Cooperative Containment of a High-speed Evader

Eloy Garcia and Shaunak D. Bopardikar

***Abstract*— A pursuit-evasion differential game where a group of slow pursuers cooperate in order to capture a high-speed evader is considered. Pursuit of a fast evader is a challenging problem and capture is not guaranteed given the arbitrary initial conditions. Further, cooperation among pursuers is necessary in order for successful encirclement and capture of the fast evader. A novel analysis based on Cartesian Ovals is provided which accurately separates the reachable regions of a pursuer and high-speed evader in the case where the pursuer is endowed with a positive capture radius.**

I. Introduction

In multi-player pursuit-evasion games, one of the most important problems is to design cooperative control schemes and coordination strategies among members of the same team in order to achieve a global objective [1]. In particular, conflict problems between two adversarial teams calls for implementation of cooperative strategies that can guarantee certain performance with respect to best actions that the possibly superior adversary can achieve.

The authors of [2] addressed the problem of cooperative pursuers which try to capture an evader and the game is played in a closed convex domain. The papers [3], [4] considered a pursuit problem with multiple pursuers and a single evader in an unbounded environment in which each player has limited-range sensing. The cooperative pursuit of a single and slower evader has been considered in [5]. The papers [6], [7] considered the differential game of protecting a border between a team of pursuers and a team of evaders where cooperative guidance among members of the same team is necessary in order to achieve the corresponding goals and best performance.

Pursuit-evasion games where an evader is faster than the pursuers have also received attention in the past. For instance, [8] addressed the pursuit-evasion problem in which a faster evader must pass between two slower pursuers. The authors of [9] considered the same problem addressed in this paper, which is the pursuit of a high-speed evader by a group of slow pursuers. The need for cooperation among the slower pursuers was emphasized in this reference. Unfortunately, the approach proposed in [9] is based on parallel guidance where the evader is required to tell the pursuers where it is going to be at the next time step. In differential games, as in any conflict between intelligent adversarial players, each party does not share its plan and strategy ahead of time. The optimal strategies should be state-feedback strategies [10], [11] and should provide a performance guarantee with respect to any behavior of the opponent. Unilateral deviations from the saddle-point solution by any given player only benefit the adversary.

It is well-known that one or a group of slow pursuers cannot point capture a fast evader. Hence, several authors have considered the case where the slow pursuers are endowed with a positive capture radius. The authors of [12], [13] considered slow pursuers with a positive capture radius as in this paper. However, they separated the reachable regions of the evader with respect to any given pursuer using the corresponding Apollonius circle. This approach may become conservative since the Apollonius circle, which is only a function of the speed ratio parameter, does not take into consideration the capture radius parameter. Similar approaches have been considered in works such as [14]–[16], wherein the slow pursuers are endowed with a positive capture radius but Apollonius circles are used. The authors of [17] also considered the same problem of pursuit of a fast evader. Although Apollonius circles are not used explicitly, their theoretical analysis is based on point capture. Then, the strategies obtained assuming point capture are used in their examples when pursuers have a positive capture radius. This is a similar situation to [12], [13], [15], [16] where the capture radius parameter is completely disregarded in the design and analysis of strategies.

In the case where the pursuers possess a positive capture radius, the reachable regions between a pursuer and an evader are not separated by an Apollonius circle. In such a case, the reachable regions are separated by the corresponding Cartesian Oval [18] between the evader and the given pursuer. The following are the major improvements with respect to [12], [13], [15]–[17]. In this paper, we will first show that the Cartesian Oval is a function of both parameters, the speed ratio and the capture radius, and that the Apollonius circle is a conservative representation of a single pursuer's reachable region. Then, we identify active and non-active pursuers and provide the optimal strategies of each type of pursuer. Additionally, we derive the strategy of the evader in order to optimally attack the weakest link in the pursuer formation. Furthermore, although our analysis is based on a more complicated function than the Apollonius circle that separates the reachable regions of the players, we nevertheless provide optimal strategies in analytical form; thereby distinguishing this work from preceding papers such as [12], [13].

The paper is organized as follows. Section II formalizes the problem of cooperative pursuit of a high-speed evader. In Section III, the function that separates the reachable regions between the evader and a pursuer with a positive

This work has been supported in part by AFOSR LRIR No. 21RQ-COR084 and in part by NSF award ECCS 2030556.

E. Garcia is with the Control Science Center of Excellence, Air Force Research Laboratory, Wright-Patterson AFB, OH 45433. eloy.garcia.2@us.af.mil

S. D. Bopardikar is with the Department of Electrical and Computer Engineering, Michigan State University, East Lansing, MI. shaunak@egr.msu.edu

capture radius is obtained. The optimal strategies of non-active pursuers are derived in Section IV. In Section V, the strategies of the pair of active pursuers and the strategy of the evader are presented. Examples are presented in Section VI and concluding remarks are made in Section VII.

II. Problem Statement

We consider a pursuit-evasion game where a fast evader, E, tries to avoid being captured by a group of pursuers, P_i, for $i = 1, ..., N$. The states of the evader and the pursuers are respectively specified by their Cartesian coordinates $\mathbf{x}_E = (x_E, y_E)$ and $\mathbf{x}_i = (x_i, y_i)$; the players have constant speeds v_E and v_i, for $i = 1, ..., N$. The evader's control is his instantaneous heading angle, $\mathbf{u}_E = \{\psi_E\}$. The team of pursuers cooperatively choose their instantaneous respective headings ψ_i, so the pursuers' control is $\mathbf{u}_P = \{\psi_i\}$ for $i = 1, ...N$. The dynamics/kinematics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_E, \mathbf{u}_P)$ are specified by the system of ordinary differential equations

$$\begin{aligned} \dot{x}_E &= v_E \cos\psi_E, & x_E(0) &= x_{E_0} \\ \dot{y}_E &= v_E \sin\psi_E, & y_E(0) &= y_{E_0} \\ \dot{x}_i &= v_i \cos\psi_i, & x_i(0) &= x_{i_0} \\ \dot{y}_i &= v_i \sin\psi_i, & y_i(0) &= y_{i_0} \end{aligned} \tag{1}$$

where the admissible controls are given by $\psi_E, \phi_i \in [-\pi, \pi)$, for $i = 1, ..., N$. We consider the case where the pursuers have the same speed, that is, $v = v_i = v_j$ for any two pursuers P_i and P_j. Also, for a fast evader, we have $v_E > v$. The speed ratio is defined as $\gamma = \frac{v}{v_E} < 1$. Without loss of generality, we normalize the speeds so that $v_E = 1$ and $v = \gamma$.

The initial state of the system is $\mathbf{x}_0 := (x_{E_0}, y_{E_0}, x_{i_0}, y_{i_0}) = \mathbf{x}(t_0)$. The terminal time t_f is defined as the time instant when the evader is captured by any one of the pursuers, that is, the game terminates if $\sqrt{(x_E - x_i)^2 + (y_E - y_i)^2} = \rho$ for some pursuer $i = 1, ..., N$, where $\rho > 0$ denotes the capture radius of the pursuers. The evader strives to avoid termination of the game by evading the group of pursuers and avoid being captured.

III. Reachable Regions

We begin the analysis of the cooperative pursuit of a high-speed evader by characterizing the reachable regions between the fast evader and any given pursuer, P_i. Let

$$\begin{aligned} d_i &= \sqrt{(x_i - x_E)^2 + (y_i - y_E)^2}, \\ \lambda_i &= \arctan\left(\tfrac{y_i - y_E}{x_i - x_E}\right), \end{aligned} \tag{2}$$

denote, respectively, the distance between E and pursuer P_i and the Line-of-sight (LOS) angle, with respect to the positive x-axis, from E to P_i, for $i = 1, ..., N$.

In the papers [12], [13], [15], [16], the Apollonius circle has been used to separate the reachable regions between a pursuer and an evader with different speeds and where the pursuer has a positive capture radius. An Apollonius circle is defined as the locus of points S such that the following holds: $\overline{P_iS} = \gamma\overline{ES}$. This means that, for any point S on the circle, the distance between P_i and S is equal to the distance between E and S multiplied by the speed ratio parameter γ. Also, if both P_i and E head directly to point S at their respective speeds v and v_E, then they reach point S at the same time instant. Therefore, $d_i = 0$ at that point. It is easy to see that the Apollonius circle separates the reachable regions between two players with different speed but only if the capture radius is equal to zero.

We are now interested in generalizing the separation of reachable regions to the case where the capture radius is a positive constant. Therefore, we need to incorporate the capture radius $\rho > 0$ into the reachable region condition, that is, $\overline{P_iS} = \gamma\overline{ES} + \rho$. In other words, we need to define the locus of points S such that the distance between P_i and S is equal distance between E and S multiplied by γ plus the capture constant ρ. In this case, if both P_i and E head directly to point S at their respective speeds v and v_E, then E reaches point S at the same time instant that P_i reaches a distance ρ from S. Therefore, $d_i = \rho$ at that point, which is exactly the capture condition we want to enforce. The following theorem characterizes the reachable regions between E and any given pursuer P_i.

Theorem 1: **Reachable regions of E and P_i.** Let (x_E, y_E) and (x_i, y_i) denote the instantaneous positions of E and P_i, respectively. Then, the reachable regions of the evader and pursuer i, for $i = 1, ..., N$, where E tries to evade P_i, are separated by the Cartesian Oval (CO)

$$\begin{aligned} x &= x_E + r(\phi)\cos(\lambda_i + \phi), \\ y &= y_E + r(\phi)\sin(\lambda_i + \phi), \end{aligned} \tag{3}$$

where

$$r(\phi) = \frac{\gamma\rho + d_i\cos\phi \pm \sqrt{(\gamma\rho + d_i\cos\phi)^2 - (1-\gamma^2)(d_i^2 - \rho^2)}}{1-\gamma^2}, \tag{4}$$

for $\phi \in [-\phi_i, \phi_i]$, where

$$\phi_i = \arccos\left(\frac{\sqrt{(1-\gamma^2)(d_i^2-\rho^2)} - \gamma\rho}{d_i}\right) \tag{5}$$

and the variables d_i and λ_i are given by (2).

Proof. The boundary of the reachable region of E with respect to pursuer P_i is the locus of points that E can reach before being captured by P_i. The slow pursuer P_i possesses a capture radius $\rho > 0$ and therefore, E's reachable region is delineated by the Cartesian Oval [18], or capture condition

$$r' = \gamma r + \rho \tag{6}$$

where r denotes the distance traveled by E to reach some point on the oval and r' is the distance from P_i to the same point on the oval, see Fig. 1. For any $\phi \in [-\phi_i, \phi_i]$, the following holds

$$r'^2 = r^2 + d_i^2 - 2d_i r\cos\phi.$$

Substituting (6) into the previous equation we obtain

$$(1-\gamma^2)r^2 - 2(\gamma\rho + d_i\cos\phi)r + d_i^2 - \rho^2 = 0. \tag{7}$$

The two roots of (7) are given by (4) and both roots are used to construct the CO. This is due to the fact that for any $\phi \in [-\phi_i, \phi_i]$, the condition (6) holds for two different sets of values of r and r'. Hence, P_i can intercept E on the front side or on the back side of the CO. These two sides are shown in solid and dashed lines in Fig. 1 to highlight the

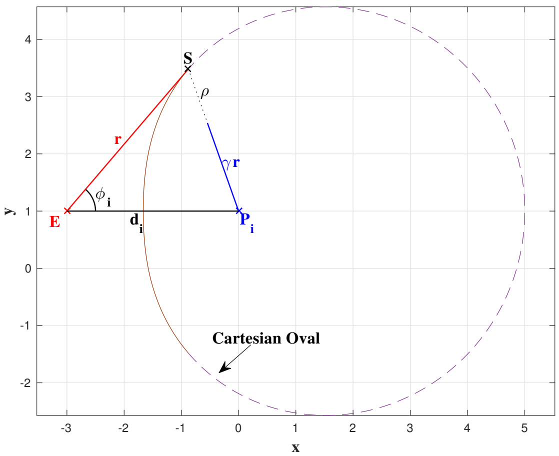

Fig. 1. Cartesian Oval for $\rho = 1$, $\gamma = 0.5$, $\lambda_i(0) = 0$ and $d_i(0) = 3$. The particular value of ϕ shown in this figure is $\phi = \phi_i$.

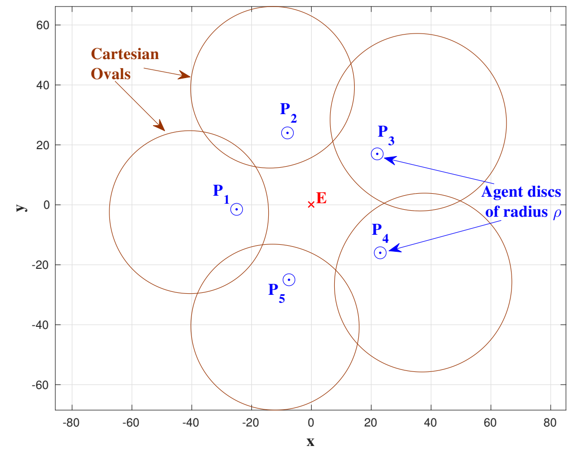

Fig. 3. Team of pursuers around evader with $\gamma = 0.6$ and $\rho = 2$. Initial configuration where $\theta_{ij} > 0$ for each pair of neighbor pursuers.

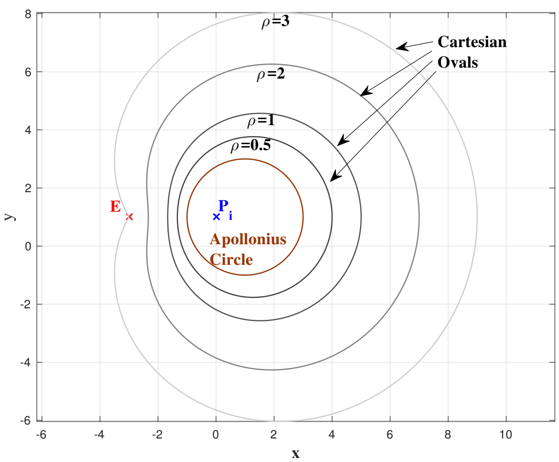

Fig. 2. Cartesian Ovals for different values of ρ including the Apollonius circle case ($\rho = 0$), for a given initial distance $d_i(0) = 3$.

difference. The two points where the front and the back sides connect are the points on the CO where the tangent of the CO passes through E. At those points, the two roots (4) are the same. Thus, the angle ϕ_i in the limiting case, before the roots become complex, can be obtained from the following equation

$$(\gamma\rho + d_i \cos\phi)^2 - (1-\gamma^2)(d_i^2 - \rho^2) = 0.$$

Solving for ϕ in the previous equation, we obtain (5). Finally, since $r = r(\phi)$ and ϕ is the angle at E, then the Cartesian Oval is obtained based on the current position of E using (3). Note that the locus of points (x, y) given by (3) denote the position of E at the time instant when $d_i = \rho$. $\square$

Remark. The Cartesian Oval generalizes the Apollonius circle to the case of positive capture radius. In the particular case where $\rho = 0$, we recover the Apollonius circle condition $r' = \gamma r$. However, if $\rho > 0$, then the Apollonius circle becomes a conservative criterion to separate the players reachable regions since it does not take into account the capture condition $\rho > 0$. It is also worth mentioning that, if $\gamma = 1$ and $\rho > 0$, then the CO becomes $r' = r + \rho$, a hyperbola. The hyperbola was used by Isaacs [10] to separate the reachable regions between a pursuer and an evader with the same speed, but the pursuer has a positive capture radius $\rho > 0$.

Figure 2 shows the Cartesian Ovals between E and P_i for $d_i(0) = 3$ and for different values of ρ. This figure clearly shows how the reachable region of P_i is in fact a function of ρ. In general, we have that as $d_i \to \rho$, the Apollonius circle becomes an extremely conservative representation of the pursuer's reachable region. Although the CO is a fourth-order curve and analysis becomes more difficult compared to the simpler quadratic curve given by the Apollonius circle, we are nevertheless able to obtain analytic solutions for the optimal strategies of the evader and the pursuers in the pursuit-evasion game under consideration in this paper.

The reachable regions of E and P_i are separated by the CO (3) where P_i seeks to block the path of E. Two cases may occur, E is blocked by P_i on the front side of (3) or E is blocked by P_i on the back side of (3). In the former, the separation $d = \rho$ is first attained when E reaches the CO. In the latter, the CO condition (6) is satisfied, and $d = \rho$, at the time instant when E reaches the CO. However, $d < \rho$ holds for a time interval prior to E reaching the CO if E aims directly at the CO. Since E needs to avoid capture, at the first time instant such that $d = \rho$, E needs to switch guidance in order to maintain a safe separation $d = \rho$ to avoid being captured by E. It is worth to note that the obtained strategies in this paper do not warrant blocking on the back side of the CO and switch guidance is not necessary.

IV. Cooperative Strategy of Non-active Pursuers

It is clear that one slow pursuer is unable to capture the faster evader, even if the pursuer is endowed with a positive capture radius. Hence, a team of slow pursuers needs to cooperate in order to strategically overlap their reachable regions, the COs described in (3), in order to deny an exit path to an already encircled evader. We assume that the team of pursuers forms a ring-like formation around the evader and we denote as neighbor pursuers, any two adjacent pursuers in the ring formation. Hence, any given pursuer has two and only two neighbors to which it looks to overlap COs. Fig. 3 shows a team of pursuers where their capture radius is $\rho = 1$. In this figure, the pursuers are drawn as circular discs of radius ρ centered at the pursuer's current position. The Cartesian Oval between the evader and each pursuer is also shown in the figure. It can be seen that, initially, there is an overlap between any two adjacent COs and the evader seems to be contained since there is no initial escape route.

Consider two neighbor pursuers P_i and P_j and assume, without loss of generality, that $\lambda_i > \lambda_j$. Let θ_{ij} denote the overlapping angle between E and any two neighbor pursuers P_i and P_j, which is given by

$$\theta_{ij} = \phi_i + \phi_j - (\lambda_i - \lambda_j), \tag{8}$$

where λ_i is given by (2) and ϕ_i is given by (5), for $i, j = 1, ..., N$.

Lemma 1: If $\theta_{ij} < 0$ for some pair of neighboring pursuers P_i and P_j, then the evader is able to escape by implementing a constant heading angle ψ_A such that

$$\psi_A \in (\lambda_j + \phi_j, \lambda_i - \phi_i). \tag{9}$$

The angles ϕ_i, ϕ_j, and θ_{ij} are shown in Fig. 4.

We consider then the case where, initially, $\theta_{ij} > 0$ for $i = 1, ..., N$ and for any neighbor P_j such that $\lambda_i - \lambda_j > 0$. Even in the case where all overlapping angles are strictly positive, capture of the evader is not guaranteed since, as the evader moves, the rate of change of each θ_{ij} changes and "gaps" may be created if the pursuers do not cooperate and do not find optimal pursuit and containment strategies. In order to exemplify this case, consider the initial conditions in Fig. 3 and also consider the case where all the pursuers implement the classical Pure Pursuit (PP) guidance, that is, at every time instant, the pursuers head directly at E. The evader chooses a route between two pursuers, say P_1 and P_2, and, since the pursuers implemented PP, a wide gap soon appears between P_1 and P_2, where now $\theta_{12} < 0$. Figure 5 shows the trajectories of the players where the current COs are also shown. Hence, E is able to escape from the pursuers since they did not cooperate. In general, advantage in number is not enough to achieve a goal if the team of players do not cooperate and coordinate their actions. The main problem, therefore, is to design cooperative strategies such that E is contained and is eventually captured by the pursuers. This problem is tackled in the remaining of the paper.

In what follows, we define two types of pursuers, active and non-active pursuers. The active pursuers is the *pair of neighbor pursuers* in charge of capturing E while protecting the weakest link which is the link between COs with the smallest value of θ_{ij}. The strategies of E and the pair of active pursuers will be discussed in the next section. In this section, the objective of the non-active pursuers is to cooperate with the active pursuers by designing and implementing containment strategies that aim at preserving the overlapping of the COs. In more detail, if P_j is not active in capturing E then its objective is to avoid a gap between its overlapping angle; its best choice is to maximize the rate of change of the overlapping angle θ_{ij} in order to avoid the creation of a gap in the link between itself and its teammate P_i.

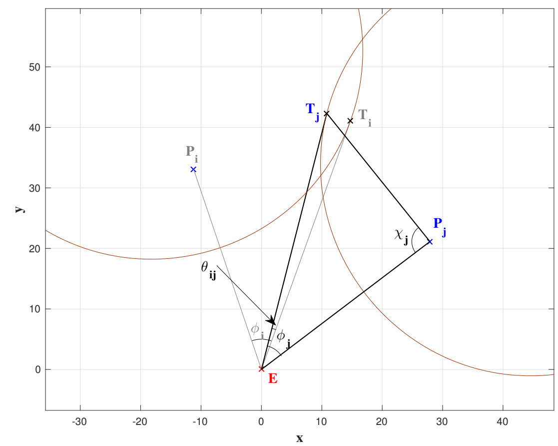

Fig. 4. Cartesian Ovals of two neighboring pursuers showing their overlap and the tangent points.

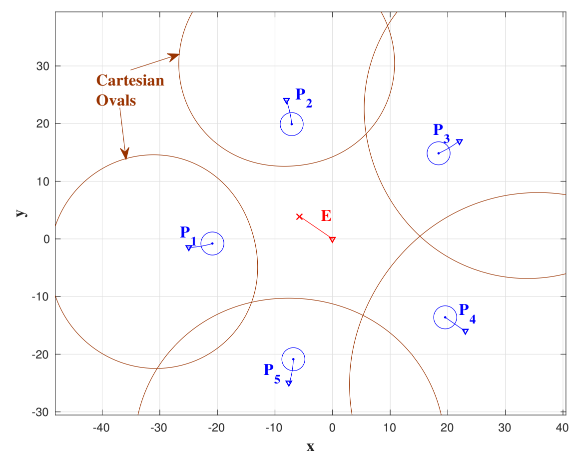

Fig. 5. All pursuers implement PP guidance and a gap is created for E to escape. $\bigtriangledown$: initial position of each player. Circular discs: current position of P_i. $\times$: current position of E.

Theorem 2: **Optimal strategies of non-active pursuers**. Consider the overlapping angle θ_{ij} between P_i and P_j where P_i is an active pursuer and P_j is non-active and suppose that P_j cooperates with P_i in order to maximize the rate of change of θ_{ij}. Then, the optimal, state-feedback strategy of P_j which maximizes $\dot{\theta}_{ij}$ is given by

$$\psi_j^* = \arccos(-\tfrac{\rho}{d_j}) + \lambda_j. \tag{10}$$

Remark. This strategy can be applied to each remaining non-active pursuer to create a chain reaction that maximizes the overlap between each CO. Every non-active pursuer cooperates with the neighbor which is the closest to the active pursuers in order to maximize the rate of change of each overlapping angle and deny any escape route to the evader.

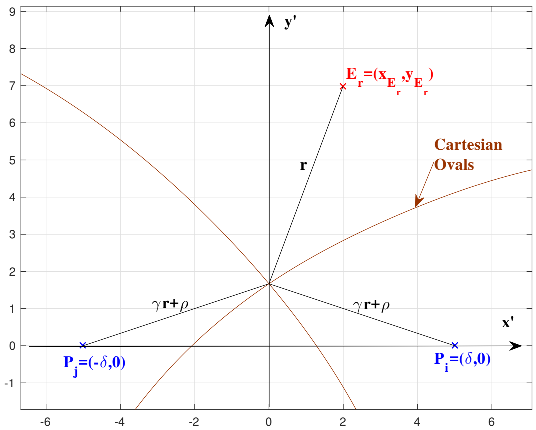

Fig. 6. Pair of active pursuers and intersection of their Cartesian Ovals.

Also note that a perfect information game is considered, where each player is aware of the position of every other player in the game. With this information, every player is able to compute all overlapping angles and there is no need for communication between pursuers.

V. Evader's Attack of Weakest Link

Throughout this section we assume, without loss of generality, that $\lambda_i > \lambda_j$ for a pair of active pursuers P_i and P_j. We also assume that $\lambda_i - \lambda_j < \pi$. The evader tries to evade the pursuers by attacking the weakest link, that is, the link between two pursuers P_i and P_j such that θ_{ij} is the smallest. Player E's strategy looks to minimize the rate of change of θ_{ij}. This is the best opportunity for E to create a gap. Then, the strategy for the pursuers is to protect the weakest link and capture E. Players P_i and P_j for which θ_{ij} is the smallest become the active pursuers and their optimal strategy is to capture E at the intersection of their COs. The active pursuers do not minimize their overlapping angle, they capture the evader at the earliest interception point according to the reachable regions of both active pursuers with respect to the evader. These reachable regions are given by the corresponding Cartesian Ovals.

Let $x_m = \frac{x_i+x_j}{2}$ and $y_m = \frac{y_i+y_j}{2}$ denote the coordinates of the midpoint between pursuers P_i and P_j. In order to obtain the optimal headings of the active pursuers we consider the relative coordinate frame, $x' - y'$, shown in Fig. 6 where the origin is located at (x_m, y_m) and the pursuers are located along the x-axis such that $P_{i_r} = (\delta, 0)$ and $P_{j_r} = (-\delta, 0)$ where $\delta = \frac{1}{2}\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$. The coordinates of E in the relative frame are denoted by (x_{E_r}, y_{E_r}).

Theorem 3: **Optimal strategies of active pursuers**. The optimal, state-feedback strategy of the active pursuers is to defend the weakest link $\theta_{i^*j^*}$ where

$$\{i^*, j^*\} = \arg\min_{i,j}\{\theta_{ij}\}, \tag{11}$$

and θ_{ij} are given by (8) for any pair of neighbor pursuers P_i and P_j. The optimal headings of the active pursuers in the relative coordinate frame are given by

$$\begin{aligned} \cos\psi_{i_r}^* &= \frac{-\delta}{\sqrt{\delta^2+y^{*2}}} \\ \sin\psi_{i_r}^* &= \frac{y^*}{\sqrt{\delta^2+y^{*2}}} \\ \cos\psi_{j_r}^* &= \frac{\delta}{\sqrt{\delta^2+y^{*2}}} \\ \sin\psi_{j_r}^* &= \frac{y^*}{\sqrt{\delta^2+y^{*2}}} \end{aligned} \tag{12}$$

where y^* is the solution of the following equation

$$\begin{aligned} &(1-\gamma^2)^2 y^4 + 4\gamma^2 y_{E_r}(1-\gamma^2)y^3 \\ &+2[(1-\gamma^2)c + 2\gamma^2(\gamma^2 y_{E_r}^2 - \rho^2)]y^2 \\ &+4\gamma^2 y_{E_r}(2\rho^2 + c)y + c^2 - 4\gamma^2\rho^2(x_{E_r}^2 + y_{E_r}^2) = 0, \end{aligned} \tag{13}$$

which minimizes

$$J(y) = \sqrt{x_{E_r}^2 + (y_{E_r} - y)^2}, \tag{14}$$

and $c = \delta^2 - \rho^2 - \gamma^2(x_{E_r}^2 + y_{E_r}^2)$.

Remark. The pursuers' strategy is a state feedback strategy; at every time instant, the weakest link is calculated and the pair of pursuers associated with the weakest link become the active pursuers according to (11) which can be computed based only on state information. In other words, the evader may decide not to aim at the initial smallest overlapping angle, if the smallest overlapping angle changes, then the pursuers' strategy switches, that is, the active pursuers will be the new pair of pursuers associated with the new smallest overlapping angle. Additionally, the optimal headings (12) only depend on the state of the players.

Theorem 4: **Optimal strategy of evader**. The evader's optimal, state-feedback strategy in order to minimize the rate of change of the weakest link, that is, minimize $\dot{\theta}_{i^*j^*}$, is given by

$$\sin\psi_E^* = \frac{\Psi_{E_s}}{\sqrt{\Psi_{E_s}^2+\Psi_{E_c}^2}}, \quad \cos\psi_E^* = \frac{\Psi_{E_c}}{\sqrt{\Psi_{E_s}^2+\Psi_{E_c}^2}} \tag{15}$$

where

$$\begin{aligned} \Psi_{E_s} &= \tfrac{1}{d_j}\Big(\cos\lambda_j - \tfrac{\rho}{\sqrt{d_j^2-\rho^2}}\sin\lambda_j\Big) \\ &\quad - \tfrac{1}{d_i}\Big(\cos\lambda_i + \tfrac{\rho}{\sqrt{d_i^2-\rho^2}}\sin\lambda_i\Big), \\ \Psi_{E_c} &= \tfrac{1}{d_i}\Big(\sin\lambda_i - \tfrac{\rho}{\sqrt{d_i^2-\rho^2}}\cos\lambda_i\Big) \\ &\quad - \tfrac{1}{d_j}\Big(\sin\lambda_j + \tfrac{\rho}{\sqrt{d_j^2-\rho^2}}\cos\lambda_j\Big). \end{aligned} \tag{16}$$

VI. Examples and Future Work

A. Example

Consider the example with five pursuers and the initial positions of E and each pursuer as shown in Fig. 3. The parameters are $\gamma = 0.6$ and $\rho = 2$. The weakest link is between pursuers P_1 and P_2 since θ_{12} is the smallest. The evader is captured as it is shown in Fig. 7. The values of each overlapping angle θ_{ij} as a function of time are shown in Fig. 8. It can be seen in this example that θ_{12} remains the smallest for $t \in [0, t_f]$; also, $\theta_{12} > 0$ for $t \in [0, t_f]$ and E is captured by P_1 and P_2.

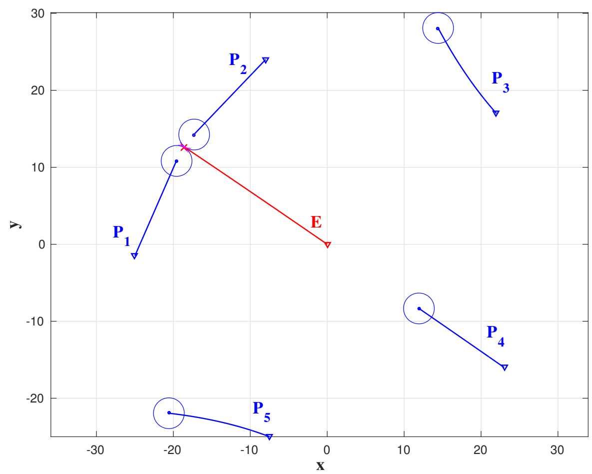

Fig. 7. Example. $\bigtriangledown$: initial position of each player. Circular discs: current position of P_i. $\times$: current position of E.

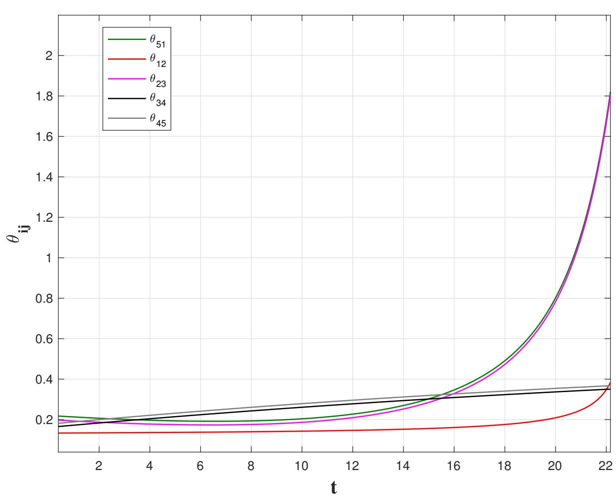

Fig. 8. Example. Overlapping angles as a function of time.

B. Future Work

In the future, we plan to extend the proposed approach to the case of irregular formations of pursuers. In such cases, there may not exist an intersection between neighbor pursuers, but $\theta_{ij} > 0$ since one pursuer is much closer to E than its neighbor is. Intuitively, E needs to go around the closest pursuer in order to create a gap, that is, to make $\theta_{ij} < 0$ and guarantee a safe escape. We expect to find the correct coordination between neighbor pursuers in this case and also obtain the corresponding strategy of E.

We note in that example that $\theta_{i^*j^*}$ is monotonically increasing. In future work, we will formally prove that if $\theta_{i^*j^*}(t_0) > 0$, then, $\dot{\theta}_{i^*j^*}(t; \psi_E^*, \psi_i^*, \psi_j^*) > 0$, for all $t \in [t_0, t_f]$. This is an important property because, once $\theta_{i^*j^*} > 0$, a gap in the weakest link cannot be created by the evader. However, while E is attacking the weakest link, other overlapping angles are not necessarily monotonically increasing. This is related to dispersal surfaces since it could occur that a different link becomes as weak as the initial weakest link. Therefore, future work should address all possible choices for the players when they arrive at the dispersal surface.

VII. Conclusions

The problem of pursuit of a superior evader by a group of pursuers was analyzed. This problem represents a quintessential example of cooperation in adversarial scenarios. The evader has advantage in speed. The pursuers, although slower than the evader, have advantage in number and a non-zero capture radius. However, in order to exploit the advantage in number, the pursuers need to devise the best way to cooperate in order to keep encirclement of and eventually capture the fast evader. This paper provided a step forward into realizing such type of cooperation by providing an accurate representation of the players reachable regions, and accordingly designing the players' strategies.

References

[1] I. E. Weintraub, M. Pachter, and E. Garcia, "An introduction to pursuit-evasion differential games," in *2020 American Control Conference (ACC)*, 2020, pp. 1049–1066.

[2] Z. Zhou, W. Zhang, J. Ding, H. Huang, D. M. Stipanović, and C. J. Tomlin, "Cooperative pursuit with Voronoi partitions," *Automatica*, vol. 72, pp. 64–72, 2016.

[3] S. D. Bopardikar, F. Bullo, and J. P. Hespanha, "Cooperative pursuit with sensing limitations," in *2007 American Control Conference*. IEEE, 2007, pp. 5394–5399.

[4] ——, "On discrete-time pursuit-evasion games with sensing limitations," *IEEE Transactions on Robotics*, vol. 24, no. 6, pp. 1429–1439, 2008.

[5] A. Von Moll, M. Pachter, E. Garcia, D. Casbeer, and D. Milutinović, "Robust policies for a multiple-pursuer single-evader differential game," *Dynamic Games and Applications*, vol. 10, no. 1, pp. 202–221, 2020.

[6] R. Yan, Z. Shi, and Y. Zhong, "Task assignment for multiplayer reach–avoid games in convex domains via analytical barriers," *IEEE Transactions on Robotics*, vol. 36, no. 1, pp. 107–124, 2019.

[7] E. Garcia, D. W. Casbeer, A. Von Moll, and M. Pachter, "Multiple pursuer multiple evader differential games," *IEEE Transactions on Automatic Control, early access*, 2020.

[8] P. Hagedorn and J. Breakwell, "A differential game with two pursuers and one evader," *Journal of Optimization Theory and Applications*, vol. 18, no. 1, pp. 15–29, 1976.

[9] X. Wang, J. B. Cruz Jr, G. Chen, K. Pham, and E. Blasch, "Formation control in multi-player pursuit evasion game with superior evaders," in *Defense Transformation and Net-Centric Systems 2007*, vol. 6578. International Society for Optics and Photonics, 2007, p. 657811.

[10] R. Isaacs, *Differential Games*. New York: Wiley, 1965.

[11] J. Lewin, *Differential Games: theory and methods for solving game problems with singular surfaces*. Springer-Verlag London Limited, 1994.

[12] M. V. Ramana and M. Kothari, "A cooperative pursuit-evasion game of a high-speed evader," in *IEEE 54th Conference on Decision and Control*. IEEE, 2015, pp. 2969–2974.

[13] ——, "Pursuit strategy to capture high-speed evaders using multiple pursuers," *Journal of Guidance, Control, and Dynamics*, vol. 40, no. 1, pp. 139–149, 2017.

[14] T. R. Vechalapu, "A trapping pursuit strategy for capturing a high speed evader," in *AIAA Scitech 2020 Forum*, 2020, p. 2069.

[15] X. Fang, C. Wang, L. Xie, and J. Chen, "Cooperative pursuit with multi-pursuer and one faster free-moving evader," *IEEE transactions on cybernetics*, 2020.

[16] M. D. Awheda and H. M. Schwartz, "A decentralized fuzzy learning algorithm for pursuit-evasion differential games with superior evaders," *Journal of Intelligent & Robotic Systems*, vol. 83, no. 1, pp. 35–53, 2016.

[17] J. Chen, W. Zha, Z. Peng, and D. Gu, "Multi-player pursuit–evasion games with one superior evader," *Automatica*, vol. 71, pp. 24–32, 2016.

[18] J. D. Lawrence, *A catalog of special plane curves*. Dover, 1972.