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Braids groups and mapping class groups: The Birman–Hilden theory

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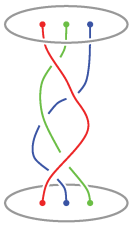
Abstract

In the 1970s, Joan Birman and Hugh Hilden wrote several papers on the problem of relating the mapping class group of a surface to that of a covering space. Their results provide a bridge between the theories of mapping class groups and braid groups. We survey the work of Birman and Hilden, give an overview of the subsequent developments, and discuss open questions and new directions.

1. *Introduction*

In the early 1970s, Birman and Hilden wrote a series of now-classic papers that give a dictionary between the theories of braid groups and mapping class groups. Their work has resulted in a fruitful flow of ideas between the two theories, in both directions. Before describing their work, we introduce braid groups and mapping class groups and discuss some of their connections to other areas of mathematics.

Braid groups. The *braid group* on n strands is the fundamental group of the space of (unordered) configurations of n points in the unit disk D^2. We may think of an element of the braid group as a motion of n points in D^2. By tracing out the paths of the points over time in $D^2 \times [0,1]$ (the [0,1] factor corresponds to time), we obtain a picture of a braid as follows:



The braid group was originally defined by Artin [3] in 1925 (in a slightly different way).

Braid groups are ubiquitous in mathematics and science because they encode the salient features of any motion of points in the plane. The points in question can be, for example, roots of polynomials in the complex plane, physical particles in a two-dimensional medium, or robots moving on a factory floor. As such, braid groups play an important role in algebraic geometry, physics, and robotics, as well as homotopy groups of spheres, operator algebras, and dynamics, to name a few; see the comprehensive survey by Birman and Brendle [12].

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Mapping class groups. In this paper, we consider orientable surfaces, each with a (possibly empty) set of marked points. The closed, orientable surface of genus g with n marked points is denoted by $S_{g,n}$ or S_g (when $n = 0$).

The *mapping class group* of any such surface is the group of homotopy classes of orientation-preserving homeomorphisms that fix the boundary pointwise and that preserve the set of marked points (homotopies must also fix the boundary and preserve the set of marked points).

The mapping class group is of central importance in mathematics. The group $\mathrm{Mod}(S_g)$, for example, has the following interpretations.

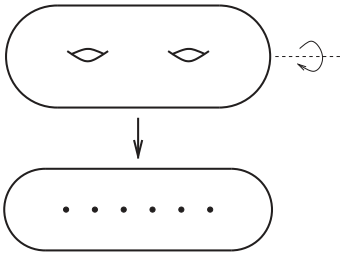
- It describes the orbifold fundamental group of moduli space $\mathcal{M}_g$.
- It classifies all surface bundles with fiber S_g.
- It is isomorphic to the outer automorphism group of $\pi_1(S_g)$.

The mapping class group was first studied by Dehn [**23**] in 1922. See the book by Farb and the first author for an introduction to mapping class groups [**29**].

There is an immediate connection between braid groups and mapping class groups, as the braid group is isomorphic to the mapping class group of the closed disk with n marked points [**29**, Theorem 9.1]. Still the two are considered as separate theories, each with their own specialized sets of tools and techniques. The Birman–Hilden theory uses covering spaces to build a bridge between these two theories.

Birman and Hilden. The initial motivation for the work of Birman and Hilden was to find a group presentation for $\mathrm{Mod}(S_2)$. (It was not until the late 1970s that Hatcher and Thurston [**35**] developed an approach for finding explicit presentations for mapping class groups.) In solving this problem, their key innovation was to relate $\mathrm{Mod}(S_2)$ to the mapping class group of $S_{0,6}$, a sphere with six marked points. Presentations for $\mathrm{Mod}(S_{0,6})$ were already known since that group is closely related to a braid group, and presentations for the latter had been well studied.

Specifically, the surfaces S_2 and $S_{0,6}$ are related by a twofold branched covering map (see Section 2 for the definitions):



The six marked points in the base surface $S_{0.6}$ are branch points. The deck transformation is called the *hyperelliptic involution* of S_2, and we denote it by ι. Every element of $\mathrm{Mod}(S_2)$ has a representative that commutes with ι (see Section 4), and so it follows that there is a map

$$\Theta : \mathrm{Mod}(S_2) \to \mathrm{Mod}(S_{0,6}).$$

The kernel of Θ is the cyclic group of order two generated by (the homotopy class of) the involution ι. One can check that each generator for $\mathrm{Mod}(S_{0,6})$ lifts to $\mathrm{Mod}(S_2)$ and so Θ is surjective. From this, we have a short exact sequence

$$1 \to \langle \iota \rangle \to \mathrm{Mod}(S_2) \xrightarrow{\Theta} \mathrm{Mod}(S_{0,6}) \to 1,$$

and hence a presentation for $\mathrm{Mod}(S_{0,6})$ can be lifted to a presentation for $\mathrm{Mod}(S_2)$.

But wait! The map Θ is not *a priori* well defined. More specifically, the problem is that elements of $\mathrm{Mod}(S_2)$ are only defined up to isotopy, and these isotopies are not required to

respect the hyperelliptic involution. The first paper by Birman and Hilden proves that in fact all isotopies can be chosen to respect the involution. Birman and Hilden realized that the theory initiated in that first paper can be generalized in various ways, and they wrote a series of papers on the subject, culminating in the paper *On Isotopies of Homeomorphisms of Riemann Surfaces* [**17**], published in *Annals of Mathematics* in 1973.

In the remainder of this article, we will give a historical account of the Birman–Hilden theory, from the original work of Birman and Hilden to the generalizations by Maclachlan–Harvey and the second author of this article. We then give several applications of the theory, explain three proof strategies for Birman–Hilden-type results, and discuss various open questions and new directions in the theory. As we will see, the Birman–Hilden theory has had influence on many areas of mathematics, from low-dimensional topology, to geometric group theory, to representation theory, to algebraic geometry and more, and it continues to produce interesting open problems and research directions.

2. *Statements of the main theorem*

A map of surface $p : S \to X$ is a *branched covering map* if there is a finite set $B \subseteq X$ so that the restriction $S \setminus p^{-1}(B) \to X \setminus B$ is a covering map. We say that p is *unbranched* if B can be taken to be the empty set, that is, if p is itself a covering map.

We say that the branched covering map p has *degree* d if the corresponding covering map does. In this case, a point of X with fewer than d preimages is called a *branch point* for p. The set of branch points is contained in B, and moreover the set of branch points is the smallest set B with the property that $S \setminus p^{-1}(B) \to X \setminus B$ is a covering map.

The *degree* of the branched covering map p is defined to be the degree of the corresponding covering map. The branched covering map p is said to be *regular* if the corresponding covering map is. We say that p is *cyclic* or *solvable* if the deck group of the corresponding covering map is cyclic or solvable, respectively.

Let $p : S \to X$ be a covering map of surfaces, possibly branched, possibly with boundary. We say that $f : S \to S$ is *fiber preserving* if for each $x \in X$ there is a $y \in X$ so that $f(p^{-1}(x)) = p^{-1}(y)$; in other words, as the terminology suggests, f takes fibers to fibers.

Given two homotopic fiber-preserving homeomorphisms of S, we can ask if they are homotopic through fiber-preserving homeomorphisms. If the answer is yes for all such pairs of homeomorphisms, we say that the covering map p has the *Birman–Hilden property*. An equivalent formulation of the Birman–Hilden property is whenever a fiber-preserving homeomorphism is homotopic to the identity, it is homotopic to the identity through fiber-preserving homeomorphisms.

We are now ready to state the main theorems of the Birman–Hilden theory. There are several versions, proved over the years by various authors, each generalizing the previous. The first version is the one that appears in the aforementioned *Annals of Mathematics* paper [**17**] by Birman and Hilden and also in the accompanying research announcement *Isotopies of Homeomorphisms of Riemann surfaces* [**15**]. Throughout, we will say that a surface is *hyperbolic* if its Euler characteristic is negative.

THEOREM 2.1 (Birman–Hilden). *Let $p : S \to X$ be a finite-sheeted regular branched covering map where S is a hyperbolic surface. Assume that p is either unbranched or is solvable. Then p has the Birman–Hilden property.*

If we apply Theorem 2.1 to the branched covering map $S_2 \to S_{0,6}$ described earlier, then it exactly says that the map $\Theta : \mathrm{Mod}(S_2) \to \mathrm{Mod}(S_{0,6})$ is well defined.

It is worthwhile to compare our Theorem 2.1 to what is actually stated by Birman and Hilden. In their paper, they state two theorems, each of which is a special case of Theorem 2.1. Their

Theorem 1 treats the case of regular covers where each deck transformation fixes each preimage of each branch point in X. This clearly takes care of the case of unbranched covers, and also the case of certain solvable branched covers (on one hand a finite group of homeomorphisms of a surface that fixes a point must be a subgroup of a dihedral group, and on the other hand there are solvable — even cyclic — branched covers that do not satisfy the condition of Theorem 1). Birman and Hilden's Theorem 2 deals with the general case of solvable covers, which includes some unbranched covers.

In early 1973, Maclachlan and Harvey [47] published a paper called *On Mapping Class Groups and Teichmüller Spaces*, in which they give the following generalization of Theorem 2.1.

THEOREM 2.2 (Maclachlan–Harvey). *Let $p : S \to X$ be a finite-sheeted regular branched covering map where S is a hyperbolic surface. Then p has the Birman–Hilden property.*

Maclachlan and Harvey's work was contemporaneous with the work of Birman and Hilden cited in Theorem 2.1, and was subsequent to the original paper by Birman and Hilden on the hyperelliptic case. Their approach is completely different, and is framed in terms of Teichmüller theory.

The 2014 PhD thesis of the second author of this article is a further generalization [56]. For the statement, a preimage of a branch point is *unramified* if some small neighborhood is mapped injectively under the covering map, and a cover is *fully ramified* if no branch point has an unramified preimage.

THEOREM 2.3 (Winarski). *Let $p : S \to X$ be a finite-sheeted branched covering map where S is a hyperbolic surface, and suppose that p is fully ramified. Then p has the Birman–Hilden property.*

Note that all regular covers are fully ramified and also that all unbranched covers are fully ramified. Thus Theorem 2.3 indeed implies Theorems 2.1 and 2.2. In Section 2.3 of her paper, Winarski gives a general construction of irregular branched covers that are fully ramified. Thus there are many examples of covering spaces that satisfy the hypotheses of Theorem 2.3 but not those of Theorem 2.2.

We will briefly remark on the assumption that S is hyperbolic. It is not hard to construct counterexamples in the other cases. For instance, suppose S is the torus T^2 and $p : S \to X$ is the branched covering map corresponding to the hyperelliptic involution of T^2. In this case, X is the sphere with four marked points. Rotation of T^2 by π in one S^1-factor is a fiber-preserving homeomorphism homotopic to the identity, but the induced homeomorphism of X acts nontrivially on the marked points and hence is not homotopic to the identity. Thus this cover fails the Birman–Hilden property. One can construct a similar example when S is the sphere S^2 and $p : S^2 \to X$ is the branched covering map induced by a finite-orde rotation.

3. *Restatement of the main theorem*

We will now give an interpretation of the Birman–Hilden property — hence all three theorems above — in terms of mapping class groups.

Let $p : S \to X$ be a branched covering map of surfaces. We treat each branch point in X as a marked point, and so homeomorphisms of X are assumed to preserve the set of branch points. Let $\mathrm{LMod}(X)$ denote the subgroup of the mapping class group $\mathrm{Mod}(X)$ consisting of all elements that have representatives that lift to homeomorphisms of S. This group is called the *liftable mapping class group* of X.

Let $\mathrm{SMod}(S)$ denote the subgroup of $\mathrm{Mod}(S)$ consisting of the homotopy classes of all fiber-preserving homeomorphisms. Here we emphasize that two homeomorphisms of S are identified in $\mathrm{SMod}(S)$ if they differ by an isotopy that is not necessarily fiber preserving (so that we have a subgroup of $\mathrm{Mod}(S)$). We also emphasize that preimages of branch points are not marked. Fiber-preserving homeomorphisms are also called *symmetric homeomorphisms*; these are exactly the lifts of liftable homeomorphisms of X. The group $\mathrm{SMod}(S)$ is called the *symmetric mapping class group* of S.

Let D denote the subgroup of $\mathrm{SMod}(S)$ consisting of the homotopy classes of the deck transformations (it is a fact that nontrivial deck transformations represent nontrivial mapping classes).

PROPOSITION 3.1. *Let $p : S \to X$ be a finite-sheeted branched covering map where S is a hyperbolic surface without boundary. Then the following are equivalent.*

- *p has the Birman–Hilden property.*
- *The natural map $\mathrm{LMod}(X) \to \mathrm{SMod}(S)/D$ is injective.*
- *The natural map $\mathrm{SMod}(S) \to \mathrm{LMod}(X)$ is well defined.*
- $\mathrm{SMod}(S)/D \cong \mathrm{LMod}(X)$.

The proposition is straightforward to prove. The main content is the equivalence of the first two statements. The other statements, while useful in practice, are equivalent by rudimentary abstract algebra. Using the proposition, one obtains several restatements of Theorems 2.1–2.3 in terms of mapping class groups.

Birman and Hilden also proved that for a regular cover $\mathrm{SMod}(S)$ is the normalizer in $\mathrm{Mod}(S)$ of the deck group D (regarded as a subgroup of $\mathrm{Mod}(S)$), and so we can also write the last statement in Proposition 3.1 as

$$N_{\mathrm{Mod}(S)}(D)/D \cong \mathrm{LMod}(X).$$

Birman and Hilden only stated the result about normalizers in the case where the deck group is cyclic. However, by combining their argument with Kerckhoff's resolution of the Nielsen realization problem [**41**], one obtains the more general version.

There is also a version of Proposition 3.1 for surfaces with boundary. Since the mapping class group of a surface with boundary is torsion free, the deck transformations do not represent elements of $\mathrm{Mod}(S_g)$. And so in this case, we can simply replace D with the trivial group. For example, in the presence of boundary the Birman–Hilden property is equivalent to the statement that $\mathrm{SMod}(S) \cong \mathrm{LMod}(X)$. This will become especially important in the discussion of braid groups below.

4. *Application to presentations of mapping class groups*

The original work on the Birman–Hilden theory concerns the case of the hyperelliptic involution and is reported in the 1971 paper *On the mapping class groups of closed surfaces as covering spaces* [**14**]. We will explain how Theorem 2.1 specializes in this case and helps to give presentations for the associated symmetric mapping class group and the full mapping class group in genus two.

Consider the covering space $S_g \to S_{0,2g+2}$ induced by a hyperelliptic involution of S_g. In general, a *hyperelliptic involution* of S_g is a homeomorphism of order two that acts by $-I$ on $H_1(S_g;\mathbb{Z})$; we remark that the hyperelliptic involution is unique up to homotopy for S_1 and S_2 but there are infinitely many distinct hyperelliptic involutions of S_g when $g \geqslant 3$.

Theorem 2.1 and Proposition 3.1 give an isomorphism

$$\mathrm{SMod}(S_g)/\langle \iota \rangle \cong \mathrm{LMod}(S_{0,2g+2}).$$

In the special case of the hyperelliptic involution, we have $\mathrm{LMod}(S_{0,2g+2}) = \mathrm{Mod}(S_{0,2g+2})$. Indeed, we can check directly that each half-twist generator for $\mathrm{Mod}(S_{0,2g+2})$ lifts to a Dehn twist in S_g.

In the case $g = 2$, we further have

$$\mathrm{SMod}(S_2) = \mathrm{Mod}(S_2).$$

In other words, every mapping class of S_2 is symmetric with respect to the hyperelliptic involution. The easiest way to see this is to note that each of the Humphries generators for $\mathrm{Mod}(S_2)$ is a Dehn twist about a curve that is preserved by the hyperelliptic involution. We thus have the following isomorphism:

$$\mathrm{Mod}(S_2)/\langle \iota \rangle \cong \mathrm{Mod}(S_{0,6}).$$

Simple presentations for $\mathrm{Mod}(S_{0,n})$ were found by Magnus, and so from his presentation for $\mathrm{Mod}(S_{0,6})$ Birman and Hilden use the above isomorphism to derive the following presentation for $\mathrm{Mod}(S_2)$. The generators are the Humphries generators for $\mathrm{Mod}(S_2)$, and we denote them by $T_1, \ldots, T_5$. The relations are:

$$\begin{aligned}
[T_i, T_j] &= 1 \qquad \text{for } |i-j| > 2 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \qquad \text{for } 1 \leqslant i \leqslant 4 \\
(T_1 T_2 T_3 T_4 T_5)^6 &= 1 \\
(T_1 T_2 T_3 T_4 T_5 T_5 T_4 T_3 T_2 T_1)^2 &= 1 \\
[T_1 T_2 T_3 T_4 T_5 T_5 T_4 T_3 T_2 T_1, T_1] &= 1.
\end{aligned}$$

The first two relations are the standard braid relations from B_6, the next relation describes the kernel of the map $B_6 \to \mathrm{Mod}(S_{0,6})$, and the last two relations come from the two-fold cover: the mapping class

$$T_1 T_2 T_3 T_4 T_5 T_5 T_4 T_3 T_2 T_1$$

is the hyperelliptic involution. This presentation is the culmination of a program begun by Bergau and Mennicke [7], who approached the problem by studying the surjective homomorphism $B_6 \to \mathrm{Mod}(S_2)$ that factors through the map $\mathrm{Mod}(S_{0,6}) \to \mathrm{Mod}(S_2)$ used here.

Birman used the above presentation to give a normal form for elements of $\mathrm{Mod}(S_2)$ and hence a method for enumerating 3-manifolds of Heegaard genus two [11].

As explained by Birman and Hilden, the given presentation for $\mathrm{Mod}(S_2)$ generalizes to a presentation for $\mathrm{SMod}(S_g)$. The latter presentation has many applications to the study of $\mathrm{SMod}(S_g)$. It was used by Meyer [50] to show that if a surface bundle over a surface has monodromy in $\mathrm{SMod}(S_g)$, then the signature of the resulting 4-manifold is zero; see also the related work of Endo [27]. Endo and Kotschick used the Birman–Hilden presentation to show that the second bounded cohomology of $\mathrm{SMod}(S_g)$ is nontrivial [28]. Also, Kawazumi [40] used it to understand the low-dimensional cohomology of $\mathrm{SMod}(S_g)$.

In 1972, Birman and Chillingworth published the paper *On the homeotopy group of a nonorientable surface* [13]. There, they determine a generating set for the mapping class group (= homeotopy group) of an arbitrary closed nonorientable surface using similar ideas, namely, they exploit the associated orientation double cover and pass information through the Birman–Hilden theorem from the orientable case. They also find an explicit finite presentation for the mapping class group of a closed nonorientable surface of genus three, which admits a degree two cover by S_2.

One other observation from the 1971 paper is that $\mathrm{Mod}(S_2)$ is both a quotient of and a subgroup of $\mathrm{Mod}(S_{2,6})$. To realize $\mathrm{Mod}(S_2)$ as a quotient, we consider the map $\mathrm{Mod}(S_{2,6}) \to$

$\mathrm{Mod}(S_2)$ obtained by forgetting the marked points/punctures; this is a special case of the Birman exact sequence studied by Birman in her thesis [10]. And to realize $\mathrm{Mod}(S_2)$ as a subgroup, we use the Birman–Hilden theorem: since every element of $\mathrm{Mod}(S_2)$ has a symmetric representative that preserves the set of preimages of the branch points in $S_{0,6}$ and since isotopies between symmetric homeomorphisms can also be chosen to preserve this set of six points, we obtain the desired inclusion. Birman and Hilden state that 'the former property is easily understood but the latter much more subtle'. As mentioned by Mess [49], the inclusion $\mathrm{Mod}(S_2) \to \mathrm{Mod}(S_{2,6})$ can be rephrased as describing a multi-section of the universal bundle over moduli space in genus two.

5. *More applications to the genus two mapping class group*

In the previous section, we saw how the Birman–Hilden theory allows us to transport knowledge about the mapping class group of a punctured sphere to the mapping class group of a surface of genus two. As the former are closely related to braid groups, we can often push results about braid groups to the mapping class group. Almost every result about mapping class groups that is special to genus two is proved in this way.

A prime example of this is the result of Bigelow–Budney [9] and Korkmaz [44] which states that $\mathrm{Mod}(S_2)$ is linear, that is, $\mathrm{Mod}(S_2)$ admits a faithful representation into $\mathrm{GL}_N(\mathbb{C})$ for some N. Bigelow and Krammer independently proved that braid groups were linear, and so the main work is to derive from this the linearity of $\mathrm{Mod}(S_{0,n})$. They then use the isomorphism $\mathrm{Mod}(S_2)/\langle\iota\rangle \cong \mathrm{Mod}(S_{0,6})$ to push the linearity up to $\mathrm{Mod}(S_2)$.

A second example is from the thesis of Whittlesey, published in 2000. She showed that $\mathrm{Mod}(S_2)$ contains a normal subgroup where every nontrivial element is pseudo-Anosov [55]. The starting point is to consider the Brunnian subgroup of $\mathrm{Mod}(S_{0,6})$. This is the intersection of the kernels of the six forgetful maps $\mathrm{Mod}(S_{0,6}) \to \mathrm{Mod}(S_{0,5})$, so it is obviously normal in $\mathrm{Mod}(S_{0,6})$. She shows that all nontrivial elements of this group are pseudo-Anosov and proves that the preimage in $\mathrm{Mod}(S_2)$ has a finite-index subgroup with the desired properties.

We give one more example. In the 1980s, before the work of Bigelow and Krammer, Vaughan Jones discovered a representation of the braid group defined in terms of Hecke algebras [39]. As in the work of Bigelow–Budney and Korkmaz, one can then derive a representation of $\mathrm{Mod}(S_{0,2g+2})$ and then — using the Birman–Hilden theory — of $\mathrm{SMod}(S_g)$. When $g = 2$, we thus obtain a representation of $\mathrm{Mod}(S_2)$ to $\mathrm{GL}_5(\mathbb{Z}[t, t^{-1}])$. This representation was used by Humphries [36] to show that the normal closure in $\mathrm{Mod}(S_2)$ of the kth power of a Dehn twist about a nonseparating curve has finite index if and only if $|k| \leqslant 3$.

There are many other examples, such as the computation of the asymptotic dimension of $\mathrm{Mod}(S_2)$ by Bell and Fujiwara [6] and the determination of the minimal dilatation in $\mathrm{Mod}(S_2)$ by Cho and Ham [20]; the list goes on, but so must we.

6. *Application to representations of the braid group*

The Birman–Hilden theorem also gives a way to embed braid groups into mapping class groups. This is probably the most oft-used application of their results.

Let S_g^1 the orientable surface of genus g with one boundary component and let D_{2g+1} denote the closed disk with $2g + 1$ marked points in the interior. Consider the covering space $S_g^1 \to D_{2g+1}$ induced by a hyperelliptic involution of S_g^1. It is well known that $\mathrm{Mod}(D_{2g+1})$ is isomorphic to the braid group B_{2g+1}. As in the closed case, it is not hard to see that $\mathrm{LMod}(D_{2g+1}) = \mathrm{Mod}(D_{2g+1})$ (again, each of the standard generators for B_{2g+1} lifts to a Dehn twist).

One is thus tempted to conclude that $\mathrm{SMod}(S_g^1)/\langle\iota\rangle \cong B_{2g+1}$. But this is not the right statement, since ι does not represent an element of $\mathrm{Mod}(S_g^1)$. Indeed, for surfaces with boundary we insist that homeomorphisms and homotopies fix the boundary pointwise. Therefore, the correct isomorphism is

$$\mathrm{SMod}(S_g^1) \cong B_{2g+1}.$$

The most salient aspect of this isomorphism is that there is an injective homomorphism

$$B_{2g+1} \to \mathrm{Mod}(S_g^1).$$

The injectivity here is sometimes attributed to Perron–Vannier [52]. It is possible that they were the first to observe this consequence of the Birman–Hilden theorem but the only nontrivial step is the Birman–Hilden theorem.

In the case of $g = 1$, the representation of B_3 is onto $\mathrm{Mod}(S_1^1)$, and so

$$\mathrm{Mod}(S_1^1) \cong B_3.$$

Similarly we have

$$\mathrm{Mod}(S_1^2) \cong B_4 \times \mathbb{Z}.$$

The point here is that B_4 surjects onto $\mathrm{SMod}(S_1^2)$ and the latter is almost isomorphic to $\mathrm{Mod}(S_1^2)$; the extra $\mathbb{Z}$ comes from the Dehn twist about a single boundary component.

One reason that the embedding of B_{2g+1} in $\mathrm{Mod}(S_g^1)$ is so important is that if we compose with the standard symplectic representation

$$\mathrm{Mod}(S_g^1) \to \mathrm{Sp}_{2g}(\mathbb{Z}),$$

then we obtain a representation of the braid group

$$B_{2g+1} \to \mathrm{Sp}_{2g}(\mathbb{Z}).$$

This representation is called the standard symplectic representation of the braid group. It is also called the *integral Burau representation* because it is the only integral specialization of the Burau representation besides the permutation representation. The symplectic representation is obtained by specializing the Burau representation at $t = -1$, while the permutation representation is obtained by taking $t = 1$.

The image of the integral Burau representation has finite index in the symplectic group: it is an extension of the level two symplectic group by the symmetric group on $2g + 1$ letters. The projection onto the symmetric group factor is the standard permutation representation of the braid group. See A'Campo's paper [1] for details.

The kernel of the integral Burau representation is known as the hyperelliptic Torelli group. This group is well studied, as it describes the fundamental group of the branch locus of the period mapping from Teichmüller space to the Siegel upper half-space; see, for instance, the paper [19] by Brendle, Putman, and the first author of this article and the references therein.

There are plenty of variations on the given representation. Most important is that if we take a surface with two boundary components S_g^2 and choose a hyperelliptic involution, that is, an order two homeomorphism that acts by $-I$ on the first homology of the surface, then the quotient is D_{2g+2} and so we obtain an isomorphism:

$$\mathrm{SMod}(S_g^2) \cong B_{2g+2}.$$

Also, since the inclusions $S_g^1 \to S_{g+1}$ and $S_g^2 \to S_{g+1}$ induce injections $\mathrm{SMod}(S_g^1) \to \mathrm{Mod}(S_{g+1})$ and $\mathrm{SMod}(S_g^2) \to \mathrm{Mod}(S_{g+1})$, we obtain embeddings of braid groups into mapping class groups of closed surfaces.

In the 1971 paper, Birman and Hilden discuss the connection with representations of the braid group. They point out the related fact that B_{2g+2} surjects onto $\mathrm{SMod}(S_g)$ (this follows

immediately from their presentation for the latter). In the special case $g = 1$, this becomes the classical fact that B_4 surjects onto $\mathrm{SMod}(S_1) = \mathrm{Mod}(S_1) \cong \mathrm{SL}_2(\mathbb{Z})$. We can also derive this fact from our isomorphism $\mathrm{Mod}(S_1^1) \cong B_3$, the famous surjection $B_4 \to B_3$, and the surjection $\mathrm{Mod}(S_1^1) \to \mathrm{Mod}(S_1)$ obtained by capping the boundary.

One useful application of the embeddings of braid groups in mapping class groups is that we can often transport relations from the former to the latter. In fact, almost all of the widely used relations in the mapping class group have interpretations in terms of braids. This is especially true in the theory of Lefschetz fibrations; see, for instance, the work of Korkmaz [45] and Hamada [34] and of Baykur and Van Horn-Morris [5].

7. *Application to a question of Magnus*

The last application we will explain is beautiful and unexpected. It is the resolution of a seemingly unrelated question of Magnus about braid groups.

As mentioned in the previous section, the braid group B_n is isomorphic to the mapping class group of a disk D_n with n marked points. Let us write D_n° for the surface obtained by removing from D_n the marked points. There is then a natural action of B_n on $\pi_1(D_n^\circ)$ (with base point on the boundary). The latter is isomorphic to the free group F_n on n letters. Basic algebraic topology tells us that this action is faithful. In other words, we have an injective homomorphism:

$$B_n \to \mathrm{Aut}(F_n).$$

This is a fruitful way to view the braid group; for instance, since the word problem in $\mathrm{Aut}(F_n)$ is easily solvable, this gives a solution to the word problem for B_n.

Let $F_{n,k}$ denote the normal closure in F_n of the elements $x_1^k, \ldots, x_n^k$. The quotient $F_n/F_{n,k}$ is isomorphic to the n-fold free product $\mathbb{Z}/k\mathbb{Z} * \cdots * \mathbb{Z}/k\mathbb{Z}$. Since the elements of B_n preserve the set of conjugacy classes $\{[x_1], \ldots, [x_n]\}$, there is an induced homomorphism

$$B_n \to \mathrm{Aut}(F_n/F_{n,k}).$$

Let $B_{n,k}$ denote the image of B_n under this map. Magnus asked:

Is B_n isomorphic to $B_{n,k}$?

In other words, is the map $B_n \to \mathrm{Aut}(F_n/F_{n,k})$ injective?

In their *Annals of Mathematics* paper [17], Birman and Hilden answer Magnus' question in the affirmative. Here is the idea. Let $H_{n,k}$ denote the kernel of the map

$$F_n \to \mathbb{Z}/k\mathbb{Z},$$

where each generator of F_n maps to 1. The covering space of D_n° corresponding to $H_{n,k}$ is a k-fold cyclic cover S°. If we consider a small neighborhood of one of the punctures in D_n°, the induced covering map is equivalent to the connected k-fold covering space of $\mathbb{C} \setminus \{0\}$ over itself (that is, the one induced by $z \mapsto z^k$). As such, we can 'plug in' to S° a total of n points in order to obtain a surface S and a cyclic branched covering map $S \to D_n$. The fundamental group of S° is $H_{n,k}$ by definition. It follows from Van Kampen's theorem that $\pi_1(S) \cong H_{n,k}/F_{n,k}$. Indeed, a simple loop around a puncture in S° projects to a loop in D_n° that circles the corresponding puncture k times.

As in the case of the hyperelliptic involution, we can check directly that each element of B_n lifts to a fiber-preserving homeomorphism of S. Therefore, to answer Magnus' question in the affirmative it is enough to check that the map $B_n \to \mathrm{Aut}\,\pi_1(S)$ is injective. Suppose $b \in B_n$ lies in the kernel. Then the corresponding fiber-preserving homeomorphism of S is homotopic, hence isotopic, to the identity. By the Birman–Hilden theorem (the version for surfaces with boundary), b is trivial, and we are done.

Bacardit and Dicks [4] give a purely algebraic treatment of Magnus' question; they credit the argument to Crisp and Paris [22]. Another algebraic argument for the case of even k was given by Johnson [38]. Yet another combinatorial proof was given by Krüger [46].

8. *A famous (but false) proof of the Birman–Hilden theorem*

When confronted with the Birman–Hilden theorem, one might be tempted to quickly offer the following easy proof: given the branched covering map $p : S \to X$, the fiber-preserving homeomorphism $f : S \to S$, the corresponding homeomorphism $\bar{f} : X \to X$, and an isotopy $H : S \times I \to S$ from f to the identity map, we can consider the composition $p \circ H$, which gives a homotopy from $\bar{f}$ to the identity. Then, since homotopic homeomorphisms of a surface are isotopic, there is an isotopy from $\bar{f}$ to the identity, and this isotopy lifts to a fiber-preserving isotopy from f to the identity. Quod erat demonstrandum.

This probably sounds convincing, but there are two problems. First of all, the composition $p \circ H$ is really a homotopy between $p \circ f$ and p which are maps from S to X; since H is not fiber preserving, there is no way to convert this to a well-defined homotopy between maps $X \to X$. The second problem is that H might send points that are not preimages of branch points to preimages of branch points; so even if we could project the isotopy, we would not obtain a homotopy of X that respects the marked points.

In the next two sections, we will outline proofs of the Birman–Hilden theorem in various cases. The reader should keep in mind the subtleties uncovered by this false proof.

9. *The unbranched (= easy) case*

Before getting to the proof of the Birman–Hilden theorem, we will warm up with the case of unbranched covers. This case is much simpler, as all of the subtlety of the Birman–Hilden theorem lies in the branch points. Still the proof is nontrivial, and later we will prove the more general case by reducing to the unbranched case.

In 1972, Birman and Hilden published the paper *Lifting and projecting homeomorphisms* [16], which gives a quick proof of Theorem 2.1 in the case of regular unbranched covers. Following along the same lines, Aramayona, Leininger, and Souto generalized their proof to the case of arbitrary (possibly irregular) unbranched covers [2]. We will now explain their proof.

Let $p : S \to X$ be an unbranched covering space of surfaces, and let $f : S \to S$ be a fiber-preserving homeomorphism that is isotopic to the identity. Without loss of generality, we may assume that f has a fixed point. Indeed, if f does not fix some point x, then we can push $p(x)$ in X by an ambient isotopy, and lift this isotopy to S until x is fixed. As a consequence, f induces a well-defined action $f_\star$ on $\pi_1(S)$. Since f is isotopic to the identity, $f_\star$ is the identity. If $\bar{f}$ is the corresponding homeomorphism of X, then it follows that $\bar{f}_\star$ is the identity on the finite-index subgroup $p_\star(\pi_1(S))$ of $\pi_1(X)$. From this, plus the fact that roots are unique in $\pi_1(X)$, we conclude that $\bar{f}_\star$ is the identity. By basic algebraic topology, $\bar{f}$ is homotopic to the identity, and hence it is isotopic to the identity, which implies that f is isotopic to the identity through fiber-preserving homeomorphisms, as desired.

10. *Three (correct) proofs of the Birman–Hilden theorem*

In this section, we present sketches of the proofs of all three versions of the Birman–Hilden theorem given in Section 2. We begin with the original proof by Birman and Hilden, which is a direct attack using algebraic and geometric topology. Then we explain the proof from Maclachlan and Harvey's Teichmüller theoretic approach, and finally the combinatorial topology approach of the second author, which gives a further generalization.

The Birman–Hilden proof: Algebraic and geometric topology. As in the statement of Theorem 2.1, let $p : S \to X$ be a regular branched covering space where S is a hyperbolic surface. As in Theorem 1 of the *Annals of Mathematics* paper by Birman and Hilden [**17**], we make the additional assumption here that each deck transformation for this cover fixes each preimage of each branch point in X. Theorem 2.1 will follow easily from this special case. Let f be a fiber-preserving homeomorphism of S and assume that f is isotopic to the identity.

Let x be the preimage in S of some branch point in X. The first key claim is that $f(x) = x$ [**17**, Lemma 1.3]. Thus if we take the isotopy H from f to the identity and restrict it to x, we obtain an element α of $\pi_1(S, x)$. Birman and Hilden argue that α must be the trivial element. The idea is to argue that α is fixed by each deck transformation (this makes sense since the deck transformations fix x), and then to argue that the only element of $\pi_1(S)$ fixed by a nontrivial deck transformation is the trivial one (to see this, regard α as an isometry of the universal cover $\mathbb{H}^2$ and regard a deck transformation as a rotation of $\mathbb{H}^2$).

Since α is trivial, we can deform it to the trivial loop, and by extension we can deform the isotopy H to another isotopy that fixes x throughout. Proceeding inductively, Birman and Hilden argue that H can be deformed so that it fixes all preimages of branch points throughout the isotopy. At this point, by deleting branch points in X and their preimages in S, we reduce to the unbranched case.

Finally, to prove their Theorem 2, which treats the case of solvable covers, Birman and Hilden reduce it to Theorem 1 by factoring any solvable cover into a sequence of cyclic covers of prime order. Such a cover must satisfy the hypotheses of their Theorem 1.

It would be interesting to use the Birman–Hilden approach to prove the more general theorem of Winarski. There is a paper by Zieschang from 1973 that uses similar reasoning to Birman and Hilden and recovers the result of Maclachlan and Harvey [**57**].

Maclachlan and Harvey's proof: Teichmüller theory. Let $p : S \to X$ be a regular branched covering space where S is a hyperbolic surface. We will give Maclachlan and Harvey's argument for Theorem 2.2 and at the same time explain why the argument gives the more general result of Theorem 2.3.

The mapping class group $\mathrm{Mod}(S)$ acts on the Teichmüller space $\mathrm{Teich}(S)$, the space of isotopy classes of complex structures on S (or conformal structures on S, or hyperbolic structures on S, or algebraic structures on S). Let X° denote the complement in X of the set of branch points. There is a map $\Xi : \mathrm{Teich}(X^\circ) \to \mathrm{Teich}(S)$ defined by lifting complex structures through the covering map p (one must apply the removable singularity theorem to extend over the preimages of the branch points).

The key point in the proof is that Ξ is injective. One way to see this is to observe that Teichmüller geodesics in $\mathrm{Teich}(X^\circ)$ map to Teichmüller geodesics in $\mathrm{Teich}(S)$ of the same length. Indeed, the only way this could fail would be if we had a Teichmüller geodesic in $\mathrm{Teich}(X^\circ)$ where the corresponding quadratic differential had a simple pole (= 1-pronged singularity) at a branch point and some preimage of that branch point was unramified (1-pronged singularities are only allowed at marked points, and preimages of branch points are not marked). This is why the most natural setting for this argument is that of Theorem 2.3, namely, where p is fully ramified.

Let Y denote the image of Ξ. The symmetric mapping class group $\mathrm{SMod}(S)$ acts on Y and the kernel of this action is nothing other than D. The liftable mapping class group $\mathrm{LMod}(X)$ acts faithfully on $\mathrm{Teich}(X^\circ)$ and hence — as Ξ is injective — it also acts faithfully on Y. It follows immediately from the definitions that the images of $\mathrm{SMod}(S)$ and $\mathrm{LMod}(X)$ in the group of automorphisms of Y are equal. It follows that $\mathrm{SMod}(S)/D$ is isomorphic to $\mathrm{LMod}(X)$, as desired.

Winarski's proof: Combinatorial topology. Let $p : S \to X$ be a fully ramified branched covering space where S is a hyperbolic surface. To prove Theorem 2.3, we will show that $\Phi : \mathrm{LMod}(X) \to \mathrm{SMod}(S)/D$ is injective.

Suppose $f \in \mathrm{LMod}(X)$ lies in the kernel of Φ. Let φ be a representative of f. Since $\Phi(f)$ is trivial, we can choose a lift $\tilde{\varphi} : S \to S$ that is isotopic to the identity; thus $\tilde{\varphi}$ fixes the isotopy class of every simple closed curve in S. The main claim is that φ fixes the isotopy class of every simple closed curve in X. From this, it follows that f has finite order in $\mathrm{LMod}(X)$. Since $\ker(\Phi)$ is torsion free [**56**, Proposition 4.2], the theorem will follow.

So let us set about the claim. Let c be a simple closed curve in X, and let $\tilde{c}$ be its preimage in S. By assumption, $\tilde{\varphi}(\tilde{c})$ is isotopic to $\tilde{c}$ and we would like to leverage this to show $\varphi(c)$ is isotopic to c. There are two stages to the argument: first dealing with the case where $\varphi(c)$ and c are disjoint, and then in the case where they are not disjoint, we reduce to the disjoint case.

If $\varphi(c)$ and c are disjoint, then $\tilde{\varphi}(\tilde{c})$ and $\tilde{c}$ are disjoint. Since the latter are isotopic, they co-bound a collection of annuli $A_1, \ldots, A_n$. Then, since orbifold Euler characteristic is multiplicative under covers, we can conclude that $p(\cup A_i)$ is an annulus with no branch points (branch points decrease the orbifold Euler characteristic), and so c and $\varphi(c)$ are isotopic.

We now deal with the second stage, where $\varphi(c)$ and c are not disjoint. In this case, $\tilde{\varphi}(\tilde{c})$ and $\tilde{c}$ are not disjoint either, but by our assumptions they are isotopic in S. Therefore, $\tilde{\varphi}(\tilde{c})$ and $\tilde{c}$ bound at least one bigon.

Consider an innermost such bigon B. Since B is innermost, $p(B)$ is an innermost bigon bounded by $\varphi(c)$ and c in X (the fact that B is innermost implies that $p|B$ is injective). If there were a branch point in $p(B)$, then since p is fully ramified, this would imply that B was a $2k$-gon with $k > 1$, a contradiction. Thus, we can apply an isotopy to remove the bigon $p(B)$ and by induction we reduce to the case where $\varphi(c)$ and c are disjoint.

For an exposition of Winarski's proof in the case of the hyperelliptic involution, see the book by Farb and the first author of this article [**29**].

11. *Open questions and new directions*

One of the most striking aspects of the Birman–Hilden story is the breadth of open problems related to the theory and the constant discovery of related directions. We mentioned a number of questions already. Perhaps the most obvious open problem is the following.

QUESTION 11.1. Which branched covers of surfaces have the Birman–Hilden property?

Based on the discussion in Section 2 above, one might hope that all branched coverings — at least where the cover is a hyperbolic surface — have the Birman–Hilden property. However, this is not true. Consider, for instance, the unique simple degree three covering map $p : S_g \to X$, where X is the sphere with $2g + 4$ branch points (a branched covering map of degree d is *simple* if each branch point has $d - 1$ preimages). As shown in Figure 1 we can find an essential curve a in X whose preimage in S_g is a union of three homotopically trivial simple closed curves. It follows that the Dehn twist T_a lies in the kernel of the map $\mathrm{LMod}(X) \to \mathrm{SMod}(S_g)$ and so p does not have the Birman–Hilden property. See Fuller's paper for further discussion of this example and the relationship to Lefschetz fibrations [**30**].

Berstein and Edmonds generalized this example by showing that no simple branched covering map of degree at least three over the sphere has the Birman–Hilden property [**8**], and Winarski further generalized this by proving that no simple cover of degree at least three over any surface has the Birman–Hilden property [**56**].

Having accepted the fact that not all covers have the Birman–Hilden property, one's second hope might be that a cover has the Birman–Hilden property if and only if it is fully ramified.

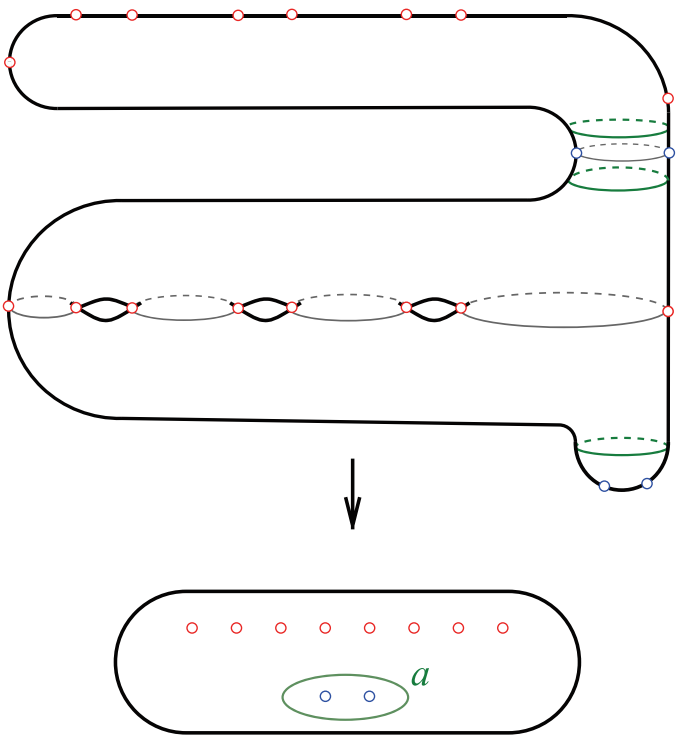


FIGURE 1 (colour online). *The simple threefold cover of S_g over the sphere. The gray curves divide S_g into three regions, each serving as a 'fundamental domain' for the cover. The preimages in S_g of the branch points in S^2 are shown as dots but are treated as unmarked points in S_g.*

However, this is also false. In his personal communication, Chris Leininger has explained to us how to construct a counterexample using the following steps. First, let S be a surface and let z be a marked point in S. Let $p : \tilde{S} \to S$ be a characteristic cover of S and let $\tilde{z}$ be one point of the full preimage $p^{-1}(z)$. Ivanov and McCarthy [**37**] observed that there is an injective homomorphism $\mathrm{Mod}(S, z) \to \mathrm{Mod}(\tilde{S}, p^{-1}(z))$ where for each element of $\mathrm{Mod}(S, z)$, we choose the lift to $\tilde{S}$ that fixes $\tilde{z}$. Aramayona–Leininger–Souto [**2**] proved that the composition of the Ivanov–McCarthy homomorphism with the forgetful map $\mathrm{Mod}(\tilde{S}, p^{-1}(z)) \to \mathrm{Mod}(\tilde{S}, \tilde{z})$ is injective. If we then take a regular branched covering map $S' \to \tilde{S}$ with branch locus $\tilde{z}$, the resulting cover $S' \to S$ is not fully ramified but it has the Birman–Hilden property.

A dynamical variant of the Birman–Hilden property was investigated by Koch–Pilgrim–Selinger. They consider a similar situation to ours, except that they allow some of the preimages of branch points to be marked. They find in this setting a class of examples that enjoy the analogue of the Birman–Hilden property [**42**, Proposition 6.2(2)]. The corresponding version of Question 11.1 is open.

Here is another basic question.

QUESTION 11.2. For which cyclic branched covering maps of S_g over the sphere is $\mathrm{SMod}(S_g)$ equal to a proper subgroup of $\mathrm{Mod}(S_g)$? When is it finite index?

Theorem 5 in the *Annals of Mathematics* paper by Birman and Hilden states for a cyclic branched covering map $S \to X$ over the sphere have $\mathrm{LMod}(X) = \mathrm{Mod}(X)$. Counterexamples to this theorem were recently discovered by Ghaswala and the second author (see the erratum [**18**]), who wrote a paper [**33**] classifying exactly which branched covers over the sphere have

$\mathrm{LMod}(X) = \mathrm{Mod}(X)$. Theorem 6 in the paper by Birman and Hilden states that for $g \geqslant 3$ and for any cyclic branched covering map from S_g to the sphere the group $\mathrm{SMod}(S_g)$ is a proper subgroup of $\mathrm{Mod}(S_g)$. The proof uses their Theorem 5, so Question 11.2 should be considered an open question. Of course this question can be generalized to other base surfaces besides the sphere and other types of covers. For simple branched covers over the sphere (which, as above, do not have the Birman–Hilden property), Berstein and Edmonds [**8**] proved that $\mathrm{SMod}(S_g)$ is equal to $\mathrm{Mod}(S_g)$.

For surfaces with boundary, Question 11.2 has been answered by Ghaswala–McLeay [**31**].

We can also ask about the Birman–Hilden theory for orbifolds and 3-manifolds.

QUESTION 11.3. Which covering spaces of two-dimensional orbifolds have the Birman–Hilden property?

Earle proved some Birman–Hilden-type results for orbifolds in his recent paper [**25**], which he describes as a sequel to his 1971 paper *On the moduli of closed Riemann surfaces with symmetries* [**24**].

QUESTION 11.4. Which covering spaces of 3-manifolds enjoy the Birman–Hilden property?

Vogt proved that certain regular unbranched covers of certain Seifert-fibered 3-manifolds have the Birman–Hilden property [**54**]. He also explains the connection to understanding foliations in codimension two, specifically for foliations of closed 5-manifolds by Seifert 3-manifolds.

A specific 3-manifold worth investigating is the connect sum of n copies of $S^2 \times S^1$; call it M_n. The outer automorphism group of the free group F_n is a quotient of the mapping class group of M_n by a finite group. Therefore, one might obtain a version of the Birman–Hilden theory for the outer automorphism group of F_n by developing a Birman–Hilden theory for M_n.

QUESTION 11.5. Does M_n enjoy the Birman–Hilden property? If so, does this give a Birman–Hilden theory for free groups?

For example, consider the hyperelliptic involution σ of F_n, the outer automorphism that (has a representative that) inverts each generator of F_n. This automorphism is realized by the homeomorphism of M_n that reverses each S^1-factor. The resulting quotient of M_n is the 3-sphere with branch locus the $(n+1)$-component unlink. This is in consonance with the fact that the centralizer of σ in the outer automorphism group of F_n is the palindromic subgroup and that the latter is closely related to the configuration space of unlinks in S^3; see the paper by Collins [**21**].

Next, there are many questions about the hyperelliptic Torelli group and its generalizations. As discussed in Section 6, the hyperelliptic Torelli group is the kernel of the integral Burau representation of the braid group. With Brendle and Putman, the first author of this article proved [**19**] that this group is generated by the squares of Dehn twists about curves that surround an odd number of marked points in the disk D_n.

QUESTION 11.6. Is the hyperelliptic Torelli group finitely generated? Is it finitely presented? Does it have finitely generated abelianization?

There are many variants of this question. By changing the branched covering map $S \to D_n$, we obtain many other representations of (the liftable subgroups of) the braid group. Each representation gives rise to its own Torelli group. Except for the hyperelliptic involution case, very little is known. One set of covers to consider is the set of superelliptic covers studied by Ghaswala and the second author of this article [**32**].

Another aspect of this question is to determine the images of the braid groups in $\mathrm{Sp}_{2N}(\mathbb{Z})$ under the various representations of (finite index subgroups of) the braid group arising from various covers $S \to D_n$. By work of McMullen [48] and Venkataramana [53], it is known that when the degree of the cover is at least three and n is more than twice the degree, the image has finite index in the centralizer of the image of the deck group.

QUESTION 11.7. For which covers $S \to D_n$ does the associated representation of the braid group have finite index in the centralizer of the image of the deck group?

There are still many aspects to the Birman–Hilden theory that we have not touched upon. Ellenberg and McReynolds [26] used the theory to prove that every algebraic curve over $\bar{\mathbb{Q}}$ is birationally equivalent over $\mathbb{C}$ to a Teichmüller curve. Nikolaev [51] uses the embedding of the braid group into the mapping class group to give cluster algebraic representations of braid groups. Kordek applies the aforementioned result of Ghaswala and the second author of this article to deduce information about the Picard groups of various moduli spaces of Riemann surfaces [43]. A Google search for 'Birman–Hilden' yields a seemingly endless supply of applications and connections (the *Annals of Mathematics* paper has 164 citations on Google Scholar at the time of this writing). We hope that the reader is inspired to learn more about these connections and pursue their own developments of the theory.

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