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Get That Basket! Deciphering Student Strategies in the Linear Algebra Game Vector Unknown

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We present results of a grounded analysis of individual interviews in which students play Vector Unknown - a video game designed to support students who are taking their first semester of linear algebra. We categorized strategies students employed while playing the game. These strategies range from less-anticipatory button-pushing to more sophisticated strategies based on approximating solutions and choosing vectors based on their direction. We also found that students focus on numeric and geometric aspects of the game interface, which provides additional insight into their strategies. These results have informed revisions to the game and also inform our team's plans for incorporating the game into classroom instruction.

Keywords: Linear Algebra, Game-Based Learning, Inquiry-Oriented Instruction

Linear algebra is an important course for students in the STEM disciplines because of its unifying power within mathematics and its applicability to areas outside of mathematics. Research in linear algebra has evolved over the last twenty years from the pioneering work at the turn of the century (e.g., Dorier & Sierpinska, 2001; Harel, 1999; Hillel, 2000) to more recent work on improvements to teaching and learning using Tall's three worlds (Stewart & Thomas, 2010), models (e.g., Trigueros, 2018), dynamic geometry systems (e.g., Sinclair & Tabaghi, 2010) and everyday examples (e.g., Adiredja, Bélanger-Rioux, & Zandieh, 2019). Our work is most closely aligned with the curriculum design work of the Inquiry-Oriented Linear Algebra (IOLA) group (e.g., Andrews-Larson, Wawro, & Zandieh, 2017; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Zandieh, Wawro, & Rasmussen, 2017). The goal of our project is to explore Linear Algebra in the context of a video game developed from the IOLA curricular materials. In recent years, game-based learning (GBL) has become popular because of its ability to motivate students. Bakker vanden Heuvel-Panhuizen, and Robitzsch (2015) found that students who played mathematical games tended to spend more time on these games when compared to the amount of time spent on their normal math homework. Part of the reason can be attributed Gee's (2003) concept of skills as strategies, which states that players are more likely to practice necessary skills when they are seen as strategies for a larger goal. We view video games as providing an avenue to support and motivate students' development of understanding Linear Algebra concepts drawing on theories from Realistic Mathematics Education (RME), IOLA, and GBL (Authors, 2018, Authors, 2019).

Literature Review

Realistic Mathematics Education

This project leverages the activities of the IOLA curriculum (http://iola.math.vt.edu), which is a set of research-based curricular and instructional materials based on RME design principles and designed to move students through the different levels of mathematical activity from an experientially realistic situation to a formal mathematical understanding (Wawro, Rasmussen, Zandieh, & Larson, 2013). The curriculum begins with a task called the Magic Carpet Ride (MCR) task, which asks students to use two forms of transportation (represented by the vectors <3,1> and <1,2>) to reach Uncle Cramer's house located at (107, 64). The primary goal of the remaining tasks in this first unit is to support students' generalization lessons learned in their earlier activity and develop the formal constructs of linear independence and span.

Game–based learning (GBL) is the use of games (including video games) for educational purposes. While GBL has gained popularity in recent years (Gee, 2003; Gresalfi & Barnes, 2016; Foster & Shah, 2015; Shih & Shih, 2015) there are few games for higher levels of mathematics. Gee (2005) notes that well-designed games present players with well-ordered problems that allow the players to learn crucial skills necessary for gameplay. According to Gee, players also engage in cycles of expertise where they encounter difficult problems, solve them, practice and master them, and then are confronted with new problems that will challenge the assumptions of the initial problems, requiring them to apply their skills in a new context. Williams-Pierce (2017) notes that players of educational mathematics games go through several stages beginning with understanding how the game works and culminating in predictive behaviors that are optimal for higher-level mathematical activity. This difference between lesser-anticipatory and more anticipatory strategies can be seen in mathematical context in Hollebrands (2007) who noted that students engaged with Geometer's Sketchpad (GSP) in two key ways, proactive and reactive. Students who engaged in a proactive way used GSP to test theories, while reactive users reacted to the information presented to them on screen, but were not able to anticipate how their actions would change the on screen environment. This indicates that understanding how students anticipate using mathematical tools and games is key to interpreting their use of tools/games.

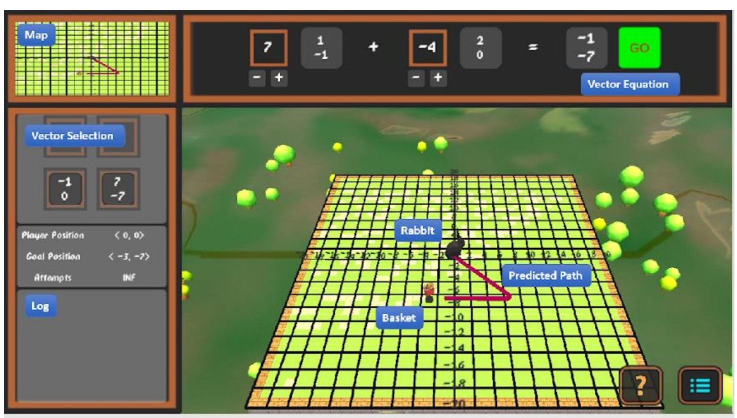

Figure 1. Sample Screen from the Game Vector Unknown.

Vector Unknown Gameplay

The game currently consists of five levels with our analysis investigating gameplay of levels 1, 2, and 5. In each level, four vectors are generated at random under the constraint that there are two pairs of linearly dependent vectors. To play, students guide the rabbit to the basket by selecting one or two vectors from the Vector Selection, placing them into the vector equation, adjusting the scalars, and pressing GO. During Level 1, players have a Predicted Path which highlights the path of the Rabbit before the player presses GO, providing players with a geometric representation of the Vector Equation. Level 2 is designed to force players who solely rely on the Predicted Path to consider the Vector Equation by removing the Predicted Path. On level 5, the Predicted Path returns, but players must now obtain several keys to unlock a lock that covers the goal forcing them to consider vectors originating from points other than the origin.

Methods

Data Collection

We interviewed five participants from a multi-purpose regional university in the southeastern United States. These participants were selected from 11 interviews comprising the total data corpus because they had not completed a first course in linear algebra. The participants came from a diverse background, including two black men, two black women, and one Asian woman. Each participant was interviewed for approximately one hour, during which they were asked to complete three levels of the *Vector Unknown* game. As needed, the interviewer provided help on how to navigate the game's screens and use the controls. The interviewer asked scripted questions along with impromptu follow-up questions. Impromptu questions were asked to further clarify and explore the participants' thinking about gameplay as well as any mathematical insights or strategies the participant developed during gameplay.

Analysis

Team members transcribed the interviews and documented student gameplay during the interview by noting significant interactions with the game's interface. Specifically, we documented moments in which the player chose or replaced vectors or when the player changed the scalar amounts for vectors in the vector equation. Consistent with grounded theory (Strauss & Corbin, 1994), team members then conducted an iterative grounded analysis of the interviews. This process began with multiple cycles of open coding and discussions during which team members agreed on interpretations of participants' statements and actions.

Table 1: Examples of codes from the first round of coding

Categories	Examples		
Focus on the vector equation	**Erp**: replaces a vector	**Es**: manipulates scalars	**Erv**: Removes a vector
Features of the game	**Fdl**: Data Log	**Fpp**: Player Position	**Fca**: Camera Angle
Focus on gameplay	**Pg**: Presses Go	**Pcomp**: Anticipates gameplay complexity	**Pu**: Undoing a Move
Conceptualizing a Vector	**Vd**: Testing the direction of a vector	**Vp**: Visualization and trace of vector path	**Vq**: Use of scalars or vectors components to reach quadrants
Student-expressed strategy	**Sc**: Connecting b/t representations	**Sp**: Previously used strategy	**Sr**: Retrospectively expresses a strategy
Focus on graphical game components	**Gdir**: Displacement (Direction)	**Gdis**: Displacement (direction and displacement)	**Gpp**: Refers to the predictive path

After initial coding, the research team grouped the codes into six broad categories (Table 1). The first category - Focus on the vector equation - includes student actions such as the introduction and removal of a vector or the manipulation of a scalar in the game interface. A second category - Features of the game - includes instances in which the student mentions using various features included in the game's interface, such as referencing the Data Log or the locations of the player or goal. The third category - Focus on gameplay - contains codes of student actions as they play the game, such as pressing GO or undoing a move. The fourth category - Conceptualizing a vector - includes codes about how the student expresses the ways in which they are Conceptualizing a vector, such as defining what the components of a vector are or testing its direction. Another category denotes Student-expressed strategies, such as focusing on one coordinate at a time or reusing a previous strategy. The final category - Focus on the graphical game components - such as the direction of the purple projected path or the graphical displacement from the goal.

Results

The initial codes are of a relatively fine grain size, but were important pieces into the analyzing the student strategies. Knowing what specific components of the game the student attended to while playing the game as well as how the student understood the vector equation and graphical component at different points in gameplay allowed us to discover several themes. These themes are seen across multiple students and can also be used to describe the strategies the student uses as they are introduced to the game and how their strategies evolve as they progress with each level. In this section, we use excerpts from four participants' gameplay to discuss the themes we developed. For each theme, we recognized that students were focusing to varying degrees on the geometric and numeric aspects of the game's interface. When a participant's strategy focused primarily on the vector equation without mentioning the Predicted Path or the graphical component of the interface, we called the strategy ***numeric***. When a strategy focused primarily on either the Predicted Path or the graph, we called the strategy ***geometric***.

There were several themes we found when analyzing student strategies. We characterized the first theme, ***button-pushing***, as rapidly adjustment of scalars and switching vectors. We denoted button-pushing as ***less-anticipatory*** when the player expressed surprise when reacting to a consequence of their actions. For example, Mouse[1] after quickly alternating the scalar keys made the following comment "what if I press the button that's ... the rabbit's going to go down. ohhh, yeah." Here, the tone of Mouse's "ohhh, yeah" comment indicated his surprise which lead us to describe his activity as *less-anticipatory* **button-pushing**. His reference to going down with the Predictive Path indicated that he was attending to the *geometric* aspects of the game. Another student Gwen, employed a less-anticipatory button-pushing strategy as evidenced from her describing her gameplay as "messing around", "mindlessly clicking", and "throwing in numbers to get the answer" while quickly adjusting vectors and scalars in Levels 1 and 2. She did not reference the Predicted Path, but instead frequently referenced the Vector Equation and its components, indicating that her ***button-pushing*** was *less-anticipatory* and *numeric*.

As participants played the game more extensively, participants began to engage in *more-anticipatory* ***button-pushing*** which involved switching out vectors and scalars, but not as quickly and the sense of surprise is replaced by a knowledge of the results prior to pushing a button. Mouse's gameplay shifted from less-anticipatory to more-anticipatory throughout the interview. This is evidenced by a decrease in the number of times Mouse switched between increasing and decreasing the scalar for a given vector when he was button-pushing. For instance, in Level 1, Mouse expressed surprise when the predictive path of the linear

[1] Students chose their own pseudonyms.

combination moved toward the goal. In contrast, while playing Level 5, Mouse put in the first vector he clicked on the "+" scalar and then uttered "Oh -" before clicking the "-" scalar. We see this as a more-anticipatory button-pushing because his actions indicate he is expecting *one* of the scalars (+ or -) to move the predictive path closer to the goal position.

We characterized ***quadrantal*** themed strategies as strategies that involve choosing vectors based upon the signs of the coordinates. This theme manifested as participants choose a starting vector in the same quadrant of the goal, or more generally a vector that could be scaled in a direction heading towards the goal. For example, Latia noted the following:

> So I know that the basket is on <-6,6>, and my initial thought was use, to start with either my <-3,9> or my <-1,3> because it is in the second quadrant and then, since I already know I'm at negative six, six, I was trying to think of what I could use so that if I multiplied those two numbers I could get to my basket.

Here, Latia made her initial vector choices by matching the signs of the goal position with that of the potential vector choices placing her strategy in the *numeric* category.

The third type of strategy was ***focus on one coordinate*** where the participant focused on trying to match one coordinate of the result of the vector equation with one coordinate of the goal position. Lance expressed this strategy as follows:

> Uhhmm, so when I started, when I saw my vector choices, I started to see the, um, almost all connections of like how certain, um, how um, I was manipulating them individually based off of like ys first and then the xs. And so I was trying to find paths and where, where I get my first, um, my first um, coordinate and my y.

Here Lance focused on reaching the y-coordinate of the goal first. The strategy is *numeric* as he did not have the Predicted Path when playing this level and did not refer to any geometric components.

Another student, Zo, noted that when she was trying to get to the basket at <-18,-1> from <-18,-9>:

> I have to go up eight but I don't know how I'm about to do that. There's no, like, straight line in these numbers so I can't do that. But, uh, I'm just going to trial and error it 'cause I don't know how else to do this.

Zo's strategy was clearly to match the -18 and then go up 8 indicating she was attending to the *geometric* aspects of the game.

The fourth strategy observed was a ***focus on one vector*****.** In this strategy, participants chose one vector, scaled it as close to the goal as possible - potentially going past the goal - and reducing the scalar, and then chose another vector and either scaled it to the answer or alternated adjusting scalars to reach the goal. In contrast, Mouse on Level 1 scaled one vector <-5,10> by 1 and then by 2 to reach the goal at <-11,8>. After which he commented that he had gone too far (referring to the Predicted Path) and reduced the scalar by 1. He then began to scale the other vector. Mouse's focus on the Predicted Path categorizes his strategy as *geometric*. Latia and Gwen also employed this strategy with the Predicted Path. While we did not observe a *numeric* example of ***focus on one vector***, we can imagine that a student would focus on adjusting the scalar on one vector until the result of the vector equation is close to the goal.

Examples of Strategies as Seen in Latia's Interview

Latia began with **less-anticipatory** ***button-pushing*** expressed by Latia as "playing with numbers", but quickly figured out that there was a link between the scalars and the geometric support noting that the predictive path doubled when she increased the scalar on a vector from 1 to 2. During the first level she relied on the geometric support to find the goal, but has clearly began to connect the two most notably stating that she made vector and scalar choices based upon the quadrant of goal expressing ***quadrantal*** reasoning. Also, of note, she mentions that she eliminated a vector choice because she did not want to go past the goal. Latia also ***focuses on one vector*** as indicated by choosing one vector, scaling it to what she believes is an appropriate distance to the goal, and then putting in a second vector and adjust it. This is followed by an adjustment to the first vector to reach the goal.

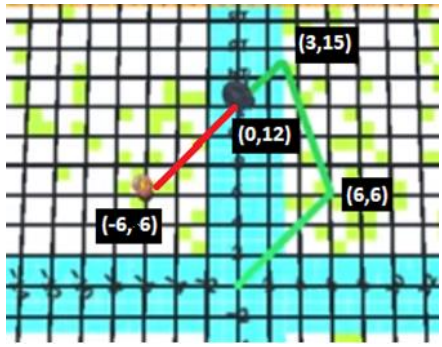

Figure 2. Latia Gameplay during Level 2.

During the second level (see Figure 2), Latia chose the vector <-3,-3> and the scalar -2 and pressed GO to take her to <6,6>. She then used a ***quadrantal*** strategy to chooses the vector <-1,3> and scales the vector by 3 attempting to ***focus on one vector*** to get close to the goal. She assumed that bunny will always start from zero and is surprised to find out it does not return to the origin after pressing GO. Noting that she wanted to go down and to the left, she utilized **more-anticipatory** ***button-pushing*** to test the direction of the vector <-1,-1>. This is an extension of **quadrantal reasoning** to a point that is not the origin. She noted that rabbit has to go down 6 and to the left 6 and scales <-1,-1> by 6 to reach the goal connecting the vector equation and its ***geometric*** interpretation without a reliance on predictive path. Latia replayed Level 2 with a goal at <-2,10> and mentioned that her strategy is to go past the goal and return, **repeating the strategy** used in a previous level playthrough and defying her intuition from the Level 1 playthrough. After not being able to visualize a solution with the given vectors, Latia solved the system of equations on a piece of paper indicating a flexibility in thinking and approach to completing the game.

Comparing conditions: Standard Basis Vectors

Both Mouse and Latia experienced Standard Basis Vectors (SBVs) on Level 2 and responded differently. On Level 1, Mouse relied more on the predictive path that resulted from ***button-pushing*** and looked at (and solved) the vector equation produced in the Data Log upon completion of the level. When he moved to Level 2, he sought to learn how the vector equation worked and focused on how the scalars related to the direction the rabbit traveled. The inclusion of SBVs allowed Mouse to focus more on the numerical components and see how the scalars affect the resultant vector. When asked about how he solved the level, he responded:

> I saw the position which is three and the other position which is negative five. So, I kinda made it to where I'm trying to get it to where the vector has three and five […]. I'm guessing that if I multiply negative one to negative three it will get three and negative one again to negative five.

Latia completed Level 1 and one attempt at Level 2 before she encounters SBVs. On opening Level 1, she noticed the vector equation and began to make sense of how the scalars and the equation relate to the graphical component of the game. On Level 2, she continued to explore this connection and gained a working understanding of how the resultant vector relates to the rabbit's motion, adjusting her strategy when starting away from the origin. When given SBVs, she ***focused on one vector*** and explained that the orientation of the vector does not matter because the negative scalar can flip directions. Latia said that the inclusion of the SBVs made the level easier to manage, which allowed her to use both vector slots comfortably.

Both students used the SBVs as a way to validate their understanding garnered from previous levels. Mouse used them to gain a better understanding on how the vector equation operates while Latia uses them to incorporate two vectors into the equation. Although this is the first time Latia used both vector slots, she began to gain the understanding in her first attempt at Level 2 when she has a starting position off the origin and needed only one vector to reach the goal from her second position.

Discussion

Throughout our analysis we found that students began with less-anticipatory gameplay. This early gameplay resembled the GBL version of reactive strategies (Hollenbrands, 2007) or Zone 1 and 2 gameplay (Williams-Pierce, 2017) which focuses on students understanding how the controls of the game function. Over time, most students shifted to more-anticipatory gameplay allowing them to develop complex strategies aligning with a shift to the GBL version proactive strategies (Hollenbrands, 2007). This is in line with Williams-Pierce's (2017) Zone 3 where players begin to anticipate what is desirable in their gameplay. This desirability is expressed in our themes as players making initial choices that they know will get them closer to the goal such as ***quadrantal strategy***, ***focusing on one vector,*** and ***focus on one coordinate***. Eventually, we found that at least one student Latia made a hypothesis about the direction of a vector and utilized the game to test the hypothesis. This aligns with students using GSP to test hypothesis (Hollenbrands, 2007) as a tenant of proactive strategies and of the characterization of Zone 4 gameplay (Williams-Pierce, 2017). Finally, Latia's reuse of a previous strategy is a key indicator that a student has moved to Zone 5 of mathematical gameplay which is ripe for mathematical discovery and advancement. A key result of our analysis is that as students spend more time playing they game they were able to anticipate what would happen in the game allowing for a greater variety of theories to form which could be valuable points of departure for educators using the game with students either inside or outside of class. about gameplay and used the game to test their theories.

In addition to analyzing student thinking and gameplay, this study served to inform further development of the videogame. For example, a new iteration of the game was created that included a difficulty setting of easy, medium, and hard. The entire easy level includes pair of vectors that are multiples of a standard basis element while higher difficulties exclude standard basis vectors. Further exploration will include interviewing more students to find some of the strategies that we speculated would exist such as the *numeric* ***focus on one vector*** strategy. Finally we wish to explore how the game can be integrated in an IOLA classroom with a particular focus on building on the student strategies discovered.

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