ALMOST MINIMIZERS FOR CERTAIN FRACTIONAL VARIATIONAL PROBLEMS

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To Nina Nikolaevna Uraltseva on the occasion of her 85th birthday.

Abstract. In this paper we introduce a notion of almost minimizers for certain variational problems governed by the fractional Laplacian, with the help of the Caffarelli-Silvestre extension. In particular, we study almost fractional harmonic functions and almost minimizers for the fractional obstacle problem with zero obstacle. We show that for a certain range of parameters, almost minimizers are almost Lipschitz or $C^{1,\beta}$-regular.

1. Introduction and Main Results

1.1. **Fractional harmonic functions.** Given $0 < s < 1$, we say that a function $u \in \mathscr{L}_s(\mathbb{R}^n) := L^1(\mathbb{R}^n, (1+|x|^{n+2s})^{-1})$ is *s-fractional harmonic* in an open set $\Omega \subset \mathbb{R}^n$ if

$$(-\Delta_x)^s u(x) := C_{n,s}\,\text{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(x+z)}{|z|^{n+2s}} = 0 \quad \text{in } \Omega, \tag{1.1}$$

where p.v. stands for Cauchy's principal value and $C_{n,s}$ is a normalization constant. The formula above is just one of many equivalent definitions of the fractional Laplacian $(-\Delta_x)^s$, another one being a pseudo-differential operator with Fourier symbol $|\xi|^{2s}$. We refer to a recent review of Garofalo [Gar19] for basic properties of $(-\Delta_x)^s$, as well as many historical remarks concerning that operator.

In recent years, there has been a surge of interest in nonlocal problems involving the fractional Laplacian, when it was discovered that the problems can be localized by the use of the so-called Caffarelli-Silvestre extension procedure [CS07]. Namely, for $a = 1-2s \in (-1,1)$, let

$$P(x,y) := C_{n,a}\frac{|y|^{1-a}}{(|x|^2+|y|^2)^{\frac{n+1-a}{2}}}, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+ = \mathbb{R}^{n+1}_+,$$

(to be called the Poisson kernel for the extension operator L_a) and consider the convolution, still denoted by u,

$$u(x,y) := u * P(\cdot, y) = \int_{\mathbb{R}^n} u(z)P(x-z,y)dz, \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

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Note that $u(x,y)$ solves the Cauchy problem

$$\begin{aligned} L_a u := \operatorname{div}(|y|^a\nabla u) = 0 \quad &\text{in } \mathbb{R}^{n+1}_+,\\ u(x,0) = u(x) \quad &\text{on } \mathbb{R}^n, \end{aligned}$$

where $\nabla = \nabla_{x,y}$ is the full gradient in x and y variables. L_a is known as the *Caffarelli-Silvestre extension operator*. Then, one can recover $(-\Delta_x)^s u$ as the fractional normal derivative on $\mathbb{R}^n$

$$(-\Delta_x)^s u(x) = -C_{n,a} \lim_{y\to 0+} y^a \partial_y u(x,y), \quad x\in\mathbb{R}^n$$

to be understood in the appropriate sense of traces. Now, going back to the definition (1.1), if we consider the even reflection of u in y-variable to all of $\mathbb{R}^{n+1}$, i.e.,

$$u(x,y) = u(x,-y), \quad x\in\mathbb{R}^n,\ y<0,$$

then the following fact holds: $u(x)$ is s-fractional harmonic in Ω if and only if $u(x,y)$ satisfies

$$L_a u = 0 \quad \text{in } \widetilde{\Omega} := \mathbb{R}^{n+1}_- \cup (\Omega\times\{0\}) \cup \mathbb{R}^{n+1}_+. \tag{1.2}$$

(We will refer to solutions of $L_a u = 0$ as L_a*-harmonic functions*.) This is essentially Lemma 4.1 in [CS07]. Since $L_a u = 0$ in $\mathbb{R}^n_\pm$ by definition, the condition (1.2) is equivalent to asking

$$L_a u = 0 \quad \text{in } B_r(x_0),$$

for any ball $B_r(x_0)$ centered at $x_0\in\Omega$ such that $B_r(x_0)\Subset\widetilde{\Omega}$, or equivalently $B'_r(x_0)\Subset\Omega$. Now, observing that the solutions of the above equation are minimizers of the weighted Dirichlet energy $\int_{B_r(x_0)} |\nabla v|^2|y|^a$, we obtain the following fact.

Proposition 1.1. *A function $u\in\mathscr{L}_s(\mathbb{R}^n)$ is s-fractional harmonic in Ω if and only if its reflected Caffarelli-Silvestre extension $u(x,y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega},|y|^a)$ and for any ball $B_r(x_0)$ with $x_0\in\Omega$ such that $B'_r(x_0)\Subset\Omega$, we have*

$$\int_{B_r(x_0)} |\nabla u|^2|y|^a \le \int_{B_r(x_0)} |\nabla v|^2|y|^a,$$

for any $v\in u + W^{1,2}_0(B_r(x_0),|y|^a)$.

We take this proposition as the starting point for the definition of almost s-fractional harmonic functions, in the spirit of Anzellotti [Anz83].

Definition 1.2 (Almost s-fractional harmonic functions)**.** Let $r_0>0$ and $\omega : (0,r_0)\to[0,\infty)$ be a modulus of continuity[1]. We say that a function $u\in\mathscr{L}_s(\mathbb{R}^n)$ is *almost s-fractional harmonic* in an open set $\Omega\subset\mathbb{R}^n$, with a gauge function ω, if its reflected Caffarelli-Silvestre extension $u(x,y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega},|y|^a)$ and for any ball $B_r(x_0)$ with $x_0\in\Omega$ and $0<r<r_0$ such that $B'_r(x_0)\Subset\Omega$, we have

$$\int_{B_r(x_0)} |\nabla u|^2|y|^a \le (1+\omega(r)) \int_{B_r(x_0)} |\nabla v|^2|y|^a, \tag{1.3}$$

for any $v\in u+W^{1,2}_0(B_r(x_0),|y|^a)$.

[1] i.e., a nondecreasing function with $\omega(0+)=0$

1.2. **Fractional obstacle problem.** A function $u \in \mathscr{L}_s(\mathbb{R}^n)$ is said to solve the s-fractional obstacle problem with obstacle ψ in an open set $\Omega \subset \mathbb{R}^n$, if

$$\min\{(-\Delta_x)^s u, u-\psi\} = 0 \quad \text{in } \Omega. \tag{1.4}$$

We refer to [Sil07, CSS08, MNS17] for general introduction and basic results on this problem. With the help of the reflected Caffarelli-Silvestre extension, we can rewrite the problem as a Signorini-type problem for the operator L_a:

$$\begin{aligned} L_a u &= 0 \quad \text{in } \mathbb{R}^{n+1}_\pm \\ \min\{-\partial^a_y u, u-\psi\} &= 0 \quad \text{in } \Omega, \end{aligned}$$

where

$$\partial^a_y u(x,0) := \lim_{y\to 0+} y^a \partial_y u(x,y).$$

This, in turn, can be written in the following variational form, see [CSS08].

Proposition 1.3. *A function $u \in \mathscr{L}_s(\mathbb{R}^n)$ solves (1.4) if and only if its reflected Caffarelli-Silvestre extension $u(x,y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega})$ and for any ball $B_r(x_0)$ with $x_0 \in \Omega$ such that $B'_r(x_0) \Subset \Omega$, we have*

$$\int_{B_r(x_0)} |\nabla u|^2 |y|^a \le \int_{B_r(x_0)} |\nabla v|^2 |y|^a,$$

for any $v \in \mathfrak{K}_{\psi,u}(B_r(x_0), |y|^a) := \{v \in u + W^{1,2}_0(B_r, |y|^a) : v \ge \psi \text{ on } B'_r(x_0)\}$.

Definition 1.4 (Almost minimizers for s-fractional obstacle problem)**.** Let $r_0 > 0$ and $\omega : (0, r_0) \to [0,\infty)$ be a modulus of continuity. We say that a function $u \in \mathscr{L}_s(\mathbb{R}^n)$ is an *almost minimizer for the s-fractional obstacle problem* in an open set $\Omega \subset \mathbb{R}^n$, with a gauge function ω, if its reflected Caffarelli-Silvestre extension $u(x,y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}, |y|^a)$ and for any ball $B_r(x_0)$ with $x_0 \in \Omega$ and $0 < r < r_0$ such that $B'_r(x_0) \Subset \Omega$, we have

$$\int_{B_r(x_0)} |\nabla u|^2 |y|^a \le (1+\omega(r)) \int_{B_r(x_0)} |\nabla v|^2 |y|^a, \tag{1.5}$$

for any $v \in \mathfrak{K}_{\psi,u}(B_r(x_0), |y|^a)$.

The notion of almost minimizers above is related to the one for the thin obstacle problem ($s = 1/2$) studied by the authors in [JP19], but there are certain important differences. In Definition 1.4, we ask the almost minimizing property (1.5) to hold only for balls centered on the "thin space" $\mathbb{R}^n$, while in [JP19], we ask that property for balls centered at any point in an open set in the "thick space" $\mathbb{R}^{n+1}$. In a sense, this means that here we think of the perturbation from minimizers as living on the thin space, while in [JP19] they live in the thick space.

1.3. **Main results and structure of the paper.** In this paper, our main concern is the regularity of almost minimizers in their original variables.

We start with examples of almost minimizers in Section 2. We then proceed to prove the following results, echoing those in [Anz83] and [JP19].

Theorem I. *Let $u \in \mathscr{L}_s(\mathbb{R}^n)$ be almost s-fractional harmonic in Ω. Then*

1. *u is almost Lipschitz in Ω, i.e, $u \in C^{0,\sigma}(\Omega)$ for any $0 < \sigma < 1$.*
2. *If $\omega(r) = r^\alpha$, then $u \in C^{1,\beta}(\Omega)$ for some $\beta = \beta_{n,a,\alpha} > 0$.*
3. *If $0 < s < 1/2$ or $s = 1/2$ and $\omega(r) = r^\alpha$ for some $\alpha > 0$, then u is actually s-fractional harmonic in Ω.*

In the case of the s-fractional obstacle problem, our results are obtained under the assumption that $1/2 \leq s < 1$. Also, because of the technical nature of the problem, we restrict ourselves to the case $\psi = 0$.

Theorem II. *Let $u \in \mathscr{L}_s(\mathbb{R}^n)$ be an almost minimizer for the s-fractional obstacle problem with obstacle $\psi = 0$ in Ω.*

(1) *If $1/2 \leq s < 1$, then $u \in C^{0,\sigma}(\Omega)$ for any $0 < \sigma < 1$.*
(2) *If $1/2 \leq s < 1$ and $\omega(r) = r^\alpha$ for some $\alpha > 0$, then $u \in C^{1,\beta}(\Omega)$ for some $\beta = \beta_{n,a,\alpha} > 0$.*

The proofs follow the general approach in [Anz83] and [JP19] by first obtaining growth estimates for minimizers (see Section 3) and then deriving their perturbed versions for almost minimizers (Section 4 for s-fractional harmonic functions and Section 5 for the s-fractional obstacle problem). The regularity then follows by an embedding theorem of a Morrey-Campanato-type space into the Hölder space, which we included in Appendix A. Finally, Appendix B contains the proof of orthogonal polynomial expansion of L_a-harmonic functions, that we rely on in deriving the growth estimates in Section 3. The polynomial expansion has other interesting corollaries such as the (known) real-analyticity of s-fractional harmonic functions, which are of independent interest.

1.4. **Notation.** Throughout the paper we use the following notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space. The points of $\mathbb{R}^{n+1}$ are denoted by $X = (x, y)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y \in \mathbb{R}$. We routinely identify $x \in \mathbb{R}^n$ with $(x, 0) \in \mathbb{R}^n \times \{0\}$. $\mathbb{R}^{n+1}_\pm$ stands for open halfspaces $\{X = (x, y) \in \mathbb{R}^{n+1} : \pm y > 0\}$.

We use the following notations for balls of radius r in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$

$$\begin{aligned}
B_r(X) &= \{Z \in \mathbb{R}^{n+1} : |X - Z| < r\}, && \text{(Euclidean) ball in } \mathbb{R}^{n+1},\\
B_r^\pm(x) &= B_r(x, 0) \cap \{\pm y > 0\}, && \text{half-ball in } \mathbb{R}^{n+1},\\
B_r'(x) &= B_r(x, 0) \cap \{y = 0\}, && \text{ball in } \mathbb{R}^n.
\end{aligned}$$

We typically drop the center from the notation if it is the origin. Thus, $B_r = B_r(0)$, $B_r' = B_r'(0)$, etc.

Next, $\nabla u = \nabla_X u = (\partial_{x_1} u, \ldots, \partial_{x_n} u, \partial_y u)$ stands for the full gradient, while $\nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$. We also use the standard notations for partial derivatives, such as $\partial_{x_i} u$, u_{x_i}, u_y etc.

In integrals, we often drop the variable and the measure of integration if it is with respect to the Lebesgue measure or the surface measure. Thus,

$$\int_{B_r} u|y|^a = \int_{B_r} u(X)|y|^a dX, \quad \int_{\partial B_r} u|y|^a = \int_{\partial B_r} u(X)|y|^a dS_X,$$

where S_X stands for the surface measure.

By $L^2(B_R, |y|^a)$ and $L^2(\partial B_R, |y|^a)$ we indicate the weighted Lebesgue spaces of functions with the norms

$$\begin{aligned}
\|u\|^2_{L^2(B_R,|y|^a)} &= \int_{B_R} u^2|y|^a\\
\|u\|^2_{L^2(\partial B_R,|y|^a)} &= \int_{\partial B_R} u^2|y|^a.
\end{aligned}$$

$W^{1,2}(B_R,|y|^a)$ is the corresponding weighted Sobolev space of functions with the norm

$$\|u\|^2_{W^{1,2}(B_R,|y|^a)}=\|u\|^2_{L^2(B_R,|y|^a)}+\|\nabla u\|^2_{L^2(B_R,|y|^a)}.$$

We also use other typical notations for Sobolev spaces. Thus, $W^{1,2}_0(B_R,|y|^a)$ stands for the closure of $C^\infty_0(B_R)$ in $W^{1,2}(B_R,|y|^a)$.

For $x\in\mathbb{R}^n$ and $r>0$, we indicate by $\langle u\rangle_{x,r}$ the $|y|^a$-weighted integral mean value of a function u over $B_r(x)$. That is,

$$\langle u\rangle_{x,r}=\fint_{B_r(x)}u|y|^a:=\frac{1}{\omega_{n+1+a}r^{n+1+a}}\int_{B_r(x)}u|y|^a,$$

where $\omega_{n+1+a}=\int_{B_1}|y|^a$ is the $|y|^a$-weighted volume of the unit ball B_1 in $\mathbb{R}^{n+1}$. (Note that here and throughout the paper, the sign $\fint$ denotes the integral mean value with respect to the weighted measure $|y|^a dX$.) Finally, similarly to the other notations, we drop the origin if it is 0 and write $\langle u\rangle_r$ for $\langle u\rangle_{0,r}$.

2. Examples of almost minimizers

Before we proceed with the proofs of the main results, we would like to give some examples of almost minimizers.

Example 2.1. Let $u\in\mathscr{L}_s(\mathbb{R}^n)$ be a solution of

$$(-\Delta_x)^s u+b(x)\cdot\nabla_x u=0\quad\text{in }\Omega,$$

where $b=(b^1,b^2,\dots,b^n)\in W^{1,\infty}(\Omega)$ and $1/2<s<1$ (or $-1<a<0$). Then u is an almost s-fractional harmonic with a gauge function $\omega(r)=Cr^{-a}$ (note that $-a>0$).

Proof. Consider a ball $B_r(x_0)$ centered at $x_0\in\Omega$ such that $B'_r(x_0)\Subset\Omega$. Without loss of generality assume that $x_0=0$. Let v be the minimizer of

$$\int_{B_r}|\nabla v|^2|y|^a$$

on $u+W^{1,2}_0(B_r,|y|^a)$. Then

$$\int_{B_r}\nabla v\nabla(u-v)|y|^a=0,$$

and as a consequence,

$$\int_{B_r}(|\nabla u|^2-|\nabla v|^2)|y|^a=\int_{B_r}|\nabla(u-v)|^2|y|^a.$$

Then, we have

$$\begin{aligned}\int_{B_r}(|\nabla u|^2-|\nabla v|^2)|y|^a&=2\int_{B_r^+}|\nabla(u-v)|^2|y|^a\\&=2\int_{B_r^+}|\nabla(u-v)|^2|y|^a+\operatorname{div}(|y|^a\nabla(u-v))\,(u-v)\\&=2\int_{B_r^+}\operatorname{div}\left(|y|^a\nabla\left(\frac{(u-v)^2}{2}\right)\right)\\&=2\int_{(\partial B_r)^+}|y|^a(u-v)(u_\nu-v_\nu)-2\int_{B'_r}(u-v)(\partial^a_y u-\partial^a_y v)\end{aligned}$$

$$= C \int_{B_r'} (u-v)(-\Delta_x)^s u$$
$$= -C \int_{B_r'} (u-v) b^i u_{x_i}$$

with $C = C_{n,a}$. Next, extending b^i to $\mathbb{R}^{n+1}$ by $b^i(x,y) := b^i(x)$, we have

$$\begin{aligned}
\int_{B_r} (|\nabla u|^2 - |\nabla v|^2)|y|^a &= -C \int_{B_r'} (u-v) b^i u_{x_i} \\
&= C \int_{B_r^+} \partial_y \left((u-v) b^i u_{x_i}\right) \\
&= C \int_{B_r^+} (u_y - v_y) b^i u_{x_i} + (u-v) b^i u_{x_i y} \\
&\le C \|b\|_{W^{1,\infty}(\Omega)} \int_{B_r^+} |\nabla u|^2 + |\nabla v|^2 \\
&\qquad + C \int_{\partial(B_r^+)} (u-v) b^i u_y \nu_{x_i} - C \int_{B_r^+} \partial_{x_i}((u-v) b^i) u_y \\
&= C \|b\|_{W^{1,\infty}(\Omega)} \int_{B_r^+} |\nabla u|^2 + |\nabla v|^2 \\
&\qquad - C \int_{B_r^+} ((u_{x_i} - v_{x_i}) b^i + (u-v) b^i_{x_i}) u_y \\
&\le C \|b\|_{W^{1,\infty}(\Omega)} \int_{B_r^+} |\nabla u|^2 + |\nabla v|^2 + |u-v|^2.
\end{aligned}$$

Using Poincare's inequality, it follows that

$$\begin{aligned}
\int_{B_r} (|\nabla u|^2 - |\nabla v|^2)|y|^a &\le C \int_{B_r} |\nabla u|^2 + |\nabla v|^2 \\
&\le C r^{-a} \int_{B_r} (|\nabla u|^2 + |\nabla v|^2)|y|^a \\
&\le C r^{-a} \int_{B_r} |\nabla u|^2 |y|^a.
\end{aligned}$$

Hence,

$$\int_{B_r(x_0)} |\nabla u|^2 |y|^a \le (1 + C r^{-a}) \int_{B_r(x_0)} |\nabla v|^2 |y|^a,$$

for $0 < r < r_0$, with C and r_0 depending on n, a, and $\|b\|_{W^{1,\infty}(\Omega)}$. $\square$

Example 2.2. Let $u \in \mathscr{L}_s(\mathbb{R}^n)$ be a solution of the obstacle problem for fractional Laplacian with drift

$$\min\{(-\Delta_x)^s u + b(x) \cdot \nabla_x u, u\} = 0 \quad \text{in } \Omega,$$

where $b = (b^1, b^2, \dots, b^n) \in W^{1,\infty}(\Omega)$ and $1/2 < s < 1$ (or $-1 < a < 0$). Then u is an almost minimizer for s-fractional obstacle problem in Ω with an obstacle $\psi = 0$ and a gauge function $\omega(r) = C r^{-a}$.

The obstacle problem above has been studied earlier in [PP15] and [GPPS17].

Proof. We argue similarly to Example 2.1. Let $B_r(x_0)$ centered at $x_0\in\Omega$ such that $B'_r(x_0)\Subset\Omega$. Without loss of generality assume that $x_0=0$. Let v be the minimizer of
$$\int_{B_r}|\nabla v|^2|y|^a$$
on $\mathfrak{K}_{0,u}(B_r,|y|^a)=\{v\in u+W^{1,2}_0(B_r,|y|^a): v\geq 0 \text{ on } B'_r(x_0)\}$. Next, we write
$$\begin{aligned}
\int_{B_r}(|\nabla u|^2-|\nabla v|^2)|y|^a &= 2\int_{B_r}\nabla u\nabla(u-v)|y|^a-\int_{B_r}|\nabla(u-v)|^2|y|^a\\
&\leq 2\int_{B_r}\nabla u\nabla(u-v)|y|^a\\
&= 4\int_{B_r^+}\nabla u\nabla(u-v)|y|^a+\operatorname{div}(|y|^a\nabla u)(u-v)\\
&= -4\int_{B'_r}(u-v)\partial^a_y u\\
&= C\int_{B'_r}(u-v)(-\Delta_x)^s u\\
&= C\left[-\int_{B'_r\cap\{u>0\}}(u-v)b^iu_{x_i}+\int_{B'_r\cap\{u=0\}}(-v)\,(-\Delta_x)^s u\right]\\
&\leq C\left[-\int_{B'_r\cap\{u>0\}}(u-v)b^iu_{x_i}-\int_{B'_r\cap\{u=0\}}(-v)b^iu_{x_i}\right]\\
&= -C\int_{B'_r}(u-v)b^iu_{x_i},
\end{aligned}$$
where we used that $(-\Delta)^s u+b^iu_{x_i}\geq 0$ and $-v\leq 0$ on $B'_r\cap\{u=0\}$ in the last inequality.

Then we complete the proof as in Example 2.1. $\square$

3. Growth estimates for minimizers

In this section we prove growth estimates for L_a-harmonic functions and solutions of the Signorini problem for L_a, i.e., minimizers v of the weighted Dirichlet integral
$$\int_{B_r}|\nabla v|^2|y|^a$$
on $v+W^{1,2}_0(B_r,|y|^a)$ or on the thin obstacle constraint set $\mathfrak{K}_{0,v}(B_r,|y|^a)$.

The idea is that these estimates will extend to almost minimizers and will ultimately imply their regularity with the help of Morrey-Campanato-type space embedding.

The proofs in this section are akin to those in [JP19] for almost minimizers of the thin obstacle problem. Yet, one has to be careful with different growth rates for tangential and normal derivatives.

3.1. Growth estimates for L_a-harmonic functions.

Lemma 3.1. *Let $v\in W^{1,2}(B_R,|y|^a)$ be a solution of $L_av=0$ in B_R. If v is even in y, then for $0<\rho<R$*
$$\int_{B_\rho}|\nabla_x v|^2|y|^a\leq\left(\frac{\rho}{R}\right)^{n+1+a}\int_{B_R}|\nabla_x v|^2|y|^a$$

$$\int_{B_\rho} |v_y|^2|y|^a \leq \left(\frac{\rho}{R}\right)^{n+3+a} \int_{B_R} |v_y|^2|y|^a.$$

Proof. Note that we can write

$$v(x,y) = \sum_{k=0}^\infty p_k(x,y),$$

where p_k's are L_a-harmonic homogeneous polynomials of degree k (see Appendix B). Then $\{\partial_{x_i} p_k\}_{k=1}^\infty$ are L_a-harmonic homogeneous polynomials of degree $k-1$, and thus orthogonal in $L^2(\partial B_1, |y|^a)$. Thus,

$$\begin{aligned}
\int_{B_\rho} |\nabla_x v|^2|y|^a &= \sum_{k=1}^\infty \int_{B_\rho} |\nabla_x p_k|^2|y|^a \\
&= \sum_{k=1}^\infty \left(\frac{\rho}{R}\right)^{n+1+a+2(k-1)} \int_{B_R} |\nabla_x p_k|^2|y|^a \\
&\leq \left(\frac{\rho}{R}\right)^{n+1+a} \sum_{k=1}^\infty \int_{B_R} |\nabla_x p_k|^2|y|^a \\
&= \left(\frac{\rho}{R}\right)^{n+1+a} \int_{B_R} |\nabla_x v|^2|y|^a.
\end{aligned}$$

Similarly, $\{|y|^a\partial_y p_k\}_{k=1}^\infty$ are L_{-a}-harmonic homogeneous functions of degree $k-1+a$, and thus orthogonal in $L^2(\partial B_1, |y|^{-a})$. Notice that since $p_1(x,y) = p_1(x)$ is independent of y variable by the even symmetry, we have $|y|^a\partial_y p_1 = 0$. Thus,

$$\begin{aligned}
\int_{B_\rho} |v_y|^2|y|^a &= \int_{B_\rho} \left||y|^a v_y\right|^2 |y|^{-a} \\
&= \sum_{k=2}^\infty \int_{B_\rho} \left||y|^a\partial_y p_k\right|^2 |y|^{-a} \\
&= \sum_{k=2}^\infty \left(\frac{\rho}{R}\right)^{n+1-a+2(k-1+a)} \int_{B_R} ||y|^a\partial_y p_k|^2|y|^{-a} \\
&\leq \left(\frac{\rho}{R}\right)^{n+3+a} \int_{B_R} |v_y|^2|y|^a.
\end{aligned}$$

□

Lemma 3.2. *Let v be a solution of $L_a v = 0$ in B_R, even in y. Then, for $0 < \rho < R$,*

$$\int_{B_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2|y|^a \leq \left(\frac{\rho}{R}\right)^{n+a+3} \int_{B_R} |\nabla_x v - \langle \nabla_x v\rangle_R|^2|y|^a. \tag{3.1}$$

Proof. First note that since $L_a(\nabla_x v) = 0$ in B_R, $\langle \nabla_x v\rangle = \nabla_x v(0)$ by the mean value theorem for L_a-harmonic functions, see [CSS08, Lemma 2.9]. If we use the expansion $v = \sum_{k=0}^\infty p_k(x,y)$ in B_R as in the proof of Lemma 3.1, then $\nabla_x v - \nabla_x v(0) = \sum_{k=2}^\infty \nabla_x p_k$ and consequently

$$\begin{aligned}
\int_{B_\rho} |\nabla_x v - \nabla_x v(0)|^2|y|^a &= \sum_{k=2}^\infty \int_{B_\rho} |\nabla_x p_k|^2|y|^a \\
&= \sum_{k=2}^\infty \left(\frac{\rho}{R}\right)^{n+a+2k-1} \int_{B_R} |\nabla_x p_k|^2|y|^a
\end{aligned}$$

$$\begin{aligned}
&\le \left(\frac{\rho}{R}\right)^{n+a+3} \sum_{k=2}^{\infty} \int_{B_R} |\nabla_x p_k|^2 |y|^a \\
&= \left(\frac{\rho}{R}\right)^{n+a+3} \int_{B_R} |\nabla_x v - \nabla_x v(0)|^2 |y|^a. \qquad \square
\end{aligned}$$

3.2. **Growth estimates for the solutions of the Signorini problem for L_a.** Our estimates for the solutions of the Signorini problem will require an assumption that $1/2 \le s < 1$, or $a \le 0$. Also, unless stated otherwise, the obstacle ψ is assumed to be zero.

The first estimate is the analogue of Lemma 3.1, but with less information of the growth of v_y.

Lemma 3.3. *Let v be a solution of the Signorini problem for L_a in B_R, even in y, with $a \le 0$. Then, for $0 < \rho < R$*

$$\int_{B_\rho} |\nabla v|^2 |y|^a \le \left(\frac{\rho}{R}\right)^{n+1+a} \int_{B_R} |\nabla v|^2 |y|^a. \tag{3.2}$$

Proof. We use the following property: if v is as in the statement of the lemma, then v_{x_i}, $i = 1, \dots, n$, and $y|y|^{a-1} v_y$ are Hölder continuous in B_R, see [CSS08]. Moreover, we have that

$$L_a(v_{x_i}^{\pm}) \ge 0, \quad L_{-a}((y|y|^{a-1} v_y)^{\pm}) \ge 0 \quad \text{in } B_R.$$

This follows from the fact that $L_a v_{x_i} = 0$ in $\{\pm v_{x_i} > 0\}$ and $L_{-a}(y|y|^{a-1} v_y) = 0$ in $\{\pm y|y|^{a-1} v_y > 0\}$, by the complementarity condition $v_y v = 0$ on B'_R, as well as an argument in Exercise 2.6 or Exercise 9.5 in [PSU12]. As a consequence, we have

$$L_a(|\nabla_x v|^2) \ge 0, \quad L_{-a}(||y|^a v_y|^2) \ge 0 \quad \text{in } B_R.$$

We next use the following $|y|^a$-weighted sub-mean value property for L_a-subharmonic functions: If $L_a w \ge 0$ weakly in B_R, $-1 < a < 1$, then

$$\rho \mapsto \frac{1}{\rho^{n+1+a}} \int_{B_\rho} w |y|^a$$

is nondecreasing. This follows by integration from the spherical sub-mean value property, see [CSS08, Lemma 2.9]. Thus, we have that

$$\begin{aligned}
&\rho \mapsto \frac{1}{\rho^{n+1+a}} \int_{B_\rho} |\nabla_x v|^2 |y|^a \\
&\rho \mapsto \frac{1}{\rho^{n+1-a}} \int_{B_\rho} v_y^2 |y|^a
\end{aligned}$$

are monotone nondecreasing for $0 < \rho < R$. This implies

$$\begin{aligned}
\int_{B_\rho} |\nabla_x v|^2 |y|^a &\le \left(\frac{\rho}{R}\right)^{n+1+a} \int_{B_R} |\nabla_x v|^2 |y|^a \\
\int_{B_\rho} v_y^2 |y|^a &\le \left(\frac{\rho}{R}\right)^{n+1-a} \int_{B_R} v_y^2 |y|^a.
\end{aligned}$$

In the case $a \le 0$, we therefore conclude that the bound (3.2) holds. $\square$

Lemma 3.4. *Let v be a solution of the Signorini problem for L_a in B_R, even in y, with $a \le 0$. If $v(0) = 0$, then there exists $C = C_{n,\alpha}$ such that for $0 < \rho < r < (3/4)R$,*

$$\int_{B_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a \leq \left(\frac{\rho}{r}\right)^{n+a+3} \int_{B_r} |\nabla_x v - \langle \nabla_x v\rangle_r|^2 |y|^a \\ + C\|v\|^2_{L^\infty(B_R)} \frac{\rho^{n+2}}{R^{2+2s}}.$$

Proof. Define

$$\varphi(r) := \frac{1}{r^{n+a+3}} \int_{B_r} |\nabla_x v - \langle \nabla_x v\rangle_r|^2 |y|^a.$$

Then,

$$\begin{aligned}\varphi(r) &= \frac{1}{r^{n+a+3}} \left[\int_{B_r} |\nabla_x v|^2 |y|^a - 2\langle \nabla_x v\rangle_r \int_{B_r} \nabla_x v |y|^a + \langle \nabla_x v\rangle_r^2 \int_{B_r} |y|^a\right] \\ &= \frac{1}{r^{n+a+3}} \left[\int_{B_r} |\nabla_x v|^2 |y|^a - \frac{1}{\omega_{n+1+a} r^{n+1+a}} \left(\int_{B_r} \nabla_x v |y|^a\right)^2\right].\end{aligned}$$

Thus, using the Cauchy-Schwarz and Young's inequality, we obtain

$$\begin{aligned}\varphi'(r) = \frac{1}{r^{n+a+3}} \Bigg[&- \frac{n+a+3}{r} \int_{B_r} |\nabla_x v|^2 |y|^a + \int_{\partial B_r} |\nabla_x v|^2 |y|^a \\ &+ \frac{n+a+3}{\omega_{n+1+a} r^{n+2+a}} \left(\int_{B_r} \nabla_x v |y|^a\right)^2 + \frac{n+1+a}{\omega_{n+1+a} r^{n+2+a}} \left(\int_{B_r} \nabla_x v |y|^a\right)^2 \\ &- \frac{2}{\omega_{n+1+a} r^{n+1+a}} \left(\int_{B_r} \nabla_x v |y|^a\right) \left(\int_{\partial B_r} \nabla_x v |y|^a\right)\Bigg] \\ \geq -\frac{C}{r^{n+a+3}} &\left[\frac{1}{r} \int_{B_r} |\nabla_x v|^2 |y|^a + \left(\frac{1}{r} \int_{B_r} |\nabla_x v|^2 |y|^a\right)^{1/2} \left(\int_{\partial B_r} |\nabla_x v|^2 |y|^a\right)^{1/2}\right] \\ \geq -\frac{C}{r^{n+a+3}} &\left[\frac{1}{r} \int_{B_r} |\nabla_x v|^2 |y|^a + \int_{\partial B_r} |\nabla_x v|^2 |y|^a\right].\end{aligned}$$

Next, we note that

$$[\nabla_x v]_{C^{0,s}(B_{3/4R})} \leq \frac{C_{n,s}}{R^{1+s}} \|v\|_{L^\infty(B_R)}.$$

Indeed, this follows from the known interior regularity for solutions of the Signorini problem for L_a in B_1 in the case $R = 1$, see e.g. [CSS08], and a simple scaling argument for all $R > 0$. Noting also that $\nabla_x v(0) = 0$, since v attains its minimum on B'_r at 0, we have that for $X \in \overline{B_r}$ with $r < (3/4)R$

$$|\nabla_x v(X)| = |\nabla_x v(X) - \nabla_x v(0)| \leq \frac{C}{R^{1+s}} \|v\|_{L^\infty(B_R)} r^s$$

and so

$$\frac{1}{r} \int_{B_r} |\nabla_x v|^2 |y|^a + \int_{\partial B_r} |\nabla_x v|^2 |y|^a \leq C\|v\|^2_{L^\infty(B_R)} \frac{r^{n+1}}{R^{2+2s}}.$$

This gives

$$\varphi'(r) \geq -\frac{C}{r^{a+2}} \|v\|^2_{L^\infty(B_R)} \frac{1}{R^{2+2s}}.$$

Thus, for $0 < \rho < r < (3/4)R$,

$$\begin{aligned}\varphi(r) - \varphi(\rho) &= \int_\rho^r \varphi'(t)\, dt \\ &\geq -C\|v\|^2_{L^\infty(B_R)} \frac{\rho^{-1-a} - r^{-1-a}}{R^{2+2s}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{B_\rho}&|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a\\ &=\rho^{n+a+3}\varphi(\rho)\\ &\le\rho^{n+a+3}\left(\varphi(r)+C\|v\|^2_{L^\infty(B_R)}\frac{\rho^{-1-a}-r^{-1-a}}{R^{2+2s}}\right)\\ &\le\left(\frac{\rho}{r}\right)^{n+a+3}\int_{B_r}|\nabla_x v-\langle\nabla_x v\rangle_r|^2|y|^a+C\|v\|^2_{L^\infty(B_R)}\frac{\rho^{n+2}}{R^{2+2s}}.\qquad\square\end{aligned}$$

Lemma 3.5. *Let v be a solution of the Signorini problem for L_a in B_R, even in y. Then there are $C_1=C_{n,a}$, $C_2=C_{n,a}$ such that for all $0<\rho<S<(3/8)R$,*

$$\begin{aligned}\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a\le C_1\left(\frac{\rho}{S}\right)^{n+a+3}\int_{B_S}|\nabla_x v-\langle\nabla_x v\rangle_S|^2|y|^a&\\ +C_2\|v\|^2_{L^\infty(B_R)}\frac{S^{n+2}}{R^{2+2s}}.&\end{aligned}$$

Proof. If $\rho\ge S/8$, then we immediately have

$$\begin{aligned}\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a&\le C\left(\frac{8\rho}{S}\right)^{n+a+3}\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a\\ &\le C\left(\frac{\rho}{S}\right)^{n+a+3}\int_{B_S}|\nabla_x v-\langle\nabla_x v\rangle_S|^2|y|^a.\end{aligned}$$

Thus we may assume $\rho<S/8$. Due to Lemma 3.4, we may assume $v(0)>0$. Let $d:=\operatorname{dist}(0,\{v(\cdot,0)=0\})>0$. Then $L_a v=0$ in B_d. Thus, if $d\ge S$, we may use Lemma 3.2 to obtain

$$\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a\le\left(\frac{\rho}{S}\right)^{n+a+3}\int_{B_S}|\nabla_x v-\langle\nabla_x v\rangle_S|^2|y|^a.$$

Thus we may also assume $d<S$.

Case 1. $S/4\le d\,(<S)$.

Case 1.1. Suppose $0<\rho<d\,(<S)$. Then using $L_a(\nabla_x v)=0$ in B_d again,

$$\begin{aligned}\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a&\le\left(\frac{\rho}{d}\right)^{n+a+3}\int_{B_d}|\nabla_x v-\langle\nabla_x v\rangle_d|^2|y|^a\\ &\le C\left(\frac{\rho}{S}\right)^{n+a+3}\int_{B_S}|\nabla_x v-\langle\nabla_x v\rangle_S|^2|y|^a.\end{aligned}$$

Case 1.2. Suppose $\rho\ge d\,(\ge S/4)$. Then

$$\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a\le\left(\frac{4\rho}{S}\right)^{n+a+3}\int_{B_S}|\nabla_x v-\langle\nabla_x v\rangle_S|^2|y|^a.$$

Case 2. $0<d<S/4$.

Case 2.1. Suppose $\rho<d/2$. Take $x_1\in\partial(B'_d)$ such that $v(x_1)=0$. Then using inclusions $B_\rho\subset B_{d/2}\subset B_{(3/2)d}(x_1)\subset B_{S/2}(x_1)\subset B_{R/2}(x_1)$, $L_a v=0$ in B_d and

the preceding Lemma 3.4, we obtain

$$
\begin{aligned}
&\int_{B_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a \\
&\quad\le \left(\frac{2\rho}{d}\right)^{n+a+3} \int_{B_{d/2}} |\nabla_x v - \langle \nabla_x v\rangle_{d/2}|^2 |y|^a \\
&\quad\le \left(\frac{2\rho}{d}\right)^{n+a+3} \int_{B_{(3/2)d}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,(3/2)d}|^2 |y|^a \\
&\quad\le \left(\frac{2\rho}{d}\right)^{n+a+3} \Bigg[\left(\frac{3d}{S}\right)^{n+a+3} \int_{B_{S/2}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,S/2}|^s |y|^a \\
&\qquad\qquad + C\|v\|^2_{L^\infty(B_{R/2}(x_1))} \frac{S^{n+2}}{R^{2+2s}}\Bigg] \\
&\quad\le C\left(\frac{\rho}{S}\right)^{n+a+3} \int_{B_S} |\nabla_x v - \langle \nabla_x v\rangle_S|^2 |y|^a + C\|v\|^2_{L^\infty(B_R)} \frac{S^{n+2}}{R^{2+2s}}.
\end{aligned}
$$

Case 2.2. Suppose $d/2 \le \rho$. Then we see that $B_\rho \subset B_{3\rho}(x_1) \subset B_{S/2}(x_1) \subset B_S$. As we did in Case 2.1, we have

$$
\begin{aligned}
&\int_{B_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a \\
&\quad\le \int_{B_{3\rho}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,3\rho}|^2 |y|^a \\
&\quad\le C\left(\frac{\rho}{S}\right)^{n+a+3} \int_{B_{S/2}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,S/2}|^2 |y|^a \\
&\qquad\qquad + C\|v\|^2_{L^\infty(B_{R/2}(x_1))} \frac{S^{n+2}}{R^{2+2s}} \\
&\quad\le C\left(\frac{\rho}{S}\right)^{n+a+3} \int_{B_S} |\nabla_x v - \langle \nabla_x v\rangle_S|^2 |y|^a + C\|v\|^2_{L^\infty(B_R)} \frac{S^{n+2}}{R^{2+2s}}. \qquad \square
\end{aligned}
$$

Corollary 3.6. *Let v be a solution of the Signorini problem for L_a in B_R, even in y. Then there are $C_1 = C_{n,a}$, $C_2 = C_{n,a}$ such that for all $0 < \rho < S < (3/16)R$,*

$$
\begin{aligned}
\int_{B_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a \le C_1 \left(\frac{\rho}{S}\right)^{n+a+3} \int_{B_S} |\nabla_x v - \langle \nabla_x v\rangle_S|^2 |y|^2 \\
+ C_2 \langle v^2\rangle_R \frac{S^{n+2}}{R^{2+2s}}.
\end{aligned}
$$

Proof. Since $v^\pm = \max(\pm v, 0) \ge 0$ and $L_a(v^\pm) = 0$ in $\{v^\pm > 0\}$, we have $L_a(v^\pm) \ge 0$ in B_R. (For this, one may follow the argument in Exercise 2.6 or Exercise 9.5 in [PSU12].) Thus, we have by Theorem 2.3.1 in [FKS82]

$$
\sup_{B_{R/2}} v^\pm \le C\left(\frac{1}{\omega_{n+1+a} R^{n+1+a}} \int_{B_R} \left(v^\pm\right)^2 |y|^a\right)^{1/2}.
$$

Hence,

$$
\|v\|^2_{L^\infty(B_{R/2})} \le C\langle v^2\rangle_R,
$$

which completes the proof. $\square$

4. Almost s-fractional harmonic functions

In this section we prove Theorem I, by deducing growth estimates for almost s-fractional harmonic functions from that of s-fractional harmonic functions and then applying the Morrey-Campanato space embedding to deduce the regularity of almost s-fractional harmonic functions.

Theorem 4.1 (Almost Lipschitz regularity)**.** *If u is an almost s-fractional harmonic function in B_1', $0<s<1$, then $u \in C^{0,\sigma}(B_1')$ for any $0<\sigma<1$.*

Besides the growth estimates for minimizers we will also need the following lemma.

Lemma 4.2. *Let $r_0>0$ be a positive number and let $\varphi:(0,r_0)\to(0,\infty)$ be a nondecreasing function. Let a, β, and γ be such that $a>0$, $\gamma>\beta>0$. There exist two positive numbers $\varepsilon=\varepsilon_{a,\gamma,\beta}$, $c=c_{a,\gamma,\beta}$ such that, if*

$$\varphi(\rho)\le a\Big[\Big(\frac{\rho}{r}\Big)^{\gamma}+\varepsilon\Big]\varphi(r)+b\,r^{\beta}$$

for all ρ, r with $0<\rho\le r<r_0$, where $b\ge 0$, then one also has, still for $0<\rho<r<r_0$,

$$\varphi(\rho)\le c\Big[\Big(\frac{\rho}{r}\Big)^{\beta}\varphi(r)+b\rho^{\beta}\Big].$$

Proof. See Lemma 3.4 in [HL97]. □

Proof of Theorem 4.1. Let K be a compact subset of B_1' containing 0. Take $\delta=\delta_{n,\omega,\sigma,K}>0$ such that $\delta<\operatorname{dist}(K,\partial B_1')$ and $\omega(\delta)\le\varepsilon$, where $\varepsilon=\varepsilon_{2,n+1+a,n-1+a+2\sigma}$ is as Lemma 4.2. For $0<R<\delta$, let v be a minimizer of

$$\int_{B_R}|\nabla v|^2|y|^a$$

on $u+W_0^{1,2}(B_R)$. Then $L_a v=0$ in B_R. In particular,

$$\int_{B_R}\nabla v\cdot\nabla(u-v)|y|^a=0,$$

and hence

$$\begin{aligned}\int_{B_R}|\nabla(u-v)|^2|y|^a&=\int_{B_R}|\nabla u|^2|y|^a-\int_{B_R}|\nabla v|^2|y|^a-2\int_{B_R}\nabla v\cdot\nabla(u-v)|y|^a\\&\le\omega(R)\int_{B_R}|\nabla v|^2|y|^a.\end{aligned}$$

Moreover, by Lemma 3.1, for $0<\rho<R$ we have

$$\int_{B_\rho}|\nabla v|^2|y|^a\le\Big(\frac{\rho}{R}\Big)^{n+1+a}\int_{B_R}|\nabla v|^2|y|^a.$$

Thus

$$\begin{aligned}\int_{B_\rho}|\nabla u|^2|y|^a&\le 2\int_{B_\rho}|\nabla v|^2|y|^a+2\int_{B_\rho}|\nabla(u-v)|^2|y|^a\\&\le 2\Big(\frac{\rho}{R}\Big)^{n+1+a}\int_{B_R}|\nabla v|^2|y|^a+2\int_{B_\rho}|\nabla(u-v)|^2|y|^a\end{aligned}$$

$$
\begin{aligned}
&\le 2\left(\frac{\rho}{R}\right)^{n+1+a}\int_{B_R}|\nabla v|^2|y|^a + 2\omega(R)\int_{B_R}|\nabla v|^2|y|^a\\
&\le 2\left[\left(\frac{\rho}{R}\right)^{n+1+a}+\varepsilon\right]\int_{B_R}|\nabla u|^2|y|^a.
\end{aligned}
$$

By Lemma 4.2,

$$
\int_{B_\rho}|\nabla u|^2|y|^a \le C_{n,a,\sigma}\left(\frac{\rho}{R}\right)^{n-1+a+2\sigma}\int_{B_R}|\nabla u|^2|y|^a,
$$

for any $0<\sigma<1$. Taking $R\nearrow\delta$ we have

$$
\int_{B_\rho}|\nabla u|^2|y|^a \le C_{n,a,\sigma,\delta}\|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n-1+a+2\sigma}. \tag{4.1}
$$

By weighted Poincaré inequality [FKS82, Theorem (1.5)]

$$
\int_{B_\rho}|u-\langle u\rangle_\rho|^2|y|^a \le C_{n,a,\sigma,\delta}\|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n+1+a+2\sigma}.
$$

Now, a similar estimates holds at all point $x_0\in K$, which implies the Hölder continuity of u (see Theorem A.1) with

$$
\|u\|_{C^{0,\sigma}(K)} \le C_{n,a,\omega,\sigma,K}\|u\|_{W^{1,2}(B_1,|y|^a)}. \qquad \square
$$

Theorem 4.3 ($C^{1,\beta}$ regularity)**.** *If u is an almost s-fractional harmonic function in B_1', $0<s<1$, with gauge function $\omega(r)=r^\alpha$, $\alpha>0$, then $\nabla_x u\in C^{0,\beta}(B_1')$ for some $\beta=\beta(n,s,\alpha)$.*

Proof. Let $K\Subset B_1'$ be a ball and take $0<\delta<\operatorname{dist}(K,\partial B_1')$. Let $B_R'(x_0)\Subset B_1'$ with $0<R<\delta$, for $x_0\in K$. For simplicity write $x_0=0$, and let v be the L_a-harmonic function in B_R with $v=u$ on ∂B_R. Then, by Jensen's inequality we have

$$
\int_{B_\rho}|\langle\nabla_x u\rangle_\rho-\langle\nabla_x v\rangle_\rho|^2|y|^a \le \int_{B_\rho}|\nabla_x u-\nabla_x v|^2|y|^a,
$$

and hence

$$
\begin{aligned}
\int_{B_\rho}|\nabla_x u-\langle\nabla_x u\rangle_\rho|^2|y|^a &\le 3\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a + 3\int_{B_\rho}|\nabla_x u-\nabla_x v|^2|y|^a\\
&\quad + 3\int_{B_\rho}|\langle\nabla_x u\rangle_\rho-\langle\nabla_x v\rangle_\rho|^2|y|^a\\
&\le 3\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a + 6\int_{B_\rho}|\nabla_x u-\nabla_x v|^2|y|^a.
\end{aligned}
$$

Similarly,

$$
\int_{B_R}|\nabla_x v-\langle\nabla_x v\rangle_R|^2|y|^a \le 3\int_{B_R}|\nabla_x u-\langle\nabla_x u\rangle_R|^2|y|^a + 6\int_{B_R}|\nabla_x u-\nabla_x v|^2|y|^a.
$$

Next let $\beta\in(0,\alpha/2)$. Then using the estimate (4.1) in the proof of Theorem 4.1 with $\sigma=1+\beta-\frac{\alpha}{2}$, we have

$$
\begin{aligned}
\int_{B_R}|\nabla u-\nabla v|^2|y|^a &= \int_{B_R}|\nabla u|^2|y|^a - \int_{B_R}|\nabla v|^2|y|^a\\
&\le R^\alpha\int_{B_R}|\nabla u|^2|y|^a\\
&\le C\|\nabla u\|^2_{L^2(B_1,|y|^a)}R^{n+1+a+2\beta}.
\end{aligned}
$$

Then, with the help of Lemma 3.2, we have that for $\rho < R$

$$\begin{aligned}
&\int_{B_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2 |y|^a \\
&\quad\le C\int_{B_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a + C\int_{B_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \\
&\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R} |\nabla_x v - \langle \nabla_x v\rangle_R|^2 |y|^a + C\int_{B_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \\
&\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2 |y|^a + C\int_{B_R} |\nabla_x u - \nabla_x v|^2 |y|^a \\
&\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2 |y|^a + C\|\nabla u\|^2_{L^2(B_1,|y|^a)} R^{n+1+a+2\beta}.
\end{aligned}$$

Hence, by Lemma 4.2, we obtain that for $\rho < R$

$$\begin{aligned}
&\int_{B_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2 |y|^a \\
&\le C\left[\left(\frac{\rho}{R}\right)^{n+1+a+2\beta}\int_{B_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2 |y|^a + \|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n+1+a+2\beta}\right].
\end{aligned}$$

Taking $R \nearrow \delta$, we have

$$\int_{B_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2 |y|^a \le C_{n,a,\alpha,\beta,K}\|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n+1+a+2\beta}.$$

Now, a similar estimate holds for any $x_0 \in K$. Fixing β and applying Theorem A.1, we have

$$\|\nabla_x u\|_{C^{0,\beta}(K)} \le C_{n,a,\alpha,K}\|u\|_{W^{1,2}(B_1,|y|^a)}. \qquad \square$$

Remark 4.4. From the assumption for almost minimizers that the Caffarelli-Silvestre extension $u \in W^{1,2}_{\mathrm{loc}}$ we know only that $\nabla_x u \in L^2_{\mathrm{loc}}$, which is not sufficient to deduce the existence of the trace of $\nabla_x u$ on B_1'. However, in the proof of Theorem 4.3 we showed that $\nabla_x u$ is in a Morrey-Campanato space, which implies the existence of the trace as the limit of averages

$$T(\nabla_x u)(x_0) = \lim_{r\to 0+}\langle \nabla_x u\rangle_{x_0,r}.$$

It is not hard to see that $T(\nabla_x u)$ is the distributional derivative $\nabla_x u$ on B_1'. Indeed, if $\eta \in C^\infty_0(B_1')$, then extending it to $\mathbb{R}^{n+1}$ by $\eta(x,y) = \eta(x)$, we have

$$\begin{aligned}
\int_{B_1'} T(\partial_{x_i} u)\eta &= \lim_{r\to 0+}\int_{B_1'}\langle \partial_{x_i} u\rangle_{x,r}\eta = \lim_{r\to 0+}\int_{B_1'}\partial_{x_i} u\langle \eta\rangle_{x,r} \\
&= \lim_{r\to 0+} -\int_{B_1'} u\langle \partial_{x_i}\eta\rangle_{x,r} = -\int_{B_1'} u\partial_{x_i}\eta.
\end{aligned}$$

Theorem 4.5. *Let u be an almost s-fractional harmonic function in B_1' for $0 < s < 1/2$ or $s = 1/2$ and a gauge function $\omega(r) = r^\alpha$ for some $\alpha > 0$. Then u is actually s-fractional harmonic in B_1'.*

Proof. We argue as in the proof Theorem 4.1. Let K, δ, R, v be as in the proof of that theorem. Then, by Lemma 3.1, for $0 < \rho < R$

$$\int_{B_\rho} |v_y|^2 |y|^a \le \left(\frac{\rho}{R}\right)^{n+3+a}\int_{B_R} |v_y|^2 |y|^a.$$

Thus, for any $0<\sigma<1$, we have

$$\begin{aligned}\int_{B_\rho}||y|^a u_y|^2|y|^{-a} &\leq 2\int_{B_\rho}|v_y|^2|y|^a+2\int_{B_\rho}|u_y-v_y|^2|y|^a\\ &\leq 2\left(\frac{\rho}{R}\right)^{n+3+a}\int_{B_R}|v_y|^2|y|^a+2\int_{B_\rho}|u_y-v_y|^2|y|^a\\ &\leq 4\left(\frac{\rho}{R}\right)^{n+3+a}\int_{B_R}|u_y|^2|y|^a+6\int_{B_R}|u_y-v_y|^2|y|^a\\ &\leq 4\left(\frac{\rho}{R}\right)^{n+3+a}\int_{B_R}||y|^a u_y|^2|y|^{-a}+6\omega(R)\int_{B_R}|\nabla u|^2|y|^a\\ &\leq 4\left(\frac{\rho}{R}\right)^{n+3+a}\int_{B_R}||y|^a u_y|^2|y|^{-a}\\ &\qquad+C_{n,a,\sigma,\delta}\,\omega(R)\|\nabla u\|^2_{L^2(B_1,|y|^a)}R^{n-1+a+2\sigma},\end{aligned}$$

where we used (4.1) in the last inequality.

Consider now the two cases in statement of the theorem.

Case 1. $0<s<1/2$ (or $a>0$). In this case by Lemma 4.2,

$$\begin{aligned}&\int_{B_\rho}||y|^a u_y|^2|y|^{-a}\\ &\quad\leq C\left[\left(\frac{\rho}{R}\right)^{n-1+a+2\sigma}\int_{B_R}||y|^a u_y|^2|y|^{-a}+\omega(\delta)\|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n-1+a+2\sigma}\right]\\ &\quad\leq C\|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n+1-a+(-2+2a+2\sigma)}.\end{aligned}$$

Now we take $\sigma=1-a/2\in(0,1)$ to have $-2+2a+2\sigma=a>0$. Varying the center, we have a similar bound at every $x\in K$. Then, by Theorem A.1, we obtain that the limit of the averages $T(y|y|^{a-1}u_y)=0$ on B_1'. This implies that $(-\Delta_x)^s u=0$ on B_1'. Indeed, arguing as in Remark 4.4, by considering the mollifications u_ε in x-variable, we note that

$$\int_{B_\rho}||y|^a(u_\varepsilon)_y|^2|y|^{-a}\leq C\rho^{n+1-a+a}$$

which implies that $T(y|y|^{a-1}(u_\varepsilon)_y)=0$ on $K\Subset B_1'$. On the other hand, $u_\varepsilon\in C^2\cap\mathscr{L}_s(\mathbb{R}^n)$, which implies that $y|y|^{a-1}(u_\varepsilon)_y$ is continuous up to $y=0$, since we can explicitly write, for $y>0$, the symmetrized formula

$$y^a(u_\varepsilon)_y(x,y)=\int_{\mathbb{R}^n}\frac{u_\varepsilon(x+z)+u_\varepsilon(x-z)-2u_\varepsilon(x)}{|z|^2}|z|^2y^a\partial_yP(z,y)dz$$

with locally integrable kernel $|z|^2|y^a\partial_yP(z,y)|\leq C/|z|^{n-1-a}$. Hence, we obtain that $(-\Delta_x)^s u_\varepsilon=\partial_y^a u_\varepsilon=0$ on the ball $K\Subset B_1'$. Then, passing to the limit as $\varepsilon\to0$, this implies that $(-\Delta_x)^s u=0$ in B_1'.

Case 2. $s=1/2$ (or $a=0$) and $\omega(r)=r^\alpha$. In this case, we have a bound

$$\int_{B_\rho}|u_y|^2\leq 4\left(\frac{\rho}{R}\right)^{n+3}\int_{B_R}|u_y|^2+C\|\nabla u\|^2_{L^2(B_1)}R^{n-1+2\sigma+\alpha}.$$

Then, by Lemma 4.2, we have

$$\begin{aligned}\int_{B_\rho}|u_y|^2 &\le C\left[\left(\frac{\rho}{R}\right)^{n-1+2\sigma+\alpha}\int_{B_R}|u_y|^2+\|\nabla u\|^2_{L^2(B_1)}\rho^{n-1+2\sigma+\alpha}\right]\\ &\le C\|\nabla u\|^2_{L^2(B_1)}\rho^{n+1+(\alpha-2+2\sigma)}.\end{aligned}$$

Taking $1-\alpha/4<\sigma<1$, we can guarantee that $\alpha-2+2\sigma>\alpha/2>0$, which implies that $T(y|y|^{-1}u_y)=0$ on B_1'. Then, arguing as at the end of Case 1, we conclude that $(-\Delta_x)^{1/2}u=0$ in B_1'. □

We finish this section with formal proof of Theorem I.

Proof of Theorem I. Parts (1), (2), and (3) are proved in Theorems 4.1, 4.3, and 4.5, respectively. □

5. Almost minimizers for s-fractional obstacle problem

In this section we investigate the regularity of almost minimizers for the s-fractional obstacle problem with zero obstacle and give a proof of Theorem II. All results in this section are proved under the assumption $1/2\le s<1$, or $-1<a\le 0$.

Theorem 5.1 (Almost Lipschitz regularity)**.** *Let u be an almost minimizer for s-fractional obstacle problem with zero obstacle in B_1', for $1/2\le s<1$. Then $u\in C^{0,\sigma}(B_1')$ for any $0<\sigma<1$ with*

$$\|u\|_{C^{0,\sigma}(K)}\le C_{n,a,\omega,\sigma,K}\|u\|_{W^{1,2}(B_1,|y|^a)}$$

for any $K\Subset B_1'$.

Proof. Let $K\Subset B_1'$ with $0\in K$. Take $\delta=\delta_{n,a,\omega,\sigma,K}>0$ such that $\delta<\operatorname{dist}(K,\partial B_1')$ and $\omega(\delta)\le\varepsilon$, where $\varepsilon=\varepsilon_{2,n+1+a,n-1+a+2\sigma}$ as in Lemma 4.2. For $0<R<\delta$, let v be the minimizer of

$$\int_{B_R}|\nabla v|^2|y|^a$$

on $\mathfrak{K}_{0,u}(B_R,|y|^a)$. Then v satisfies the variational inequality

$$\int_{B_R}\nabla v\nabla(w-v)|y|^a\ge 0$$

for any $w\in\mathfrak{K}_{0,u}(B_R,|y|^a)$. Particularly, taking $w=u$, we have

$$\int_{B_R}\nabla v\nabla(u-v)|y|^a\ge 0.$$

As a consequence,

$$\begin{aligned}\int_{B_R}|\nabla(u-v)|^2|y|^a &=\int_{B_R}|\nabla u|^2|y|^a-\int_{B_R}|\nabla v|^2|y|^a-2\int_{B_R}\nabla v\cdot\nabla(u-v)|y|^a\\ &\le\omega(R)\int_{B_R}|\nabla v|^2|y|^a.\end{aligned}$$

Next, we use (3.2) to derive a similar estimate for u. We have,

$$\begin{aligned}\int_{B_\rho}|\nabla u|^2|y|^a &\le 2\int_{B_\rho}|\nabla v|^2|y|^a+2\int_{B_\rho}|\nabla(u-v)|^2|y|^a\\ &\le 2\left(\frac{\rho}{R}\right)^{n+1+a}\int_{B_R}|\nabla v|^2|y|^a+2\omega(R)\int_{B_R}|\nabla v|^2|y|^a\end{aligned}$$

$$\leq 2\left[\left(\frac{\rho}{R}\right)^{n+1+a}+\varepsilon\right]\int_{B_R}|\nabla u|^2|y|^a.$$

Hence, by Lemma 4.2,

$$\int_{B_\rho}|\nabla u|^2|y|^a \leq C_{n,a,\sigma}\left(\frac{\rho}{R}\right)^{n-1+a+2\sigma}\int_{B_R}|\nabla u|^2|y|^a.$$

As we have seen in Theorem 4.1, this implies

$$\int_{B_\rho}|\nabla u|^2|y|^a \leq C_{n,a,\sigma,\delta}\|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n-1+a+2\sigma} \tag{5.1}$$

then

$$\int_{B_\rho}|u-\langle u\rangle_\rho|^2|y|^a \leq C_{n,a,\sigma,\delta}\|\nabla u\|^2_{L^2(B_1,|y|^a)}\rho^{n+1+a+2\sigma}$$

and ultimately

$$\|u\|_{C^{0,\sigma}(K)} \leq C_{n,a,\omega,\sigma,K}\|u\|_{W^{1,2}(B_1,|y|^a)}.$$

□

Theorem 5.2 ($C^{1,\beta}$ regularity)**.** *Let u be an almost minimizer for the s-fractional obstacle problem with zero obstacle in B_1', $1/2 \leq s < 1$, and a gauge function $\omega(r) = r^\alpha$. Then $\nabla_x u \in C^{0,\beta}(B_1')$ for $\beta < \frac{\alpha s}{8(n+1+a+\alpha/2)}$ and for any $K \Subset B_1'$ there holds*

$$\|\nabla_x u\|_{C^{0,\beta}(K)} \leq C_{n,a,\alpha,\beta,K}\|u\|_{W^{1,2}(B_1,|y|^a)}.$$

Proof. Let K be a thin ball centered at 0 such that $K \Subset B_1$. Let $\varepsilon := \frac{\alpha}{4(n+1+a+\alpha/2)}$ and $\gamma := 1-\frac{s\varepsilon}{2(1-\varepsilon)}$. We fix $R_0 = R_0(n,a,\alpha,K) > 0$ small so that $R_0^{1-\varepsilon} \leq d/2$, where $d := \operatorname{dist}(K,\partial B_1')$ and $R_0 < \left(\frac{3}{16}\right)^{1/\varepsilon}$. Then $\widetilde{K} := \{x \in B_1' : \operatorname{dist}(x,K) \leq R_0^{1-\varepsilon}\} \Subset B_1$. We claim that for $x_0 \in K$ and $0<\rho<R<R_0$,

$$\begin{aligned}
&\int_{B_\rho(x_0)}|\nabla_x u - \langle\nabla_x u\rangle_{x_0,\rho}|^2|y|^a\\
&\quad\leq C_{n,a}\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R(x_0)}|\nabla_x u-\langle\nabla_x u\rangle_{x_0,R}|^2|y|^a\\
&\qquad + C_{n,a,\alpha,K}\|u\|^2_{W^{1,2}(B_1,|y|^a)}R^{n+1+a+s\varepsilon}.
\end{aligned} \tag{5.2}$$

Note that once we have this bound, the proof will follow by the application of Lemma 4.2 and Theorem A.1.

For simplicity we may assume $x_0 = 0$, and fix $0<R<R_0$. Let $\overline{R} := R^{1-\varepsilon}$. Let v be the minimizer of

$$\int_{B_{\overline{R}}}|\nabla v|^2|y|^a$$

on $\mathfrak{K}_{0,u}(B_{\overline{R}},|y|^a)$. Then by (3.2) and (5.1) with $\sigma=\gamma$, for $0<\rho\leq\overline{R}$

$$\begin{aligned}
\int_{B_\rho}|\nabla v|^2|y|^a &\leq \left(\frac{\rho}{\overline{R}}\right)^{n+1+a}\int_{B_{\overline{R}}}|\nabla v|^2|y|^a\\
&\leq \left(\frac{\rho}{\overline{R}}\right)^{n+1+a}\int_{B_{\overline{R}}}|\nabla u|^2|y|^a\\
&\leq C_{n,a,\alpha,K}\left(\frac{\rho}{\overline{R}}\right)^{n+1+a}\|u\|^2_{W^{1,2}(B_1,|y|^a)}\overline{R}^{n-1+a+2\gamma}\\
&\leq C_{n,a,\alpha,K}\|u\|^2_{W^{1,2}(B_1,|y|^a)}\rho^{n-1+a+2\gamma}.
\end{aligned} \tag{5.3}$$

This gives

$$\fint_{B_\rho} |v - v_\rho|^2 |y|^a \le C_1 \|u\|^2_{W^{1,2}(B_1,|y|^a)} \rho^{2\gamma}, \quad C_1 = C_{n,a,\alpha,K}. \tag{5.4}$$

Since this estimate holds for any $0 < \rho < \overline{R}$, the standard dyadic argument gives

$$|v(0) - \langle v \rangle_{\overline{R}}| \le C_2 \|u\|_{W^{1,2}(B_1,|y|^a)} \overline{R}^{\gamma}, \quad C_2 = C_{n,a,\alpha,K}. \tag{5.5}$$

Moreover, using (3.2) and (5.1) again, we have for any $x_1 \in B'_{\overline{R}/2}$, $0 < \rho < \overline{R}/2$,

$$\begin{aligned}
\int_{B_\rho(x_1)} |\nabla v|^2 |y|^a &\le \left(\frac{2\rho}{\overline{R}}\right)^{n+1+a} \int_{B_{\overline{R}/2}(x_1)} |\nabla v|^2 |y|^a \\
&\le \left(\frac{2\rho}{\overline{R}}\right)^{n+1+a} \int_{B_{\overline{R}}} |\nabla u|^2 |y|^a \\
&\le C_{n,a,\alpha,K} \|u\|^2_{W^{1,2}(B_1,|y|^a)} \rho^{n-1+a+2\gamma},
\end{aligned} \tag{5.6}$$

which implies

$$[v]_{C^{0,\gamma}(\overline{B'_{R/2}})} \le C_3 \|u\|_{W^{1,2}(B_1,|y|^a)}, \quad C_3 = C_{n,a,\alpha,K}. \tag{5.7}$$

Now we define

$$C_4 := C_1 + C_2^2 + C_3^2.$$

Our analysis then distinguishes the following two cases

$$\langle v^2 \rangle_{\overline{R}} \le 6\, C_4 \|u\|^2_{W^{1,2}(B_1,|y|^a)} \overline{R}^{2\gamma} \qquad \text{or} \qquad \langle v^2 \rangle_{\overline{R}} > 6\, C_4 \|u\|^2_{W^{1,2}(B_1,|y|^a)} \overline{R}^{2\gamma}.$$

Case 1. Suppose first that

$$\langle v^2 \rangle_{\overline{R}} \le 6 C_4 \|u\|^2_{W^{1,2}(B_1,|y|^a)} \overline{R}^{2\gamma}.$$

Note that $R_0 < \left(\frac{3}{16}\right)^{1/\varepsilon}$ implies $R < \frac{3}{16}\overline{R}$. Then, using Corollary 3.6, we see that for $0 < \rho < R$,

$$\begin{aligned}
\int_{B_\rho} |\nabla_x u - \langle \nabla_x u \rangle_\rho|^2 |y|^a &\le 3 \int_{B_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a + 6 \int_{B_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \, dx \\
&\le C_{n,a} \left(\frac{\rho}{R}\right)^{n+a+3} \int_{B_R} |\nabla_x v - \langle \nabla_x v \rangle_R|^2 |y|^a \\
&\qquad + C_{n,a} \langle v^2 \rangle_{\overline{R}} \frac{R^{n+2}}{\overline{R}^{2+2s}} + 6 \int_{B_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \\
&\le C \left(\frac{\rho}{R}\right)^{n+a+3} \int_{B_R} |\nabla_x u - \langle \nabla_x u \rangle_R|^2 |y|^a \\
&\qquad + C \langle v^2 \rangle_{\overline{R}} \frac{R^{n+2}}{\overline{R}^{2+2s}} + C \int_{B_R} |\nabla_x u - \nabla_x v|^2 |y|^a.
\end{aligned}$$

Note that for $\sigma := 1 - \alpha/4$

$$\begin{aligned}
\int_{B_R} |\nabla_x u - \nabla_x v|^2 |y|^a &\le \int_{B_{\overline{R}}} |\nabla_x u - \nabla_x v|^2 |y|^a \\
&\le \overline{R}^{\alpha} \int_{B_{\overline{R}}} |\nabla v|^2 |y|^a
\end{aligned}$$

$$\begin{aligned}&\le \overline{R}^{\alpha}\int_{B_{\overline{R}}}|\nabla u|^2|y|^a\\&\le C_{n,a,\alpha,K}\overline{R}^{\alpha}\|u\|^2_{W^{1,2}(B_1,|y|^a)}\overline{R}^{n-1+a+2\sigma}\\&=C\|u\|^2_{W^{1,2}(B_1,|y|^a)}R^{n+1+a+\alpha/4}.\end{aligned}$$

Moreover by the assumption

$$\begin{aligned}C\langle v^2\rangle_{\overline{R}}\frac{R^{n+2}}{\overline{R}^{2+2s}}&\le C_{n,a,\alpha,K}\|u\|^2_{W^{1,2}(B_1,|y|^a)}R^{n+2}\overline{R}^{2\gamma-2-2s}\\&=C\|u\|^2_{W^{1,2}(B_1,|y|^a)}R^{n+1+a+s\varepsilon}.\end{aligned}$$

Hence, we obtain (5.2) in this case.

Case 2. Now we assume

$$\langle v^2\rangle_{\overline{R}}>6\,C_4\|u\|^2_{W^{1,2}(B_1,|y|^a)}\overline{R}^{2\gamma}.$$

Then, by (5.4) and (5.5) we obtain

$$\begin{aligned}⨍_{B_{\overline{R}}}|v-v(0)|^2|y|^a&\le 2⨍_{B_{\overline{R}}}|v-v_{\overline{R}}|^2|y|^a+2⨍_{B_{\overline{R}}}|v_{\overline{R}}-v(0)|^2|y|^a\\&\le 2C_4\|u\|^2_{W^{1,2}(B_1,|y|^a)}\overline{R}^{2\gamma}.\end{aligned}$$

Combining the latter bound and the assumption,

$$\begin{aligned}v(0)^2&=⨍_{B_{\overline{R}}}|v(0)|^2|y|^a\\&\ge\frac{1}{2}⨍_{B_{\overline{R}}}|v(X)|^2|y|^a-⨍_{B_{\overline{R}}}|v(X)-v(0)|^2|y|^a\\&\ge C_4\|u\|^2_{W^{1,2}(B_1,|y|^a)}\overline{R}^{2\gamma}.\end{aligned}$$

Since $C_4\ge C_3^2$, we have $v>0$ on $B'_{\overline{R}/2}$ by (5.7). Thus, $L_av=0$ in $B_{\overline{R}/2}$, and by Lemma 3.2 we have for $0<\rho<R$

$$\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a\le\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R}|\nabla_x v-\langle\nabla_x v\rangle_R|^2|y|^a.$$

Thus,

$$\begin{aligned}&\int_{B_\rho}|\nabla_x u-\langle\nabla_x u\rangle_\rho|^2|y|^a\\&\quad\le 3\int_{B_\rho}|\nabla_x v-\langle\nabla_x v\rangle_\rho|^2|y|^a+6\int_{B_\rho}|\nabla_x u-\nabla_x v|^2|y|^a\\&\quad\le 3\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R}|\nabla_x v-\langle\nabla_x v\rangle_R|^2|y|^a+6\int_{B_\rho}|\nabla_x u-\nabla_x v|^2|y|^a\\&\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R}|\nabla_x u-\langle\nabla_x u\rangle_R|^2|y|^a+C\int_{B_R}|\nabla_x u-\nabla_x v|^2|y|^a\\&\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3}\int_{B_R}|\nabla_x u-\langle\nabla_x u\rangle_R|^2|y|^a+C\|u\|^2_{W^{1,2}(B_1,|y|^a)}R^{n+1+a+\alpha/4}.\end{aligned}$$

This implies (5.2) and completes the proof. □

Proof of Theorem II. Parts (1) and (2) are contained in Theorems 5.1 and 5.2, respectively. ◻

Appendix A. Morrey-Campanato-type Space

Theorem A.1. *Let $u \in L^2(B_1, |y|^a)$ and M be such that $\|u\|_{L^2(B_1,|y|^a)} \le M$ and for some $\sigma \in (0,1)$*

$$\int_{B_r(x)} |u - \langle u\rangle_{x,r}|^2 |y|^a \le M^2 r^{n+1+a+2\sigma}, \quad \langle u\rangle_{x,r} = \frac{1}{\omega_{n+1+a} r^{n+1+a}} \int_{B_r(x)} u\,|y|^a$$

for any ball $B_r(x)$ centered at $x = (x,0) \in B'_{1/2}$ and radius $0 < r < r_0 \le 1/2$. Then for any $x \in B'_{1/2}$ there exists the limit of averages

$$Tu(x) := \lim_{r\to 0} \langle u\rangle_{x,r},$$

which will also satisfy

$$\int_{B_r(x)} |u - Tu(x)|^2 |y|^a \le C_{n,a,\sigma} M^2 r^{n+1+a+2\sigma}.$$

Moreover, $Tu \in C^{0,\sigma}(B'_{1/2})$ with

$$\|Tu\|_{C^{0,\sigma}(B'_{1/2})} \le C_{n,a,\sigma,r_0} M.$$

Remark A.2. Note, we can redefine $u(x,0) = Tu(x)$ for any $x \in B'_{1/2}$, making $(x,0)$ a Lebesgue point for u.

Proof. Let $x, z \in B'_{1/2}$ and $0 < \rho < r < r_0$ be such that $B_\rho(x) \subset B_r(z)$. Then

$$\begin{aligned}
|\langle u\rangle_{x,\rho} - \langle u\rangle_{z,r}| &\le \fint_{B_\rho(x)} |u - \langle u\rangle_{z,r}||y|^a \\
&\le \left(\frac{r}{\rho}\right)^{n+1+a} \fint_{B_r(z)} |u - \langle u\rangle_{z,r}||y|^a \\
&\le \left(\frac{r}{\rho}\right)^{n+1+a} \left(\fint_{B_r(z)} |u - \langle u\rangle_{z,r}|^2 |y|^a\right)^{1/2} \left(\fint_{B_r(z)} |y|^a\right)^{1/2} \\
&\le C_{n,a} \left(\frac{r}{\rho}\right)^{n+1+a} M r^\sigma.
\end{aligned}$$

Now, taking $x = z$ and using a dyadic argument, we can conclude that

$$|\langle u\rangle_{x,\rho} - \langle u\rangle_{x,r}| \le C_{n,a,\sigma} M r^\sigma, \quad \text{for any } 0 < s = \rho < r < r_0.$$

Indeed, let $k = 0, 1, 2, \ldots$ be such that $r/2^{k+1} \le \rho < r/2^k$. Then

$$\begin{aligned}
|\langle u\rangle_{x,\rho} - \langle u\rangle_{x,r}| &\le \sum_{j=1}^{k} |\langle u\rangle_{x,r/2^{j-1}} - \langle u\rangle_{x,r/2^j}| + |\langle u\rangle_{x,r/2^k} - \langle u\rangle_{x,\rho}| \\
&\le C_{n,a} M \sum_{j=1}^{k+1} (r/2^{j-1})^\sigma \le C_{n,a,\sigma} M r^\sigma.
\end{aligned}$$

This implies that the limit

$$Tu(x) = \lim_{r\to 0} \langle u\rangle_{x,r}$$

exists and
$$|Tu(x) - \langle u\rangle_{x,r}| \leq C_{n,a,\sigma} M r^\sigma.$$
Hence, we also have the Hölder integral bound
$$\int_{B_r(x)} |u - Tu(x)|^2 |y|^a \leq C_{n,a,\sigma} M^2 r^{n+1+a+2\sigma}.$$
Besides, we have
$$|Tu(x)| \leq \langle u\rangle_{x,r_0} + C_{n,a,\sigma} M r_0^\sigma \leq C_{n,a,\sigma,r_0} M.$$
It remains to estimate the Hölder seminorm of Tu on $B'_{1/2}$. Let $x, z \in B'_{1/2}$ and consider two cases.

Case 1. If $|x-z| < r_0/4$, let $r = 2|x-z|$. Then note that $B_{r/2}(x) \subset B_r(z)$ and therefore we can write
$$\begin{aligned}|Tu(x) - Tu(z)| &\leq |Tu(x) - \langle u\rangle_{x,r/2}| + |Tu(z) - \langle u\rangle_{z,r}| + |\langle u\rangle_{x,r/2} - \langle u\rangle_{z,r}|\\ &\leq C_{n,a,\sigma} M r^\sigma = C_{n,a,\sigma} M |x-z|^\sigma.\end{aligned}$$

Case 2. If $|x-z| \geq r_0/4$, then
$$\begin{aligned}|Tu(x) - Tu(z)| &\leq |Tu(x)| + |Tu(z)|\\ &\leq C_{n,a,\sigma,r_0} M\\ &\leq C_{n,a,\sigma,r_0} M |x-z|^\sigma.\end{aligned}$$
Thus, we conclude
$$\|Tu\|_{C^{0,\sigma}(B'_{1/2})} \leq C_{n,a,\sigma,r_0} M. \qquad \square$$

Appendix B. Polynomial expansion for Caffarelli-Silvestre extension

Some of the results in Section 3 rely on polynomial expansion theorem for L_a-harmonic functions given below.

Theorem B.1. *Let $u \in W^{1,2}(B_1, |y|^a)$, $-1 < a < 1$, be a weak solution of the equation $L_a u = 0$ in B_1, even in y. Then we have the following polynomial expansion:*
$$u(x,y) = \sum_{k=0}^\infty p_k(x,y)$$
locally uniformly in B_1, where $p_k(x,y)$ are L_a-harmonic polynomials, homogeneous of degree k and even in y. Moreover, the polynomials p_k above are orthogonal in $L^2(\partial B_1, |y|^a)$, i.e.,
$$\int_{\partial B_1} p_k p_m |y|^a = 0, \quad k \neq m.$$
In particular, u is real analytic in B_1.

This theorem has the following immediate corollaries, which are of independent interest and are likely known in the literature. We state them here for reader's convenience and for possible future reference.

Corollary B.2. *Let $u \in W^{1,2}(B_1, |y|^a)$, $-1 < a < 1$, be a weak solution of the equation $L_a u = 0$ in B_1. Then, we have a representation*
$$u(x,y) = \varphi(x,y) + y|y|^{-a}\psi(x,y), \quad (x,y) \in B_1,$$
where $\varphi(x,y)$ and $\psi(x,y)$ are real analytic functions, even in y.

Corollary B.3. *Let $u \in \mathscr{L}_s(\mathbb{R}^n)$ satisfies $(-\Delta)^s u = 0$ in the unit ball $B_1' \subset \mathbb{R}^n$. Then u is real analytic in B_1'.*

Corollary B.4. *Let $u \in W^{1,2}(B_1, |y|^a)$, $-1 < a < 1$, be a weak solution of the equation $L_a u = 0$ in B_1, even in y. If $u(\cdot, 0) \equiv 0$ in B_1', then $u \equiv 0$ in B_1.*

The proof of Theorem B.1 and subsequently those of Corollaries B.2, B.3, and B.4 are based on the following lemmas. We follow the approach of [ABR01] for harmonic functions.

Let $\mathscr{P}_m^* = \{p : p(x, y) \text{ polynomial of degree} \le m, \text{ even in } y\}$.

Lemma B.5. *Let $p \in \mathscr{P}_m^*$. Then there exists $\tilde{p} \in \mathscr{P}_m^*$ such that*

$$L_a \tilde{p} = 0 \quad \text{in } B_1, \quad \tilde{p} = p \quad \text{on } \partial B_1.$$

In other words, the solution of the Dirichlet problem for L_a in B_1 with boundary values in $\mathscr{P}_m^*$ on ∂B_1 is itself in $\mathscr{P}_m^*$.

Proof. For $m = 0, 1$, we simply have $\tilde{p} = p$. For $m \ge 2$, we proceed as follows.

For $q \in \mathscr{P}_{m-2}^*$ define $Tq \in \mathscr{P}_{m-2}^*$ by

$$(Tq)(x, y) = |y|^{-a} L_a((1 - x^2 - y^2) q(x, y)).$$

(It is straightforward to verify that Tq is indeed in $\mathscr{P}_{m-2}^*$). We now claim that the mapping $T : \mathscr{P}_{m-2}^* \to \mathscr{P}_{m-2}^*$ is bijective. Since T is clearly linear and $\mathscr{P}_{m-2}^*$ is finite dimensional it is equivalent to showing that T is injective. To this end, suppose that $Tq = 0$ for some $q \in \mathscr{P}_{m-2}^*$. This means that $Q(x, y) = (1 - x^2 - y^2) q(x, y)$ is L_a-harmonic in B_1:

$$L_a Q = 0 \quad \text{in } B_1.$$

On the other hand $Q = 0$ on ∂B_1 and therefore, by the maximum principle $Q = 0$ in B_1. But this implies that $q = 0$ in B_1, or that $q \equiv 0$. Hence, the mapping T is injective, and consequently bijective. It is now easy to see that

$$\tilde{p} = p - (1 - x^2 - y^2) T^{-1}(|y|^{-a} L_a(p)) \in \mathscr{P}_m^*$$

satisfies the required properties. □

Lemma B.6. *Polynomials, even in y, are dense in the subspace of functions in $L^2(\partial B_1, |y|^a)$, even in y.*

Proof. Polynomials, even in y are dense in the space of continuous functions in $C(\partial B_1)$, even in y, with the uniform norm. The claim now follows from the observation that the embedding $C(\partial B_1) \hookrightarrow L^2(\partial B_1, |y|^a)$ is continuous:

$$\|v\|_{L^2(\partial B_1, |y|^a)} \le \|v\|_{L^\infty(\partial B_1)} \left(\int_{\partial B_1} |y|^a \right)^{1/2} \le C \|v\|_{L^\infty(\partial B_1)}. \qquad \square$$

Lemma B.7. *The subspace of functions in $L^2(\partial B_1, |y|^a)$, even in y, has an orthonormal basis $\{p_k\}_{k=0}^\infty$ consisting of homogeneous L_a-harmonic polynomials p_k, even in y.*

Proof. If p is a polynomial, even in y, then restricted to ∂B_1 it can be replaced with an L_a-harmonic polynomial $\tilde{p}$. On the other hand, if we decompose

$$\tilde{p} = \sum_{i=0}^m q_i$$

where q_i is a homogeneous polynomial of order i, even in y, then

$$|y|^{-a}L_a\tilde{p} = \sum_{i=2}^{m} |y|^{-a}L_a q_i$$

where $|y|^{-a}L_a q_i$ is a homogeneous polynomial of order $i-2$, $i = 2, \dots, m$. Hence, $L_a\tilde{p} = 0$ iff $L_a q_i = 0$, for all $i = 0, \dots, m$ (for $i = 0, 1$ this holds automatically).

We further note that if q_i and q_j are two homogeneous L_a-harmonic polynomials of degrees $i \neq j$, then they are orthogonal in $L^2(\partial B_1, |y|^a)$. Indeed,

$$\begin{aligned} 0 = \int_{B_1} q_i \operatorname{div}(|y|^a\nabla q_j) - \operatorname{div}(|y|^a\nabla q_i)q_j &= \int_{\partial B_1} (q_i\partial_\nu q_j - q_j\partial_\nu q_i)|y|^a \\ &= (j-i)\int_{\partial B_1} q_i q_j |y|^a. \end{aligned}$$

Using this and following the standard orthogonalization process, we can construct a basis consisting of homogeneous L_a-harmonic polynomials. □

Lemma B.8. *Let $u \in W^{1,2}(B_1, |y|^a) \cap C(\overline{B_1})$ is a weak solution of $L_a u = 0$ in B_1. Then*

$$\|u\|_{L^\infty(K)} \leq C_{n,a,K}\|u\|_{L^2(\partial B_1,|y|^a)}.$$

for any $K \Subset B_1$.

Proof. First, we note that by [FS87]

$$\|u\|_{L^\infty(K)} \leq C_{n,a,K}\|u\|_{L^2(B_1,|y|^a)}.$$

So we just need to show that

$$\|u\|_{L^2(B_1,|y|^a)} \leq C_{n,a}\|u\|_{L^2(\partial B_1,|y|^a)}.$$

This follows from the fact that u^2 is a subsolution: $L_a(u^2) \geq 0$ in B_1 and therefore the weighted spherical averages

$$r \mapsto \frac{1}{\omega_{n,a}r^{n+a}}\int_{\partial B_r} u^2|y|^a, \quad 0 < r < 1$$

are increasing. Integrating, we easily obtain that

$$\|u\|_{L^2(B_1,|y|^a)} \leq C_{n,a}\|u\|_{L^2(\partial B_1,|y|^a)}.$$

□

We are now ready to prove Theorem B.1.

Proof of Theorem B.1. Without loss of generality we may assume $u \in W^{1,2}(B_1, |y|^a) \cap C(\overline{B_1})$, otherwise we can consider a slightly smaller ball. Now, using the orthonormal basis $\{p_k\}_{k=0}^\infty$ from Lemma B.7 we represent

$$u = \sum_{k=0}^\infty a_k p_k \quad \text{in } L^2(\partial B_1, |y|^a).$$

We then claim that

$$u(x,y) = \sum_{k=0}^\infty a_k p_k(x,y) \quad \text{uniformly on any } K \Subset B_1.$$

Indeed, if $u_m(x,y) = \sum_{k=0}^m a_k p_k(x,y)$, then $\|u - u_m\|_{L^2(\partial B_1,|y|^2)} \to 0$ as $m \to \infty$ and therefore by Lemma B.8

$$\|u - u_m\|_{L^\infty(K)} \leq C_{n,a,K}\|u - u_m\|_{L^2(\partial B_1,|y|^a)} \to 0.$$

□

We now give the proofs of the corollaries.

Proof of Corollary B.2. Write $u(x,y)$ in the form

$$u(x,y) = u_{\mathrm{even}}(x,y) + u_{\mathrm{odd}}(x,y),$$

where u_{even} and u_{odd} are even and odd in y, respectively. Clearly, both functions are L_a-harmonic. Moreover, by Theorem B.1, u_{even} is real analytic and we take $\varphi = u_{\mathrm{even}}$. On the other hand, consider

$$v(x,y) = |y|^a \partial_y u_{\mathrm{odd}}(x,y).$$

Then, v is L_{-a}-harmonic in B_1 and again by Theorem B.1, v is real analytic. We can now represent

$$u_{\mathrm{odd}}(x,y) = y|y|^{-a}\psi(x,y), \quad \psi(x,y) = y^{-1}|y|^a \int_0^y |s|^{-a} v(x,s)ds.$$

It is not hard to see that $\psi(x,y)$ is real analytic, which completes our proof. $\square$

Proof of Corollary B.3. The proof follows immediately from Theorem B.1 by considering the Caffarelli-Silvestre extension

$$u(x,y) = u * P(\cdot,y) = \int_{\mathbb{R}^n} P(x-z,y)u(z)dz, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+$$

where $P(x,y) = C_{n,a} \frac{y^{1-a}}{(|x|^2+y^2)^{(n+1-a)/2}}$ is the Poisson kernel for L_a, and noting that its extension to $\mathbb{R}^{n+1}$ by even symmetry in y (still denoted u) satisfies $L_a u = 0$ in B_1. $\square$

Proof of Corollary B.4. Represent $u(x,y)$ as a locally uniformly convergent in B_1 series

$$u(x,y) = \sum_{k=0}^{\infty} q_k(x,y),$$

where $q_k(x,y)$ is a homogeneous of degree k L_a-harmonic polynomial, even in y. We have

$$u(x,0) = \sum_{k=0}^{\infty} q_k(x,0) \equiv 0$$

from which we conclude that $q_k(x,0) \equiv 0$. We now want to show that $q_k \equiv 0$. To this end represent

$$q_k(x) = \sum_{j=0}^{[k/2]} p_{k-2j}(x) y^{2j},$$

where $p_{k-2j}(x)$ is a homogeneous polynomial of order $k-2j$ in x. Clearly $p_k(x) \equiv 0$. Taking partial derivatives $\partial_x^\alpha q_k(x)$ of order $|\alpha| = k-2$, we see that

$$\partial_x^\alpha q_k(x) = c_\alpha y^2, \quad c_\alpha = \partial_x^\alpha p_{k-2}$$

is L_a-harmonic, which can happen only when $c_\alpha = 0$. Hence $D_x^{k-2} p_{k-2}(x) \equiv 0$ and therefore $p_{k-2} \equiv 0$. Then taking consequently derivatives of orders $k-2j$, $j = 2, \ldots$, we conclude that $p_{k-2j}(x) \equiv 0$ for all $j = 0, \ldots, [k/2]$ and hence $q_k(x,y) \equiv 0$. $\square$

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