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Skew key polynomials and the key poset

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Abstract. We generalize Young's lattice on integer partitions to a new partial order on weak compositions called the key poset. Saturated chains in this poset correspond to standard key tableaux, the combinatorial objects that generate the key polynomial basis for the polynomial ring, a generalization of the Schur basis for symmetric functions. Generalizing skew Schur functions, we define skew key polynomials in terms of the poset, and, using weak dual equivalence, we give a nonnegative composition Littlewood–Richardson rule for the key expansion of skew key polynomials.

Keywords: Young's lattice, composition poset, key polynomials, skew Schur functions

1 Preliminaries

Schur polynomials are central to the study of the representation theories of the general linear group and of the symmetric group, as well as to the geometry of the Grassmannian. The combinatorics of Young tableaux associated with Schur polynomials often sheds light on important representation theoretic or geometric properties such as tensor products, induction and restriction of modules, and intersection numbers.

The celebrated Littlewood–Richardson rule [15] gives a combinatorial description for the Schur expansion of a product of two Schur polynomials or, equivalently, for the Schur expansion of a *skew Schur polynomial* as

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu, \quad \text{or} \quad s_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda,\mu} s_\mu,$$

where $c^\nu_{\lambda,\mu}$ is the number of saturated chains in *Young's lattice* from λ to ν satisfying certain conditions depending on μ. Here Young's lattice is the partial order on integer partitions given by containment of Young diagrams. These coefficients have deep interpretations in representation theory as the irreducible multiplicities for the tensor product of two irreducible representations for the general linear group and as the irreducible multiplicities for the induced tensor product of two irreducible representations

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for the symmetric group. They also appear geometrically, giving the number of points lying in a suitable intersection of three Grassmannian Schubert varieties.

One of myriad generalizations of the Littlewood–Richardson rule comes in [6] where Bessenrodt, Luoto and van Willigenburg define a partial order on strong compositions that gives rise to a nonnegative Littlewood–Richardson rule for a skew analog of the quasisymmetric Schur functions of Haglund, Luoto, Mason and van Willigenburg [10]. Another example comes in [11], where the latter authors use the quasisymmetric Schur functions to derive a nonnegative Littlewood–Richardson rule for the product of a *key polynomial* and a Schur polynomial with sufficiently many variables.

The key polynomials are polynomial generalizations of Schur polynomials first studied by Demazure [7] in connection with Schubert varieties. Key polynomials are irreducible characters of Demazure modules for the general linear group [8] and represent Schubert classes for vexillary permutations [14]. Their structure constants are not, in general, nonnegative, though Assaf and Quijada [1] have made progress on understanding the signs in the Pieri case. Assaf [4] considered a diagram containment-based skew analog of key polynomials, but obtained nonnegativity results only in very special cases.

In this abstract, we generalize Young's lattice to a partial order on *weak* compositions that we call the *key poset*. In contrast with the generalization to strong compositions in [6], we give explicit cover relations as well as explicit criteria for comparability in the poset, though as with the strong composition poset, the key poset is not a lattice. Using this paradigm, we re-define skew key polynomials with respect to the poset and give a general nonnegative Littlewood–Richardson rule for skew key polynomials, vastly generalizing [4]. Moreover, we show that under the more general containment definition for skew key polynomials considered in [4], all key polynomial coefficients are nonnegative if and only if the shapes are comparable in the key poset.

2 Posets

An integer ***partition*** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. The ***Young diagram*** of a partition λ is the collection of λ_i unit cells left-justified in row i indexed from the bottom. Integer partitions become a ***poset***, or partially ordered set, under containment of diagrams, i.e. $\lambda \subseteq \mu$, if $\lambda_i \leq \mu_i$ for all i or, equivalently, if the diagram for λ is a subset of the diagram for μ. We call this poset ***Young's lattice***.

The ***cover relations*** for Young's lattice may be described by $\lambda \prec\!\!\cdot\, \mu$ if and only if μ is obtained from λ by incrementing a single part λ_i for which $i = 1$ or $\lambda_{i-1} > \lambda_i$ by 1 or, equivalently, by adding a single box to the end of a row for which the higher row is strictly shorter. The cover relations show that Young's lattice is ranked by the number of cells of the diagrams. Fig. 1 depicts Young's lattice up to rank 4.

Young's lattice is a ***lattice*** with least upper bound and greatest lower bound given by

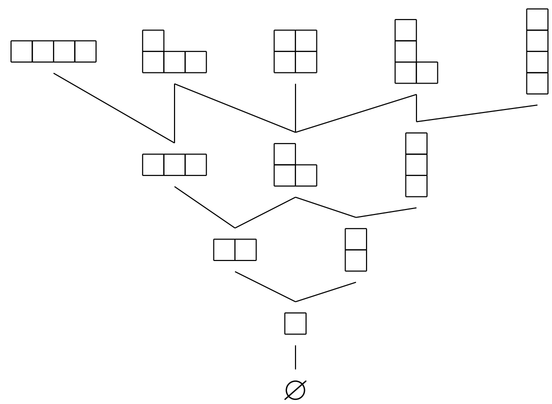

Figure 1: The Hasse diagram of Young's lattice up to rank 4.

the set-theoretic union and the set-theoretic intersection of the diagrams, respectively.

Young's lattice is a prominent tool in algebraic combinatorics, used to study symmetric functions, representations of finite and affine Lie groups, and intersection numbers for finite and affine Grassmannians. We generalize the construction from integer partitions to weak compositions in such a way that maintains these connections.

A ***weak composition*** $a = (a_1, a_2, \ldots, a_n)$ is a sequence of nonnegative integers. The ***key diagram*** of a weak composition a is the collection of a_i unit cells left-justified in row i indexed from the bottom.

Definition 2.1. The ***key poset*** is the partial order $\prec$ on weak compositions of length n defined by the relation $a \preceq b$ if and only if $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ and for any indices $1 \leq i < j \leq n$ for which $b_j > a_j$ and $a_i > a_j$, we have $b_i > b_j$.

In terms of key diagrams, we have $a \prec b$ if and only if $a \subset b$ and whenever a cell of b/a lies above a cell of a, the lower row is strictly longer in b; see Fig. 2.

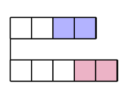

Figure 2: An illustration of the partial order on weak compositions in terms of key diagrams. Here, cells $\square$ lie in $a \subseteq b$, cells $\square$ lie in b/a, and cells $\square$ lie in $b = a \cup b$.

This partial order is not given simply by containment of diagrams. For example, $a = (2,1,1) \subset b = (2,1,2)$, but $a \not\prec b$ since $b_3 > a_3$ and $a_1 > a_3$ but $b_1 = b_3$.

As with Young's lattice, this partial order on weak compositions is ranked by the number of boxes, and we may describe the covering relations in terms of adding cells.

Theorem 2.2. *The key poset is ranked by number of cells with covering relation $a \prec\!\!\cdot\, b$ if and only if b is obtained from a by incrementing a_j by 1 where for any $i < j$ we have $a_i \neq a_j + 1$.*

On key diagrams, $a \prec\!\!\cdot\, b$ if and only if $a \subset b$, there is a single cell of b/a, and this cell does not sit above any cell at the end of its row. Fig. 3 shows the key poset up to rank 3.

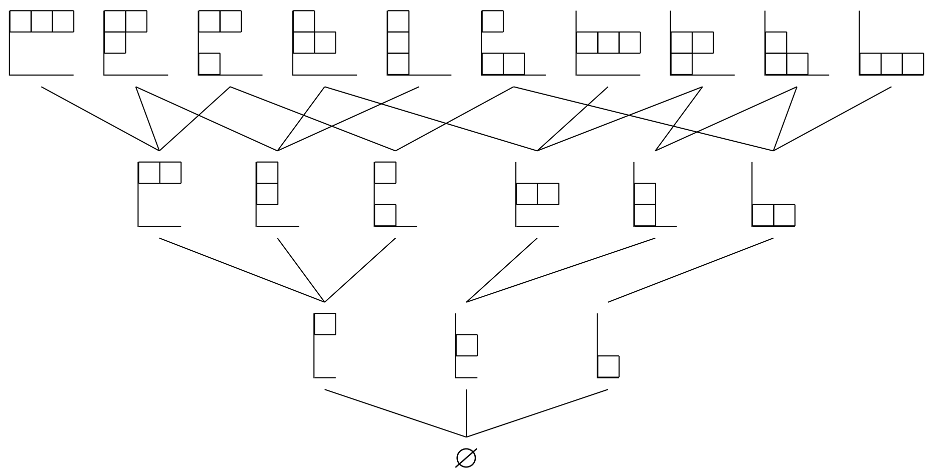

Figure 3: The Hasse diagram of the key poset up to rank 3.

While containment is not sufficient for covering in general, it is for the partition case.

Proposition 2.3. *Given a, b with a weakly increasing, we have $a \subseteq b$ if and only if $a \preceq b$.*

In particular, any finite subposet of Young's lattice is a subposet of the key poset.

If $a, b \preceq a \cup b$, then $a \cup b$ is the unique least upper bound, and if $a \cap b \preceq a, b$, then $a \cap b$ is the unique greatest lower bound. However, in general, this is not the case.

Proposition 2.4. *The key poset is not a lattice.*

This is demonstrated by the example illustrated in Fig. 4, where $a = (2,1,1)$, $b = (2,1,2)$, $c = (1,1,1)$, and $d = (1,0,1)$. Here both c and d are greatest lower bounds for a, b, and, conversely, both a and b are least upper bounds for c, d.

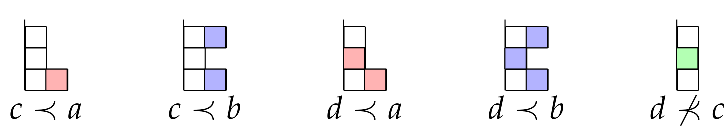

Figure 4: An example showing that the key poset is not a lattice. Here $\square \in c$ or d, $\square \in a/c$ or a/d, $\square \in b/c$ or b/d, and $\square \in c/d$.

While this paper does not consider topological properties of the key poset, questions such as shellability are certainly worth exploring.

3 Polynomials

Given a partition λ of rank n, the set $\mathrm{SYT}(\lambda)$ of ***standard Young tableaux of shape*** λ consists of bijective fillings of the Young diagram of λ with numbers $1, 2, \ldots, n$ such that row entries increase left to right and column entries increase bottom to top.

We may identify standard Young tableaux of shape λ with ***saturated chains*** in Young's lattice from $\varnothing$ to λ, i.e. $\varnothing = \lambda^{(0)} \prec\!\!\cdot\, \lambda^{(1)} \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, \lambda^{(n)} = \lambda$, by the correspondence that places i in the unique cell of $\lambda^{(i)}/\lambda^{(i-1)}$. For example, the two saturated chains from $\varnothing$ to $(2, 1)$ are shown in Fig. 5. The cover relations for Young's lattice are precisely equivalent to the increasing rows and columns conditions.

Figure 5: Two saturated chains in Young's lattice from $\varnothing$ to $(2, 1)$.

Schur functions are ubiquitous throughout mathematics, arising as irreducible characters for polynomial representations of the general linear group, Frobenius characters for irreducible representations of the symmetric group, and polynomial representatives for the cohomology classes of Schubert cycles in Grassmannians. Combinatorially, Schur functions are the ***quasisymmetric generating functions*** for standard Young tableaux.

Gessel introduced the ***fundamental quasisymmetric functions*** [9], indexed by strong compositions, that form an important basis for quasisymmetric functions, given by

$$F_\alpha(X) = \sum_{\mathrm{flat}(b) \text{ refines } \alpha} x_1^{b_1} x_2^{b_2} \cdots, \tag{3.1}$$

where the sum is over weak compositions b whose nonzero parts, $\mathrm{flat}(b)$, refine α.

For a standard Young tableau T, say that i is a ***descent*** of T if $i+1$ lies weakly left of i. The ***descent composition of*** T, denoted by $\mathrm{Des}(T)$, is the strong composition given by maximal length runs between descents.

For example, the left tableau T in Fig. 5 has a descent only at 2, and so $\mathrm{Des}(T) = (2, 1)$, whereas the right tableau U has a descent only at 1, and so $\mathrm{Des}(U) = (1, 2)$.

By [9], we may define a Schur function as follows using the fundamental basis,

$$s_\lambda(X) = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}(X). \tag{3.2}$$

For example, from Fig. 5, we have $s_{(2,1)}(X) = F_{(2,1)}(X) + F_{(1,2)}(X)$.

Using this paradigm, Young's lattice becomes a powerful tool in studying Schur functions as well as the many contexts in which they arise.

Based on the ***quasi-Yamanouchi tableaux*** of Assaf and Searles [3], Assaf defined ***standard key tableaux*** as follows [4, Definition 3.10].

Definition 3.1 ([4])**.** A ***standard key tableau*** is a bijective filling of a key diagram with $1, 2, \ldots, n$ such that rows weakly decrease from left to right, and if some entry i is above and in the same column as an entry k with $i < k$, then there is an entry j immediately right of k and $i < j$. Denote the set of standard key tableaux of shape a by $\mathrm{SKT}(a)$.

Parallel to the case for Young's lattice, saturated chains from $\varnothing$ to a in the key poset precisely correspond to standard key tableaux of shape a.

Theorem 3.2. *Saturated chains from $\varnothing$ to a in the key poset are in bijection with standard key tableaux of shape a by the correspondence placing $n - i + 1$ into the unique cell of $a^{(i)}/a^{(i-1)}$.*

Figure 6: Two saturated chains in the key poset from $\varnothing$ to $(0, 2, 1)$.

For example, Fig. 6 shows the two saturated chains in the key poset from $\varnothing$ to $(0, 2, 1)$. Notice, as well, the two saturated chains in Fig. 5 are also saturated chains in the key poset from $\varnothing$ to $(0, 1, 2)$ under the label reversing map $i \mapsto n - i + 1$.

The ***key polynomials***, indexed by weak compositions, form an important basis for the full polynomial ring. Key polynomials arise as characters of Demazure modules [7] for the general linear group and coincide with Schubert polynomials [13] in the vexillary case [14]. Key polynomials are nonsymmetric generalizations of Schur functions, studied combinatorially by Reiner and Shimozono [17] and later by Mason [16], though our perspective follows that of Assaf and Searles [3] and Assaf [4] who define them as the ***fundamental slide generating polynomial*** for standard key tableaux.

Assaf and Searles introduced the ***fundamental slide polynomials*** [2], indexed by weak compositions, that form a basis for the full polynomial ring.

Definition 3.3 ([2])**.** The ***fundamental slide polynomial*** $\mathfrak{F}_a$ is given by

$$\mathfrak{F}_a = \sum_{\substack{\mathrm{flat}(b) \text{ refines } \mathrm{flat}(a) \\ b_1 + \cdots + b_k \geq a_1 + \cdots + a_k \ \forall k}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}, \tag{3.3}$$

where the sum is over weak compositions b that dominate a in lexicographic order.

For a standard key tableau T, say that i is a ***descent*** of T if $i + 1$ lies weakly *right* of i. We assign a ***weak descent composition*** for T, defined in [4, Definition 3.12], that will index the corresponding fundamental slide polynomial.

Definition 3.4 ([4])**.** For a standard key tableau T, let $(\tau^{(k)}| \ldots |\tau^{(1)})$ be the partitioning of the decreasing word $n \cdots 21$ broken between $i + 1$ and i precisely whenever i is a descent of T. Set t'_i to be the lowest row index in T of a letter in $\tau^{(i)}$. Set $t_k = t'_k$ and, for $i < k$, set $t_i = \min(t'_i, t_{i+1} - 1)$. Define the ***weak descent composition of*** T, denoted by $\mathrm{des}(T)$, by $\mathrm{des}(T)_{t_i} = |\tau^{(i)}|$ and all other parts are zero if all $t_i > 0$; and $\mathrm{des}(T) = \varnothing$ otherwise.

For example, the left tableau T in Fig. 6 has a descent only at 2, so $(\tau^{(2)}|\tau^{(1)}) = (3|21)$. Thus $t_2 = t'_2 = 3$, $t'_1 = 2$, and $t_1 = \min(t'_1, t_2 - 1) = \min(2,2) = 2$, giving $\mathrm{des}(T) = (0, |\tau^{(1)}|, |\tau^{(2)}|) = (0,2,1)$. The right tableau U in Fig. 6 has a descent only at 1, so $(\tau^{(2)}|\tau^{(1)}) = (32|1)$. Thus $t_2 = t'_2 = 2$, $t'_1 = 3$, and $t_1 = \min(t'_1, t_2 - 1) = \min(3,1) = 1$, giving $\mathrm{des}(U) = (|\tau^{(1)}|, |\tau^{(2)}|, 0) = (1,2,0)$.

We take [4, Corollary 3.16] as our definition for key polynomials.

Definition 3.5 ([4])**.** For a weak composition a, the ***key polynomial*** κ_a is given by

$$\kappa_a = \sum_{T \in \mathrm{SKT}(a)} \mathfrak{F}_{\mathrm{des}(T)}, \tag{3.4}$$

where the sum is over $T \in \mathrm{SKT}(a)$ for which $\mathrm{des}(T) \neq \varnothing$.

For example, from Fig. 6, we have $\kappa_{(0,2,1)} = \mathfrak{F}_{(0,2,1)} + \mathfrak{F}_{(1,2,0)}$.

The key polynomials generalize Schur polynomials in that Schur polynomials are key polynomials and Schur functions are the ***stable limits*** of key polynomials,

$$\kappa_{(0,\ldots,0,\lambda_\ell,\ldots,\lambda_1)}(x_1, x_2, \ldots, x_n) = s_\lambda(x_1, x_2, \ldots, x_n) \tag{3.5}$$

$$\lim_{m\to\infty} \kappa_{0^m \times a}(x_1, \ldots, x_m, 0, \ldots, 0) = s_{\mathrm{sort}(a)}(X). \tag{3.6}$$

Thus we hope to use the key poset similar to Young's lattice to study important properties of key polynomials that relate them to representation theory and geometry.

4 Positivity

Standard Young tableaux are saturated chains in Young's lattice beginning at $\varnothing$. We may define a more general class of objects, called standard skew Young tableaux, as saturated chains between any two partition $\lambda \subset \nu$. The direct combinatorial characterization is completely analogous: the ***standard skew Young tableaux*** of shape ν/λ are bijective fillings of the set-theoretic difference ν/λ, called the ***skew diagram***, with $1, 2, \ldots, n$ such that rows increase left to right and columns increase bottom to top.

Directly extending the notion of descents and descent compositions allows us to define ***skew Schur functions*** for partitions $\lambda \subset \nu$ by

$$s_{\nu/\lambda}(X) = \sum_{T \in \mathrm{SYT}(\nu/\lambda)} F_{\mathrm{Des}(T)}(X). \tag{4.1}$$

Skew Schur functions are symmetric functions, and so we may consider the ***Littlewood–Richardson coefficients*** $c^{\nu}_{\lambda,\mu}$ that give their Schur expansion,

$$s_{\nu/\lambda}(X) = \sum_{\mu} c^{\nu}_{\lambda,\mu} s_{\mu}(X). \tag{4.2}$$

A priori, these coefficients are integers. It is a deep result in algebraic combinatorics, with myriad beautiful proofs, that these coefficients are *nonnegative*. This nonnegativity manifests in representation theory multiplicities of irreducible representations in tensor products of polynomial representations for the general linear group and in geometry as intersection numbers for Grassmannian Schubert varieties.

One (of many) proofs of the nonnegativity of $c^{\nu}_{\lambda,\mu}$ from the skew Schur function perspective utilizes ***dual equivalence*** [5, Definition 4.1] to consolidate skew standard Young tableaux into equivalence classes, each corresponding to a single Schur function.

Given a strong composition α of n and integers $1 \leq h \leq i \leq n$, let $\alpha_{(h,i)}$ be the composition obtained by deleting the first $h-1$ and last $n-i$ pieces from α.

For example, for $\alpha = (3,2,3,1)$, a strong composition of 9, we have $\alpha_{(3,7)} = (1,2,2)$ by deleting the first 2 pieces from α_1 and the last 2 pieces from α_4 and α_3.

Definition 4.1 ([5])**.** Let $\mathcal{A}$ be a finite set, and Des a map from $\mathcal{A}$ to strong compositions of n. A ***dual equivalence for*** $(\mathcal{A}, \mathrm{Des})$ is a family of involutions $\{\varphi_i\}_{1<i<n}$ on $\mathcal{A}$ such that

i. For all $0 \leq i-h \leq 3$ and all $T \in \mathcal{A}$, there exists a partition λ such that

$$\sum_{U \in [T]_{(h,i)}} F_{\mathrm{Des}_{(h-1,i+1)}(U)}(X) = s_{\lambda}(X),$$

where $[T]_{(h,i)}$ is the equivalence class generated by $\varphi_h, \ldots, \varphi_i$.

ii. For all $|i-j| \geq 3$ and all $T \in \mathcal{A}$, we have $\varphi_j\varphi_i(T) = \varphi_i\varphi_j(T)$.

Define ***simple involutions*** s_i for $i = 1, 2, \ldots, n-1$ on standard fillings of rank n that interchange i and $i+1$. On standard Young tableaux, we combine these into ***elementary dual equivalence involutions***, denoted by d_i, that act by

$$d_i(T) = \begin{cases} T & \text{if } i \text{ lies between } i-1 \text{ and } i+1 \text{ in reading order,} \\ s_{i-1}\cdot T & \text{if } i+1 \text{ lies between } i \text{ and } i-1 \text{ in reading order,} \\ s_i \cdot T & \text{if } i-1 \text{ lies between } i \text{ and } i+1 \text{ in reading order,} \end{cases} \tag{4.3}$$

where we may take either row or column reading order in any direction; see Fig. 7.

Haiman [12] showed that these are well-defined on standard Young tableaux and that all standard Young tableaux of fixed shape fall into a single equivalence class. Assaf [5, Proposition 3.3] showed that they give an example of a dual equivalence, and, more importantly, the converse holds. That is, by [5, Theorem 3.7], any dual equivalence is essentially this. At the level of generating functions, we have [5, Corollary 4.4].

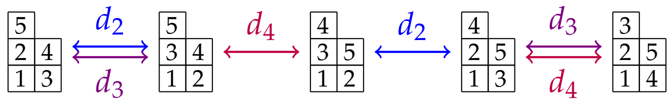

Figure 7: The elementary dual equivalence involutions on SYT$(2, 2, 1)$.

Theorem 4.2 ([5])**.** *If there exists a dual equivalence for* $(\mathcal{A}, \mathrm{Des})$*, then* $\sum_{T \in \mathcal{A}} F_{\mathrm{Des}(T)}(X)$ *is symmetric and Schur positive.*

It follows directly from the local nature of Definition 4.1 that extending the elementary dual equivalences to standard skew Young tableaux using Eq. (4.3) results in a dual equivalence for SYT(ν/λ), thus giving a combinatorial proof of the following.

Corollary 4.3 ([5])**.** *For* $\lambda \subset \nu$*,* $c^{\nu}_{\lambda,\mu}$ *is the number of dual equivalence classes of* SYT(ν/λ) *isomorphic to* SYT(μ)*. In particular, skew Schur functions are Schur positive.*

Assaf considered ***standard skew key tableaux*** in [4, Definition 4.7], defined for any pair of weak compositions $a \subset d$. However, the positivity result for the corresponding ***skew key polynomials*** [4, Theorem 4.10], generalizing Corollary 4.3, is proved only for the limited case when the smaller weak composition is a *partition*. Examples show that these are not the only cases where nonnegativity holds, with many overlooked examples arising naturally from geometric contexts. As we shall see, the key to positivity lies in the key poset. We begin by generalizing Theorem 3.2 to skew diagrams.

Theorem 4.4. *Saturated chains from* a *to* d *in the key poset are in bijection with* SKT(d/a) *by the correspondence placing* $n - i + 1$ *into the unique cell of* $a^{(i)}/a^{(i-1)}$*.*

Following [4], extend Definition 3.4 directly to skew standard key tableaux.

Definition 4.5. For weak compositions $a \prec d$, the ***skew key polynomial*** $\kappa_{d/a}$ is given by

$$\kappa_{d/a} = \sum_{T \in \mathrm{SKT}(d/a)} \mathfrak{F}_{\mathrm{des}(T)}. \tag{4.4}$$

Skew key polynomials generalize skew Schur polynomials in that every skew Schur polynomial is a skew key polynomial and every skew Schur function is the stable limit of a skew key polynomial. However, skew key polynomials are more general, with examples of stable limits of skew key polynomials that are not skew Schur functions.

Note that unlike [4, Definition 4.8], we define skew key polynomials only for comparable elements of the key poset. However, the special case of skewing by an increasing composition, for which the positivity in [4, Theorem 4.10] holds, conforms with this more restrictive definition by Proposition 2.3.

Since key polynomials are a basis for all polynomials, we may define ***weak composition Littlewood–Richardson coefficients*** $c^{d}_{a,b}$ as the key expansion of skew key polynomials,

$$\kappa_{d/a} = \sum_{b} c^{d}_{a,b} \kappa_b. \tag{4.5}$$

A priori, these coefficients are *integers*. To prove nonnegativity of the weak composition Littlewood–Richardson coefficients, we utilize ***weak dual equivalence*** [4], a polynomial generalization of dual equivalence, to consolidate skew standard key tableaux into equivalence classes, each of which corresponds to a single key polynomial.

For a weak composition a of rank n and integers $1 \leq h \leq i \leq n$, let $a_{(h,i)}$ be the weak composition obtained by deleting the first $h-1$ and last $n-i$ pieces from a.

For example, let $a = (0,3,2,0,3,1)$, a weak composition of rank 9. Then $a_{(3,7)} = (0,1,2,0,2,0)$ by deleting the first 2 pieces from a_2 and the last 2 pieces from a_6 and a_5.

Definition 4.6. Let $\mathcal{A}$ be a finite set and des a map from $\mathcal{A}$ to weak compositions of n. A ***weak dual equivalence for*** $(\mathcal{A}, \mathrm{des})$ is a family of involutions $\{\psi_i\}_{1<i<n}$ on $\mathcal{A}$ such that

i. For all $i - h \leq 3$ and all $T \in \mathcal{A}$, there exists a weak composition a such that

$$\sum_{U \in [T]_{(h,i)}} \mathfrak{F}_{\mathrm{des}_{(h-1,i+1)}(U)} = \kappa_a,$$

where $[T]_{(h,i)}$ is the equivalence class generated by $\psi_h, \ldots, \psi_i$.

ii. For all $|i-j| \geq 3$ and all $T \in \mathcal{A}$, we have $\psi_j \psi_i(T) = \psi_i \psi_j(T)$.

Define the ***braid involutions*** b_i for $i = 2, \ldots, n-1$ on standard fillings of rank n for which exactly one of $i-1$ or $i+1$ lies in the same row as i by cycling entries $i-1, i, i+1$ in the unique way that maintains this condition. For example, b_2 will exchange the two standard key tableaux of shape $(0,2,1)$ shown in Fig. 6. Using these, we define the elementary weak dual equivalence involutions from [4, Definition 3.21].

Definition 4.7 ([4])**.** Define ***elementary weak dual equivalence involutions***, denoted by d_i, on skew standard key tableaux that act by

$$d_i(T) = \begin{cases} b_i \cdot T & \text{if exactly one of } i-1, i+1 \text{ lies in the row of } i, \\ s_{i-1} \cdot T & \text{else if } i+1 \text{ lies between } i \text{ and } i-1 \text{ in reading order,} \\ s_i \cdot T & \text{else if } i-1 \text{ lies between } i \text{ and } i+1 \text{ in reading order,} \\ T & \text{otherwise} \end{cases} \tag{4.6}$$

where we take *column* reading order, bottom to top and left to right.

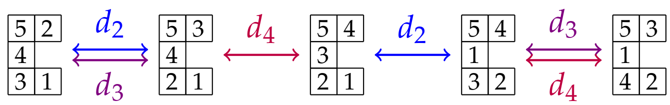

Figure 8: The elementary dual equivalence involutions on SKT$(2, 1, 2)$.

For examples of the elementary weak involutions, see Fig. 8.

Assaf [4, Theorem 3.25] showed these are well-defined involutions on standard key tableaux, all standard key tableaux of fixed shape fall into a single equivalence class, and this is an example of a weak dual equivalence. Moreover, under certain stability conditions, the converse holds. That is, by [4, Theorem 3.29], any weak dual equivalence is essentially this and, on the level of generating polynomials, we have the following.

Theorem 4.8 ([4])**.** *If there exists a weak dual equivalence for* $(\mathcal{A}, \mathrm{des})$ *for which* $\mathrm{des}(T) \neq \varnothing$ *for every* $T \in \mathcal{A}$*, then* $\sum_{T\in\mathcal{A}} \mathfrak{F}_{\mathrm{des}(T)}$ *is a nonnegative sum of key polynomials.*

The condition that the weak descent composition is nonempty for every element can often be circumvented if the polynomials under consideration stabilize. Using this along with the same elementary weak dual equivalence involutions, the local nature of Definition 4.6 allows us to give a combinatorial proof of the following.

Theorem 4.9. *For* $a \prec d$ *in the key poset,* $c^d_{a,b}$ *is the number of weak dual equivalence classes of* SKT(d/a) *isomorphic to* SKT(b)*. In particular, skew key polynomials are key positive.*

The special case when a is weakly increasing is proved in [4, Theorem 4.10] and follows from Theorem 4.9 by Proposition 2.3. In fact, we use the key poset to prove that Theorem 4.9 is tight, outside of a few accidental cases.

Theorem 4.10. *For* $a \subset d$ *for which* $a \not\prec d$ *in the key poset, if* $\mathrm{des}(T) \neq \varnothing$ *for every* $T \in$ SKT(d/a)*, then there exists a weak composition* b *for which* $c^d_{a,b}$ *is negative.*

Thus the key poset precisely characterizes nonnegativity of skew key polynomials.

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