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THERE IS NO FINITELY ISOMETRIC KRIVINE'S THEOREM

JAMES KILBANE AND MIKHAIL I. OSTROVSKII

Abstract. We prove that for every $p \in (1, \infty)$, $p \neq 2$, there exist a Banach space X isomorphic to ℓ_p and a finite subset U in ℓ_p, such that U is not isometric to a subset of X. This result shows that the finite isometric version of the Krivine theorem (which would be a strengthening of the Krivine theorem (1976)) does not hold.

Keywords: isometric embedding, isomorphism of Banach spaces, Krivine theorem, Orlicz space, modular space

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1. Introduction

One of the most fundamental results on the structure of the general infinite-dimensional Banach spaces is the following theorem of Dvoretzky.

Theorem 1.1 (Dvoretzky [5]). *For each infinite-dimensional Banach space X, each $n \in \mathbb{N}$, and each $\varepsilon > 0$, there is an n-dimensional subspace $X_n \subset X$ and an isomorphism $T : X_n \to \ell_2^n$ such that $||T|| \cdot ||T^{-1}|| \leq 1 + \varepsilon$.*

It is well known that $(1 + \varepsilon)$ cannot be replaced by 1 in this theorem. This follows, for example, from the fact that the unit ball of any finite-dimensional subspace in c_0 is a polytope. The fact that ℓ_p does not contain all of ℓ_2^n isometrically, unless p is an even integer, was proved in [4].

Recently embeddability of finite metric spaces into Banach spaces became a very important direction in the Banach space theory. One of the main reasons for this is the discovery that such embeddings have important algorithmic applications, see [16, 1]. Low-distortion embedding of finite metric spaces into Banach spaces became a very powerful toolkit for designing efficient algorithms, the interested reader can find more information in surveys such as [8, 9, 15, 17, 18, 19, 24].

In this connection it is worthwhile to observe that Theorem 1.1 can be derived from the following seemingly weaker theorem. Our terminology follows [19].

Theorem 1.2 (Finite Dvoretzky Theorem). *For each infinite-dimensional Banach space X, each finite subset $F \subset \ell_2$, and each $\varepsilon > 0$, there is a bilipschitz embedding of F into X with distortion at most $(1 + \varepsilon)$.*

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Proof of (Theorem 1.2)⇒(Theorem 1.1). We are going to use ultrapowers of Banach spaces (see [19, Section 2.2]). We need to show that if the conclusion of Theorem 1.2 holds for a Banach space X, then there exists an ultrapower of X containing an isometric copy of ℓ_2. This can be done as follows (this is a slightly modified version of [19, Proposition 2.33]).

Denote by J the set of all finite subsets of ℓ_2 containing 0. Consider the set $I = J\times(0,1)$ as an ordered set: $(j_1,\varepsilon_1) \succeq (j_2,\varepsilon_2)$ if and only if $j_1 \supseteq j_2$ and $\varepsilon_1 \le \varepsilon_2$. Consider an ultrafilter $\mathcal{U}$ on I containing the filter generated by sets of the form $\{(j,\varepsilon) : \ (j,\varepsilon) \succeq (j_0,\varepsilon_0)\}$, where $j_0 \in J$, $\varepsilon_0 \in (0,1)$.

The conclusion of Theorem 1.2 implies that for each pair $(j,\varepsilon) \in J \times (0,1)$ there is a map $T_{(j,\varepsilon)} : j \to X$ with distortion $\le 1+\varepsilon$ satisfying $T(0) = 0$.

It remains to observe that the maps

$$z \mapsto \begin{cases} T_{(j,\varepsilon)}(z) & \text{if } z \in j \\ 0 & \text{if } z \notin j \end{cases}$$

(parameterized by pairs $(j,\varepsilon) \in I$) induce an isometric embedding of ℓ_2 into $X^{\mathcal{U}}$. □

An important difference between Theorem 1.1 and Theorem 1.2 is that there are no known examples showing the necessity of the $+\varepsilon$ in Theorem 1.2. The examples above showing the necessity of the $+\varepsilon$ in Theorem 1.1 do not serve as examples in the finite case. In fact, Fréchet [6] (see also [19, Proposition 1.17]) proved that each n-element set embeds isometrically into ℓ_∞^n, and thus, into c_0. Ball [2] proved that each n-element subset of L_p embeds isometrically into $\ell_p^{\binom{n}{2}}$. Since, as is well known [10, p. 16] ℓ_2 embeds isometrically into $L_p[0,1]$ for every p, it follows that for $X = \ell_p$ the statement of finite Dvoretzky theorem remains true if we replace $(1+\varepsilon)$ by 1.

In this connection, the second-named author asked whether the result which can be called "finite isometric Dvoretzky theorem" is true for all infinite-dimensional Banach spaces X, that is,

Problem 1.3 ([20]). *Does there exist a finite subset F of ℓ_2 and an infinite-dimensional Banach space X such that F does not admit an isometric embedding into X?*

This problem remains open. In this paper we show that the result which could be called "finite isometric Krivine theorem" does not hold for any $p \in [1,\infty]$, $p \neq 2$. More precisely, we answer in the negative, for every $p \in (1,\infty)$, $p \neq 2$, the following problem suggested in [21]:

Problem 1.4 ([21]). *Let Y be a Banach space isomorphic to ℓ_p, $1 < p < \infty$. Is it true that any finite subset of ℓ_p is isometric to some finite subset of Y?*

To justify the term "finite isometric Krivine theorem" let us recall the following landmark result of Krivine.

Theorem 1.5 (Krivine [13]). *For each $p \in [1,\infty]$, each Banach space X isomorphic to ℓ_p, each $n \in \mathbb{N}$, and each $\varepsilon > 0$, there is an n-dimensional subspace $X_n \subset X$ and an isomorphism $T : X_n \to \ell_p^n$ such that $||T|| \cdot ||T^{-1}|| \le 1+\varepsilon$.*

The cases $p = 1$ and $p = \infty$ were not included in Problem 1.4 because Bill Johnson had already described examples in these cases in his answer to [11]. The examples are the following: both ℓ_1 and ℓ_∞ are isomorphic to strictly convex spaces. On the other hand, both ℓ_1 and ℓ_∞ contain quadruples of points a, b, c, d such that $b \neq c$ and both b and c are metric midpoints between a and d. It is easy to see that such quadruples do not exist in strictly convex Banach spaces.

It is worth mentioning that although we prove that the answer to Problem 1.4 is negative, there exist "many" subsets of the unit sphere of ℓ_p for which the result is positive, see the paper [12] of the first-named author for precise statement.

Our main result is the following (we denote by $\{e_i\}$ the unit vector basis of ℓ_p):

Theorem 1.6. (a) *For each* $1 < p < 2$ *there exist a Banach space* X *isomorphic to* ℓ_p *such that the set* $U := \{e_1, e_2, -e_1, -e_2, 0\}$*, considered as a subset of* ℓ_p*, does not embed isometrically into* X*.*

(b) *For each* $2 < p < \infty$ *there exist a Banach space* X *isomorphic to* ℓ_p *such that the set* $V := \{\pm 2^{-1/p}(e_1 + e_2), \pm 2^{-1/p}(e_1 - e_2), 0\}$*, considered as a subset of* ℓ_p*, does not embed isometrically into* X*.*

Remark. The coefficient $2^{-1/p}$ in the statement of (b) is needed to make the vectors normalized (this will be convenient), of course the result holds without this coefficient.

The main technical tools we will use in the proof of Theorem 1.6 are the Clarkson inequalities. In the following theorem, if $q \in (1, \infty)$ we set q' to be the so-called conjugate index of q, defined by $\frac{1}{q} + \frac{1}{q'} = 1$. We recall the following (see [7, Theorem 9.7.2] for generalized Clarkson inequalities):

Theorem 1.7 ([3, Theorem 2])**.** *Suppose that* $x, y \in \ell_p$*, where* $1 < p < \infty$*, and* $r = \min(p, p')$*. Then,*

(1) $2(\|x\|_p^{r'} + \|y\|_p^{r'}) \leq \|x + y\|_p^{r'} + \|x - y\|_p^{r'} \leq 2^{r'-1}(\|x\|_p^{r'} + \|y\|_p^{r'})$
(2) $2^{r-1}(\|x\|_p^r + \|y\|_p^r) \leq \|x + y\|_p^r + \|x - y\|_p^r \leq 2(\|x\|_p^r + \|y\|_p^r)$.

Remark. The following remark is for the reader who knows the definition of the *James constant* of a Banach space, which is defined as $J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$. One can unify parts (a) and (b) of Theorem 1.6 in terms of the following:

Theorem 1.8. *For each* $p \neq 2$ *there is a Banach space* X *such that* X *is isomorphic to* ℓ_p*,* $J(X) = J(\ell_p)$*, but the supremum in the definition of the James constant is not attained.*

Proving Theorem 1.8 one can use some of the results of [23] and [25]. To make our argument as elementary and self-contained we prefer to present a direct argument in terms of the metric spaces U and V.

2. The case $p \in (1, 2)$

We show that in this case we can choose X to be an Orlicz sequence space ℓ_M for a suitably chosen function $M(t)$. Let us recall the definition of an Orlicz sequence space.

Definition 1. Let $M : [0,\infty) \to [0,\infty)$ be a continuous, non-decreasing and convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We define the sequence space ℓ_M to be the collection of sequences $x = (x_1, x_2, \dots)$ such that $\sum M(|x_n|/\rho) < \infty$ for some ρ and define the norm $\|x\|_M$ to be

$$\|x\|_M = \inf\left\{\rho > 0 : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) \leq 1\right\}$$

We refer to [14, Section 4.a] for basic properties of Orlicz sequence spaces, however, our proof will require very little of this theory to understand.

Let $p \in (1,2)$, pick any $r \in (p,2)$ and let $M(t) = t^p + t^r$. We show that the corresponding Orlicz space ℓ_M has all of the desired properties. The fact that ℓ_M is isomorphic to ℓ_p follows immediately from [14, Proposition 4.a.5].

Assume that ℓ_M does not have the second property, that is, assume that U admits an isometric embedding f into ℓ_M. Without loss of generality we may assume that $f(0) = 0$. Denote $f(e_1)$ by x and $f(e_2)$ by y. It is easy to see that, since ℓ_M is a strictly convex space [22, Chapter VII], we have $f(-e_1) = -x$ and $f(-e_2) = -y$. We have $||x|| = ||y|| = 1$. So we need to get a contradiction by showing that it is not possible that both of the vectors: $x + y$ and $x - y$ have norm $2^{\frac{1}{p}}$ in ℓ_M.

Since $x, y \in S_{\ell_M}$, we have $\sum_{i=1}^{\infty} |x_i|^p + \sum_{i=1}^{\infty} |x_i|^r = 1$ and $\sum_{i=1}^{\infty} |y_i|^p + \sum_{i=1}^{\infty} |y_i|^r = 1$. We can write this as $\|x\|_p^p + \|x\|_r^r = 1$ and $\|y\|_p^p + \|y\|_r^r = 1$, where by $\|\cdot\|_p$ we denote the norm of a sequence in ℓ_p. Adding these equalities we get

$$2 = \|x\|_p^p + \|y\|_p^p + \|x\|_r^r + \|y\|_r^r.$$

By the Clarkson inequality ((2) of Theorem 1.7), we get

$$4 \geq \|x+y\|_p^p + \|x-y\|_p^p + \|x+y\|_r^r + \|x-y\|_r^r \tag{1}$$

Denote $u = \|x+y\|_M$ and $v = \|x-y\|_M$, and note that

$$\frac{\|x+y\|_p^p}{u^p} + \frac{\|x+y\|_r^r}{u^r} = 1$$

and

$$\frac{\|x-y\|_p^p}{v^p} + \frac{\|x-y\|_r^r}{v^r} = 1.$$

Suppose that $u = v = 2^{1/p}$, i.e., the embedding described above is isometric. We get that

$$\frac{1}{2}\,\|x+y\|_p^p + \frac{1}{2^{r/p}}\|x+y\|_r^r = 1$$

and

$$\frac{1}{2}\,\|x-y\|_p^p + \frac{1}{2^{r/p}}\|x-y\|_r^r = 1.$$

Doubling and adding gives

$$\|x+y\|_p^p + \|x-y\|_p^p + 2^{1-\frac{r}{p}}\|x+y\|_r^r + 2^{1-\frac{r}{p}}\|x-y\|_r^r = 4$$

Since $2^{1-r/p} < 1$, we get a contradiction with (1).

3. Case $p \in (2, \infty)$

This case is more difficult. The reason is the following: to get a counterexample in the above we "perturbed" ℓ_p slightly "in the direction of ℓ_2". While for $p < 2$ this is achievable by addition of t^r to the Orlicz function corresponding to ℓ_p, this is no longer possible for $p > 2$ (the space corresponding to $t^p + t^r$ with $r < p$ is isomorphic to ℓ_r, and not to ℓ_p), for this reason we have to consider more complicated, so-called *modular spaces.*

Let us recall the definition of modular spaces:

Definition 2. For each $i \in \mathbf{N}$, let $M_i : [0, \infty) \to [0, \infty)$ be a continuous, convex, non-decreasing and convex function such that $M(0) = 0$ and $\lim_{t \to \infty} M(t) = \infty$. Then the *modular sequence space* $\ell_{\{M_i\}}$ is the Banach space of all sequences $x = \{x_i\}_{i=1}^\infty$ with $\sum_{i=1}^\infty M_i(|x_i|/\rho) < \infty$ for some $\rho > 0$, equipped with the norm

$$||x|| = \inf \left\{ \rho > 0 : \quad \sum_{i=1}^\infty M_i \left(\frac{|x_i|}{\rho} \right) \leq 1 \right\}.$$

We refer to [14, Section 4.d] for basic information on modular sequence spaces.

Let $p > 2$. We introduce a sequence $\{M_i\}_{i=1}^\infty$ of functions given by $M_i(t) = t^p + t^{p_i}$ where $2 < p_i < p$ for any i, and the sequence p_i converges to p rapidly enough, so that the obtained modular space $\ell_{\{M_i\}}$ is isomorphic to ℓ_p. To see that this is achievable, we recall the following criterion from [14, p. 167]: if M_i and N_i are two collections of functions having the properties in Definition 2, then $\ell_{\{M_i\}}$ and $\ell_{\{N_i\}}$ are isomorphic with the identity map being an isomorphism if there exist numbers $K > 0$, $t_i \geq 0$, $i = 1, 2, \ldots$, and an integer i_0 so that

(a) $K^{-1} N_i(t) \leq M_i(t) \leq K N_i(t)$ for all $i \geq i_0$ and $t \geq t_i$.
(b) $\sum_{i=1}^\infty N_i(t_i) < \infty$.

We are going to apply this criterion with $N_i(t) = t^p$ for every $i \in \mathbb{N}$. We choose $t_i > 0$ to be a convergent to 0 sequence for which $\sum_{i=1}^\infty t_i^p < \infty$ (so (b) is satisfied). Finally, we let $i_0 = 1$ and choose the sequence $\{p_i\} \in (2, p)$ so rapidly approaching p, that the condition (a) is satisfied with $K = 3$. It is easy to see that this is possible.

We wish to use an argument similar to the argument in Section 2. To do this we will need to first prove the strict convexity of $\ell_{\{M_i\}}$. To show this, suppose that we pick two distinct elements $u, v \in S_{\ell_{\{M_i\}}}$. This means that

$$\sum_{i=1}^\infty (|u_i|^p + |u_i|^{p_i}) = ||u||_p^p + \sum_{i=1}^\infty |u_i|^{p_i} = 1 \tag{2}$$

and

$$\sum_{i=1}^\infty (|v_i|^p + |v_i|^{p_i}) = ||v||_p^p + \sum_{i=1}^\infty |v_i|^{p_i} = 1 \tag{3}$$

Adding these together, and using the Clarkson inequality ((1) of Theorem 1.7), we get

$$2^{1-p}(\|u+v\|_p^p + \|u-v\|_p^p) + \sum_{i=1}^\infty 2^{1-p_i}(|u_i + v_i|^{p_i} + |u_i - v_i|^{p_i}) \leq 2 \tag{4}$$

To show that $\|u+v\|<2$, assume that $\|u+v\|=2$, that is,

$$2^{-p}\|u+v\|_p^p+\sum_{i=1}^{\infty}2^{-p_i}|u_i+v_i|^{p_i}=1.$$

Multiplying by 2 and comparing with Equation (4) is a contradiction, therefore $\|u+v\|<2$.

We now continue, and show that V does not isometrically embed into $\ell_{\{M_i\}}$. Observe that the distance in ℓ_p between any of the vector $\pm 2^{-1/p}(e_1+e_2)$ and any of the vectors $\pm 2^{-1/p}(e_1-e_2)$ is equal to $2^{1-\frac{1}{p}}$. Assume that V admits an isometric embedding f into $\ell_{\{M_i\}}$. Without loss of generality we may assume that $f(0)=0$. Set $x=f(2^{-1/p}(e_1+e_2))$ and $y=f(2^{-1/p}(e_1-e_2))$. By the strict convexity we get $-x=f(-2^{-1/p}(e_1+e_2))$ and $-y=f(-2^{-1/p}(e_1-e_2))$. We have $||x||_{\{M_i\}}=||y||_{\{M_i\}}=1$. To complete the proof it suffices to show that

$$||x-y||_{\{M_i\}}=||x+y||_{\{M_i\}}=2^{1-\frac{1}{p}}$$

leads to a contradiction. This gives us that

$$\frac{\|x+y\|_p^p}{2^{p-1}}+\sum_{i=1}^{\infty}\frac{|x+y|^{p_i}}{2^{p_i(1-\frac{1}{p})}}=1$$

and

$$\frac{\|x-y\|_p^p}{2^{p-1}}+\sum_{i=1}^{\infty}\frac{|x-y|^{p_i}}{2^{p_i(1-\frac{1}{p})}}=1$$

Adding and rearranging we get

$$2^{1-p}(\|x+y\|_p^p+\|x-y\|_p^p)+\sum_{i=1}^{\infty}2^{p_i(\frac{1}{p}-1)}(|x_i+y_i|^{p_i}+|x_i-y_i|^{p_i})=2 \tag{5}$$

Since $2<p_i<p$, and therefore $1-p_i>p_i(\frac{1}{p}-1)$, the equations (4) (which was valid for any elements of $S_{\ell_{\{M_i\}}}$, so we are free to set $u=x$ and $v=y$) and (5) contradict each other.

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(J.K.) Department of Pure Maths and Mathematical Statistics, University of Cambridge

E-mail address: `jk511@cam.ac.uk`

(M.O.) Department of Mathematics and Computer Science, St. John's University, 8000 Utopia Parkway, Queens, NY 11439, USA

E-mail address: `ostrovsm@stjohns.edu`