2020 59th IEEE Conference on Decision and Control (CDC)
Jeju Island, Republic of Korea, December 14-18, 2020

Convex model to evaluate worst-case performance of local search in the Optimal Power Flow problem

Elizabeth Glista and Somayeh Sojoudi
University of California, Berkeley

Abstract— **The optimal power flow (OPF) problem is a well-known non-convex optimization problem that aims to minimize the cost of electric power generation subject to consumer demand, the physics of power flow, and technological constraints. To find an optimal solution to this problem, local search techniques such as interior point methods are typically used. However, due to the non-convex nature of the problem, these methods are likely to result in a sub-optimal solution. The goal of this paper is to characterize the worst-case performance of local search on the OPF problem. To accomplish this, we formulate the OPF problem as a canonical quadratically-constrained quadratic program (QCQP). Then, we study the problem of finding the worst-case local minimum of this QCQP, which is non-convex and hard to solve in general. We find a relaxation of this problem into a semidefinite program (SDP) and show that it is exact for certain cases. Using some test cases which are known to have multiple local minima, we demonstrate the effectiveness of the proposed relaxation to bound the worst-case local minimum. We compare the obtained upper bound on local minima to the lower bound provided by the standard SDP relaxation of the OPF problem to understand how much SDP outperforms local search for a given problem.**

I. Introduction

The fundamental problem of minimizing the cost to generate and transport electricity over the power grid is called the optimal power flow (OPF) problem. By solving the OPF problem, we find the optimal steady-state operating point for the electric grid. The solution to the OPF problem must satisfy consumer demand for power as well as the physical and technological constraints of power flow through the network. Due to the nonlinear nature of the physics of alternating current, coupled with the existence of lower bounds on voltage magnitude, the OPF problem is non-convex. As a result, there are no universal methods to solve the problem to global optimality in polynomial time.

Nevertheless, the issue of solving the OPF problem to global optimality remains highly critical for the operation of the U.S. power grid, which connects 145 million customers to over 7,300 power plants [1]. The cost of a sub-optimal solution to the OPF problem is estimated at billions of dollars annually in the U.S. [2]. Furthermore, adding renewable energy sources to the power system increases the complexity of the OPF problem. For example, consumers with solar panels on their homes may add power back into the power network, making the network structure more cyclic and leading to a greater number of local optima in the OPF problem. As renewable energy sources generate an increasing share of U.S. electrical power [3], the importance of solving the OPF problem will grow.

The OPF problem is the basis for a whole range of other optimization problems, such as security-constrained optimal power flow (SCOPF) and unit commitment, which are the problems typically solved in industry practice. In this paper, we examine the OPF problem, noting that the results developed here could be applied to any of these OPF-based problems. Additionally, since we develop results for the OPF problem formulated as a canonical quadratically-constrained quadratic program (QCQP), it is clear that many of these results can apply to the broader class of non-convex problems that can be formulated as QCQPs.

A. Survey of Existing Methods

Two categories of well-studied techniques for solving the OPF problem are interior point methods and conic relaxations. Modern interior point methods, which are contained in the broader class of local search methods, have been shown to efficiently find a solution to the OPF problem, if a feasible point is available. These methods provide no guarantee on the quality of the solution and may find a local minimum. They are highly dependent on a good choice of the initial point, which can be hard to find in general. There have been some advances in using numerical methods to improve the solution quality of interior point methods, such as in [4].

Another area of research concerns optimization methods with global guarantees, typically in the form of conic relaxations of the original OPF problem. The paper [5] first developed the semidefinite programming (SDP) relaxation for the OPF problem in the real domain. In the following years, the SDP relaxation of the OPF problem has gained attention due to several papers which show that the relaxation has a zero duality gap in certain cases [6], implying that the SDP relaxation yields an exact solution to the original problem for many real-world networks (see [7], [8], [9], [10]). However, these conditions are not met for all power networks. Thus, the solution to the SDP relaxation may not be feasible for the original OPF problem, although it provides a lower bound on the original solution.

Other types of conic relaxations, such as quadratic programming and second-order cone programming relaxations, of the OPF problem have been proposed [11], but these are typically dominated by the SDP relaxation [8]. More recent

Elizabeth Glista and Somayeh Sojoudi are with the Department of Mechanical Engineering at the University of California, Berkeley. Emails: {glista, sojoudi}@berkeley.edu.
This work was supported by grants from NSF, AFOSR and ONR.

978-1-7281-7447-1/20/$31.00 ©2020 IEEE

work has aimed to strengthen existing conic relaxations by adding valid inequalities or using branch-and-cut approaches [12], [13]. However, these methods cannot promise an exact solution to the original problem in general.

B. Existence of Local Optima

In practice, OPF solutions have been shown to be unique, i.e. there is typically only one globally optimal solution to the problem [14]. Due to the non-convex nature of the OPF problem, there also may be many local optima. It has been shown that several realistic OPF test cases have many local minima [14]. Typical networks with many local minima include cyclic networks with losses or networks with large phase angle differences. While conic relaxation methods may work for many cases with local minima, they fail in some of these cases, such as when there is excess real power in the network or when the system is under stress [15], [16].

There is no efficient way to characterize all the local minima of the OPF problem. (Note that if there were, the original problem would be easy to solve to global optimality.) To find a collection of local minima, Monte-Carlo simulations with a randomization of the initial point used in the local search method could be implemented. With this procedure, one may find many local minima but cannot guarantee that all the local minima of the problem have been detected.

C. Contributions

This paper proposes a new method to characterize the worst-case local minimum of the OPF problem. This allows us to quantify the quality of the solution obtained from a local search method, independent of the initial point. In order to find the worst-case local minimum of the OPF problem, we formulate a new maximization problem based on the first-order and second-order optimality conditions applied to a QCQP model of the OPF. Since this problem is also non-convex, we propose an SDP relaxation to find an upper bound on the worst-case local minimum of the original OPF problem. We prove that this SDP relaxation is exact in a special case. The proposed SDP relaxation is tested on different networks. By comparing this upper bound on the unknown local minima with the lower bound on the global minimum obtained from the SDP relaxation of the original problem, one can bound the range of solutions obtained from a local search method. We interpret this distance as a measure of the hardness of the problem, i.e. an estimate of how far apart the global minimum is from the worst-case local minimum, and thus as a measure for the usefulness of convex relaxation techniques to improve solution quality.

D. Notations

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{R}^N$ and $\mathbb{C}^N$ denote the spaces of N-dimensional real and complex vectors, respectively. The symbol $\mathbb{S}^N$ denotes the space of $N \times N$ symmetric real matrices. The symbols $(\cdot)^T$ and $(\cdot)^*$ denote the transpose and conjugate transpose of a vector or matrix. $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ denote the real and imaginary part of a given scalar

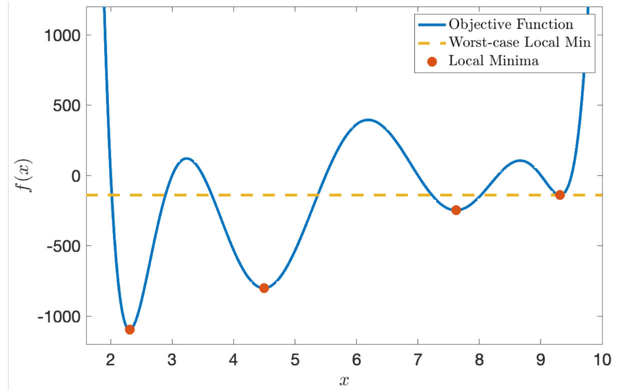

Fig. 1. The worst-case local minimum problem for a one-dimensional unconstrained optimization problem. The feasible set of the worst-case local minimum problem is the collection of local minima, shown in orange. The objective of the worst-case local minimum problem is the maximum of these minima, highlighted by a dashed yellow line.

or matrix. The symbol $|\cdot|$ is the absolute value operator if the argument is a scalar, vector, or matrix; otherwise, it is the cardinality of a measurable set. The imaginary unit is denoted by $\mathbf{j} = \sqrt{-1}$.

II. Problem Formulation

In this section, we present the mathematical formulations for the OPF problem and the new problem of the worst-case local minimum OPF.

A. Classical AC-OPF Problem

Let the power network be defined by a graph $\mathcal{N} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of buses and $\mathcal{E}$ is the set of transmission lines. Let $\mathcal{G} \subseteq \mathcal{V}$ be the buses that are attached to generators. The classical OPF problem can be written as:

$$\min_{v \in \mathbb{C}^{|\mathcal{V}|}} \sum_{i \in \mathcal{G}} f_i(p_i^g) \tag{1a}$$

$$\text{s.t. } \underline{P_i^g} \leq p_i^g \leq \overline{P_i^g} \qquad \forall i \in \mathcal{G} \tag{1b}$$

$$\underline{Q_i^g} \leq q_i^g \leq \overline{Q_i^g} \qquad \forall i \in \mathcal{G} \tag{1c}$$

$$\underline{V_i} \leq |v_i| \leq \overline{V_i} \qquad \forall i \in \mathcal{V} \tag{1d}$$

$$p_i^g - P_i^d = \sum_{(i,j) \in \mathcal{E}} \mathrm{Re}\{v_i(v_i - v_j)^* Y_{ij}^*\}, \forall i \in \mathcal{G} \tag{1e}$$

$$q_i^g - Q_i^d = \sum_{(i,j) \in \mathcal{E}} \mathrm{Im}\{v_i(v_i - v_j)^* Y_{ij}^*\}, \forall i \in \mathcal{G} \tag{1f}$$

$$-P_i^d = \sum_{(i,j) \in \mathcal{E}} \mathrm{Re}\{v_i(v_i - v_j)^* Y_{ij}^*\}, \forall i \in \mathcal{V} \setminus \mathcal{G} \tag{1g}$$

$$-Q_i^d = \sum_{(i,j) \in \mathcal{E}} \mathrm{Im}\{v_i(v_i - v_j)^* Y_{ij}^*\}, \forall i \in \mathcal{V} \setminus \mathcal{G} \tag{1h}$$

where $f_i(\cdot)$ is the power generation cost at bus i, typically a convex polynomial or piecewise linear function. The decision variable v is a vector of complex voltages, where v_i is the complex voltage at bus i. The variables p_i^g and q_i^g are the real and reactive power generated at bus i and can be derived from the vector v. The fixed quantities P_i^d and Q_i^d are the real and reactive power demanded at bus i. The parameters

$\underline{P_i^g}$, $\overline{P_i^g}$, $\underline{Q_i^g}$, $\overline{Q_i^g}$, $\underline{V_i}$, $\overline{V_i}$ are respectively the minimum real power generated, maximum real power generated, minimum reactive power generated, maximum reactive power generated, minimum voltage magnitude, and maximum voltage magnitude at bus i. The network parameters Y_{ij}, G_{ij}, and B_{ij} are respectively the complex admittance, conductance, and susceptance for the transmission line between buses i and j, where $Y_{ij} = G_{ij} + \mathbf{j}B_{ij}$.

Note that we can also add constraints on line flow capacity to Problem (1). To be consistent with the following QCQP formulation, these constraints should have the form $\underline{P_{ij}} \leq p_{ij} \leq \overline{P_{ij}}$, where p_{ij} is the real power flow over line $(i,j) \in \mathcal{E}$ and $\underline{P_{ij}}$ and $\overline{P_{ij}}$ are its upper and lower bounds.

B. OPF Problem as a Canonical QCQP

It is well-known that the OPF problem has a QCQP formulation [6]. In order to streamline the reformulation of the OPF problem (1) as a canonical QCQP, we assume that the cost functions $f_i(\cdot)$ are linear in p_i^g for all $i \in \mathcal{G}$ and thus are quadratic in terms of the decision vector v. A similar approach could be used to develop a reformulation for cost functions that are polynomial in p_i^g.

By considering linear costs in p_i^g, both the objective function and all constraints can be written as quadratic functions of the decision vector $u \in \mathbb{R}^{2|\mathcal{V}|}$ defined as:

$$u = \begin{bmatrix}\text{Re}\{v_1\} & \ldots\text{Re}\{v_{|\mathcal{V}|}\} & \text{Im}\{v_1\} & \ldots & \text{Im}\{v_{|\mathcal{V}|}\}\end{bmatrix}^T$$

We also introduce a vector of slack variables $z \in \mathbb{R}^{4|\mathcal{G}|+2|\mathcal{V}|}$ whose entries are associated with the inequality constraints in the OPF formulation. Using the slack variables, one can convert inequality constraints to equality constraints (by adding z_i^2) and rewrite the OPF problem (1) as the general non-convex QCQP:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & x^T M_0 x + k \\ \text{subject to:} \quad & x^T M_i x = a_i \qquad \forall i = 1,\ldots,p \end{aligned} \tag{2}$$

where $x = \begin{bmatrix} u \\ z \end{bmatrix}$, $n = 4|\mathcal{V}| + 4|\mathcal{G}|$ and $p = 4|\mathcal{V}| + 2|\mathcal{G}|$. The matrices $M_0, \ldots, M_p \in \mathbb{R}^{n\times n}$ and the scalars $k, a_1, \ldots, a_p \in \mathbb{R}$ can be easily derived from the power flow equations in the OPF formulation. Note that the matrices $M_0, \ldots, M_p$ are symmetric by construction.

We use this canonical QCQP as the baseline problem throughout the rest of the paper. Note that any arbitrary QCQP can be reformulated in this form. Therefore, the results of this paper extend beyond the OPF problem to a wide variety of non-convex problems that can be formulated as QCQPs. The QCQP in (2) may have many local minima, local maxima, and saddle points and is NP-hard to solve in general.

C. Worst-case Local Minimum

To formulate the problem of finding the worse-case local minimum, which is the focus of this paper, we note that any regular local minimum will satisfy the Karush-Kuhn-Tucker (KKT) conditions. A local minimum x^* is called a regular point if the gradients of the constraints, namely M_1x^*,..., M_px^*, are linearly independent. We make the mild assumption that all local minima of the OPF problem are regular. The Lagrangian of Problem (2) is:

$$L(x,\lambda) = x^T M_0 x + k + \sum_{i=1}^{p} \lambda_i (x^T M_i x - a_i) \tag{3}$$

where λ_i are the Lagrange multipliers associated with the equality constraints. Then, the KKT conditions are given by the following equations:

$$0 = \nabla_x L(x,\lambda) = 2M_0 x + 2\sum_{i=1}^{p} \lambda_i M_i x \tag{4a}$$

$$0 = \frac{\partial L(x,\lambda)}{\partial \lambda_i} = x^T M_i x - a_i \qquad \forall i = 1,\ldots,p \tag{4b}$$

The points that satisfy these KKT conditions may be local minima, local maxima, or saddle points. Thus, we add a second-order necessary condition which only local minima and certain saddle points will satisfy. Such points are called second-order critical points. The second-order condition ensures that the feasible space of the worst-case local minimum problem contains only second-order critical points of the original problem (2). From [17], we have the second-order necessary condition:

$$\begin{aligned} & y^T(\nabla^2_{xx} L(x,\lambda))y \geq 0 \\ & \quad \text{for all } y \text{ such that } y^T M_i x = 0, \ \forall i = 1,\ldots,p \end{aligned} \tag{5}$$

where $\nabla^2_{xx} L(x,\lambda) = 2M_0 + 2\sum_{i=1}^{p} \lambda_i M_i$. This second-order condition involves a possibly infinite number of constraints. Thus, it is more useful to reformulate the second-order necessary condition as a finite-dimensional constraint. Combining the KKT first-order necessary conditions with a modified second-order necessary condition for a local minimum, we define the feasible space of the worst-case local minimum problem. In this worst-case problem, the objective is to maximize the local minimum, as stated below.

Theorem 1. *Let x^* be a second-order critical point of the OPF problem with the highest objective value. Then, x^* is a globally optimal solution of the following optimization problem if c is selected to be greater than a certain threshold:*

$$\max_{x\in\mathbb{R}^n, \lambda\in\mathbb{R}^p} \quad x^T M_0 x + k \tag{6a}$$

$$\textit{subject to:} \quad x^T M_i x = a_i \qquad \forall i = 1,\ldots,p \tag{6b}$$

$$\left(M_0 + \sum_{i=1}^{p} \lambda_i M_i\right) x = 0 \tag{6c}$$

$$M_0 + \sum_{i=1}^{p} \lambda_i M_i + c\sum_{i=1}^{p} M_i x x^T M_i \succeq 0 \tag{6d}$$

Proof: Condition (5) $\implies$ Condition (6d): Define $P = M_0 + \sum_{i=1}^{p} \lambda_i M_i$ and $Q = \sum_{i=1}^{p} M_i(x^*)(x^*)^T M_i$, where P is positive semidefinite on the nullspace of Q. It follows from Lemma 4.2.1 in [17] that (5) and (6d) are equivalent for all large values of c.

Condition (6d) $\implies$ Condition (5): (6d) is equivalent to:

$$y^T\left(M_0+\sum_{i=1}^{p}\lambda_i M_i + c\sum_{i=1}^{p} M_i x x^T M_i\right)y \geq 0, \ \ \forall y\in\mathbb{R}^n$$

If y is selected to satisfy the equations $y^T M_i x = 0$ for all $i=1,\ldots,p$, then the above inequality reduces to:

$$y^T\left(M_0+\sum_{i=1}^{p}\lambda_i M_i\right)y\geq 0$$

which is (5).

Note that Problem (6), which is non-convex, finds the worst-case second-order critical point. As a result, its optimal objective value serves as an upper bound on the objective value at the worst-case local minimum.

D. Decision Version of Worst-case Local Min Problem

In the study of the worst-case local minimum problem, it is useful to also examine the related feasibility problem:

$$\max_{x\in\mathbb{R}^n,\lambda\in\mathbb{R}^p} \quad 0 \tag{7a}$$

$$\text{subject to:} \quad x^T M_i x = a_i \qquad \forall i=1,\ldots,p \tag{7b}$$

$$\left(M_0+\sum_{i=1}^{p}\lambda_i M_i\right)x = 0 \tag{7c}$$

$$M_0+\sum_{i=1}^{p}\lambda_i M_i + c\sum_{i=1}^{p} M_i x x^T M_i \succeq 0 \tag{7d}$$

$$x^T M_0 x + k \geq \alpha \tag{7e}$$

If there is a local minimum to the OPF problem whose corresponding cost is greater than or equal to α, then the optimal value of the above problem will be 0. Otherwise, the optimal value of this problem will be $-\infty$. Thus, the problem (7) asks whether or not there exists any local minima to the OPF problem above a threshold α. The interpretation of this problem is to certify that any local search solution is a "near-global" solution, i.e. below a given threshold.

For the rest of this paper, we will focus on the optimization version (6) of the worst-case local minimum problem since the decision version (7) can be easily deduced from the result of the optimization version. However, we remark that notions from algebraic geometry such as sum of squares and Positivstellensatz could be used to certify that there is no solution to the decision problem. If there is no solution to the above problem, then there exists a certificate that no real solution exists. However, the degree of this certificate may be arbitrarily large, thus those techniques are not efficient in general. See [18]–[20] for more details.

III. SDP Relaxation of Worst-case Local Min Problem

To bound the worst-case local minimum, Theorem 1 requires solving the non-convex problem (6) to global optimality, which cannot be achieved using local search methods. However, any upper bound on the optimal objective value will still serve the same purpose, and this can be accomplished using convex relaxations. In this paper, we develop a tightened SDP relaxation of the worst-case local minimum problem.

We define a matrix $W\in\mathbb{S}^{n+p+1}$ based on $x\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}^p$ as follows:

$$W=\begin{bmatrix}1\\ x\\ \lambda\end{bmatrix}\begin{bmatrix}1 & x^T & \lambda^T\end{bmatrix}=\begin{bmatrix}1 & x^T & \lambda^T\\ x & xx^T & x\lambda^T\\ \lambda & \lambda x^T & \lambda\lambda^T\end{bmatrix} \tag{8}$$

We regard W as a 3×3 block matrix with the block entries W_{ij} for all $i,j\in\{1,2,3\}$.

Theorem 2. *Having selected a sufficiently large c, the optimal objective value of the following SDP provides an upper bound on the cost of the worst-case local minimum of the OPF problem:*

$$\max_{W\in\mathbb{S}^{n+p+1}} \quad \text{trace}\{M_0 W_{22}\}+k \tag{9a}$$

$$\text{subject to:} \quad \text{trace}\{M_i W_{22}\} = a_i \qquad \forall i=1,\ldots,p \tag{9b}$$

$$M_0 W_{21}+\sum_{i=1}^{p} M_i (W_{23})_i = 0 \tag{9c}$$

$$M_0+\sum_{i=1}^{p}(W_{31})_i M_i + c\sum_{i=1}^{p} M_i W_{22} M_i \succeq 0 \tag{9d}$$

$$\text{trace}\{M_0 W_{22}\}+\sum_{i=1}^{p} a_i (W_{31})_i = 0 \tag{9e}$$

$$W_{11}=1 \tag{9f}$$

$$W\succeq 0 \tag{9g}$$

where $(W_{23})_i$ is the i^{th} column of W_{23} and $(W_{31})_i$ is the i^{th} entry of W_{31} for all $i\in\{1,\ldots,p\}$.

Proof: The objective (9a) and constraints (9b), (9c) and (9d) follow directly from the objective (6a) and constraints (6b), (6c) and (6d), along with the definition of W (8). To control the structure of matrix W, we have the constraints (9f) and (9g), as well as the non-convex constraint $\text{rank}(W)=1$. We drop the constraint $\text{rank}(W)=1$ to obtain a convex feasible region for the problem. Thus, the obtained relaxation (9) is an upper bound on (6), which is an upper bound on the worst-case local minimum. Finally, to strengthen the SDP relaxation of the worst-case local minimum problem, we also add the valid constraint, which follows from constraints (6b) and (6c):

$$\text{trace}\{M_0 x x^T\}+\sum_{i=1}^{p}\lambda_i a_i = 0 \tag{10}$$

This constraint becomes constraint (9e) in the strengthened SDP relaxation.

IV. Analysis of SDP Relaxation

Because the worst-case local minimum problem is a non-convex problem, there could be a nonzero gap between the optimal objective values of the non-convex problem (6) and the tightened convex relaxation (9). However, it is desirable to show that the gap is zero in a fundamental class of QCQPs, and therefore the SDP relaxation yields an exact solution to the worst-case local minimum problem for this class.

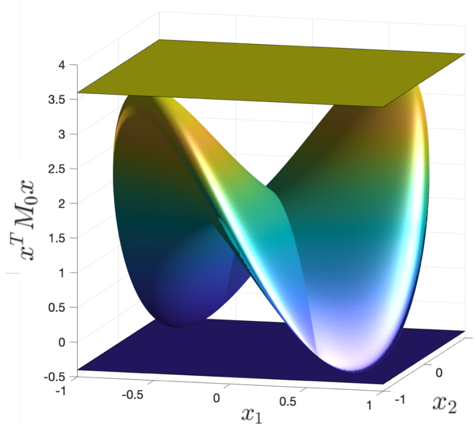

Fig. 2. The three-dimensional curve shows the non-convexity of a particular case of the canonical QCQP, given by Equation (11), for a 3×3 symmetric matrix M_0 with the eigenvalues -0.4, 1.6, and 3.6. The maximum and minimum eigenvalues are plotted as surfaces in the plane. There are also saddle points in between. The objective value of the worst-case local min problem is the same as the minimum eigenvalue, i.e. -0.4.

A. Particular Case with Exact SDP Relaxation

Consider a particular case of the canonical QCQP in (2):

$$\min_{x\in\mathbb{R}^n} \; x^T M_0 x \quad \text{subject to: } x^T x = 1 \tag{11}$$

where M_0 is an arbitrary symmetric matrix with n distinct eigenvalues, ordered as $\mu_1 < \cdots < \mu_n$. associated with the normalized eigenvectors $y_1, \ldots, y_n \in \mathbb{R}^n$. For this problem, one can analytically compute all of the points that satisfy the KKT conditions of (11). It can be shown that there are $2n$ KKT points: $\pm y_1$ are local minima, $\pm y_n$ are local maxima, and $\pm y_2, ..., \pm y_{n-1}$ are saddle points. Thus, the cost corresponding to the worst-case local minimum problem is equal to $y_1^T M_0 y_1$. The optimization problem (6) provided in Theorem 1 can be written as:

$$\max_{x\in\mathbb{R}^n,\lambda\in\mathbb{R}} \; x^T M_0 x \tag{12a}$$
$$\text{subject to: } x^T x = 1 \tag{12b}$$
$$(M_0 + \lambda I)x = 0 \tag{12c}$$
$$M_0 + \lambda I + c x x^T \succeq 0 \tag{12d}$$

In light of (12b) and (12c), the only possible solutions are $x = \pm y_i$ and $\lambda = -\mu_i$ for $i = 1, 2, ..., n$. However, the only solution satisfying (12d) is $\lambda = -\mu_1$ and $x = \pm y_1$. In addition, (12d) is satisfied for any c greater than or equal to zero. This leads to the following result.

Theorem 3. *The SDP relaxation (9) returns the cost corresponding to the worst-case local minimum of the QCQP (12) for $c = 0$, and provides an upper bound for $c > 0$.*

Proof: The SDP relaxation turns out to be:

$$\max_{W\in\mathbb{S}^{n+2}} \; \text{trace}\{M_0 W_{22}\} \tag{13a}$$
$$\text{subject to: } \text{trace}\{W_{22}\} = 1 \tag{13b}$$
$$M_0 W_{21} + W_{23} = 0 \tag{13c}$$
$$M_0 + W_{31} I + c W_{22} \succeq 0 \tag{13d}$$
$$\text{trace}\{M_0 W_{22}\} + W_{31} = 0 \tag{13e}$$
$$W_{11} = 1 \tag{13f}$$
$$W \succeq 0 \tag{13g}$$

Since (13) is a relaxation of (12) we have (12) $\leq$ (13). For the constraint $M_0 + W_{31} I \succeq 0$ to hold, W_{31} must be greater than or equal to $-\mu_1$. Since trace$\{M_0 W_{22}\} = -W_{31}$, the optimal objective value of (13) is less than or equal to μ_1. In addition, the optimal objective value of (12) is equal to μ_1. Combining these inequalities yields the fact that the SDP relaxation (13) meant to provide an upper bound on the optimal value of (12) also provides a lower bound. Thus, the relaxation is exact.

Figure 2 exemplifies the non-convexity of the QCQP (11) for $n = 3$. This problem has 6 KKT points, and an SDP relaxation that does not incorporate the second-order optimality condition will return the global maximum. However, in light of Theorem 3, the SDP relaxation (13) will be able to correctly eliminate the local maxima and saddle points with a negative curvature when $c = 0$.

B. Choice of Parameter "c"

The parameter c is used to convert the infinite-dimensional second-order optimality condition to a finite-dimensional one. The exact value of c is not needed in the non-convex model (6) since Theorem 2 states that every sufficiently large c enables finding the worst-case second-order critical point. However, since (9) is a relaxation of (6), selecting an exorbitantly large value for c affects the quality of the solution to the SDP relaxation of the worst-case local min problem (9).

For example, in the particular QCQP case described above, the relaxation is exact at $c = 0$ and gradually becomes loose as c increases. This is due to the fact that the second-order necessary condition (6d) holds for $c = 0$ at the local minima $\pm y_1$. In general, the smallest c needed in Theorem 2 or 3 coincides with the smallest number c satisfying the second-order condition (6d) at the worst-case second-order critical point x^*. Note that x^* is high dimensional in general, whereas c is a single scalar. Finding a good upper bound on this scalar requires a careful analysis of the matrices $M_1, ..., M_p$ and is left as future work.

V. Simulations

To test the tightened SDP relaxation of the worst-case local minimum problem on benchmark systems, we use MATPOWER to compute the line admittance values and then formulate the matrices $M_0, \ldots, M_p$ and scalars $k, a_1, \ldots, a_p$ based on the given OPF constraints (see Section II-A). We then solve the SDP relaxation of the worst-case local minimum problem in MATLAB using the solver SDPT3. Note that each of the simulations took less than 30 seconds on a standard laptop. We also remark that sparse SDP constraints, as described in [21] and [22], could be implemented to efficiently solve problem (9) on larger networks.

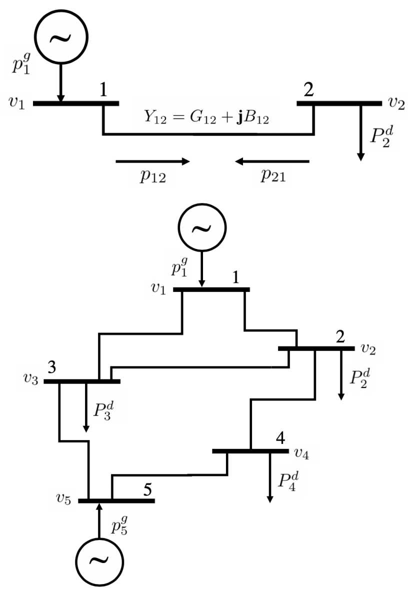

Fig. 3. WB2 2-bus (top) and WB5 5-bus (bottom) networks.

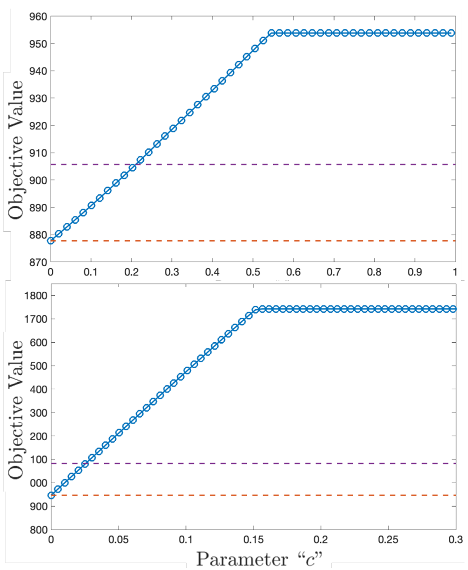

Fig. 4. Objective value of the worst-case local min SDP relaxation on the WB2 network (top) and WB5 network (bottom) for varying values of the parameter c, shown by blue circles. The simulation results are compared to the objective values at two known local minima for these networks, shown as dashed lines.

A. Cases with Known Local Optima

First, we test the worst-case local minimum SDP relaxation on small networks with known local minima. Using test networks from the online database [23], which are known to have multiple local minima, we run a series of simulations of the worst-case local min SDP relaxation, sweeping over a range of the parameter c (the networks are given in Figure 3). From the simulations, it can be observed that the objective value of the relaxation increases with c, until a saturation point is reached. The objective value at this saturation point provides an upper bound on the worst-case local minimum,

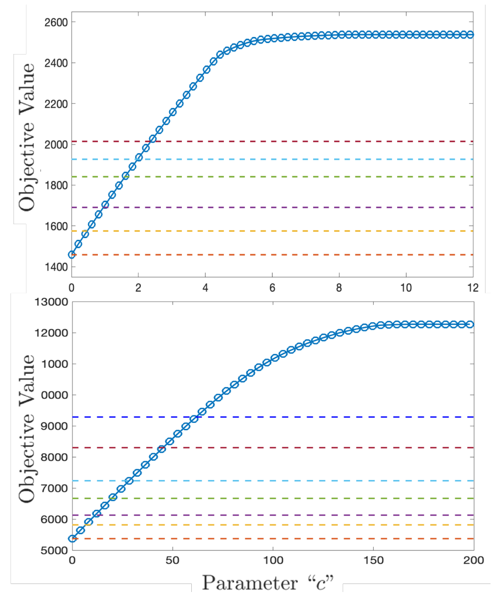

Fig. 5. Objective value of the worst-case local min SDP relaxation for the IEEE 9-bus (top) and 14-bus (bottom) networks for varying values of the parameter c, shown by blue circles, compared to "discovered" local minima from randomized initializations of local search, shown as dashed lines.

and it is not tight for the WB2 and WB5 networks given in Figure 4. With an understanding of a good choice of c, one can tighten this bound.

B. IEEE Test Cases

Next, we test the worst-case local min SDP relaxation on some IEEE test networks. Note that for these networks, we have removed the quadratic cost terms so that the costs are linear in terms of real power generation (as described in section II-B). We run a series of simulations of the worst-case local min SDP relaxation, sweeping over a range of the parameter c. We also run 200 simulations of local search on the canonical QCQP in Equation (2) with 200 random, feasible initial points. Out of these 200 simulations, we take the solutions of those simulations that converged as the "discovered" local minima. Note that some of these discovered local minima may in fact be saddle points, depending on solver performance. For these local search simulations, we use the FMINCON solver in MATLAB.

It can be observed from the simulation results in Figure 5 that the objective value increases with c until a saturation point is reached, at which point the value of c is too high and the relaxation is not exact.

C. Comparison of Worst-case SDP with Original SDP

We compare the objective value of the worst-case local min SDP relaxation (9) with that of the SDP relaxation of the original problem (the SDP relaxation of (2)). Table I shows the objective value of the worst-case local min SDP relaxation at the saturation point of c for each of the four test cases. These values provide an upper bound on the worst-case local minimum.

TABLE I

OBJECTIVE VALUE OF SDP RELAXATION OF WORST-CASE LOCAL MIN PROBLEM AT SATURATION WITH RESPECT TO c

Case	Worst-case SDP Objective	c
WB2	953.85	1
WB5	1742.55	0.3
case9	2537.59	12
case14	12268.8	200

TABLE II

OBJECTIVE VALUE OF SDP RELAXATION OF ORIGINAL PROBLEM

Case	SDP Objective value
WB2	877.78
WB5	946.53
case9	1458.91
case14	5371.58

By comparing the worst-case local min SDP relaxation to the SDP relaxation of the original problem, we can compute a lower bound on the global optimality degree, which is defined as:

$$\text{Global optimality degree} = 100\% \times \left(1 - \frac{\text{upper bound - lower bound}}{|\text{upper bound}|}\right) \quad (14)$$

The SDP relaxation of the original problem provides a lower bound on the optimal value of the original non-convex problem and is given in Table II. The SDP relaxation of the worst-case local min problem provides an upper bound on the objective value at any local minima. Thus, by computing the global optimality degree for these two bounds, we can find a lower bound on the global optimality degree for the problem.

This lower bound on optimality degree provides a metric of how useful the original SDP relaxation is for the given problem. If the lower bound on optimality degree is high, then any local search solution will be relatively close to the SDP solution. For these cases, such as the WB2 case, the more expensive SDP relaxation is less useful since local search solutions are relatively high quality. For cases with a larger gap between the SDP solution and the worst-case local minimum, there is some benefit in using a convex relaxation instead of local search for solving the OPF problem.

VI. CONCLUSIONS

This paper formulates the problem of finding the worst-case local minimum for a canonical QCQP, with a focus on the application in optimal power flow. Since the problem is non-convex, an SDP relaxation is designed to find an upper bound on objective value at the worst-case local minimum. We show that this SDP relaxation is exact in a particular case with many saddle points. Additionally, we find that the tightness of this upper bound depends on the choice of a parameter in the second-order necessary optimality condition. By comparing the objective value obtained from the SDP relaxation of the worst-case local minimum problem to the objective value of the SDP relaxation of the original problem, we provide a metric on how much SDP can outperform local search. These two SDP relaxations for the upper and lower bounds allow us to evaluate the projected performance of local search methods when good initial points are not available. The worst-case local minimum problem is a useful tool to bound the range of possible solutions to a given non-convex QCQP.

REFERENCES

[1] "U.s. electric system is made up of interconnections and balancing authorities," *U.S. Energy Information Administration*, July 2016.

[2] M. B. Cain, R. P. O'Neill, and A. Castillo, "History of optimal power flow and formulations," *Federal Energy Regulatory Commision*, December 2012.

[3] "U.s. renewable electricity generation has doubled since 2008," *U.S. Energy Information Administration*, March 2019.

[4] S. Park, E. Glista, J. Lavaei, and S. Sojoudi, "Homotopy method for finding the global solution of post-contingency optimal power flow," *American Control Conference*, 2020.

[5] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite programming for optimal power flow problems," *International Journal of Electrical Power & Energy Systems*, vol. 30, no. 6-7, pp. 383–392, July 2008.

[6] J. Lavaei and S. H. Low, "Zero duality gap in optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, February 2012.

[7] B. Zhang and D. Tse, "Geometry of feasible injection region of power networks," *Allerton Conference*, September 2011.

[8] S. Sojoudi and J. Lavaei, "Physics of power networks makes hard optimization problems easy to solve," *2012 IEEE Power and Energy Society General Meeting*, pp. 1–8, 2012.

[9] J. Lavaei, D. Tse, and B. Zhang, "Geometry of power flows and optimization in distribution networks," *IEEE Transactions on Power Systems*, vol. 29, no. 2, March 2014.

[10] S. Sojoudi and J. Lavaei, "Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure," *SIAM Journal on Optimization*, vol. 24, no. 4, pp. 1746–1778, 2014.

[11] R. Jabr, "Radial distribution load flow using conic programming," *IEEE Transactions on Power Systems*, vol. 21, no. 3, pp. 1458 – 1459, August 2006.

[12] B. Kocuk, S. S. Dey, and X. A. Sun, "Strong socp relaxations for the optimal power flow problem," October 2015.

[13] C. Coffrin, H. Hijazi, and P. Van Hentenryck, "Strengthening the sdp relaxation of ac power flows with convex envelopes, bound tightening, and valid inequalities," *IEEE Transactions on Power Systems*, vol. 32, no. 5, September 2017.

[14] W. Bukhsh, A. Grothey, K. McKinnon, and P. Trodden, "Local solutions of the optimal power flow problem," *IEEE Transactions on Power Systems*, November 2013.

[15] B. Lesieutre, D. Molzahn, A. Borden, and C. DeMarco, "Examining the limits of the application of semidefinite programming to power flow problems," *Allerton Conference*, September 2011.

[16] R. Madani, S. Sojoudi, and J. Lavaei, "Convex relaxation for optimal power flow problem: Mesh networks," *IEEE Transactions on Power Systems*, vol. 30, no. 1, pp. 199–211, January 2015.

[17] D. P. Bertsekas, *Nonlinear programming*, 3rd ed. Athena scientific, 2016.

[18] B. Sturmfels, "Solving systems of polynomial equations," in *American Mathematical Society, CBMS Regional Conferences Series*, no. 97, 2002.

[19] P. A. Parrilo, "Sum of squares programs and polynomial inequalities," 2004.

[20] I. A. Fotiou, P. Rostalski, P. A. Parrilo, and M. Morari, "Parametric optimization and optimal control using algebraic geometry methods," *International Journal of Control*, vol. 79, no. 11, February 2006.

[21] R. Madani, M. Ashraphijuo, and J. Lavaei, "Promises of conic relaxation for contingency-constrained optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1297 – 1307, March 2016.

[22] R. Madani, S. Sojoudi, G. Fazelnia, and J. Lavaei, "Finding low-rank solutions of sparse linear matrix inequalities using convex optimization," *SIAM*, vol. 27, no. 2, pp. 725–758, 2017.

[23] W. Bukhsh, May 2013. [Online]. Available: https://www.maths.ed.ac.uk/optenergy/LocalOpt/