LOCAL WELL-POSEDNESS FOR THE HALL-MHD SYSTEM IN OPTIMAL SOBOLEV SPACES

MIMI DAI

Abstract. We show that the viscous resistive magnetohydrodynamics system with Hall effect is locally well-posed in $H^s(\mathbb{R}^3)\times H^{s+1-\varepsilon}(\mathbb{R}^3)$ with $s>\frac12$ and any small enough $\varepsilon>0$ such that $s-\varepsilon>\frac12$. This space is to date the largest local well-posedness space in the class of Sobolev spaces for the system. It is also optimal according to the predominant scalings of the two equations in the system.

KEY WORDS: Magnetohydrodynamics; Hall effect; local well-posedness; scaling structure.
CLASSIFICATION CODE: 76D03, 35Q35.

1. Introduction

Considered in this paper is the three dimensional incompressible viscous resistive magnetohydrodynamics with Hall effect (Hall-MHD) governed by the system:

$$\begin{aligned} u_t+u\cdot\nabla u-b\cdot\nabla b+\nabla p-\nu\Delta u&=0,\\ b_t+u\cdot\nabla b-b\cdot\nabla u+\eta\nabla\times((\nabla\times b)\times b)-\mu\Delta b&=0,\\ \nabla\cdot u&=0, \end{aligned}\tag{1.1}$$

accompanied with the initial conditions

$$u(x,0)=u_0(x), \qquad b(x,0)=b_0(x), \qquad \nabla\cdot u_0=\nabla\cdot b_0=0,\tag{1.2}$$

for $x\in\mathbb{R}^3$ and $t\geq 0$. In the system, u represents the fluid velocity, p is the fluid pressure and b stands for the magnetic field. The parameters ν,μ and η denote the fluid viscosity, resistivity (electrical diffusivity) and the Hall effect coefficient, respectively. It is important to observe that, if $\nabla\cdot b_0=0$, the divergence free condition for b is propagated by the second equation of (1.1), see [4]. The Hall term $\nabla\times((\nabla\times b)\times b)$ distinguishes (1.1) from the usual MHD system (system (1.1) with $\eta=0$). In contrast to the latter one, the Hall-MHD model is more advantageous due to the fact that it can capture the essential characteristics of the magnetohydrodynamics with strong magnetic reconnection where the Hall effect plays a significant role. Magnetic reconnection is a fundamental dynamical process in highly conductive plasmas in astrophysics, allowing for explosive and efficient magnetic to kinetic energy conversion. For a more comprehensive physical background of the magnetic reconnection phenomena and the Hall-MHD model, we refer the readers to [11, 14, 16] and references therein.

Despite its increasing popularity among the astrophysicists community, the mathematical understanding of the Hall-MHD model is very limited. Conceptually, we can have a peek about the barriers from various perspectives. First, the Hall term

The author was partially supported by NSF grants DMS–1815069 and DMS–2009422.

launches new physics into the system at small length scales and hence intrinsically challenging into the mathematical analysis. Second, it is well-known that the main obstacle to understand the turbulent flows governed by the Navier-Stokes equation (NSE) relies on the nonlinearity in the form of $(u \cdot \nabla)u$. One can imagine that system (1.1) is more intricate than the NSE, for the former one contains the NSE and a magnetic field equation with the Hall term which appears more singular than $(u \cdot \nabla)u$. Third, the natural scaling structure is a strong motivation in the study of both the NSE and the MHD system, who share the same scaling. However, the Hall term destroys such natural scaling. Into more details, for the MHD system, if $(u(x,t), p(x,t), b(x,t))$ solves (1.1) with $\eta = 0$ with the initial data $(u_0(x), b_0(x))$, then the triplet $(u_\lambda(x,t), p_\lambda(x,t), b_\lambda(x,t))$ defined by

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad b_\lambda(x,t) = \lambda b(\lambda x, \lambda^2 t) \tag{1.3}$$

solves the same system with the data

$$u_{0\lambda}(x,t) = \lambda u_0(\lambda x), \quad b_{0\lambda}(x,t) = \lambda b_0(\lambda x).$$

The scaling (1.3) no longer holds for system (1.1) with $\eta > 0$. On the other hand, we can extract the so-called electron MHD

$$b_t + \nabla \times ((\nabla \times b) \times b) = \Delta b \tag{1.4}$$

which has the scaling

$$b_\lambda(x,t) = b(\lambda x, \lambda^2 t). \tag{1.5}$$

Since the Hall term is the most singular nonlinearity in system (1.1), it suggests that the predominant scaling for (1.1) could be

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad b_\lambda(x,t) = b(\lambda x, \lambda^2 t). \tag{1.6}$$

In fact, based on scaling (1.6), we obtained a regularity criterion for (1.1) in three dimension which improves various criteria in the literature, see [9].

In this paper our interest is to find the largest possible (optimal) Sobolev space where system (1.1) is locally well-posed. On this topic, it was first shown in [6] that system (1.1) in three dimension is locally well-posed in $H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ with $s > \frac{5}{2}$. By taking (1.5) as the dominant scaling, in [8], we obtained the local well-posedness of (1.1) in $H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ with $s > \frac{3}{2}$, which improves the result of [6]. In fact, the NSE is known to be locally well-posed in $H^s(\mathbb{R}^3)$ with $s > \frac{1}{2}$; according to scaling (1.5), one can show that the electron MHD (1.4) is locally well-posed in $H^s(\mathbb{R}^3)$ with $s > \frac{3}{2}$ by standard arguments of local well-posedness. Thereby, motivated by scaling (1.6), one would expect that system (1.1) may be locally well-posed in $H^s(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3)$ with $s > \frac{1}{2}$. In order to justify this conjecture, we need to treat the energy estimates for u and b separately, namely, u in H^s and b in H^{s+1}. In this situation, beside estimating the flux contribution from the Hall term, we encounter the essential difficulty: no cancelation can be employed to deal with the two terms $b \cdot \nabla b$ and $b \cdot \nabla u$. As a matter of fact, cancelation among the coupling terms plays a vital role in estimating both u and b in the same space H^s, as done in [8]. To overcome this barrier due to the lack of cancelation among the coupling terms, it comes to our mind that we need to optimize the estimates of the flux contributed from the two terms by fully employing the diffusion of both the u and the b. Techniques based on the paradifferential calculus enables us to operate such optimizations. Namely, we prove the main result below.

Theorem 1.1. *Let* $(u_0, b_0) \in H^s(\mathbb{R}^3) \times H^{s+1-\varepsilon}(\mathbb{R}^3)$ *with* $s > \frac{1}{2}$ *and any small enough* $\varepsilon > 0$ *such that* $s - \varepsilon > \frac{1}{2}$*. Assume* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$*. There exists a time* $T = T(\nu, \mu, \|u_0\|_{H^s}, \|b_0\|_{H^{s+1-\varepsilon}}) > 0$ *and a unique solution* (u, b) *of (1.1) on* $[0, T]$ *such that*

$$(u, b) \in C([0, T]; H^s(\mathbb{R}^3)) \times C([0, T]; H^{s+1-\varepsilon}(\mathbb{R}^3)).$$

Remark 1.2. The proof of Theorem 1.1 will be carried out for space $\mathbb{R}^n$ with general $n \geq 3$, although the Hall term is not physically relevant in space with dimension greater than 3. Ignoring the meaning of the Hall term, the proof in Section 3 shows local well-posedness of the system in the space $H^s(\mathbb{R}^n) \times H^{s+1-\varepsilon}(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$ and any small enough $\varepsilon > 0$ such that $s + 1 - \varepsilon > \frac{n}{2}$.

Regarding the result, the fact that b needs to be in a space with higher regularity is determined by the Hall term. Based on the scaling (1.5) of the electron MHD (1.4), the optimal Sobolev space of well-posedness for b would be $H^{s+1}(\mathbb{R}^3)$ with $s > \frac{1}{2}$. However, as stated in Theorem 1.1, the obtained well-posdness space for b is $H^{s+1-\varepsilon}(\mathbb{R}^3)$ for any small $\varepsilon > 0$. It may be explained by getting a closer look at the term $b \cdot \nabla u$. While estimating $\|b \cdot \nabla u\|_{H^r}$ by applying both diffusions of u and b, it happens that we need to take r slightly smaller than $s + 1$.

2. Preliminaries

2.1. **Notation.** In order to avoid confusion, we specify a few notations. We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some absolute constant C, and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with absolute constants C_1, C_2. For simplification, it is understood that $\|\cdot\|_p = \|\cdot\|_{L^p}$. We use C_ν to denote a constant which depends on the viscosity ν and may vary from line to line. The same convention applies to notations C_μ and $C_{\nu,\mu}$.

2.2. **Littlewood-Paley decomposition.** As in our previous articles on the local well-posedness of magnetohydrodynamics systems, the main tool is paradifferential calculus. To be self-contained, we recall the Littlewood-Paley decomposition theory briefly, even though it appears in our earlier work on related topics. For a more detailed description on this theory we refer the readers to [2] and [12].

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively. Define $\lambda_q = 2^q$ for integers q. A nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ is chosen such that

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4} \\ 0, & \text{for } |\xi| \geq 1. \end{cases}$$

Let

$$\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$$

and

$$\varphi_q(\xi) = \begin{cases} \varphi(\lambda_q^{-1}\xi) & \text{for } q \geq 0, \\ \chi(\xi) & \text{for } q = -1. \end{cases}$$

For a tempered distribution vector field u we define the Littlewood-Paley projection

$$\begin{aligned}
&h = \mathcal{F}^{-1}\varphi, \qquad \tilde{h} = \mathcal{F}^{-1}\chi,\\
&u_q := \Delta_q u = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1}\xi)\mathcal{F}u) = \lambda_q^n \int h(\lambda_q y)u(x-y)dy, \qquad \text{for } q \geq 0,\\
&u_{-1} = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}u) = \int \tilde{h}(y)u(x-y)dy.
\end{aligned}$$

By the Littlewood-Paley theory, the identity

$$u = \sum_{q=-1}^{\infty} u_q$$

holds in the distributional sense. For brevity, we agree with the notations

$$u_{\leq Q} = \sum_{q=-1}^{Q} u_q, \qquad \tilde{u}_q = \sum_{|p-q|\leq 1} u_p.$$

Definition 2.1. A tempered distribution u belongs to the Besov space $B^s_{p,\infty}$ if and only if

$$\|u\|_{B^s_{p,\infty}} = \sup_{q\geq -1} \lambda_q^s \|u_q\|_p < \infty.$$

We can identify the Sobolev space H^s by the Besov space $B^s_{2,2}$, i.e.

$$\|u\|_{H^s} \sim \left(\sum_{q=-1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{1/2}$$

for each $u \in H^s$ and $s \in \mathbb{R}$.

Lemma 2.2. *(Bernstein's inequality. See* [13]*.) Let n be the space dimension and $r \geq s \geq 1$. Then for all tempered distributions u, we have*

$$\|u_q\|_r \lesssim \lambda_q^{n(\frac{1}{s}-\frac{1}{r})} \|u_q\|_s. \tag{2.7}$$

2.3. **Bony's paraproduct and commutator.** Bony's paraproduct formula

$$\begin{aligned}
\Delta_q(u\cdot\nabla v) = &\sum_{|q-p|\leq 2} \Delta_q(u_{\leq p-2}\cdot\nabla v_p) + \sum_{|q-p|\leq 2} \Delta_q(u_p \cdot \nabla v_{\leq p-2})\\
&+ \sum_{p\geq q-2} \Delta_q(\tilde{u}_p \cdot \nabla v_p),
\end{aligned} \tag{2.8}$$

will be used constantly to decompose the nonlinear terms in energy estimate. We will also use the notation of the commutator

$$[\Delta_q, u_{\leq p-2}\cdot\nabla]v_p := \Delta_q(u_{\leq p-2}\cdot\nabla v_p) - u_{\leq p-2}\cdot\nabla\Delta_q v_p. \tag{2.9}$$

Lemma 2.3. *The commutator satisfies the following estimate, for any $1 < r < \infty$*

$$\|[\Delta_q, u_{\leq p-2}\cdot\nabla]v_p\|_r \lesssim \|\nabla u_{\leq p-2}\|_\infty \|v_p\|_r.$$

2.4. **Auxiliary estimates.** To handle the Hall term $\nabla\times((\nabla\times b)\times b)$, more preparation is needed. We first introduce two more commutators and their estimates. We define that, for vector valued functions F and G,

$$[\Delta_q, F\times\nabla\times]G=\Delta_q(F\times(\nabla\times G))-F\times(\nabla\times G_q), \tag{2.10}$$

$$[\Delta_q, \nabla\times F\times]G=\Delta_q(\nabla\times F\times G)-\nabla\times F\times G_q. \tag{2.11}$$

In principle, the commutators will be used to reveal certain cancellation; and to shift derivative from high modes to low modes. It was shown in [9] they satisfy the following estimates.

Lemma 2.4. *Let F and G be vector valued functions. Assume $\nabla\cdot F=0$ and F, G vanish at large $|x|\in\mathbb{R}^3$. For any $1<r<\infty$, we have*

$$\|[\Delta_q, F\times\nabla\times]G\|_r\lesssim\|\nabla F\|_\infty\|G\|_r;$$

$$\|[\Delta_q, \nabla\times F\times]G\|_r\lesssim\|\nabla F\|_\infty\|G\|_r.$$

Lemma 2.5. *Let F, G and H be vector valued functions. Assume F, G and H vanish at large $|x|\in\mathbb{R}^3$. For any $1<r_1,r_2<\infty$ with $\frac{1}{r_1}+\frac{1}{r_2}=1$, we have*

$$\left|\int_{\mathbb{R}^3}[\Delta_q,\nabla\times F\times]G\cdot\nabla\times H\,dx\right|\lesssim\|\nabla^2F\|_\infty\|G\|_{r_1}\|H\|_{r_2}.$$

3. A priori estimate

In this section, we establish a priori estimate for regular solutions in $H^s(\mathbb{R}^n)\times H^r(\mathbb{R}^n)$ with appropriate index s and r. Such estimate is the most crucial ingredient in the argument of local well-posedness, which is rather standard for dissipative equations, see [15]. Thus we only present the following theorem and its proof.

Theorem 3.1. *Let $(u_0,b_0)\in H^s(\mathbb{R}^n)\times H^r(\mathbb{R}^n)$ with $s>\frac{n}{2}-1$ and $\frac{n}{2}<r\le s+1-\varepsilon$ for small enough $\varepsilon>0$. There exists a time $T=T(\nu,\mu,\|u_0\|_{H^s},\|b_0\|_{H^r})>0$ such that the Hall-MHD system (1.1) has a solution (u,b) satisfying*

$$u\in L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^2(0,T;H^{s+1}(\mathbb{R}^n)),$$

$$b\in L^\infty(0,T;H^r(\mathbb{R}^n))\cap L^2(0,T;H^{r+1}(\mathbb{R}^n)).$$

The proof involves certain amount of computations and estimates which will be divided into several lemmas, each carrying an estimate for a flux term. To start, multiplying the first equation of (1.1) by $\lambda_q^{2s}\Delta_q u_q$ and the second one by $\lambda_q^{2r}\Delta_q b_q$, and adding up for all $q\ge -1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{q\ge-1}\lambda_q^{2s}\|u_q\|_2^2+\nu\sum_{q\ge-1}\lambda_q^{2s+2}\|u_q\|_2^2\le -I_1-I_2, \tag{3.12}$$

$$\frac{1}{2}\frac{d}{dt}\sum_{q\ge-1}\lambda_q^{2r}\|b_q\|_2^2+\mu\sum_{q\ge-1}\lambda_q^{2r+2}\|b_q\|_2^2\le -I_3-I_4-I_5, \tag{3.13}$$

with

$$I_1 = \sum_{q\geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u\cdot\nabla u)\cdot u_q\,dx, \qquad I_2 = -\sum_{q\geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b\cdot\nabla b)\cdot u_q\,dx,$$
$$I_3 = \sum_{q\geq -1} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(u\cdot\nabla b)\cdot b_q\,dx, \qquad I_4 = -\sum_{q\geq -1} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(b\cdot\nabla u)\cdot b_q\,dx,$$
$$I_5 = \sum_{q\geq -1} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q((\nabla\times b)\times b)\cdot\nabla\times b_q\,dx.$$

To fully exploit cancelations in the flux terms I_1, I_3 and I_5, we will apply commutator estimates along with Bony's paraproduct and some fundamental inequalities. While $r\neq s$, there is no cancelation in I_2+I_4, and hence I_2 and I_4 will be treated in slightly different ways.

Lemma 3.2. *Let $s>\frac{n}{2}-1$. We have that, for some absolute constants $\gamma_1,\gamma_2>0$,*

$$|I_1|\leq \frac{\nu}{8}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + C_\nu\|u\|_{H^s}^{2+\gamma_1} + C_\nu\|u\|_{H^s}^{2+\gamma_2}.$$

Proof: Using Bony's paraproduct (2.8) followed by the commutator notation (2.9), I_1 is decomposed as

$$\begin{aligned} I_1 =& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_{\leq p-2}\cdot\nabla u_p)\cdot u_q\,dx\\ &+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla u_{\leq p-2})\cdot u_q\,dx\\ &+\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla \tilde{u}_p)\cdot u_q\,dx\\ =&I_{11}+I_{12}+I_{13}, \end{aligned}$$

with

$$\begin{aligned} I_{11} =& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, u_{\leq p-2}\cdot\nabla]u_p\cdot u_q\,dx\\ &+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}u_{\leq q-2}\cdot\nabla\Delta_q u_p\cdot u_q\,dx\\ &+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}(u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q u_p\cdot u_q\,dx\\ =&I_{111}+I_{112}+I_{113}. \end{aligned}$$

Thanks to the facts $\sum_{q-2\leq p\leq q+2}\Delta_q u_p = u_q$ and $\nabla\cdot u_{\leq q-2}=0$, the term I_{112} vanishes. Notice that I_{12} and I_{13} can be treated in the analogous way as I_{111} and I_{113}, respectively. Thus we will only show the estimates of I_{111} and I_{113}. Applying the commutator estimate in Lemma 2.3 and Bernstein's inequality to I_{111} gives rise to

$$
\begin{aligned}
|I_{111}| \leq & \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\|\nabla u_{\leq p-2}\|_\infty\|u_p\|_2\|u_q\|_2\\
\lesssim & \sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2\sum_{p\leq q}\lambda_p^{\frac{n}{2}+1}\|u_p\|_2\\
\lesssim & \sum_{q\geq -1}\lambda_q^{(s+1)\theta}\|u_q\|_2^\theta\lambda_q^{s(2-\theta)}\|u_q\|_2^{2-\theta}\sum_{p\leq q}\lambda_p^{(s+1)\delta}\|u_p\|_2^\delta\lambda_p^{s(1-\delta)}\|u_p\|_2^{1-\delta}\left(\lambda_q^{-\theta}\lambda_p^{\frac{n}{2}+1-s-\delta}\right)\\
\lesssim & \sum_{q\geq -1}\lambda_q^{(s+1)\theta}\|u_q\|_2^\theta\lambda_q^{s(2-\theta)}\|u_q\|_2^{2-\theta}\sum_{p\leq q}\lambda_p^{(s+1)\delta}\|u_p\|_2^\delta\lambda_p^{s(1-\delta)}\|u_p\|_2^{1-\delta}\lambda_{p-q}^{\theta}
\end{aligned}
$$

with constants θ and δ satisfying $0<\theta<2$, $0<\delta<1$ and

$$
s\geq \frac{n}{2}+1-\theta-\delta. \tag{3.14}
$$

It then follows from Young's inequality with $(r_1,r_2,r_3,r_4)\in(1,\infty)^4$ satisfying

$$
\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=1,\quad r_1=\frac{2}{\theta},\quad r_3=\frac{2}{\delta} \tag{3.15}
$$

such that for some $\theta_1>0,\theta_2>0$

$$
\begin{aligned}
|I_{111}| \leq & \frac{\nu}{64}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\sum_{q\geq -1}\left(\lambda_q^{2s}\|u_q\|_2^2\right)^{\frac{(2-\theta)r_2}{2}}\\
&+\frac{\nu}{64}\sum_{q\geq -1}\sum_{p\leq q}\lambda_p^{2s+2}\|u_p\|_2^2\lambda_{p-q}^{\theta_1}+C_\nu\sum_{q\geq -1}\sum_{p\leq q}\left(\lambda_p^{2s}\|u_p\|_2^2\right)^{\frac{(1-\delta)r_4}{2}}\lambda_{p-q}^{\theta_2}\\
\leq & \frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\left(\sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{\frac{(2-\theta)r_2}{2}}+C_\nu\left(\sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{\frac{(1-\delta)r_4}{2}}
\end{aligned}
$$

Notice that (3.14) and (3.15) imply that $s>\frac{n}{2}-1$.

To estimate I_{113}, it follows from Hölder, Bernstein and Young's inequalities that

$$
\begin{aligned}
|I_{113}| \leq & \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\|u_{\leq p-2}-u_{\leq q-2}\|_2\|\nabla u_p\|_\infty\|u_q\|_2\\
\lesssim & \sum_{q\geq -1}\lambda_q^{2s+\frac{n}{2}+1}\|u_q\|_2^3\\
\lesssim & \sum_{q\geq -1}\lambda_q^{(s+1)\theta}\|u_q\|_2^\theta\lambda_q^{s(3-\theta)}\|u_q\|_2^{3-\theta}\lambda_q^{\frac{n}{2}+1-s-\theta}\\
\leq & \frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\left(\sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{\frac{3-\theta}{2-\theta}}
\end{aligned}
$$

for $s\geq\frac{n}{2}+1-\theta$ and $0<\theta<2$. Thus

$$
I_1\leq\frac{\nu}{8}\|\nabla u\|_{H^s}^2+C_\nu\|u\|_{H^s}^{2+\gamma_1}+C_\nu\|u\|_{H^s}^{2+\gamma_2}
$$

for $s>\frac{n}{2}-1$ and some $\gamma_1,\gamma_2>0$.

$\square$

Lemma 3.3. *Let $\frac{n}{2}+s-2r\leq 0$ and $s<r$. The following estimate holds*

$$|I_2|\leq \frac{\nu}{8}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\|b\|_{H^r}^4.$$

Proof: We first decompose I_2 by using Bony's paraproduct,

$$\begin{aligned}
I_2=&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_{\leq p-2}\cdot\nabla b_p)\cdot u_q\,dx\\
&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\cdot\nabla b_{\leq p-2})\cdot u_q\,dx\\
&-\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\cdot\nabla \tilde{b}_p)\cdot u_q\,dx\\
=&I_{21}+I_{22}+I_{23}.
\end{aligned}$$

Due to the lack of cancelation, I_{21} is the worst term which can be estimated as

$$\begin{aligned}
|I_{21}|\leq&\sum_{q\geq -1}\lambda_q^{2s+1}\|u_q\|_2\sum_{|q-p|\leq 2}\|b_{\leq p-2}\|_\infty\|b_p\|_2\\
\lesssim&\sum_{q\geq -1}\lambda_q^{2s+1}\|u_q\|_2\left(\sum_{|q-q'|\leq 2}\|b_{q'}\|_2\right)\sum_{p\leq q}\lambda_p^{\frac{n}{2}}\|b_p\|_2\\
\lesssim&\sum_{q\geq -1}\lambda_q^{s+1}\|u_q\|_2\left(\sum_{|q-q'|\leq 2}\lambda_q^r\|b_{q'}\|_2\right)\sum_{p\leq q}\lambda_p^r\|b_p\|_2\lambda_{q-p}^{s-r}\lambda_p^{\frac{n}{2}+s-2r}\\
\lesssim&\sum_{q\geq -1}\lambda_q^{s+1}\|u_q\|_2\left(\sum_{|q-q'|\leq 2}\lambda_q^r\|b_{q'}\|_2\right)\sum_{p\leq q}\lambda_p^r\|b_p\|_2\lambda_{q-p}^{s-r}
\end{aligned}$$

for $\frac{n}{2}+s-2r\leq 0$. As a result, Young's inequality gives rise to

$$\begin{aligned}
|I_{21}|\leq&\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\sum_{q\geq -1}\left(\left(\sum_{|q-q'|\leq 2}\lambda_q^r\|b_{q'}\|_2\right)\sum_{p\leq q}\lambda_p^r\|b_p\|_2\lambda_{q-p}^{s-r}\right)^2\\
\leq&\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\sum_{q\geq -1}\left(\left(\sum_{|q-q'|\leq 2}\lambda_{q'}^r\|b_{q'}\|_2\right)\sum_{p\leq q}\lambda_p^r\|b_p\|_2\lambda_{q-p}^{s-r}\right)^2
\end{aligned}$$

where we used the fact $\lambda_q^r\sim\lambda_{q'}^r$ for $|q-q'|\leq 2$ to obtain the last step. We also point out that the constant C_ν in the second line is different from that in the first line. Then we apply Jensen's inequality, if $s<r$,

$$\begin{aligned}|I_{21}| \leq & \frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + C_\nu \sum_{q\geq -1}\left(\sum_{|q-q'|\leq 2}\lambda_{q'}^{r}\|b_{q'}\|_2\right)^2 \sum_{p\leq q}\lambda_p^{2r}\|b_p\|_2^2\lambda_{q-p}^{s-r}\\ \leq & \frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + C_\nu \sum_{q\geq -1}\sum_{|q-q'|\leq 2}\lambda_{q'}^{2r}\|b_{q'}\|_2^2\sum_{p\leq q}\lambda_p^{2r}\|b_p\|_2^2\\ \leq & \frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + C_\nu \sum_{q\geq -1}\sum_{|q-q'|\leq 2}\lambda_{q'}^{2r}\|b_{q'}\|_2^2\sum_{p\geq -1}\lambda_p^{2r}\|b_p\|_2^2\\ \leq & \frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + C_\nu \left(\sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2^2\right)^2.\end{aligned}$$

To analyze the term I_{22}, we recall that

$$|I_{21}| \leq \sum_{q\geq -1}\lambda_q^{2s+1}\|u_q\|_2\sum_{|q-p|\leq 2}\|b_{\leq p-2}\|_\infty\|b_p\|_2.$$

On the other hand, the following inequality holds

$$\begin{aligned}|I_{22}| &\leq \sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2\sum_{|p-q|\leq 2}\|b_p\|_2\|\nabla b_{\leq p-2}\|_\infty\\ &\leq \sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2\sum_{|p-q|\leq 2}\|b_p\|_2\lambda_q\|b_{\leq p-2}\|_\infty.\end{aligned}$$

Thus, we claim that I_{22} shares the same estimate as I_{21}.

In order to estimate I_{23}, we first move the derivative from high modes to low modes in I_{23}, by applying integration by parts

$$|I_{23}| = \left|\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b_p\otimes\tilde{b}_p)\cdot\nabla u_q\,dx\right|.$$

It then follows from Hölder's and Bernstein's inequalities

$$\begin{aligned}|I_{23}| &\lesssim \sum_{q\geq -1}\lambda_q^{2s+1}\|u_q\|_2\sum_{p\geq q-4}\|b_p\|_2\|b_p\|_\infty\\ &\lesssim \sum_{q\geq -1}\lambda_q^{2s+1}\|u_q\|_2\sum_{p\geq q-4}\lambda_p^{\frac{n}{2}}\|b_p\|_2^2\\ &\lesssim \sum_{q\geq -1}\lambda_q^{s+1}\|u_q\|_2\sum_{p\geq q-4}\lambda_p^{2r}\|b_p\|_2^2\lambda_{q-p}^{s}\lambda_p^{\frac{n}{2}+s-2r}\\ &\lesssim \sum_{q\geq -1}\lambda_q^{s+1}\|u_q\|_2\sum_{p\geq q-4}\lambda_p^{2r}\|b_p\|_2^2\lambda_{q-p}^{s}\end{aligned}$$

for $\frac{n}{2}+s-2r\leq 0$. Applying Young's inequality, Jensen's inequality and changing order of the summations yields

$$\begin{aligned}|I_{23}|\leq&\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\sum_{q\geq -1}\left(\sum_{p\geq q-4}\lambda_p^{2r}\|b_p\|_2^2\lambda_{q-p}^{s}\right)^2\\ \leq&\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\sum_{q\geq -1}\sum_{p\geq q-4}\lambda_p^{4r}\|b_p\|_2^4\lambda_{q-p}^{s}\\ \leq&\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\sum_{p\geq -1}\lambda_p^{4r}\|b_p\|_2^4\sum_{q\leq p+4}\lambda_{q-p}^{s}\\ \leq&\frac{\nu}{16}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+C_\nu\left(\sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2^2\right)^2.\end{aligned}$$

Combining the estimates of I_{21}, I_{22} and I_{23} above, we conclude the proof.

□

Lemma 3.4. *Let $s>\frac{n}{2}-1$ and $\frac{n}{4}+\frac{s}{2}<r<s+2-\varepsilon$ with small enough $\varepsilon>0$. We have the estimate*

$$|I_3|\leq\frac{\nu}{8}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+\frac{\mu}{8}\sum_{q\geq -1}\lambda_q^{2r+2}\|b_q\|_2^2+C_{\nu,\mu}\|u\|_{H^s}^{2+\gamma_3}+C_{\nu,\mu}\|b\|_{H^r}^{2+\gamma_4}$$

for some constants $\gamma_3,\gamma_4>0$.

Proof: As for I_1, we first decompose I_3 by Bony's paraproduct

$$\begin{aligned}I_3=&\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(u_{\leq p-2}\cdot\nabla b_p)\cdot b_q\,dx\\ &+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla b_{\leq p-2})\cdot b_q\,dx\\ &+\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde b_p)\cdot b_q\,dx\\ =&I_{31}+I_{32}+I_{33},\end{aligned}$$

and further decompose I_{31} by using the commutator to

$$\begin{aligned}I_{31}=&-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}[\Delta_q,u_{\leq p-2}\cdot\nabla]b_p\cdot b_q\,dx\\ &-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}(u_{\leq q-2}\cdot\nabla\Delta_q b_p)\cdot b_q\,dx\\ &-\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}((u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q b_p)\cdot b_q\,dx\\ =&I_{311}+I_{312}+I_{313}.\end{aligned}$$

It is not hard to see that $I_{312} = 0$. By the commutator estimate in Lemma 2.3, we infer

$$
\begin{aligned}
|I_{311}| \leq & \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2r}\|\nabla u_{\leq p-2}\|_\infty\|b_p\|_2\|b_q\|_2\\
\lesssim & \sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2^2\sum_{p\leq q}\lambda_p^{\frac{n}{2}+1}\|u_p\|_2\\
\lesssim & \sum_{q\geq -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^{\theta}\lambda_q^{r(2-\theta)}\|b_q\|_2^{2-\theta}\sum_{p\leq q}\lambda_p^{(s+1)\delta}\|u_p\|_2^{\delta}\lambda_p^{s(1-\delta)}\|u_p\|_2^{1-\delta}\left(\lambda_q^{-\theta}\lambda_p^{\frac{n}{2}+1-s-\delta}\right)\\
\lesssim & \sum_{q\geq -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^{\theta}\lambda_q^{r(2-\theta)}\|b_q\|_2^{2-\theta}\sum_{p\leq q}\lambda_p^{(s+1)\delta}\|u_p\|_2^{\delta}\lambda_p^{s(1-\delta)}\|u_p\|_2^{1-\delta}\lambda_{p-q}^{\theta}
\end{aligned}
$$

for parameters θ and δ satisfying $0 < \theta < 2$, $0 < \delta < 1$ and

$$
s \geq \frac{n}{2} + 1 - \theta - \delta. \tag{3.16}
$$

It then follows from Young's inequality with $(r_1, r_2, r_3, r_4) \in (1, \infty)^4$ satisfying

$$
\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=1, \quad r_1=\frac{2}{\theta}, \quad r_3=\frac{2}{\delta} \tag{3.17}
$$

such that

$$
\begin{aligned}
|I_{311}| \leq & \frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+\frac{\mu}{32}\sum_{q\geq -1}\lambda_q^{2r+2}\|b_q\|_2^2\\
&+C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{1+\gamma_3}+C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2^2\right)^{1+\gamma_4}
\end{aligned}
$$

for some constants $\gamma_3, \gamma_4 > 0$. Notice that (3.16) and (3.17) imply for large enough r_2 and r_4, and δ, θ close enough to 1, there exists a small $\varepsilon > 0$ such that

$$
s \geq \frac{n}{2} - \theta + \varepsilon > \frac{n}{2} - 1.
$$

The term I_{313} can be treated similarly as I_{113}. However, since $r > s$ and hence $\lambda_q^{2r} > \lambda_q^{2s}$, it involves more effort to distribute the wavenumber to achieve an appropriate estimate. Therefore, we choose to carry out the details in the following. Applying Hölder's inequality yields

$$
|I_{313}| \leq \sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2\sum_{|p-q|\leq 2}\|u_{\leq p-2}-u_{\leq q-2}\|_2\|\nabla b_p\|_\infty.
$$

To shorten the presentation, we identify the finite sum $\sum_{|p-q|\leq 2}\|u_{\leq p-2}-u_{\leq q-2}\|_2$ by $\|u_q\|_2$ and $\sum_{|p-q|\leq 2}\|\nabla b_p\|_\infty$ by $\|\nabla b_q\|_\infty$, which only changes the estimate up to a constant multiple. Thus, we have

$$\begin{aligned}|I_{313}| &\lesssim \sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2\|u_q\|_2\|\nabla b_q\|_\infty\\ &\lesssim \sum_{q\geq -1}\lambda_q^{2r+\frac{n}{2}+1}\|u_q\|_2\|b_q\|_2^2\\ &\lesssim \sum_{q\geq -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(2-\theta)}\|b_q\|_2^{2-\theta}\lambda_q^{(s+1)\delta}\|u_q\|_2^\delta\lambda_q^{s(1-\delta)}\|u_q\|_2^{1-\delta}\left(\lambda_q^{\frac{n}{2}+1-s-\theta-\delta}\right)\\ &\lesssim \sum_{q\geq -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(2-\theta)}\|b_q\|_2^{2-\theta}\lambda_p^{(s+1)\delta}\|u_p\|_2^\delta\lambda_p^{s(1-\delta)}\|u_p\|_2^{1-\delta}\end{aligned}$$

for parameters θ and δ satisfying $0<\theta<2$, $0<\delta<1$ and

$$s\geq \frac{n}{2}+1-\theta-\delta.$$

We can finish the estimate of I_{313} as that of I_{311} and hence

$$\begin{aligned}|I_{313}| \leq&\frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+\frac{\mu}{32}\sum_{q\geq -1}\lambda_q^{2r+2}\|b_q\|_2^2\\ &+C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2\right)^{1+\gamma_3}+C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2^2\right)^{1+\gamma_4}\end{aligned}$$

for some constants $\gamma_3,\gamma_4>0$.

Following the similar strategy as for I_{311}, we estimate I_{32} as follows,

$$\begin{aligned}|I_{32}| &\leq \sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2\sum_{|q-p|\leq 2}\|u_p\|_2\|\nabla b_{\leq p-2}\|_\infty\\ &\lesssim \sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2\left(\sum_{|q-q'|\leq 2}\|u_{q'}\|_2\right)\sum_{p\leq q}\lambda_p^{\frac{n}{2}+1}\|b_p\|_2\\ &\lesssim \sum_{q\geq -1}\lambda_q^{s+1}\left(\sum_{|q-q'|\leq 2}\|u_{q'}\|_2\right)\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(1-\theta)}\|b_q\|_2^{1-\theta}\\ &\quad\cdot\sum_{p\leq q}\lambda_p^r\|b_p\|_2\lambda_{q-p}^{r-s-1-\theta}\lambda_p^{\frac{n}{2}-s-\theta}\\ &\lesssim \sum_{q\geq -1}\lambda_q^{s+1}\left(\sum_{|q-q'|\leq 2}\|u_{q'}\|_2\right)\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(1-\theta)}\|b_q\|_2^{1-\theta}\cdot\sum_{p\leq q}\lambda_p^r\|b_p\|_2\lambda_{q-p}^{r-s-1-\theta}\end{aligned}$$

for $0<\theta<1$ and

$$s\geq \frac{n}{2}-\theta. \tag{3.18}$$

It then follows from Young's inequality and Jensen's inequality, with the triplet $(2,\frac{2}{\theta},\frac{2}{1-\theta})$ satisfying

$$r-s-1-\theta<0 \tag{3.19}$$

such that

$$
\begin{aligned}
|I_{32}| \lesssim & \frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\left(\sum_{|q-q'|\leq 2}\|u_{q'}\|_2\right)^2 + \frac{\mu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|b_q\|_2^2 \\
& + C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{\frac{1}{1-\theta}} \\
\leq & \frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + \frac{\mu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|b_q\|_2^2 + C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{\frac{1}{1-\theta}}
\end{aligned}
$$

The constraints (3.18) and (3.19) implies that for $\theta = 1-\varepsilon$

$$
s > r-1-\theta, \quad s \geq \frac{n}{2} - 1 + \varepsilon > \frac{n}{2} - 1.
$$

The term I_{33} can be estimated in an analogous way as for I_{23}. To not over burden the analysis with computations, we omit the details and claim

$$
|I_{33}| \leq \frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + \frac{\mu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|b_q\|_2^2 + C_{\nu,\mu}\left(\sum_{q\geq -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{1+\gamma_4/2}
$$

for some constant $\gamma_4 > 0$.

□

Lemma 3.5. *Let the index r and s satisfy conditions in Lemma 3.4. In addition, assume $r \leq s+1-\varepsilon$ for a small enough constant $\varepsilon > 0$. We have*

$$
\begin{aligned}
|I_4| \leq & \frac{\nu}{32}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + \frac{\mu}{32}\sum_{q\geq -1}\lambda_q^{2r+2}\|b_q\|_2^2 \\
& + C_{\nu,\mu}\|u\|_{H^s}^{2+\gamma_5} + C_{\nu,\mu}\|b\|_{H^r}^{2+\gamma_6} + C_{\nu,\mu}\|b\|_{H^r}^{2+\gamma_7}
\end{aligned}
$$

for various constants $C_{\nu,\mu}$ depending on ν, μ, and some constants $\gamma_5, \gamma_6, \gamma_7 > 0$.

Proof: As usual, using Bony's paraproduct, I_4 can be written as

$$
\begin{aligned}
I_4 = & -\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(b_{\leq p-2}\cdot\nabla u_p)\cdot b_q\,dx \\
& -\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(b_p\cdot\nabla u_{\leq p-2})\cdot b_q\,dx \\
& -\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(\tilde{b}_p\cdot\nabla u_p)\cdot b_q\,dx \\
= & I_{41} + I_{42} + I_{43}.
\end{aligned}
$$

First we notice that I_{42} and I_{43} can be estimated as I_{311} and I_{33}, respectively. While I_{41} needs to be treated in a different way, since cancellation is not available here. Applying Hölder's inequality and Bernstein's inequality first, we get

$$
\begin{aligned}
|I_{41}| &\leq \sum_{q\geq -1}\lambda_q^{2r}\|b_q\|_2\sum_{|q-p|\leq 2}\|b_{\leq p-2}\|_\infty\|\nabla u_p\|_2\\
&\lesssim \sum_{q\geq -1}\lambda_q^{2r+1}\|b_q\|_2\left(\sum_{|q-q'|\leq 2}\|u_{q'}\|_2\right)\sum_{p\leq q}\lambda_p^{\frac{n}{2}}\|b_p\|_2.
\end{aligned}
$$

Normally, we would carry the finite sum $\left(\sum_{|q-q'|\leq 2}\|u_{q'}\|_2\right)$ of five terms to the end of the estimate, as what we did in estimating I_{32}. We realize that replacing $\left(\sum_{|q-q'|\leq 2}\|u_{q'}\|_2\right)$ by $\|u_q\|_2$ would give the same estimate up to a constant multiple. Thus, to avoid lengthy inequalities, we proceed to estimate I_{41} as follows

$$
\begin{aligned}
|I_{41}| \lesssim &\sum_{q\geq -1}\lambda_q^{2r+1}\|b_q\|_2\|u_q\|_2\sum_{p\leq q}\lambda_p^{\frac{n}{2}}\|b_p\|_2\\
\lesssim &\sum_{q\geq -1}\left(\lambda_q^{(r+1)\delta}\|b_q\|_2^{\delta}\right)\left(\lambda_q^{r(1-\delta)}\|b_q\|_2^{1-\delta}\right)\left(\lambda_q^{(s+1)\eta}\|u_q\|_2^{\eta}\right)\left(\lambda_q^{s(1-\eta)}\|u_q\|_2^{1-\eta}\right)\\
&\cdot\left(\sum_{p\leq q}\lambda_p^{r}\|b_p\|_2\lambda_{q-p}^{r+1-s-\delta-\eta}\lambda_p^{\frac{n}{2}+1-s-\delta-\eta}\right)\\
\lesssim &\sum_{q\geq -1}\left(\lambda_q^{(r+1)\delta}\|b_q\|_2^{\delta}\right)\left(\lambda_q^{r(1-\delta)}\|b_q\|_2^{1-\delta}\right)\left(\lambda_q^{(s+1)\eta}\|u_q\|_2^{\eta}\right)\left(\lambda_q^{s(1-\eta)}\|u_q\|_2^{1-\eta}\right)\\
&\cdot\left(\sum_{p\leq q}\lambda_p^{r}\|b_p\|_2\lambda_{q-p}^{r+1-s-\delta-\eta}\right)
\end{aligned}
$$

provided that $\frac{n}{2}+1-s-\delta-\eta\leq 0$. We apply Young's inequality with parameters $1\leq r_1,r_2,r_3,r_4,r_5\leq\infty$ satisfying

$$
\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}+\frac{1}{r_5}=1,\quad r_1=\frac{2}{\delta},\quad r_3=\frac{2}{\eta},
$$

for some $\delta,\eta\in(0,1)$. It yields that

$$
\begin{aligned}
|I_{41}| \leq &\frac{\nu}{64}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2+\frac{\mu}{64}\sum_{q\geq -1}\lambda_q^{2r+2}\|b_q\|_2^2+C_{\nu,\mu}\sum_{q\geq -1}\lambda_q^{r(1-\delta)r_2}\|b_q\|_2^{(1-\delta)r_2}\\
&+C_{\nu,\mu}\sum_{q\geq -1}\lambda_q^{s(1-\eta)r_4}\|u_q\|_2^{(1-\eta)r_4}+C_{\nu,\mu}\sum_{q\geq -1}\left(\sum_{p\leq q}\lambda_p^{r}\|b_p\|_2\lambda_{q-p}^{r+1-s-\delta-\eta}\right)^{r_5}.
\end{aligned}
$$

Assume $r < s-1+\delta+\eta$. Using Jensen's inequality to the last term and exchanging the order of summation gives rise to

$$\begin{aligned}\sum_{q\geq -1}\left(\sum_{p\leq q}\lambda_p^r\|b_p\|_2\lambda_{q-p}^{r+1-s-\delta-\eta}\right)^{r_5} &\lesssim \sum_{q\geq -1}\sum_{p\leq q}\lambda_p^{rr_5}\|b_p\|_2^{r_5}\lambda_{q-p}^{r+1-s-\delta-\eta}\\ &\lesssim \sum_{p\leq -1}\lambda_p^{rr_5}\|b_p\|_2^{r_5}\sum_{q\geq p}\lambda_{q-p}^{r+1-s-\delta-\eta}\\ &\lesssim \left(\sum_{p\leq -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{\frac{r_5}{2}}.\end{aligned}$$

Thus one can choose δ and η close enough to 1 and r_2, r_4, r_5 large enough such that $(1-\delta)r_2 = 2+\gamma_5$, $(1-\eta)r_4 = 2+\gamma_6$ and $r_5/2 = 1+\gamma_7/2$ with $\gamma_5, \gamma_6, \gamma_7 > 0$. It then follows that

$$\begin{aligned}|I_{41}| \leq &\frac{\nu}{64}\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2 + \frac{\mu}{64}\sum_{q\geq -1}\lambda_q^{2r+2}\|b_q\|_2^2\\ &+ C_{\nu,\mu}\|u\|_{H^s}^{2+\gamma_5} + C_{\nu,\mu}\|b\|_{H^r}^{2+\gamma_6} + C_{\nu,\mu}\|b\|_{H^r}^{2+\gamma_7}\end{aligned}$$

Indeed, one can choose $\delta+\eta = 2-\varepsilon$ with $\varepsilon = \frac{1}{2}[s-(\frac{n}{2}-1)]$.

□

Lemma 3.6. *Let $r > \frac{n}{2}$. Then I_5 satisfies*

$$|I_5| \leq \frac{\mu}{16}\sum_{q\geq -1}\lambda_q^{2r+2}\|b_q\|_2^2 + C_\mu\|b\|_{H^r}^{2+\gamma_8} + C_\mu\|b\|_{H^r}^{2+\gamma_9}$$

for some constants $\gamma_8, \gamma_9 > 0$.

Proof: Applying Bony's paraproduct first, we decompose I_5 to

$$\begin{aligned}I_5 = &\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(b_{\leq p-2}\times(\nabla\times b_p))\cdot\nabla\times b_q\,dx\\ &+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(b_p\times(\nabla\times b_{\leq p-2}))\cdot\nabla\times b_q\,dx\\ &+\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(b_p\times(\nabla\times\tilde{b}_p))\cdot\nabla\times b_q\,dx\\ =&I_{51}+I_{52}+I_{53}.\end{aligned}$$

Using the commutator notation (2.10), I_{51} can be further decomposed as

$$\begin{aligned}I_{51} = &\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}[\Delta_q, b_{\leq p-2}\times\nabla\times]b_p\cdot\nabla\times b_q\,dx\\ &+\sum_{q\geq -1}\lambda_q^{2r}\int_{\mathbb{R}^3}b_{\leq q-2}\times(\nabla\times b_q)\cdot\nabla\times b_q\,dx\\ &+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2r}\int_{\mathbb{R}^3}(b_{\leq p-2}-b_{\leq q-2})\times(\nabla\times(b_p)_q)\cdot\nabla\times b_q\,dx\\ =&I_{511}+I_{512}+I_{513},\end{aligned}$$

where we used the fact $\sum_{q-2\le p\le q+2}\Delta_q b_p = b_q$. It is clear that $I_{512}=0$ due to the cross product property. By the commutator estimate in Lemma 2.5, we infer

$$
\begin{aligned}
|I_{511}| &\lesssim \sum_{q\ge -1}\sum_{|p-q|\le 2}\lambda_q^{2r+1}\|\nabla b_{\le p-2}\|_\infty\|b_p\|_2\|b_q\|_2\\
&\lesssim \sum_{q\ge -1}\lambda_q^{2r+1}\|b_q\|_2^2\sum_{p\le q}\lambda_p\|b_p\|_\infty\\
&\lesssim \sum_{q\ge -1}\lambda_q^{2r+1}\|b_q\|_2^2\sum_{p\le q}\lambda_p^{1+\frac{n}{2}}\|b_p\|_2\\
&\lesssim \sum_{q\ge -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(2-\theta)}\|b_q\|_2^{2-\theta}\sum_{p\le q}\lambda_p^{(r+1)\delta}\|b_p\|_2^\delta\lambda_p^{r(1-\delta)}\|b_p\|_2^{1-\delta}\lambda_p^{1+\frac{n}{2}-r-\delta}\lambda_q^{1-\theta}\\
&\lesssim \sum_{q\ge -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(2-\theta)}\|b_q\|_2^{2-\theta}\sum_{p\le q}\lambda_p^{(r+1)\delta}\|b_p\|_2^\delta\lambda_p^{r(1-\delta)}\|b_p\|_2^{1-\delta}\lambda_{q-p}^{1-\theta}
\end{aligned}
$$

for $0<\theta<2$, $0<\delta<1$ and

$$
r\ge \frac{n}{2}+2-(\theta+\delta),\quad 1-\theta<0. \tag{3.20}
$$

It then follows from Young's inequality with $(r_1,r_2,r_3,r_4)\in(1,\infty)^4$ satisfying

$$
\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=1,\quad r_1=\frac{2}{\theta},\quad r_3=\frac{2}{\delta} \tag{3.21}
$$

such that

$$
|I_{511}|\le \frac{\mu}{16}\sum_{q\ge -1}\lambda_q^{2r+2}\|b_q\|_2^2 + C_\mu\left(\sum_{q\ge -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{1+\bar\gamma_1} + C_\mu\left(\sum_{q\ge -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{1+\bar\gamma_2}
$$

for some constants $\bar\gamma_1,\bar\gamma_2>0$. The conditions (3.20) and (3.21) imply that

$$
r\ge\frac{n}{2}+2-2+\varepsilon>\frac{n}{2},\quad \alpha>\frac{1}{\theta}=\frac{1}{2-\varepsilon}>\frac12 \tag{3.22}
$$

provided θ close enough to 2 and δ close enough to 0.

The term I_{513} is estimated as follows,

$$
\begin{aligned}
|I_{513}| &\le \sum_{q\ge -1}\sum_{|p-q|\le 2}\lambda_q^{2r}\int_{\mathbb{R}^3}|(b_{\le p-2}-b_{\le q-2})\times(\nabla\times(b_p)_q)\cdot\nabla\times b_q|\,dx\\
&\lesssim \sum_{q\ge -1}\sum_{|p-q|\le 2}\lambda_q^{2r}\|\nabla b_q\|_\infty\|b_{\le p-2}-b_{\le q-2}\|_2\|\nabla b_p\|_2\\
&\lesssim \sum_{q\ge -1}\lambda_q^{2r+\frac{n}{2}+2}\|b_q\|_2^3\\
&\lesssim \sum_{q\ge -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(3-\theta)}\|b_q\|_2^{3-\theta}\lambda_q^{\frac{n}{2}+2-r-\theta}\\
&\lesssim \sum_{q\ge -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(3-\theta)}\|b_q\|_2^{3-\theta}
\end{aligned}
$$

for $0<\theta<2$ and

$$
r\ge\frac{n}{2}+2-\theta=\frac{n}{2}+2-2+\varepsilon>\frac{n}{2} \tag{3.23}
$$

provided $\theta = 2-\varepsilon$ with small enough ε. Thus, we have by Young's inequality that

$$|I_{513}| \le \frac{\mu}{16}\sum_{q\ge -1}\lambda_q^{2r+2}\|b_q\|_2^2 + C_\mu\left(\sum_{q\ge -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{1+\bar\gamma_3}$$

for some constant $\bar\gamma_3>0$.

Notice that

$$\begin{aligned}|I_{52}| &= \left|\sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2r}\int_{\mathbb{R}^3}\Delta_q(\nabla\times b_{\le p-2}\times b_p)\cdot\nabla\times b_q\,dx\right|\\ &\lesssim \sum_{q\ge -1}\sum_{|q-p|\le 2}\lambda_q^{2r+1}\|b_p\|_2\|\nabla b_{\le p-2}\|_\infty\|b_q\|_2,\end{aligned}$$

thus I_{52} enjoys the same estimate as for I_{511}.

To estimate I_{53}, we proceed as, by using Hölder's inequality and Bernstein's inequality

$$\begin{aligned}|I_{53}| \le& \sum_{q\ge -1}\sum_{p\ge q-2}\lambda_q^{2r}\int_{\mathbb{R}^3}|\Delta_q(b_p\times\nabla\times\tilde b_p)\cdot\nabla\times b_q|\,dx\\ \lesssim& \sum_{q\ge -1}\lambda_q^{2r}\|\nabla b_q\|_\infty\sum_{p\ge q-3}\|b_p\|_2\|\nabla b_p\|_2\\ \lesssim& \sum_{q\ge -1}\lambda_q^{2r+1+\frac{n}{2}}\|b_q\|_2\sum_{p\ge q-3}\lambda_p\|b_p\|_2^2\\ \lesssim& \sum_{q\ge -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(1-\theta)}\|b_q\|_2^{1-\theta}\sum_{p\ge q-3}\lambda_p^{(r+1)\delta}\|b_p\|_2^\delta\lambda_p^{r(2-\delta)}\|b_p\|_2^{2-\delta}\\ &\cdot\lambda_{p-q}^{1-2r-\delta}\lambda_q^{\frac{n}{2}+2-r-(\theta+\delta)}\\ \lesssim& \sum_{q\ge -1}\lambda_q^{(r+1)\theta}\|b_q\|_2^\theta\lambda_q^{r(1-\theta)}\|b_q\|_2^{1-\theta}\sum_{p\ge q-3}\lambda_p^{(r+1)\delta}\|b_p\|_2^\delta\lambda_p^{r(2-\delta)}\|b_p\|_2^{2-\delta}\lambda_{p-q}^{1-2r-\delta}\end{aligned}$$

for $0<\theta<1$, $0<\delta<2$ and

$$r\ge\frac{n}{2}+2-(\theta+\delta),\quad 1-2r-\delta<0. \tag{3.24}$$

Then by Young's inequality with $(r_1,r_2,r_3,r_4)\in(1,\infty)^4$ satisfying

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=1,\quad r_1=\frac{2}{\theta},\quad r_3=\frac{2}{\delta} \tag{3.25}$$

and Jensen's inequality, we have

$$|I_{53}| \le \frac{\mu}{16}\sum_{q\ge -1}\lambda_q^{2r+2}\|b_q\|_2^2 + C_\mu\left(\sum_{q\ge -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{1+\bar\gamma_4} + C_\mu\left(\sum_{q\ge -1}\lambda_p^{2r}\|b_p\|_2^2\right)^{1+\bar\gamma_5}$$

for some constants $\bar\gamma_4,\bar\gamma_5>0$. Again, (3.24) and (3.25) imply

$$r>\frac{n}{2}$$

provided r_2, r_4 are large enough. To summarize, we have for $r > \frac{n}{2}$

$$|I_5| \leq \frac{\mu}{16} \sum_{q\geq -1} \lambda_q^{2r+2} \|b_q\|_2^2 + C_\mu \left(\sum_{q\geq -1} \lambda_p^{2r} \|b_p\|_2^2 \right)^{1+\gamma_8/2} + C_\mu \left(\sum_{q\geq -1} \lambda_p^{2r} \|b_p\|_2^2 \right)^{1+\gamma_9/2}$$

for some constants $\gamma_8, \gamma_9 > 0$. In fact, we can take $\gamma_8/2$ as the smallest number of $\bar\gamma_1, ..., \bar\gamma_5$ and $\gamma_9/2$ as the largest one of these constants.

□

We are ready to show the uniform estimate for $\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2$ on a short time interval.

Lemma 3.7. *Assume r and s satisfy*

$$s > \frac{n}{2} - 1, \quad r > \frac{n}{2}, \quad \frac{n}{4} + \frac{s}{2} < r \leq s + 1 - \varepsilon$$

for a small enough constant $\varepsilon > 0$. There exists a time $T = T(\nu, \mu, \|u_0\|_{H^s}, \|b_0\|_{H^r})$ and a constant $C_{\nu,\mu}$ depending on ν and μ such that

$$\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2 \leq C_{\nu,\mu} \left(\|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2 \right), \quad \forall t \in [0, T].$$

Proof: Combining (3.12), (3.13), and the estimates in Lemma 3.2 to Lemma 3.6, there exist various constants $C_{\nu,\mu}$ depending on ν and μ such that

$$\begin{aligned}
&\frac{d}{dt} \left(\|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right) + \nu \|\nabla u\|_{H^s}^2 + \mu \|\nabla b\|_{H^r}^2 \\
\leq & C_{\nu,\mu} \left(\|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right)^{1+\underline{\gamma}} + C_{\nu,\mu} \left(\|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right)^{1+\overline{\gamma}}
\end{aligned}$$

with constants $\underline{\gamma} = \min\{\gamma_1, ..., \gamma_9\} > 0$ and $\overline{\gamma} = \max\{\gamma_1, ..., \gamma_9\} > 0$. It thus follows that, there exists a time $T = T(\nu, \mu, \|u_0\|_{H^s}, \|b_0\|_{H^r}) > 0$ and constant $C = C(\nu, \mu, \|u_0\|_{H^s}, \|b_0\|_{H^r})$ depending on ν, μ, $\|u_0\|_{H^s}$ and $\|b_0\|_{H^r}$ such that for $t \in [0, T]$,

$$\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2 \leq C(\nu, \mu, \|u_0\|_{H^s}, \|b_0\|_{H^r}) \left(\|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2 \right).$$

It completes the proof of the lemma and concludes the proof of Theorem 3.1.

4. Uniqueness and continuity

In this section, we establish the uniqueness of solutions stated in Theorem 1.1. The continuity in time can be obtained through a rather standard procedure, see [15]; hence we omit the proof.

Theorem 4.1. *Let $\varepsilon > 0$ be small enough. Assume (u_1, b_1, p_1) and (u_2, b_2, p_2) are solutions of (1.1)-(1.2) in $H^s(\mathbb{R}^n) \times H^{s+1-\varepsilon}(\mathbb{R}^n)$ satisfying the estimates in Theorem 3.1. Then $(u_1, b_1) = (u_2, b_2)$.*

Proof: The difference $(U, B, \pi) = (u_1 - u_2, b_1 - b_2, p_1 - p_2)$ satisfies the equations

$$\begin{aligned}
&U_t + u_2 \cdot \nabla U - b_2 \cdot \nabla B + U \cdot \nabla u_1 - B \cdot \nabla b_1 + \nabla \pi = \nu \Delta U, \\
&B_t + u_2 \cdot \nabla B - b_2 \cdot \nabla U + U \cdot \nabla b_1 - B \cdot \nabla u_1 - \nabla \times ((\nabla \times b_2) \times B) \\
&\qquad + \nabla \times ((\nabla \times B) \times b_1) = \mu \Delta B.
\end{aligned} \tag{4.26}$$

The goal is to obtain a Grönwall type of inequality for the L^2 energy of (U, B). Thus, we take inner product of the equations of U and B in (4.26) with U and B, respectively, to arrive at

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|U\|_2^2+\frac{1}{2}\|B\|_2^2\right)+\nu\|\nabla U\|_2^2+\mu\|\nabla B\|_2^2\\
=&\int_{\mathbb{R}^n}(b_2\cdot\nabla)B\cdot U\,dx+\int_{\mathbb{R}^n}(B\cdot\nabla)b_1\cdot U\,dx-\int_{\mathbb{R}^n}(u_2\cdot\nabla)U\cdot U\,dx\\
&-\int_{\mathbb{R}^n}(U\cdot\nabla)u_1\cdot U\,dx+\int_{\mathbb{R}^n}(b_2\cdot\nabla)U\cdot B\,dx+\int_{\mathbb{R}^n}(B\cdot\nabla)u_1\cdot B\,dx\\
&-\int_{\mathbb{R}^n}(u_2\cdot\nabla)B\cdot B\,dx-\int_{\mathbb{R}^n}(U\cdot\nabla)b_1\cdot B\,dx\\
&+\int_{\mathbb{R}^n}\nabla\times((\nabla\times b_2)\times B)\cdot B\,dx-\int_{\mathbb{R}^n}\nabla\times((\nabla\times B)\times b_1)\cdot B\,dx.
\end{aligned}
\tag{4.27}
$$

Since (u_1, b_1) and (u_2, b_2) are in $H^s(\mathbb{R}^n)\times H^{s+1-\varepsilon}(\mathbb{R}^n)$ with $s > \frac{n}{2}-1$, so is (U, B). Thus it can be justified that many terms on the right hand side vanish, i.e.

$$
\begin{aligned}
\int_{\mathbb{R}^n}(u_2\cdot\nabla)U\cdot U\,dx=0,\quad &\int_{\mathbb{R}^n}(u_2\cdot\nabla)B\cdot B\,dx=0,\\
&\int_{\mathbb{R}^n}\nabla\times((\nabla\times B)\times b_1)\cdot B\,dx=0\\
\int_{\mathbb{R}^n}(b_2\cdot\nabla)B\cdot U\,dx+&\int_{\mathbb{R}^n}(b_2\cdot\nabla)U\cdot B\,dx=0.
\end{aligned}
$$

We are left to estimate the five non-zero flux terms. The first one is estimated as

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n}(B\cdot\nabla)b_1\cdot U\,dx\right|=&\left|\int_{\mathbb{R}^n}(B\cdot\nabla)U\cdot b_1\,dx\right|\\
\le&\|B\|_2\|\nabla U\|_2\|b_1\|_\infty\\
\le&\frac{\nu}{8}\|\nabla U\|_2^2+C_\nu\|B\|_2^2\|b_1\|_\infty^2\\
\le&\frac{\nu}{8}\|\nabla U\|_2^2+C_\nu\|B\|_2^2\|b_1\|_{H^{s+1-\epsilon}}^2
\end{aligned}
$$

where we used the embedding $H^{s+1-\varepsilon}\subset L^\infty$ for $s+1-\varepsilon > \frac{n}{2}$ (since we can choose $\varepsilon = \frac{1}{2}[s-(\frac{n}{2}-1)]$ and $s > \frac{n}{2}-1$). Analogous computation shows

$$
\left|\int_{\mathbb{R}^n}(U\cdot\nabla)u_1\cdot U\,dx\right|\le\frac{\nu}{8}\|\nabla U\|_2^2+C_\nu\|U\|_2^2\|u_1\|_{H^{s+1}}^2,
$$

$$
\left|\int_{\mathbb{R}^n}(B\cdot\nabla)u_1\cdot B\,dx\right|\le\frac{\mu}{8}\|\nabla B\|_2^2+C_\mu\|B\|_2^2\|u_1\|_{H^{s+1}}^2,
$$

$$
\left|\int_{\mathbb{R}^n}(U\cdot\nabla)b_1\cdot B\,dx\right|\le\frac{\mu}{8}\|\nabla B\|_2^2+C_\mu\|U\|_2^2\|b_1\|_{H^{s+1-\varepsilon}}^2.
$$

In the end, we estimate the Hall term as follows

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \nabla\times((\nabla\times b_2)\times B)\cdot B\,dx\right| =&\left|\int_{\mathbb{R}^n} ((\nabla\times b_2)\times B)\cdot \nabla\times B\,dx\right|\\ \leq&\|\nabla\times B\|_2\|\nabla\times b_2\|_\infty\|B\|_2\\ \leq&\frac{\mu}{8}\|\nabla B\|_2^2+C_\mu\|\nabla\times b_2\|_\infty^2\|B\|_2^2\\ \leq&\frac{\mu}{8}\|\nabla B\|_2^2+C_\mu\|\nabla b_2\|_{H^{s+1-\varepsilon}}^2\|B\|_2^2.\end{aligned}$$

The estimates above along with (4.27) give us

$$\begin{aligned}&\frac{d}{dt}\left(\|U\|_2^2+\|B\|_2^2\right)+\nu\|\nabla U\|_2^2+\mu\|\nabla B\|_2^2\\ \leq&C_{\nu,\mu}\left(\|u_1\|_{H^{s+1}}^2+\|\nabla b_2\|_{H^{s+1-\varepsilon}}^2+\|b_1\|_{H^{s+1-\varepsilon}}^2\right)\left(\|U\|_2^2+\|B\|_2^2\right)\\ \leq&C_{\nu,\mu}\left(\|u_1\|_{H^{s+1}}^2+\|\nabla b_2\|_{H^{s+1-\varepsilon}}^2+C\right)\left(\|U\|_2^2+\|B\|_2^2\right).\end{aligned}$$

It follows from Grönwall's inequality that

$$\begin{aligned}&\|U(t)\|_2^2+\|B(t)\|_2^2\\ \leq&\left(\|U(0)\|_2^2+\|B(0)\|_2^2\right)e^{CC_{\nu,\mu}t}\exp\left\{C_{\nu,\mu}\int_0^t\|u_1(\tau)\|_{H^{s+1}}^2+\|\nabla b_2(\tau)\|_{H^{s+1-\varepsilon}}^2\,d\tau\right\}.\end{aligned}$$

Since $U(0)=B(0)=0$, $u_1\in L^2(0,T;H^{s+1})$ and $b_2\in L^2(0,T;H^{s+2-\varepsilon})$, we infer

$$\|U(t)\|_2^2+\|B(t)\|_2^2=0,\quad \forall t\in[0,T].$$

□

References

[1] M. Acheritogaray, P. Degond, A. Frouvelle and J-G. Liu. *Kinetic formulation and global existence for the Hall-Magnetohydrodynamic system.* Kinetic and Related Models, 4: 901–918, 2011.

[2] H. Bahouri, J. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations.* Grundlehrender Mathematischen Wissenschaften, 343. Springer, Heidelberg, 2011.

[3] D. Chae, P. Degond and J-G. Liu. *Well-posedness for Hall–magnetohydrodynamics.* arXiv:1212.3919, 2012.

[4] D. Chae and J. Lee. *On the blow-up criterion and small data global existence for the Hall-magneto-hydrodynamics.* J. Differential Equations, 256: 3835–3858, 2014.

[5] D. Chae, and M. Schonbek. *On the temporal decay for the Hall-magnetohydrodynamic equations.* Journal of Differential Equations, Vol.255, Iss. 11: 3971–3982, 2013.

[6] D. Chae, R. Wan and J. Wu. *Local well-posedness for the Hall–MHD equations with fractional magnetic diffusion.* Journal of Mathematical Fluid Mechanics, Vol. 17, Iss. 4: 627–638, 2015.

[7] D. Chae and J. Wolf. *On partial regularity for the 3D non-stationary Hall magnetohydrodynamics equations on the plane.* SIAM Journal on Mathematical Analysis, Vol. 48, No.1: 443–469.

[8] M. Dai. *Local well-posedness of the Hall-MHD system in $H^s(\mathbb{R}^n)$ with $s>\frac{n}{2}$.* Mathematische Nachrichten. DOI:10.1002/mana.201800107, 2018.

[9] M. Dai. *Regularity criterion for the 3D Hall-magneto-hydrodynamics.* Journal of Differential Equations. Vol. (261): 573–591, 2016.

[10] E. Dumas and F. Sueur. *On the weak solutions to the Maxwell-Landau-Lifshitz equations and to the Hall-magnetohydrodynamic equations.* Comm. Math. Phys., 330: 1179–1225, 2014.

[11] T. G. Forbes. *Magnetic reconnection in solar flares.* Geophys. astropphys. fluid dynamics, 62: 15–36, 1991.

[12] L. Grafakos. *Modern Fourier analysis*. Second edition. Graduate Texts in Mathematics, 250. Springer, New York, 2009.

[13] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*. Chapman and Hall/CRC Research Notes in Mathematics, 431. Chapman and Hall/CRC, Boca Raton, FL, 2002.

[14] M. J. Lighthill. *Studies on magnetohydrodynamic waves and other anisotropic wave motions*. Philos. Trans. R. Soc. Lond., Ser. A : 397–430, 1960.

[15] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*. Cambridge University Press, 2001.

[16] A. N. Simakov and L. Chacon. *Quantitative, analytical model for magnetic reconnection in Hall magnetohydrodynamics*. Phys. Rev. Lett, 101, 105003, 2008.

Department of Mathematics, Stat. and Comp.Sci., University of Illinois at Chicago, Chicago, IL 60607,USA

Email address: `mdai@uic.edu`