Damping with Varying Regularization in Optimal Decentralized Control

Han Feng and Javad Lavaei

***Abstract*—We study the design of an optimal static decentralized controller with a quadratic cost. The method involves a combination of the classical local search in the space of control policies, a gradual damping of the system dynamics and a gradual variation of the objective parameter. The proposed strategy is a particular type of homotopy continuation method that generates a series of optimal distributed control (ODC) problems via a continuous variation of some parameters. Instead of focusing on tracking a specific trajectory of locally optimal controllers for these ODC problems, we focus on the merging phenomenon of several locally optimal controller trajectories with the aim of finding the global solution of the original ODC problem. We prove continuity and asymptotic properties of this method. In particular, we prove that with enough damping, there is no spurious locally optimal controller for a block-diagonal control structure. This leads to a sufficient condition under which an iterative algorithm can find a global solution to a class of optimal decentralized control problems. The "damping property" introduced in this analysis is shown to be unique for general system matrices. To demonstrate the effectiveness of the proposed technique, we present empirical observations for instances with an exponential number of connected components, where damping could merge all local solutions to the one global solution.**

***Index Terms*—Decentralized control, optimal control, homotopy continuation method, damping, local search method.**

I. Introduction

THE optimal decentralized control problem (ODC) adds controller constraints to the classical centralized optimal control problem. This addition breaks down the separation principle and the classical solution formulas culminated in [1]. Although ODC has been proved intractable in general [2], [3], the problem has convex formulations under assumptions such as partially nestedness [4], positiveness [5], and quadratic invariance [6]. A recently proposed System Level Approach [7] has convexified the problem in the space of system response matrices. Convex relaxation techniques have been extensively documented in [8], though it is challenging to solve large-scale optimization problems with linear matrix inequalities and those relaxations might not be exact.

As an alternative to convexification techniques with a high computational complexity, local search methods are extensively used in the practice of optimization. This approach stands out for many problems in machine learning, where it is empirically and theoretically shown that simple policy search methods with stochastic gradient descent are able to effectively solve non-convex optimization or learning problem in practical scenarios [9]–[11]. Many efficiency statements of local search from the machine learning literature, however, are unlikely to directly carry over to ODC, due to the recent investigation of the topological properties of ODC in [12] showing that — unlike many problems in machine learning — ODC can have an exponential number of locally optimal solutions, and therefore, the landscape of optimization is highly complex.

This paper attempts to delineate the boundary of tractable ODC instances that are solvable by local-search methods, by studying the evolution of locally optimal decentralized controllers as the system dynamics and the objective cost vary. We have recently proved that one variation of the system dynamics called "damping" effectively reduces the topological complexity of the set of stabilizing decentralized controllers [12]. The main objective of the present paper is to show how damping reduces the number of locally optimal decentralized controllers. It is known that a large regularization term may help to convexify and approximate the solution of many control and optimization problems [13], [14]. We show in this paper how damping can be combined with varying regularization to improve a locally optimal decentralized controller. The variation of the damping and regularization parameters necessitates a study of the continuity and asymptotic properties of the trajectories of the locally optimal solutions. Notably, the analysis leads to the result that if the system dynamics is dampened enough, as long as the condition number of the regularization matrices remains bounded, there is no spurious locally optimal controller, by which we mean all locally optimal controllers are globally optimal for the damped system. The damped system, therefore, is a tractable approximate ODC problem. Furthermore, we show that this globally optimal controller in the damped system can be continuously connected to the globally optimal controller in the original system via a variation of the homotopy method, if the globally optimal decentralized controllers are unique in the damping process. The observations of this study shall shed light on the properties of local minima in reinforcement learning, whose aim is to design optimal control policies in an uncertain environment, and different local minima have different practical behaviors.

This work is closely related to homotopy continuation methods. They are known to be appealing yet theoretically poorly understood [15]. There is a limited literature of homotopy methods in solving problems in control theory: in [16], the author has mentioned the idea of gradually moving from a stable system to the original system to obtain a stabilizing

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H. Feng and J. Lavaei are with Industrial Engineering and Operations Research Department at the University of California, Berkeley, CA, 94720 USA (e-mail: han_feng@berkeley.edu; lavaei@berkeley.edu).

controller. The paper [17] has considered the H_2 reduced-order problem and proposed several homotopy maps and initialization strategies; in its numerical experiments, initialization with a large multiple of $-I$ was found appealing. However, no theoretical results are known for the optimal decentralized control that explains when and what homotopy strategies are effective. The difficulty of obtaining a convergence theory for a general constrained optimal control problem can be appreciated from the examples in [18]. Compared with those earlier works, we analyze a specific type of continuation, namely, damping with varying regularization, with the aim of eliminating some local minima in the ODC problem. Our setting avoids some ill-behaviors of the general homotopy setting mentioned in [18], such as stable-unstable interlaces and discontinuous solution paths. Moreover, instead of following a specific path during the homotopy process, we focus on the evolution of several paths and the movement of locally optimal solutions from one path to another in the tracking process. The proposed technique allows for (i) obtaining an approximate ODC that can be solved using local-search to global optimality, (ii) obtaining a sequential local-search method that can solve the original ODC problem via starting from a fictitious ODC that is easy to solve and gradually moving to the desirable ODC problem. Our method relies on the crucial "damping property", which will be shown unique in preserving the stability constraints.

The remainder of this paper is organized as follows. Notations and problem formulations are given in Section II. Continuity and asymptotic properties of the proposed damping strategies are outlined in Section III and Section IV, respectively. The details of the proofs are collected in Section V. Numerical experiments are detailed in Section VI, followed by concluding remarks in Section VII.

II. Problem Formulation

We study the optimal decentralized control problem (ODC) with a static controller and a quadratic cost. Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are real matrices of compatible sizes. The vector $x(t)$ is the state of the system with an unknown initialization $x(0) = x_0$, where x_0 is modeled as a random variable with zero mean and a positive definite covariance $\mathbb{E}[x(0)x(0)^\top] = D_0$ (where $\mathbb{E}[\cdot]$ denotes the expectation operator). The control input $u(t)$ is to be determined via a static state-feedback law $u(t) = Kx(t)$ with the gain $K \in \mathbb{R}^{m\times n}$ such that some quadratic performance measure is maximized. Given a controller K, the closed-loop system is

$$\dot{x}(t) = (A + BK)x(t).$$

A matrix is said to be stable if all its eigenvalues lie in the open left half of the complex plane. The controller K is said to stabilize the system (A, B) if $A + BK$ is stable. ODC optimizes over the set of structured stabilizing controllers

$$\mathcal{K}_S = \{K : A + BK \text{ is stable}, K \in \mathcal{S}\}, \tag{1}$$

where $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ is a linear subspace of matrices, often specified by fixing certain entries of the matrix to zero. In that case, the sparsity pattern can be equivalently described with the indicator matrix $I_{\mathcal{S}}$, whose (i, j)-entry is defined to be

$$[I_{\mathcal{S}}]_{ij} = \begin{cases} 1, & \text{if } K_{ij} \text{ is free} \\ 0, & \text{if } K_{ij} = 0. \end{cases}$$

The structural constraint $K \in \mathcal{S}$ is then equivalent to $K \circ I_{\mathcal{S}} = K$, where $\circ$ denotes entry-wise multiplication. In the following, we will consider a sequence of damped cost functions with a varying regularization, which is defined as

$$\begin{aligned} J(K,\alpha) =& \mathbb{E}\int_0^\infty \left[e^{-2\alpha t}\left(\hat{x}^\top(t)Q\hat{x}(t) + \hat{u}^\top(t)R_\alpha\hat{u}(t)\right)\right] dt \\ s.t. \quad & \dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t) \\ & \hat{u}(t) = K\hat{x}(t). \end{aligned} \tag{2}$$

where $Q \succeq 0$ is positive semi-definite and the varying regularization $R_\alpha \succ 0$ is positive definite for all $\alpha \geq 0$. The expectation is taken over x_0. By a change of variable $x(t) = e^{-\alpha t}\hat{x}(t)$ and $u(t) = e^{-\alpha t}\hat{u}(t)$, the cost $J(K, \alpha)$ can be equivalently written as

$$\begin{aligned} J(K,\alpha) =& \mathbb{E}\int_0^\infty \left[x^\top(t)Qx(t) + u^\top(t)R_\alpha u(t)\right] dt \\ s.t. \quad & \dot{x}(t) = (A-\alpha I)x(t) + Bu(t) \\ & u(t) = Kx(t), \end{aligned} \tag{3}$$

ODC is commonly defined for $\alpha = 0$ as optimizing (3) over the set of stabilizing structured controllers (1). Formally

$$\begin{aligned} \min_K \quad & J(K, 0) \\ s.t. \quad & K \text{ stabilizes } (A, B) \\ & K \in \mathcal{S}. \end{aligned}$$

In our setting, the notion of stability is relaxed for a positive α. We define K as a stabilizing solution to (3) if K stabilizes the system $(A - \alpha I, B)$, in which case formulation (2) is also meaningful. Formally, we define ODC with damping and varying regularization as

$$\begin{aligned} \min_K \quad & J(K, \alpha) \\ s.t. \quad & K \text{ stabilizes } (A - \alpha I, B) \\ & K \in \mathcal{S}. \end{aligned} \tag{4}$$

Our relaxed notion of stability coincides with ODC when $\alpha = 0$. We emphasize that the relaxation of stability in the damped regime is a solution method, while the aim remains in obtaining an optimal stabilizing controller for the undamped system with $\alpha = 0$. We shall denote the problem (4) by ODC(α). We write ODC(α, K_0) if additionally a stabilizing controller K_0 is given.

The two equivalent formulations (2) and (3) motivate the notion of "damping property". We make a formal statement below.

Lemma 1. *The function $J(K, \alpha)$ defined in* (2) *and* (3) *satisfies the following "damping property": assuming that K*

stabilizes the system $(A - \alpha I, B)$, the following statements hold for all $\beta > \alpha$:

- *K stabilizes the system $(A - \beta I, B)$,*
- *$J(K, \beta) < J(K, \alpha)$ if $R_\beta \preceq R_\alpha$.*

Proof. From the formulation (4), when $A - \alpha I + BK$ is stable and $\beta > \alpha$, it holds that $A - \beta I + BK = (A - \alpha I + BK) - (\beta - \alpha) I$ is stable. Therefore, $J(K, \beta)$ is well-defined. From formulation (2), $J(K, \beta) < J(K, \alpha)$ when $R_\beta \preceq R_\alpha$. □

We define R_α to be monotonically decreasing if $R_\beta \preceq R_\alpha$ for all $\beta > \alpha \geq 0$. We use $K^*(\alpha)$ to denote the set of globally optimal solutions of (4). We further introduce the set of locally optimal solutions $K^\dagger(\alpha)$, which contains those controllers K that satisfy first-order optimality conditions (5a)-(5d) (see [19] for their derivation):

$$\begin{aligned}&(A - \alpha I + BK)^\top P_\alpha(K) + \\ &\qquad P_\alpha(K)(A - \alpha I + BK) + K^\top R_\alpha K + Q = 0\end{aligned} \tag{5a}$$

$$\begin{aligned}&L_\alpha(K)(A - \alpha I + BK)^\top + \\ &\qquad (A - \alpha I + BK) L_\alpha(K) + D_0 = 0\end{aligned} \tag{5b}$$

$$\left[(B^\top P_\alpha(K) + R_\alpha K) L_\alpha(K)\right] \circ I_S = 0 \tag{5c}$$

$$K \circ I_S = K. \tag{5d}$$

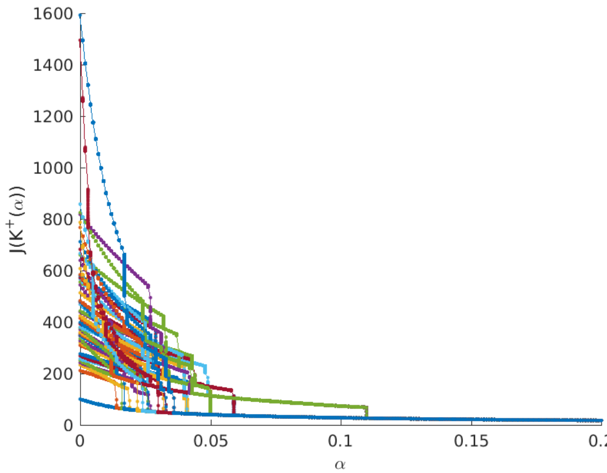
The matrices $P_\alpha(K)$ and $L_\alpha(K)$ are the closed-loop Gramians. The above conditions provide a closed-form expression for the cost

$$J(K, \alpha) = \mathrm{tr}(D_0 P_\alpha(K)), \tag{6}$$

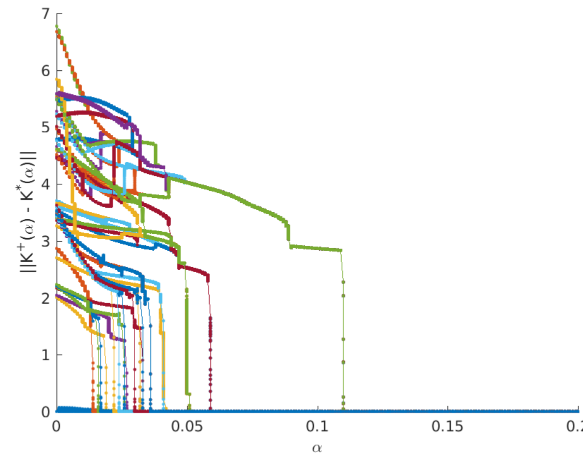
where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. Given α, the equations (5a)-(5d) and (6) are algebraic, involving only polynomial functions of the unknown matrices K, P_α and L_α. The matrices P_α and L_α are written as a function of K because they are uniquely determined from (5a) and (5b) given a stabilizing controller K. When the context is clear, we drop the implicit dependence on K in the notations P_α and L_α.

The paper studies the properties of $K^*(\alpha)$, $K^\dagger(\alpha)$, and $J(K, \alpha)$ for any control K belonging to $K^*(\alpha)$ or $K^\dagger(\alpha)$. To motivate the study of $K^\dagger(\alpha)$, Figure 1 illustrates the evolution of many locally optimal distributed controllers for a particular system as α varies (see Section VI for details on the experiment). It is known that systems of this type have a large number of locally optimal controllers [12]. Figure 1a plots selected trajectories of $J(K, \alpha)$ against α, where $K \in K^\dagger(\alpha)$. The selected trajectories are connected to a stabilizing controller in $K^\dagger(0)$. The lowest curve corresponds to $J(K^*(\alpha), \alpha)$. Figure 1b plots the distance of the selected $K \in K^\dagger(\alpha)$ from the controller $K \in K^*(\alpha)$.

Figure 1 illustrates the property that modest damping causes the locally optimal trajectories to "collapse" to each other. This attractive phenomenon suggests an improvement strategy for ODC by varying the damping parameter and an initialization strategy by continuation from a highly damped ODC problem. The two strategies are detailed in Algorithm 1 and Algorithm 2. Algorithm 1 has the potential to improve the locally optimal controllers obtained from many other methods. The outcome of the algorithm is plotted in Figure 2. Algorithm 2 avoids many unnecessary local optima and has been used in H_2 reduced-order model [17]. Algorithm 2 starts with a large



(a) Locally optimal cost trajectory against the damping parameter



(b) Distance between $K^\dagger(\alpha)$ and $K^*(\alpha)$

Fig. 1. Samples of locally optimal cost and locally optimal controller trajectories of system in equation (27) as the damping parameter α varies.

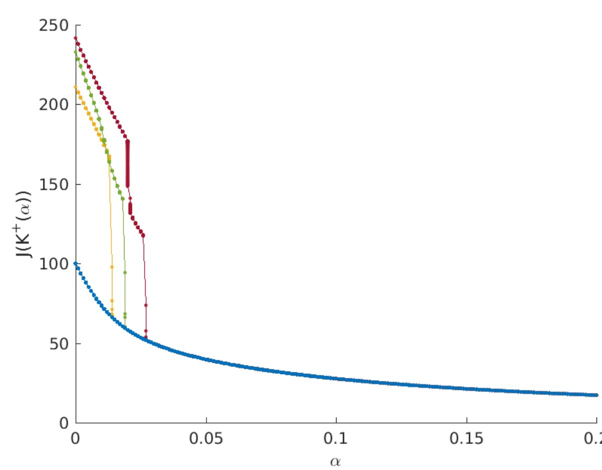


Fig. 2. Selected cost trajectories of Algorithm 1 applied to several locally optimal controllers. The system is described in equation (27). All curves are merged to the blue curve after the damping parameter α is increased beyond 0.05. When decreasing α to 0, no matter where the inital optimal controller is, the algorithm tracks the best blue curve.

Algorithm 1 Improving an Existing Stabilizing Controller: The Forward-Backward Method

Input: $J(K,\alpha)$ and $K_0 \in S$ that stabilizes the system (A,B).
Output: A potentially improved $K_0 \in K^\dagger(0)$.
Select a list of parameters $0=\alpha_0<\alpha_1,\ldots,<\alpha_T$.
for $t \leftarrow 1,\ldots,T$ **do**
  Obtain a $K_t \in K^\dagger(\alpha_t)$ by solving ODC(α_t, K_{t-1}) using local search.
end for
for $t \leftarrow T-1, T-2, \ldots, 0$ **do**
  Obtain a $K_t \in K^\dagger(\alpha_t)$ by solving ODC(α_t, K_{t+1}) using local search.
end for

Algorithm 2 Obtain a Stabilizing Controller: The Backward Method

Input: $J(K,\alpha)$
Output: A potentially stabilizing $K_0 \in K^\dagger(0)$.
Select a list of parameters $0=\alpha_0<\alpha_1,\ldots,<\alpha_T$, where α_T is large enough such that $K_T=0$ stabilizes the system $(A-\alpha_T I, B)$.
for $t \leftarrow T-1, T-2, \ldots, 0$ **do**
  Obtain a $K_t \in K^\dagger(\alpha_t)$ by solving ODC(α_t, K_{t+1}) using local search.
end for

enough α for which $K=0$ is an initial stabilizing controller in the set $\mathcal{S}$ and iteratively solves for a better controller while reducing the damping parameter α. The improvement at $\alpha=\alpha_t$ is achieved using local-search and the initialization K_{t+1} from the previous step. Algorithm 1 is different from Algorithm 2 in that it starts with a potentially undesirable controller for $\alpha=0$ and gradually increases α to obtain an improved optimal controller for a highly-damped system and then applies a variant of Algorithm 2 to backtrack that controller to a globally optimal controller for $\alpha=0$.

The granularity of the of the space for α, namely $\{\alpha_0,\alpha_1,\ldots,\alpha_T\}$, is not essential as long as the discretization step is small enough so that the algorithm can follow the continuous paths. Admittedly, the literature of numerical continuation methods is rich with appealing predictor-corrector and piecewise-linear methods [20], and they can be applied in the tracking of $K^\dagger(\alpha)$ and $K^*(\alpha)$. Nevertheless, the paper aims to analyze the possibility of using local search to locate a better path, as opposed to following all paths closely. The potential improvement of the above strategies with more sophisticated numerical continuation methods is left as a future direction of research.

Due to the NP-hardness of ODC, one cannot expect any guarantee for producing a globally optimal, or even a stabilizing, decentralized controller, unless certain conditions are met, which will be discussed later. The breakdown of these strategies will be discussed in Section VI. In Section III, we first prove the continuity of the trajectories, which is the prerequisite for trajectory tracking.

III. CONTINUITY

This section studies the continuity properties of $K^*(\alpha)$ and $K^\dagger(\alpha)$. The key notion of hemi-continuity captures the evolution of parametrized optimization problems.

Definition 1. *The set valued map $\Gamma:\mathcal{A}\to\mathcal{B}$ is said to be upper hemi-continuous at a point a if for any open neighborhood V of $\Gamma(a)$ there exists a neighborhood U of a such that $\Gamma(U)\subseteq V$.*

A related notion of lower hemi-continuity is provided in Section V. A set-valued map is said to be continuous if it is both upper and lower hemi-continuous. A single-valued function is continuous if and only if it is upper hemi-continuous. We restate a version of Berge Maximum Theorem with a compactness assumption from [21].

Lemma 2 (Berge Maximum Theorem). *Let $\mathcal{A}\subseteq\mathbb{R}$ and $\mathcal{S}\subseteq\mathbb{R}^{m\times n}$. Assume that $J:\mathcal{S}\times\mathcal{A}\to\mathbb{R}$ is jointly continuous and $\Gamma:\mathcal{A}\to\mathcal{S}$ is a compact-valued correspondence. Define*

$$K^*(\alpha)=\arg\min\{J(K,\alpha)|K\in\Gamma(\alpha)\},\ \textit{for}\ \alpha\in\mathcal{A},$$
$$J(K^*(\alpha),\alpha)=\min\{J(K,\alpha)|K\in\Gamma(\alpha)\},\ \textit{for}\ \alpha\in\mathcal{A}.$$

If Γ is continuous at some $\alpha\in\mathcal{A}$, then $J(K^*(\alpha),\alpha)$ is continuous at α. Furthermore, K^* is non-empty, compact-valued, closed, and upper hemi-continuous.

Berge Maximum Theorem does not trivially apply to ODC: the set of stabilizing controllers is open and often unbounded. The difficulty is not essential and can be overcome by restricting the relevant map to a lower level-set.

Theorem 1. *Assume that R_α is continuous in α and that $K^*(0)$ is non-empty. Then, the set $K^*(\alpha)$ is non-empty for all $\alpha>0$. Furthermore, $K^*(\alpha)$ is upper hemi-continuous and the optimal cost $J(K^*(\alpha),\alpha)$ is continuous. If R_α is monotonically decreasing, $J(K^*(\alpha),\alpha)$ is strictly decreasing in α.*

Proof. When $K^*(0)$ is non-empty, there is an optimal decentralized controller for the undamped system. With the set of stabilizing controller non-empty, we can apply $K^*(0)$ to the damped system and conclude that

$$J(K^*(\alpha),\alpha)\le J(K^*(0),\alpha)<\infty.$$

The inequality above assumes the existence of the globally controller for all values of the damping parameter α. This is true because the lower-level set of $J(K,\alpha)$ is compact [22]. Precisely, define $\Gamma_M(\alpha)$ to be

$$\Gamma_M(\alpha)=\{K\in S: A-\alpha I+BK\ \text{stable},\ J(K,\alpha)\le M\}. \quad (7)$$

The set-valued function Γ_M is compact-valued for all fixed α given a fixed M. We select any $M>J(K^*(0),\alpha)$ and optimize $J(K,\alpha)$ instead over $K\in\Gamma_M(\alpha)$ without losing any globally optimal controller. The continuity of $\Gamma_M(\alpha)$ at α for almost all M is proved in Section V. Berge maximum theorem then yields the desired continuity of $K^*(\alpha)$ and $J(K^*(\alpha),\alpha)$. When R_α is monotonically decreasing, the "damping property" ensures that $J(K^*(\alpha),\alpha)$ is monotonically decreasing. □

The above argument can be extended to characterize all locally optimal controllers. A caveat is the possible existence of locally optimal controllers whose costs approaching infinity in the damped problem. Their existence does not contradict the damping property — damping can introduce locally optimal controllers that are not stabilizing without the damping.

Theorem 2. *Assume that R_α is continuous in α and that $K^\dagger(0)$ is non-empty. Then, the set $K^\dagger(\alpha)$ is nonempty for all $\alpha > 0$. Suppose furthermore that at an $\alpha_0 > 0$, we have*

$$\lim_{\epsilon \to 0^+} \sup_{\alpha \in [\alpha_0 - \epsilon, \alpha_0 + \epsilon]} \sup_{K \in K^\dagger(\alpha)} J(K, \alpha) < \infty.$$

Then, $K^\dagger(\alpha)$ is upper hemi-continuous at α_0 and the optimal cost $J(K^\dagger(\alpha), \alpha)$ is upper hemi-continuous at α_0.

Proof. The fact that $K^\dagger(\alpha)$ is non-empty follows from the existence of globally optimal controllers in Theorem 1. Consider the parametrized optimization problem

$$\begin{aligned} \min \quad & \|\nabla J(K, \alpha)\| \\ s.t. \quad & K \in \Gamma_M(\alpha), \end{aligned} \tag{8}$$

where $\|\cdot\|$ denotes the 2-norm of a vector. The assumption of the theorem ensures the existence of a real number M and $\epsilon > 0$ such that $M > J(K, \alpha)$ for $K \in K^\dagger(\alpha)$ where $\alpha \in [\alpha_0 - \epsilon, \alpha_0 + \epsilon]$. This choice of M guarantees that the formulation (8) does not cut off any locally optimal controller. As proved in the Section V, $\Gamma_M(\alpha)$ is continuous at α_0 for almost all M, and a large M can be selected to make $\Gamma_M(\alpha)$ continuous at α_0. Berge Maximum Theorem applies to conclude that $K^\dagger(\alpha)$ is upper hemi-continuous. Since $J(K, \alpha)$ is jointly continuous in (K, α), the map $J(K^\dagger(\alpha), \alpha)$ is upper hemi-continuous. □

IV. Asymptotic Properties

In this section, we state asymptotic properties of the local solutions $K^\dagger(\alpha)$. They shed light on the general shape of the trajectories illustrated in Figure 1.

The following theorem characterizes the evolution of locally optimal controllers for a specific sparsity pattern. It also justifies the practice of random initialization around zero and our initialization strategy in Algorithm 2.

Theorem 3. *Suppose that the sparsity pattern I_S is block-diagonal with square blocks and that R_α has the same sparsity pattern as I_S for all α. If the smallest eigenvalue of R_α is bounded away from zero for all α, then, all points in $K^\dagger$ converge to the zero matrix as $\alpha \to \infty$. Furthermore, if R_α is monotonically decreasing, then $J(K, \alpha) \to 0$ as $\alpha \to \infty$ for all $K \in K^\dagger(\alpha)$.*

Proof. Refer to Section V. □

Not only do all locally optimal controllers approach zero, the problem is also convex over bounded regions with enough damping. We use $\|K\|$ to denote the operator 2-norm of the matrix K, which is equal to K's largest singular value.

Theorem 4. *Suppose that the condition number of R_α is uniformly bounded for all $\alpha \geq 0$. Then, for any given $r > 0$, the Hessian matrix $\nabla^2 J(K, \alpha)$ is positive definite over $\|K\| \leq r$ for all large α.*

Proof. Refer to Section V. □

Corollary 1. *Under the assumption of Theorem 3 and Theorem 4, there is no spurious locally optimal controller for large α, that is, $K^\dagger(\alpha) = K^*(\alpha)$ for all large values of α.*

Proof. For any given $r > 0$, all controllers in the ball $\mathcal{B} = \{K : \|K\| \leq r\}$ are stabilizing when α is large. As a result, stability constraints can be relaxed over $\mathcal{B}$. Furthermore, from Theorem 3, when α is large, all locally optimal controllers will be inside $\mathcal{B}$. From Theorem 4, the objective function becomes convex over $\mathcal{B}$ for large enough α. These observations imply that local and global solutions coincide. □

Note that with damping, the regularization matrix R_α does not need to go to infinity in order to convexify the problem. Corollary 1 implies that with a large damping and a well-conditioned R_α, the problem is tractable.

Corollary 2. *Under the same assumption of Theorem 3 and Theorem 4, suppose further that the globally optimal solution is unique for all damping parameters, namely, $K^*(\alpha)$ is a singleton set for all $\alpha \geq 0$. Then, the trajectory $K^*(\alpha)$ is continuous. Moreover, if there is an $\epsilon > 0$ such that the local search method initialized at ϵ distance away from $K^*(\alpha)$ converges to $K^*(\alpha)$, then Algorithm 1 and Algorithm 2 output the globally optimal stabilizing controller in $K^*(0)$ with a proper discretization of the α space.*

A proper discretization $0 = \alpha_0 < \alpha_1, \ldots, < \alpha_T$ has a large α_T for which the "no spurious property" of Corollary 1 holds. A proper discretization further requires α_t and α_{t+1} to be reasonably close to guarantee that the local search method initialized at K_{t+1} is able to converge to K_t in Algorithm 1 and Algorithm 2.

Proof. We have shown in Theorem 1 that $K^*(\alpha)$ is upper hemi-continuous. With the singleton assumption, we conclude the continuity of $K^*(\alpha)$ because a single-valued function is continuous if and only if it is upper hemi-continuous. We choose a discretization $0 = \alpha_0 < \alpha_1, \ldots, < \alpha_T$, where α_T is large enough for which the "no spurious property" of Corollary 1 holds. As a result, Algorithm 1 and Algorithm 2 are able to locate the continuous globally optimal trajectory $K^*(\alpha)$ at $\alpha = \alpha_T$. To obtain $K^*(0)$, we follow the continuous $K^*(\alpha)$ in the manner of Algorithm 1 and Algorithm 2, where α_t and α_{t+1} are close enough so K_{t+1} lies in the region where the local search method initialized at K_{t+1} converges to K_t. This discretization inductively yields a serious of controllers K_t, for $t = T, T-1, \ldots, 0$ that all lie on the path $K^*(\alpha)$, for $\alpha \in [0, \alpha_T]$. □

All the theorems above rely on the "damping property" in Lemma 1. It is worth commenting that damping the system with $-I$ is almost the only continuation method for general system matrices "A" that achieves the monotonic increasing of stable sets. This will be formalized below.

Theorem 5. *When $n \geq 3$, for any n-by-n real matrix H that is not a multiple of $-I$, there exists a stable matrix A for which $A + H$ is unstable.*

The proof is given in Section V. This theorem justifies the use of $-\alpha I$ as the continuation parameter and is the reason that our setting avoids the undesirable behaviors of homotopy documented in [18]. Note, however, matrices other than $-I$ may be used if the system is structured; if A has certain structures, there are non-trivial matrices H for which $A + tH$ is always stable when $t > 0$.

A. Discrete-time Stochastic Systems

We detour briefly to discuss damping with varying regularization in discrete-time stochastic systems. This shall illustrate the difference between discrete- and continuous-time systems. Consider the stochastic system

$$x[t+1] = Ax[t] + Bu[t] + d[t]$$

under a static feedback policy $u[t] = Kx[t]$, where K is to be designed such that the damped objective

$$J(K, \alpha) = \lim_{t\to\infty} \mathbb{E}\left[\alpha^{2t}\left(x[t]^\top Q x[t] + u[t]^\top R_\alpha u[t]\right)\right]$$

is minimized. The damping parameter α belongs to the interval $[0, 1]$. Assume that the random variables $d[t], t = 0, 1, 2, \ldots,$ are independent and $d[t]$ has the covariance matrix $\Sigma_{\alpha,d}[t]$. After closing the loop, one can write

$$x[t+1] = (A + BK)^{t+1}x[0] + \sum_{\tau=0}^{t}(A + BK)^{(t-\tau)}d[\tau].$$

When $\|\alpha A + \alpha BK\| < 1$, we have

$$\begin{aligned} J(K, \alpha) &= \lim_{t\to\infty} \mathbb{E}\,\mathrm{tr}[(Q + K^\top R_\alpha K)x[t]x[t]^\top \alpha^{2t}] \\ &= \mathrm{tr}\Bigg[(Q + K^\top R_\alpha K)\cdot \\ &\lim_{t\to\infty}\sum_{\tau=0}^{t}(\alpha A+\alpha BK)^{t-\tau}\Sigma_{\alpha,d}[\tau]\alpha^{2\tau}(\alpha A+\alpha BK)^{\top(t-\tau)}\Bigg]. \end{aligned}$$

Assuming that $\Sigma_{\alpha,d}[\tau]\alpha^{2\tau} = \Sigma_d$, we have the simplified expression of the problem as follows,

$$\begin{aligned} &\min_K J(K, \alpha) = \mathrm{tr}[(K^\top R_\alpha K + Q)P_\alpha(K)], \\ &s.t.\ (\alpha A+\alpha BK)P_\alpha(K)(\alpha A+\alpha BK)^\top - P_\alpha(K) + \Sigma_d = 0, \\ &\quad \alpha\|(A + BK)\| < 1. \end{aligned} \tag{9}$$

Note that we scaled the matrices A, B and the covariances matrices at the same time. Moreover, the formulation is not linear in K or in P_α. Still, we deduce weaker asymptotic results with an additional bounded assumption. The proof of the lemma is given in Section V. We use $\lambda_{\min}(\cdot)$ to denote the minimum eigenvalue of a symmetric matrix.

Lemma 3. *Suppose that $\lambda_{\min}(R_\alpha) \geq \epsilon > 0$ for all $\alpha \in [0, 1]$. Assume further that a locally optimal solution K_α to* (9) *exists and is uniformly bounded for all $\alpha \in [0, 1]$. Then, as $\alpha \to 0$, it holds that $P_\alpha(K_\alpha) \to \Sigma_d$ and $K_\alpha \to 0$.*

The above lemma suggests an analogue of Algorithm 1 and Algorithm 2 in the discrete setting, where the damping parameter α is discretalized over $[0, 1]$.

V. Proofs

This section collects the proofs of the results in the previous sections.

Lemma 4 and Lemma 5 below prove the continuity of the lower level-set map Γ_M defined in (7). The continuity of Γ_M is the pre-requisite for applying the Berge Maximum Theorem. The reader is referred to [21] for an accessible treatment of relevant definitions.

Recall the notion of upper hemi-continuity of a set valued map $\Gamma : A \to B$ in Definition 1. If B is compact, upper hemi-continuity is equivalent to the graph of Γ being closed, that is, if $a_n \to a^*$ and $b_n \in \Gamma(a_n) \to b^*$, then $b^* \in \Gamma(a^*)$. Lemma 4 resolves the upper hemi-continuity of Γ_M.

Lemma 4. *Assume that R_α is continuous in α and that for a given $M > 0$, $\Gamma_M(\alpha)$ is not empty for all $\alpha \geq 0$. Then, $\Gamma_M(\alpha)$ is an upper hemi-continuous set-valued map.*

Proof. From [22], $\Gamma_M(\alpha)$ is compact for all α. To characterize the continuity of Γ at a point $\alpha^* \geq 0$, it suffices to assume that the range of Γ_M is compact and, therefore, the sequence characterization of upper hemi-continuity applies. Suppose that $\alpha_i \to \alpha^*$, select a sequence of $K_i \in \Gamma_M(\alpha_i)$ that converges to K^*. The continuity of $J(K, \alpha)$ implies $J(K^*, \alpha^*) \leq M$. The fact that the cost is bounded implies that $A - \alpha^* I + BK^*$ is stable. Since subspaces of matrices are closed, $K^* \in \mathcal{S}$. We have verified all conditions for $K^* \in \Gamma_M(\alpha^*)$, and therefore Γ_M is upper hemi-continuous. □

A complementary notion of upper hemi-continuity is lower hemi-continuity, which is stated below.

Definition 2. *The set valued map $\Gamma : A \to B$ is said to be lower hemi-continuous at a point a if for any open neighborhood V intersecting $\Gamma(a)$ there exists a neighborhood U of a such that $\Gamma(x)$ intersects V for all $x \in U$.*

Equivalently, for all $a_m \to a \in A$ and $b \in \Gamma(a)$, there exists a_{m_k} subsequence of a_m and a corresponding $b_k \in \Gamma(a_{m_k})$, such that $b_k \to b$. The map Γ_M is lower hemi-continuous for almost all M.

Lemma 5. *At any given $\alpha^* \geq 0$, $\Gamma_M(\alpha)$ is lower hemi-continuous at α^* except when $M \in \{J(K, \alpha^*) : K \in K^\dagger(\alpha^*)\}$, which is a finite set of locally optimal costs.*

Proof. To prove by contradiction, consider a sequence $\alpha_i \to \alpha^*$ and a matrix $K^* \in \Gamma_M(\alpha^*)$, for which there exists no subsequence of α_i and $K_i \in \Gamma_M(\alpha_i)$ such that $K_i \to K^*$. We must have

- $J(K^*, \alpha^*) = M$ — otherwise by the continuity of J, $J(K^*, \alpha_i) < M$ for large i and, since the set of stabilizing controllers is open, $K^* \in \Gamma_M(\alpha_i)$ for large i, which is a contradiction.
- K^* must be a local minimum of $J(K, \alpha^*)$ — otherwise there exists a sequence $K_j \to K^*$ with $J(K_j, \alpha^*) < M$ and, by the continuity of J, there exists a sequence of large enough indices $n_j, j = 1, 2, \ldots,$ such that $J(K_j, \alpha_{n_j}) < M$; the sequence $K_j \in \Gamma_M(\alpha_{n_j})$ converges to K^*.

The argument above implies that M is the cost of some locally optimal controllers at α^*. Because given α^*, $J(K, \alpha^*)$ can be described as a linear function in terms of K over an algebraic set given by (6), the cost of locally optimal controller can take finitely many values. □

Proof of Theorem 3. Recall the expression of the objective function (2), the first-order necessary conditions (5a)-(5d), and (6). As α increases, some local solutions may disappear, some new local solutions may appear. The appearance cannot occur infinitely often because the equations (5a)-(5d) are algebraic. Suppose that when α is greater than α_0, the number of local solutions does not change. The damping property ensures the following for $\beta > \alpha > \alpha_0$:

$$\max_{K\in K^\dagger(\beta)} J(K,\beta) \leq \max_{K\in K^\dagger(\alpha)} J(K,\beta)$$

The right-hand side of the above inequality optimizes over a fixed, finite set of controllers and approaches zero as $\beta \to \infty$ due to (2) and the dominated convergence theorem. The left-hand side, therefore, also converges to zero as $\beta \to \infty$. From (6) and the assumption that D_0 is positive definite, we have $\|P_\beta(K)\| \to 0$ for all $K \in K^\dagger(\beta)$ as $\beta \to \infty$.

The assumption on sparsity allows the expression of the locally optimal controllers in (5c) as

$$K = -R_\alpha^{-1}((B^\top P_\alpha(K)L_\alpha(K)) \circ I_S)(L_\alpha(K) \circ I_S)^{-1}.$$

Especially, we bound

$$\|BK\| \leq e_\alpha(K) \cdot \lambda_{\min}(L_\alpha(K))^{-1},$$

where

$$e_\alpha(K) = \|BR_\alpha^{-1}\| \cdot \|B^\top P_\alpha(K)L_\alpha(K)\|.$$

The term $\|BR_\alpha^{-1}\|$ is bounded due to the assumption that the minimum eigenvalue of R_α is bounded away from zero. Pre- and post-multiply (5b) by the unit eigenvector v of the smallest eigenvalue of $L_\alpha(K)$ yields

$$\lambda_{\min}(L_\alpha(K))(2a - 2v^\top(A+BK)v) = v^\top D_0 v. \tag{10}$$

Therefore,

$$\begin{aligned}\lambda_{\min}(L_\alpha(K)) &\geq \frac{\lambda_{\min}(D_0)}{2\alpha + 2\|A+BK\|} \\ &\geq \frac{\lambda_{\min}(D_0)}{2\alpha + 2\|A\| + 2\|BK\|} \\ &\geq \frac{\lambda_{\min}(D_0)}{2\alpha + 2\|A\| + 2e_\alpha(K)\lambda_{\min}(L_\alpha(K))^{-1}},\end{aligned}$$

which simplifies to

$$\lambda_{\min}(L_\alpha(K)) \geq \frac{\lambda_{\min}(D_0) - 2e_\alpha(K)}{(2\alpha + 2\|A\|)} \tag{11}$$

Take the trace of (5b), consider the estimate

$$\begin{aligned}&2n\|A\|\|L_\alpha\| + \operatorname{tr}(D_0) \geq 2\|A\| \operatorname{tr}(L_\alpha) + \operatorname{tr}(D_0) \\ &\geq 2\alpha \operatorname{tr}(L_\alpha) + 2\operatorname{tr}[BR_\alpha^{-1}((B^\top P_\alpha L_\alpha)\circ I_S)(L_\alpha \circ I_S)^{-1}L_\alpha] \\ &\geq 2\alpha \operatorname{tr}(L_\alpha) - 2e_\alpha(K)\operatorname{tr}[(L_\alpha \circ I_S)^{-1}L_\alpha] \\ &= 2\alpha \operatorname{tr}(L_\alpha) - 2e_\alpha(K)n \\ &\geq 2\alpha\|L_\alpha\| - 2n\|BR_\alpha^{-1}\|\|B^\top\|\|P_\alpha\|\|L_\alpha\|,\end{aligned} \tag{12}$$

where for clarity we drop the implicit dependence on K in L_α and P_α. The second and the third inequalities use the bound $|\operatorname{tr}(AL)| \leq \|A\| \operatorname{tr}(L)$ for a positive definite matrix L and any matrix A. The next equality in the above sequence follows from the assumption that I_S is block diagonal. The estimate (12), combined with the previous argument that $\|P_\alpha\| \to 0$, implies that $\|L_\alpha\| \to 0$ and thereby, $e_\alpha(K) \to 0$. The inequality (12) further implies

$$\|L_\alpha\| \leq \frac{\operatorname{tr}(D_0)}{2a - 2n\|A\| - 2n\|BR_\alpha^{-1}\|\|B^\top\|\|P_\alpha\|}, \tag{13}$$

for a small enough P_α. Combining (11) and (13) leads to

$$\begin{aligned}\|K\| &\leq \|R_\alpha^{-1}\| \cdot \|(B^\top P_\alpha L_\alpha)\circ I_S\| \cdot \|(L_\alpha \circ I_S)^{-1}\| \\ &\leq \|R_\alpha^{-1}\| \cdot \|B^\top\| \cdot \|P_\alpha\| \cdot \|L_\alpha\| \cdot \lambda_{\min}(L_\alpha)^{-1} \\ &\leq \|R_\alpha^{-1}\| \cdot \|B^\top\| \cdot \|P_\alpha\| \\ &\quad\times \frac{\operatorname{tr}(D_0)}{2\alpha - 2n\|A\| - 2n\|BR_\alpha^{-1}\|\|B^\top\|\|\|P_\alpha\|} \\ &\quad\times \frac{(2\alpha + 2\|A\|)}{\lambda_{\min}(D_0) - 2e_\alpha(K)},\end{aligned}$$

which converges to 0 as $\alpha \to \infty$. □

Proof of Theorem 4. We use $\otimes$ to denote the Kronecker project of two matrices and vec to denote the vectorized operation that stack the columns of a matrix together into a vector. We make use of the vectorized Hessian formula in the following lemma.

Lemma 6 (From [19])**.** *Define $j_\alpha : \mathbb{R}^{m\cdot n} \to \mathbb{R}$ by $j_\alpha(vec(K)) = J(K,\alpha)$. The Hessian of j_α is given by the formula*

$$H_\alpha(K) = 2\left\{(L_\alpha(K) \otimes R_\alpha) + G_\alpha(K)^\top + G_\alpha(K)\right\}, \tag{14}$$

where

$$\begin{aligned}G_\alpha(K) =&[I \otimes (B^\top P_\alpha(K) + R_\alpha K)]\times \\ &[I \otimes (A - \alpha I + BK) + (A - \alpha I + BK) \otimes I]^{-1} \\ &(I_{n,n} + P(n,n))[L_\alpha(K) \otimes B]\end{aligned}$$

and $P(n,n)$ is an $n^2 \times n^2$ permutation matrix.

We first show that $H_\alpha(K)$ in Lemma 6 is positive definite for any fixed K when α is large. Recall the definition of L_α and P_α in (5a)-(5b) and apply the triangle inequality:

$$\begin{aligned}2\alpha\|L_\alpha(K)\| &\leq \|D_0\| + 2\|A+BK\|\|L_\alpha(K)\|, \\ 2\alpha\|P_\alpha(K)\| &\leq \|Q\| + 2\|A+BK\|\|P_\alpha(K)\| + \|R_\alpha\|\|K\|^2.\end{aligned}$$

The above inequalities imply $\|P_\alpha(K)\|/\|R_\alpha\| \to 0$ and $\|L_\alpha(K)\| \to 0$ as $\alpha \to \infty$. We now bound the minimum eigenvalue of $L_\alpha(K)$. Let v be the unit eigenvector of $L_\alpha(K)$ corresponding to $\lambda_{\min}(L_\alpha(K))$; pre- and post-multiply (5b) by v; we obtain

$$\begin{aligned}\lambda_{\min}(L_\alpha(K)) &\geq \frac{v^\top D_0 v}{2\alpha - 2v^\top(A+BK)v} \\ &\geq \frac{\lambda_{\min}(D_0)}{2\alpha + 2\|A+BK\|}.\end{aligned} \tag{15}$$

The first Hessian term $L_\alpha(K) \otimes R_\alpha$ in (14) can be lower bounded with (15). Due to the assumption that R_α has a uniformly bounded condition number, there exists a constant $\delta > 0$ such that $\lambda_{\min}(R_\alpha)/\|R_\alpha\| \geq \delta$ for all $\alpha \geq 0$. Therefore,

$$\begin{aligned}\lambda_{\min}\left(L_\alpha(K) \otimes R_\alpha\right) &= \lambda_{\min}(L_\alpha(K)) \cdot \lambda_{\min}(R_\alpha) \\ &\geq \frac{\lambda_{\min}(D_0)}{2\alpha + 2\|A+BK\|} \cdot \delta \cdot \|R_\alpha\|.\end{aligned}$$

We bound the norm of the second and the third Hessian terms $\|G_\alpha(K)\|$ as follows:

$$\begin{aligned}&\|G_\alpha(K)\| \leq \|I \otimes (B^\top P_\alpha(K) + R_\alpha K)\| \\ &\quad \times \| \left[I \otimes (A - \alpha I + BK) + (A - \alpha I + BK) \otimes I\right]^{-1} \| \\ &\quad \times \| \left[I_{n,n} + P(n,n)\right] \left[L_\alpha(K) \otimes B\right] \| \\ &\lesssim \|R_\alpha\|(1 + \|P_\alpha\|/\|R_\alpha\|) \times \\ &\quad (-\lambda_{\max}\left(I \otimes (A - \alpha I + BK) + (A - \alpha I + BK) \otimes I\right))^{-1} \times \\ &\quad \|L_\alpha(K)\| \\ &\lesssim \|R_\alpha\|(2\alpha)^{-1}\|L_\alpha(K)\|,\end{aligned}$$

where $\lesssim$ hides constants that do not depend on α. Comparing the two estimates above, we find that the first term $L_\alpha(K) \otimes R_\alpha$ in (14) dominates the following $G_\alpha(K)^\top + G_\alpha(K)$ with a large α for all bounded K. Therefore, the Hessian $H_\alpha(K)$ is positive definite over bounded K when α is large. Note that $H_\alpha(K)$ is the Hessian of the objective function when the controller is centralized. The conclusion carries over the decentralized controller because the Hessian for the decentralized controller is a principal sub-matrix of the Hessian for the centralized controller. □

Proof of Lemma 3. We use the Einstein notation where subscript variables that appear twice in a monomial are summed over and the subscripts that appear once are free over the corresponding set of indices. We use the lower-case letters to denote the entries of the corresponding upper-case letter matrices and write $A = (a_{ij}), B = (b_{ij}), K_\alpha = (k_{ij}), \Sigma_d = (\sigma_{ij}), P_\alpha = (p_{ij}), R_\alpha = (r_{ij}), Q = (q_{ij})$. The optimal solution K_α satisfies the first-order necessary condition to be derived below:

$$\begin{aligned}0 = \frac{\partial J}{\partial k_{ij}} &= \frac{\partial[(k_{ba}r_{bc}k_{cd} + q_{ad})p_{ad}]}{\partial k_{ij}} \\ &= (r_{ic}k_{cd})p_{jd} + (k_{ba}r_{bi})p_{aj} + (k_{ba}r_{bc}k_{cd} + q_{ad})\frac{\partial p_{ad}}{\partial k_{ij}}.\end{aligned} \tag{16}$$

The constraints in (9) may be written as

$$\alpha^2(a_{ab} + b_{ac}k_{cb})p_{bd}(a_{ed} + b_{ef}k_{fd}) - p_{ae} + \sigma_{ae} = 0 \tag{17}$$

Taking its partial derivatives of k_{ij} yields

$$\begin{aligned}&2\alpha^2 b_{ai}p_{jd}(a_{ed} + b_{ef}k_{fd}) + \\ &\alpha^2(a_{ab} + b_{ac}k_{cb})\frac{\partial p_{bd}}{\partial k_{ij}}(a_{ed} + b_{ef}k_{fd}) - \frac{\partial p_{ae}}{\partial k_{ij}} = 0\end{aligned} \tag{18}$$

By assumption, the entries of the controller k_{ij} are bounded as $\alpha \to 0$. Hence, (17) implies that $P_\alpha(K_\alpha) \to \Sigma_d$ as $\alpha \to 0$ and is consequently bounded. This, combined with (18), implies that the partial derivatives of $P_\alpha(K)$ with respect to K vanish as $\alpha \to 0$. This implies that the first two terms in (16), which are both $R_\alpha K_\alpha P_\alpha(K)^\top$ in matrix form, converge to zero. Because $P_\alpha(K)$ and R_α are invertible, $K_\alpha \to 0$ as $\alpha \to 0$. □

To prove Theorem 5, define the set of stable directions as

$$\mathcal{H} = \{H : A + tH \text{ is stable for all stable } A \text{ and } t \geq 0\}, \tag{19}$$

where A and H are n-by-n real matrices.

Lemma 7. *All matrices in $\mathcal{H}$ are similar to a diagonal matrix with non-positive diagonal entries. Especially, they cannot have complex eigenvalues.*

Proof. When t is large, $A + tH$ is a small perturbation of tH. Thus, the eigenvalues of H must be in the closed left half-plane. With a suitable similar transformation, assume that H is in the real Jordan form. We first consider the case when the dimension $n = 2$, and we emphasize the dimension in the subscript in H_2 and A_2. To prove for contradiction, assume that H_2 is not diagonalizable. The non-diagonal real Jordan form of H_2 has the following possibilities:

- $H_2 = \begin{bmatrix} h & 1 \\ 0 & h \end{bmatrix}$, where H_2 has real eigenvalue $h < 0$ of multiplicity 2: Let $A_2 = \begin{bmatrix} 4h & -2 \\ 10h^2 & -3h \end{bmatrix}$, which is stable because $\operatorname{tr}(A_2) = h < 0$ and $\det(A_2) = 8h^2 > 0$. We have $A_2 + tH_2 = \begin{bmatrix} ht + 4hby & t - 2 \\ 10h^2 & ht - 3h \end{bmatrix}$, whose stability criteria $\operatorname{tr}(A_2 + tH_2) < 0$ and $\det(A_2 + tH_2) > 0$ amount to
$$2ht + h < 0 \text{ and } h^2(t^2 - 9t + 8) > 0,$$
or equivalently $t \in (-1/2, 1) \cup (8, +\infty)$. In particular, when $t = 2$, the matrix $A_2 + tH_2$ is not stable.
- $H_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$: Consider the stable matrix $A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$, for which $A_2 + tH_2$ is not stable when $t = 2$.
- $H_2 = \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix}$, where $f > 0$: by selecting $A_2 = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$, the matrix $A_2 + \frac{2}{f}H_2 = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$ is not stable.
- $H_2 = \begin{bmatrix} h & f \\ -f & h \end{bmatrix}$, where $h < 0$ and $f > 0$: by rescaling, that assume $f = 1$. Consider the matrix function
$$G(t) = \begin{bmatrix} 0 & \frac{1}{2} + (u+w)h \\ -\frac{1}{2} + (u-w)h & h \end{bmatrix} + t\begin{bmatrix} h & 1 \\ -1 & h \end{bmatrix}. \tag{20}$$

We have

$$\begin{aligned}\operatorname{tr}(G(t)) &= h + 2ht, \\ \det(G(t)) &= (1 + h^2)t^2 + (1 + h^2 + 2hw)t \\ &\quad + h^2(w^2 - u^2) + hw + \frac{1}{4}.\end{aligned}$$

Therefore,

$$\operatorname{tr}(G(-\frac{1}{2})) = 0,$$
$$\frac{d}{dt}\operatorname{tr} G(t) = 2h,$$
$$\det(G(-\frac{1}{2})) = h^2(-\frac{1}{4} - u^2 + w^2),$$
$$\frac{d}{dt}\det G(t)\Big|_{t=-\frac{1}{2}} = 2hw.$$

Hence, as long as

$$w > 0 \text{ and } -\frac{1}{4} - u^2 + w^2 > 0, \tag{21}$$

for a small enough $\epsilon > 0$, the matrix $A_2 = G(-\frac{1}{2}+\epsilon)$ is a stable matrix and there is a matrix $G(t)$ with $t > -\frac{1}{2}$ whose trace is negative and whose determinant is smaller than $\det(A_2)$. Consider the minimal value of $\det G(t)$, which is obtained at $-\frac{1}{2} - \frac{hw}{1+h^2}$,

$$\det G\left(-\frac{1}{2}-\frac{hw}{1+h^2}\right) = h^2\left(-\frac{1}{4}-u^2+\frac{h^2}{1+h^2}w^2\right).$$

As a result, when

$$-\frac{1}{4} - u^2 + \frac{h^2}{1+h^2}w^2 < 0, \tag{22}$$

the matrix $G(t)$ with $t = -\frac{1}{2} - \frac{hw}{1+h^2}$ is unstable. The parameters u and w that satisfy (21) and (22) always exist.

For the higher dimension $n > 2$, the real Jordan form of H is a block upper-triangular matrix

$$H = \begin{bmatrix} H_2 & * \\ 0 & * \end{bmatrix},$$

where H_2 can take the four possibilities mentioned above ("$*$" denotes an arbitrary sub-matrix). We take the corresponding stable A_2 constructed above, which has the property that $A_2 + t_0H_2$ is not stable for some $t_0 > 0$. Select a block diagonal matrix

$$A = \begin{bmatrix} A_2 & 0 \\ 0 & -I \end{bmatrix}.$$

Then, A is stable, while $A + t_0H = \begin{bmatrix} A_2 + t_0H_2 & * \\ 0 & * \end{bmatrix}$ is unstable. □

We can strengthen the argument above and further characterize $\mathcal{H}$ in the case $n \geq 3$.

Lemma 8. *When $n \geq 3$, the set of stable directions $\mathcal{H}$ does not contain any matrices of rank 1, 2, …, $n-2$.*

Proof. From lemma 7, it suffices to consider a diagonal matrix H with negative diagonal entries. Assume that there is an $H \in \mathcal{H}$ whose rank is in $\{1, 2, \ldots, n-2\}$, write

$$H = \begin{bmatrix} H_3 & 0 \\ 0 & * \end{bmatrix},$$

where $H_3 = \operatorname{diag}(-1, 0, 0)$. We will construct a stable 3-by-3 matrix A_3 such that $A_3 + t_0H_3$ is unstable for some $t_0 > 0$, and then carry the instability to $A + t_0H$ with the extended matrix

$$A = \begin{bmatrix} A_3 & 0 \\ 0 & -I \end{bmatrix}.$$

From [12], the set

$$T = \left\{ t : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 0.85 & 0.2 & 0.2 \end{bmatrix} \text{ is stable} \right\}$$

has two disconnected components. Consider the Jordan decomposition of the matrix

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 0.85 & 0.2 & 0.2 \end{bmatrix} = W \operatorname{diag}(-0.2, 0, 0)W^{-1},$$

where W is some invertible matrix. Write

$$G(t) = 5W^{-1}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & -1 \end{bmatrix} W + t \times \operatorname{diag}(-1, 0, 0).$$

After this similar transformation, the set T can be written in terms of $G(t)$ as

$$T = \{t : G(t) \text{ is stable}\}.$$

Since T is disconnected, there exists some $t_1 < t_2$ such that $G(t_1)$ is stable while $G(t_2)$ is unstable with some eigenvalue in the open right half-plane. Setting $A_3 = G(t_1)$ and $t_0 = t_2 - t_1$ completes the proof. □

Since we can perturb the direction and make H full-rank, the restrictions on the rank of H is not essential. The following lemma confirms this observation, and it completes the proof of Theorem 5.

Lemma 9. *When $n \geq 3$, $\mathcal{H} = \{-\lambda I, \lambda \geq 0\}$.*

Proof. From lemma 7, it suffices to consider the case where H is diagonal with negative diagonal entries. Write

$$H = \begin{bmatrix} H_3 & 0 \\ 0 & * \end{bmatrix},$$

where $H_3 = \operatorname{diag}(h_1, h_2, h_3)$. The diagonal entries $h_i, i = 1, 2, 3$ are non-positive and not all equal. We will construct a stable A_3 and a corresponding t_0 such that $A_3 + t_0H_3$ is not stable, and extend to the general A as in Lemma 8. The case with a rank-1 matrix H_3 has been considered in Lemma 8. In what follows we prove the case for rank-2 and rank-3 matrix H_3. Without loss of generality we rescale H_3 and assume that $h_1 = -1$, consider the following two standard forms of H_3:

- $H_3 = \operatorname{diag}(-1, h_2, 0)$, where $h_2 < 0$. Consider the matrix function

$$G(t) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -h_2 \\ 2 & 1 & 0 \end{bmatrix} + tH_3 = \begin{bmatrix} -t & -1 & 0 \\ 0 & th_2 & -h_2 \\ 2 & 1 & 0 \end{bmatrix}.$$

  The characteristic polynomial of $G(t)$, which we denote by $\phi_{G(t)}(x)$, can be written as

$$\phi_{G(t)}(x) = x^3 + (t - th_2)x^2 + (h_2 - t^2h_2)x + (t-2)h_2.$$

The Routh-Hurwitz Criterion states that the stability of $G(t)$ is equivalent to the following system of inequalities:

$$t(1-h_2) > 0,$$
$$(t-2)h_2 > 0,$$
$$t(1-h_2)h_2(1-t^2) > (t-2)h_2.$$

which can be simplified with $h_2 < 0$ to

$$0 < t < 2, \tag{23a}$$
$$(1-h_2)t^3 + th_2 - 2 > 0. \tag{23b}$$

When $t = \frac{3}{2}$, (23b) simplifies to the obvious expression $\frac{1}{8}(11 - 15h_2) > 0$; when $t = 3$, (23a) implies that $G(t)$ is not stable. Setting $A_3 = G(\frac{3}{2})$ and $t_0 = \frac{3}{2}$ completes the proof.

- $H_3 = \operatorname{diag}(-1, h_2, h_3)$, where without loss of generally we assume that

$$-1 \leq h_2, h_3 < 0, \text{ and one of them is not } -1. \tag{24}$$

Consider the matrix

$$G(t) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & h_2 \\ ah_3 & h_3 & 0 \end{bmatrix} + tH_3 = \begin{bmatrix} -t & -1 & 0 \\ 0 & th_2 & h_2 \\ ah_3 & h_3 & th_3 \end{bmatrix}.$$

The Routh-Hurwitz Criterion states that the stability of $G(t)$ is equivalent to the following system of inequalities:

$$t > 0, \tag{25a}$$
$$f_1(t) = a - t + t^3 > 0, \tag{25b}$$
$$f_2(t) = -ah_2h_3 + th_2h_3(h_2+h_3) + t^3(1-h_2)(1-h_3)(-h_2-h_3) > 0. \tag{25c}$$

We claim that when

$$\sqrt{\frac{h_2h_3(h_2+h_3)^2}{(-h_2-h_3+h_2h_3)^3}} < a < \sqrt{\frac{4}{27}}, \tag{26}$$

the set of t that satisfy the Routh-Hurwitz Criterion is disconnected. To prove this, we write the positive local minimum of $f_1(t)$ in (25b) as $t_1 = \sqrt{\frac{1}{3}}$ and write the positive local minimum of $f_2(t)$ in (25c) as $t_2 = \sqrt{\frac{h_2h_3}{3(1-h_1)(1-h_2)}}$. The condition (24) implies that $t_1 < t_2$ and the condition (26) implies that $f_1(t_1)$ and $f_2(t_2)$ are negative. Furthermore, consider $t_0 = a\frac{h_2+h_3-h_2h_3}{h_2+h_3}$, which is the root of $(1-h_2)(1-h_3)(-h_2-h_3)f_1(t) - f_2(t)$. It holds that $t_1 < t_0 < t_2$ and both $f_1(t_0)$ and $f_2(t_0)$ are positive. We conclude that when $t = t_0$, the matrix $G(t_0)$ is stable, and when t is large, $G(t)$ is again stable. Yet, when $t = t_2 \in (t_0, \infty)$, the matrix $G(t_2)$ is not stable.

□

VI. Numerical Experiments

In this section, we catalogue various homotopy behaviors as the damping parameter α varies. The focus is on the evolution of locally optimal trajectories, which can be tracked by any local search or path-following methods. The experiments are performed on small-sized systems so the random initialization can find a reasonable number of distinct locally optimal solutions. Despite the small system dimension, the existence of many locally optimal solutions and their convoluted trajectories demonstrates the power and the limit of using homotopy methods in optimal decentralized control.

For the local search method, we use the projected gradient descent. At a controller K^i, we perform line search along the direction $\tilde{K}^i = -\nabla J(K) \circ I_S$. The step size is determined with backtracking and Armijo rule, namely, we select s^i as the largest number in $\{\bar{s}, \bar{s}\beta, \bar{s}\beta^2, ...\}$ such that $K^i + s^i\tilde{K}^i$ is stabilizing while

$$J(K^i + s^i\tilde{K}^i) < J(K^i) + \gamma s^i \langle \nabla J(K^i), \tilde{K}^i \rangle.$$

We select the parameters $\gamma = 0.001$, $\beta = 0.5$, and $\bar{s} = 1$. We terminate the iteration when the norm of the gradient is less than 10^{-2}.

A. Systems with a Large Number of Local Minima

We first consider the examples from [12], where the feasible set is highly disconnected and admits many local minima. The system matrices are

$$A = \begin{bmatrix} -1 & 2 & 0 & & \\ -2 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 2 & \ddots \\ & 0 & -2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & & \\ -1 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 1 & \ddots \\ & 0 & -1 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{27}$$
$$D_0 = I, \quad I_{\mathcal{S}} = I, \quad Q = I, \quad R_\alpha = I.$$

When the dimension n is equal to 9, it is known that the set of stabilizing decentralized controllers has at least 55 connected components, each of them containing at least one locally optimal controller. We track 50 of those locally optimal solutions. The damping parameter α is gradually increased from 0 to 0.2 with a 0.001 increment. The trajectories of locally optimal solutions are tracked by solving the newly damped system with the previous local optimal solution as the initialization, in the same spirit of Algorithm 1. The evolution of the optimal cost and the distance from the best known optimal controller is plotted in Figure 1. Notice that all sub-optimal local trajectories terminate after a modest damping $\alpha \approx 0.12$. After that, the minimization algorithm always tracks a single trajectory. This illustrates the prediction of Corollary 1. Especially, if we start tracking a sub-optimal controller trajectory from $\alpha = 0$, we will be on a better trajectory when $\alpha \approx 0.2$. At that time, if we gradually decrease α to zero, we will obtain a stabilizing controller with a lower cost.

B. Experiments on Small Random Systems

With the same initialization and optimization procedure, we perform the experiments on 3-by-3 system matrices A and B randomly generated from the normal distribution with zero mean and unit variance. For 92 out of 100 samples, we are not able to find more than one locally optimal trajectory. Examples with more than one local trajectories are provided in Figure 3, 4, and 5. The top plot in each figure shows the cost of locally optimal controllers. The bottom plot shows the distance of the locally optimal controllers to the controller with the lowest cost. Note that the order of the cost trajectories may be preserved during the damping (Figure 3) or may be disrupted (Figure 4 and Figure 5). In Figure 4, at the intersection of the two curves, there are two distinct global solutions and therefore Algorithm 1 may fail to obtain the globally optimal decentralized controller. More than one trajectory may have the lowest cost as the damping increases (Figure 5), but with high damp, there is only one trajectory that has the lowest cost. If Algorithm 1 is applied with an initialization on the purple curve, whose cost is around 180, after the damping parameter α is increased to around 2, the purple curve merges with the orange curve. When the damp is reduced to $\alpha = 0$, Algorithm 1 will return to the orange curve with cost around 80, which is a sub-optimal decentralized controller. This illustrates the necessity of assuming the uniqueness of the globally optimal controller in Corollary 2.

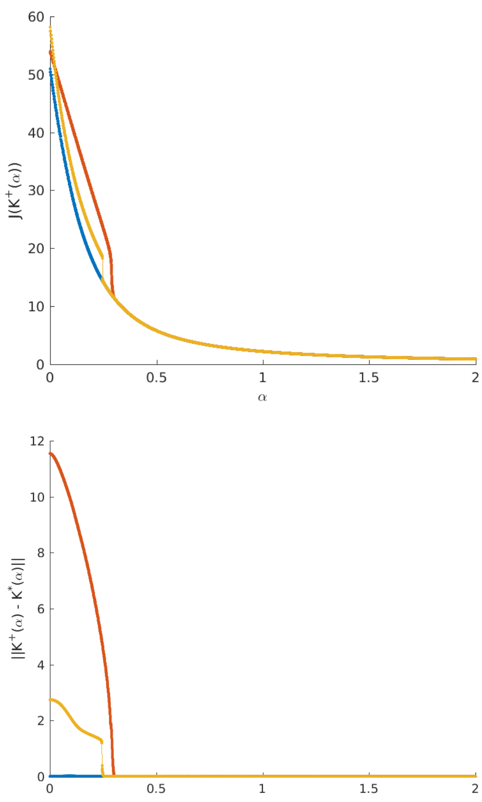


Fig. 3. Trajectories of a randomly generated system where the order of locally optimal controllers is preserved as the damping parameter α changes.

VII. Conclusion

This paper studied the optimal distributed control problem with a large number of locally optimal solutions. To be able to find a globally optimal control policy, we proposed a homotopy method that gradually changed the control problem. We investigated the trajectories of the locally and globally optimal solutions to the optimal decentralized control problem as the damping parameter and the regularization of the decentralized

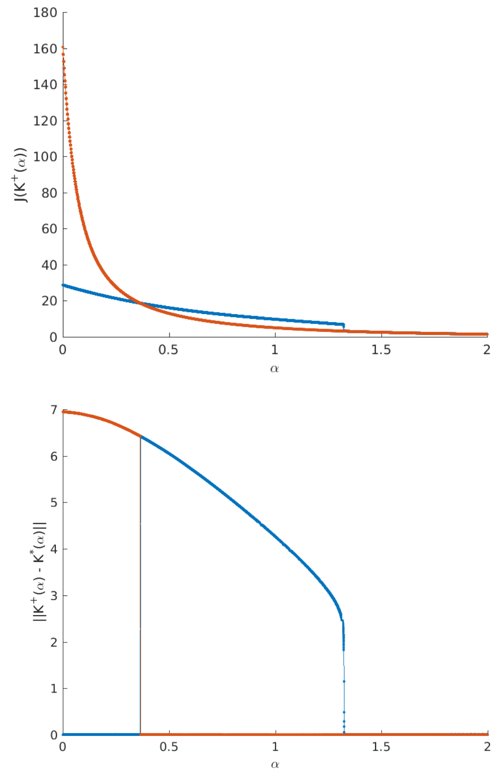


Fig. 4. Trajectories of a randomly generated system where the order of locally optimal controllers is disrupted as the damping parameter α changes.

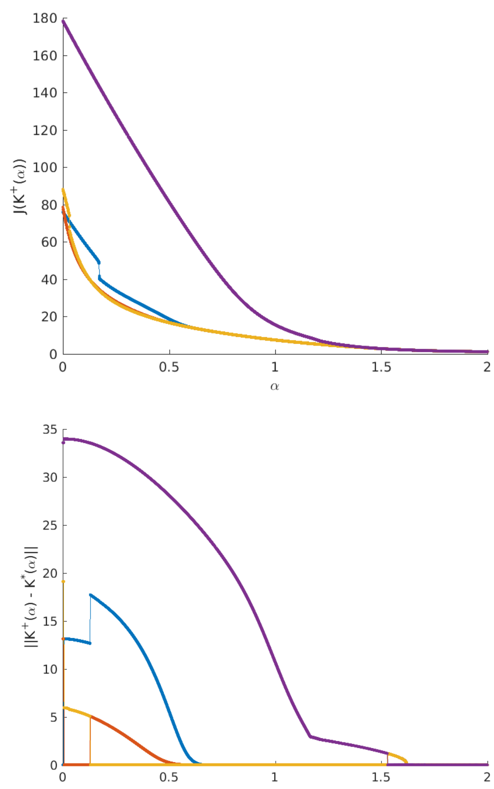


Fig. 5. Trajectories of a randomly generated system with a complicated behavior.

control problem varied. Asymptotic and continuity properties of trajectories were proved, which were based on the notion of "damping property". A sufficient condition was developed together with an algorithm based on local search for finding the global solution of the optimal distributed control problem. The complicated behavior of numerical continuation methods was illustrated with numerical examples with many local minima.

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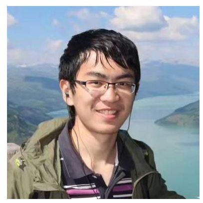

Han Feng is a Ph.D student in the Department of Industrial Engineering and Operations Research at UC Berkeley. He obtained his B.Sc. degree in applied mathematics from the University of Science and Technology of China in 2016.

Javad Lavaei is an Associate Professor in the Department of Industrial Engineering and Operations Research at UC Berkeley. He obtained the Ph.D. degree in Control & Dynamical Systems from California Institute of Technology in 2011. He has worked on different interdisciplinary problems in power systems, optimization theory, control theory, and data science. He has won several awards, including Presidential Early Career Award for Scientists and Engineers given by the White House, DARPA Young Faculty Award, Office of Naval Research Young Investigator Award, Air Force Office of Scientific Research Young Investigator Award, NSF CAREER Award, DARPA Director's Fellowship, Office of Naval Research's Director of Research Early Career Grant, Google Faculty Award, Donald P. Eckman Award, Resonate Award, INFORMS Optimization Society Prize for Young Researchers, INFORMS ENRE Energy Best Publication Award, and SIAM Control and Systems Theory Prize. He is an associate editor of the IEEE Transactions on Automatic Control, the IEEE Transactions on Smart Grid, and the IEEE Control System Letters. He serves on the conference editorial boards of the IEEE Control Systems Society and European Control Association.