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Incompressible viscous fluids in $\mathbb{R}^2$ and SPDEs on graphs, in presence of fast advection and non smooth noise

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Abstract. The asymptotic behavior of a class of stochastic reaction-diffusion-advection equations in the plane is studied. We show that as the divergence-free advection term becomes larger and larger, the solutions of such equations converge to the solution of a suitable stochastic PDE defined on the graph associated with the Hamiltonian. Firstly, we deal with the case that the stochastic perturbation is given by a singular spatially homogeneous Wiener process taking values in the space of Schwartz distributions. As in previous works, we assume here that the derivative of the period of the motion on the level sets of the Hamiltonian does not vanish. Then, in the second part, without assuming this condition on the derivative of the period, we study a weaker type of convergence for the solutions of a suitable class of linear SPDEs.

Résumé. Le comportement asymptotique d'une classe d'équations stochastiques de réaction-diffusion-advection dans le plan est étudié. Nous montrons qu'à mesure que le terme d'advection sans divergence devient de plus en plus grand, les solutions de telles équations convergent vers la solution d'une EDP stochastique appropriée définie sur le graphe associé à l'Hamiltonien. Tout d'abord, nous traitons le cas où la perturbation stochastique est donnée par un processus de Wiener spatialement homogène singulier prenant des valeurs dans l'espace des distributions de Schwartz. Comme dans les travaux précédents, nous supposons ici que la dérivée de la période du mouvement sur les level sets de l'Hamiltonien ne s'évanouit pas. Puis, dans la seconde partie, sans supposer cette condition sur la dérivée de la période, nous étudions un type de convergence plus faible pour les solutions d'une classe appropriée de EDPS linéaires.

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1. Introduction

In this paper we are interested in studying the limiting behavior of some particles that move together with an incompressible flow in $\mathbb{R}^2$, with stream function $H(x)$, under the assumption that the flow has a small viscosity and the particles are subject to a slow chemical reaction, which consists of a deterministic and a stochastic component. The density $v_\epsilon(t,x)$ of the particles, at time $t \geq 0$ and position $x \in \mathbb{R}^2$, satisfies the equation

$$\begin{cases}\partial_t v_\epsilon(t,x) = \frac{\epsilon}{2}\Delta v_\epsilon(t,x) + \langle \nabla^\perp H(x), \nabla v_\epsilon(t,x)\rangle + \epsilon b(v_\epsilon(t,x)) + \sqrt{\epsilon}\sigma(v_\epsilon(t,x))\partial_t \mathcal{W}(t,x), \\ v_\epsilon(0,x) = \varphi(x), \quad x \in \mathbb{R}^2,\end{cases} \tag{1.1}$$

for some parameter $0 < \epsilon \ll 1$. Throughout the paper, we assume that the Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ is a generic function, having four continuous derivatives, with bounded second derivative, such that $H(x) \to \infty$, as $|x| \to \infty$. The nonlinearities $b, \sigma : \mathbb{R} \to \mathbb{R}$ are assumed to be Lipschitz continuous and $\mathcal{W}(t,x)$ is a spatially homogeneous Wiener process (see below for all details).

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It is immediate to check that, under these conditions, on any finite time interval $[0,T]$ the solutions v_ϵ of equation (1.1) converge to the solution v of the Liouville equation

$$\partial_t v(t,x) = \langle \nabla^\perp H(x), \nabla v(t,x)\rangle, \quad v(0,x) = \varphi(x).$$

However, on time intervals of order ϵ^{-1} the difference $v_\epsilon - v$ is of order 1, as $\epsilon \to 0$. Actually, on such a time interval, the limiting behavior of v_ϵ is described by a non-standard SPDE defined on the graph Γ associated with the Hamiltonian H, which is obtained by identifying all points on the same connected component of each level set of H (see Section 2.1 for the precise definition). Such an asymptotic behavior of v_ϵ has been studied in [3], under quite restrictive conditions on the regularity of the noise $\mathcal{W}(t)$ and under the assumption that the derivative of the period of the motion on the level sets of the Hamiltonian H does not vanish. In the present paper we want to understand what happens when these conditions are not satisfied.

To this purpose, before proceeding with the description of the content of the paper, we would like to remark that the study of SPDEs on graphs is still a quite new field of investigation and very few results are available in the existing literature. In addition to the already mentioned paper [3], in [2] a class of SPDEs on graphs, obtained as limits of SPDEs in narrow tubes, is studied. In [1] first and then, more recently, in [5], suitable classes of SPDEs on graphs have been also considered. In [8], small stochastic perturbations of Hamiltonian systems are studied by using deterministic tools.

With the time change $t \mapsto t/\epsilon$, for every fixed $\epsilon > 0$ the function $u_\epsilon(t,x) := v_\epsilon(t/\epsilon, x)$ satisfies the equation

$$\begin{cases} \partial_t u_\epsilon(t,x) = L_\epsilon u_\epsilon(t,x) + b(u_\epsilon(t,x)) + \sigma(u_\epsilon(t,x))\partial_t \mathcal{W}(t,x), \\ u_\epsilon(0,x) = \varphi(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{1.2}$$

where

$$L_\epsilon \varphi(x) = \frac{1}{2}\Delta\varphi(x) + \frac{1}{\epsilon}\langle \nabla^\perp H(x), \nabla\varphi(x)\rangle.$$

The operator L_ϵ is the generator of the Markov semigroup $S_\epsilon(t)$, $t \geq 0$, associated with the stochastic differential equation

$$dX_\epsilon(t) = \frac{1}{\epsilon}\nabla^\perp H\big(X_\epsilon(t)\big)\,dt + dB(t),$$

where $B(t)$ is a Brownian motion in $\mathbb{R}^2$, defined on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$. More precisely, for every Borel and bounded function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and every $x \in \mathbb{R}^2$

$$S_\epsilon(t)\varphi(x) = \mathbf{E}_x \varphi\big(X_\epsilon(t)\big), \quad t \geq 0. \tag{1.3}$$

This means, in particular, that u_ϵ is a mild solution to equation (1.2) if

$$u_\epsilon(t) = S_\epsilon(t)\varphi(x) + \int_0^t S_\epsilon(t-s)B\big(u_\epsilon(s)\big)\,ds + \int_0^t S_\epsilon(t-s)\Sigma\big(u_\epsilon(s)\big)\,d\mathcal{W}(s), \tag{1.4}$$

where B and Σ are the composition/multiplication operators associated with b and σ, respectively.

In [3], together with M. Freidlin, the first named author proved that for every $p \geq 1$ and $0 < \tau < T$

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[\tau,T]} \big|u_\epsilon(t) - \bar{u}(t) \circ \Pi\big|_{H_\gamma}^p = 0, \tag{1.5}$$

where $\bar{u}$ is the solution of an *averaged* SPDE defined on the graph Γ and H_γ is a suitable weighted space of square integrable functions on $\mathbb{R}^2$, with respect to a finite measure $\gamma^\vee(x)\,dx$.

Due to (1.4), it is evident that the proof of (1.5) is based on the analysis of the limiting behavior of the semigroups $S_\epsilon(t)$, as $\epsilon \downarrow 0$, for every $t \in [\tau, T]$. To this purpose, in [7, Chapter 8], it is proved that if Π is the projection of $\mathbb{R}^2$ onto Γ, the slow process $Y_\epsilon(\cdot) := \Pi(X_\epsilon(\cdot))$, defined on the graph Γ, converges weakly in $C([0,T];\Gamma)$ to a continuous Markov process $\bar{Y}(\cdot)$ on Γ, whose generator $\bar{L}$ is explicitly given in terms of differential operators on each edge and suitable gluing conditions at the vertices. Hence, starting from such result, in [3, Appendix A] it has been shown that for every $\varphi \in C_b(\mathbb{R}^2)$ and for every $x \in \mathbb{R}^2$ and $0 < \tau < T$

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \big|S_\epsilon(t)\varphi(x) - \big(\bar{S}(t)\varphi^\wedge\big) \circ \Pi(x)\big| = 0, \tag{1.6}$$

where

$$\varphi^\wedge(z,k) := \frac{1}{T_k(z)} \oint_{C_k(z)} \frac{\varphi(x)}{|\nabla H(x)|}\, dl_{z,k}, \quad (z,k) \in \Gamma,$$

$dl_{z,k}$ is the length element on $C_k(z)$, the k-th connected component of $C(z) := \{x \in \mathbb{R}^2 : H(x) = z\}$, and

$$T_k(z) := \oint_{C_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k},$$

(for all details see Section 2.1). Once identified the right weighted spaces H_γ and proved limit (1.6), it can be shown that for every $\varphi \in H_\gamma$

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|S_\epsilon(t)\varphi - \left(\bar{S}(t)\varphi^\wedge\right) \circ \Pi\right|_{H_\gamma} = \lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\left(S_\epsilon(t)\varphi\right)^\wedge - \bar{S}(t)\varphi^\wedge\right|_{\bar{H}_\gamma} = 0. \tag{1.7}$$

Here the choice of the weight $\gamma^\vee$ requires a non-trivial analysis, as it has to be admissible with respect to all semigroups $S_\epsilon(t)$ and its projection γ on Γ has to be admissible with respect to $\bar{S}(t)$. Moreover, the space $H_\gamma = L^2(\mathbb{R}^2, \gamma^\vee(x)\, dx)$ has to be properly projected into the space $\bar{H}_\gamma = L^2(\Gamma, \nu_\gamma)$, where ν_γ is the projection on Γ of $\gamma^\vee(x)\, dx$ (see Section 2.3 and [3] for all details).

In [3], limit (1.7) is then used in (1.4), to obtain limit (1.5). Taking the limit, as $\epsilon \to 0$, in the first two terms on the right-hand side in (1.4) is an immediate consequence of (1.7) and the Lipschitz-continuity of the non-linearity b. On the other hand, taking the limit in the last term, the stochastic integral, requires some extra effort and, most importantly, requires the spatially homogeneous Wiener process $\mathcal{W}$ to be smooth. In particular, in [3] it is assumed that its spectral measure is finite, so that $\mathcal{W}(t,\cdot)$ takes values in the functional space H_γ. Moreover, the proof of (1.5) requires the condition

$$\frac{dT_k(z)}{dz} \neq 0, \quad (z,k) \in \Gamma. \tag{1.8}$$

This assumption is needed for the proof of (1.6). Actually, (1.6) and hence (1.5) still stand if (1.8) is true except for a finite number of points on the graph Γ. But it is easy to check that important examples such as $H(x) = |x|^2$, for which the graph is $[0,\infty)$ and the period $T(z) \equiv \pi$, are still excluded by such an assumption.

In the first part of the present paper, we are interested in understanding if limit (1.5) is still valid, under the minimal assumptions on the spectral measure μ that assure the well posedness of equation (1.2) in the space H_γ (see [9] and Assumption 2). In Section 3, assuming that the spectral measure to the singular spatially homogeneous Wiener process $\mathcal{W}(t)$ in $\mathbb{R}^2$ has a density function m in $L^p(\mathbb{R}^2)$ for some $p \in (1,\infty)$ and (1.8) holds, we prove that (1.5) is still valid (see Theorem 3.10). Actually, with little modification to our proof, we can further extend Theorem 3.10 to singular spatially homogeneous Wiener processes with spectral measure

$$\mu = \mu_1 + \mu_2,$$

where μ_1 is a finite measure and μ_2 has density function $m \in L^p(\mathbb{R}^2)$ for some $p \in (1,\infty)$. This combines the results of [3] and Section 3, and covers a large class of spatially homogeneous Wiener processes (for specific examples of the processes, we refer to [9]).

To understand the convergence of the solutions to the SPDEs under singular spatially homogeneous Wiener process, in Section 3 we first study the properties of the semigroups $S_\epsilon(t)$ and their limit $\bar{S}(t)$. For this purpose, we introduce the kernel $G_\epsilon(t,x,y)$ of the semigroup $S_\epsilon(t)$, and we prove that

$$\sup_{\epsilon>0} G_\epsilon(t,x,y) \leq \frac{C}{t} \exp\left(-\frac{(\sqrt{H(y)+1} - \sqrt{H(x)+1})^2}{4Ct}\right), \tag{1.9}$$

for any $(t,x,y) \in (0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$. Notice that due to (1.6) we have that the semigroup $\bar{S}(t)^\vee$, defined by

$$\bar{S}(t)^\vee \varphi(x) := \left(\bar{S}(t)\varphi^\wedge\right) \circ \Pi(x), \quad x \in \mathbb{R}^2, t \geq 0,$$

admits a kernel $\bar{G}(t,x,y)$, which satisfies estimate (1.9) as well.

Now, given a spatially homogeneous Wiener process $\mathcal{W}(t)$ in $\mathbb{R}^2$ with spectral measure $m \in L^p(\mathbb{R}^2)$ for some $p \in (1,\infty)$, we define $\bar{\mathcal{W}}(t)$ to be the projection of $\mathcal{W}(t)$ on Γ. We denote by $\mathscr{S}'_q$ and $\bar{\mathscr{S}}'_q$ the reproducing kernels of the

Wiener processes $\mathcal{W}(t)$ and $\bar{\mathcal{W}}(t)$, respectively. Using (1.9), we prove that for every $T>0$ there exists a constant $C_T>0$ such that

$$\sum_{j=1}^{\infty}\big|S_\epsilon(t)(\psi e_j)\big|^2_{H_\gamma}\le C_T\|m\|_{L^p}t^{-(p-1)/p}|\psi|^2_{H_\gamma},\quad t\in(0,T],$$

and

$$\sum_{j=1}^{\infty}\big|\bar{S}(t)^\vee(\psi e_j)\big|^2_{H_\gamma}\le C_T\|m\|_{L^p}t^{-(p-1)/p}|\psi|^2_{\bar{H}_\gamma},\quad t\in(0,T],$$

where $\{e_j\}_{j\in\mathbb{N}}$ is the orthonormal basis of $\mathscr{S}'_q$. This, in particular, allows us to prove the well-posedness of the SPDEs (1.2) in H_γ. Next, for the convergence of the solutions u_ϵ to $\bar{u}$, we need a stronger type of convergence for the semigroups. In fact, by using a suitable decomposition of the density function m of the spectral measure, we prove that for any $\psi\in H_\gamma$

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}\sum_{j=1}^{\infty}\big|\big(S_\epsilon(t)-\bar{S}(t)^\vee\big)(\psi e_j)\big|^2_{H_\gamma}=0. \tag{1.10}$$

Thanks to (1.10), we can then handle the convergence of the stochastic integral in (1.4) and prove (1.5).

In the second part of this paper we try to understand what happens when condition (1.8) does not hold. We recall that such condition is needed in both [3] and Section 3. This assumption is necessary for proving (1.6) and hence (1.5), i.e. the convergence of $S_\epsilon(t)\varphi$ to $\bar{S}(t)^\vee\varphi$ for any fixed time $t>0$ and $\varphi\in H_\gamma$. Thanks to (1.3), it is easy to see that (1.5) is equivalent to

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}\big|\mathbf{E}_x u\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar{Y}(t)\big)\big|=0. \tag{1.11}$$

Without assuming (1.8), clearly (1.11) is no longer true, as can be shown in the case $H(x)=|x|^2$. Nevertheless, in Section 4, (see Theorem 4.1) we prove that a weaker type of convergence holds. Namely,

$$\lim_{\epsilon\to 0}\sup_{x\in K}\left|\int_\tau^T\big[\mathbf{E}_x u\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar{Y}(t)\big)\big]\theta(t)\,dt\right|=0, \tag{1.12}$$

for any compact set $K\subset\mathbb{R}^2$, $u\in C_b(\mathbb{R}^2)$ and $\theta\in C_b([\tau,T])$.

Using (1.12), we further study the convergence of the SPDEs. Since limit (1.12) is not preserved by the nonlinearities b and σ, we restrict our consideration to the linear case

$$\begin{cases}\partial_t u_\epsilon(t,x)=\frac{1}{2}\Delta u_\epsilon(t,x)+\frac{1}{\epsilon}\langle\nabla^\perp H(x),\nabla u_\epsilon(t,x)\rangle+\partial_t\mathcal{W}(t,x),\\ u_\epsilon(0,x)=\varphi(x),\quad x\in\mathbb{R}^2.\end{cases}$$

In this case, we show that

$$\lim_{\epsilon\to 0}\mathbb{E}\left|\int_0^T\big[u_\epsilon(t)-\bar{u}(t)^\vee\big]\theta(t)\,dt\right|^q_{H_\gamma}=0,$$

(see Theorem 4.6).

The structure of the paper is as follows. In Section 2, we introduce the necessary notations and preliminaries from previous works. In Section 3 we prove our first main result stated in Theorem 3.10. Under the assumption that the density of the spectral measure is in $L^p(\mathbb{R}^2)$, for some $p\in(1,\infty)$, we first study the properties of the semigroups and the well posedness of the SPDEs. Then we prove Theorem 3.10. In Section 4, we prove that if condition (1.8) is not satisfied, then a weaker type of convergence of the semigroups $S_\epsilon(t)$ holds. Next, we prove that this implies a weaker type of convergence for the solutions of a class of linear SPDEs.

2. Notations and preliminaries

In this section, we introduce the notations that will be used in later sections. For the completeness of the paper, we also briefly recall the results in previous works, which will be used in our work here.

To study the convergence of the SPDEs, we first need to understand the convergence of the semigroups $S_\epsilon(t)$. In Section 2.2, we briefly recall the Freidlin-Wentzell averaging results in [7]. Then in Section 2.3, we recall some properties of the weighted spaces H_γ and $\bar{H}_\gamma$ proved in [3], which will be used when studying the solutions to the SPDEs that fall in the weighted spaces. Finally, the random forcing $\mathcal{W}(t,x)$ in the SPDEs are assumed to be *spatially homogeneous Wiener processes* with positive-symmetric spectral measure μ on $\mathbb{R}^2$. We recall the main definitions and properties of the spatially homogeneous Wiener process in Section 2.4 following [9].

2.1. *The Hamiltonian and the associated graph*

Throughout this paper, we consider the Hamiltonian system

$$dx(t) = \nabla^\perp H\big(x(t)\big), \quad x \in \mathbb{R}^2, \tag{2.1}$$

where

$$\nabla^\perp H(x) = \left(\frac{\partial H(x)}{\partial x_2}, -\frac{\partial H(x)}{\partial x_1}\right), \quad x \in \mathbb{R}^2.$$

We shall assume that the Hamiltonian H satisfies the following conditions.

Assumption 1. The Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ satisfies that

1. H is four times continuously differentiable, with bounded second derivatives. It has only a finite number of critical points $x_1, \dots, x_n$, and they are all non-degenerate. Moreover,

$$H(x_i) \neq H(x_j), \quad \text{if } i \neq j;$$

2. There exists $a > 0$ such that for all $x \in \mathbb{R}^2$ with $|x|$ large enough, we have

$$H(x) \geq a|x|^2, \qquad \big|\nabla H(x)\big| \geq a|x|, \qquad \Delta H(x) \geq a;$$

3. We have $\min_{x\in\mathbb{R}^2} H(x) = 0$.

For any $z \geq 0$, we denote by $C(z)$ the z-level set of the Hamiltonian H

$$C(z) := \big\{x \in \mathbb{R}^2 : H(x) = z\big\} = \bigcup_{k=1}^{N(z)} C_k(z),$$

where $C_k(z)$, $k = 1, \dots, N(z)$, are all the connected components of $C(z)$. If we denote by $k(x)$ the number of the connected component of $C(H(x))$ containing x, then

$$x(0) = x \quad \Longrightarrow \quad x(t) \in C_{k(x)}\big(H(x)\big), \quad t \geq 0.$$

If z is not a critical value, each $C_k(z)$ is a one periodic trajectory of the Hamiltonian system (2.1), and

$$T_k(z) := \oint_{C_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k} \tag{2.2}$$

is the period of the motion along the level set $C_k(z)$ (here $dl_{z,k}$ is the length element on $C_k(z)$). Moreover, the probability measure

$$d\mu_{z,k} := \frac{1}{T_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k}$$

is invariant for the Hamiltonian equation (2.1) on the level set $C_k(z)$

Now, by identifying the points on the same connected components $C_k(z)$, we obtain a graph Γ. We denote by $\Pi : \mathbb{R}^2 \to \Gamma$ the identification map. The graph Γ consists of edges $I_0, \dots, I_n$ and vertices $O_0, \dots, O_m$. The vertices are of two types, external and internal vertices. External vertices correspond to local extrema of H, while internal vertices correspond to saddle points of H. Among external vertices, we denote by O_0 the vertex corresponding to the point at infinity and by I_0 the only unbounded edge connected to O_0 (see Figure 1).

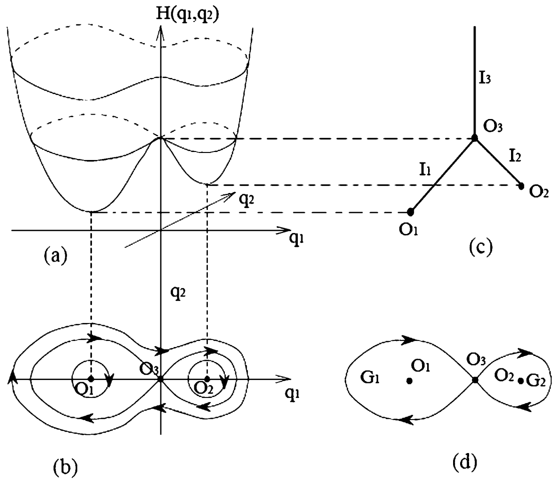

Fig. 1. The Hamiltonian, the level sets, the projection and the graph.

On graph Γ, a distance can be introduced as follows. If two points y_1 and y_2 on the graph are on the same edge I_k, i.e. $y_1 = (z_1, k)$ and $y_2 = (z_2, k)$, then $d(y_1, y_2) = |z_1 - z_2|$. If y_1 and y_2 are on different edges, then

$$d(y_1, y_2) = \min\{d(y_1, O_{i_1}) + d(O_{i_1}, O_{i_2}) + \cdots + d(O_{i_j}, y_2)\},$$

where the minimum is taken over all possible paths from y_1 to y_2, through every possible sequences of vertices $O_{i_1}, \dots, O_{i_j}$, connecting y_1 and y_2. Corresponding to each edge I_k, there is an open set

$$G_k = \{x \in \mathbb{R}^2 : \Pi(x) \in \mathring{I}_k\}.$$

For $0 \le z_1 < z_2$, we can define

$$G(z_1, z_2) = \{x \in \mathbb{R}^2 : z_1 < H(x) < z_2\},$$

and

$$G_k(z_1, z_2) = \{x \in G_k : z_1 < H(x) < z_2\}.$$

Given $\delta > 0$, we set

$$G(\pm\delta) = \bigcup_{i=1}^{m} G^i(\pm\delta) = \bigcup_{i=1}^{m} \{x \in \mathbb{R}^2 : H(O_i) - \delta < H(x) < H(O_i) + \delta\}.$$

For each vertex O_i, we denote

$$D^i = \{x \in \mathbb{R}^2 : \Pi(x) = O_i\}.$$

In addition, given any edge I_k connected to the vertex O_i, we denote

$$D^i_k = D^i \cap \bar{G}_k.$$

If an edge I_k is connected to a vertex O_i, we write $I_k \sim O_i$. For each $\delta > 0$ and $I_k \sim O_i$, we set

$$D(\pm\delta) = \bigcup_{i=1}^{m} \bigcup_{k:I_k \sim O_i} D^i_k(\pm\delta) = \bigcup_{i=1}^{m} \bigcup_{k:I_k \sim O_i} \{x \in G_k : d(\Pi(x), O_i) = \delta\}.$$

For further details, we refer to [7, Chapter 8] and [3].

2.2. *The Freidlin–Wentzell averaging result*

With a change of time in (2.1), for every $\epsilon > 0$, the function $x_\epsilon(t) := x(t/\epsilon)$ satisfies the equation

$$dx_\epsilon(t) = \frac{1}{\epsilon} \nabla^\perp H\big(x_\epsilon(t)\big). \tag{2.3}$$

Now, suppose B_t is a standard Brownian motion on $\mathbb{R}^2$. For every $\epsilon > 0$, we denote by $X_\epsilon(t)$ the solution of the stochastic differential equation

$$dX_\epsilon(t) = \frac{1}{\epsilon} \nabla^\perp H\big(X_\epsilon(t)\big)\, dt + dB(t). \tag{2.4}$$

The second order differential operator associated with (2.4) is

$$L_\epsilon u(x) = \frac{1}{2} \Delta u(x) + \frac{1}{\epsilon} \big\langle \nabla^\perp H(x), \nabla u(x) \big\rangle.$$

In what follows, we shall denote by $S_\epsilon(t)$ the corresponding Markov transition semigroup. We recall that, for every Borel bounded $u : \mathbb{R}^2 \to \mathbb{R}$, there is

$$S_\epsilon(t) u(x) = \mathbf{E}_x u\big(X_\epsilon(t)\big), \quad \text{for } x \in \mathbb{R}^2, t \geq 0.$$

Now, for every $x \in \mathbb{R}^2$, we consider the process $\Pi(X_\epsilon(t))$, $t \geq 0$, defined on the graph Γ, with $X_\epsilon(0) = x$. In [7, Chapter 8], is studied the limiting behavior, as $\epsilon \downarrow 0$, of the process $\Pi(X_\epsilon)$ in the space $C([0, T]; \Gamma)$, for any fixed $T > 0$ and $x \in \mathbb{R}^2$. Namely, in [7, Theorem 8.2.2] it has been proved that if the Hamiltonian H satisfies Assumption 1, the process $\Pi(X_\epsilon)$, which describes the slow motion of X_ϵ, converges, in the sense of weak convergence of distributions in the space of continuous Γ-valued functions, to a diffusion process $\bar{Y}$ on Γ.

The process $\bar{Y}$ has been described in [7, Theorem 8.2.1] in terms of its generator $\bar{L}$. The operator $(\bar{L}, D(\bar{L}))$ is a non-standard operator, which is given by suitable differential operators $\bar{L}_k$ within each edge I_k of the graph and by certain gluing conditions at the interior vertices O_i of the graph. Moreover, it is degenerate at the vertices of the graph. Nevertheless, in [7, Theorem 8.2.1] it is shown that it is the generator of a Markov process $\bar{Y}$ on the graph Γ. In what follows, we shall denote by $\bar{S}(t)$ the semigroup associated with $\bar{Y}$, defined by

$$\bar{S}(t) f(z, k) = \mathbf{E}_{(z,k)} f\big(\bar{Y}(t)\big),$$

for every bounded Borel function $f : \Gamma \to \mathbb{R}$.

2.3. *The weighted spaces H_γ and $\bar{H}_\gamma$*

For any $u : \mathbb{R}^2 \to \mathbb{R}$ and $0 \leq z_1 < z_2$, we have

$$\int_{G(z_1,z_2)} u(x)\, dx = \sum_{k=0}^{n} \int_{I_{k,z_1,z_2}} \oint_{C_k(z)} \frac{u(x)}{|\nabla H(x)|}\, dl_{z,k}\, dz,$$

where

$$I_{k,z_1,z_2} := \big\{(z, k) \in I_k : z \in [z_1, z_2]\big\}.$$

In particular, it holds that

$$\int_{\mathbb{R}^2} u(x)\, dx = \sum_{k=0}^{n} \int_{I_k} \oint_{C_k(z)} \frac{u(x)}{|\nabla H(x)|}\, dl_{z,k}\, dz.$$

In what follows, for every $u : \mathbb{R}^2 \to \mathbb{R}$, we shall define

$$u^\wedge(z, k) = \frac{1}{T_k(z)} \oint_{C_k(z)} \frac{u(x)}{|\nabla H(x)|}\, dl_{z,k} = \oint_{C_k(z)} u(x)\, d\mu_{z,k}, \quad (z, k) \in \Gamma.$$

Moreover, for every $f:\Gamma\to\mathbb{R}$, we shall define

$$f^\vee(x)=f\big(\Pi(x)\big),\quad x\in\mathbb{R}^2.$$

With these notations, given a positive continuous function γ on the graph Γ, if we assume that

$$\sum_{k=0}^n\int_{I_k}\gamma(z,k)T_k(z)\,dz<\infty,$$

then $\gamma^\vee\in L^1(\mathbb{R}^2)\cap C_b(\mathbb{R}^2)$. For any such function γ, we define

$$H_\gamma=\left\{u:\mathbb{R}^2\to\mathbb{R}:|u|^2_{H_\gamma}=\int_{\mathbb{R}^2}\big|u(x)\big|^2\gamma^\vee(x)\,dx<\infty\right\},$$

and

$$\bar{H}_\gamma=\left\{f:\Gamma\to\mathbb{R}:|f|^2_{\bar{H}_\gamma}=\sum_{k=0}^n\int_{I_k}\big|f(z,k)\big|^2\gamma(z,k)T_k(z)\,dz<\infty\right\}.$$

We recall the following results proved in [3].

Proposition 2.1. *For every $u\in H_\gamma$, we have $u^\wedge\in\bar{H}_\gamma$ and for every $f\in\bar{H}_\gamma$, we have $f^\vee\in H_\gamma$. Moreover,*

$$\big|u^\wedge\big|_{\bar{H}_\gamma}\le|u|_{H_\gamma},\qquad\big|f^\vee\big|_{H_\gamma}=|f|_{\bar{H}_\gamma}.\tag{2.5}$$

Finally, if $u\in H_\gamma$ and $f\in\bar{H}_\gamma$, then

$$\big\langle f,u^\wedge\big\rangle_{\bar{H}_\gamma}=\big\langle f^\vee,u\big\rangle_{H_\gamma},\qquad\big(f^\vee u\big)^\wedge=fu^\wedge.\tag{2.6}$$

Now, for every linear operator $Q\in\mathcal{L}(H_\gamma)$ and $A\in\mathcal{L}(\bar{H}_\gamma)$, we define

$$Q^\wedge f:=\big(Qf^\vee\big)^\wedge,\qquad A^\vee u:=\big(Au^\wedge\big)^\vee$$

for $f\in\bar{H}_\gamma$ and $u\in H_\gamma$. Moreover, It can be proved that

$$\big\|Q^\wedge\big\|_{\mathcal{L}(\bar{H}_\gamma)}\le\|Q\|_{\mathcal{L}(H_\gamma)},\qquad\big\|A^\vee\big\|_{\mathcal{L}(H_\gamma)}\le\|A\|_{\mathcal{L}(\bar{H}_\gamma)}.\tag{2.7}$$

2.4. *Spatially homogeneous Wiener processes*

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with filtration $(\mathcal{F}_t)_{t\ge0}$ and let $\mathscr{S}$ be the Schwartz space with its dual space $\mathscr{S}'$ (the space of Schwartz or tempered distributions). We say that $\mathcal{W}(t)$ is a *Wiener process*, defined on Ω and taking values in $\mathscr{S}'$, if for each $\psi\in\mathscr{S}$, the mapping $t\to\langle\mathcal{W}(t),\psi\rangle$ defines a Wiener process. In particular, there exists a bilinear continuous symmetric positive-definite form $Q:\mathscr{S}\times\mathscr{S}\to\mathbb{R}$ such that

$$\mathbb{E}\big\langle\mathcal{W}(t),\psi\big\rangle\big\langle\mathcal{W}(t),\varphi\big\rangle=t\wedge s\,Q(\psi,\varphi).$$

In addition, we say that the Wiener process $\mathcal{W}(t)$ is *spatially homogeneous* if the law of $\mathcal{W}(t)$ is invariant under all translations $\tau_h(f)(x):=f(x+h)$ with $h\in\mathbb{R}^2$. This implies that the bilinear form Q must be of the form

$$Q(\psi,\varphi)=\langle\Lambda,\psi*\varphi_{(s)}\rangle,$$

where $\Lambda\in\mathscr{S}'$ is the Fourier transform of a positive-symmetric tempered measure μ on $\mathbb{R}^d$, and $\varphi_{(s)}(x)=\varphi(-x)$. μ is called the *spectral measure* of $\mathcal{W}(t)$.

In what follows, we shall introduce in $\mathcal{S}$ the norm $q([\psi])=\sqrt{Q(\psi,\psi)}$ and we shall denote by $\mathscr{S}_q$ the completion of the set $\mathscr{S}/\operatorname{Ker}Q$ under the norm q. The space $\mathscr{S}_q'$ is dual to $\mathscr{S}_q$ and can be represented by

$$\mathscr{S}_q'=\big\{\xi\in\mathscr{S}':\exists C>0\text{ with }\big|\langle\xi,\psi\rangle\big|\le Cq\big([\psi]\big),\ \text{for all }\psi\in\mathscr{S}\big\}.$$

It turns out that $\mathscr{S}_q'$ is the reproducing kernel of the Wiener process $\mathcal{W}(t)$.

Now, suppose $L^2_{(s)}(\mathbb{R}^2, d\mu)$ is the space of all functions $u \in L^2(\mathbb{R}^2, d\mu)$ such that $u_{(s)} = u$. As shown in [9, Proposition 1.2], a distribution ξ belongs to $\mathscr{S}'_q$ iff there exists a $u \in L^2_{(s)}(\mathbb{R}^2, d\mu)$ such that $\xi = \widehat{u\mu}$. Moreover, for every $u, v \in L^2_{(s)}(\mathbb{R}^2, d\mu)$

$$\langle \widehat{u\mu}, \widehat{v\mu} \rangle_{\mathscr{S}'_q} = \langle u, v \rangle_{L^2(\mathbb{R}^2, d\mu)}. \tag{2.8}$$

In what follows, we shall assume the following.

Assumption 2. The spectral measure μ of the spatially homogeneous Wiener process has density function $m \in L^p(\mathbb{R}^2)$, with $p \in (1, \infty)$.

In particular, for any $u \in L^2_{(s)}(\mathbb{R}^2, d\mu)$ we have that

$$\|um\|_{2p/(p+1)} \leq \|u\|_{L^2(\mathbb{R}^2, d\mu)} \|m\|_p^{1/2}.$$

Notice that $1 \leq 2p/(p+1) \leq 2$, then by the Hausdorff–Young inequality we have that

$$\|\widehat{um}\|_{2p/(p-1)} \leq C_p \|u\|_{L^2(\mathbb{R}^2, d\mu)} \|m\|_p^{1/2}.$$

This implies that $\mathscr{S}'_q \subset L^{2p/(p-1)}(\mathbb{R}^2)$. Let $\{u_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2_{(s)}(\mathbb{R}^2, \mu)$. According to (2.8), the functions $e_j := \widehat{u_j m}$ define an orthonormal complete system in $\mathscr{S}'_q$, and the spatially homogeneous Wiener processes can be represented as

$$\mathcal{W}(t, x) = \sum_{j=1}^{\infty} \widehat{u_j m}(x) \beta_j(t),$$

where $\{\beta_j\}_{j \in \mathbb{N}}$ is a sequence of independent Brownian motions. In particular, the corresponding Wiener process on the graph can be written as

$$\bar{\mathcal{W}}(t, z, k) = \sum_{j=1}^{\infty} (\widehat{u_j m})^\wedge(z, k) \beta_j(t). \tag{2.9}$$

We shall denote the reproducing kernel of $\bar{\mathcal{W}}$ by $\bar{\mathscr{S}}'_q$.

3. The SPDE on $\mathbb{R}^2$ and the SPDE on the graph Γ

In this section, we consider the SPDE in $\mathbb{R}^2$

$$\begin{cases} \partial_t u_\epsilon(t, x) = L_\epsilon u_\epsilon(t, x) + b(u_\epsilon(t, x)) + \sigma(u_\epsilon(t, x)) \partial_t \mathcal{W}(t, x), \\ u_\epsilon(0, x) = \varphi(x), \end{cases} \tag{3.1}$$

where we recall

$$L_\epsilon u(x) = \frac{1}{2} \Delta u(x) + \frac{1}{\epsilon} \langle \bar{\nabla} H(x), \nabla u(x) \rangle, \quad x \in \mathbb{R}^2.$$

In what follows, we shall assume the following condition on the coefficients b and σ.

Assumption 3. The nonlinearities $b, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous.

For every $u \in H_\gamma$ and $v \in \mathcal{S}'_q$, we shall denote by

$$B(u)(x) = b\big(u(x)\big), \qquad \big[\Sigma(u)v\big](x) = \sigma\big(u(x)\big) v(x), \quad \text{for } x \in \mathbb{R}^2.$$

With these notations, we say that an adapted process $u_\epsilon \in L^p(\Omega, C([0,T]; H_\gamma))$ is a *mild solution* to equation (3.1) if it satisfies

$$u_\epsilon(t) = S_\epsilon(t)\varphi + \int_0^t S_\epsilon(t-s)B\big(u_\epsilon(s)\big)\,ds + \int_0^t S_\epsilon(t-s)\Sigma\big(u_\epsilon(s)\big)\,d\mathcal{W}(s). \tag{3.2}$$

If we denote by M the *multiplication operator* defined by

$$M(\psi)\xi = \psi\xi, \quad \psi \in H_\gamma, \xi \in \mathcal{S}'_q,$$

we have

$$\Sigma(u)v = M\big(\sigma(u)\big)v.$$

As in [3], where the noise in equation (3.1) was a smooth Wiener process $\mathcal{W}$, having finite spectral measure μ, we are here interested in studying the limiting behavior of u_ϵ, as $\epsilon \to 0$, in the space $L^p(\Omega; C([0,T]; H_\gamma))$. The limiting process will be the solution $\bar{u}$ of the following SPDE on the graph Γ

$$\begin{cases} \partial_t \bar{u}(t,z,k) = \bar{L}\bar{u}(t,z,k) + b(\bar{u}(t,z,k)) + \sigma(\bar{u}(t,z,k))\partial_t \bar{\mathcal{W}}(t,z,k), \\ \bar{u}(0,z,k) = \varphi^\wedge(z,k), \quad (z,k) \in \Gamma, \end{cases} \tag{3.3}$$

where $\bar{\mathcal{W}}$ is the Wiener process on the graph Γ corresponding to $\mathcal{W}$, as defined in (2.9). We say $\bar{u}$ is a mild solution to (3.3) if it is an adapted process in $L^p(\Omega; C([0,T]; \bar{H}_\gamma))$ that satisfies the integral equation

$$\bar{u}(t) = \bar{S}(t)\varphi^\wedge + \int_0^t \bar{S}(t-s)B\big(\bar{u}(s)\big)\,ds + \int_0^t \bar{S}(t-s)\Sigma\big(\bar{u}(s)\big)\,d\bar{\mathcal{W}}(s). \tag{3.4}$$

3.1. *The semigroups $S_\epsilon(t)$ and $\bar{S}(t)$*

Here, we investigate the properties of the semigroups $S_\epsilon(t)$ and their limit $\bar{S}(t)$. Firstly, we review a few results obtained in previous works, where the following condition on the Hamiltonian H is assumed.

Assumption 4. For any $(z,k) \in \Gamma$, we assume that

$$\frac{dT_k(z)}{dz} \neq 0.$$

In [3, Theorem A.2] it is shown that under Assumption 4, for any $u \in C_b(\mathbb{R}^2)$, $x \in \mathbb{R}^2$ and $0 < \tau \le T$

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \big|S_\epsilon(t)u(x) - \bar{S}(t)^\vee u(x)\big| = 0. \tag{3.5}$$

Furthermore, in [3, Corollary B.1] it is shown that for any $u \in H_\gamma$ and $0 < \tau \le T$

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \big|S_\epsilon(t)u - \bar{S}(t)^\vee u\big|^2_{H_\gamma} = \lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \big|\big(S_\epsilon(t)u\big)^\wedge - \bar{S}(t)u^\wedge\big|^2_{\bar{H}_\gamma} = 0. \tag{3.6}$$

Suppose $G_\epsilon(t,x,y)$ is the kernel corresponding to $S_\epsilon(t)$, i.e.

$$S_\epsilon(t)u(x) = \int_{\mathbb{R}^2} G_\epsilon(t,x,y)u(y)\,dy, \quad x \in \mathbb{R}^2.$$

Limit (3.5) implies that for any fixed (t,x), kernels $G_\epsilon(t,x,\cdot)$ converge weakly to some $\bar{G}(t,x,\cdot)$, which satisfies that

$$\bar{S}(t)^\vee u(x) = \int_{\mathbb{R}^2} \bar{G}(t,x,y)u(y)\,dy.$$

Next, we determine the weighted space H_γ, on which the semigroups $S_\epsilon(t)$ and $\bar{S}^\vee(t)$ are bounded. To determine the weight γ, we have the following result from [3, Proposition 4.1].

Proposition 3.1. *There exists a strictly positive decreasing function $h \in C^2([0,\infty))$ and a constant $C \geq 0$, such that the function $\gamma : \Gamma \to (0,\infty)$ defined by $\gamma(z,k) = h(z)$, for every $(z,k) \in \Gamma$, satisfies*

$$\int_{\mathbb{R}^2} G_\epsilon(t,x,y)\gamma^\vee(x)\,dx \leq e^{Ct}\gamma^\vee(y), \quad y \in \mathbb{R}^2, \tag{3.7}$$

for every $t > 0$. Moreover, for the same constant C, we have that

$$\left|S_\epsilon(t)u\right|^2_{H_\gamma} \leq e^{Ct}|u|^2_{H_\gamma}. \tag{3.8}$$

Remark 3.2. The constant C in Proposition 3.1 is independent of ϵ. Therefore, by (3.5) and (3.6), for the limit semigroup $\bar{S}(t)$, we also have that

$$\int_{\mathbb{R}^2} \bar{G}(t,x,y)\gamma^\vee(x)\,dx \leq e^{Ct}\gamma^\vee(y), \quad y \in \mathbb{R}^2, \tag{3.9}$$

and

$$\left|\bar{S}(t)^\vee u\right|^2_{H_\gamma} \leq e^{Ct}|u|^2_{H_\gamma} \tag{3.10}$$

for the same constant C. Throughout the rest of the paper, we will always assume γ to be a weight that satisfies (3.7)–(3.10) as proved in Proposition 3.1.

In addition to the weak convergence of the kernels $G_\epsilon(t,x,y)$ to $\bar{G}(t,x,y)$, we are now proving the following uniform upper bound to the kernels $G_\epsilon(t,x,y)$.

Theorem 3.3. *Suppose the Hamiltonian H satisfies Assumption 1. Then, there exists a constant $C > 0$ independent of ϵ such that*

$$G_\epsilon(t,x,y) \leq \frac{C}{t}\exp\left(-\frac{(\sqrt{H(y)+1}-\sqrt{H(x)+1})^2}{4Ct}\right), \tag{3.11}$$

for any $(t,x,y) \in (0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$. Due to the weak convergence of $G_\epsilon(t,x,y)$ to $\bar{G}(t,x,y)$, as $\epsilon \to 0$, the same point-wise upper bound as in (3.11) is valid for $\bar{G}(t,x,y)$.

Before proving Theorem 3.3, we introduce some notation and prove a preliminary lemma.

To this purpose, we define $\psi(x) = \alpha\sqrt{H(x)+1}$, where the constant $\alpha \in \mathbb{R}$ is to be determined later. Since $|\nabla H(x)| \leq C|x|$ and $H(x)+1 \geq C|x|^2$, we have that

$$\left|\nabla\psi(x)\right| = \frac{\alpha|\nabla H(x)|}{2\sqrt{H(x)+1}} \leq \alpha C$$

for some $C > 0$. Now, for any $\epsilon > 0$ we consider the linear problem

$$\begin{cases} \partial_t z_\epsilon(t,x) = \frac{1}{2}\Delta z_\epsilon(t,x) + \frac{1}{\epsilon}\langle\nabla^\perp H(x), \nabla z_\epsilon(t,x)\rangle, \\ z_\epsilon(0,x) = z_0(x), \end{cases} \tag{3.12}$$

whose solution has representation

$$z_\epsilon(t,x) = \int_{\mathbb{R}^2} G_\epsilon(t,x,y)z_0(y)\,dy.$$

Now, we introduce the transformed kernel

$$G^T_\epsilon(t,x,y) := e^{-\psi(x)}G_\epsilon(t,x,y)e^{\psi(y)},$$

and we define

$$z^T_\epsilon(t,x) := \int_{\mathbb{R}^2} G^T_\epsilon(t,x,y)z_0(y)\,dy.$$

The following result holds.

Lemma 3.4. *For any $p \geq 1$, we have*

$$\frac{d}{dt}\|z_\epsilon^T(t,\cdot)\|_{L^{2p}}^{2p} \leq p^2\alpha^2C^2\|z_\epsilon^T(t,\cdot)\|_{L^{2p}}^{2p} - \|\nabla(z_\epsilon^T(t,\cdot))^p\|_{L^2}^2. \tag{3.13}$$

Proof. By the definition of z_ϵ^T and G_ϵ^T

$$\begin{aligned}\frac{d}{dt}\|z_\epsilon^T(t,\cdot)\|_{L^{2p}}^{2p} &= 2p\int_{\mathbb{R}^2} z_\epsilon^T(t,x)^{2p-1}\left(\int_{\mathbb{R}^2}\frac{d}{dt}G_\epsilon^T(t,x,y)z_0(y)\,dy\right)dx\\ &= 2p\int_{\mathbb{R}^2} z_\epsilon^T(t,x)^{2p-1}\left(\int_{\mathbb{R}^2} e^{-\psi(x)+\psi(y)}\frac{1}{2}\Delta_x G_\epsilon^T(t,x,y)z_0(y)\,dy\right)dx\\ &\quad+2p\int_{\mathbb{R}^2} z_\epsilon^T(t,x)^{2p-1}\left(\int_{\mathbb{R}^2} e^{-\psi(x)+\psi(y)}\frac{1}{\epsilon}\langle\nabla^\perp H(x),\nabla_x G_\epsilon^T(t,x,y)\rangle z_0(y)\,dy\right)dx.\end{aligned}$$

If we integrate by part

$$\begin{aligned}\frac{d}{dt}\|z_\epsilon^T(t,\cdot)\|_{L^{2p}}^{2p} &= -2p\int_{\mathbb{R}^2}\frac{1}{2}\langle\nabla(z_\epsilon^T(t,x)^{2p-1}e^{-\psi(x)}),\nabla(z_\epsilon^T(t,x)e^{\psi(x)})\rangle\,dx\\ &\quad+2p\int_{\mathbb{R}^2} z_\epsilon^T(t,x)^{2p-1}e^{-\psi(x)}\frac{1}{\epsilon}\langle\nabla^\perp H(x),\nabla(z_\epsilon^T(t,x)e^{\psi(x)})\rangle\,dx\\ &= p\int_{\mathbb{R}^2} z_\epsilon^T(t,x)^{2p}|\nabla\psi(x)|^2\,dx-(2p-2)\int_{\mathbb{R}^2}\langle\nabla(z_\epsilon^T(t,x))^p,\nabla\psi(x)\rangle z_\epsilon^T(t,x)^p\,dx\\ &\quad-\frac{2p-1}{p}\int_{\mathbb{R}^2}|\nabla(z_\epsilon^T(t,x))^p|^2\,dx+2p\int_{\mathbb{R}^2} z_\epsilon^T(t,x)^{2p}\frac{1}{\epsilon}\langle\nabla^\perp H(x),\nabla\psi(x)\rangle\,dx\\ &\quad+2p\int_{\mathbb{R}^2} z_\epsilon^T(t,x)^{2p-1}\frac{1}{\epsilon}\langle\nabla^\perp H(x),\nabla z_\epsilon^T(t,x)\rangle\,dx\\ &=: I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

The definition of $\nabla^\perp H(x)$ and ψ clearly implies that $I_4 = 0$. Moreover, since $\mathrm{div}\nabla^\perp H = 0$, we have

$$I_5 = \frac{1}{\epsilon}\int_{\mathbb{R}^2}\langle\nabla^\perp H(x),\nabla(z_\epsilon^T(t,x)^{2p})\rangle\,dx = 0.$$

Since $|\nabla\psi(x)| \leq \alpha C$

$$\begin{aligned}I_2+I_3 &= -\int_{\mathbb{R}^2}|\nabla(z_\epsilon^T(t,x))^p|^2\,dx+p(p-1)\int_{\mathbb{R}^2}|\nabla\psi(x)|^2 z_\epsilon^T(t,x)^{2p}\,dx\\ &\quad-\frac{p-1}{p}\int_{\mathbb{R}^2}|\nabla(z_\epsilon^T(t,x))^p+p\nabla\psi(x)z_\epsilon^T(t,x)^p|^2\,dx\\ &\leq p(p-1)\alpha^2C^2\|z_\epsilon^T(t,\cdot)\|_{L^{2p}}^{2p}-\|\nabla(z_\epsilon^T(t,\cdot))^p\|_{L^2}^2.\end{aligned}$$

Together with

$$I_1 \leq p\alpha^2C^2\|z_\epsilon^T(t,\cdot)\|_{L^{2p}}^{2p},$$

we complete the proof. □

Proof of Theorem 3.3. If we apply Nash's inequality and inequality (3.13) as in [4, Lemma 1.4], we can deduce that

$$\|z_\epsilon^T(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{1/2}}\exp(C\alpha^2 t)\|z_0\|_{L^2}. \tag{3.14}$$

The dual equation to (3.12) only changes the sign of the first order coefficient $\nabla^\perp H(x)$, which means (3.14) is also true for the dual equation. By duality, there is

$$\|z_\epsilon^T(t,\cdot)\|_{L^2} \leq \frac{C}{t^{1/2}}\exp(C\alpha^2 t)\|z_0\|_{L^1}.$$

Together with (3.14), this implies that

$$\left\| z_\epsilon^T(t,\cdot)\right\|_{L^\infty} \le \frac{C}{t}\exp\bigl(C\alpha^2 t\bigr)\|z_0\|_{L^1}.$$

By the definition of $z_\epsilon^T(t,x)$, we obtain that

$$G_\epsilon^T(t,x,y) \le \frac{C}{t}\exp\bigl(C\alpha^2 t\bigr),$$

and hence

$$G_\epsilon(t,x,y) \le \frac{C}{t}\exp\bigl(C\alpha^2 t + \alpha\sqrt{H(x)+1} - \alpha\sqrt{H(y)+1}\bigr)$$

for any $\alpha \in \mathbb{R}$, $t \in (0,\infty)$ and $x, y \in \mathbb{R}^2$. Here we can take $\alpha = \frac{\sqrt{H(y)+1}-\sqrt{H(x)+1}}{2Ct}$ to minimize the right-hand side to obtain

$$G_\epsilon(t,x,y) \le \frac{C}{t}\exp\biggl(-\frac{(\sqrt{H(y)+1}-\sqrt{H(x)+1})^2}{4Ct}\biggr).$$

□

Corollary 3.5. *Given any compact subset $K \subset \mathbb{R}^2$, there exist C and R depending on K such that*

$$\sup_{x\in K} G_\epsilon(t,x,y) \le \begin{cases} \frac{C}{t} & |y| \le R, \\ \frac{C}{t}\exp(-\frac{|y|^2}{Ct}) & |y| > R \end{cases} \tag{3.15}$$

for any $t \in (0,\infty)$ and $y \in \mathbb{R}^2$. Moreover, the limit $\bar{G}(t,x,y)$ satisfies the same upper bound as in (3.15).

Proof. Actually we always have $G_\epsilon(t,x,y) \le \frac{C}{t}$. Then since $H(x)$ is bounded for $x \in K$, by Assumption 1 we have that

$$G_\epsilon(t,x,y) \le \frac{C}{t}\exp\biggl(-\frac{H(y)+H(x)+2-2\sqrt{H(y)+1}\sqrt{H(x)+1}}{4Ct}\biggr) \le \frac{C}{t}\exp\biggl(-\frac{|y|^2}{Ct}\biggr)$$

for large enough $|y|$. □

Now we consider the stochastic convolutions

$$\int_0^t S_\epsilon(t-s)\Sigma\bigl(u_\epsilon(s)\bigr)\,d\mathcal{W}(s) \tag{3.16}$$

and

$$\int_0^t \bar{S}(t-s)\Sigma\bigl(\bar{u}(s)\bigr)\,d\bar{\mathcal{W}}(s), \tag{3.17}$$

as in the definition of mild solutions, and show that they are well-defined in H_γ and $\bar{H}_\gamma$, respectively, when the spectral measure μ of the spatially homogeneous Wiener process $\mathcal{W}$ has density function m in $L^p(\mathbb{R}^2)$, with $p \in (1,\infty)$.

To be more precise, as stated in the following lemma, we show that the semigroup $S_\epsilon(t)$ improves the regularity of (3.16) following the proof of [9, Proposition 4.1].

Lemma 3.6. *Under Assumption 2, $S_\epsilon(t)M(\psi)$ are Hilbert–Schmidt operators from $\mathscr{S}'_q$ to H_γ, for all $\psi \in H_\gamma$. Moreover, for each $T > 0$ there exists a constant $C_T > 0$ such that*

$$\bigl\|S_\epsilon(t)M(\psi)\bigr\|^2_{L_{(\mathrm{HS})}(\mathscr{S}'_q,H_\gamma)} \le C_T\|m\|_{L^p} t^{-(p-1)/p}|\psi|^2_{H_\gamma}, \quad t \in [0,T].$$

Proof. Let $\{v_j\}$ be an orthonormal basis of $L^2_{(s)}(\mathbb{R}^2, dx)$. Thanks to (2.8), if we define $e_j = \widehat{v_j m^{1/2}}$, we have that $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal complete system in $\mathscr{S}'_q$. Then, for any $\psi \in H_\gamma$

$$I := \sum_{j=1}^\infty \bigl|S_\epsilon(t)\psi e_j\bigr|^2_{H_\gamma} = \sum_{j=1}^\infty \bigl|S_\epsilon(t)\bigl[\psi\bigl(\widehat{m^{1/2}} * \widehat{v_j}\bigr)\bigr]\bigr|^2_{H_\gamma}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} G_\epsilon(t,x,y)\psi(y)\big(\widehat{m^{1/2}} * \widehat{v_j}\big)(y)\, dy \right]^2 \gamma^\vee(x)\, dx \\
&\le \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \big|\widehat{m^{1/2}} * \big(G_\epsilon(t,x,\cdot)\psi\big)(y)\big|^2 \, dy \gamma^\vee(x)\, dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \big|m^{1/2}(y)\big|^2 \big|\big(G_\epsilon \widehat{(t,x,\cdot)}\psi\big)(y)\big|^2 \, dy \gamma^\vee(x)\, dx \\
&\le \|m\|_{L^p} \int_{\mathbb{R}^2} \big\| \big(G_\epsilon \widehat{(t,x,\cdot)}\psi\big)\big\|^2_{L^{2p^*}} \gamma^\vee(x)\, dx,
\end{aligned}
$$

where p^* is the Hölder conjugate of p. The Hausdorff–Young inequality implies that $\|(G_\epsilon \widehat{(t,x,\cdot)}\psi)\|_{L^{2p^*}} \le \|(G_\epsilon(t,x,\cdot)\psi)\|_{L^{2p/(p+1)}}$ and we obtain

$$
\begin{aligned}
I &\le \|m\|_{L^p} \int_{\mathbb{R}^2} \big\| \big(G_\epsilon(t,x,\cdot)\psi\big)\big\|^2_{L^{2p/(p+1)}} \gamma^\vee(x)\, dx \\
&= \|m\|_{L^p} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \big|G_\epsilon(t,x,y)\psi(y)\big|^{2p/(p+1)} \, dy \right]^{(p+1)/p} \gamma^\vee(x)\, dx.
\end{aligned}
$$

By Theorem 3.3, we have that

$$
I \le C\|m\|_{L^p} t^{-(p-1)/p} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} G_\epsilon(t,x,y)\big|\psi(y)\big|^{2p/(p+1)} \, dy \right]^{(p+1)/p} \gamma^\vee(x)\, dx.
$$

Since $2p/(p+1) \le 2$ and $G_\epsilon(t,x,y)\, dy$ is a probability measure,

$$
\left[\int_{\mathbb{R}^2} G_\epsilon(t,x,y)\big|\psi(y)\big|^{2p/(p+1)} \, dy \right]^{(p+1)/p} \le \left[\int_{\mathbb{R}^2} G_\epsilon(t,x,y)\big|\psi(y)\big|^2 \, dy \right],
$$

and then, using Proposition 3.1, we conclude

$$
\begin{aligned}
I &\le C\|m\|_{L^p} t^{-(p-1)/p} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} G_\epsilon(t,x,y)\big|\psi(y)\big|^2 \, dy \right] \gamma^\vee(x)\, dx \\
&= C\|m\|_{L^p} t^{-(p-1)/p} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\epsilon(t,x,y)\gamma^\vee(x)\, dx \big|\psi(y)\big|^2 \, dy \\
&\le C\|m\|_{L^p} t^{-(p-1)/p} e^{CT} \int_{\mathbb{R}^2} \big|\psi(y)\big|^2 \gamma^\vee(y)\, dy \\
&= Ce^{CT}\|m\|_{L^p} t^{-(p-1)/p} |\psi|^2_{H_\gamma}. \qquad \square
\end{aligned}
$$

Now we consider the limit semigroup $\bar{S}(t)$ and show that an analogous result holds.

Lemma 3.7. *Under Assumption 2, $\bar{S}(t)M(\psi)$ are Hilbert–Schmidt operators from $\bar{\mathscr{S}}_q'$ to $\bar{H}_\gamma$. For each $T > 0$ there exists a constant $C_T > 0$ such that for all $\psi \in \bar{H}_\gamma$*

$$
\big\|\bar{S}(t)M(\psi)\big\|^2_{L_{(\mathrm{HS})}(\bar{\mathscr{S}}_q', \bar{H}_\gamma)} \le C_T \|m\|_{L^p} t^{-(p-1)/p} |\psi|^2_{\bar{H}_\gamma}, \quad t \in [0,T].
$$

Proof. We have

$$
I := \sum_{j=1}^{\infty} \big|\bar{S}(t)\psi e_j^\wedge\big|^2_{\bar{H}_\gamma} = \sum_{j=1}^{\infty} \big|\bar{S}(t)\big[\psi\big(\widehat{m^{1/2}} * \widehat{v_j}\big)^\wedge\big]\big|^2_{\bar{H}_\gamma}.
$$

Then by Proposition 2.1 and the definition of $\bar{S}(t)^\vee$ and $\bar{G}(t,x,y)$,

$$
I = \sum_{j=1}^{\infty} \big|\bar{S}(t)^\vee\big[\psi^\vee\big(\widehat{m^{1/2}} * \widehat{v_j}\big)\big]\big|^2_{H_\gamma}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \bar{G}(t,x,y)\psi^{\vee}(y)\big(\widehat{m^{1/2}} * \widehat{v_j}\big)(y)\,dy \right]^2 \gamma^{\vee}(x)\,dx \\
&\le \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \big|\widehat{m^{1/2}} * \big(\bar{G}(t,x,\cdot)\psi^{\vee}\big)(y)\,dy\big|^2 \gamma^{\vee}(x)\,dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \big|m^{1/2}(y)\big|^2 \big|\big(\widehat{\bar{G}(t,x,\cdot)\psi^{\vee}}\big)(y)\big|^2\,dy\gamma^{\vee}(x)\,dx \\
&\le \|m\|_{L^p} \int_{\mathbb{R}^2} \big\|\big(\widehat{\bar{G}(t,x,\cdot)\psi^{\vee}}\big)\big\|^2_{L^{2p^*}} \gamma^{\vee}(x)\,dx.
\end{aligned}
$$

Now, with the same arguments used in the proof of Lemma 3.6, using (3.9) and the bound $\bar{G}(t,x,y) \le Ct^{-1}$, we have that

$$
I \le Ce^{CT}\|m\|_{L^p} t^{-(p-1)/p}\big|\psi^{\vee}\big|^2_{H_\gamma} = Ce^{CT}\|m\|_{L^p} t^{-(p-1)/p}|\psi|^2_{\bar{H}_\gamma},
$$

where the last equality follows from Proposition 2.1. □

Using classical arguments, in Section 3.2 we will show that Lemma 3.6 and Lemma 3.7 imply that SPDEs (3.1) and (3.3) admit a unique mild solution.

Next, to prove the convergence of mild solutions u_ϵ of equations (3.1) to the mild solution $\bar{u}$ of equation (3.3), we show that the three terms in the definition of mild solutions (3.2) converge to that of (3.4). Among these three terms, the most difficult one is the convergence of the stochastic integrals (3.16) to (3.17), for which we will need the following approximation result.

Lemma 3.8. *Given any $\psi \in H_\gamma$, for any fixed $0 < \tau < T$*

$$
\lim_{\epsilon \to 0} \sup_{t \in [\tau,T]} \sum_{j=1}^{\infty} \big|\big(S_\epsilon(t) - \bar{S}(t)^{\vee}\big)(\psi e_j)\big|^2_{H_\gamma} = 0. \tag{3.18}
$$

Proof. We show that for any given $\delta > 0$, there exists $\epsilon_\delta > 0$ such that for any $0 < \epsilon \le \epsilon_\delta$,

$$
\sum_{j=1}^{\infty} \big|\big(S_\epsilon(t) - \bar{S}(t)^{\vee}\big)(\psi e_j)\big|^2_{H_\gamma} \le \delta, \quad t \in [\tau, T]. \tag{3.19}
$$

The spectral measure m belongs to $L^p(\mathbb{R}^2)$, for $p \in [1, \infty)$, which means that $m^{1/2} \in L^{2p}(\mathbb{R}^2)$. Given any $\eta > 0$, we write $m = m_1 + m_2$, where

$$
m_1 = m1_{\{m<\eta\}}, \qquad m_2 = m1_{\{m\ge\eta\}}.
$$

Then $m^{1/2} = m_1^{1/2} + m_2^{1/2}$ and

$$
\begin{aligned}
I &= \sum_{j=1}^{\infty} \big|\big(S_\epsilon(t) - \bar{S}(t)^{\vee}\big)\big(\psi \widehat{v_j m^{1/2}}\big)\big|^2_{H_\gamma} \\
&= \sum_{j=1}^{\infty} \big|\big(S_\epsilon(t) - \bar{S}(t)^{\vee}\big)\big(\psi \widehat{v_j m_1^{1/2}}\big) + \big(S_\epsilon(t) - \bar{S}(t)^{\vee}\big)\big(\psi \widehat{v_j m_2^{1/2}}\big)\big|^2_{H_\gamma} \\
&\le 2\sum_{j=1}^{\infty} \big|\big(S_\epsilon(t) - \bar{S}(t)^{\vee}\big)\big(\psi \widehat{v_j m_1^{1/2}}\big)\big|^2_{H_\gamma} + 2\sum_{j=1}^{\infty} \big|\big(S_\epsilon(t) - \bar{S}(t)^{\vee}\big)\big(\psi \widehat{v_j m_2^{1/2}}\big)\big|^2_{H_\gamma} \\
&=: I_1(\epsilon, t, \eta) + I_2(\epsilon, t, \eta).
\end{aligned}
$$

For the first term, due to (3.15), we have

$$\begin{aligned} I_1(\epsilon,t,\eta) &\leq \sum_{j=1}^{\infty}\int_{\mathbb{R}^2}\Big[\int_{\mathbb{R}^2}\big(G_\epsilon(t,x,y)-\bar{G}(t,x,y)\big)\psi(y)\big(\widehat{m_1^{1/2}}*\widehat{v_j}\big)(y)\,dy\Big]^2\gamma^\vee(x)\,dx \\ &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big|m_1^{1/2}(y)\big|^2\big|\big(\big(G_\epsilon(t,x,\widehat{\cdot)-\bar{G}}(t,x,\cdot)\big)\psi\big)(y)\big|^2\,dy\gamma^\vee(x)\,dx \\ &\leq \eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big|\big(G_\epsilon(t,x,y)-\bar{G}(t,x,y)\big)\psi(y)\big|^2\,dy\gamma^\vee(x)\,dx \\ &\leq C\eta t^{-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}G_\epsilon(t,x,y)\big|\psi(y)\big|^2+\bar{G}(t,x,y)\big|\psi(y)\big|^2\,dy\gamma^\vee(x)\,dx \\ &= C\eta t^{-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big[G_\epsilon(t,x,y)\gamma^\vee(x)+\bar{G}(t,x,y)\gamma^\vee(x)\big]\,dx\big|\psi(y)\big|^2\,dy. \end{aligned}$$

Then, thanks to (3.7) and (3.9), we get

$$I_1(\epsilon,t,\eta)\leq C\eta t^{-1}2e^{Ct}\int_{\mathbb{R}^2}\big|\psi(y)\big|^2\gamma^\vee(y)\,dy=C\eta t^{-1}2e^{Ct}|\psi|^2_{H_\gamma}.$$

This means that we can fix $\eta_\delta=\eta(\delta,\tau,T,\psi)>0$ such that

$$\sup_{\epsilon>0}\sup_{t\in[\tau,T]}I_1(\epsilon,t,\eta_\delta)\leq\frac{\delta}{2}. \tag{3.20}$$

Now, concerning the second term $I_2(\epsilon,t,\eta)$, we have

$$\begin{aligned} \sum_{j=1}^{\infty}\big|\psi\widehat{v_jm_2^{1/2}}\big|^2_{H_\gamma} &= \sum_{j=1}^{\infty}\int_{\mathbb{R}^2}\big|\psi(x)\widehat{v_jm_2^{1/2}}(x)\big|^2\gamma^\vee(x)\,dx \\ &= (2\pi)^{-2}\int_{\mathbb{R}^2}\sum_{j=1}^{\infty}\Big|\int_{\mathbb{R}^2}\exp(i\xi\cdot x)\psi(x)v_j(\xi)m_2^{1/2}(\xi)\,d\xi\Big|^2\gamma^\vee(x)\,dx \\ &\leq (2\pi)^{-2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big|\exp(i\xi\cdot x)\psi(x)\big|^2m_2(\xi)\,d\xi\gamma^\vee(x)\,dx \\ &= (2\pi)^{-2}\|m_2\|_{L^1}|\psi|^2_{H_\gamma}. \end{aligned}$$

Then, since $\|m_2\|_{L^1}\leq\|m\|_{L^p}/\eta^{p-1}$, if we take $\eta=\eta_\delta$ we get

$$\sum_{j=1}^{\infty}\big|\psi\widehat{v_jm_2^{1/2}}\big|^2_{H_\gamma}\leq(2\pi)^{-2}\eta_\delta^{-(p-1)}\|m\|_{L^p}|\psi|^2_{H_\gamma}.$$

Due to (3.8) and (3.10), this implies that we can choose N_δ large enough such that

$$\sup_{\epsilon>0}\sup_{t\in[\tau,T]}\sum_{j=N_\delta+1}^{\infty}\big|\big(S_\epsilon(t)-\bar{S}(t)^\vee\big)\big(\psi\widehat{v_jm_2^{1/2}}\big)\big|^2_{H_\gamma}\leq\frac{\delta}{4}. \tag{3.21}$$

Moreover, by (3.6), we can choose $0<\epsilon_\delta$ small enough such that

$$\sup_{t\in[\tau,T]}\sum_{j=1}^{N_\delta}\big|\big(S_\epsilon(t)-\bar{S}(t)^\vee\big)\big(\psi\widehat{v_jm_1^{1/2}}\big)\big|^2_{H_\gamma}\leq\frac{\delta}{4} \tag{3.22}$$

for any $\epsilon\leq\epsilon_\delta$. These two inequalities (3.21) and (3.22), together with (3.20), imply (3.19). □

3.2. *Existence and uniqueness*

Here we state the existence and uniqueness of mild solutions to SPDEs (3.1) and (3.3) using Lemma 3.6 and Lemma 3.7. We state it in the following theorem.

Theorem 3.9. *Suppose the Hamiltonian H satisfies Assumption 1, coefficients b and σ satisfy Assumption 3. We assume that the spectral measure of the spatially homogeneous Wiener process $\mathcal{W}(t)$ satisfies Assumption 2, i.e. there is a density function $m(x) = d\mu/dx \in L^p(\mathbb{R}^2)$ for some $p \in (1,\infty)$. Given $q \geq 1$, $\mathscr{H}_q := L^q(\Omega; C([0,T]; H_\gamma))$ and $\bar{\mathscr{H}_q} := L^q(\Omega; C([0,T]; \bar{H}_\gamma))$ are Banach spaces with norms*

$$\|u\|_{\mathscr{H}_q} = \Big(\mathbb{E}\sup_{t\in[0,T]}\big|u(t)\big|^q_{H_\gamma}\Big)^{1/q}, \qquad \|\bar{u}\|_{\bar{\mathscr{H}_q}} = \Big(\mathbb{E}\sup_{t\in[0,T]}\big|\bar{u}(t)\big|^q_{\bar{H}_\gamma}\Big)^{1/q},$$

respectively. Then for any $\epsilon > 0$ and $q > 2p$, there is a unique mild solution u_ϵ to (3.1) satisfying that

$$\sup_{\epsilon\in(0,1)} \|u_\epsilon\|^q_{\mathscr{H}_q} \leq C_{T,q}\big(1 + |\varphi|^q_{H_\gamma}\big). \tag{3.23}$$

Moreover, there is also a unique mild solution $\bar{u}$ to (3.3) satisfying

$$\|\bar{u}\|^q_{\bar{\mathscr{H}_q}} \leq C_T\big(1 + \big|\varphi^\wedge\big|^q_{\bar{H}_\gamma}\big). \tag{3.24}$$

Remark. As discussed in [3], the existence and uniqueness of the mild solutions stated in Theorem 3.2 is also true if the spectral measure μ is finite. Together, Theorem 3.2 is actually true when the spectral measure can be written as $\mu = \mu_1 + \mu_2$, where μ_1 is a finite measure and μ_2 has density function $m \in L^p(\mathbb{R}^2)$ for some $p \in (1,\infty)$.

The proof of Theorem 3.2 follows the arguments in [9], which is essentially to show that all terms in the definition of the mild solutions (3.2) and (3.4) are contraction mappings on Banach spaces $\mathscr{H}_q$ and $\bar{\mathscr{H}_q}$, respectively. The condition that $q > 2p$ is required for the construction of contraction mappings. By Hölder's inequality, actually the mild solutions are in $\mathscr{H}_q$ and $\bar{\mathscr{H}_q}$ for any $q \geq 1$. Here we omit the detailed proof of Theorem 3.9, since it is standard.

3.3. *Convergence of the mild solutions*

In this section, we study the convergence of u_ϵ to $\bar{u}$. The main result of this section is stated in the following theorem.

Theorem 3.10. *Suppose the Hamiltonian H satisfies Assumption 1 and 4, coefficients b and σ satisfy Assumption 3 and the spectral measure of the spatially homogeneous Wiener process $\mathcal{W}(t)$ satisfies Assumption 2. Let u_ϵ be the unique mild solution to (3.1) and $\bar{u}$ be the unique mild solution to (3.3), with the same initial conditions φ and $\varphi^\wedge$, respectively. Then, for any fixed $q \geq 1$ and $0 < \tau < T$, we have that*

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[\tau,T]}\big|u_\epsilon(t) - \bar{u}(t)^\vee\big|^q_{H_\gamma} = \lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[\tau,T]}\big|u_\epsilon(t)^\wedge - \bar{u}(t)\big|^q_{\bar{H}_\gamma} = 0. \tag{3.25}$$

Proof. Without loss of generality, it is enough to prove (3.25) for large enough $q > 2p$. For any fixed $0 < \tau < T$ and $q > 2p$, we denote by

$$\Delta_{\epsilon,q}(\tau,t) := \mathbb{E}\sup_{s\in[\tau,t]}\big|u_\epsilon(s) - \bar{u}(s)^\vee\big|^q_{H_\gamma}, \quad t \in [\tau,T], \epsilon > 0.$$

Then there is

$$\begin{aligned}
u_\epsilon(s) - \bar{u}(s)^\vee &= \big[S_\epsilon(s)\varphi - \bar{S}(s)^\vee\varphi\big] + \Big[\int_0^s S_\epsilon(s-r)B\big(u_\epsilon(r)\big)\,dr - \Big(\int_0^s \bar{S}(s-r)B\big(\bar{u}(r)\big)\,dr\Big)^\vee\Big] \\
&\quad + \Big[\int_0^s S_\epsilon(s-r)\Sigma\big(u_\epsilon(r)\big)\,d\mathcal{W}(r) - \Big(\int_0^s \bar{S}(s-r)\Sigma\big(\bar{u}(r)\big)\,d\bar{\mathcal{W}}(r)\Big)^\vee\Big] \\
&= \big[S_\epsilon(s)\varphi - \bar{S}(s)^\vee\varphi\big] + \Big[\int_0^s S_\epsilon(s-r)B\big(u_\epsilon(r)\big)\,dr - \int_0^s \bar{S}(s-r)^\vee B\big(\bar{u}(r)^\vee\big)\,dr\Big]
\end{aligned}$$

$$+\left[\int_0^s S_\epsilon(s-r)\Sigma\big(u_\epsilon(r)\big)\,d\mathcal{W}(r)-\int_0^s \bar{S}(s-r)^\vee\Sigma\big(\bar{u}(r)^\vee\big)\,d\mathcal{W}(r)\right]$$
$$=: I_{\epsilon,1}(s)+I_{\epsilon,2}(s)+I_{\epsilon,3}(s).$$

Therefore, due to Lemma 3.11 and Lemma 3.12 below, we have

$$\begin{aligned}\Delta_{\epsilon,q}(\tau,t)&\le\sum_{i=1}^3\mathbb{E}\sup_{s\in[\tau,t]}\big|I_{\epsilon,i}(s)\big|_{H_\gamma}^q\\&\le C_{q,T}\int_\tau^t\Delta_{\epsilon,q}(\tau,s)\,ds+C_{q,T}\tau+\mathbb{E}\sup_{s\in[\tau,T]}\big|I_{\epsilon,1}(s)\big|_{H_\gamma}^q+H_{\epsilon,1}(\tau,T)+H_{\epsilon,2}(T),\end{aligned}$$

and, thanks to the Grönwall lemma, this implies

$$\Delta_{\epsilon,q}(\tau,t)\le C_{q,T}\Big[\tau+\mathbb{E}\sup_{s\in[\tau,T]}\big|I_{\epsilon,1}(s)\big|_{H_\gamma}^q+H_{\epsilon,1}(\tau,T)+H_{\epsilon,2}(T)\Big].$$

Firstly, it is enough to prove (3.25) for small enough τ. Hence, for any $\delta>0$ fixed, we can choose τ_δ small enough so that $C_{q,T}\tau<\delta/2$ for every $\tau\le\tau_\delta$. Next, we notice that by (3.6), we have

$$\lim_{\epsilon\to0}\mathbb{E}\sup_{s\in[\tau,t]}\big|I_{\epsilon,1}(s)\big|_{H_\gamma}^q=\lim_{\epsilon\to0}\mathbb{E}\sup_{s\in[\tau,T]}\big|I_{\epsilon,1}(s)^\wedge\big|_{\bar{H}_\gamma}^q=0.$$

Thus, thanks to (3.27) and (3.31), we can find $\epsilon_\delta>0$ such that

$$C_{T,q}\Big[\mathbb{E}\sup_{s\in[\tau,T]}\big|I_{\epsilon,1}(s)\big|_{H_\gamma}^q+H_{\epsilon,1}(\tau,T)+H_{\epsilon,2}(T)\Big]<\delta/2,$$

for every $\epsilon\le\epsilon_\delta$ and $0\le\tau<T$. This clearly implies our theorem. □

Lemma 3.11. *For every $q\ge1$ and for every $0<\tau<T$ there exists $C_{q,T}>0$ such that for every $0<\tau\le t\le T$*

$$\mathbb{E}\sup_{s\in[0,t]}\big|I_{\epsilon,2}(s)\big|_{H_\gamma}^q\le C_{q,T}\left(\int_\tau^t\mathbb{E}\sup_{r\in[\tau,s]}\big|u_\epsilon(r)-\bar{u}(r)^\vee\big|_{H_\gamma}^q\,ds+\tau\right)+H_{\epsilon,1}(\tau,T),\tag{3.26}$$

where $H_{\epsilon,1}(\tau,T)$ satisfies that

$$\lim_{\epsilon\to0}H_{\epsilon,1}(\tau,T)=0.\tag{3.27}$$

Proof. We have

$$\begin{aligned}I_{\epsilon,2}(t)&=\int_0^t S_\epsilon(t-s)\big[B\big(u_\epsilon(s)\big)-B\big(\bar{u}(s)^\vee\big)\big]\,ds+\int_0^t\big[S_\epsilon(t-s)-\bar{S}(t-s)^\vee\big]B\big(\bar{u}(s)^\vee\big)\,ds\\&=: J_{\epsilon,1}(t)+J_{\epsilon,2}(t).\end{aligned}$$

Then, since $|B(u)|_{H_\gamma}\le c(1+|u|_{H_\gamma})$, for any $t,\tau>0$ we have that

$$\begin{aligned}\big|J_{\epsilon,1}(t)\big|_{H_\gamma}^q&\le C_q\left|\int_0^\tau S_\epsilon(t-s)\big[B\big(u_\epsilon(s)\big)-B\big(\bar{u}(s)^\vee\big)\big]\,ds\right|_{H_\gamma}^q+C_q\left|\int_\tau^t S_\epsilon(t-s)\big[B\big(u_\epsilon(s)\big)-B\big(\bar{u}(s)^\vee\big)\big]\,ds\right|_{H_\gamma}^q\\&\le C_{q,T}\int_0^\tau\big(1+\big|u_\epsilon(s)\big|_{H_\gamma}^q+\big|\bar{u}(s)^\vee\big|_{H_\gamma}^q\big)\,ds+C_{q,T}\int_\tau^t\big|u_\epsilon(s)-\bar{u}(s)^\vee\big|_{H_\gamma}^q\,ds\\&\le C_{q,T}\tau\sup_{s\in[0,T]}\big(1+\big|u_\epsilon(s)\big|_{H_\gamma}^q+\big|\bar{u}(s)^\vee\big|_{H_\gamma}^q\big)+C_{q,T}\int_\tau^t\sup_{r\in[\tau,s]}\big|u_\epsilon(r)-\bar{u}(r)^\vee\big|_{H_\gamma}^q\,ds.\end{aligned}\tag{3.28}$$

As shown in (3.23) and (3.24), we have

$$\sup_{\epsilon>0}\mathbb{E}\sup_{s\in[0,T]}\big(1+\big|u_\epsilon(s)\big|_{H_\gamma}^q+\big|\bar{u}(s)^\vee\big|_{H_\gamma}^q\big)\le C$$

Thus, after taking supremum over time and expectation in (3.28), we obtain

$$\mathbb{E}\sup_{s\in[0,t]}\left|J_{\epsilon,1}(s)\right|_{H_\gamma}^q \leq C_{q,T}\tau + C_{q,T}\int_\tau^t \mathbb{E}\sup_{r\in[\tau,s]}\left|u_\epsilon(r)-\bar{u}(r)^\vee\right|_{H_\gamma}^q ds. \tag{3.29}$$

For the second term $J_{\epsilon,2}(t)$, using again the linear growth of B in H_γ, we have

$$\begin{aligned}\left|J_{\epsilon,2}(t)\right|_{H_\gamma}^q &\leq C_q\left|\int_{t-\tau}^t \left[S_\epsilon(t-s)-\bar{S}(t-s)^\vee\right]B\left(\bar{u}(s)^\vee\right)ds\right|_{H_\gamma}^q + C_q\left|\int_0^{t-\tau}\left[S_\epsilon(t-s)-\bar{S}(t-s)^\vee\right]B\left(\bar{u}(s)^\vee\right)ds\right|_{H_\gamma}^q \\ &\leq C_{q,T}\tau\sup_{s\in[0,T]}\left(1+\left|\bar{u}(s)^\vee\right|_{H_\gamma}^q\right) + C_{q,T}\int_0^T \sup_{r\in[\tau,T]}\left|\left[S_\epsilon(r)-\bar{S}(r)^\vee\right]B\left(\bar{u}(s)^\vee\right)\right|_{H_\gamma}^q ds.\end{aligned}$$

This implies

$$\mathbb{E}\sup_{s\in[0,t]}\left|J_{\epsilon,2}(s)\right|_{H_\gamma}^q \leq C_{q,T}\tau + C_{q,T}\int_0^T \mathbb{E}\sup_{r\in[\tau,T]}\left|\left[S_\epsilon(r)-\bar{S}(r)^\vee\right]B\left(\bar{u}(s)^\vee\right)\right|_{H_\gamma}^q ds.$$

Together with (3.29), we proved (3.26) with

$$H_{\epsilon,1}(\tau,T) := C_{q,T}\int_0^T \mathbb{E}\sup_{r\in[\tau,T]}\left|\left[S_\epsilon(r)-\bar{S}(r)^\vee\right]B\left(\bar{u}(s)^\vee\right)\right|_{H_\gamma}^q ds.$$

By (3.6) and (3.8), using the dominated convergence theorem we have that

$$\lim_{\epsilon\to 0}\int_0^T \mathbb{E}\sup_{r\in[\tau,T]}\left|\left[S_\epsilon(r)-\bar{S}(r)^\vee\right]B\left(\bar{u}(s)^\vee\right)\right|_{H_\gamma}^q ds = 0$$

for any $0<\tau<T$ and this implies (3.27). □

Lemma 3.12. *For every $q>2p$ and for every $0<\tau<T$, we have that*

$$\mathbb{E}\sup_{s\in[0,t]}\left|I_{\epsilon,3}(s)\right|_{H_\gamma}^q \leq C_{q,T}\int_\tau^t \mathbb{E}\sup_{r\in[\tau,s]}\left|u_\epsilon(r)-\bar{u}(r)^\vee\right|_{H_\gamma}^q ds + H_{\epsilon,2}(T), \tag{3.30}$$

where $H_{\epsilon,2}(T)$ satisfies that

$$\lim_{\epsilon\to 0} H_{\epsilon,2}(T) = 0. \tag{3.31}$$

Proof. We have

$$\begin{aligned} I_{\epsilon,3}(t) &= \int_0^t S_\epsilon(t-s)\left[\Sigma\left(u_\epsilon(s)\right)-\Sigma\left(\bar{u}(s)^\vee\right)\right]d\mathcal{W}(s) + \int_0^t \left[S_\epsilon(t-s)-\bar{S}(t-s)^\vee\right]\Sigma\left(\bar{u}(s)^\vee\right)d\mathcal{W}(s) \\ &=: J_{\epsilon,1}(t)+J_{\epsilon,2}(t).\end{aligned}$$

By the factorization formula, for every $\alpha\in(0,1)$ we have

$$J_{\epsilon,1}(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t (t-s)^{\alpha-1}S_\epsilon(t-s)Y_\alpha(s)\,ds,$$

where

$$Y_\alpha(s) = \int_0^s (s-r)^{-\alpha}S_\epsilon(s-r)\left[\Sigma\left(u_\epsilon(r)\right)-\Sigma\left(\bar{u}(r)^\vee\right)\right]d\mathcal{W}(r).$$

Now, since $m\in L^p(\mathbb{R}^2)$, for some $p\in(1,\infty)$, we can find $\alpha\in(0,1)$ and $q>1$ such that $2p<1/\alpha<q$. Then, using Proposition 3.1 and Hölder's inequality, we have

$$\mathbb{E}\sup_{s\in[0,t]}\left|J_{\epsilon,1}(s)\right|_{H_\gamma}^q \leq C_{q,T}\int_0^t \mathbb{E}\left|Y_\alpha(s)\right|_{H_\gamma}^q ds.$$

By Lemma 3.6, due to the Lipschitz continuity of σ, we have that

$$\mathbb{E}\big|Y_\alpha(s)\big|_{H_\gamma}^q \le C_{q,T,\alpha} C^q \mathbb{E}\bigg(\int_0^s (s-r)^{-2\alpha}(s-r)^{-(p-1)/p}\big|u_\epsilon(r)-\bar{u}(r)^\vee\big|_{H_\gamma}^2\,dr\bigg)^{q/2}.$$

Then, from Young's inequality and estimates (3.23) and (3.24), we obtain

$$\begin{aligned}\int_0^t \mathbb{E}\big|Y_\alpha(s)\big|_{H_\gamma}^q\,ds &\le C_{q,T}\bigg(\int_0^t s^{-2\alpha-(p-1)/p}\,ds\bigg)^{q/2}\mathbb{E}\int_0^t\big|u_\epsilon(s)-\bar{u}(s)^\vee\big|_{H_\gamma}^q\,ds\\ &\le C_{q,T}\mathbb{E}\int_0^t\big|u_\epsilon(s)-\bar{u}(s)^\vee\big|_{H_\gamma}^q\,ds\\ &\le C_{q,T}\tau\Big(\mathbb{E}\sup_{s\in[0,T]}\big|u_\epsilon(s)\big|_{H_\gamma}^q+\mathbb{E}\sup_{s\in[0,T]}\big|\bar{u}(s)^\vee\big|_{H_\gamma}^q\Big)+C_{q,T}\bigg(\int_\tau^t\mathbb{E}\sup_{r\in[\tau,s]}\big|u_\epsilon(r)-\bar{u}(r)^\vee\big|_{H_\gamma}^q\,ds\bigg)\\ &\le C_{q,T}\bigg(\tau+\int_\tau^t\mathbb{E}\sup_{r\in[\tau,s]}\big|u_\epsilon(r)-\bar{u}(r)^\vee\big|_{H_\gamma}^q\,ds\bigg).\end{aligned}$$

This implies

$$\mathbb{E}\sup_{s\in[0,t]}\big|J_{\epsilon,1}(s)\big|_{H_\gamma}^q \le C_{q,T}\bigg(\tau+\int_\tau^t\mathbb{E}\sup_{r\in[\tau,s]}\big|u_\epsilon(r)-\bar{u}(r)^\vee\big|_{H_\gamma}^q\,ds\bigg).$$

Again, using the factorization formula

$$J_{\epsilon,2}(t)=\frac{\sin\pi\alpha}{\pi}\int_0^t(t-s)^{\alpha-1}S_\epsilon(t-s)Y_{\alpha,1}(s)\,ds+\frac{\sin\pi\alpha}{\pi}\int_0^t(t-s)^{\alpha-1}\big[S_\epsilon(t-s)-\bar{S}(t-s)^\vee\big]Y_{\alpha,2}(s)\,ds,$$

where

$$Y_{\alpha,1}(s)=\int_0^s(s-r)^{-\alpha}\big[S_\epsilon(s-r)-\bar{S}(s-r)^\vee\big]G\big(\bar{u}(r)^\vee\big)\,d\mathcal{W}(r),$$

and

$$Y_{\alpha,2}(s)=\int_0^s(s-r)^{-\alpha}\bar{S}(s-r)^\vee G\big(\bar{u}(r)^\vee\big)\,d\mathcal{W}(r).$$

Then

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\big|J_{\epsilon,2}(s)\big|_{H_\gamma}^q &\le C_T\int_0^t\mathbb{E}\big|Y_{\alpha,1}(s)\big|_{H_\gamma}^q\,ds+C_T\mathbb{E}\sup_{s\in[0,t]}\int_0^s\big|\big[S_\epsilon(s-r)-\bar{S}(s-r)^\vee\big]Y_{\alpha,2}(r)\big|_{H_\gamma}^q\,dr\\ &=: K_{\epsilon,1}(t)+K_{\epsilon,2}(t).\end{aligned}$$

Here

$$\mathbb{E}\big|Y_{\alpha,1}(s)\big|_{H_\gamma}^q \le C^q\mathbb{E}\bigg(\int_0^s(s-r)^{-2\alpha}\sum_{j=1}^\infty\big|\big[S_\epsilon(s-r)-\bar{S}(s-r)^\vee\big]G\big(\bar{u}(r)^\vee\big)e_j\big|_{H_\gamma}^2\,dr\bigg)^{q/2}.$$

By Lemma 3.8 and (3.8), using the dominated convergence theorem, we have that

$$\lim_{\epsilon\to0}\mathbb{E}\big|Y_{\alpha,1}(s)\big|_{H_\gamma}^q=0.$$

This implies that

$$\lim_{\epsilon\to0}\sup_{t\in[0,T]}K_{\epsilon,1}(t)=0.$$

For $K_{\epsilon,2}$, by (3.8) and the dominated convergence theorem, we again have

$$\lim_{\epsilon\to 0}\sup_{t\in[0,T]} K_{\epsilon,2}(t)=0.$$

Therefore, if we define

$$H_{\epsilon,2}(T):=\sup_{t\in[0,T]} K_{\epsilon,1}(t)+\sup_{t\in[0,T]} K_{\epsilon,2}(t),$$

our proof is complete. □

4. A weaker type convergence if $dT/dz=0$

In [3], it has been shown that if Assumption 4 is verified, that is

$$\frac{dT_k(z)}{dz}\neq 0, \quad (z,k)\in\Gamma,$$

then for any $u\in H_\gamma$ and $0<\tau<T$

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}\left|\mathbf{E}_x u\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar{Y}(t)\big)\right|=0. \tag{4.1}$$

In [3], Assumption 4 is actually used to say that, as shown in [6, Lemma 4.3], if $\alpha\in(4/7,2/3)$ then for every $u\in C_b^2(\mathbb{R}^2)$ and for every compact set $K\in\mathbb{R}^2$

$$\lim_{\epsilon\to 0}\sup_{x\in K}\left|\mathbf{E}_x u\big(X_\epsilon\big(\epsilon^\alpha\big)\big)-\big(u^\wedge\big)^\vee(x)\right|=0. \tag{4.2}$$

When Assumption 4 is not satisfied, we don't have a way to prove (4.2), which is a key ingredient in the proof of (4.1). In this section, we will show that when Assumption 4 is not verified and hence we cannot prove (4.2), then limit (4.1) can be replaced by the following weaker type of convergence.

Theorem 4.1. *Under Assumptions 1, 2 and 3, for any $0\leq\tau<T$ and any compact set $K\subset\mathbb{R}^2$, we have*

$$\lim_{\epsilon\to 0}\sup_{x\in K}\left|\int_\tau^T\big[\mathbf{E}_x u\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar{Y}(t)\big)\big]\theta(t)\,dt\right|=0 \tag{4.3}$$

for any $u\in C_b(\mathbb{R}^2)$ and $\theta\in C_b([\tau,T])$.

To prove Theorem 4.1, we need the following notations. For $\Pi(x)=(z,k)$ in the interior of edge I_k, we set $T(x)=T_k(z)$. Given a compact set $K\subset\mathbb{R}^2$ and $\delta>0$, we denote

$$T_{M,\delta}(K):=\sup_{x\in K\setminus G(\pm\delta)} T(x),\qquad T_{m,\delta}(K):=\inf_{x\in K\setminus G(\pm\delta)} T(x).$$

Here we remove a small neighborhood of all the vertices, $G(\pm\delta)$, when taking the supremum and infimum. Therefore we always have that $T_{M,\delta}(K)<\infty$ and $T_{m,\delta}(K)>0$ (see [7, Chapter 8]).

Now, suppose $(z,0)\in\Gamma$ is such that

$$z\geq\max_{i=1,\dots,m} H(x_i)+1, \tag{4.4}$$

where $x_1,\dots,x_m$ are the critical points of the Hamiltonian H. We define the stopping time

$$\rho_{\epsilon,z}:=\inf\big\{t\geq 0: H\big(X_\epsilon(t)\big)\geq z\big\},$$

which is finite almost surely by Theorem 3.3. It is proved in [7, Lemma 8.3.2] that for any compact set $K\in\mathbb{R}^2$, there exists a $\epsilon_0>0$ such that the family of distributions corresponding to the processes $\{\Pi(X_\epsilon(\cdot)):\epsilon\in(0,\epsilon_0),X_\epsilon(0)\in K\}$ is tight in $C([0,T];\Gamma)$ for every $T>0$. This implies that for any given $\eta>0$ and $T>0$, there exists $(z_\eta,0)\in\Gamma$ satisfying (4.4) such that

$$\sup_{x\in K,0<\epsilon\le\epsilon_0}\mathbf{P}_x\Big\{\sup_{t\le T}H\big(X_\epsilon(t)\big)\ge z_\eta\Big\}\le\eta,$$

which is equivalent to

$$\sup_{x\in K,0<\epsilon\le\epsilon_0}\mathbf{P}_x\{\rho_{\epsilon,z_\eta}\le T\}\le\eta,\tag{4.5}$$

i.e., the probability of processes $X_\epsilon(t)$ hitting the level curve $C(z_\eta)$ before time T are uniformly less than η for any initial data $x\in K$ and $1<\epsilon<\epsilon_0$. Given $\eta>0$ and z_η as in (4.5), for any $0<\delta'<\delta$, define

$$\sigma_n^{\epsilon,\eta,\delta,\delta'}:=\inf\big\{t\ge\tau_n^{\epsilon,\eta,\delta,\delta'}:X_\epsilon(t)\in G(\pm\delta)^c\big\},\tag{4.6}$$

and

$$\tau_n^{\epsilon,\eta,\delta,\delta'}:=\inf\big\{t\ge\sigma_{n-1}^{\epsilon,\eta,\delta,\delta'}:X_\epsilon(t)\in D\big(\pm\delta'\big)\cup C(z_\eta)\big\}.\tag{4.7}$$

We set $\tau_0^{\epsilon,\eta,\delta,\delta'}=0$. After the process $X_\epsilon(t)$ reaches $C(z_\eta)$, all $\tau_n^{\epsilon,\eta,\delta,\delta'}$ and $\sigma_n^{\epsilon,\eta,\delta,\delta'}$ are taken equal ρ_{ϵ,z_η}.

4.1. *A weaker type of convergence for the semigroup*

In order to prove Theorem 4.1, it is sufficient to prove

$$\lim_{\epsilon\to0}\sup_{x\in K}\left|\int_0^T\big[\mathbf{E}_xu\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar Y(t)\big)\big]\theta(t)\,dt\right|=0.\tag{4.8}$$

Actually, if this is the case we can use $\int_\tau^T=\int_0^T-\int_0^\tau$ to obtain (4.3).

Thanks to [3, Lemma A.3], for any $0<\tau<T$ and $x\in\mathbb{R}^2$, we have

$$\lim_{\epsilon\to0}\sup_{t\in[\tau,T]}\big|\mathbf{E}_x\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar Y(t)\big)\big|=0.$$

In fact, given a compact subset $K\subset\mathbb{R}^2$, we further have that

$$\lim_{\epsilon\to0}\sup_{x\in K,t\in[\tau,T]}\big|\mathbf{E}_x\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar Y(t)\big)\big|=0.\tag{4.9}$$

Notice that we have the decomposition below

$$\begin{aligned}\int_0^T\big[\mathbf{E}_xu\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar Y(t)\big)\big]\theta(t)\,dt&=\int_0^T\big[\mathbf{E}_xu\big(X_\epsilon(t)\big)-\mathbf{E}_x\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\\&\quad+\int_0^\tau\big[\mathbf{E}_x\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar Y(t)\big)\big]\theta(t)\,dt\\&\quad+\int_\tau^T\big[\mathbf{E}_x\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)-\bar{\mathbf{E}}_{\Pi(x)}u^\wedge\big(\bar Y(t)\big)\big]\theta(t)\,dt\\&=:I_1+I_2+I_3.\end{aligned}$$

Since u, $(u^\wedge)^\vee$ and φ are bounded, we can choose τ small enough to control I_2. Then using (4.9) we can control I_3. Hence, in order to obtain (4.3), it is enough to prove the following result.

Lemma 4.2. *Suppose K is a compact subset of $\mathbb{R}^2$. Then, for every $T>0$ and $\theta\in C([0,T])$ it holds that*

$$\lim_{\epsilon\to0}\sup_{x\in K}\left|\int_0^T\big[\mathbf{E}_xu\big(X_\epsilon(t)\big)-\mathbf{E}_x\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|=0.\tag{4.10}$$

Proof. Actually we can assume that $u \in C_b^1(\mathbb{R}^2)$ (and $\theta \in C^1([0,T])$), because for any $u \in C_b(\mathbb{R}^2)$ (and $\theta \in C([0,T])$) we can find an approximation sequence $\{u_n\} \subset C_b^1(\mathbb{R}^2)$ (and $\{\theta_n\} \subset C^1([0,T])$) such that $u_n \to u$ in $L^\infty(\mathbb{R}^2)$ (and $\theta_n \to \theta$ in $L^\infty([0,T])$.

Since $u \in C_b^1(\mathbb{R}^2)$ and $\theta \in C^1([0,T])$, we can define

$$M_1 := \| u - (u^\wedge)^\vee \|_{L^\infty}, \qquad M_2 := \max\{ \|\theta\|_{L^\infty}, \|\theta'\|_{L^\infty} \}, \qquad M_3 := \|\nabla u\|_{L^\infty}.$$

By (4.5), for every $\eta > 0$ we can choose z_η large enough such that $\sup_{x\in K} \mathbf{P}_x(\rho_{\epsilon,z_\eta} \le T) \le \eta$. Hence,

$$\begin{aligned}\mathbf{E}_x \int_0^T \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt &= \mathbf{E}_x \int_0^{T\wedge\rho_{\epsilon,z_\eta}} \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt \\ &\quad + \mathbf{E}_x \int_{T\wedge\rho_{\epsilon,z_\eta}}^T \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt.\end{aligned}$$

For the second term on the right hand side,

$$\sup_{x\in K}\left| \mathbf{E}_x \int_{T\wedge\rho_{\epsilon,z_\eta}}^T \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt \right| \le T M_1 M_2 \mathbf{P}_x(\rho_{\epsilon,z_\eta} \le T) \le T M_1 M_2 \eta.$$

Now we fixed z_η and the stoping time ρ_{ϵ,z_η}. For the stopping times $\tau_i^{\epsilon,\eta,\delta,\delta/2}$ and $\sigma_i^{\epsilon,\eta,\delta,\delta/2}$ defined in (4.6) and (4.7) with $\delta' = \frac{\delta}{2}$, we set $\tau_i = \tau_i^{\epsilon,\eta,\delta,\delta/2} \wedge T \wedge \rho_{\epsilon,z_\eta}$ and $\sigma_i = \sigma_i^{\epsilon,\eta,\delta,\delta/2} \wedge T \wedge \rho_{\epsilon,z_\eta}$. Then, recalling that $\tau_0 = 0$, we have

$$\begin{aligned}\mathbf{E}_x \int_0^{T\wedge\rho_{\epsilon,z_\eta}} \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt &= \sum_{i=0}^\infty \mathbf{E}_x \int_{\tau_i}^{\sigma_i} \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt \\ &\quad + \sum_{i=0}^\infty \mathbf{E}_x \int_{\sigma_i}^{\tau_{i+1}} \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt.\end{aligned}$$

For the first term on the right hand side, we have

$$\begin{aligned}&\left| \sum_{i=0}^\infty \mathbf{E}_x \int_{\tau_i}^{\sigma_i} \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt \right| \\ &\quad \le M_1 M_2 \sum_{i=0}^\infty \mathbf{E}_x[\sigma_i - \tau_i] \\ &\quad \le M_1 M_2 \sum_{i=0}^\infty \mathbf{P}_x\big(\tau_i^{\epsilon,\eta,\delta,\delta/2} < T\big)\Big[\sup_{y\in D(\pm\delta/2)} \mathbf{E}_y \sigma_0^{\epsilon,\eta,\delta,\delta/2}\Big] \\ &\quad \le M_1 M_2 \Big[\sup_{y\in D(\pm\delta/2)} \mathbf{E}_y \sigma_0^{\epsilon,\eta,\delta,\delta/2}\Big] e^T \sum_{i=0}^\infty \mathbf{E}_x e^{-\tau_i^{\epsilon,\eta,\delta,\delta/2}}.\end{aligned}$$

Recall that from [7, (8.3.14)] and [3, A.19] we have

$$\sup_{x\in K} \sum_{i=0}^\infty \mathbf{E}_x e^{-\tau_i^{\epsilon,\eta,\delta,\delta/2}} \le \frac{C}{\delta},$$

and from [7, (8.5.17)] and [3, A.21] we have

$$\sup_{y\in D(\pm\delta/2)} \mathbf{E}_y \sigma_0^{\epsilon,\eta,\delta,\delta/2} \le C\delta^2 |\log\delta|.$$

Since there exists $\delta_0 > 0$ such that $C\delta|\log\delta| < \eta$ for all $0 < \delta < \delta_0$, from the inequality above we get

$$\sup_{x\in K}\left| \sum_{i=0}^\infty \mathbf{E}_x \int_{\tau_i}^{\sigma_i} \big[u(X_\epsilon(t)) - (u^\wedge)^\vee(X_\epsilon(t))\big]\theta(t)\,dt \right| \le C M_1 M_2 e^T \delta|\log\delta| < M_1 M_2 e^T \eta. \tag{4.11}$$

Using Lemma 4.3 below we have

$$\sup_{x\in K}\left|\sum_{i=0}^{\infty}\mathbf{E}_x\int_{\sigma_i}^{\tau_{i+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|$$
$$\le\sum_{i=0}^{\infty}\mathbf{P}_x\big(\sigma_i^{\epsilon,\eta,\delta,\delta/2}\le T\big)\sup_{x\in K}\left|\mathbf{E}_x\int_{\sigma_i}^{\tau_{i+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|$$
$$\le C\sum_{i=0}^{\infty}\mathbf{P}_x\big(\sigma_i^{\epsilon,\eta,\delta,\delta/2}\le T\big)\sqrt{\epsilon}$$
$$\le C\left(1+\sum_{i=0}^{\infty}\mathbf{P}_x\big(\tau_i^{\epsilon,\eta,\delta,\delta/2}\le T\big)\right)\sqrt{\epsilon}$$
$$\le Ce^T\left(\frac{C}{\delta}+1\right)\sqrt{\epsilon}.$$

This implies that we can find $\epsilon_0>0$ small enough such that for all $0<\epsilon\le\epsilon_0$

$$\sup_{x\in K}\left|\sum_{i=0}^{\infty}\mathbf{E}_x\int_{\sigma_i}^{\tau_{i+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|\le\eta.$$

This, together with (4.11) gives (4.10). □

Lemma 4.3. *For every given $\delta>0$ and each $i\in\mathbf{N}$, we have*

$$\sup_{x\in K}\left|\mathbf{E}_x\int_{\sigma_i}^{\tau_{i+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|\le C\sqrt{\epsilon}.$$

where the constant C depends on M_j with $j=1,2,3$, T, $T_{M,\delta/2}:=T_{M,\delta/2}(\overline{G(0,z_\eta)})$, and $T_{m,\delta/2}:=T_{m,\delta/2}(\overline{G(0,z_\eta)})$ and

$$M_{4,\delta}:=\sup_{x\in G(0,z_\eta)\setminus G(\pm\delta/2)}\big|\nabla\big(\big(u^\wedge\big)^\vee\big)(x)\big|<\infty.$$

Proof. We introduce the following sequence of stoping times $\sigma_i=s_0\le s_1\le s_2\le\cdots\le s_\nu=\tau_{i+1}$, by setting

$$s_{k+1}=\big[s_k+\epsilon T\big(X_\epsilon(s_k)\big)\big]\wedge\tau_{i+1},\quad k=1,\dots,\nu-1.$$

Notice that we must have $\nu\le N:=[\epsilon^{-1}T/T_{m,\delta/2}]+1$, since $|\tau_{i+1}-\sigma_i|\le T$. Then we have

$$\left|\mathbf{E}_x\int_{\sigma_i}^{\tau_{i+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|\le\sum_{k=0}^{N-1}\left|\mathbf{E}_x\int_{s_k}^{s_{k+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|.$$

For each k, we have that

$$\left|\mathbf{E}_x\int_{s_k}^{s_{k+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\right|$$
$$\le\left|\mathbf{E}_x\int_{s_k}^{s_k+\epsilon T(X_\epsilon(s_k))}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\cdot 1_{\{s_k+\epsilon T(X_\epsilon(s_k))<\tau_{i+1}\}}\right|$$
$$+\left|\mathbf{E}_x\int_{s_k}^{\tau_{i+1}}\big[u\big(X_\epsilon(t)\big)-\big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big]\theta(t)\,dt\cdot 1_{\{s_k+\epsilon T(X_\epsilon(s_k))\ge\tau_{i+1},s_k<\tau_{i+1}\}}\right|$$
$$=:I_1+I_2.$$

By the definition of σ_i and τ_{i+1}, $X_\epsilon(s_k) \in G(0, z_\eta)\backslash G(\pm\delta/2)$. For I_2, we have

$$|\tau_{i+1} - s_k| \le \epsilon T\big(X_\epsilon(s_k)\big) \le \epsilon T_{M,\delta/2}\big(G(0, z_\eta)\big),$$

which implies that

$$I_2 \le \mathbf{P}_x\big\{s_k + \epsilon T\big(X_\epsilon(s_k)\big) \ge \tau_{i+1}, s_k < \tau_{i+1}\big\} M_1 M_2 T_{M,\delta/2}\epsilon.$$

For I_1, we use the decomposition

$$\begin{aligned} u\big(X_\epsilon(t)\big) - \big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big) &= \big[u\big(X_\epsilon(t)\big) - u\big(x_\epsilon(t)\big) + \big(u^\wedge\big)^\vee\big(x_\epsilon(t)\big) - \big(u^\wedge\big)^\vee\big(X_\epsilon(t)\big)\big] \\ &\quad + \big[u\big(x_\epsilon(t)\big) - \big(u^\wedge\big)^\vee\big(x_\epsilon(t)\big)\big] \\ &=: U_1(t) + U_2(t), \end{aligned}$$

where $x_\epsilon(t)$ is the deterministic fast motion defined by (2.3), with initial condition $x_\epsilon(s_k) = X_\epsilon(s_k)$. Then

$$\begin{aligned} I_1 &\le \left|\mathbf{E}_x \int_{s_k}^{s_k+\epsilon T(X_\epsilon(s_k))} U_1(t)\theta(t)\,dt \cdot 1_{\{s_k+\epsilon T(X_\epsilon(s_k))<\tau_{i+1}\}}\right| \\ &\quad + \left|\mathbf{E}_x \int_{s_k}^{s_k+\epsilon T(X_\epsilon(s_k))} U_2(t)\theta(t)\,dt \cdot 1_{\{s_k+\epsilon T(X_\epsilon(s_k))<\tau_{i+1}\}}\right| \\ &=: I_{11} + I_{12}, \end{aligned}$$

Since $u = (u^\wedge)^\vee$ on the level set $C_{H(x)}$ and $x_\epsilon(t)$ moves on the same connected components of $C_{H(x)}$, for all $t \in [0, \epsilon T(x)]$,

$$\int_0^{\epsilon T(x)} \big[u\big(x_\epsilon(t)\big) - \big(u^\wedge\big)^\vee\big(x_\epsilon(t)\big)\big]\,dt = 0.$$

Therefore, we have that

$$I_{12} \le \left|\mathbf{E}_x \int_{s_k}^{s_k+\epsilon T(X_\epsilon(s_k))} U_2(t)\big(\theta(t) - \theta(s_k)\big)\,dt \cdot 1_{\{s_k+\epsilon T(X_\epsilon(s_k))<\tau_{i+1}\}}\right| \le M_1 M_2 T_{M,\delta/2}^2 \epsilon^2.$$

Since processes $X_\epsilon(t)$ and $x_\epsilon(t)$ always stay in the region $G(0, z_\eta)\backslash G(\pm\delta/2)$, we have

$$I_{11} \le \left|\mathbf{E}_x \int_{s_k}^{s_k+\epsilon T(X_\epsilon(s_k))} (M_3 + M_{4,\delta})\big|X_\epsilon(t) - x_\epsilon(t)\big| M_2\,dt \cdot 1_{\{s_k+\epsilon T(X_\epsilon(s_k))<\tau_{i+1}\}}\right|.$$

It is not difficult to check that

$$\sup_{x\in G(0,z_\eta)\backslash G(\pm\delta/2)} \mathbf{E}_x\big|X_\epsilon\big(\epsilon s \wedge \tau_1^{\epsilon,\eta,\delta,\delta/2}\big) - x_\epsilon\big(\epsilon s \wedge \tau_1^{\epsilon,\delta,\delta/2}\big)\big| \le C(\epsilon s)^{\frac{1}{2}}$$

for any $s \in [0, T(x)]$. Then, by the Strong Markov property of the diffusion $X_\epsilon(t)$ we have that

$$\begin{aligned} I_{11} &\le \mathbf{P}_x\big(s_k + \epsilon T\big(X_\epsilon(s_k)\big) < \tau_{i+1}\big) \sup_{x\in G(0,z_\eta)\backslash G(\pm\delta/2)} \mathbf{E}_x \int_0^{\epsilon T(x)} (M_3 + M_{4,\delta}) M_2 \big|X_\epsilon(t) - x_\epsilon(t)\big|\,dt \\ &\le (M_3 + M_{4,\delta}) M_2 C T_{M,\delta/2}^{\frac{3}{2}} \epsilon^{\frac{3}{2}}. \end{aligned}$$

Notice that

$$\sum_{k=0}^{N-1} \mathbf{P}\big\{s_k + \epsilon T\big(X_\epsilon(s_k)\big) \ge \tau_{i+1}, s_k < \tau_{i+1}\big\} = 1.$$

Now we have

$$\begin{aligned}
&\left|\mathbf{E}_x \int_{\sigma_i}^{\tau_{i+1}} \left[u\left(X_\epsilon(t)\right) - \left(u^\wedge\right)^\vee\left(X_\epsilon(t)\right)\right]\theta(t)\,dt\right| \\
&\quad \le \sum_{k=0}^{N-1}\left|\mathbf{E}_x \int_{s_k}^{s_{k+1}} \left[u\left(X_\epsilon(t)\right) - \left(u^\wedge\right)^\vee\left(X_\epsilon(t)\right)\right]\theta(t)\,dt\right| \\
&\quad \le \sum_{k=0}^{N-1} \mathbf{P}\left\{s_k + \epsilon T\left(X_\epsilon(s_k)\right) \ge \tau_{i+1}, s_k < \tau_{i+1}\right\} M_1 M_2 T_{M,\delta/2}\epsilon \\
&\qquad + N\left[M_1 M_2 T_{M,\delta/2}^2 \epsilon^2 + (M_3 + M_{4,\delta}) M_2 C T_{M,\delta/2}^{\frac{3}{2}} \epsilon^{\frac{3}{2}}\right] \\
&\quad \le M_1 M_2 T_{M,\delta/2}\epsilon + T M_1 M_2 T_{M,\delta/2}\epsilon + T(M_3 + M_{4,\delta}) M_2 C T_{M,\delta/2}^{\frac{1}{2}} \epsilon^{\frac{1}{2}}.
\end{aligned}$$

Finally, as all of the estimates for I_{11}, I_{12} and I_2 are uniform for initial data $x \in K$, our proof is complete. □

4.2. *The corresponding weaker convergence of the SPDEs*

Now we consider the convergence of the SPDEs based on the convergence of the semigroups obtained in Section 4.1 without Assumption 4. Notice that in equation (3.1), the nonlinear functions b and σ are assumed to be Lipschitz and hence preserve the strong convergence in H_γ. In this section, the semigroups converge in a weak sense, and the nonlinear functions no longer preserve it. This indicates that we can not obtain the same convergence result we obtained earlier. Here, we consider the special case when $b = 0$ and the noise is additive, i.e.

$$\begin{cases} \partial_t u_\epsilon(t,x) = \frac{1}{2}\Delta u_\epsilon(t,x) + \frac{1}{\epsilon}\langle \nabla^\perp H(x), \nabla u_\epsilon(t,x)\rangle + \partial_t \mathcal{W}(t,x), \\ u_\epsilon(0,x) = \varphi(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{4.12}$$

and

$$\begin{cases} \partial_t \bar{u}(t,z,k) = \bar{L}\bar{u}(t,z,k) + \partial_t \bar{\mathcal{W}}(t,z,k), \\ \bar{u}(0,z,k) = \varphi^\wedge(z,k), \quad (z,k) \in \Gamma. \end{cases} \tag{4.13}$$

Similar to (3.2), mild solutions to (4.12) and (4.13) are defined to be

$$u_\epsilon(t) = S_\epsilon(t)\varphi + \int_0^t S_\epsilon(t-s)\,d\mathcal{W}(s),$$

and

$$\bar{u}(t) = \bar{S}(t)\varphi^\wedge + \int_0^t \bar{S}(t-s)\,d\bar{\mathcal{W}}(s).$$

In what follows, we define operators

$$R_\epsilon^\theta(\tau,T) = \int_\tau^T S_\epsilon(t)\theta(t)\,dt,$$

and

$$\bar{R}^\theta(\tau,T) = \int_\tau^T \bar{S}(t)^\vee \theta(t)\,dt.$$

Then (4.3) is equivalent to

$$\lim_{\epsilon\to 0}\sup_{x\in K}\left|R_\epsilon^\theta(\tau,T)u(x) - \bar{R}^\theta(\tau,T)u(x)\right| = 0.$$

Recall that u is assumed to be in $C_b(\mathbb{R}^2)$ in Theorem 4.1. In the following proposition, we will extend it for $u \in H_\gamma$.

Proposition 4.4. *Under Assumptions 1 and 2, we have*

$$\lim_{\epsilon\to 0}\big|\big[R^\theta_\epsilon(\tau,T)-\bar R^\theta(\tau,T)\big]u\big|_{H_\gamma}=0$$

for any $u\in H_\gamma(\mathbb{R}^2)$, $0\le\tau<T$ *and* $\theta\in C_b([\tau,T])$.

Proof. Since $[R^\theta_\epsilon(\tau,T)-\bar R^\theta(\tau,T)]u$ converges point-wise for every $u\in C_b(\mathbb{R}^2)$, due to the dominated convergence theorem, we have

$$\lim_{\epsilon\to 0}\big|\big[R^\theta_\epsilon(\tau,T)-\bar R^\theta(\tau,T)\big]u\big|_{H_\gamma}=0.$$

The function γ introduced in Proposition 3.1 satisfies that $\inf_{x\in K}\gamma^\vee(x)>0$ for every compact set $K\subset\mathbb{R}^2$. Then, using a localization argument, it is possible to prove that for any $u\in H_\gamma$ there exists a sequence $\{u_n\}_{n\ge 1}\subset C_b(\mathbb{R}^2)$ such that $u_n\to u$ in H_γ. Thanks to (3.8) and (3.9), we have

$$\big|\big[R^\theta_\epsilon(\tau,T)-\bar R^\theta(\tau,T)\big](u-u_n)\big|_{H_\gamma}\le C_T|u-u_n|_{H_\gamma}.$$

This implies that

$$\big|\big[R^\theta_\epsilon(\tau,T)-\bar R^\theta(\tau,T)\big]u\big|_{H_\gamma}\le\big|\big[R^\theta_\epsilon(\tau,T)-\bar R^\theta(\tau,T)\big]u_n\big|_{H_\gamma}+C_T|u-u_n|_{H_\gamma},$$

and the proof is done. □

Next we will show that the mild solutions u_ϵ to the SPDEs (4.12) converges to the mild solution $\bar u$ to (4.13), for which we need the following lemma.

Lemma 4.5. *Under Assumptions 1 and 2, for any fixed* $T>0$ *and* $\theta\in C([0,T])$ *we have*

$$\lim_{\epsilon\to 0}\sum_{j=1}^\infty\bigg|\int_0^t\big(S_\epsilon(s)-\bar S(s)^\vee\big)e_j\theta(s)\,ds\bigg|^2_{H_\gamma}=0. \tag{4.14}$$

Proof. We will prove that for any $\delta>0$, there exists $\epsilon_\delta>0$ such that for any $0<\epsilon<\epsilon_\delta$

$$\sum_{j=1}^\infty\bigg|\int_0^t\big(S_\epsilon(s)-\bar S(s)^\vee\big)e_j\theta(s)\,ds\bigg|^2_{H_\gamma}\le\delta.$$

The spectral measure $m\in L^p(\mathbb{R}^2)$ for $p\in(1,\infty)$, which means that $m^{1/2}\in L^{2p}(\mathbb{R}^2)$. Given $\eta>0$, we write $m=m_1+m_2$,

$$m_1:=m1_{\{m<\eta^2\}},\qquad m_2:=m1_{\{m\ge\eta^2\}}.$$

Then $m^{1/2}=m_1^{1/2}+m_2^{1/2}$ and

$$\begin{aligned}\sum_{j=1}^\infty\bigg|\int_0^t\big(S_\epsilon(s)-\bar S(s)^\vee\big)e_j\theta(s)\,ds\bigg|^2_{H_\gamma}&\le 2\sum_{j=1}^\infty\bigg|\int_0^t\big(S_\epsilon(s)-\bar S(s)^\vee\big)\big(\widehat{v_jm_1^{1/2}}\big)\theta(s)\,ds\bigg|^2_{H_\gamma}\\&\quad+2\sum_{j=1}^\infty\bigg|\int_0^t\big(S_\epsilon(s)-\bar S(s)^\vee\big)\big(\widehat{v_jm_2^{1/2}}\big)\theta(s)\,ds\bigg|^2_{H_\gamma}\\&=:I^\eta_{1,\epsilon}+I^\eta_{2,\epsilon}.\end{aligned}$$

For the first term, since $\|m_1\|_{L^{2p}}\le\eta\|m\|^{1/2}_{L^p}$, due to Lemma 3.6 we have

$$I^\eta_{1,\epsilon}\le 2\sum_{j=1}^\infty\bigg|\int_0^t\big(S_\epsilon(s)-\bar S(s)^\vee\big)\big(\widehat{v_jm_1^{1/2}}\big)\theta(s)\bigg|^2_{H_\gamma}\,ds$$

$$\leq C_T \int_0^t \|m_1\|_{L^{2p}} s^{-(2p-1)/2p}\, ds \|\theta\|^2_{L^\infty}$$

$$\leq C_T \eta \|m\|^{1/2}_{L^p} \int_0^T s^{-(2p-1)/2p}\, ds \|\theta\|^2_{L^\infty}.$$

Hence we can choose $\eta_\delta > 0$ small enough such that

$$\sup_{\epsilon>0} I^{\eta_\delta}_{1,\epsilon} < \frac{\delta}{3}. \tag{4.15}$$

For the second term, for every $N \in \mathbb{N}$ we have

$$I^{\eta}_{2,\epsilon} = 2\sum_{j=1}^{\infty} \left| \int_0^t \left(S_\epsilon(s) - \bar{S}(s)^\vee\right)\left(\widehat{v_j m_2^{1/2}}\right)\theta(s)\, ds \right|^2_{H_\gamma}$$

$$\leq C \sum_{j=1}^{N} \left| \int_0^t \left(S_\epsilon(s) - \bar{S}(s)^\vee\right)\left(\widehat{v_j m_2^{1/2}}\right)\theta(s)\, ds \right|^2_{H_\gamma} + C_T \sum_{j=N+1}^{\infty} \int_0^t \left|\left(S_\epsilon(s) - \bar{S}(s)^\vee\right)\left(\widehat{v_j m_2^{1/2}}\right)\right|^2_{H_\gamma} \left|\theta(s)\right|^2 ds$$

$$=: J^{N,\eta}_{1,\epsilon} + J^{N,\eta}_{2,\epsilon}.$$

Since

$$\|m_2\|_{L^1} \leq \eta^{-(2p-2)} \|m\|^p_{L^p},$$

by proceeding as in the proof of Lemma 3.8, once fixed $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that

$$\sup_{\epsilon>0} J^{N_\delta,\eta_\delta}_{2,\epsilon} < \frac{\delta}{3}. \tag{4.16}$$

Then, once fixed N_δ, due to Proposition 4.4 we have that there exists $\epsilon_\delta > 0$ such that

$$J^{N_\delta,\eta_\delta}_{1,\epsilon} < \frac{\delta}{3}, \quad \text{for } \epsilon < \epsilon_\delta.$$

This inequality, together with (4.16) and (4.15), implies (4.14). □

Theorem 4.6. *Suppose the Hamiltonian H satisfies Assumption 1. The spectral measure to the spatially homogeneous Wiener process $\mathcal{W}(t)$ satisfies Assumption 2. Let $u_\epsilon \in \mathcal{H}_q$ be the unique mild solutions to (4.12) and $\bar{u} \in \bar{\mathcal{H}}_q$ be the unique mild solution to (4.13) with the same initial condition φ and $\varphi^\wedge$, respectively. Then for any fixed $T > 0$, $q \geq 1$ and $\theta \in C([0,T])$, we have that*

$$\lim_{\epsilon\to 0} \mathbb{E}\left| \int_0^T \left[u_\epsilon(t) - \bar{u}(t)^\vee\right]\theta(t)\, dt \right|^q_{H_\gamma} = \lim_{\epsilon\to 0} \mathbb{E}\left| \int_0^T \left[u_\epsilon(t)^\wedge - \bar{u}(t)\right]\theta(t)\, dt \right|^q_{\bar{H}_\gamma} = 0. \tag{4.17}$$

Proof. We have

$$\int_0^T \left[u_\epsilon(t) - \bar{u}(t)^\vee\right]\theta(t)\, dt = \int_0^T \left[S_\epsilon(t)\varphi - \bar{S}(t)^\vee\varphi\right]\theta(t)\, dt + \int_0^T \int_0^t \left[S_\epsilon(t-s) - \bar{S}(t-s)^\vee\right] d\mathcal{W}(s)\theta(t)\, dt$$

$$=: I_{\epsilon,1} + I_{\epsilon,2}.$$

By Proposition 4.4

$$\lim_{\epsilon\to 0} |I_{\epsilon,1}|_{H_\gamma} = \lim_{\epsilon\to 0} \left|\left[R^\theta_\epsilon(0,T) - \bar{R}^\theta(0,T)\right]u\right|_{H_\gamma} = 0. \tag{4.18}$$

For the second term, using Lemma 3.6 and Lemma 3.7

$$\mathbb{E}\left| \int_0^t \left[S_\epsilon(t-s) - \bar{S}(t-s)^\vee\right] d\mathcal{W}(s) \right|_{H_\gamma} \leq \mathbb{E}\left(\int_0^t \left|S_\epsilon(t-s)\right|^2_{L_{(\mathrm{HS})}(\mathscr{S}'_q, H_\gamma)} + \left|\bar{S}(t-s)^\vee\right|^2_{L_{(\mathrm{HS})}(\mathscr{S}'_q, H_\gamma)}\, ds \right)^{1/2}$$

$$\le C_T \|m\|_p^{1/2} \left(\int_0^t (t-s)^{-(p-1)/p} \, ds \right)^{1/2}$$
$$= C_T \|m\|_p^{1/2} t^{1/2p},$$

which is finite. Hence we can use the Burkholder–Davis–Gundy inequality to obtain

$$\begin{aligned}
\mathbb{E}|I_{\epsilon,2}|_{H_\gamma}^q &= \mathbb{E}\left| \int_0^T \int_s^T \left[S_\epsilon(t-s) - \bar{S}(t-s)^\vee \right] \theta(t) \, dt \, d\mathcal{W}(s) \right|_{H_\gamma}^q \\
&= \mathbb{E}\left| \int_0^T \int_0^{T-s} \left[S_\epsilon(t) - \bar{S}(t)^\vee \right] \theta(t+s) \, dt \, d\mathcal{W}(s) \right|_{H_\gamma}^q \\
&\le C_{T,q} \left(\int_0^T \sum_{j=1}^\infty \left| \int_0^{T-s} \left[S_\epsilon(t) e_j - \bar{S}(t)^\vee e_j \right] \theta(t+s) \, dt \right|_{H_\gamma}^2 ds \right)^{q/2}.
\end{aligned}$$

By Lemma 4.5 and the dominated convergence theorem, we have that

$$\lim_{\epsilon \to 0} \mathbb{E}|I_{\epsilon,2}|_{H_\gamma}^q = 0.$$

This, together with (4.18), implies (4.17). □

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