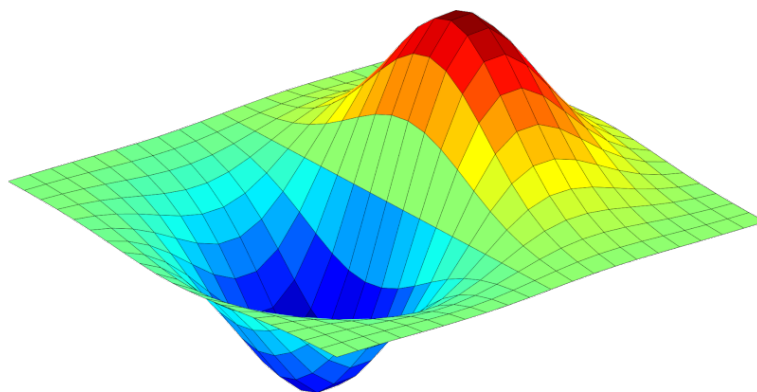


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Simple finite elements and multigrid for efficient mass-consistent wind downscaling in a coupled fire-atmosphere model

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1 Introduction

In the coupled atmosphere-fire model WRF-SFIRE [6, 5], the weather model runs at 300–1km horizontal resolution, while the fire model runs at the resolution of 30m or finer. The wind has a fundamental effect on fire behavior and the topography details have a strong effect on the wind, but WRF does not see the topography on the fire grid scale. We want to downscale the wind from WRF to account for the fine-scale terrain. For this purpose, we fit the wind from WRF with a divergence-free flow over the detailed terrain. Such methods, called mass-consistent approximations, were originally proposed on regular grids [9, 10] for urban and complex terrain modeling, with terrain and surface features modeled by excluding entire grid cells from the domain. For fire applications, WindNinja [11] uses finite elements on a terrain-following grid. The resulting equations are generally solved by iterative methods such as SOR, which converge slowly, so use of GPUs is of interest [2]. A multigrid method with a terrain-following grid by a change of coordinates was proposed in [13].

The method proposed here is to be used in every time step of WRF-SFIRE in the place of interpolation to the fire model grid. Therefore, it needs to have the potential to (1) scale to hundreds or thousands of processors using WRF parallel infrastructure [12]; (2) scale to domains size at least 100km by 100km horizontally, with $3000 \times 3000 \times 15$ grid cells and more; (3) have reasonable memory requirements per grid point; (4) not add to the cost of the time step significantly when started from the solution in the previous time step; and, (5) adapt to the problem automatically, with minimum or no parameters to be set by the user.

2 Finite element formulation

Given vector field $\mathbf{u}_0$ on domain $\Omega \subset \mathbb{R}^d$, subset $\Gamma \subset \partial\Omega$, and $d \times d$ symmetric positive definite coefficient matrix $\mathbf{A} = \mathbf{A}(\mathbf{x})$, we want to find the closest divergence-free vector field $\mathbf{u}$ by solving the problem

$$\min_{\mathbf{u}} \frac{1}{2} \int_{\Omega} (\mathbf{u} - \mathbf{u}_0) \cdot \mathbf{A} (\mathbf{u} - \mathbf{u}_0) d\mathbf{x} \quad \text{subject to } \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \quad (1)$$

where Γ is the bottom of the domain (the surface), and $\mathbf{A}(\mathbf{x})$ is a 3×3 diagonal matrix with penalty constants a_1^2, a_2^2, a_3^2 on the diagonal. Enforcing the constraints in (1) by a Lagrange

multiplier λ , we obtain the solution $(\mathbf{u}, \lambda)$ as a stationary point of the Lagrangean

$$\mathcal{L}(\mathbf{u}, \lambda) = \frac{1}{2} \int_{\Omega} \mathbf{A}(\mathbf{u} - \mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0) d\mathbf{x} + \int_{\Omega} \lambda \operatorname{div} \mathbf{u} d\mathbf{x} - \int_{\Gamma} \lambda \mathbf{n} \cdot \mathbf{u} ds. \quad (2)$$

Eliminating $\mathbf{u}$ from the stationarity conditions $\partial \mathcal{L}(\mathbf{u}, \lambda)/\partial \lambda = 0$ and $\partial \mathcal{L}(\mathbf{u}, \lambda)/\partial \mathbf{u} = 0$ by

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{A}^{-1} \operatorname{grad} \lambda \quad (3)$$

leads to the generalized Poisson equation for Lagrange multiplier λ ,

$$-\operatorname{div} \mathbf{A}^{-1} \operatorname{grad} \lambda = \operatorname{div} \mathbf{u}_0 \text{ on } \Omega, \quad \lambda = 0 \text{ on } \partial\Omega \setminus \Gamma, \quad \mathbf{n} \cdot \mathbf{A}^{-1} \operatorname{grad} \lambda = -\mathbf{n} \cdot \mathbf{u}_0 \text{ on } \Gamma. \quad (4)$$

Multiplication of (4) by a test function μ , $\mu = 0$ on $\partial\Omega \setminus \Gamma$, and integration by parts yields the variational form to find λ such that $\lambda = 0$ on $\partial\Omega \setminus \Gamma$ and

$$\int_{\Omega} \mathbf{A}^{-1} \operatorname{grad} \lambda \cdot \operatorname{grad} \mu d\mathbf{x} = - \int_{\Omega} \operatorname{grad} \mu \cdot \mathbf{u}_0 d\mathbf{x} \quad (5)$$

for all μ such that $\mu = 0$ on $\partial\Omega \setminus \Gamma$. The solution is then recovered from (3). We proceed formally and avoid the language of functional spaces; the theory will be presented elsewhere.

The variational problem (5) is discretized by standard isoparametric 8-node hexahedral finite elements, e.g., [4]. The integral on the left-hand side of (5) is evaluated by tensor-product Gauss quadrature with two nodes in each dimension, while for the right-hand side, one-node quadrature at the center of the element is sufficient. The same code for the derivatives of a finite element function is used to evaluate $\operatorname{grad} \lambda$ in (3) at the center of each element.

The unknown λ is represented by its values at element vertices, and the wind vector is represented naturally by its values at element centers. No numerical differentiation of λ from its nodal values, computation of the divergence of the initial wind field $\mathbf{u}_0$, or explicit implementation of the boundary condition on $\operatorname{grad} \lambda$ in (4) is needed. These are all taken care of by the finite elements naturally.

3 Multigrid iterations

The finite element method for (5) results in a system of linear equations $Ku = f$. The values of the solution are defined on a grid, which we will call a *fine grid*. One cycle of the multigrid method consists of several iterations of a basic iterative method, such as Gauss-Seidel, called a *smoother*, followed by a *coarse-grid correction*. A prolongation matrix P is constructed to interpolate values from a coarse grid, in the simplest case consisting of every other node, to the fine grid. For a given approximate solution u after the smoothing, we seek an improved solution in the form $u + Pu_c$ variationally, by solving

$$P^{\top} K (u + Pu_c) = P^{\top} f \quad (6)$$

for u_c , and obtain the coarse-grid correction procedure as

$$\begin{aligned} f_c &= P^{\top} (f - Ku) && \text{form the coarse right-hand side} \\ K_c &= P^{\top} K P && \text{form the coarse stiffness matrix} \\ K_c u_c &= f_c && \text{solve the coarse-grid problem} \\ u &\leftarrow u + Pu_c && \text{insert the coarse-grid correction} \end{aligned} \quad (7)$$

The coarse grid correction is followed by several more smoothing steps, which completes the multigrid cycle.

In the simplest case, P is a linear interpolation and the coarse stiffness matrix K_c is the stiffness matrix for a coarse finite element discretization on a grid with each coarse-grid element taking the place of a $2 \times 2 \times 2$ agglomeration of fine-grid elements. That makes it possible to apply the same method to the coarse-grid problem (7) recursively. This process creates a hierarchy of coarser grids. Eventually, the coarsest grid problem is solved by a direct method, or one can just do some more iterations on it.

Multigrid methods gain their efficiency from the fact that simple iterative methods like Gauss-Seidel change the values of the solution at a node from differences of the values between this and neighboring nodes. When the error values at neighboring nodes become close, the error can be well approximated in the range of the prolongation P and the coarse-grid correction can find u_c such that $u + Pu_c$ is a much better approximation of the solution. For analysis of variational multigrid methods and further references, see [1, 7].

Multigrid methods are very efficient. For simple elliptic problems, such as the Poisson equation on a regular grid, convergence rates of about 0.1 (reduction of the error by a factor of 10) at the cost of 4 to 5 Gauss-Seidel sweeps on the finest grid are expected [3]. However, the convergence rates get worse on more realistic grids, and adaptations are needed. We choose as the smoother vertical sweeps of Gauss-Seidel from the bottom up to the top, with red-black ordering horizontally into 4 groups. For the base method, we use $2 \times 2 \times 2$ coarsening and construct P so that the vertices of every $2 \times 2 \times 2$ agglomeration of elements interpolate to the fine-grid nodes in the agglomeration, with the same weights as the trilinear interpolation on a regular grid. The interpolation is still trilinear on a stretched grid, but only approximately trilinear on a deformed terrain-following grid.

The base method works as expected as long as some grid layers are not tightly coupled. If they are, we mitigate the slower convergence by semicoarsening [8]: After smoothing, the error is smoother in the tightly coupled direction(s), which indicates that we should not coarsen the other direction(s). When the grid is stretched vertically away from the ground, the nodes are relatively closer and thus tightly coupled in the horizontal direction. Similarly, when the penalty coefficient a_3 in the vertical direction is larger than a_1 and a_2 in the horizontal directions, the neighboring nodes in the vertical direction are tightly coupled numerically. The algorithm to decide on coarsening we use is: Suppose that the penalty coefficients are $a_1 = a_2 = 1$ and $a_3 \geq 1$, and at the bottom of the grid, the grid spacing is $h_1 = h_2$ (horizontal) and h_3 (vertical). If $h_3/(h_1 a_3) > 1/3$, coarsen in the horizontal directions by 2, otherwise do not coarsen. Then, replace h_1 and h_2 by their new values, coarsened (multiplied by 2) or not, and for every horizontal layer from the ground up, if $h_3/(h_1 a_3) < 3$, coarsen about that layer vertically, otherwise do not coarsen. This algorithm retains the coarse grids as logically cartesian, which is important for computational efficiency and keeping the code simple, and it controls the convergence rate to remain up to about 0.28 with four smoothing steps per cycle.

4 Conclusion

We have presented a simple and efficient finite element formulation of mass-consistent approximation, and a multigrid iterative method with adaptive semicoarsening, which maintains the convergence of iteration over a range of grids and penalty coefficients. A prototype code is available at: <https://github.com/openwfm/wrf-fire-matlab/tree/femwind/femwind>

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