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A GEOMETRIC REALIZATION OF SOCLE-PROJECTIVE CATEGORIES FOR POSETS OF TYPE $\mathbb{A}$

RALF SCHIFFLER AND ROBINSON-JULIAN SERNA

To the memory of A.G. Zavadskij (1946–2012)

Abstract. This paper establishes a link between the theory of cluster algebras and the theory of representations of partially ordered sets. We introduce a class of posets by requiring avoidance of certain types of peak-subposets and show that these posets can be realized as the posets of quivers of type $\mathbb{A}$ with certain additional arrows. This class of posets is therefore called *posets of type* $\mathbb{A}$. We then give a geometric realization of the category of finitely generated socle-projective modules over the incidence algebra of a poset of type $\mathbb{A}$ as a combinatorial category of certain diagonals of a regular polygon. This construction is inspired by the realization of the cluster category of type $\mathbb{A}$ as the category of all diagonals by Caldero, Chapoton and the first author [10].

We also study the subalgebra of the cluster algebra generated by those cluster variables that correspond to the socle-projectives under the above construction. We give a sufficient condition for when this subalgebra is equal to the whole cluster algebra.

Contents

Key words and phrases. category of diagonals, cluster category, cluster algebra, poset of type $\mathbb{A}$, socle-projective representation, Auslander Reiten quiver.

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1. Introduction

Geometric realizations of algebraic structures using the combinatorial geometry of surfaces have been developed by different authors in recent years (for instance see [2–6, 8, 10, 14–16, 26, 28]). This approach provides geometric and combinatorial tools for the study of the objects and morphisms in the category. It plays an important role in cluster-tilting theory and in representation theory in general. For example, the category $\mathcal{C}$ of diagonals (not including boundary edges) in a regular polygon Π_{n+3} with $n+3$ vertices introduced by Caldero, Chapoton and the first author [10] is a geometric realization of the cluster category of type $\mathbb{A}_n$; which, in greater generality, was defined simultaneously by Buan-Marsh-Reiten-Reineke-Todorov [9]. They defined cluster categories as orbit categories of the bounded derived category of hereditary algebras. As an application in [10], the module category of a cluster-tilted algebra of type $\mathbb{A}_n$ is described by a category of diagonals $\mathcal{C}_T$ in Π_{n+3}, where T is a triangulation of Π_{n+3}.

The present work links the theory of cluster algebras [17] and cluster categories with the theory of representations of partially ordered sets (in short, posets) through a geometric realization inspired by the one in [10].

The representation theory of posets was established parallel to the development of the representation theory of Artin algebras; the notion of a matrix representation of a poset $\mathcal{P}$ over an algebraically closed field k was introduced in the 1970s by Nazarova and Roiter [25]. Aside from matrix representations of a poset $\mathcal{P}$, the concept of $\mathcal{P}$-space (or representation of $\mathcal{P}$) over a field k was introduced by Gabriel [18] in connection with the investigation of representations of quivers. If $(\mathcal{P}, \preceq)$ is a finite poset, the category of $\mathcal{P}$-spaces of the poset $\mathcal{P}$ is nothing else than the category $\mathrm{mod}_{sp}(k\mathcal{P}^\star)$ of finitely generated socle-projective modules over the incidence algebra $k\mathcal{P}^\star$ of the enlarged poset $\mathcal{P}^\star = \mathcal{P} \cup \{\star\}$ such that $x \prec \star$ for each $x \in \mathcal{P}$; however, there are genuine methods in representation theory of posets such as the differentiation algorithms [11, 12, 33, 36]. In a more general situation, Simson studied the category of peak $\mathcal{P}$-spaces which is identified with the category $\mathrm{mod}_{sp}(k\mathcal{P})$ of finitely generated socle-projective $k\mathcal{P}$-modules, where $k\mathcal{P}$ is the incidence algebra of a poset $\mathcal{P}$ [31, 34]. He gave the finiteness criterion for those categories while his student J. Kosakowska classified the sincere posets of finite representation type [19–21]. Moreover, the tameness criterion was given by Kasjan and Simson in [22]. In general, the theory of representations of posets plays an important role in the study of lattices over orders, in the classification of indecomposable lattices over some simple curve singularities and in the classification of abelian groups of finite rank (see [1, 33]).

In this paper, we introduce a class of posets which we call posets of type $\mathbb{A}$. Roughly speaking, they are posets with $n \geq 1$ elements whose category of socle-projective representations is embedded in the category of representations of a Dynkin quiver of type $\mathbb{A}_n$. We characterize these posets as those not allowing a peak-subposet of one of four types, see Definition 3.1. Then, we define a subcategory $\mathcal{C}_{(T,F)}$ of the category $\mathcal{C}_T$ of diagonals of a triangulated polygon Π_{n+3} with $n+3$ vertices to give a geometric realization of the category $\mathrm{mod}_{sp}(k\mathcal{P})$ of posets $\mathcal{P}$ of type $\mathbb{A}$, where T is a triangulation of Π_{n+3} associated to a Dynkin quiver Q of type $\mathbb{A}_n$ and F

is a set of additional arrows for Q. We show that there is an equivalence of categories $\mathcal{C}_{(T,F)} \to \text{mod}_{sp}(k\mathcal{P})$ in Theorem 4.5. Moreover, we define a subalgebra $\mathcal{A}(\mathcal{P})$ of the cluster algebra $\mathcal{A} = \mathcal{A}(\boldsymbol{x}, Q)$ generated by the cluster variables associated to diagonals in $\mathcal{C}_{(T,F)}$ and diagonals in T; then, we establish that if $\mathcal{P}$ is the poset whose Hasse quiver is a Dynkin quiver Q of type $\mathbb{A}_n$ then $\mathcal{A} = \mathcal{A}(\mathcal{P})$ in Theorem 5.2.

The paper is organized as follows: In section 2, we recall some notation and results about categories of diagonals in regular polygons and categories of socle-projective representations of posets. In section 3, we define and study posets of type $\mathbb{A}$. Section 4 is devoted to proving our main result, Theorem 4.5. Finally, the last section deals with the subalgebras $\mathcal{A}(\mathcal{P})$ of the cluster algebra $\mathcal{A}$.

2. Preliminaries

2.1. **Category of diagonals $\mathcal{C}_T$.** We recall some results and notation of [10] (see also Chapter 3 in [29]) which are used in this work. A *diagonal* in a regular polygon is a straight line segment that joins two of the vertices and goes through the interior of the polygon. A *triangulation* of the polygon is a maximal set of non-crossing diagonals. Such a triangulation cuts the polygon into triangles.

Let $T = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of a regular polygon Π_{n+3} (or $(n+3)$-gon) with $n+3$ vertices and let γ and γ' be diagonals that are not in T. The diagonal γ is related to the diagonal γ' by a *pivoting elementary move* if they share a vertex on the boundary (this vertex is called *pivot*), the other vertices of γ and γ' are the vertices of a boundary edge of the polygon and the rotation around the pivot is positive (for the trigonometric direction) from γ to γ'. Let $P_v : \gamma \to \gamma'$ denote the pivoting elementary move from γ to γ' with pivot v. Compositions of pivoting elementary moves are called *pivoting paths.*

The combinatorial k-linear additive category $\mathcal{C}_T$ of diagonals is defined as follows: The objects are positive integral linear combinations of diagonals that are not in T. By additivity, it is enough define morphisms between diagonals. To do that, we recall that the *mesh relations* are the equivalence relation between pivoting paths induced by identifying every couple of pivoting paths of the form

$$\gamma \xrightarrow{P_{v_1}} \beta \xrightarrow{P_{v_2'}} \gamma' \text{ and } \gamma \xrightarrow{P_{v_2}} \beta' \xrightarrow{P_{v_1'}} \gamma'$$

where $v_1 \neq v_2'$ and $v_2 \neq v_1'$ (see Figure 1). In these relations, diagonals in T or boundary edges are allowed with the following convention: If one of the intermediate edges (β or β') is either boundary edge or diagonal in T, the corresponding term in the mesh relation is replaced by zero. Thus, the space of morphisms from a diagonal $\gamma \notin T$ to a diagonal $\gamma' \notin T$ is the quotient of the vector space over k spanned by pivoting paths from γ to γ' modulo the mesh relations.

The following lemma describes the relative positions of diagonals γ and γ', when there exist a nonzero morphism between them.

Lemma 2.1. [10, Lemma 2.1] *The vector space* $\text{Hom}_{\mathcal{C}_T}(\gamma, \gamma')$ *is nonzero if and only if there exists a diagonal* $\tau_i \in T$ *such that* τ_i *crosses the diagonals* γ *and* γ' *and the relative positions of them are as in Figure 2. That is, let* v_1, v_2 *be the endpoints of* τ_i *and* u_1, u_2 *(respectively* u_1', u_2'*) be the endpoints of* γ *(respectively* γ'*). Then*

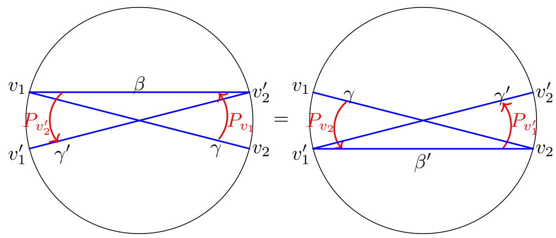


FIGURE 1. Mesh relations $P_{v_2'}P_{v_1} = P_{v_1'}P_{v_2}$ in $\mathcal{C}_T$

ordering the vertices of the polygon in the positive trigonometric direction starting at v_1*, we have* $v_1 < u_1 \leq u_1' < v_2 < u_2 \leq u_2'$*. In this case,* $\mathrm{Hom}_{\mathcal{C}_T}(\gamma, \gamma')$ *is of dimension one.*

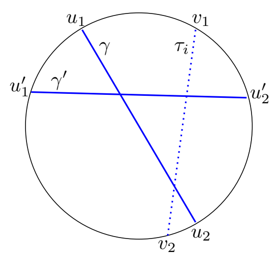


FIGURE 2. Relative position

A triangulation T of the $(n+3)$-gon is said to be *triangulation without internal triangles* if each triangle has at least one side on the boundary of the polygon. It is important to recall that every triangulation T of the $(n+3)$-gon gives rise to a cluster-tilted algebra kQ_T/I of type $\mathbb{A}_n$, where I is the two-sided ideal generated by all length two subpath of oriented 3-cycles in Q_T, and every cluster-tilted algebra is of this form. In particular, every Dynkin quiver of type $\mathbb{A}_n$ corresponds to a triangulation without internal triangles. The map associates a quiver Q_T to the triangulation $T = \{\tau_1, \dots, \tau_n\}$ of Π_{n+3} as follows: The vertices of Q_T are $(Q_T)_0 = \{1, 2, \dots, n\}$ and there is an arrow $x \to y$ in $(Q_T)_1$ precisely if the diagonals τ_x and τ_y bound a triangle in which τ_y lies counter-clockwise from τ_x (see Figure 3 and Example 4.3).

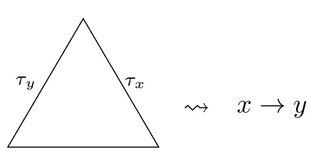


FIGURE 3. τ_y is counter-clockwise from τ_x

A vertex $x \in (Q_T)_0$ belongs to the *support* $\mathrm{supp}\,\gamma$ of a diagonal $\gamma \notin T$ if the diagonal $\tau_x \in T$ crosses γ. The following lemma permits to see diagonals that are not in

T as indecomposable objects in the module category of the cluster-tilted algebra kQ_T/I of type $\mathbb{A}$.

Lemma 2.2. [10, Lemma 3.2] *Let γ be a diagonal which does not belong to T. The set* $\operatorname{supp}\gamma$ *is connected as a subset of the quiver Q_T.*

In [10], the authors defined a k-linear additive functor Θ from $\mathcal{C}_T$ to the category $\operatorname{mod} kQ_T/I$ of finitely generated kQ_T/I-modules. The image of a diagonal $\gamma \notin T$ is the representation $M^\gamma = (M_x^\gamma, f_\alpha^\gamma)$ defined as follows: For each vertex x in Q_T,

$$M_x^\gamma = \begin{cases} k & \text{if } x \in \operatorname{supp}\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

For any arrow $\alpha : x \to y$ in Q_T,

$$f_\alpha^\gamma = \begin{cases} \mathrm{id}_k & \text{if } M_x^\gamma = M_y^\gamma = k, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for any pivoting elementary move $P : \gamma \to \gamma'$ they defined the morphism $\Theta(P)$ from $(M_x^\gamma, f_\alpha^\gamma)$ to $(M_x^{\gamma'}, f_\alpha^{\gamma'})$ to be id_k whenever possible and 0 otherwise. The category $\mathcal{C}_T$ of diagonals gives a geometric realization of the category of finitely generated kQ_T/I-modules in the following sense.

Theorem 2.3. [10, Theorems 4.4 and 5.1]

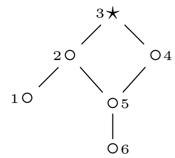
(a) *The functor Θ is an equivalence of categories.*
(b) *The irreducible morphisms of $\mathcal{C}_T$ are direct sums of the generating morphisms given by pivoting elementary moves.*
(c) *The mesh relations of $\mathcal{C}_T$ are the mesh relations of the AR-quiver of $\mathcal{C}_T$.*
(d) *The AR-translation is given on diagonals by r^-. Here, r^- (respectively r^+) denotes the clockwise (respectively counter-clockwise) elementary rotation of the regular polygon Π_{n+3}.*
(e) *The projective indecomposable objects of $\mathcal{C}_T$ are the diagonals in $r^+(T)$.*
(f) *The injective indecomposable objects of $\mathcal{C}_T$ are the diagonals in $r^-(T)$.*

2.2. **Socle-projective modules over incidence algebras.** In this section, we recall some of the main results regarding finitely generated socle-projective modules over incidence algebras of posets due to Simson [30, 31, 33, 34].

We denote by $(\mathcal{P}, \preceq)$ a finite partially ordered set (in short, poset) with respect to the partial order $\preceq$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. For the sake of simplicity we write $\mathcal{P}$ instead of $(\mathcal{P}, \preceq)$. Let $\max\mathcal{P}$ (respectively $\min\mathcal{P}$) be the set of all maximal (respectively minimal) points of $\mathcal{P}$. A poset $\mathcal{P}$ is called an *r-peak poset* if $|\max\mathcal{P}| = r$. We recall that a full subposet $\mathcal{P}'$ of $\mathcal{P}$ is said to be a *peak-subposet* if $\max\mathcal{P}' \subseteq \max\mathcal{P}$. In the sequel, we denote $\mathcal{P}^- = \mathcal{P} \setminus \max\mathcal{P}$.

The *Hasse diagram* of $\mathcal{P}$ is obtained as follows: One represents each element of $\mathcal{P}$ as a vertex in the plane and draws a line segment or curve that goes upward from x to y whenever *y covers x*, that is, whenever $x \prec y$ and there is no z such that $x \prec z \prec y$. These lines may cross each other but must not touch any vertices other than their endpoints. Such a diagram, with labeled vertices, uniquely determines its partial order.

Example 2.4. Given the one-peak poset $\mathcal{P}$ whose Hasse diagram is



the subposet $\{2, 4, 5, 6\}$ is not a peak-subposet of $\mathcal{P}$, whereas $\{1, 6, 3\}$ is a peak-subposet of $\mathcal{P}$.

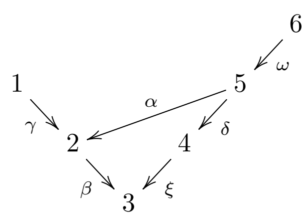
For a point $a \in \mathcal{P}$, the subposets of $\mathcal{P}$

$$a^{\nabla} = \{x \in \mathcal{P} \mid a \preceq x\}, \quad a_{\vartriangle} = \{x \in \mathcal{P} \mid x \preceq a\}$$

are called *up-cone and down-cone* respectively. In the literature, the up-cone of a is also called the *principal filter* of a and its down-cone is its *principal ideal.* A poset $\mathcal{P}$ is called a *chain* (or a *totally ordered set* or a *linearly ordered set*) if and only if for all $x, y \in \mathcal{P}$ we have $x \preceq y$ or $y \preceq x$. On the other hand, an ordered set $\mathcal{P}$ is called an *antichain* if and only if for all $x, y \in \mathcal{P}$ we have $x \preceq y$ in $\mathcal{P}$ only if $x = y$. The cardinality of a maximal antichain in a poset $\mathcal{P}$ is called the *width* $w(\mathcal{P})$ of $\mathcal{P}$. If some subsets $X_1, \ldots, X_n$ of $\mathcal{P}$ do not intersect mutually (but may have comparable points), then their union $X_1 \cup \cdots \cup X_n$ is called a *sum* and is denoted by $X_1 + \cdots + X_n$. We recall that according to the Dilworth's theorem if the width $w(\mathcal{P}) = n$ then $\mathcal{P}$ is a sum of n chains. For details on posets, we refer to [13, 35].

Given a finite poset $\mathcal{P}$, by $k\mathcal{P}$ we mean the *incidence algebra* of the poset $\mathcal{P}$. $k\mathcal{P}$ can be described as a bound quiver algebra kQ/I induced by the *Hasse quiver* Q of $\mathcal{P}$ whose vertices are the points of $\mathcal{P}$ and there is an arrow $\alpha : x \to y$ for each pair $x, y \in \mathcal{P}$ such that y covers x. The ideal I is generated by all the commutativity relations $\gamma - \gamma'$ with γ and γ' parallel paths in Q. In this case, the category $\mathrm{mod}(k\mathcal{P})$ of the finitely generated $k\mathcal{P}$-modules is identified with the well known category $\mathrm{rep}(Q, I)$ of representations of the bound quiver (Q, I).

Example 2.5. The incidence algebra $k\mathcal{P}$ of the poset $\mathcal{P}$ defined in Example 2.4 is the bound quiver algebra kQ/I, where Q is the is the quiver



and the ideal I is generated by the relation $\alpha\beta - \delta\xi$.

Recall that the *socle* $\mathrm{soc}\, M$ of a module M is the semisimple submodule generated by all simple submodules of M. A module M is called *socle-projective* if $\mathrm{soc}\, M$ is a projective module. We denote by $\mathrm{mod}_{sp}(k\mathcal{P})$ the full subcategory of $\mathrm{mod}(k\mathcal{P})$ whose objects are the socle-projective $k\mathcal{P}$-modules. We have an explicit description of the objects in $\mathrm{mod}_{sp}(k\mathcal{P})$ as follows.

Proposition 2.6. [30, Section 3] *Each $k\mathcal{P}$-module M in $\mathrm{mod}(k\mathcal{P})$ is identified with a collection $M = (M_x, {}_yh_x)_{x,y \in \mathcal{P}}$ of finite-dimensional k-vector spaces M_x, one for* *each point $x \in \mathcal{P}$, and a collection of k-linear maps ${}_y h_x : M_x \to M_y$, one for each relation $x \preceq y$ in $\mathcal{P}$, such that*

(a) *${}_x h_x$ is the identity of M_x for all $x \in \mathcal{P}$ and ${}_w h_y \cdot {}_y h_x = {}_w h_x$ for all $x \preceq y \preceq w$ in $\mathcal{P}$.*

Furthermore, $M = (M_x, {}_y h_x)_{x,y\in\mathcal{P}}$ is a socle-projective module if it also holds that

(b) *For all $x \in \mathcal{P}^-$, the k-subpace*

$$I_x = \bigcap_{\substack{z\in \max \mathcal{P} \\ z \succ x}} \ker {}_z h_x$$

of M_x is the zero subspace.

Note that it is enough to define the linear maps ${}_y h_x$ when y covers x, that is, one for each arrow in the Hasse quiver of $\mathcal{P}$ because if $x \prec y$ but y does not cover x then for any chain $x = x_0 \prec x_1 \prec \cdots \prec x_l = y$ in $\mathcal{P}$ such that x_{i+1} covers x_i we have that ${}_y h_x = {}_y h_{x_{l-1}} \cdots {}_{x_1} h_x$. The condition (a) implies that it is well defined.

Let $M = (M_x, {}_y h_x)_{x,y\in\mathcal{P}}$ and $N = (N_x, {}_y h'_x)_{x,y\in\mathcal{P}}$ be two objects in $\mathrm{mod}(k\mathcal{P})$. A *morphism* of $k\mathcal{P}$-modules $f : M \to N$ is a collection $f = (f_x)_{x\in\mathcal{P}}$ of linear maps

$$f_x : M_x \to N_x$$

such that for each relation $x \preceq y$ in $\mathcal{P}$ the diagram

$$\begin{array}{ccc} M_x & \xrightarrow{{}_y h_x} & M_y \\ \downarrow{\scriptstyle f_x} & & \downarrow{\scriptstyle f_y} \\ N_x & \xrightarrow{{}_y h'_x} & N_y \end{array}$$

commutes, that is,

$$f_y \circ {}_y h_x = {}_y h'_x \circ f_x.$$

In the context of the representations of posets introduced by Nazarova and Roiter [25], the category $\mathrm{mod}_{sp}(k\mathcal{P})$ is identified with the category $\mathcal{P}$-spr of peak $\mathcal{P}$-spaces over the field k defined by Simson (see [34]). Thus, following [27,31,34], $\mathrm{mod}_{sp}(k\mathcal{P})$ is an additive Krull–Schmidt category of finite global dimension which is closed under taking kernels and extensions. Furthermore, it has enough projective objects, AR-sequences, source maps, and sink maps.

The category $\mathrm{mod}_{sp}(k\mathcal{P})$ (or the poset $\mathcal{P}$) is said to be of *finite representation type* if it has only a finite number of nonisomorphic indecomposable socle-projective $k\mathcal{P}$-modules, otherwise, it is of *infinite representation type.* We have the following criterion of finite representation type due Simson.

Theorem 2.7. [34, Theorem 3.1] *The category $\mathrm{mod}_{sp}(k\mathcal{P})$ is of finite representation type if and only if the poset $\mathcal{P}$ does not contain as a peak-subposet any of the posets $\mathcal{P}_1, \ldots, \mathcal{P}_{110}$ presented in* [34, Section 5]*.*

For a poset $\mathcal{P}$ of finite representation type, all the indecomposable socle-projective $k\mathcal{P}$-modules can be obtained via the sincere peak-subposets of $\mathcal{P}$.

Given a poset $\mathcal{P}$ and an object $M = (M_x, {}_y h_x)_{x,y\in\mathcal{P}}$ in $\mathrm{mod}(k\mathcal{P})$, the *Jacobson radical* of M is the $k\mathcal{P}$-module given by $\mathrm{rad}\ M = ((\mathrm{rad}\ M)_x, {}_y \overline{h}_x)_{x,y\in\mathcal{P}}$, where

$(\text{rad } M)_x = \sum_{a \prec x} \text{Im}({}_x h_a)$ and ${}_y\overline{h}_x$ is the restriction of ${}_y h_x$ to $(\text{rad } M)_x$ for each $x \preceq y$ in $\mathcal{P}$. By *top* of M we mean the semisimple quotient $k\mathcal{P}$-module top $M = M/\text{rad } M$ given by $((\text{top } M)_x, {}_y f_x)_{x,y \in \mathcal{P}}$, where $(\text{top } M)_x = M_x/(\text{rad } M)_x$ and ${}_y f_x = 0$ for each $x \prec y$ in $\mathcal{P}$. Moreover, the *coordinate vector* of M is given by the vector

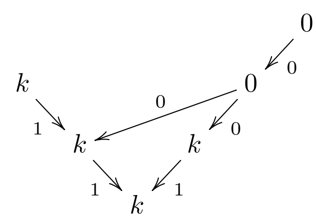
$$d = \underline{\text{cdim}}\, M = (d_x)_{x \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}$$

such that $d_x = \dim_k M_x$ if $x \in \max \mathcal{P}$ and $d_x = \dim_k(\text{top } M)_x$ otherwise. If M is an indecomposable socle-projective $k\mathcal{P}$-module, the *coordinate support*

$$\text{csupp}\, M = \{x \in \mathcal{P} \mid (\underline{\text{cdim}}\, M)_x \neq 0\}$$

of M is a peak-subposet of $\mathcal{P}$. In particular, if $\text{csupp}\, M = \mathcal{P}$, M is called a *sincere socle-projective* $k\mathcal{P}$-module. Furthermore, if there exists a sincere socle-projective $k\mathcal{P}$-module, we say that $\mathcal{P}$ is a *sincere poset.*

Example 2.8. Let $\mathcal{P}$ be the one-peak poset given in Example 2.4 whose Hasse quiver is shown in Example 2.5. The $k\mathcal{P}$-module M given by the system



is an indecomposable socle-projective $k\mathcal{P}$-module. Indeed, since $\mathcal{P}$ has a unique maximal point $z = 3$, we have that the k-subspace I_x of M_x defined in Proposition 2.6 is given by $I_x = \ker {}_3h_x = 0$ for all $x \in \mathcal{P}^-$. Note that, replacing 1 on arrow γ by 0 would result in a representation that is not socle-projective. Moreover the coordinate vector of M is $\underline{\text{cdim}}\, M = (d_1, \ldots, d_6) = (1, 0, 1, 1, 0, 0)$ and the coordinate support of M is the peak-subposet given by $\text{csupp}\, M = \{1, 3, 4\}$. Since the restriction of M to the poset $\text{csupp}\, M$ is a sincere socle-projective $k(\text{csupp}\, M)$-module, $\text{csupp}\, M$ is a sincere poset.

The classification of all sincere r-peak posets of representation finite type and the sincere socle-projective representations of them ñ was given by M. Kleiner [24], for the case $r = 1$, and by J. Kosakowska [19–21] in the remaining cases. Such lists are important because we can get all indecomposable objects in $\text{mod}_{sp}(k\mathcal{P})$ of a given poset $\mathcal{P}$ of finite representation type lifting all sincere socle-projective $k\mathcal{S}$-modules of all sincere peak-subposets $\mathcal{S}$ of $\mathcal{P}$ via the *subposet induced functor* [23].

$$T_{\mathcal{S}} : \text{mod}_{sp}\, k\mathcal{S} \to \text{mod}_{sp}\, k\mathcal{P} \tag{2.1}$$

that assigns to the socle-projective $k\mathcal{S}$-module M the socle-projective $k\mathcal{P}$-module $X \otimes_{k\mathcal{S}} (e_L(k\mathcal{P})e_{P^- \cup \max \mathcal{P}})$, where $e_J = \sum_{i \in J} e_i$ for any subposet $J \subseteq \mathcal{P}$.

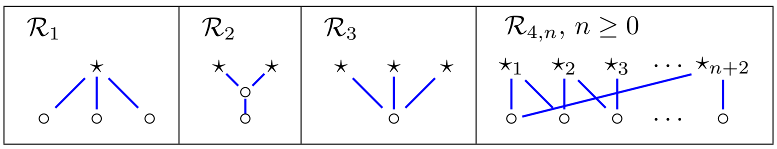
Proposition 2.9. [32, Proposition 2.11] *Up to isomorphism, any indecomposable object M in $\text{mod}_{sp}(k\mathcal{P})$ is the image $T_{\mathcal{S}}(L)$ of a sincere socle-projective $k\mathcal{S}$-module L, where $\mathcal{S}$ is a sincere peak-subposet of $\mathcal{P}$. In this case, $\mathcal{S} = \text{csupp}\, M$ and L is the restriction of M to $\mathcal{S}$.*

In particular, if $\mathcal{P}$ is a one-peak poset the above result was presented in [33, Section 5.3].

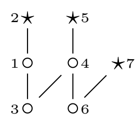
3. Posets of type $\mathbb{A}$

In this section, we introduce a family of posets which we call posets of type $\mathbb{A}$ because of a characterization using a type $\mathbb{A}$ quiver given in Proposition 3.8.

Definition 3.1. A finite connected poset $\mathcal{P}$ is said to be *poset of type* $\mathbb{A}$ if $\mathcal{P}$ does not contain as a peak-subposet any of the following posets:



Example 3.2. The poset $\mathcal{P}$ given in Example 2.4 is a poset of type $\mathbb{A}$ because although $\{2, 4, 5, 6\}$ is a subposet of type $\mathcal{R}_2$ it is not a peak-subposet of $\mathcal{P}$. On the other hand, the poset



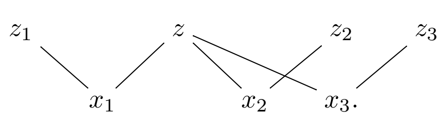
is a three-peak poset of type $\mathbb{A}$ which can be viewed as a Dynkin quiver of type $\mathbb{E}_7$.

We say that two maximal points z and z' in a poset $\mathcal{P}$ are *neighbors* if $z_\vartriangle \cap z'_\vartriangle \neq \emptyset$. Then, we describe this notion when $\mathcal{P}$ is a poset of type $\mathbb{A}$ as follows.

Lemma 3.3. *Let $\mathcal{P}$ be an r-peak poset of type $\mathbb{A}$ with $r \geq 2$. The following statements hold:*

(a) *The points $z, z' \in \max \mathcal{P}$ are neighbors if and only if $z_\vartriangle \cap z'_\vartriangle = \{x\}$, for some $x \in \min \mathcal{P}$.*
(b) *For all $z \in \max \mathcal{P}$, z has at most two neighbors. Moreover, there exists at least a point $z \in \max \mathcal{P}$ such that z has a unique neighbor.*

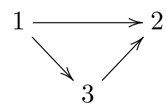
Proof. Since $\mathcal{R}_2$ is not peak-subposet of $\mathcal{P}$ then $x \in z_\vartriangle \cap z'_\vartriangle$ implies that $x \in \min \mathcal{P}$ because otherwise there exists $y \prec x$ and $\mathcal{R}_2 = \{y, x, z, z'\}$ would be a peak-subposet of $\mathcal{P}$. Now, if $x \neq x' \in z_\vartriangle \cap z'_\vartriangle$ then $\mathcal{R}_{4,0}$ is peak-subposet of $\mathcal{P}$ which is a contradiction. Thus, the set $z_\vartriangle \cap z'_\vartriangle$ is a singleton. Clearly the converse implication is true. On other hand, since $\mathcal{P}$ is a connected poset then each maximal point z has at least one neighbor, but z does not have three neighbor points. Indeed, if z_1, z_2, z_3 are distinct neighbors of z with $x_i \in z_\vartriangle \cap (z_i)_\vartriangle$ then we have the subposet



If x_1, x_2, x_3 are three distinct points then $\{z, x_1, x_2, x_3\}$ is a peak-subposet of type $\mathcal{R}_1$, a contradiction. Suppose that two of the x_i are equal, for instance $x_1 = x_2$. Then $\{z_1, z, z_2, x_1\}$ is a peak-subposet of type $\mathcal{R}_3$, a contradiction. Thus z has at most two neighbors. Finally, if each maximal point has exactly two neighbor points then $\mathcal{R}_{4,n}$ is peak-subposet of $\mathcal{P}$ for some $n \geq 0$, which is contradictory, and we are done. $\square$

Actually, the posets of type $\mathbb{A}$ can be viewed as posets associated to certain quivers which are obtained from Dynkin quivers of type $\mathbb{A}$ by adding some new arrows. To explain this, we need the following definitions:

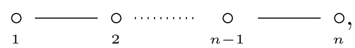
Let Q be an acyclic quiver and let $\mathcal{P}_Q = Q_0$ be its set of vertices. We define an order on $\mathcal{P}_Q$ by $x \preceq y$ if and only if there exists a path from x to y in Q. We say that $\mathcal{P}_Q$ is the *poset associated to the quiver* Q. Note that there is a unique poset associated to a finite acyclic quiver, but the converse is false in general. As an example, the poset associated to the quiver



is $\{1 < 3 < 2\}$. However, the Hasse quiver of this poset is $1 \longrightarrow 3 \longrightarrow 2$. Thus, the two quivers have the same associated poset. As another example, corresponding to the poset $\mathcal{P} = \{1, 2\}$ together with the usual ordering $1 < 2$, we get countably many quivers with n arrows from 1 to 2 for any natural number $n \in \mathbb{N}$.

Recall that a vertex x in a quiver Q is said to be a *sink vertex* (respectively *source vertex*) if there is no arrow α in Q such that $s(\alpha) = x$ (respectively $t(\alpha) = x$), where $s(\alpha)$ is the starting vertex and $t(\alpha)$ is the target vertex of the arrow α.

Given a Dynkin quiver Q of type $\mathbb{A}$, its underlying graph $\overline{Q}$ has the form



and the vertices 1 and n are called *extreme vertices* of Q. Moreover, if z is a sink vertex in Q, the maximal full subquiver $Q^{(z)}$ of Q such that z is the unique sink vertex in $Q^{(z)}$ is said to be the *z-subquiver* of Q. In other words, the vertices of $Q^{(z)}$ are the vertices in the support $\operatorname{Supp} I(z)$ of the indecomposable injective representation $I(z)$ at vertex z.

Example 3.4. The quiver $Q = \; 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longrightarrow 5 \longleftarrow 6 \longrightarrow 7$ of type $\mathbb{A}_7$ contains the 2-subquiver $1 \longrightarrow 2 \longleftarrow 3$, the 5-subquiver $3 \longrightarrow 4 \longrightarrow 5 \longleftarrow 6$, and the 7-subquiver $6 \longrightarrow 7$.

We will now add new arrows to our quiver Q as follows.

Definition 3.5. A set $F = \{\alpha_1, \ldots, \alpha_t\}$ of new arrows for Q is called an *alien set* for Q if the following conditions hold.

(a) For all $\alpha \in F$, there exists a sink vertex z in Q such that $s(\alpha), t(\alpha) \in \operatorname{Supp} I(z)$.
(b) For all $\alpha \in F$, $t(\alpha)$ is not a source vertex in Q unless it is an extreme vertex in Q.
(c) For all $\alpha \in F$, the arrow α is the unique path from $s(\alpha)$ to $t(\alpha)$ in Q^F, where Q^F is the quiver such that $Q^F_0 = Q_0$ and $Q^F_1 = Q_1 \cup F$.
(d) The quiver Q^F is acyclic.

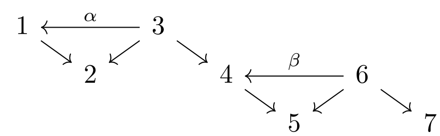
The arrows in an alien set for Q will be called *alien arrows*.

Example 3.6. If Q is the quiver $1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \longleftarrow 6$ of type $\mathbb{A}_6$ then $F = \{\ \alpha : 5 \to 2\}$ is an alien set for Q and Q^F is the quiver in Example 2.5. Note that the poset $\mathcal{P}_{Q^F}$ is the one-peak poset of type $\mathbb{A}$ defined in Example 2.4.

Example 3.7. Let Q be the quiver in Example 3.4. The set

$$F = \{\alpha : 3 \to 1,\ \beta : 6 \to 4\}$$

is an alien set for Q. Moreover, the quiver Q^F is equal to



Note that the poset $\mathcal{P}_{Q^F}$ associated to Q^F is the three-peak poset of type $\mathbb{A}$ defined in Example 3.2.

The following proposition characterizes posets of type $\mathbb{A}$.

Proposition 3.8. *A poset $\mathcal{P}$ is of type $\mathbb{A}$ if and only if there exists a Dynkin quiver Q of type $\mathbb{A}$ and an alien set F for Q such that $\mathcal{P} = \mathcal{P}_{Q^F}$ is the poset associated to the quiver Q^F.*

Proof. In order to prove the necessary condition we proceed by induction on the number r of peaks in $\mathcal{P}$. First we suppose that $\mathcal{P}$ is a one-peak poset with a maximal point z. Since $\mathcal{R}_1$ is not peak-subposet of $\mathcal{P}$ we conclude that $w(\mathcal{P}) \leq 2$. Thus, if $w(\mathcal{P}) = 1$ then $\mathcal{P}$ is a chain and it can be viewed as a linearly oriented quiver Q of type $\mathbb{A}$. Clearly, if $F = \emptyset$ then $\mathcal{P}$ is the poset associated to the quiver Q^F. On the other hand, if $w(\mathcal{P}) = 2$ then by Dilworth's theorem $\mathcal{P}^-$ is a sum of two chains $\mathcal{P}_1 = \{x_1 \prec \cdots \prec x_s\}$ and $P_2 = \{y_1 \prec \cdots \prec y_t\}$. Given the quiver

$$Q = \ x_1 \longrightarrow \cdots \longrightarrow x_s \longrightarrow z \longleftarrow y_t \longleftarrow \cdots \longleftarrow y_1\ ,$$

the set $F = F_1 \cup F_2$ such that $F_1 = \{\alpha : x \to y \mid y \in \mathcal{P}_2 \text{ covers } x \in \mathcal{P}_1\}$ and $F_2 = \{\alpha : y \to x \mid x \in \mathcal{P}_1 \text{ covers } y \in \mathcal{P}_2\}$ is an alien set for Q. Let $\alpha : x \to y$ be an alien arrow in F. We suppose that there is another path from x to y in Q^F, then there exists an alien arrow $\alpha' : x' \to y'$ in Q^F such that $x \preceq x'$, $y' \preceq y$ and $x \neq x'$ or $y \neq y'$. However, in this case, y does not cover x which is a contradiction. Thus, F is an alien set for Q and $\mathcal{P}$ is the poset $\mathcal{P}_{Q^F}$ associated to the quiver Q^F.

Now, we suppose that the assertion is true for any h-peak poset of type $\mathbb{A}$, for all $1 \leq h \leq r-1$. Let $\mathcal{P}$ be a r-peak poset of type $\mathbb{A}$. By Lemma 3.3 part (b) we can choose a point $z \in \max \mathcal{P}$ such that z has a unique neighbor. The peak-subposets $\bar{\mathcal{P}} = \{z_1, \ldots, z_{r-1}\}_\vartriangle$ and $\mathcal{P}_z = z_\vartriangle$ of $\mathcal{P}$ are two posets of type $\mathbb{A}$, where $\max \mathcal{P} = \{z_1, \ldots, z_{r-1}, z\}$. By induction there are two Dynkin quivers Q' and Q'' of type $\mathbb{A}$ and two alien sets F' and F'' for Q' and Q'' respectively such that $\bar{\mathcal{P}}$ is the poset associated to the quiver $Q'^{F'}$ and $\mathcal{P}_z$ is the poset associated to the quiver $Q''^{F''}$. We suppose that $z' \in (\max \mathcal{P}) \backslash \{z\}$ is the neighbor of the point z. By Lemma 3.3 part (a) we conclude that $z_\vartriangle \cap z'_\vartriangle = \{x\}$, where $x \in \min \mathcal{P}$, in other words, x is a source vertex in Q' and Q''. Clearly $\bar{\mathcal{P}} \cap \mathcal{P}_z = \{x\}$, otherwise z would have two neighbors. Now we are going to prove that the point x is an extreme vertex of both quivers Q' and Q''. Since Q'' has a unique sink vertex z and x is a source vertex in Q'' then x is an extreme vertex in Q''. Moreover, if x is a source vertex which is not an extreme vertex in Q' then $\mathcal{R}_3$ would be a peak-subposet of $\mathcal{P}$ and in this way we get a contradiction. Then the quiver $Q = (Q_0, Q_1)$ such that $Q_0 = Q'_0 \cup Q''_0$ and $Q_1 = Q'_1 \cup Q''_1$ is a Dynkin quiver of type $\mathbb{A}$. Note also that $F = F' \cup F''$ is an alien set for Q because there is no alien arrow ending at x, otherwise $\mathcal{R}_2$ would be a peak-subposet of $\mathcal{P}$. Furthermore, $\mathcal{P}$ is the poset associated to the quiver Q^F.

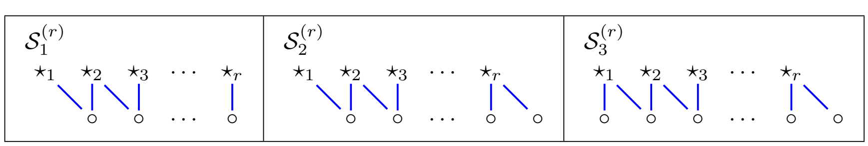
The sufficiency of the assertion is proved as follows; let us suppose that $\mathcal{P}$ is the poset $\mathcal{P}_{Q^F}$ associated to a quiver Q^F, where Q is a Dynkin quiver of type $\mathbb{A}$ and F is an alien set for Q, we shall prove that $\mathcal{P}$ is of type $\mathbb{A}$. Locally an alien arrow $\alpha \in F$ with $s(\alpha), t(\alpha) \in \operatorname{Supp} I(z)$, where z is a sink vertex in Q is such that $s(\alpha) \neq z$, otherwise the quiver Q^F would be cyclic. Then the maximal points in $\mathcal{P}$ are exactly the sink vertices in the quiver Q. For each $z \in \max \mathcal{P}$, Definition 3.5 implies that the peak-subposet $z_\vartriangle = Q_0^{(z)}$ of $\mathcal{P}$ is a poset of width at most two and then $\mathcal{R}_1$ is not a peak-subposet of $\mathcal{P}$. Note that the poset $\mathcal{P}_Q$ is a poset of type $\mathbb{A}$ because Q is a Dynkin quiver of type $\mathbb{A}$. Thus, by Lemma 3.3 part (b), if $z, z' \in \max \mathcal{P}_Q$ are neighbors and $x \in z_\vartriangle \cap z'_\vartriangle$ then $x \in \min \mathcal{P}_Q$. Since Q is a Dynkin quiver of type $\mathbb{A}$, x is a source vertex in Q. However, x is a non-extreme vertex in Q because an alien arrow always connects two vertices in the same z-subquiver. Definition 3.5 part (b) implies that $\mathcal{P}$ does not contain $\mathcal{R}_2$ as peak-subposet. Now, we suppose that $\mathcal{P}$ contains $\mathcal{R}_3$ as peak-subposet, that is, there are three maximal points z, z', z'' in $\mathcal{P}$ and a point $x \in \mathcal{P}$ such that $x \in z_\vartriangle \cap z'_\vartriangle \cap z''_\vartriangle$. Thus, by the same arguments as above, z, z' and z'' are sink vertices in Q. Moreover, since $\mathcal{R}_2$ is not a peak-subposet of $\mathcal{P}$, then x is a minimal point in $\mathcal{P}$ which implies that x is a source vertex in the quiver Q. Moreover, since Q is a Dynkin quiver of type $\mathbb{A}$, we can suppose that there is no path in Q from x to z''; thus, Definition 3.5 implies that $x \not\prec z''$ in $\mathcal{P}$, a contradiction. These arguments allow us to conclude that $\mathcal{R}_3$ is not peak-subposet of $\mathcal{P}$. In the same way, we can see that for all $n \geq 0$, $\mathcal{R}_{4,n}$ is not a peak-subposet of $\mathcal{P}$. □

A poset $\mathcal{P}$ is said to be *locally of width* n or *have local width* n if n is the minimum integer such that for each $z \in \max \mathcal{P}$ it holds that $w(z_\vartriangle) \leq n$. Clearly a poset of type $\mathbb{A}$ has local width less than or equal to two. The following lemma describes sincere posets of type $\mathbb{A}$ and their socle-projective indecomposable modules.

Lemma 3.9. *Let $\mathcal{P}$ be a poset of type $\mathbb{A}$. Then*

(a) $\operatorname{mod}_{sp} k\mathcal{P}$ *is of finite representation type.*

(b) $\mathcal{P}$ *is a sincere poset if and only if $\mathcal{P}$ is isomorphic to one of the following posets:*



for some $r \geq 1$. Furthermore, the module $M = (M_x, {}_y h_x)_{x,y \in \mathcal{P}}$ in $\operatorname{mod}_{sp}(k\mathcal{P})$ such that $M_x = k$ for all $x \in \mathcal{P}$ and ${}_y h_x = id_k$ for each $x \preceq y$ in $\mathcal{P}$ is the unique sincere object in $\operatorname{mod}_{sp}(k\mathcal{P})$.

Proof. According to Theorem 2.7 part (b), to prove the part (a) is enough to observe that no poset listed in [34, section 5] is a peak-subposet of $\mathcal{P}$. Indeed, the posets of the series $\mathcal{P}_{2,n+1}$, $\mathcal{P}''_{2,n}$, $\mathcal{P}_{3,n}$, $n \geq 0$ and the poset $\mathcal{P}_{2,0}$ contain $\mathcal{R}_3$ as peak-subposet. Moreover, the posets of the series $\mathcal{P}'_{2,n+1}$, $\mathcal{P}''_{3,n}$, $n \geq 0$ contain $\mathcal{R}_1$ as peak-subposet and the posets of the series $\mathcal{P}'_{3,n}$, $n \geq 0$ contain $\mathcal{R}_2$ as peak-subposet. Note that by definition $\mathcal{P}$ does not contain as a peak-subposet a poset of the series $\mathcal{P}_{1,n}$, $n \geq 0$. Moreover, we note that any poset of the form $\{\mathcal{P}_4, \ldots, \mathcal{P}_{110}\}$ contains as peak-subposet to $\mathcal{R}_i$ for some $i = 1, 2, 3$.

In order to prove (b), first we consider that $\mathcal{P}$ is one-peak poset. In this case, according to the list of sincere one-peak-posets (see [24]) we have that $\mathcal{P} = \mathcal{S}_i^{(1)}$ for some $i = 1, 2, 3$. Moreover, we observe in the known lists of sincere r-peak posets of finite type that $\mathcal{F}_1^{(2)} = \mathcal{S}_1^{(2)}, \mathcal{F}_2^{(2)} = \mathcal{S}_2^{(2)}, \mathcal{F}_5^{(2)} = \mathcal{S}_3^{(2)}$ are the sincere two-peak posets of type $\mathbb{A}$ (see [19]), $\mathcal{F}_{44}^{(3)} = \mathcal{S}_1^{(3)}$, $\mathcal{F}_{46}^{(3)} = \mathcal{S}_2^{(3)}, \mathcal{F}_{53}^{(3)} = \mathcal{S}_3^{(3)}$ are the sincere three-peak posets of type $\mathbb{A}$ (see [20]) and $\mathcal{F}_8^{(r)} = \mathcal{S}_1^{(3)}, \mathcal{F}_{10}^{(r)} = \mathcal{S}_2^{(3)}, \mathcal{F}_{13}^{(r)} = \mathcal{S}_3^{(3)}$ are the sincere r-peak posets of type $\mathbb{A}$, with $r \geq 4$ (see [21]). Thus, the first part of (b) is true. Now, we observe in the mentioned lists that for each $i = 1, 2, 3.$ and for each $r \geq 1$ the sincere r-peak poset $\mathcal{S}_i^{(r)}$ has only one sincere socle-projective $k\mathcal{S}_i^{(r)}$-module $M = (M_x, {}_y h_x)_{x,y \in \mathcal{S}_i^{(r)}}$ such that $M_x = k$ and ${}_y h_x = \mathrm{id}_k$ for each $x \preceq y$. □

Recall that the *support* $\operatorname{supp} M$ of a representation $M = (M_x, {}_y h_x)_{x,y\in\mathcal{P}}$ in $\mathrm{mod}_{sp}(k\mathcal{P})$ is given by $\{x \in \mathcal{P} \mid M_x \neq 0\}$. The following lemma will be used to prove the categorical equivalence proposed in Theorem 4.5.

Lemma 3.10. *Let $\mathcal{P}$ be a poset of type $\mathbb{A}$ associated to the quiver Q^F as in Proposition 3.8. Then*

(a) *Up to isomorphism, any indecomposable $k\mathcal{P}$-module $M = (M_x, {}_y h_x)_{x,y\in\mathcal{P}}$ in $\mathrm{mod}_{sp}(k\mathcal{P})$ is such that $M_x = k$ and ${}_y h_x = id_k$ for all $x \preceq y$ in $\operatorname{supp} M$.*
(b) *The support $\operatorname{supp} M$ of an indecomposable object in the category $\mathrm{mod}_{sp}\, k\mathcal{P}$ is connected as a subset of the quiver Q.*

Proof. Let $M = (M_x, {}_y h_x)_{x,y\in\mathcal{P}}$ be an indecomposable object in $\mathrm{mod}_{sp}(k\mathcal{P})$. Proposition 2.9 implies that $M = T_{\mathcal{S}}(L)$, where $L = (L_x, {}_y g_x)$ is a sincere object in $\mathrm{mod}_{sp}(k\mathcal{S})$, $\mathcal{S}$ is a sincere peak-subposet of $\mathcal{P}$ and $T_{\mathcal{S}}$ is the subposet induced functor defined in Equation (2.1). Thus, Definition 3.1 implies that $\mathcal{S}$ is a poset of type $\mathbb{A}$. Indeed, if $\mathcal{R} \in \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_{4,n}\}$ is peak-subposet of $\mathcal{S}$ then $\mathcal{R}$ is peak-subposet of $\mathcal{P}$, a contradiction. By Lemma 3.9, we have $\mathcal{S} = \mathcal{S}_i^{(r)}$ for some $r \geq 1$ and some $i = 1, 2, 3$, $L_x = k$ for all $x \in \mathcal{S}$, and ${}_y g_x = \mathrm{id}_k$ for each $x \preceq y$ in $\mathcal{S}$. Thus, M is the representation described in (a).

To prove (b) it is enough to see that the set $(\max \mathcal{S})_{\vartriangle} \cap \mathcal{S}^{\triangledown}$ is connected as a subset of the quiver Q for any sincere peak-subposet $\mathcal{S}$ of $\mathcal{P}$. Note that the poset $\mathcal{S}_1^{(r)}$ is a peak-subposet of $\mathcal{S} = \mathcal{S}_i^{(r)}$ for all $i = 1, 2, 3$. We suppose that

$$\mathcal{S}_1^{(r)} = \{z_1 \succ x_2 \prec z_2, z_2 \succ x_3 \prec z_3, \ldots, z_{r-1} \succ x_r \prec z_r\}$$

then $\{z_1, \ldots, z_r\} \subseteq \max \mathcal{P}$ and since $\mathcal{R}_2 \nsubseteq \mathcal{P}$ we have that $\{x_2, \ldots, x_r\} \subseteq \min \mathcal{P}$. Thus, for each z_i, with $2 \leq i \leq r-1$ the z_i-subquiver $Q^{(z_i)}$ of Q has the form $x_i \longrightarrow \cdots \longrightarrow z_i \longleftarrow \cdots \longleftarrow x_{i+1}$. Since each vertex in $Q_0^{(z_i)}$ belongs to the set $\{z_i\}_{\vartriangle} \cap \{x_i, x_{i+1}\}^{\triangledown}$, then $Q_0^{(z_i)} \subset \operatorname{supp} M$ for each $2 \leq i \leq r-1$. Let w (respectively

w') be the left (respectively right) extreme vertex of the quiver associated to $\mathcal{S}$ and let x (respectively y) be minimal element in $(x_2^\triangledown \cap (\mathcal{P} \setminus \mathcal{S})) \cup \{w\}$ (respectively $(x_r^\triangledown \cap (\mathcal{P} \setminus \mathcal{S})) \cup \{w'\}$) then it is easy to see that $\operatorname{supp} M = [x, y]_Q$, where $[x, y]_Q$ denote a interval of Q which is a connected subset of Q. □

4. Category of sp-diagonals

In this section, we define a category $\mathcal{C}_{(T,F)}$ of diagonals associated to a poset $\mathcal{P}$ of type $\mathbb{A}$ and we prove in Theorem 4.5 and Corollary 4.6 that this category gives a geometric realization of the category of finitely generated socle-projective modules over the incidence $k-$algebra $k\mathcal{P}$.

Let $\mathcal{P}$ be a poset of type $\mathbb{A}$ associated to the quiver Q^F as in Proposition 3.8. Thus, Q is a Dynkin quiver of type $\mathbb{A}_n$ and F is an alien set for Q. Let $T = \{\tau_1, \ldots, \tau_n\}$ be the triangulation of a $(n+3)$-gon Π_{n+3} such that $Q_T = Q$. A *fan* in T is a maximal subset $\Sigma_v \subseteq T$ of at least two diagonals such that all the diagonals in Σ_v share the vertex v of Π_{n+3}. A diagonal $\tau \in \Sigma_v$ is said to be the *peak-diagonal* of Σ_v if it is maximal in Σ_v in accordance with the order $\tau_x \leq \tau_y$ if and only if there is a path from the vertex x to the vertex y in the quiver Q. Geometrically, the peak-diagonal of a fan Σ_v is the diagonal that can be obtained from each other diagonal in Σ_v by a clockwise rotation around the vertex v (see Figure 4).

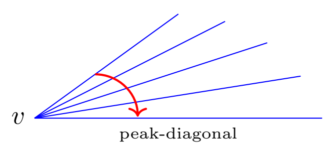


Figure 4. Fan of a triangulation

Lemma 4.1. *Let $T = \{\tau_x \mid x \in Q_0\}$ be a triangulation associated to a Dynkin quiver Q of type $\mathbb{A}$. Then*

(i) *A diagonal $\tau_z \in T$ is the peak-diagonal of a fan in T if and only if z is a sink vertex in the quiver Q.*
(ii) *If $\tau_x \in T$, there are at most two fans in T containing τ_x.*
(iii) *There are exactly two fans containing the diagonal $\tau_x \in T$ if and only if x is a non-extremal sink vertex or a non-extremal source vertex in Q.*

Proof. The property (i) is a consequence of the order $\leq$ defined in T and the definition of sink vertex. Note that if x is a vertex in Q and z is a sink vertex in Q such that $\tau_x \leq \tau_z$, there exists a source vertex x' such that $\tau_{x'} \leq \tau_x$. Thus, a fan containing τ_x is given by $\Sigma = \{\tau_y \in T \mid y \in Q'_0\}$, where Q' is the full subquiver $x' \to \cdots \to x \to \cdots \to z$ of Q. Clearly, the fan Σ is unique if x is neither a source vertex nor a sink vertex. If $x = z$ is a sink vertex we have two possibilities, either z is an extreme vertex or it is not. In the first case, there exists a unique source vertex x' in Q such that $\tau_{x'} < \tau_z$ which determines a unique fan $\Sigma = \{\tau_y \in T \mid y \in Q'_0\}$ containing τ_z, where Q' is the full subquiver $x' \to \cdots \to z$ of Q. In the second case, there exist two source vertices x' and x'' in Q such that $\tau_{x'} < \tau_z$ and $\tau_{x''} < \tau_z$. Thus, there are exactly two fans $\Sigma = \{\tau_y \in T \mid y \in Q'_0\}$ and

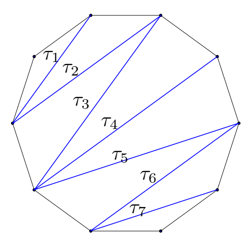
$\Sigma' = \{\tau_y \in T \in T \mid x \in Q''_0\}$ containing τ_z, where Q' and Q'' are full subquivers of Q given by $x' \to \cdots \to z$ o and $z \leftarrow \cdots \leftarrow x''$ respectively. Analogous to the proof of the above case, we can prove the result when x is a source vertex in Q. As a consequence, the properties (ii) and (iii) are true. □

Definition 4.2. A diagonal $\gamma \notin T$ is an *sp-diagonal* if it satisfies the following conditions:

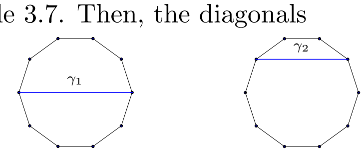
(a) If γ crosses $\tau \in T$ then γ crosses the peak-diagonal of at least one fan in T containing τ. Henceforth, any diagonal $\gamma \notin T$ satisfying this condition will be called a $\star$-*diagonal*.

(b) For all alien arrows $\alpha \in F$ with $s(\alpha), t(\alpha) \in \operatorname{supp} I(z)$, if γ crosses $\tau_{s(\alpha)}$ and τ_z then γ also crosses $\tau_{t(\alpha)}$. Diagonals $\gamma \notin T$ satisfying this condition will be called *non-frozen* diagonals. Moreover, if there exists $\alpha \in F$ with $s(\alpha), t(\alpha) \in \operatorname{supp} I(z)$ such that γ crosses $\tau_{s(\alpha)}$ and τ_z but not $\tau_{t(\alpha)}$, we say that γ is *frozen by* α.

Example 4.3. Let Q be the quiver in Example 3.4 then $Q = Q_T$, where T is the following triangulation



In this case, the sets $\{\tau_1, \boldsymbol{\tau_2}\}$, $\{\boldsymbol{\tau_2}, \tau_3\}$, $\{\tau_3, \tau_4, \boldsymbol{\tau_5}\}$, $\{\boldsymbol{\tau_5}, \tau_6\}$ and $\{\tau_6, \boldsymbol{\tau_7}\}$ are fans of T. We have used bold font for the peak-diagonal of each fan. Moreover, let Q^F be the quiver in the Example 3.7. Then, the diagonals



are such that $\operatorname{supp}\gamma_1 = \{3, 4\}$ and $\operatorname{supp}\gamma_2 = \{1, 2, 3\}$. Thus, γ_1 is not a $\star$-diagonal because it crosses τ_4 but it does not cross the peak-diagonal τ_5 in the unique fan $\{\tau_3, \tau_4, \boldsymbol{\tau_5}\}$ of τ_4, whereas γ_2 is a $\star$-diagonal because it crosses τ_2 which is the peak-diagonal in the fans $\{\tau_1, \boldsymbol{\tau_2}\}$ and $\{\boldsymbol{\tau_2}, \tau_3\}$ for τ_1, τ_2 and τ_3.

Given the alien arrows $\alpha : 3 \to 1$ and $\beta : 6 \to 4$ (see Example 3.7), a diagonal γ is frozen by α if γ crosses τ_3 and τ_2 but not τ_1; whereas the diagonals frozen by β cross τ_6 and τ_5 but not τ_4 (see Figure 5). Note that, γ_2 is an sp-diagonal.

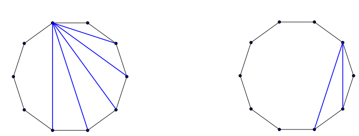


FIGURE 5. Diagonals frozen by α (left) and by β (right).

The following lemma describes the relation between $\star$-diagonals and socle-projective modules in $\operatorname{mod} kQ_T$.

Lemma 4.4. *Let $\Theta : \mathcal{C}_T \to \operatorname{mod} kQ_T$ be the equivalence of categories of Theorem 2.3, where Q_T is a Dynkin quiver of type $\mathbb{A}$. Then γ is a $\star$-diagonal if and only if $\Theta(\gamma)$ is socle-projective.*

Proof. Since Q_T is a Dynkin quiver of type $\mathbb{A}$, then $T = \{\tau_x \mid x \in (Q_T)_0\}$ is a triangulation without internal triangles. First, we suppose that γ is a $\star$-diagonal. Let x be a vertex in Q_T such that the indecomposable simple kQ_T-module $S(x)$ at vertex x is a submodule of $\Theta(\gamma) = M^\gamma$. We shall prove that $S(x)$ is a projective kQ_T-module. Since $\operatorname{Hom}(S(x), M^\gamma) \neq 0$, then $M^\gamma_x = k$, that is, τ_x crosses γ. By hypothesis, there exists a fan Σ containing τ_x such that γ crosses the peak-diagonal τ_z of Σ. If $x \neq z$ then $\tau_x < \tau_z$, that is, there is a path p in Q_T from x to z whose vertices are in $\operatorname{supp}\gamma$. Moreover, a nonzero morphism $f = (f_x)_{x\in(Q_T)_0}$ of representations from $S(x)$ to M^γ is such that $f_t = 0$ for all $t \neq x$ because $S(x)$ is the simple representation at vertex x. Let $x \to y$ be the arrow in p starting in x, then the diagram

$$\begin{array}{ccc} S(x)_x & \xrightarrow{\ 0\ } & S(x)_y \\ \big\downarrow{\scriptstyle f_x} & & \big\downarrow{\scriptstyle 0} \\ M^\gamma_x & \xrightarrow{\ 1\ } & M^\gamma_y \end{array}$$

commutes because f is a morphism of representations of the quiver Q_T. Since $S(x)_y$ is zero and $M^\gamma_x = M^\gamma_y = k$, then $f_x = 0$. Therefore, the morphism f is zero, a contradiction. Thus, we conclude that $x = z$, that is, τ_x is a peak-diagonal. By Lemma 4.1, x is a sink vertex in Q_T and then $S(x)$ is projective. Since all simple submodules of M^γ are projectives, we have that soc M is projective.

In the other direction, we have that $\Theta(\gamma)$ is socle-projective. Let τ_x be a diagonal in T crossing γ. If τ_x is a peak-diagonal then the definition of $\star$-diagonal is trivially satisfied. If τ_x is not a peak-diagonal, we suppose that for all fans Σ containing τ_x, γ does not cross the peak-diagonal in Σ. By Lemma 4.1, we have that the number s of fans containing τ_x is either one or two. In the case $s = 1$, let τ_y be the maximal diagonal in the fan Σ which crosses γ. Then $\tau_x \leq \tau_y < \tau_z$, where τ_z is the peak-diagonal in Σ. In other words, there is a path p from x to z in Q_T passing by y, such that the vertices $x, \ldots, y$ in p belong to $\operatorname{supp}\gamma$, whereas the others vertices in p are not in $\operatorname{supp}\gamma$. In particular, $M^\gamma_x = M^\gamma_y = k$ and $M^\gamma_z = 0$. Let $S(y)$ be the simple representation of Q_T at vertex y. Because the diagram

$$\begin{array}{ccccc} S(y)_x & \xrightarrow{\ 0\ } & S(y)_y & \xrightarrow{\ 0\ } & S(y)_z \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle \lambda} & & \big\downarrow{\scriptstyle 0} \\ M^\gamma_x & \xrightarrow{\ 1\ } & M^\gamma_y & \xrightarrow{\ 0\ } & M^\gamma_z \end{array}$$

commutes, we conclude that there is a nonzero injective morphism from $S(y)$ to M^γ. Therefore, $S(y)$ is a non-projective module which is a submodule of M^γ, a contradiction to the hypothesis. In the case $s = 2$, if τ_y (respectively $\tau_{y'}$) is the maximal diagonal in Σ (respectively Σ') crossing γ. By the above arguments, we conclude that $S(y)$ and $S(y')$ are non-projective summands of soc M^γ, a contradiction to the hypothesis. Therefore, γ is a $\star$-diagonal. □

Let $\mathcal{C}_{(T,F)}$ be the full subcategory of the category of diagonals $\mathcal{C}_T$ generated by all sp-diagonals in $\mathcal{C}_T$. We denote by $E(T,F)$ the set whose elements are the sp-diagonals in $\mathcal{C}_{(T,F)}$, the diagonals in T, and the boundary edges in Π_{n+3}.

Since the irreducible morphisms in $\mathcal{C}_{(T,F)}$ cannot be factorized through sp-diagonals, we introduce the notion of a *pivoting sp-move* from $\gamma \in E(T,F)$ to $\gamma' \in E(T,F)$, that is, a composition of pivoting elementary moves of the form

$$P : \gamma = \gamma_0 \xrightarrow{P_v^{(1)}} \gamma_1 \xrightarrow{P_v^{(2)}} \dots \xrightarrow{P_v^{(s)}} \gamma_s = \gamma'$$

with the same pivot v such that $\gamma_1, \dots, \gamma_{s-1}$ are not sp-diagonals in Π_{n+3}. Note that the irreducible morphisms in $\mathcal{C}_{(T,F)}$ are precisely the pivoting sp-moves between sp-diagonals.

Next, we analyze the relations in the category $\mathcal{C}_{(T,F)}$. These come from the mesh relations in $\mathcal{C}_T$. We suppose that γ and γ' are sp-diagonals and that the compositions $\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma'$ and $\gamma \xrightarrow{P_3} \beta' \xrightarrow{P_4} \gamma'$ of two pivoting sp-moves are as in Figure 6. We have that the maps $\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma'$ and $\gamma \xrightarrow{P_3} \beta' \xrightarrow{P_4} \gamma'$ are equal if we take into account the following convention: If one of the intermediate edges (β or β') is either a boundary edge or a diagonal in T, the corresponding term in the identity is replaced by zero.

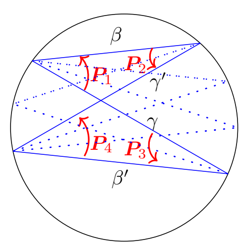


FIGURE 6. Mesh relations in $\mathcal{C}_{(T,F)}$.

4.1. **The functor Ω.** Let $\mathcal{P}$ be the poset of type $\mathbb{A}$ associated to the quiver Q^F, where Q is a quiver of Dynkin type $\mathbb{A}$ and F an alien set for Q and denote by T a triangulation associated to Q. Let us define a k-linear additive functor

$$\Omega : \mathcal{C}_{(T,F)} \to \mathrm{mod}_{sp}(k\mathcal{P})$$

from the category of sp-diagonals to the category of finitely generated socle-projective $k\mathcal{P}$-modules such that for any sp-diagonal γ we have $\Omega(\gamma) = M^\gamma = (M_x^\gamma, {}_y h_x^\gamma)$ where M^γ is defined by the following identities:

$$M_x^\gamma = \begin{cases} k & \text{if } x \in \operatorname{supp}\gamma, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and if } x \preceq y \in \mathcal{P} \text{ then } {}_y h_x^\gamma = \begin{cases} \mathrm{id}_k & \text{if } x, y \in \operatorname{supp}\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we define the functor Ω on morphisms. By additivity, it is sufficient to define the functor on morphisms between sp-diagonals. Our strategy is to define the functor on pivoting sp-moves and then check that the mesh relations in $\mathcal{C}_{(T,F)}$ hold. For any pivoting sp-move $P : \gamma \to \gamma'$, we define the morphism

$$\Omega(P) = \ (\Omega(P)_x)_{x \in \mathcal{P}} : (M_x^{\gamma}, {}_y h_x^{\gamma}) \to (M_x^{\gamma'}, {}_y h_x^{\gamma'})$$

by the formula

$$\Omega(P)_x = \begin{cases} \mathrm{id}_k, & \text{if } M_x^{\gamma} = M_x^{\gamma'} = k, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, Ω maps compositions of pivoting sp-moves to compositions of the images of the pivoting sp-moves. Note that if $\mathcal{P}$ is the poset $\mathcal{P}_Q$ associated to a Dynkin quiver Q of type $\mathbb{A}$ (without alien arrows) then the functor Ω is the restriction of the functor Θ defined in section 2.1 to the full subcategory of $\mathcal{C}_T$ generated by the $\star$-diagonals in $\mathcal{C}_T$.

Now, we prove that the functor Ω is well-defined and that it is an equivalence of categories.

Theorem 4.5. *Ω is an equivalence of categories.*

Proof. Recall here that $\mathcal{P}$ and $\mathcal{P}_Q$ are two different posets, that they have the same vertices and that $\mathcal{P}$ is obtained from $\mathcal{P}_Q$ by adding edges to the Hasse diagram corresponding to the alien arrows in F. In particular, $x \preceq y$ in $\mathcal{P}_Q$ implies $x \preceq y$ in $\mathcal{P}$. In order to prove that $M^{\gamma} \in \mathrm{mod}_{sp}(k\mathcal{P})$ we have to proof the conditions (a) and (b) in Proposition 2.6. To prove (a) it is enough to consider the non trivial situation when $x \prec y \prec w$ in $\mathcal{P}$ such that $x, w \in \operatorname{supp}\gamma$ and $y \notin \operatorname{supp}\gamma$. First we note that, by Lemma 2.2, if $x \prec y \prec w$ in $\mathcal{P}_Q$ then $y \in \operatorname{supp}\gamma$ which is contradictory. Therefore, we have that $x \not\preceq y$ or $y \not\preceq w$ in $\mathcal{P}_Q$. We consider the following cases (recall that we always suppose $x \prec y \prec w$ in $\mathcal{P}$ such that $x, w \in \operatorname{supp}\gamma$ and $y \notin \operatorname{supp}\gamma$).

Case 1 $y \prec w$ in $\mathcal{P}_Q$ and $x \not\preceq y$ in $\mathcal{P}_Q$. In this case, there exists an alien arrow $\alpha : x' \to y'$ on vertices of a z-subquiver $Q^{(z)}$ of Q such that $x \preceq x' \prec z$ and $y' \preceq y \prec w \preceq z$ in $\mathcal{P}_Q$. Indeed, if $w \preceq z'$ where $z' \in \max\mathcal{P}$ and $z \neq z'$ then $\mathcal{R}_2 = \{x, x', z, z'\}$ would be a peak-subposet of $\mathcal{P}$, which contradicts Definition 3.1. Since w is not source vertex in Q, Lemma 4.1 implies that there is a unique fan Σ in T containing τ_w and that τ_z is the peak-diagonal in Σ. Thus, γ crosses τ_z because $w \in \operatorname{supp}\gamma$ and γ is a $\star-$diagonal. Since $x, z \in \operatorname{supp}\gamma$, Lemma 2.2 implies that γ crosses $\tau_{x'}$, and since γ is non-frozen then γ crosses $\tau_{y'}$. Again using Lemma 2.2, we obtain that γ crosses τ_y, that is, $y \in \operatorname{supp}\gamma$ which is contradictory.

Case 2 $x \prec y$ in $\mathcal{P}_Q$ and $y \not\preceq w$ in $\mathcal{P}_Q$. In this case, there exists an alien arrow $\alpha : y' \to w'$ on vertices of a z-subquiver $Q^{(z)}$ of Q such that $x \prec y \preceq y' \prec z$ and $w' \preceq w \prec z$ in $\mathcal{P}_Q$. Indeed, if $w \preceq z'$ where $z' \in \max\mathcal{P}$ and $z \neq z'$ then $\mathcal{R}_2 = \{x, y, z, z'\}$ would be a peak-subposet of $\mathcal{P}$, which contradicts Definition 3.1. Since w is not source vertex in Q, the same arguments in Case 1 imply that γ crosses τ_z. Thus, since $x, z \in \operatorname{supp}\gamma$ Lemma 2.2 implies that γ crosses τ_y, in other words, $y \in \operatorname{supp}\gamma$ which cannot be.

Case 3 $x \not\preceq y$ and $y \not\preceq w$ in $\mathcal{P}_Q$. Equal arguments to the previous cases imply that there exist two alien arrows $\alpha : x' \to y'$ and $\alpha : y'' \to w'$ on vertices of a same z-subquiver $Q^{(z)}$ of Q such that $x \preceq x' \prec w' \preceq w \prec z$ and $y' \preceq y \preceq y'' \prec z$ in $\mathcal{P}_Q$

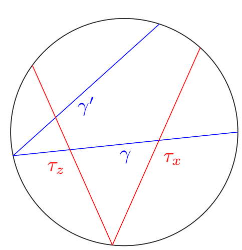
and that γ crosses τ_z. Thus, Lemma 2.2 implies that γ crosses $\tau_{x'}$, and since γ is non-frozen we conclude that γ crosses $\tau_{y'}$. Again using Lemma 2.2 we obtain that γ crosses τ_y, that is, $y \in \operatorname{supp}\gamma$ which cannot be.

We have shown that if $x \prec y \prec w$ in $\mathcal{P} = \mathcal{P}_{Q^F}$ such that $x, w \in \operatorname{supp}\gamma$ then $y \in \operatorname{supp}\gamma$. Thus, ${}_wh^\gamma_y = {}_yh^\gamma_x = {}_wh^\gamma_x = \mathrm{id}_k$ and condition (a) holds. To prove condition (b), let x be an element of $\mathcal{P}^- = \mathcal{P} \setminus \max\mathcal{P}$. If $x \notin \operatorname{supp}\gamma$ then clearly $\ker {}_zh^\gamma_x = 0$ for all $z \in \max\mathcal{P}$ such that $x \prec z$. If $x \in \operatorname{supp}\gamma$ then ${}_{z'}h^\gamma_x = \mathrm{id}_k$ for some $z' \in \max\mathcal{P}$, where $\tau_{z'}$ is the peak-diagonal in some fan containing τ_x. Thus, $\bigcap\limits_{z\in\max\mathcal{P}} \ker {}_zh_x = 0$ for all $x \in \mathcal{P}^-$ such that $x \prec z$. This shows that $\Omega(\gamma) = M^\gamma$ is indeed an object in $\mathrm{mod}_{sp}(k\mathcal{P})$.

Let us now check that $\Omega(P)$ is well defined for every pivoting sp-move $P : \gamma \to \gamma'$. Indeed, it is enough to show that for any relation $x \prec y$ such that y covers x in $\mathcal{P}$ the diagram

$$\begin{array}{ccc} M^\gamma_x & \xrightarrow{{}_yh^\gamma_x} & M^\gamma_y \\ \Big\downarrow{\scriptstyle \Omega(P)_x} & & \Big\downarrow{\scriptstyle \Omega(P)_y} \\ M^{\gamma'}_x & \xrightarrow{{}_yh^{\gamma'}_x} & M^{\gamma'}_y \end{array}$$

commutes. Note that the result holds if $M^\gamma_x = 0$ or $M^{\gamma'}_y = 0$ and also if both M^γ_y and $M^{\gamma'}_x$ are null spaces. Suppose now that $M^\gamma_x = M^{\gamma'}_y = k$. If $M^\gamma_y = M^{\gamma'}_x = k$, then all four maps are id_k and the diagram commutes. The only remaining case is if exactly one of M^γ_y, $M^{\gamma'}_x$ is nonzero. We will show that this cannot happen. Suppose that $M^{\gamma'}_x = 0$ and $M^\gamma_y = k$, that is, $x, y \in \operatorname{supp}\gamma$, $y \in \operatorname{supp}\gamma'$ and $x \notin \operatorname{supp}\gamma'$. Since y covers x in $\mathcal{P}$, there exists an arrow $\alpha : x \to y$ in Q^F. If $x \prec y$ in $\mathcal{P}_Q$ then α is an arrow in Q, that is, τ_x and τ_y share a vertex of the polygon and are connected by a pivoting elementary move. Since $P : \gamma \to \gamma'$ is a pivoting sp-move we get that τ_x crosses γ, that τ_x and γ' have a common point on the boundary of the polygon and that τ_y crosses γ and γ'. This implies that τ_y is clockwise from τ_x and that contradicts the orientation $x \to y$ in the quiver Q (see Figure 3). Next, we suppose that $x \not\prec y$ in $\mathcal{P}_Q$, then $\alpha : x \to y$ is an alien arrow in F with x and y in $\operatorname{Supp} I(z)$ for some sink vertex z in Q. Now, by Definition 3.5 part (b), y is not a source vertex in Q unless y is an extreme vertex in Q. Thus, there is at most one arrow in Q with starting point y, and therefore there is exactly one fan Σ containing τ_y and τ_z is its peak-diagonal. By Definition 4.2, both γ and γ' cross τ_z, because they are $\star$-diagonals crossing τ_y.



On the other hand, there is a pivoting path from τ_z to τ_x in Π_{n+3}, since x belongs to Supp $I(z)$. But this is impossible, because if $\tau \to \tau_x$ is a pivot, then τ does not cross γ'. The other case where $M_x^{\gamma'} = k$ and $M_y^{\gamma} = 0$ is proved in a similar way.

To show that the functor Ω is well defined, it only remains to check the mesh relations. Indeed, let $\gamma \xrightarrow{P_1} \beta$, $\beta \xrightarrow{P_2} \gamma'$, $\gamma \xrightarrow{P_3} \beta'$, $\beta' \xrightarrow{P_4} \gamma'$ be pivoting sp-moves as in Figure 6 with γ,γ' sp-diagonals and $\beta \neq \beta'$ sp-diagonals, diagonals in T or boundary edges. Note that, we can exclude the case where β and β' are both diagonals in the triangulation T or both boundary edges because in this case either γ or γ' is a diagonal in T. Without loss of generality, we may assume from now on that β is an sp-diagonal. Suppose first that β' is an sp-diagonal; then one has to check the commutativity of the following diagram

$$\begin{array}{ccc} M_x^{\gamma} & \xrightarrow{\Omega(P_1)_x} & M_x^{\beta} \\ \downarrow{\scriptstyle \Omega(P_3)_x} & & \downarrow{\scriptstyle \Omega(P_2)_x} \\ M_x^{\beta'} & \xrightarrow{\Omega(P_4)_x} & M_x^{\gamma'} \end{array}$$

for all $x \in \mathcal{P}$. Again, the only non trivial case happens when $M_x^{\gamma} = M_x^{\gamma'} = k$. In this case we also have $M_x^{\beta} = M_x^{\beta'} = k$ because any diagonal crossing both γ and γ' must also crosses β and β'. Thus all maps are id_k and the diagram commutes. Suppose now that β' is a boundary edge or diagonal in T. Then we have to show that the composition $M_x^{\gamma} \xrightarrow{\Omega(P_1)} M_x^{\beta} \xrightarrow{\Omega(P_2)} M_x^{\gamma'}$ is zero for all $x \in \mathcal{P}$. Clearly if β' is a boundary edge or diagonal in T then no diagonal $\tau \in T$ can cross both γ and γ' then $\mathrm{Hom}(\Omega(\gamma), \Omega(\gamma')) = 0$.

In order to prove that Ω is dense we fix an indecomposable $M \in \mathrm{mod}_{sp}(k\mathcal{P})$. Then by Lemma 3.10 part (b), Lemma 2.2 and Theorem 2.3 part (a) there exists a diagonal $\gamma \notin T$ such that $\mathrm{supp}\,\gamma = \mathrm{supp}\,M$. We show that γ is an sp-diagonal. Indeed, since the socle of M is projective, Lemma 4.4 implies that γ is a $\star$-diagonal. Moreover, given an alien arrow $\alpha : x \to y$ in F, with x and y in Supp $I(z)$ for some sink vertex z in Q_0 such that $x, z \in \mathrm{supp}\,M$ then ${}_zh_x = \mathrm{id}_k$. By Proposition 2.6 part (a), we have that ${}_zh_x = {}_zh_y \cdot {}_yh_x$, thus $y \in \mathrm{supp}\,M$. Therefore γ crosses τ_y and thus γ is a non-frozen diagonal. We conclude that γ is an sp-diagonal and that $\Omega(\gamma) = M$.

To show that Ω is faithful, it is enough to prove that the image of a nonzero morphism between sp-diagonals is a nonzero morphism in $\mathrm{mod}_{sp}(k\mathcal{P})$. Indeed, let $P \in \mathrm{Hom}_{\mathcal{C}_{(T,F)}}(\gamma, \gamma')$ be a nonzero morphism in $\mathcal{C}_{(T,F)}$. Then P also is a nonzero morphism in $\mathcal{C}_T$. Lemma 2.1 implies that there exists a diagonal $\tau_x \in T$ crossing γ and γ' as in Figure 2. In particular, $M_x^{\gamma} = M_x^{\gamma'} = k$, and therefore $\Omega(P)_x = \mathrm{id}_k \neq 0$.

Finally, we show that functor Ω is full. To do so, let $\Omega(\gamma) \xrightarrow{g} \Omega(\gamma')$ be a nonzero morphism in $\mathrm{mod}_{sp}(k\mathcal{P})$. Then $g = (g_x)_{x \in Q_0}$, where g_x is a linear map from $\Omega(\gamma)_x$ to $\Omega(\gamma')_x$. The map $\hat{g} = (\hat{g}_x)_{x \in Q_0}$ from $\Theta(\gamma)$ to $\Theta(\gamma')$ such that $\hat{g}_x = g_x$ is a morphism of representations in $\mathrm{mod}\,kQ$. Indeed, for each arrow $\alpha : x \to y$ in Q_1, we have $x \prec y$ in $\mathcal{P}$. Since g is morphism in $\mathrm{mod}_{sp}\,k\mathcal{P}$, then the diagram

$$\begin{array}{ccc} \Omega(\gamma)_x & \xrightarrow{\;{}_y h_x^{\gamma}\;} & \Omega(\gamma)_y \\ \downarrow{\scriptstyle g_x} & & \downarrow{\scriptstyle g_y} \\ \Omega(\gamma')_x & \xrightarrow{\;{}_y h_x^{\gamma'}\;} & \Omega(\gamma')_y \end{array}$$

commutes. Note that the elements in $\mathcal{P}$ are the vertices in Q_0. Moreover, if γ is an sp-diagonal then the representations $\Theta(\gamma) = (\Theta(\gamma)_x, f^{\gamma}_{\alpha})$ in $\operatorname{mod} kQ$ and $\Omega(\gamma) = (\Omega(\gamma)_x, {}_y h^{\gamma}_x)$ in $\operatorname{mod}_{sp}(k\mathcal{P})$ have the same k-vector spaces $\Omega(\gamma)_x = \Theta(\gamma)_x$ for all $x \in \mathcal{P}$ and the same maps ${}_y h^{\gamma}_x = f^{\gamma}_{\alpha}$ for each $\alpha : x \to y$ in Q_1 (the map f^{γ}_{α} is not defined when α is an alien arrow for Q). Thus we have a commutative diagram

$$\begin{array}{ccc} \Theta(\gamma)_x & \xrightarrow{\;f_{\alpha}^{\gamma}\;} & \Theta(\gamma)_y \\ \downarrow{\scriptstyle \hat{g}_x} & & \downarrow{\scriptstyle \hat{g}_y} \\ \Theta(\gamma')_x & \xrightarrow{\;f_{\alpha}^{\gamma'}\;} & \Theta(\gamma')_y \end{array}$$

and hence the map $\hat{g}$ is a morphism in $\operatorname{mod} kQ$. Under the equivalence of categories $\Theta : \mathcal{C}_T \to \operatorname{mod} kQ_T$ of Theorem 2.3, the morphism $\hat{g}$ corresponds to a morphism $P \in \operatorname{Hom}_{\mathcal{C}_T}(\gamma, \gamma')$, with $\Theta(P) = \hat{g}$. Since γ and γ' are sp-diagonals in $\mathcal{C}_T$, P also is a morphism in the full subcategory $\mathcal{C}_{(T,F)}$ of $\mathcal{C}_T$. The definition of the functors Θ and Ω on morphisms implies that $\Omega(P) = g$. $\square$

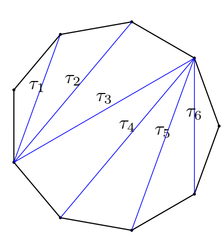
The following corollary is an direct consequence of the arguments used in Theorem 4.5 and section 4.

Corollary 4.6. *Let $\mathcal{P}$ be a poset of type $\mathbb{A}$ associated to the quiver Q^F as in Proposition 3.8 and let $\mathcal{C}_{(T,F)}$ be the corresponding category of sp-diagonals. Then*

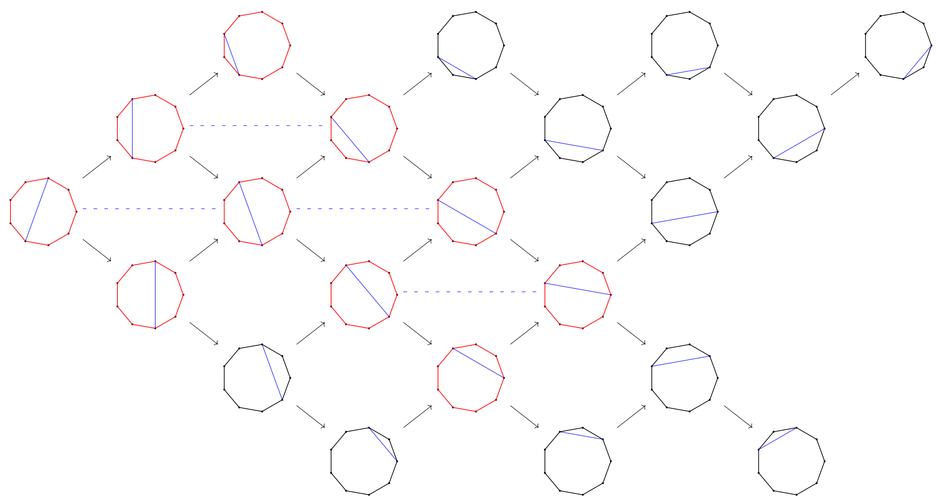
(a) *The irreducible morphisms of $\mathcal{C}_{(T,F)}$ are direct sums of the generating morphisms given by pivoting sp-moves.*

(b) *Let $\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma'$ and $\gamma \xrightarrow{P_3} \beta' \xrightarrow{P_4} \gamma'$ be compositions of two pivoting sp-moves as in Figure 6, where γ, γ', and β are sp-diagonals. Then*
  - (i) *The sequence $0 \longrightarrow \gamma \longrightarrow \beta \oplus \beta' \longrightarrow \gamma' \longrightarrow 0$ is an AR-sequence if β' is a sp-diagonal.*
  - (ii) *The sequence $0 \longrightarrow \gamma \longrightarrow \beta \longrightarrow \gamma' \longrightarrow 0$ is an AR-sequence if β' is either a boundary edge or a diagonal in T.*
  - (iii) *If $\beta' \notin E(T, F)$ then γ' is an indecomposable projective in $\mathcal{C}_{(T,F)}$ and γ is an indecomposable injective in $\mathcal{C}_{(T,F)}$.*

Example 4.7. Let Q and F as in Example 3.6. Then the triangulation T associated to Q has the form

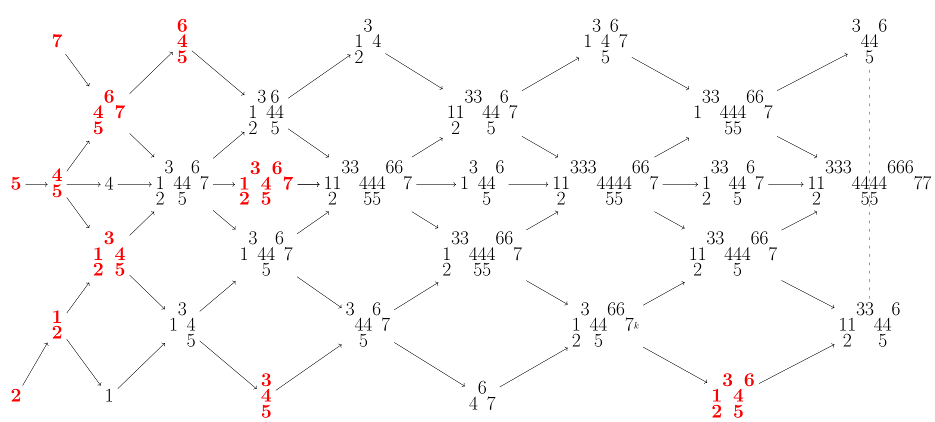


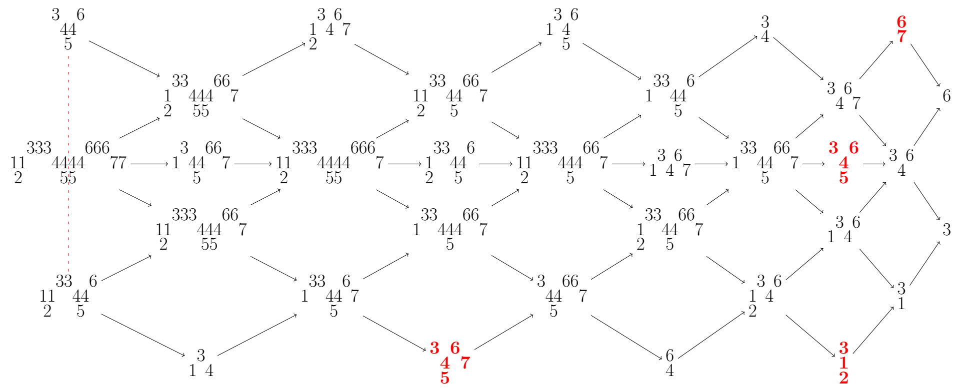
The AR-quiver $\Gamma(\mathcal{C}_T)$ of the category $\mathcal{C}_T$ has the shape



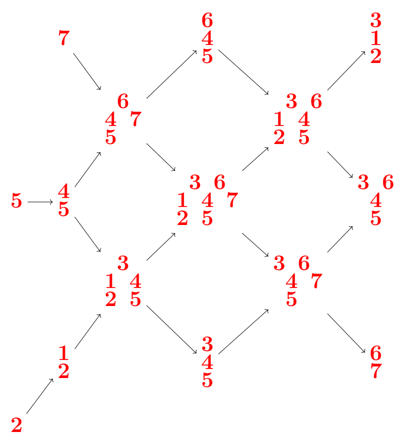
Here, we have drawn the polygons with sp-diagonals using red color, that is, the diagonals γ such that γ crosses τ_3 and if γ crosses τ_5 then γ crosses τ_2. Hence, the AR-quiver $\Gamma(\mathcal{C}_{(T,F)})$ of the category $\mathcal{C}_{(T,F)}$ is the red part of $\Gamma(\mathcal{C}_T)$, where dotted lines have been drawn to describe the action of the AR-translation in $\Gamma(\mathcal{C}_{(T,F)})$.

Example 4.8. Let $\mathcal{P} = \mathcal{P}_{Q^F}$ be the three-peak poset of type $\mathbb{A}$ defined in Example 3.2 which is the poset associated to the quiver Q^F in Example 3.7. Recall that, $\mathcal{P}$ can be viewed as a Dynkin quiver of type $\mathbb{E}_7$. Thus, the AR-quiver $\Gamma(\text{mod}(k\mathcal{P}))$ of the module category $\text{mod}(k\mathcal{P})$ has the form

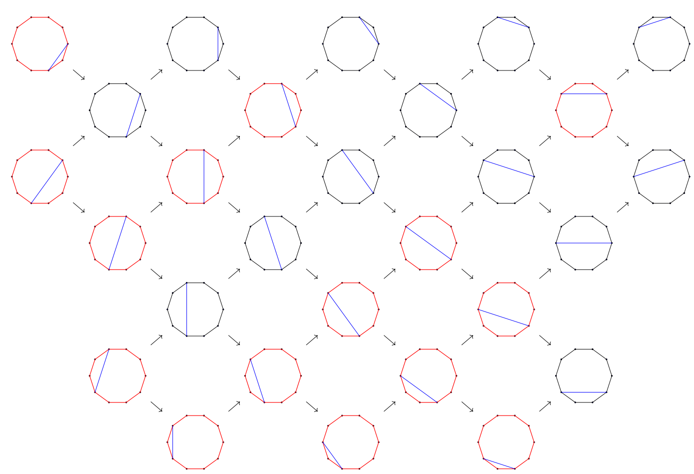




In the diagram, we have drawn the dimensions of indecomposable socle-projective modules with red color. Hence, the AR-quiver $\Gamma(\mathrm{mod}_{sp}(k\mathcal{P}))$ of the category of socle-projective modules $\mathrm{mod}_{sp}(k\mathcal{P})$ has the form



On the other hand, the triangulation T associated to Q was described in Example 4.3 and the AR-quiver $\Gamma(\mathcal{C}_T)$ of the category $\mathcal{C}_T$ has the form



Here, we have drawn the polygons with sp-diagonals using red color. Hence, the AR-quivers $\Gamma(\mathcal{C}_{(T,F)})$ and $\Gamma(\mathrm{mod}_{sp}(k\mathcal{P}))$ are identified.

5. Associated subalgebra of the cluster algebra

Let $\mathcal{P}$ be a poset of type $\mathbb{A}$ and let Q^F be the quiver associated to $\mathcal{P}$ as in Proposition 3.8. We denote by $\mathcal{A} = \mathcal{A}(\boldsymbol{x}, Q)$ the cluster algebra associated to the initial seed $(\boldsymbol{x}, Q)$ [17]. It is well known that the initial cluster variables in $\boldsymbol{x}$ correspond to the shift of indecomposable projectives in the cluster category (see [9]). Let $\mathcal{A}(\mathcal{P})$ be the subalgebra of $\mathcal{A}$ generated by the cluster variables x_γ such that γ is an sp-diagonal in the category $\mathcal{C}_{(T,F)}$ together with the cluster variables in the initial cluster $\boldsymbol{x}$. It is a natural to ask under which conditions we have $\mathcal{A}(\mathcal{P}) = \mathcal{A}$. A partial answer is given in Theorem 5.2.

Lemma 5.1. *Let Q be a quiver of tree type with n vertices and let $\mathcal{A} = \mathcal{A}(\boldsymbol{x}, Q)$ be the cluster algebra associated to Q with initial cluster $\boldsymbol{x} = \{x_1, \dots, x_n\}$. If $\mathcal{A}'$ is the subalgebra of $\mathcal{A}$ generated by the cluster variables $x_1, \dots, x_n, x_{P_1}, \dots, x_{P_n}$, where for all $i = 1, \dots, n$, x_{P_i} is the cluster variable associated to the indecomposable projective kQ-module P_i in $\operatorname{mod} kQ$, then $\mathcal{A}' = \mathcal{A}$.*

Proof. Because of [7, Corollary 1.21] it suffices to show that $\mathcal{A}'$ contains the initial cluster $x_1, \dots, x_n$ as well as the n cluster variables $x'_1, \dots, x'_n$ obtained from the initial cluster by a single mutation. We proceed by induction on the number n of vertices in Q. The case $n = 1$ is trivial. Now, let us consider Q a tree with n vertices, then Q has $n-1$ arrows. Let w be a leaf of Q and define Q' to be the full subquiver of Q whose vertices are $Q_0 \setminus \{w\}$. Then Q is obtained from Q by adding one vertex w and one arrow α_w that starts or ends at w. We have two cases: either (i) $\alpha_w : t \to w$ or (ii) $\alpha_w : w \to t$ for some $t \in Q_0$. We recall the so-called exchange relation

$$x'_k x_k = p_k^- + p_k^+, \tag{5.1}$$

defined for any vertex k in Q, where $p_k^- = \prod_{\alpha : r \to k} x_r$ and $p_k^+ = \prod_{\beta : k \to r} x_r$. Here the product p_k^- (respectively p_k^+) is taken over all arrows $\alpha \in Q_1$ (respectively $\beta \in Q_1$) that terminate (respectively start) in vertex k. We shall proof that the variables x'_w and x'_t belong to $\mathcal{A}'$. In case (i), w is a sink vertex and then $x'_w = x_{P_w}$. Hence, $x'_w \in \mathcal{A}'$. Additionally, following the knitting algorithm, we have that

$$x_{P_t} x_t = 1 + p_t^- \prod_{\beta : t \to r} x_{P_r}, \tag{5.2}$$

where the product is taken over all arrows $\beta \in Q_1$ that start in vertex t. We multiply (5.2) by p_t^+ and we obtain $x_{P_t} x_t p_t^+ = p_t^+ + p_t^- \prod_{\beta : t \to r} x_r x_{P_r}$. Since α_w is an arrow from t to w, then $x_{P_t} x_t p_t^+ = p_t^+ + p_t^- x_w x_{P_w} \delta$ where the product $\delta = \prod_{\beta : t \to r \neq w} x_r x_{P_r}$ is taken over all arrows $\beta \in Q_1$ that start in vertex t and terminate in a vertex $r \neq w$. Also, $x_{P_t} x_t p_t^+ = p_t^+ + p_t^- (1 + x_t)\delta$ because $x_w x_{P_w} = x'_w x_w = 1 + x_t$. Since $x_{P_t}, p_t^+ \in \mathcal{A}'$ we have

$$x_{P_t} p_t^+ = \frac{p_t^+ + p_t^- (1 + x_t)\delta}{x_t} = \frac{p_t^+ + p_t^-}{x_t} + p_t^- \delta \in \mathcal{A}'.$$

Since $p_t^- \delta$ belongs to $\mathcal{A}'$, equation (5.1) implies $x'_t \in \mathcal{A}'$. In case (ii), we have

$$x_w x_{P_w} = 1 + x_{P_t}. \tag{5.3}$$

Multiplying (5.3) by x_t and using (5.2) we deduce that

$$x_w x_{P_w} x_t = x_t + 1 + p_t^- \prod_{\beta : t \to r} x_{P_r}.$$

Since $x_{P_w}, x_t \in \mathcal{A}'$ then

$$x_{P_w}x_t = \frac{x_t + 1 + p_t^- \prod_{\beta:t\to r} x_{P_r}}{x_w} \in \mathcal{A}'.$$

Moreover, since there is an arrow α_w from w to t then x_w is a factor of p_t^-; thus, $x'_w = \frac{1+x_t}{x_w} \in \mathcal{A}'$. Analogous to the proof of the case (i), we can prove that $x'_t \in \mathcal{A}'$. As a consequence of the hypothesis of induction on the quiver Q' the variables x'_s with vertex $s \neq t$ in Q'_0 belong to $\mathcal{A}'$. Thus, [7, Corollary 1.21] implies the result. □

Theorem 5.2. *Let $\mathcal{P}$ be a poset of type $\mathbb{A}$ associated to the quiver $Q^{\emptyset}$ as in Proposition 3.8 and let $\mathcal{A}(\mathcal{P})$ be the subalgebra of $\mathcal{A}$ associated to $\mathcal{P}$. Then $\mathcal{A}(\mathcal{P}) = \mathcal{A}$.*

Proof. In this case, the poset $\mathcal{P}$ is viewed as the quiver Q of type $\mathbb{A}$. Then, the subcategory $\mathcal{C}_{(T,F)}$ of $\mathcal{C}_T$ is given by $\star$-diagonals because $F = \emptyset$ and it is equivalent to the category $\operatorname{mod}_{sp} kQ$ of socle-projective kQ-modules (see Theorem 4.5). By Theorem 2.3 part (e) the indecomposable projectives in $\operatorname{mod} kQ$ can be identified with diagonals $r^+(T)$ in the category $\mathcal{C}_T$ which are clearly $\star$-diagonals. Hence, the category $\mathcal{A}(\mathcal{P})$ contains the clusters variables described in the hypothesis of the above Lemma. Moreover, Q is a tree quiver. As a consequence, $\mathcal{A}(\mathcal{P}) = \mathcal{A}$. □

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Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA
Email address: `schiffler@math.uconn.edu`

School of Mathematics and Statistics, Pedagogical and Technological University of Colombia, Tunja, Boyacá 150001
Email address: `robinson.serna@uptc.edu.co`