k-2}\|u\|_{k+1}^2 + C\epsilon s_h(e_h,e_h)
\end{aligned} \tag{50}$$

where we used Young's inequality with ϵ being sufficiently small such that $1 - C\epsilon > 0$, which gives

$$(1-C\epsilon)s_h(e_h,e_h) + c_h(\gamma_h,\gamma_h) \le Ch^{2k-2}\|u\|_{k+1}^2,$$

which gives

$$s_h(e_h,e_h) + c_h(\gamma_h,\gamma_h) \le Ch^{2k-2}\|u\|_{k+1}^2. \tag{51}$$

Using (51), (45) gives

$$\|\gamma_h\|_0 \le Ch^{k-1}\|u\|_{k+1}, \tag{52}$$

which, from (48) and (51), gives

$$|\!|\!| e_h |\!|\!|_2 \le Ch^{k-1}\|u\|_{k+1}. \tag{53}$$

Combining (52) and (53) and using Lemma 6.3 completes the proof of the theorem.

□

9. Error Estimates in H^1 and L^2

In this section, we shall establish the error estimates in H^1 and L^2 norm for the M-PDWG solution arising from the scheme (17)-(18).

Lemma 9.1. [6] *There exists a constant C such that for any $v \in W_k(T)$, we have*

$$\|\partial^2_{ij,w}v\|_T^2 \le C\left(\|\partial^2_{ij}v_0\|_T^2 + s_T(v,v)\right). \tag{54}$$

Consider an auxiliary problem: Find w satisfying

$$\sum_{i,j=1}^d \partial^2_{ji}(a_{ij}w) = \theta, \qquad \text{in } \Omega, \tag{55}$$

$$w = 0, \qquad \text{on } \partial\Omega, \tag{56}$$

where θ is a given function. The variational formulation for (55)-(56) seeks $w \in L^2(\Omega)$ such that

$$b(v,w) = (\theta, v), \qquad \forall v \in H^2(\Omega)\cap H^1_0(\Omega), \tag{57}$$

where the bilinear form $b(\cdot,\cdot)$ is given by (5).

The problem (55)-(56) is assumed to be H^{1+s}-regular ($s \in [0,1]$) in the sense that for any $\theta \in H^{s-1}(\Omega)$, there exists a unique $w \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ satisfying (57) and a priori estimate:

$$\|w\|_{1+s} \le C\|\theta\|_{s-1}. \tag{58}$$

Lemma 9.2. [6] *Assume that the coefficient tensor* $a = (a_{ij}) \in [C^1(\Omega)]^{d\times d}$. *For any* $v = \{v_0, v_b, \mathbf{v}_g\} \in W_{h,k}^0$, *there holds*

$$\begin{aligned}(v_0,\theta) = &\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(a_{ij}\partial_{ij,w}^2 v, w)_T - \langle (v_{gi}-\partial_i v_0)n_j, (\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T}\\ &+\langle (v_b - v_0)n_i, \partial_j(\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T}.\end{aligned} \tag{59}$$

Lemma 9.3. [6] *Assume that the coefficient matrix* $a = (a_{ij}) \in [\Pi_{T\in\mathcal{T}_h} W^{1,\infty}(T)]^{d\times d}$. *There exists a constant C such that for any* $v \in W_{h,k}^0$, *we have*

$$\left|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle (v_{gi}-\partial_i v_0)n_j, (\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T}\right| \le Ch\ \|v\|_{2,h}\|\theta\|_{-1}, \tag{60}$$

$$\left|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle (v_b - v_0)n_i, \partial_j(\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T}\right| \le Ch\ \|v\|_{2,h}\|\theta\|_{-1}, \tag{61}$$

provided that the dual problem (57) has the regularity estimate (58) with $s = 0$.

Lemma 9.4. *Assume that the coefficient matrix* $a = (a_{ij}) \in \Pi_{T\in\mathcal{T}_h}[W^{2,\infty}(T)]^{d\times d}$ *and* $P_1(T) \subset V_k(T)$ *for each element* $T \in \mathcal{T}_h$. *There exists a constant C such that for any* $v \in W_{h,k}^0$, *we have*

$$\left|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle (v_{gi}-\partial_i v_0)n_j, (\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T}\right| \le Ch^2\ \|v\|_{2,h}\|\theta\|_{0}, \tag{62}$$

$$\left|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle (v_b - v_0)n_i, \partial_j(\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T}\right| \le Ch^2\ \|v\|_{2,h}\|\theta\|_{0}, \tag{63}$$

provided that the regularity estimate (58) holds true with $s = 1$.

For convenience of analysis, in what follows of this paper, for any $\rho, \sigma \in V_{h,k}$, we shall employ the specific $c_h(\rho,\sigma)$ define in (19).

Theorem 9.5. *Let* $u_h = \{u_0, u_b, \mathbf{u}_g\} \in W_{h,k}^0$ *be the M-PDWG solution arising from the numerical scheme (17)-(18). Assume that* $a = (a_{ij}) \in [C^1(\Omega)]^{d\times d}$ *and the exact solution of the model problem (1) is sufficiently regular such that* $u \in H^{k+1}(\Omega)$. *There exists a constant C such that*

$$\left(\sum_{T\in\mathcal{T}_h}\|\nabla u_0 - \nabla u\|_T^2\right)^{\frac{1}{2}} \le Ch^k\|u\|_{k+1}, \tag{64}$$

provided that the meshsize h is sufficiently small and the dual problem (55)-(56) has H^1-regularity estimate (58) with $s = 0$.

Proof. Given $\theta = -\nabla \cdot \eta$ with $\eta \in [C^1(\Omega)]^d$ satisfying $\eta = 0$ on $\mathcal{E}_h$, assume w is the solution of the dual problem (55)-(56). Taking $v = e_h$ in Lemma (9.2) yields

(65)
$$\begin{aligned}-(e_0, \nabla \cdot \eta) =& \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (a_{ij}\partial^2_{ij,w} e_h, w)_T - \langle (e_{gi} - \partial_i e_0) n_j, (\mathcal{Q}_h - I)(a_{ij} w)\rangle_{\partial T} \\ &+ \langle (e_b - e_0) n_i, \partial_j(\mathcal{Q}_h - I)(a_{ij} w)\rangle_{\partial T} \\ =& I_1 - I_2 + I_3,\end{aligned}$$

where $I_j (j = 1, 2, 3)$ are defined accordingly. Due to $\eta = 0$ on $\mathcal{E}_h$, using the integration by parts to (65) gives

$$(\nabla e_0, \eta) = I_1 - I_2 + I_3. \tag{66}$$

From Lemma 9.3 and H^1-regularity estimate (58) with $s = 0$, the terms I_2 and I_3 are bounded as follows

$$|I_2| + |I_3| \le Ch\|\theta\|_{-1}\|e_h\|_{2,h} \le Ch\|\eta\|_0\|e_h\|_{2,h}. \tag{67}$$

Regarding to the term I_1, from the error equation (31), we have

(68)
$$\begin{aligned}I_1 =& \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (a_{ij}\partial^2_{ij,w} e_h, w)_T \\ =& \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (a_{ij}\partial^2_{ij,w} e_h, \mathcal{Q}_h w)_T + (a_{ij}\partial^2_{ij,w} e_h, (I - \mathcal{Q}_h) w)_T \\ =& \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} ((I - \mathcal{Q}_h)\partial^2_{ij} u, a_{ij}\mathcal{Q}_h w)_T + c_h(\gamma_h, \mathcal{Q}_h w) \\ &+ \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (a_{ij}\partial^2_{ij,w} e_h, (I - \mathcal{Q}_h) w)_T \\ =& J_1 + J_2 + J_3,\end{aligned}$$

where J_i for $i = 1, 2, 3$ are defined accordingly. As to the term J_1, from the Cauchy-Schwarz inequality, we have

(69)
$$\begin{aligned}|J_1| =& \Big|\sum_{T\in\mathcal{T}_h} ((I - \mathcal{Q}_h)\partial^2_{ij} u, a_{ij}\mathcal{Q}_h w)_T\Big| \\ =& \Big|\sum_{T\in\mathcal{T}_h} |((I - \mathcal{Q}_h)\partial^2_{ij} u, (I - \mathcal{Q}_h) a_{ij}\mathcal{Q}_h w)_T\Big| \\ \le& \Big(\sum_{T\in\mathcal{T}_h} \|(I - \mathcal{Q}_h)\partial^2_{ij} u\|_T^2\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|(I - \mathcal{Q}_h) a_{ij}\mathcal{Q}_h w\|_T^2\Big)^{\frac{1}{2}} \\ \le& Ch\|(I - \mathcal{Q}_h)\partial^2_{ij} u\|\|w\|_1.\end{aligned}$$

As to the term J_2, using the Cauchy-Schwarz inequality, the inverse inequality and (19) gives

$$|J_2| = |c_h(\gamma_h, \mathcal{Q}_h w)| \le Ch^4\|\gamma_h\|_{2,h}\|\mathcal{Q}_h w\|_{2,h} \le Ch\|\gamma_h\|_0\|w\|_1. \tag{70}$$

As to the term J_3, using the Cauchy-Schwarz inequality and (54), we have

$$
\begin{aligned}
|J_3| =& \Big| \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (a_{ij}\partial^2_{ij,w} e_h, (I-\mathcal{Q}_h)w)_T \Big| \\
=& \Big| \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} ((a_{ij}-\bar{a}_{ij})\partial^2_{ij,w} e_h, (I-\mathcal{Q}_h)w)_T \Big| \\
\le& \Big(\sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} \|a_{ij}-\bar{a}_{ij}\|^2_{L^\infty(T)} \|\partial^2_{ij,w} e_h\|^2_T \Big)^{\frac12} \Big(\sum_{T\in\mathcal{T}_h} \|(I-\mathcal{Q}_h)w\|^2_T \Big)^{\frac12} \\
\le& Ch\|w\|_1 \Big(\sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} (\varepsilon(h_T))^2 (\|\partial^2_{ij} e_0\|^2_T + s_T(e_h,e_h)) \Big)^{\frac12},
\end{aligned} \tag{71}
$$

where $\bar{a}_{ij}$ is the average of a_{ij} on the element T and $\varepsilon(h_T)\to 0$ as $h\to 0$. Substituting (69) - (71) into (68) yields

$$
\begin{aligned}
|I_1| \le& Ch\Big(\varepsilon(h)\|\nabla^2 e_0\|_0 + \varepsilon(h)\|e_h\|_{2,h} + \sum_{i,j=1}^{d} \|(I-\mathcal{Q}_h)\partial^2_{ij} u\|_0 + \|\gamma_h\|_0 \Big)\|w\|_1 \\
\le& C\Big(\varepsilon(h)\|\nabla e_0\|_0 + h\varepsilon(h)\|e_h\|_{2,h} + h\sum_{i,j=1}^{d} \|(I-\mathcal{Q}_h)\partial^2_{ij} u\|_0 + h\|\gamma_h\|_0 \Big)\|\eta\|_0,
\end{aligned} \tag{72}
$$

where we used the inverse inequality and the estimate $\|w\|_1 \le C\|\theta\|_{-1} \le C\|\eta\|_0$. Substituting (72) and (67) into (66) gives

$$
\begin{aligned}
|(\nabla e_0, \eta)| \le& C\Big(\varepsilon(h)\|\nabla e_0\|_0 + h(1+\varepsilon(h))\|e_h\|_{2,h} \\
&+ h\sum_{i,j=1}^{d} \|(I-\mathcal{Q}_h)\partial^2_{ij} u\|_0 + h\|\gamma_h\|_0 \Big)\|\eta\|_0.
\end{aligned}
$$

Note that the set of all such η is dense in $L^2(\Omega)$. The above inequality implies

$$
\|\nabla e_0\|_0 \le C\Big(\varepsilon(h)\|\nabla e_0\|_0 + h(1+\varepsilon(h))\|e_h\|_{2,h} + h\sum_{i,j=1}^{d} \|(I-\mathcal{Q}_h)\partial^2_{ij} u\|_0 + h\|\gamma_h\|_0 \Big).
$$

Therefore, we have

$$
\|\nabla e_0\|_0 \le Ch\Big(\|e_h\|_{2,h} + \sum_{i,j=1}^{d} \|(I-\mathcal{Q}_h)\partial^2_{ij} u\|_0 + \|\gamma_h\|_0 \Big) \tag{73}
$$

provided that the meshsize h is sufficiently small such that $1 - C\varepsilon(h) > 0$ and $\varepsilon(h) \to 0$. The inequality (73), the error estimate (38), and the estimate (37) with $m = k$ completes the proof of the estimate (64) using the usual triangle inequality and the estimate (35) with $m = k$. □

We further present the L^2 error estimate for the primal variable u_h.

Theorem 9.6. *Assume that (1) the coefficients* $a_{ij} \in C^1(\Omega) \cap \big[\Pi_{T\in\mathcal{T}_h} W^{2,\infty}(T)\big]$ *for* $i, j = 1, \cdots, d$; *(2) the dual problem (55)-(56) satisfies* H^2*-regularity estimate*

(58) with $s = 1$; and (3) $P_1(T) \subset V_k(T)$ for any $T \in \mathcal{T}_h$. There exists a constant C such that

$$\|u_0 - u\|_0 \le Ch^{k+1}\|u\|_{k+1}, \tag{74}$$

provided that the meshsize h is sufficiently small.

Proof. Let w be the solution of the dual problem (55)-(56) for a given $\theta \in L^2(\Omega)$. Choosing $v = e_h$ in Lemma 9.2 yields

$$\begin{aligned}(e_0, \theta) =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(a_{ij}\partial^2_{ij,w}e_h, w)_T - \langle (e_{gi} - \partial_i e_0)n_j, (\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T} \\ &+ \langle (e_b - e_0)n_i, \partial_j(\mathcal{Q}_h - I)(a_{ij}w)\rangle_{\partial T} \\ =& J_1 - J_2 + J_3,\end{aligned} \tag{75}$$

where J_i are defined accordingly for $i = 1, 2, 3$. Using Lemma 9.4, we obtain

$$|J_2| + |J_3| \le Ch^2\|\theta\|_0\|e_h\|_{2,h}. \tag{76}$$

As to the term J_1, using the error equation (31) gives rise to

$$\begin{aligned}J_1 =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(a_{ij}\partial^2_{ij,w}e_h, w)_T \\ =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(a_{ij}\partial^2_{ij,w}e_h, \mathcal{Q}_h w)_T + (a_{ij}\partial^2_{ij,w}e_h, (I - \mathcal{Q}_h)w)_T \\ =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}((I - \mathcal{Q}_h)\partial^2_{ij}u, a_{ij}\mathcal{Q}_h w)_T + c_h(\gamma_h, \mathcal{Q}_h w) \\ &+ \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(a_{ij}\partial^2_{ij,w}e_h, (I - \mathcal{Q}_h)w)_T \\ =& I_1 + I_2 + I_3,\end{aligned} \tag{77}$$

where $I_i (i = 1, 2, 3)$ are defined accordingly. Recall that $P_1(T) \subseteq V_k(T)$ and $\mathcal{Q}_h$ is the L^2 projection onto $V_k(T)$. As to the term I_1, using Cauchy-Schwarz inequality gives

$$\begin{aligned}|I_1| =& \Big|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}((I - \mathcal{Q}_h)\partial^2_{ij}u, a_{ij}\mathcal{Q}_h w)_T\Big| \\ =& \Big|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}((I - \mathcal{Q}_h)\partial^2_{ij}u, (I - \mathcal{Q}_h)a_{ij}\mathcal{Q}_h w)_T\Big| \\ \le& \Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\|(I - \mathcal{Q}_h)\partial^2_{ij}u\|_T^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\|(I - \mathcal{Q}_h)a_{ij}\mathcal{Q}_h w\|_T^2\Big)^{\frac{1}{2}} \\ \le& Ch^2\sum_{i,j=1}^{d}\|(I - \mathcal{Q}_h)\partial^2_{ij}u\|_0\|w\|_2.\end{aligned} \tag{78}$$

As to the term I_2, using Cauchy-Schwarz inequality, the inverse inequality and (19) gives

$$
\begin{aligned}
I_2 = &\sum_{T\in\mathcal{T}_h} h_T^4 \left\{ (\gamma_h, \mathcal{Q}_h w)_T + (\nabla\gamma_h, \nabla\mathcal{Q}_h w)_T + \sum_{i,j=1}^d (\partial_{ij}^2\gamma_h, \partial_{ij}^2\mathcal{Q}_h w)_T \right\} \\
\le & Ch^2\|\gamma_h\|_0\|w\|_2,
\end{aligned} \tag{79}
$$

As to the term I_3, using (54) yields

$$
\begin{aligned}
|I_3| =& |\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d (a_{ij}\partial_{ij,w}^2 e_h, (I-\mathcal{Q}_h)w)_T| \\
=& |\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d ((a_{ij}-\bar{a}_{ij})\partial_{ij,w}^2 e_h, (I-\mathcal{Q}_h)w)_T| \\
\le& \Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d \|a_{ij}-\bar{a}_{ij}\|_{L^\infty(T)}^2 \|\partial_{ij,w}^2 e_h\|_T^2\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h}\|(I-\mathcal{Q}_h)w\|_T^2\Big)^{\frac{1}{2}} \\
\le& Ch^3\|w\|_2\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d \|\partial_{ij}^2 e_0\|_T^2 + s_T(e_h,e_h)\Big)^{\frac{1}{2}},
\end{aligned} \tag{80}
$$

where $\bar{a}_{ij}$ is the average of a_{ij} on the element $T\in\mathcal{T}_h$ such that $\|a_{ij}-\bar{a}_{ij}\|_{L^\infty(T)} \le h_T$.

Using (78)-(80), the inverse inequality and the regularity assumption (58) for $s=1$, we have

$$
\begin{aligned}
|J_1| \le& C(h^3\|\nabla^2 e_0\|_0 + h^3\|e_h\|_{2,h} + h^2\sum_{i,j=1}^d \|(I-\mathcal{Q}_h)\partial_{ij}^2 u\|_0 + h^2\|\gamma_h\|_0)\|w\|_2 \\
\le& C(h^2\|\nabla e_0\|_0 + h^3\|e_h\|_{2,h} + h^2\sum_{i,j=1}^d \|(I-\mathcal{Q}_h)\partial_{ij}^2 u\|_0 + h^2\|\gamma_h\|_0)\|\theta\|_0.
\end{aligned} \tag{81}
$$

Substituting (81) and (76) into (75) gives

$$
|(e_0,\theta)| \le Ch^2\big(\|\nabla e_0\|_0 + \|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I-\mathcal{Q}_h)\partial_{ij}^2 u\|_0 + \|\gamma_h\|_0\big)\|\theta\|_0.
$$

This indicates

$$
\begin{aligned}
\|e_0\|_0 \le& Ch^2\big(\|\nabla e_0\|_0 + \|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I-\mathcal{Q}_h)\partial_{ij}^2 u\|_0 + \|\gamma_h\|_0\big) \\
\le& Ch\|\nabla e_0\|_0 + Ch^2\big(\|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I-\mathcal{Q}_h)\partial_{ij}^2 u\|_0 + \|\gamma_h\|_0\big),
\end{aligned}
$$

where we used the inverse inequality, which gives

$$
\|e_0\|_0 \le Ch^2\big(\|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I-\mathcal{Q}_h)\partial_{ij}^2 u\|_0 + \|\gamma_h\|_0\big), \tag{82}
$$

provided that the meshsize h is sufficiently small. Combining (82), (38), and (37) with $m = k$, completes the proof of the theorem. □

Remark 9.1. [6] *The optimal order error estimate (74) is based on the assumption that $P_1(T) \subseteq V_2(T)$, which is used to derive (76) and (78)-(80). When it comes to the case of $P_1(T) \nsubseteq V_2(T)$, those inequalities are modified by replacing $\|w\|_{2,T}$ by $h_T^{-1}\|w\|_{1,T}$. The conclusion is stated as follows: We assume (1) the coefficients $a_{ij} \in C^1(\Omega)$ for $i, j = 1, \cdots, d$, (2) the meshsize h is sufficiently small, and (3) the dual problem (55)-(56) satisfies the H^1-regularity estimate (58) for $s = 0$. The sub-optimal order error estimate holds true*

$$\|u_0 - u\|_0 \le Ch^k\|u\|_{k+1}.$$

We introduce the following norms for the two boundary components u_b and $\mathbf{u}_g$; i.e.,

$$\|e_b\|_0 := \Big(\sum_{T\in\mathcal{T}_h} h_T\|e_b\|_{\partial T}^2\Big)^{\frac{1}{2}}, \quad \|\mathbf{e}_g\|_0 := \Big(\sum_{T\in\mathcal{T}_h} h_T\|\mathbf{e}_g\|_{\partial T}^2\Big)^{\frac{1}{2}}.$$

Theorem 9.7. [6] *Under the assumptions of Theorem 9.6, there exists a constant C such that*

$$\begin{aligned} \|u_b - Q_b u\|_0 &\le Ch^{k+1}\|u\|_{k+1}, \\ \|\mathbf{u}_g - \mathbf{Q}_b\nabla u\|_0 &\le Ch^k\|u\|_{k+1}. \end{aligned}$$

10. Numerical Experiments

A series of the numerical results are illustrated to verify the accuracy of the theory developed for the M-PDWG method (17)-(18).

We shall take the lowest order WG element with $k = 2$ on triangular partitions as an example in the implementation. The finite element spaces are thus respectively given by

$$W_{h,2} = \{v = \{v_0, v_b, \mathbf{v}_g\} : \ v_0 \in P_2(T), v_b \in P_2(e), \mathbf{v}_g \in [P_1(e)]^2, \forall T \in \mathcal{T}_h, e \in \mathcal{E}_h\},$$

$$V_{h,2} = \{\sigma : \ \sigma|_T \in V_2(T), \ \forall T \in \mathcal{T}_h\},$$

where both $V_2(T) = P_1(T)$ and $V_2(T) = P_0(T)$ are considered. A finite element function $v \in W_{h,2}$ is named C^0-type if $v_b = v_0|_{\partial T}$ for each element T. The C^0-type WG element leads to a linear system with less computational complexity compared with the general WG elements. However, the C^0 continuity does not permit the availability of polygonal elements. Note that the theoretical results developed in this paper could be generalized to C^0-type triangular elements without any difficulty. The C^0-type WG element with $V_2(T) = P_1(T)$ is called the $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element; and the C^0-type WG element with $V_2(T) = P_0(T)$ is called the $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element.

Three domains are used in our numerical experiments: (1) the unit square domain $\Omega_1 = (0, 1)^2$; (2) the square domain $\Omega_2 = (-1, 1)^2$; and (3) the non-convex L-shaped domain Ω_3 with vertices $A_0 = (0, 0)$, $A_1 = (2, 0)$, $A_2 = (1, 1)$, $A_3 = (1, 2)$, and $A_4 = (0, 2)$. Starting from a given initial coarse triangulation of the domain, the triangular partition is obtained by successively dividing each coarse level triangle into four congruent sub-triangles through connecting the mid-points on each edge of each triangle.

Let $u_h = \{u_0, \mathbf{u}_g\} \in W_{h,2}$ and $\lambda_h \in V_{h,2}$ be the M-PDWG solution arising from the scheme (17)-(18). Recall that the exact solution of Lagrange multiplier is $\lambda = 0$. These numerical solutions are compared with the interpolants of the corresponding exact solutions; i.e.,

$$e_h = \{e_0, \mathbf{e}_g\} = \{u_0 - I_h u,\ \mathbf{u}_g - \mathbf{I}_g(\nabla u)\}, \quad \gamma_h = \lambda_h - 0,$$

where $I_h u$ is the Lagrange interpolation of the exact solution u on each triangular element using three vertices and three mid-points on the edges, and $\mathbf{I}_g(\nabla u)$ is the linear interpolant of ∇u on each edge $e \in \mathcal{E}_h$. The following L^2 norms are employed to measure the errors:

$$\|e_0\|_0 = \Big(\sum_{T\in\mathcal{T}_h} \int_T |e_0|^2 dT\Big)^{\frac{1}{2}}, \qquad \|\mathbf{e}_g\|_0 = \Big(\sum_{T\in\mathcal{T}_h} h_T \int_{\partial T} |\mathbf{e}_g|^2 ds\Big)^{\frac{1}{2}},$$
$$\|\gamma_h\|_0 = \Big(\sum_{T\in\mathcal{T}_h} \int_T |\gamma_h|^2 dT\Big)^{\frac{1}{2}}.$$

Test Case 1. Find u such that

$$\begin{aligned} \sum_{i,j=1}^{2} a_{ij}\partial_{ij}^2 u &= f, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega, \end{aligned} \tag{83}$$

where $\Omega = \Omega_i (i = 1, 3)$, the coefficients are $a_{11} = 3$, $a_{12} = a_{21} = 1$ and $a_{22} = 2$, and the exact solution is given by $u = \sin(x_1)\sin(x_2)$.

Tables 1-2 show the numerical results for the M-PDWG method (17)-(18) for the test problem (83) when the C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element is applied. We observe from Tables 1-2 that the convergence rates for e_0 in the discrete L^2-norm are of orders $\mathcal{O}(h^4)$ and $\mathcal{O}(h^{3.6})$ on the unit square domain Ω_1 and on the L-shaped domain Ω_3, respectively. The convergence rates for $\mathbf{e}_g$ and γ_h in the discrete L^2 norm are of orders $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$ on both Ω_1 and Ω_3 respectively. Note that the expected optimal convergence rates for e_0, $\mathbf{e}_g$ and γ_h in the discrete L^2-norm on the convex domain Ω_1 are of orders $\mathcal{O}(h^3)$, $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$, respectively. When it comes to the non-convex L-shaped domain Ω_3, the theoretical order of convergence for e_0 in the discrete L^2-norm should be between $\mathcal{O}(h^2)$ and $\mathcal{O}(h^3)$ due to the lack of H^2-regularity required for the dual problem (55)-(56). However, the theoretical rates of convergence for $\mathbf{e}_g$ and γ_h remain to be of orders $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$, respectively. It is clear that the numerical results are greatly consistent with the theory for $\mathbf{e}_g$ and γ_h in the discrete L^2-norm, and outperform the theory for e_0 in the discrete L^2-norm for the case of smooth solutions with smooth coefficients on uniform triangular partitions.

Test Case 2. Find u such that

$$\begin{aligned} \sum_{i,j=1}^{2} (1+\delta_{ij})\frac{x_i}{|x_i|}\frac{x_j}{|x_j|}\partial_{ij}^2 u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{84}$$

where $\Omega_2 = (-1, 1)^2$, and the exact solution is $u = x_1 x_2(1 - e^{1-|x_1|})(1 - e^{1-|x_2|})$. It is easy to check the Cordès condition (3) is satisfied for the test problem (84) with $\varepsilon = 3/5$ and the coefficient matrix $a = (a_{ij})$ is discontinuous across the $x_i (i = 1, 2)$ axis.

TABLE 1. Test Case 1: Convergence rates for C^0-C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element on Ω_1.

$1/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	0.006248		0.1260		3.36E-04	
2	0.001470	2.087	0.04477	1.493	6.51E-04	-0.9546
4	1.39E-04	3.399	0.01157	1.952	2.84E-04	1.195
8	1.03E-05	3.753	0.002843	2.025	1.32E-04	1.102
16	6.97E-07	3.891	7.02E-04	2.017	6.43E-05	1.043
32	4.54E-08	3.940	1.75E-04	2.007	3.17E-05	1.018

TABLE 2. Test Case 1: Convergence rates for C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element on Ω_3.

$1/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	0.01676		0.4804		0.004498	
2	0.002489	2.751	0.1248	1.945	0.001956	1.201
4	2.30E-04	3.435	0.03100	2.009	8.76E-04	1.160
8	1.94E-05	3.572	0.007674	2.014	4.13E-04	1.082
16	1.61E-06	3.585	0.001907	2.008	2.02E-04	1.035
32	1.37E-07	3.557	4.75E-04	2.006	9.99E-05	1.015

TABLE 3. Test Case 2: Convergence rates for C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element on Ω_2.

$2/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	0.6160		2.554		1.000	
2	0.4621	0.4148	1.676	0.6074	0.8970	0.1572
4	0.1389	1.734	1.006	0.7369	3.270	-1.866
8	0.02019	2.782	0.1339	2.909	0.6337	2.368
16	0.006505	1.634	0.03229	2.052	0.2249	1.494
32	0.001640	1.988	0.007814	2.047	0.09469	1.248

TABLE 4. Test Case 2: Convergence rates for C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element on Ω_2.

$2/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	0.1590		0.7950		0.07950	
2	0.2253	-0.5027	1.383	-0.7982	0.3321	-2.062
4	0.1963	0.1984	0.7627	0.8582	0.2444	0.4423
8	0.06727	1.545	0.2109	1.854	0.1349	0.8577
16	0.01536	2.130	0.04616	2.192	0.05452	1.307
32	0.003276	2.230	0.01020	2.178	0.02134	1.354

Table 3 presents the numerical performance of the M-PDWG scheme (17)-(18) for the test problem (84) when the C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element is employed. The numerical results indicate that the convergence rate for $\mathbf{e}_g$ in the discrete L^2 norm is of an expected optimal order $\mathcal{O}(h^2)$. The convergence rate for the Lagrange multiplier in the discrete L^2 norm seems to be of an order higher than the expected order $\mathcal{O}(h)$. The convergence order for e_0 in the discrete L^2 norm seems to be of an order $\mathcal{O}(h^2)$. Note that it is not clear to us whether the dual problem (55)-(56) has the regularity required for the convergence analysis. There are no theoretical results on the convergence rate for e_0 in the discrete L^2 norm. Table 4 shows the numerical results for the test problem (84) when the C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element is applied. We observe from Table 4 that the convergence rates for e_0, $\mathbf{e}_g$ and γ_h in the discrete L^2 norm seem to be a little higher than the convergence order corresponding to the case when the C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element is employed.

Figures 1-2 illustrate the numerical error for the Lagrange multiplier λ_h when the C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element and the C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element are employed respectively, compared with the PDWG scheme proposed in [6].

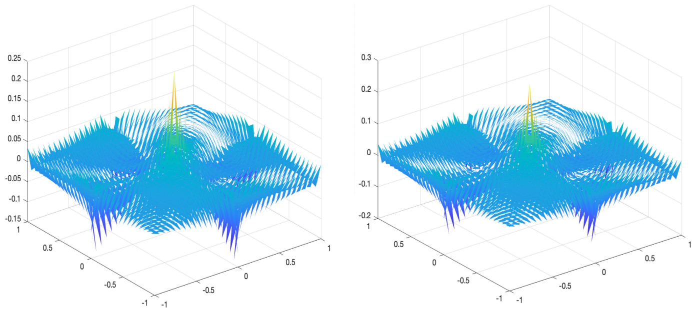

FIGURE 1. Test Case 2: Numerical error for Lagrange multiplier when C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element is applied: left figure is without the term $c(\cdot,\cdot)$ proposed in [6]; right figure is with the term $c(\cdot,\cdot)$ proposed in this paper.

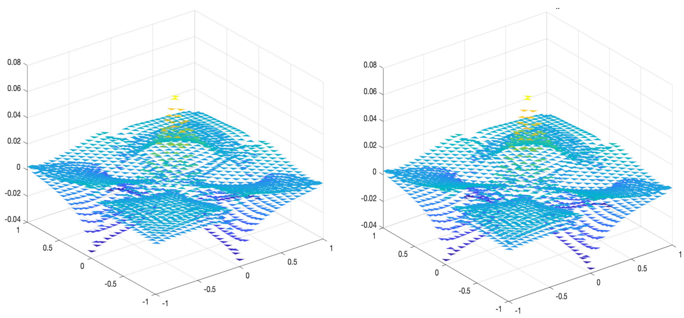

FIGURE 2. Test Case 2: Numerical error for Lagrange multiplier when C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element is applied: left figure is without the term $c(\cdot,\cdot)$ proposed in [6]; right figure is with the term $c(\cdot,\cdot)$ proposed in this paper.

Test Case 3. Find u satisfying

$$\sum_{i,j=1}^{2}\left(\delta_{ij}+\frac{x_i x_j}{x_1^2+x_2^2}\right)\partial_{ij}^2 u = f, \qquad \text{in } \Omega_i \ (i=1,2). \tag{85}$$

For the case of $\alpha > 1$, the exact solution $u = |x|^\alpha$ has $H^{1+\alpha-\tau}(\Omega)$ regularity for arbitrarily small $\tau > 0$ and the load function is $f = (2\alpha^2 - \alpha)|x|^{\alpha-2}$. The Cordès condition holds true with $\varepsilon = 4/5$.

TABLE 5. Test Case 3: Convergence rates for C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element on Ω_1.

$1/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	0.06193		0.7395		1.408	
2	0.008210	2.915	0.1116	2.729	0.3570	1.980
4	0.001760	2.222	0.04270	1.385	0.2169	0.7190
8	4.30E-04	2.034	0.01483	1.526	0.1351	0.6833
16	1.05E-04	2.035	0.005024	1.562	0.08752	0.6260
32	2.55E-05	2.042	0.001681	1.580	0.05735	0.6098

TABLE 6. Test Case 3: Convergence rates for C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element on Ω_1.

$1/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	0.003403		0.4903		0.0650	
2	0.007769	-1.1911	0.1774	1.467	0.06253	0.05684
4	0.002576	1.593	0.06160	1.526	0.04782	0.3870
8	7.83E-04	1.719	0.02099	1.554	0.03270	0.5482
16	2.19E-04	1.839	0.007048	1.574	0.02183	0.5832
32	5.84E-05	1.906	0.002349	1.585	0.01447	0.5930

TABLE 7. Test Case 3: Convergence rates for C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element on Ω_2.

$2/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	0.8998		1.207		0.4146	
2	0.7142	0.3333	1.808	-0.5834	2.289	-2.465
4	0.1928	1.889	1.244	0.5394	4.685	-1.034
8	0.04503	2.098	0.0967	3.685	0.5329	3.136
16	0.02497	0.8506	0.05352	0.8540	0.3078	0.7919
32	0.01242	1.007	0.02806	0.9316	0.1958	0.6526

Tables 5-6 present the numerical results of the M-PDWG scheme on the domain $\Omega_1 = (0,1)^2$. It is clear that the coefficient matrix $a = (a_{ij})_{2\times 2}$ is continuous in the interior of the domain Ω_1, but it fails to be continuous at the corner point $(0,0)$. Note that the exact solution $u = |x|^{1.6}$ has $H^{2.6-\tau}(\Omega)$ regularity for arbitrarily small $\tau > 0$. The numerical approximation indicates that the convergence rates for $\mathbf{e}_g$ and γ_h in the discrete L^2 norm are of orders $\mathcal{O}(h^{1.6})$ and $\mathcal{O}(h^{0.6})$, respectively, which are consist with the theoretical results. The convergence rate for e_0 in the discrete L^2 norm seems to be of an order $\mathcal{O}(h^2)$, for which there is no theory available to apply.

TABLE 8. Test Case 3: Convergence rates for C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element on Ω_2.

$2/h$	$\|e_0\|_0$	order	$\|\mathbf{e}_g\|_0$	order	$\|\gamma_h\|_0$	order
1	6.82E-01		0.5800		0.1091	
2	6.13E-01	0.1518	0.7084	-0.2884	0.08120	0.4271
4	2.54E-01	1.273	0.4067	0.8004	0.05057	0.6831
8	1.12E-01	1.175	0.2177	0.9018	0.04179	0.2753
16	5.12E-02	1.137	0.1101	0.9829	0.02969	0.4930
32	0.02354	1.120	0.05402	1.028	0.02011	0.5620

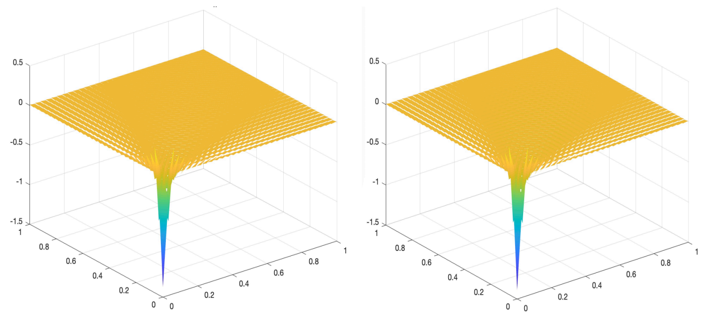

FIGURE 3. Test Case 3: Numerical error for Lagrange multiplier when C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element is applied on Ω_1: left figure is without the term $c(\cdot,\cdot)$ proposed in [6]; right figure is with the term $c(\cdot,\cdot)$ proposed in this paper.

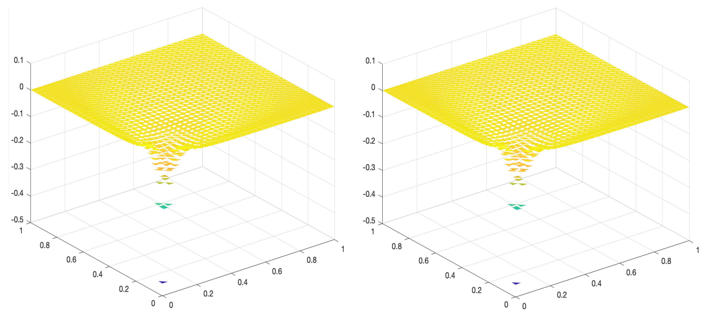

FIGURE 4. Test Case 3: Numerical error for Lagrange multiplier when C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element is applied on Ω_1: left figure is without the term $c(\cdot,\cdot)$ proposed in [6]; right figure is with the term $c(\cdot,\cdot)$ proposed in this paper.

Figures 3-4 shows the numerical error γ_h for the C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element and the C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element on the domain Ω_1 respectively, compared with the PDWG scheme proposed in [6].

Tables 7-8 demonstrate the numerical performance of the M-PDWG scheme (17)-(18) for the test equation (85) in the domain $\Omega_2 = (-1,1)^2$. The coefficient matrix $a = (a_{ij})_{2\times 2}$ is discontinuous at the center point $(0,0)$ of the domain Ω_2

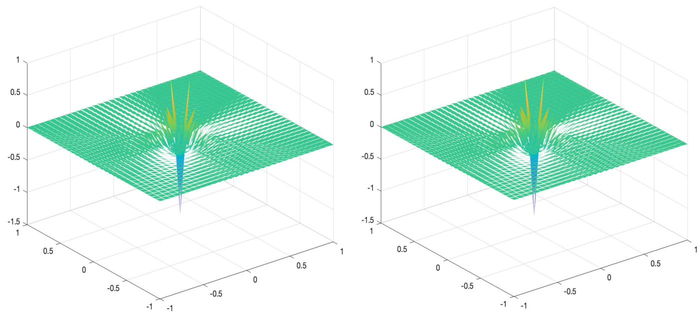

Figure 5. Test Case 3: Numerical error for Lagrange multiplier when C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element is applied on Ω_2: left figure is without the term $c(\cdot,\cdot)$ proposed in [6]; right figure is with the term $c(\cdot,\cdot)$ proposed in this paper.

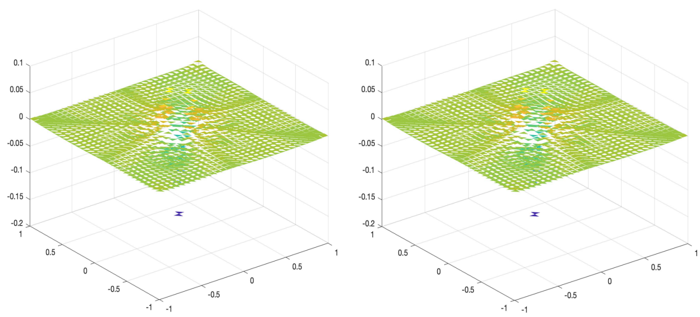

Figure 6. Test Case 3: Numerical error for Lagrange multiplier when C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element is applied on Ω_2: left figure is without the term $c(\cdot,\cdot)$ proposed in [6]; right figure is with the term $c(\cdot,\cdot)$ proposed in this paper.

so that the duality argument in the convergence theory is not applicable. We observe from Tables 7-8 that the numerical results are less accurate than the case of $\Omega_1 = (0,1)^2$ presented in Tables 5-6. The convergence rate for γ_h in the L^2 norm is of an order $\mathcal{O}(h^{0.6})$, which is consistent with the theory; while the convergence rates for e_0 and $\mathbf{e}_g$ in the L^2 norm are both of an order $\mathcal{O}(h)$ or slightly higher.

Figures 5-6 shows the numerical error γ_h for the C^0-$P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element and the C^0-$P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element on the domain Ω_2 respectively, compared with the PDWG scheme proposed in [6].

Acknowledgement

I would like to express my gratitude to Dr. Junping Wang for his valuable discussion and suggestions. The research of Chunmei Wang was partially supported by National Science Foundation Award DMS-1849483.

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