Cooperative Evasion by Translating Targets with Variable Speeds

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***Abstract*— We consider a problem of cooperative evasion between a single pursuer and multiple evaders in which the evaders are constrained to move in the positive Y direction. The evaders are slower than the vehicle and can choose their speeds from a bounded interval. The pursuer aims to intercept all evaders in a given sequence by executing a Manhattan pursuit strategy of moving parallel to the X axis, followed by moving parallel to the Y axis. The aim of the evaders is to cooperatively pick their individual speeds so that the total time to intercept all evaders is maximized. We first obtain conditions under which evaders should cooperate in order to maximize the total time to intercept as opposed to each moving greedily to optimize its own intercept time. Then, we propose and analyze an algorithm that assigns evasive strategies to the evaders in two iterations as opposed to performing an exponential search over the choice of evader speeds. We also characterize a fundamental limit on the total time taken by the pursuer to capture all evaders when the number of evaders is large. Finally, we provide numerical comparisons against random sampling heuristics.**

I. Introduction

We consider a single pursuer multi-evader pursuit evasion problem in which the aim of the pursuer is to intercept all of the evaders in a fixed given sequence. The evaders are constrained to move along the positive Y direction. The pursuer follows the Manhattan distance, i.e., moving parallel to the X-axis followed by moving parallel to the Y-axis. The aim of the evaders is to cooperatively maximize the total time to intercept all evaders. Such a set-up arises in riot control or border protection scenarios in which a ground or air vehicle would like to optimally visit mobile locations headed toward a boundary/asset, or in UAV monitoring of vehicles along a highway. This setup is also applicable in multiple robotic decoy deployment [1].

A. Related work

Since the seminal work by Isaacs in [2], much has been done in the field of pursuit evasion with a lot of focus on multi-agent pursuit evasion [3], [4], [5]. The case of a single pursuer and 2 evaders has been extensively analyzed [6], [7]. Protector-Prey-Predator [8] and Target-Attacker-Defender differential game [9] are some examples of this scenario. With more than two evaders, the complexity of the problem grows exponentially with number of evaders. The problem of successive pursuit with cooperative multiple evaders is considered in [10], [11], [12] and [13]. Our problem differs as the evaders are constrained to move in a fixed direction and can choose their individual speeds from a bounded interval to maximize the total intercept time. Thus, the evasive strategies are based on the range of evader speeds.

B. Contributions

We consider an optimal evasion problem between a single pursuer and n evaders. The pursuer moves with unit speed. The evaders are constrained to move in the positive Y direction such that their speeds $v_i, i \in \{1, \dots, n\}$, lie in the interval $[u_{\min}, u_{\max}]$ with $0 < u_{\min} < u_{\max} < 1$. The evaders need to choose their speeds in order to maximize the total intercept time. We first present a complete solution to the optimal evasion problem for $n \leq 2$. We then show, for general n, that the optimal choice of the speed for each evader is one of the extremes, i.e., $u_{\min}$ or $u_{\max}$. We further show that, by enforcing cooperation among evaders, they are able to maximize the total intercept time. In order to implement the cooperative strategies, it is important to determine the conditions under which cooperation is optimal. Such conditions are also provided in this paper. We present an algorithm which assigns the evasive strategies to the evaders in two iterations as opposed to performing an exponential search over the choice of evader speeds. For sufficiently large n, for which the global optimum is difficult to compute, we establish a fundamental upper bound to the total intercept time taken by the pursuer to capture all evaders. Finally, we provide comparisons through numerical results.

C. Organization

The paper is organized as follows. Section II comprises the formal problem definition. In section III, we derive an evasive strategy for multiple evaders and provide a Sequential-Greedy-Cooperation algorithm. Section IV establishes a fundamental upper bound on the total time to intercept all evaders. Section V presents the numerical simulations. Finally, section VI summarizes this paper and outlines directions for future work.

II. Problem formulation

We consider an optimal evasion problem played between a single pursuer with simple motion and n mobile evaders. We denote the pursuer as P and evaders as E_i with $i \in \{1, \dots, n\}$. The pursuer with an arbitrary initial location at (X, Y) is assumed to be moving with unit speed either along the X or the Y axis. We term this pursuit strategy as *Manhattan pursuit*, and is formally defined as follows.

Definition 1 (Simple Manhattan pursuit) *Given initial locations (x_i, y_i) and (X, Y) of an evader E_i and the pursuer P respectively, the pursuer*

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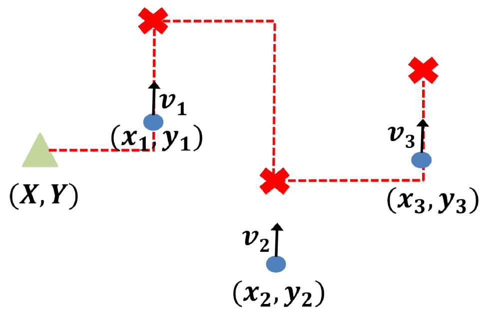

Fig. 1. Problem setup. The triangle represents the pursuer and the blue dots represent the evaders. The red dashed line represents the path taken by the pursuer to intercept the evaders. The cross represents the intercept locations.

1) *moves with unit speed along the positive or negative X direction until $X(t) = x_i$ and then,*
2) *moves with unit speed along positive or negative Y axis to intercept the evader.*

The evaders, initially located at $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, are constrained to move along the positive Y direction with simple motion such that their instantaneous speeds v_i, $i \in \{1, \ldots, n\}$, lie in the interval $[u_{\text{min}}, u_{\text{max}}]$ with $0 < u_{\text{min}} < u_{\text{max}} < 1$ (Fig. 1). The pursuer is said to *intercept* the i^{th} target when its location coincides with that of the i^{th} target. The game *terminates* when the pursuer intercepts the last evader. A *strategy* for an evader E_i is a measurable function, defined as $v_i(\{X(t), Y(t)\}, \{x_i(t), y_i(t)\}_{i=1}^{n}) \to [u_{\text{min}}, u_{\text{max}}]$, where the notation $\{X(t), Y(t)\}$ denotes the set of all locations $(X(\tau), Y(\tau)), \forall \tau \in [0, t]$. The goal is to solve the following problem.

Problem II.1 (Optimal evasion) *Given that the pursuer follows a fixed order to intercept the evaders, determine strategies $v_1^*, v_2^*, \ldots, v_n^*$ for the evaders that maximizes the total time T_n taken by the pursuer to intercept all n evaders.*

III. Evasive Strategy

We begin with the simple case of a single evader followed by two evaders, and then present the more general case. We start by defining the following simple Manhattan pursuit strategy.

A. Single Evader

In this section, we first consider the case of a single evader and a pursuer, located at (x_1, y_1) and (X, Y) respectively. We first present a result on the time taken to intercept a single evader. This will be used in deriving the optimal strategy for the evader. We denote $\Delta x_i^{i-1} := |x_{i-1} - x_i|$, where $x_0 = X$ and $T_i^{i-1}(v_i)$ as the time taken by the pursuer to intercept evader i moving with speed v_i after intercepting evader $i-1$. Specifically, $T_1^0(v)$ is the time taken to intercept the first evader moving with speed v. For brevity, we omit the proofs for the single evader case as they are similar to the general case presented later.

Lemma III.1 (Time to intercept a single evader) *The time T_1^0 taken by P to intercept E_1 is*

$$T_1^0(v) = \begin{cases} \frac{\Delta x_1^0 + y_1 - Y}{1-v}, & \text{if } \Delta x_1^0 > (Y - y_1)/v, \\ \frac{\Delta x_1^0 + Y - y_1}{1+v}, & \text{otherwise.} \end{cases}$$

Lemma III.2 (Monotonicity of time to intercept) *The time $T_1^0(v)$ is a monotonically increasing function of v if $\Delta x_1^0 > \frac{Y - y_1}{v}$. Otherwise, $T_1^0(v)$ is a monotonically decreasing function of v.*

Remark 1 *The time to intercept is monotonic even when the pursuer follows a Euclidean strategy, i.e., given the initial locations of E_1 and P, as (x_1, y_1) and (X, Y) respectively, the vehicle moves towards $(x_1, y_1 + vT_1^0)$, where T_1^0 is*

$$\frac{(y_1 - Y)v}{1 - v^2} + \sqrt{\frac{(X - x_1)^2}{1 - v^2} + \frac{(Y - y_1)^2}{(1 - v^2)^2}}.$$

Lemma III.2 characterizes the monotonic nature of $T_1^0(v)$. This only means that the maximum is achieved at one of the extremes. The next theorem characterizes the evader's optimal choice of speed.

Theorem III.3 (Single evader optimal strategy) *Given the initial locations (x_1, y_1) and (X, Y) of the evader and the pursuer respectively, the optimal strategy v^* for the evader is*

$$v^* = \begin{cases} u_{\text{min}}, & \text{if } y_1 < Y - \Delta x_1^0 \left(\frac{u_{\text{min}} + u_{\text{max}}}{2 + u_{\text{min}} - u_{\text{max}}}\right) \\ u_{\text{max}}, & \text{otherwise.} \end{cases}$$

We now consider the case of two evaders and derive the optimal evasion strategies for both evaders. We say that an evader E_i moves *greedy* if it moves with speed that maximizes its own intercept time. An evader *cooperates* if it moves with a speed that maximizes the total intercept time. We denote the greedy strategy of evader i as $v_{i_g}^*$ and the cooperative strategy as $v_{i_c}^*$. In what follows, we only provide an outline to our proofs. The detailed proofs are contained in [18].

B. Two evaders

Similar to previous section, we first derive an expression for the time taken to intercept the evaders followed by the optimal strategy for both evaders.

Let the first evader E_1 be located at (x_1, y_1) and move with speed v_1 and the second evader E_2 be located at (x_2, y_2) and move with speed v_2. Then, the following result summarizes the time to intercept E_2 after intercepting E_1. For ease of reference, we introduce the following condition:

$$\Delta x_2^1 > \frac{y_1 - y_2 + (v_1 - v_2)T_1^0(v_1)}{v_2}. \tag{1}$$

Lemma III.4 (Time to intercept E_2) *The time $T_2^1(v_1, v_2)$ taken by P to intercept E_2 after intercepting E_1 is*

$$T_2^1(v_1, v_2) = \begin{cases} \frac{\Delta x_2^1 + y_2 - y_1 + (v_2 - v_1)T_1^0(v_1)}{1 - v_2}, & \text{if (1) holds,} \\ \frac{\Delta x_2^1 + y_1 - y_2 + (v_1 - v_2)T_1^0(v_1)}{1 + v_2}, & \text{otherwise.} \end{cases}$$

Proof: The outline of the proof is as follows. We first assume that equation (1) holds which implies that after the completion of stage (1) for the pursuit of E_2, the evader's y-coordinate strictly exceeds the pursuer's location, thus, yielding the expression. The case when equation (1) does not hold is analogous. ■

Lemma III.5 (Monotonicity of time to intercept E_2) *Given that E_1 moves with v_1, the time $T_2^1(v_1, v_2)$ is monotonically increasing function of v_2 if condition (1) holds. Otherwise, $T_2^1(v_1, v_2)$ is a monotonically decreasing function of v_2.*

Proof: From Lemma III.4, $\frac{dT_2^1(v_1,v_2)}{dv_2} > 0$ if condition (1) holds and $\frac{dT_2^1(v_1,v_2)}{dv_2} < 0$, otherwise. ■

We now characterize an optimal greedy strategy for E_2. In what follows, we denote $V := \frac{u_{\min}+u_{\max}}{2+u_{\min}-u_{\max}}$.

Lemma III.6 (E_2's greedy strategy) *The greedy strategy v_{2g}^* for E_2 for a greedy E_1 moving with v_{1g}^* is*

$$v_{2g}^* = \begin{cases} u_{\max}, & \text{if } y_2 \geq y_1 - \Delta x_2^1 V + (v_{1g}^* - V)T_1^0(v_{1g}^*), \\ u_{\min}, & \text{otherwise.} \end{cases}$$

Proof: The central idea is to determine a location y_2' such that if $y_2 = y_2'$, then irrespective of E_2's choice of $u_{\min}$ or $u_{\max}$, the time to intercept E_2 will be the same and will be maximum at both $u_{\min}$ and $u_{\max}$. Then it follows that the time to intercept is maximized for $v_{2g}^* = u_{\min}$ when $y_2 < y_2'$, and $v_{2g}^* = u_{\max}$ when $y_2 > y_2'$. ■

Lemma III.6 yields a *greedy* strategy for E_2 when E_1 and E_2 both move greedily. However, it might be better for the evaders to cooperate to maximize the total intercept time. We now characterize the conditions on cooperation between the two evaders.

We define that a point A, located at (x_A, y_A), is *above* point B, located at (x_B, y_B), if $y_A > y_B$ and we define point A is *below* point B if $y_A < y_B$.

Lemma III.7 (Conditions on cooperation) *Given the initial locations of E_1, E_2, and P as (x_1, y_1), (x_2, y_2), and (X, Y) respectively, E_1 cooperates with E_2 if*

(i) Case 1:

$$\begin{aligned} &Y - \Delta x_1^0 V \leq y_1 \leq Y - \Delta x_1^0 u_{\min}, \text{ and} \\ &y_2 > y_1 - \Delta x_2^1 V + (v_{1g}^* - V)T_1^0(v_{1g}^*) + \\ &2(\frac{u_{\min}\Delta x_1^0 + y_1 - Y}{2 + u_{\min} - u_{\max}}), \end{aligned} \tag{2}$$

(ii) Case 2:

$$\begin{aligned} &Y - \Delta x_1^0 u_{\max} \leq y_1 \leq Y - \Delta x_1^0 V, \text{ and} \\ &y_2 \leq y_1 - \Delta x_2^1 V + (v_{1g}^* - V)T_1^0(v_{1g}^*) + \\ &2(\frac{u_{\max}\Delta x_1^0 + y_1 - Y}{2 + u_{\min} - u_{\max}}). \end{aligned} \tag{3}$$

Proof: The main idea for this proof is as follows. We observe that there are only two ways for E_1 and E_2 to cooperate. First, E_1 moves with a speed v_1 such that E_1 is intercepted below the pursuer and E_2 moves greedily and second, when E_1 moves greedily and E_2 with speed v_2 such that it is intercepted below the intercept point of E_1. Thus, to determine which of the two scenarios yield the greater time to intercept, we arrive at a condition $T_1^0(u_{\min}) + T_2^1(u_{\min}, u_{\max}) > T_1^0(u_{\max}) + T_2^1(u_{\max}, u_{\min})$ which yields the result. ■

Theorem III.8 (Optimal cooperative strategy for E_1) *Given the initial locations (x_1, y_1), (x_2, y_2), and (X, Y) of E_1, E_2 and P, respectively, if the conditions for cooperation in Lemma III.7 hold, then the optimal cooperative strategy v_{1c}^* for E_1 is*

$$v_{1c}^* = \begin{cases} u_{\min}, & \text{for case 1 from Lemma III.7 or,} \\ u_{\max}, & \text{for case 2 from Lemma III.7.} \end{cases}$$

Proof: Consider that case 1 of Lemma III.7 holds. Then E_1 moves with speed v_1 such that $\Delta x_1^0 < (Y - y_1)/v_1$. We know from Lemma III.7 that the conditions on cooperation ensure that the total time to intercept during cooperation is higher than the greedy choice. Since $T_1^0(v_1)$ is monotonic in v_1, from Lemma III.2, $v_{1c}^* = u_{\min}$. The second case is derived analogously. ■

In this subsection, we analyzed the case of 2 evaders, primarily to highlight the underlying problem structure. Next, we will consider the case of n evaders. Similar to the two evader case, we will first present a result on the time taken to intercept the k^{th} evader after intercepting the $k-1^{th}$ evader. Then we will present results on the greedy and cooperative strategies between E_k and E_{k-1}.

C. n Evaders

For ease of presentation, we will denote $(y_i - y_j)$ as Δy_j^i for some i, j and for brevity, we denote $T_i^{i-1}(v_1, \ldots, v_i)$ as $T_i^{i-1}(v_{-i}, v_i)$. We present the following condition for ease of reference.

$$\Delta x_k^{k-1} > \frac{\Delta y_k^{k-1} + (v_{k-1} - v_k)\sum_{i=1}^{k-1} T_i^{i-1}}{v_k}. \tag{4}$$

Lemma III.9 (Time to intercept E_k) *The time $T_k^{k-1}(v_{-k}, v_k)$ taken by P to intercept E_k, moving with v_k, after intercepting E_{k-1}, moving with v_{k-1}, is*

$$T_k^{k-1} = \begin{cases} \frac{\Delta x_k^{k-1} + \Delta y_{k-1}^k + (v_k - v_{k-1})\sum_{i=1}^{k-1} T_i^{i-1}}{1 - v_k}, & \text{if (4) holds,} \\ \frac{\Delta x_k^{k-1} + \Delta y_k^{k-1} + (v_{k-1} - v_k)\sum_{i=1}^{k-1} T_i^{i-1}}{1 + v_k}, & \text{otherwise.} \end{cases}$$

Proof: The main idea is to prove this result by using the method of induction on k, the base case of which was established in Lemma III.4 for $k = 2$. ■

Lemma III.10 (Monotonicity of time to intercept) *Given that each E_i, $i \in 1, \ldots, k-1$ moves with v_i, the time T_k^{k-1} is monotonically increasing function of v_k if*

Algorithm 1: Seq-GreC Algorithm

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1 Assign greedy speeds to all evaders
2 if E_i and E_{i+1} can cooperate, ∀ 1 < i < n, then
3 |  Assign optimal cooperative strategy
4 else
5 |  Assign optimal greedy strategy.
6 end
7 Repeat from step 2.
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condition (4) *holds. Otherwise,* T_k^{k-1} *is a monotonically decreasing function of* v_2.

Proof: We use induction to establish the result. Lemma III.5 yields the base of the induction. Assuming that the result holds for some $k = \bar{k}$, it can be checked that $\frac{dT^{\bar{k}}_{\bar{k}+1}}{dv_k} > 0$ if condition (4) holds. Otherwise, $\frac{dT^{\bar{k}}_{\bar{k}+1}}{dv_k} < 0$. This concludes the proof. ∎

Since Lemma III.10 establishes that the time to intercept a k^{th} evader is maximized at either $u_{\min}$ or $u_{\max}$, finding an optimal strategy for all evaders would require analyzing all 2^n possibilities in the worst case.

We now present an algorithm that assigns respective strategies to the evaders in just two iterations. The algorithm, summarized in Algorithm 1, first assigns the greedy strategies to all evaders. Then, it assigns cooperative strategies by considering two sequentially paired evaders at a time.

Now, we will present the results that the algorithm uses in assigning the strategies.

Lemma III.11 (Evader k's greedy strategy) *The greedy strategy* v^*_{kg} *for* E_k*, when each* E_i*,* $i \in \{1, \ldots, k-1\}$ *moves with* v^*_{1g} *is*

$$v^*_{kg} = \begin{cases} u_{\max}, & \text{if } y_k \geq y_{k-1} - \Delta x_k^{k-1} V + (v^*_{(k-1)g} - V) \sum_{i=1}^{k-1} T_i^{i-1}, \\ u_{\min}, & \text{otherwise.} \end{cases}$$

Proof: Suppose the result holds for some $k = \bar{k}$. Consider the next evader, $E_{\bar{k}+1}$. Similar to the proof of Lemma III.6, we find $y'_{\bar{k}+1} = y_{\bar{k}} - \Delta x^{\bar{k}}_{\bar{k}+1} V + (v^*_{\bar{k}g} - V) \sum_{i=1}^{\bar{k}} T_i^{i-1}$. If $y_{\bar{k}+1} < y'_{\bar{k}+1}$, then, from Lemma III.10, the time will be maximized at either $u_{\min}$ or $u_{\max}$. Thus, assuming $T_k^{k-1}(v^*_{1g}, \ldots, v^*_{(k-1)g}, u_{\max}) > T_k^{k-1}(v^*_{1g}, \ldots, v^*_{(k-1)g}, u_{\min})$ yields $y_{\bar{k}+1} > y'_{\bar{k}+1}$ which contradicts our assumption and hence $v^*_{kg} = u_{\min}$. By induction, the result holds for any value of $\bar{k}$. Case 2 is proved analogously. ∎

The previous lemma presented a result on the greedy strategy of any evader E_k. This result is the first step of the Algorithm 1. As the second step of Algorithm 1 requires to check the conditions of cooperation between two consecutive evaders, we will now present a result on the conditions if two evaders should cooperate or not. We introduce the notation, $U := \frac{2}{2 + u_{\min} - u_{\max}}$.

Lemma III.12 (Cooperation conditions for E_{k-1}) *Given the initial locations of* E_{k-1}*,* E_k*, and* P *as* (x_{k-1}, y_{k-1})*,* (x_k, y_k)*, and* (X, Y) *respectively, then* E_{k-1} *will cooperate with* E_k *if*

(i) Case 1:

$$y_{k-2} + (v^*_{(k-2)a} - V) \sum_{i=1}^{k-2} T_i^{i-1} - \Delta x_{k-1}^{k-2} V \leq y_{k-1} \leq y_{k-2} + (v^*_{(k-2)a} - u_{\min}) \sum_{i=1}^{k-2} T_i^{i-1} - \Delta x_{k-1}^{k-2} u_{\min}$$

and

$$\Delta y_k^{k-1} > -\Delta x_{k-1}^{k} V + (v^*_{(k-1)g} - V) \sum_{i=1}^{k} T_i^{i-1} + U\big(u_{\min} \Delta x_{k-1}^{k-2} + \Delta y_{k-2}^{k-1} - (v^*_{(k-2)a} - u_{\min}) \sum_{i=1}^{k-2} T_i^{i-1}\big) \quad (5)$$

(ii) Case 2:

$$y_{k-2} + (v^*_{(k-2)a} - u_{\max}) \sum_{i=1}^{k-2} T_i^{i-1} - \Delta x_{k-1}^{k-2} u_{\max} \leq y_{k-1} \leq y_{k-2} + (v^*_{(k-2)a} - V) \sum_{i=1}^{k-2} T_i^{i-1} - \Delta x_{k-1}^{k-2} V$$

and

$$\Delta y_{k-1}^{k} < -\Delta x_k^{k-1} V + (v^*_{(k-1)g} - V) \sum_{i=1}^{k} T_i^{i-1} + U\big(u_{\max} \Delta x_{k-1}^{k-2} + \Delta y_{k-2}^{k-1} - (v^*_{(k-2)a} - u_{\max}) \sum_{i=1}^{k-2} T_i^{i-1}\big), \quad (6)$$

where $v^*_{(k-2)a}$ *determined by Algorithm 1.*

Proof: We use induction hypothesis to establish this result following similar steps in the proof of Lemma III.7. ∎

Lemma III.12 establishes the conditions for cooperation between any two consecutive evaders. The next result characterizes the cooperative strategies of the evaders.

Theorem III.13 (Cooperative strategy for E_{k-1}) *If the conditions on cooperation in Lemma III.12 hold, then the optimal cooperative strategy* $v^*_{(k-1)c}$ *for* E_{k-1} *is*

$$v^*_{(k-1)c} = \begin{cases} u_{\min}, & \text{for case 1 of Lemma III.12}, \\ u_{\max}, & \text{for case 2 of Lemma III.12.} \end{cases}$$

Proof: We use induction hypothesis to establish this result using Lemma III.8 as the base case. ∎

Remark 2 (Sandwiched evader) *For some* $i \in \{1, \ldots, n\}$*, if Lemma III.12 holds for evader* E_{i-1} *and* E_i *as well as* E_i *and* E_{i+1}*, then evader* E_i *moves greedy.*

IV. Fundamental Limit

In the previous sections, we considered that the pursuer followed a fixed strategy to capture all evaders. We now establish a fundamental upper bound, for *a large number of evaders*, on the total time taken to intercept all evaders by the pursuer following *any* strategy. We first provide some existing results that will be useful in establishing the bound.

Given a set of m points, a *Euclidean minimum Hamiltonian path* (EMHP) is the shortest path through m points such that each point is visited exactly once. When the points are translating with some constant speed $v \in (0, 1)$, then the shortest tour though the points is called *Translational minimum Hamiltonian path* (TMHP) [19].

Lemma IV.1 (Length of EMHP tour) *Given m points in a $l \times h$ rectangle in the plane, where $h \in \mathbb{R}_{>0}$ and $l \in \mathbb{R}_{>0}$, there exists a path that starts from a unit length edge of the rectangle, passes through each of the m points exactly once, and terminates on the opposite unit length edge, with length upper bounded by* $\sqrt{2lhm} + h + 2.5$

Proof: The proof is similar to the proof provided in [20] for a $1 \times h$ rectangle and thus, has been omitted. ■

To calculate the EMHP tour through translating points $s, s_1, \ldots, s_f, f$ that move with speed v, the points are scaled by defining a conversion map $C_v : \mathbb{R}^2 \to \mathbb{R}^2$ such that $C_v(x, y) = (\frac{x}{\sqrt{(1-v^2)}}, \frac{y}{1-v^2})$ [19].

Lemma IV.2 (Length of TMHP tour [19]) *Let the initial and final point be denoted as $s = (x_s, y_s)$ and $f = (x_f, y_f)$ respectively, and $v \in (0, 1)$ denote a constant speed of all evaders, then the length of the TMHP tour is $\frac{v(y_f - y_s)}{1-v^2} + \mathcal{L}_E(C_v(s), C_v(s_1), \ldots, C_v(s_f), C_v(f))$ where, $\mathcal{L}_E(C_v(s), C_v(s_1), \ldots, C_v(s_f), C_v(f))$ denotes the length of the EMHP starting with point s, moving through points $s_1, \ldots, s_f$ and ending at point f.*

The optimal order followed by the vehicle in the TMHP solution is the same as the optimal order followed by the vehicle in the EMHP solution.

Denote $n_{\text{max}} \in \mathbb{Z}_0^+$ as the total number of evaders that move with u_{max} and $n_{\text{min}} = n - n_{\text{max}}$ as the total number of evaders that move with u_{min}. Let $\mathcal{A}_{\text{max}}$ and $\mathcal{A}_{\text{min}}$ denote the area of the smallest enclosing rectangular environment that the n_{max} and n_{min} evaders occupy initially. We assume that all of the evaders are initially located within a rectangular environment of area $\mathcal{A}$. The pursuer's strategy is to capture all the n_{max} evaders first, followed by capturing all the evaders moving with u_{min}. This is because if the pursuer captures the n_{min} evaders first then naturally, the evaders moving with u_{max} will be further away from the pursuer.

Let $T_{n_{\text{max}}}$ be the time taken by the vehicle to capture all of the n_{max} evaders and $T_{n_{\text{min}}}^{n_{\text{max}}}$ be the time taken to intercept the last evader that moves with u_{max} and the first evader that moves with u_{min} after capturing all of the n_{max} evaders respectively. Let $T_{n_{\text{min}}}$ be the total time taken by the vehicle to capture all of the remaining $n_{\text{min}} - 1$ evaders. The next result characterizes an upper bound on the time taken by the pursuer to capture all evaders following any strategy.

Theorem IV.3 (Upper bound on intercept time) *Let Δy and Δx be the difference between the initial y and x-coordinate of the last evader captured moving with u_{max} and the first evader that is captured moving with u_{min}. Then, from Lemma IV.1 and Lemma IV.2, the total time taken by the pursuer to capture all evaders is $T = T_{n_{\text{max}}} + T_{n_{\text{min}}}^{n_{\text{max}}} + T_{n_{\text{min}}}$ where,*

$$T_{n_{\text{max}}} = \sqrt{\frac{2\mathcal{A}_{\text{max}} n_{\text{max}}}{(1 - u_{\text{max}}^2)^{3/2}}},\ T_{n_{\text{min}}} = \sqrt{\frac{2\mathcal{A}_{\text{min}}(n_{\text{min}} - 1)}{(1 - u_{\text{min}}^2)^{3/2}}}$$

$$T_{n_{\text{min}}}^{n_{\text{max}}} = \frac{u_{\text{min}}}{1 - u_{\text{min}}^2}(\Delta y + (u_{\text{min}} - u_{\text{max}})T_{n_{\text{max}}}) + \sqrt{\frac{\Delta x^2}{1 - u_{\text{min}}^2} + \frac{(\Delta y + (u_{\text{min}} - u_{\text{max}})T_{n_{\text{max}}})^2}{(1 - u_{\text{min}}^2)^2}}.$$

Moreover, for large n, T is maximum for

$$n_{\text{max}}^* = \left\lfloor \frac{(u_{\text{min}} - u_{\text{max}})^2 n}{(1 - u_{\text{min}}^2)^{\frac{1}{2}}(1 - u_{\text{max}}^2)^{\frac{3}{2}} + (u_{\text{min}} - u_{\text{max}})^2} \right\rceil,$$

where $\lfloor x \rceil$ denotes the integer nearest to x.

Proof: The outline of the proof is as follows. The expression for $T_{n_{\text{max}}}$ and $T_{n_{\text{min}}}$ follows directly from Lemma IV.1 and noting that n is large. The expression for $T_{n_{\text{min}}}^{n_{\text{max}}}$ follows from [19]. Then we use the derivative test to establish the result. ■

V. Simulation Results

We first present the numerical results for Algorithm 1. We compare the mean of the total time to intercept all evaders using Algorithm 1 to the mean of the total time to intercept all evaders by randomly sampling over the evader speeds of either u_{min} or u_{max} (see Figure 2). For each value of n, we randomly generate the initial locations of the evaders and the pursuer and we consider 50 Monte Carlo trials. To select the best evader speeds, we choose $10nln(2/\delta)$ samples uniformly randomly over the set, which guarantees that the violation probability is less than a small quantity δ [21], where $\delta = 0.1$. We compute the maximum over the samples and then report the mean value in Figure 2. We observe that Algorithm 1 outperforms random sampling.

Figure 3 shows a comparison when n_{max} is selected uniformly randomly to the upper bound obtained by n_{max}^* for given initial locations. To obtain the EMHP tour required for the time to intercept evaders, the `linkern`[1] solver was used. We consider 50 Monte Carlo trials for each value of n and report the mean and standard deviation. It is observed that the total time to intercept all evaders by randomly selecting n_{max} is well below the upper bound obtained from n_{max}^*. Thus,

[1]The TSP solver `linkern` is freely available for academic research use at `http://www.math.uwaterloo.ca/tsp/concorde/`.

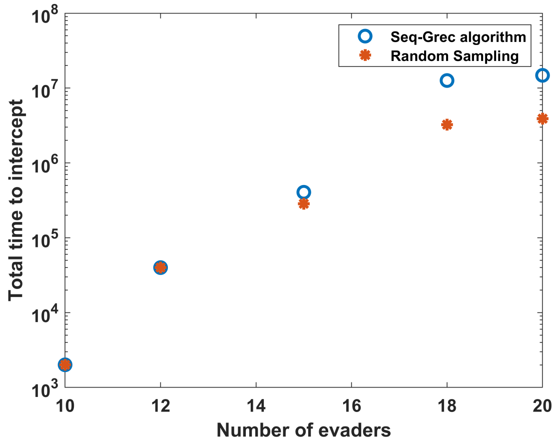

Fig. 2. Comparison of Seq-Grec with Random Sampling. The blue circles represent the mean of the total time to intercept of Seq-Grec Algorithm. The orange star represents the mean over the samples of Random Sampling

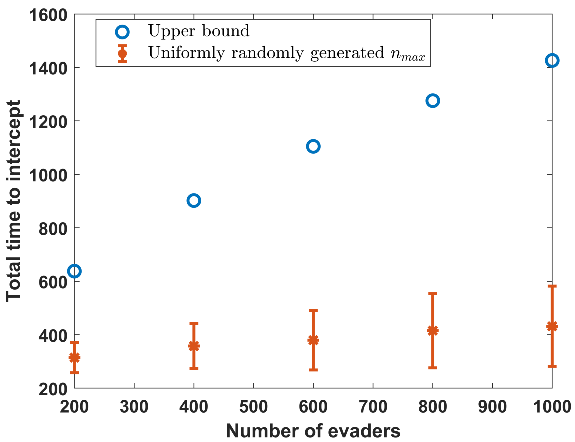

Fig. 3. Comparison of the total time to intercept when $n_{\max}$ is randomly uniformly selected to that of the upper bound obtained from $n^*_{\max}$.

by performing an additional optimization to select $n_{\max}$ the evaders can reach the upper bound on time to intercept. This means that a strategy that only depends on $n_{\max}$ may be sub-optimal for the evaders.

VI. Conclusions and Future Work

An optimal evasion problem between single a pursuer and multiple evaders was addressed. It is shown that by enforcing cooperation among evaders, they are able to maximize the total interception time. Conditions where cooperation is optimal are also presented which are crucial to implement the cooperative strategies. An upper bound on the total time to intercept all evaders is also presented.

In subsequent work, a generalized setup of multiple pursuers and evaders will be considered. Constant factor approximations for both, the evaders and the pursuers will also be addressed. Identifying which evaders should move with $u_{\max}$ is another possible extension.

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