ON A LOCAL INVERSION OF THE X-RAY TRANSFORM FROM ONE SIDED DATA

HIROSHI FUJIWARA, KAMRAN SADIQ, AND ALEXANDRU TAMASAN

ABSTRACT. We explain how the theory of A-analytic maps of A. Bukhgeim can apply to a local CT inversion problem, in which the data is restricted to lines leaning on a given arc.

1. INTRODUCTION

By the time the commercial CT became a widespread diagnostic method in medicine, it was also apparent that X-ray radiation is harmful to human body. In mitigation, the engineering and mathematics communities have proposed various methods to lower the radiation dosage, in particular by inverting the Radon transform from a restricted set of lines. It is well known, that discretization of the set of directions leads to non-unique image reconstructions, see [11]. Moreover, in two dimensions, the inversion of the classical Radon transform is non-local, and, thus, the usage of only those lines that pass through the region of interest may not be enough to uniquely invert it. Several works identify specific subsets of lines which still provide unique reconstruction in the region of interest. Among the mathematics works, which use, roughly, half the data set, we refer to [4, 23, 5] or, in the constant attenuation case to [17, 16, 13, 20, 22]; see also references below.

In this brief note, we are concerned with the inversion question in which the (fan beam) data is collected from "one side". More precisely, let $\Omega \subset \mathbb{R}^2$ be a convex domain and Λ be an arc of its boundary $\varGamma$, see Figure 1 (left) below. The chord L joining the endpoints of the arc Λ partitions the domain in two subdomains $\Omega^{\pm}$, where Ω^+ denotes the domain enclosed by $\Lambda \cup L$. For a function f compactly supported in Ω, we explain that unique determination of $f|_{\Omega^+}$ from its attenuated X-ray transform over lines leaning on Λ is theoretically possible. Note that, if f happens to also be supported in Ω^-, then its X-ray data is incomplete: while the measurements are affected by the possible nonzero values of $f|_{\Omega^-}$, an entire cone of directions through points in Ω^- are missing in the data.

The unique determination of $f|_{\Omega^+}$ does follow from the support theorem in [7]; also in the attenuated case provided the attenuation is analytic. However. those arguments have yet to yield a method of reconstruction. In here we use the theory of A-analytic maps originally developed by A. Bukhgeim in [8] to address the inversion of the attenuated X-ray transform from complete data set; see [3, 24] for the application to the attenuated case and [25, 26] for extensions to higher order tensors. For different approaches to the inversion of the attenuated X-ray transform from complete data we refer to the original work in [18, 19], and further developments in [15, 6, 4, 12, 14].

The unique determination result here follows from a Carleman type formula for A-analytic maps as in [1, 2]. The novelty of this work is in the explicit Carleman weight-operator, see

Date: January 31, 2021.

2010 *Mathematics Subject Classification.* Primary 35J56, 30E20; Secondary 45E05.

Key words and phrases. Attenuated X-ray transform, Attenuated Radon transform, partial data, A-analytic maps, Hilbert transform.

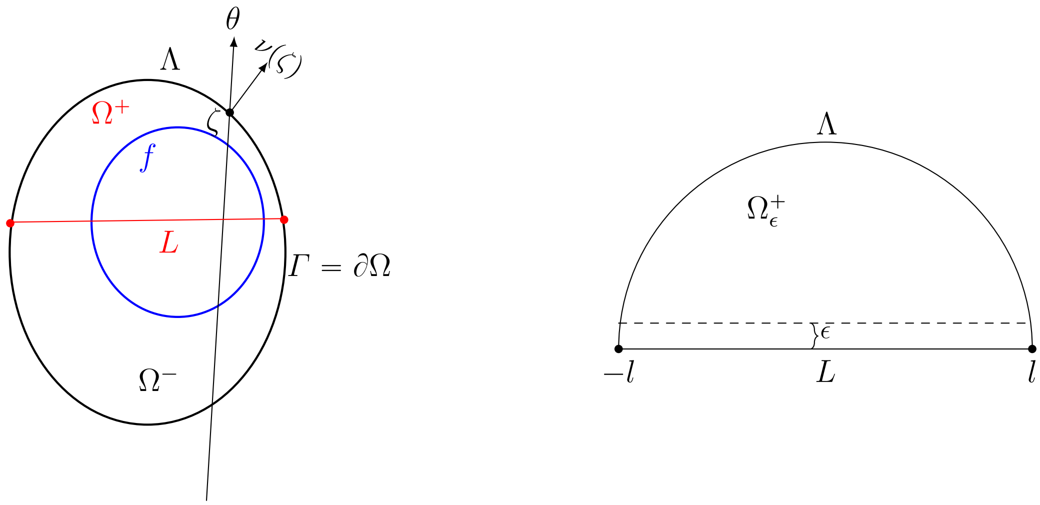

FIGURE 1. (left) Geometric setup: $\partial\Omega^+ = \Lambda \cup L$. (right) Domain Ω_ϵ^+.

equation (6) below, specifically tailored for the convex hull of the arc Λ. The arguments here have been recently refined by the authors to yield a reconstruction method in [10] via the explicit Bukhgeim-Cauchy operator in [9].

The X-ray transform of f is given by,

$$Xf(z,\theta) := \int_{-\infty}^{\infty} f(z+s\theta)ds, \quad (z,\theta) \in \Omega \times \mathbf{S}^1. \tag{1}$$

The reconstruction of f from its ray data is approached through the known equivalence between the X-ray transform and the boundary value problems for the transport equation: Let $\varGamma_\pm := \{(\zeta,\theta) \in \varGamma \times \mathbf{S}^1 : \pm\nu(\zeta)\cdot\theta > 0\}$ denote the outgoing $(+)$, respectively incoming $(-)$ submanifolds of the unit tangent bundle of $\varGamma$, with $\nu(\zeta)$ being the outer normal at $\zeta \in \varGamma$ and θ is a direction in the unit sphere $\mathbf{S}^1$. If $u(z,\theta)$ is the unique solution to

$$\theta \cdot \nabla u(z,\theta) = f(z) \quad (z,\theta) \in \Omega \times \mathbf{S}^1, \tag{2a}$$

$$u|_{\varGamma_-} = 0, \tag{2b}$$

then its trace on $\varGamma_+$ satisfies

$$u|_{\varGamma_+}(\zeta,\theta) = Xf(\zeta,\theta), \quad (\zeta,\theta) \in \varGamma_+. \tag{3}$$

In our problem here the data Xf is only available on

$$\Lambda_\pm := \{(\zeta,\theta) \in \Lambda \times \mathbf{S}^1 : \pm\nu(\zeta)\cdot\theta > 0\}.$$

Upon a rotation and translation of the domain Ω, we assume without loss of generality that the arc Λ lies in the upper half plane with the endpoints on the real axis lying symmetrically about the origin. In particular, $\Omega \cap \{\mathbb{I}\mathrm{m}\, z = 0\} = L = (-l, l)$, for some $l > 0$. For $\epsilon > 0$, define

$$\Omega_\epsilon^+ = \{z \in \Omega : \mathbb{I}\mathrm{m}\, z > \epsilon\}; \tag{4}$$

see Figure 1 on the right.

2. A CARLEMAN TYPE FORMULA FOR $\mathcal{L}$-ANALYTIC MAPS IN Ω^+

In this section we breifly recall some known properties of A-analytic functions, on which our reconstruction method is based, and present an explicit Carleman weight operator tailored for Ω^+_ϵ. For $z = x + \mathrm{i}y$, let $\overline{\partial} = (\partial_x + \mathrm{i}\partial_y)/2$, and $\partial = (\partial_x - \mathrm{i}\partial_y)/2$ be the Cauchy-Riemann operators.

A sequence valued map $\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), ...\rangle$ in $C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ is called $\mathcal{L}$-*analytic*, if

$$\left[\overline{\partial} + \mathcal{L}\partial\right]\mathbf{u}(z) = 0, \quad z \in \Omega, \tag{5}$$

where $\mathcal{L}$ is the left shift operator, $\mathcal{L}\langle u_0, u_{-1}, u_{-2}, \cdots\rangle = \langle u_{-1}, u_{-2}, \cdots\rangle$, and l_∞ is the space of bounded sequences. Note that we use the sequences of non-positive indexes to conform with the notation in Bukhgeim's original work [8].

Unique determination of f follows via a Carleman type formula as in [1], provided a suitable quenching function is known. In here we made explicit such a function tailored for the subdomain Ω^+. More precisely, for $\lambda > 0$, we consider the Carleman weight operator function

$$\Phi_\lambda(z) = e^{-\mathrm{i}\lambda z} e^{\mathrm{i}\lambda \overline{z}\mathcal{L}}, \quad z \in \overline{\Omega^+}, \tag{6}$$

and its inverse $\Phi_\lambda^{-1}(z) = \Phi_\lambda(-z)$. By direct computation, one can check that Φ_λ satisfies the operator valued equation

$$\overline{\partial}\Phi_\lambda(z) + \mathcal{L}\partial\Phi_\lambda(z) = e^{-\mathrm{i}\lambda z} e^{\mathrm{i}\lambda\overline{z}\mathcal{L}}\mathrm{i}\lambda\mathcal{L} + \mathcal{L}(-\mathrm{i}\lambda)e^{-\mathrm{i}\lambda z}e^{\mathrm{i}\lambda\overline{z}\mathcal{L}} = 0.$$

Consequently, if $\mathbf{u}(z)$ is $\mathcal{L}$-analytic in Ω^+, then $\Phi_\lambda(z)\mathbf{u}(z)$ is also $\mathcal{L}$-analytic in Ω^+, so its values can be determined from the boundary $\partial\Omega^+ = \Lambda \cup L$ by

$$\Phi_\lambda(z)\mathbf{u}(z) = \frac{1}{2\pi\mathrm{i}}\int_{\Lambda\cup L}(d\zeta - \mathcal{L}d\overline{\zeta})G(\zeta - z)\Phi_\lambda(\zeta)\mathbf{u}(\zeta),$$

where $G(z) = (z - \mathcal{L}\overline{z})^{-1}$ is the Green kernel for the differential operator in (5); see [8]. By using the commutating properties $[\Phi_\lambda^{-1}(z), \Phi_\lambda(\zeta)] = 0$, and $[\Phi_\lambda^{-1}(z), \mathcal{L}] = 0$, for $z \in \Omega^+$ and $\zeta \in \Lambda \cup L$, we obtain

$$\mathbf{u}(z) = \frac{1}{2\pi\mathrm{i}}\int_{\Lambda\cup L}(d\zeta - \mathcal{L}d\overline{\zeta})G(\zeta - z)\Phi_\lambda(\zeta - z)\mathbf{u}(\zeta) \tag{7}$$

We consider for $s \in (0, 1)$, the following space

$$l^{2,s}_\infty(\overline{\Omega}) := \left\{\mathbf{u} = \langle u_{-1}, u_{-2}, ...\rangle : \sup_{\xi\in\overline{\Omega}}\sum_{j=1}^{\infty} s^{-2j}|u_{-j}(\xi)|^2 < \infty\right\}.$$

The left shift operator $\mathcal{L} : l^{2,s}_\infty \to l^{2,s}_\infty$ is bounded, and the operator norm $\|\mathcal{L}\| = s$.

Theorem 2.1. *For $\epsilon > 0$, let $\Omega^+_\epsilon \subset \mathbb{R}^2$ be the subdomain in* (4)*, d be diameter of Ω^+, and let $s < \dfrac{\epsilon}{d}$. If $\mathbf{u}$ is $\mathcal{L}$-analytic in Ω^+ with $\mathbf{u}|_L \in l^{2,s}_\infty(L)$, then it is uniquely determined by its trace on Λ by*

$$\mathbf{u}(z) = \lim_{\lambda\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{\Lambda}(d\zeta - \mathcal{L}d\overline{\zeta})G(\zeta - z)\Phi_\lambda(\zeta - z)\mathbf{u}(\zeta), \quad z \in \Omega^+_\epsilon, \tag{8}$$

where Φ_λ is the Carleman weight operator in (6)*.*

Proof. For $\lambda > 0$, we consider the Carleman weight operator function Φ_λ in (6). We argue that the integral over the segment L in (7) vanishes in the limit with $\lambda \to \infty$,

$$\lim_{\lambda\to\infty}\int_L (d\zeta - \mathcal{L}d\overline{\zeta})G(\zeta - z)\Phi_\lambda(\zeta - z)\mathbf{u}(\zeta) = 0. \tag{9}$$

For $z = x + iy \in \Omega_\epsilon^+$ and $\zeta \in L$, we have $\epsilon \leqslant |\overline{z} - \overline{\zeta}| \leqslant d$ and $\left|e^{i\lambda(z-\zeta)}\right| = e^{-\lambda y} \leqslant e^{-\lambda\epsilon}$. If $\|\cdot\|$ denotes the operator norm in $l^{2,s}$, we obtain for any $z \in \Omega^\epsilon$ that

$$\sup_{\zeta\in L}\|\Phi_\lambda(\zeta - z)\| = \|e^{\mathrm{i}\lambda(z-\zeta)}e^{-\mathrm{i}\lambda(\overline{z}-\overline{\zeta})\mathcal{L}}\| \leqslant e^{-\lambda(\epsilon - sd)}.$$

Since $s < \dfrac{\epsilon}{d}$, by letting $\lambda \to \infty$, we conclude (9) □

The source f is recovered in Ω_ϵ^+, by

$$f(z) = 2\,\mathbb{R}\mathrm{e}\{\partial u_{-1}(z)\},$$

where u_{-1} is the first component in (8). The reconstruction to Ω^+ can be completed by a layer stripping argument.

Acknowledgment

This work was triggered by Prof. H. Kudo's talk at the workshop on "Recent developments on Inverse Problems for PDEs and their applications", Jan 6-8, 2021 organized and supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The work of H. Fujiwara was supported by JSPS KAKENHI Grant Numbers JP16H02155 and JP20H01821. The work of K. Sadiq was supported by the Austrian Science Fund (FWF), Project P31053-N32. The work of A. Tamasan was supported in part by the NSF grant DMS-1907097.

References

[1] E. V. Arbuzov and A. L. Bukhgeim, *Carleman's formulas for A-analytic functions in a half-plane*, J. Inv. Ill-Posed Problems, **5(6)**, (1997), 491–505.

[2] E. V. Arbuzov and A. L. Bukhgeim, *Carleman's formulas for the Laplace and Poisson equations with operator coefficients*, J. Inv. Ill-Posed Problems, **9(4)**, (2001), 319–326.

[3] E. V. Arbuzov, A. L. Bukhgeim and S. G. Kazantsev, *Two-dimensional tomography problems and the theory of A-analytic functions*, Siberian Adv. Math., **8** (1998), 1–20.

[4] G. Bal, *On the attenuated Radon transform with full and partial measurements*, Inverse Problems **20** (2004), 399–418.

[5] J. Boman, *On stable inversion of the attenuated Radon transform with half data*, Integral geometry and tomography, Contemp. Math., **405**, Amer. Math. Soc., Providence, RI, (2006), 19–26.

[6] J. Boman and J.-O. Strömberg, *Novikov's inversion formula for the attenuated Radon transform–a new approach*, J. Geom. Anal., **14** (2004), 185–198.

[7] J. Boman and E. T. Quinto, *Support theorems for real-analytic Radon transforms*, Duke Math. J., **55** (1987), no.4, 943–948.

[8] A. L. Bukhgeim, Inversion Formulas in Inverse Problems, in Linear Operators and Ill-Posed Problems by M. M. Lavrentiev and L. Ya. Savalev, Plenum, New York, 1995.

[9] D. V. Finch, *The attenuated x-ray transform: recent developments*, in Inside out: inverse problems and applications, Math. Sci. Res. Inst. Publ., **47**, Cambridge Univ. Press, Cambridge, 2003, 47–66.

[10] H. Fujiwara, K. Sadiq and A. Tamasan, *Partial inversion of the 2D attenuated X-ray transform with data on an arc*, 2021, under review.

[11] S. Helgason, The Radon Transform, Birkhäuser, Boston, 1999.

[12] S. G. Kazantsev and A. A. Bukhgeim, *Inversion of the scalar and vector attenuated X-ray transforms in a unit disc*, J. Inverse Ill-Posed Probl., **15** (2007), 735–765.
[13] P. Kuchment and I. Shneiberg, *Some inversion formulae in the single photon emission computed tomography*, Appl. Anal., **53** (1994), 221–231.
[14] F. Monard, *Efficient tensor tomography in fan-beam coordinates. II: Attenuated transforms*, Inverse Probl. Imaging, **12(2)** (2018), 433–460.
[15] F. Natterer, *Inversion of the attenuated Radon transform*, Inverse Problems **17** (2001), 113–119.
[16] F. Noo, M. Defrise, J. D. Pack, and R. Clackdoyle, *Image reconstruction from truncated data in single-photon emission computed tomography with uniform attenuation*, Inverse Problems **23**(2007), 645–667.
[17] F. Noo, and J. M. Wagner, *Image reconstruction in 2D SPECT with* 180° *acquisition*, Inverse Problems **17**(2001), 1357–1371.
[18] R. G. Novikov, *Une formule d'inversion pour la transformation d'un rayonnement X atténué*, C. R. Acad. Sci. Paris Sér. I Math., **332** (2001), 1059–1063.
[19] R. G. Novikov, *On the range characterization for the two-dimensional attenuated x-ray transformation*, Inverse Problems **18** (2002), no. 3, 677–700.
[20] X. Pan, C. Kao and C. Metz, *A family of* π*-scheme exponential Radon transforms and the uniqueness of their inverses*, Inverse Problems **18** (2002), 825–836.
[21] J. Radon, *Uber die Bestimmung von Funktionen durch ihre Integralwerte langs gewisser Mannigfaltigkeiten*, Ber. Verh. Sachs. Akad. Wiss. Leipzig, Math-Nat., **K** 1 **69** (1917), 262–267.
[22] H. Rullgård, *An explicit inversion formula for the exponential Radon transform using data from* 180°, Ark. Mat., **42** (2004), 353–362.
[23] H. Rullgård, *Stability of the inverse problem for the attenuated Radon transform with* 180° *data*, Inverse Problems **20** (2004), 781–797.
[24] K. Sadiq and A. Tamasan, *On the range of the attenuated Radon transform in strictly convex sets*, Trans. Amer. Math. Soc., **367(8)** (2015), 5375–5398.
[25] K. Sadiq and A. Tamasan, *On the range characterization of the two dimensional attenuated Doppler transform*, SIAM J. Math.Anal., **47(3)** (2015), 2001–2021.
[26] K. Sadiq, O. Scherzer, and A. Tamasan, *On the X-ray transform of planar symmetric 2-tensors*, J. Math. Anal. Appl., **442(1)** (2016), 31–49.

GRADUATE SCHOOL OF INFORMATICS, KYOTO UNIVERSITY, YOSHIDA HONMACHI, SAKYO-KU, KYOTO 606-8501, JAPAN
Email address: fujiwara@acs.i.kyoto-u.ac.jp

JOHANN RADON INSTITUTE OF COMPUTATIONAL AND APPLIED MATHEMATICS (RICAM), ALTENBERGERSTRASSE 69, 4040 LINZ, AUSTRIA
Email address: kamran.sadiq@ricam.oeaw.ac.at

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, 32816 FLORIDA, USA
Email address: tamasan@math.ucf.edu