

Presentations for cusped arithmetic hyperbolic lattices

Alice Mark, Julien Paupert*

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Abstract

We present a general method to compute a presentation for any cusped arithmetic hyperbolic lattice Γ , applying a classical result of Macbeath to a suitable Γ -invariant horoball cover of the corresponding symmetric space. As applications we compute presentations for the Picard modular groups $\mathrm{PU}(2, 1, \mathcal{O}_d)$ for $d = 1, 3, 7$ and the quaternion hyperbolic lattice $\mathrm{PU}(2, 1, \mathcal{H})$ with entries in the Hurwitz integer ring \mathcal{H} . The implementation of the method for these groups is computer-assisted.

1 Introduction

Discrete subgroups and lattices in semisimple Lie groups form a rich and well-studied class of finitely generated groups acting on non-positively curved metric spaces. The case of real rank one, where the associated symmetric space is negatively curved, is of special interest. There are essentially two main families of constructions of such lattices, arithmetic on one hand and geometric on the other. Arithmetic lattices are roughly speaking obtained by taking matrices with entries lying in the integer ring of some number field; the general definition is more complicated and we will not give it here, as the arithmetic lattices that we consider in this paper are of this simplest type. By Margulis' celebrated superrigidity and arithmeticity theorems, all (irreducible) lattices in G are of this arithmetic type when G is a semisimple Lie group of real rank at least 2.

The other family involves geometric constructions such as polyhedra, reflections or other types of involutions or other finite-order isometries. A prototype of this type of construction is given by Coxeter groups in the constant curvature geometries E^n , S^n and H^n , which are generated by reflections across hyperplanes. These groups are classical and were classified by Coxeter in the spaces E^n and S^n , whereas their hyperbolic counterparts (studied by Vinberg and others) are still not completely understood. However by construction these groups come equipped with data including a presentation (as an abstract Coxeter group) and a fundamental domain for their action on the symmetric space.

Arithmetic lattices are given by a global description and their global structure is in some sense well understood by work of Siegel, Borel, Tits, Prasad and others. However concrete information such as a presentation and a fundamental domain are not readily accessible from the arithmetic construction. One can obtain geometric information such as volume by Prasad's celebrated volume formula ([Pr]) but computing the constants appearing in this formula usually involves some non-trivial work (see for example [Be] and [Sto]).

Very few presentations of arithmetic lattices, and of lattices in general, are known. Presentations can provide useful geometric and algebraic information about groups, such as explicit index of torsion-free subgroups (effective Selberg lemma, as used for example in [Sto]), cohomology of the group Γ or quotient space X/Γ , see for instance [Y] (for the Picard modular groups with $d = 1, 3$) and of course representations of Γ , for instance if one is interested in deformations of Γ into a larger Lie group.

Presentations for $\mathrm{SL}(n, \mathbb{Z})$ with $n \geq 3$ were given by Steinberg ([Ste], following Magnus); the case of $\mathrm{SL}(2, \mathbb{Z})$ is classical and possibly dates to Gauss; see also Siegel [Si]. In rank one, Swan gave in [Sw] presentations for the Bianchi groups $\mathrm{PGL}(2, \mathcal{O}_d)$ (where \mathcal{O}_d denotes the ring of integers of $\mathbb{Q}[i\sqrt{d}]$ for d a positive square-free integer), following Bianchi's original construction in [Bi]. These act as isometries of (real) hyperbolic 3-space, as they are lattices in $\mathrm{PGL}(2, \mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3_{\mathbb{R}})$.

Presentations for the related Picard modular groups $\mathrm{PU}(2, 1, \mathcal{O}_d)$ were found only recently in the simplest cases of $d = 3$ ([FP]) and $d = 1$ ([FFP]). One of the reasons for this is that the associated symmetric space,

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complex hyperbolic 2-space $H_{\mathbb{C}}^2$, is more complicated and in particular has non-constant (pinched) negative curvature. A particular feature of such spaces, the absence of totally geodesic real hypersurfaces, makes constructions of fundamental domains difficult as there are no obvious walls to use to bound such domains. The presentations obtained for $d = 1, 3$ were in fact obtained by constructing fundamental domains and using the Poincaré polyhedron theorem. This approach seems to become too complicated when considering more complicated groups, such as Picard modular groups with higher values of d , and no further such constructions have appeared. Using a similar strategy, Zhao gave in [Zh] generating sets for the Picard modular groups with $d = 1, 2, 3, 7, 11$ but he does not go as far as obtaining a presentation, finding a set whose translates covers the space but without control over intersections and cycles. (We will in fact use a covering argument closely related to the one he uses to cover a fundamental prism on the ideal boundary by isometric spheres, see Lemma 13).

In this paper we present a method for obtaining presentations for cusped hyperbolic lattices, i.e. non-cocompact lattices in semisimple Lie groups of real rank one, based on a classical result of Macbeath (Theorem 1 below) which gives a presentation for a group Γ acting by homeomorphisms on a topological space X , given an open subset V whose Γ -translates cover X . We apply this by finding a suitable horoball V based at a cusp point of Γ whose Γ -translates cover X , then analyzing the triple intersections and associated cycles to obtain a presentation for Γ . The main tools for this analysis come from the additional arithmetic structure that we get by assuming that Γ is in fact an *integral lattice* in the sense that it is contained in $GL(n+1, \mathcal{O}_E)$ for some number field E (or finitely generated division algebra over \mathbb{Q}). The crucial such tool that we use is the notion of *level* between two E -rational boundary points in $\partial_{\infty} X$ (see Definition 1) which gives a notion of distance between such points using only algebraic data. More importantly for us, levels measure the relative sizes of horospheres based at the corresponding boundary points, which allows us to control whether or not such horospheres intersect at a given height (see Lemma 4).

As applications of this method we compute presentations for the Picard modular groups $PU(2, 1, \mathcal{O}_d)$ with $d = 3, 1, 7$, given in the appendix and Propositions 2, 3 respectively. The cases $d = 2, 11$ can be treated with the same method but are computationally more intensive; David Polletta has treated these in [Po]. We also compute a presentation for the quaternion hyperbolic lattice which we call the *Hurwitz modular group* $PU(2, 1, \mathcal{H})$, where \mathcal{H} is the ring of Hurwitz integers $\mathcal{H} = \mathbb{Z}[i, j, k, \frac{1+i+j+k}{2}] \subset \mathbb{H}$. This is a lattice in $PU(2, 1, \mathbb{H})$ (also denoted $PSp(2, 1)$), acting on the 8-dimensional symmetric space $H_{\mathbb{H}}^2$. As far as we know this is the first presentation ever found for a higher-dimensional quaternion hyperbolic lattice. (In dimension 1, $H_{\mathbb{H}}^1 \simeq H_{\mathbb{R}}^4$ and such groups have been studied e.g. in [DVV], see also [A], [W] and [Ph]). The implementation of our method for this group turned out to be computationally much more intensive than anticipated, and in fact the resulting presentation is too large for many purposes (see Section 5). It does however allow us to compute the abelianization of the lattice, and to find a nice generating set, see Theorems 2 and 3.

Using a somewhat similar principle Cartwright and Steger found presentations for some cocompact arithmetic lattices in $PU(2, 1)$ in their classification of the so-called fake projective planes, see [CaS1] and [CaS2].

The paper is organized as follows. In Section 2 we discuss generalities about horoball coverings of hyperbolic spaces, levels for cusp points of integral lattices, and outline how we apply Macbeath's theorem in this context. In Section 3 we discuss horosphere intersections in more detail, in particular the quantitative relation between levels and heights of horospheres for integral lattices. In Sections 4 and 5 we apply this method to compute presentations for the Picard and Hurwitz modular groups respectively.

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2 Horoball coverings and lattice presentations

2.1 Adapted horoball coverings and covering complex

Let X be a negatively curved symmetric space, i.e. a hyperbolic space $H_{\mathbb{K}}^n$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} (and $n \geq 2$ if $\mathbb{K} = \mathbb{R}$, $n = 2$ if $\mathbb{K} = \mathbb{O}$). We refer the reader to [CG] for general properties of these spaces and their isometry groups. In particular isometries of such spaces are roughly classified into the following 3 types: *elliptic* (having

a fixed point in X), *parabolic* (having no fixed point in X and exactly one on $\partial_\infty X$) or *loxodromic* (having no fixed point in X and exactly two on $\partial_\infty X$).

Let Γ be a lattice in $\text{Isom}(X)$; the well-known Godement compactness criterion states that Γ contains parabolic isometries if and only if it is non-cocompact, which we now assume. A *cusp point* of Γ is a point of $\partial_\infty X$ fixed by a parabolic element of Γ ; a *cusp group* of Γ is a subgroup of the form $\text{Stab}_\Gamma(p)$ where $p \in \partial_\infty X$ is a cusp point of Γ .

Assume that we are given a Γ -invariant covering of X by (open) horoballs (see Definition 3), i.e. a collection \mathcal{B} of horoballs such that:

$$\begin{cases} \gamma B \in \mathcal{B} \text{ for all } \gamma \in \Gamma \text{ and } B \in \mathcal{B} \\ \bigcup_{B \in \mathcal{B}} B = X \end{cases}$$

We will moreover assume that each horoball $B \in \mathcal{B}$ is based at a cusp point of Γ , and that each cusp point of Γ is the basepoint of a unique horoball in \mathcal{B} (giving a bijection between cusp points of Γ and horoballs in \mathcal{B}); we will call such a covering \mathcal{B} a *Γ -adapted horoball covering*. Since the lattice Γ has only finitely many cusp points modulo the action of Γ , it follows that such a horoball covering is a finite union of Γ -orbits of horoballs.

Given a Γ -adapted horoball covering \mathcal{B} , the *covering complex* $\mathcal{C}(\mathcal{B})$ associated to \mathcal{B} is the simplicial 2-complex with vertex set \mathcal{B} , with an edge connecting each pair of vertices B_1 and B_2 such that $B_1 \cap B_2 \neq \emptyset$, and a triangle for each triple of vertices B_1, B_2, B_3 such that $B_1 \cap B_2 \cap B_3 \neq \emptyset$. This the 2-skeleton of a simplicial complex sometimes called the nerve of the covering. By the above remark the quotient of the covering complex by the action of Γ is a finite simplicial 2-complex.

We will use the following classical result of Macbeath (M):

Theorem 1 (M) *Let Γ be a group acting by homeomorphisms on a topological space X . Let V be an open subset of X whose Γ -translates cover X .*

- (1) *If X is connected then the set $E(V) = \{\gamma \in \Gamma \mid V \cap \gamma V \neq \emptyset\}$ generates Γ .*
- (2) *If moreover X is simply-connected and V path-connected, then Γ admits a presentation with generating set $E(V)$ and relations $\gamma \cdot \gamma' = \gamma \gamma'$ for all $\gamma, \gamma' \in E(V)$ such that $V \cap \gamma V \cap \gamma \gamma' V \neq \emptyset$.*

We kept the notation from M; to clarify the notation in part (2), let $S = \{e_\gamma \mid \gamma \in E(V)\}$ be a set labelled by the generating set $E(V)$ from part (1). The claim is that Γ has presentation $\langle S \mid R \rangle$, where R consists of the relations $e_\gamma \cdot e_{\gamma'} = e_{\gamma \gamma'}$ for all $\gamma, \gamma' \in E(V)$ such that $V \cap \gamma V \cap \gamma \gamma' V \neq \emptyset$. Note that the latter condition implies that $\gamma \gamma' \in E(V)$.

Now if as above Γ is a lattice in $\text{Isom}(X)$ and \mathcal{B} is a Γ -adapted horoball covering of X , we may as remarked above write \mathcal{B} as a finite union of Γ -orbits of horoballs B_1, \dots, B_k (say, minimally). One can then apply Macbeath's theorem with $V = B_1 \cup \dots \cup B_k$, after possibly enlarging each horoball B_i in order for this union to be (path-)connected. For simplicity of exposition, we henceforth assume that Γ has a single cusp, so that the Γ -adapted horoball covering consists of a single Γ -orbit of horoballs (this is the case in all examples considered in this paper). In that case the process of obtaining a presentation from the covering complex is closely related to a complex of groups structure on the quotient of the covering complex, the only difference being that we need to take into account non-trivial edge and face stabilizers.

2.2 Levels and proximal cusp complex

Recall that if X is a hyperbolic space $H_{\mathbb{K}}^n$ (with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H}) then X admits the following *projective model* which we briefly recall.

Consider $\mathbb{K}^{n,1}$, the vector space \mathbb{K}^{n+1} endowed with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n, 1)$. (When $\mathbb{K} = \mathbb{H}$ we will use the convention that scalars act on vectors on the right, whereas matrices act on vectors on the left.) Let $V^- = \{Z \in \mathbb{K}^{n,1} \mid \langle Z, Z \rangle < 0\}$, $V^0 = \{Z \in \mathbb{K}^{n,1} \mid \langle Z, Z \rangle = 0\}$ and let $\pi : \mathbb{K}^{n+1} - \{0\} \rightarrow \mathbb{K}\mathbb{P}^n$ denote projectivization. One then defines $H_{\mathbb{K}}^n$ to be $\pi(V^-) \subset \mathbb{K}\mathbb{P}^n$, endowed with the distance d (Bergman metric) given by, for $Z, W \in V^-$:

$$\cosh^2\left(\frac{d(\pi(Z), \pi(W))}{2}\right) = \frac{|\langle Z, W \rangle|^2}{\langle Z, Z \rangle \langle W, W \rangle}. \quad (1)$$

Note that the right-hand side is independent of the choice of lifts Z, W . Then $\text{Isom}^0(X) = \text{PU}(n, 1, \mathbb{K})$, the (projectivization of) the matrix group preserving the Hermitian form (see [CG]). Note that $\text{PU}(n, 1, \mathbb{K})$ is usually denoted $\text{PO}(n, 1)$ when $\mathbb{K} = \mathbb{R}$, $\text{PU}(n, 1)$ when $\mathbb{K} = \mathbb{C}$ and $\text{PSp}(n, 1)$ when $\mathbb{K} = \mathbb{H}$. The boundary at infinity $\partial_\infty X$ is then identified with $\pi(V^0) \subset \mathbb{K}P^n$. We would like to measure distances between points of $\partial_\infty X$ using the Hermitian form as in [1]; one way to do this is to use integral lifts of vectors with rational coordinates as follows.

We now assume that Γ is an *integral lattice* in the sense that it is contained in $\text{U}(H, \mathcal{O}_E)$ for some number field E (or finite-degree division algebra over \mathbb{Q} when $\mathbb{K} = \mathbb{H}$) with ring of integers \mathcal{O}_E , and Hermitian form $H = \langle \cdot, \cdot \rangle$ defined over E . We say that an integral vector $P_0 = (p_1, \dots, p_{n+1}) \in \mathcal{O}_E^{n+1}$ is *primitive* if it has no integral submultiple in the following sense: if $P_0\lambda^{-1} \in \mathcal{O}_E^{n+1}$ for some $\lambda \in \mathcal{O}_E$ then λ is a unit in \mathcal{O}_E .

If p is an E -rational point in $\mathbb{K}P^n$, i.e. the projective image of a vector $P = (p_1, \dots, p_{n+1}) \in E^{n+1}$, a *primitive integral lift* of p is any lift P_0 of p to \mathcal{O}_E^{n+1} which is a primitive integral vector.

Lemma 1 *If \mathcal{O}_E is a principal ideal domain then primitive integral lifts are unique up to multiplication by a unit.*

Lemma 2 *(a) Any column-vector of a matrix $A \in \text{U}(H, \mathcal{O}_E)$ is a primitive integral vector. (b) If moreover E is imaginary quadratic with \mathcal{O}_E a principal ideal domain and one of the standard basis vectors B_i is H -isotropic, then for any H -isotropic primitive integral vector V and $A \in \text{U}(H, \mathcal{O}_E)$, AV is a primitive integral vector.*

Proof. (a) Let $A \in \text{U}(H, \mathcal{O}_E)$ and V a column-vector of A . Then V is integral; assuming that it is not primitive, there would exist a non-unit $\lambda \in \mathcal{O}_E$ such that $V\lambda^{-1}$ is also integral. But then the matrix A' obtained from A by replacing the column-vector V by $V\lambda^{-1}$ would also be in $\text{GL}(n+1, \mathcal{O}_E)$, with $\det A' = \det A\lambda^{-1}$, a contradiction since the latter is not an integer, as $\det A$ is a unit and λ is not.

(b) Let V be an H -isotropic primitive integral vector and $A \in \text{U}(H, \mathcal{O}_E)$. If \mathcal{O}_E is a principal ideal domain then $\text{PU}(H, \mathcal{O}_E)$ has a single cusp (see [Zi]), therefore there exists $M \in \text{U}(H, \mathcal{O}_E)$ mapping B_i to $AV\lambda$ for some $\lambda \in E$. Then as in (a) λ must be a unit, hence AV is a column-vector of $M\lambda^{-1} \in \text{U}(H, \mathcal{O}_E)$ and we conclude by (a). \square

Definition 1 *Given two E -rational points $p, q \in \partial_\infty X$, the level between p and q , denoted $\text{lev}(p, q)$, is $|\langle P_0, Q_0 \rangle|^2$ for any two primitive integral lifts P_0, Q_0 of p, q respectively. When we are given a preferred E -rational point $\infty \in \partial_\infty X$, the depth of an E -rational point $p \in \partial_\infty X$ is the level between p and ∞ .*

By Lemma 1 this is well-defined when \mathcal{O}_E is a principal ideal domain. The *proximal cusp complex of level n* , denoted $\mathcal{C}_n(\Gamma)$, is the complex whose vertices are cusp points of Γ , with an edge connecting 2 vertices p, q whenever $\text{lev}(p, q) \leq n$, and a triangle for each triple of distinct edges.

Levels give a convenient way to distinguish orbits of edges and triangles in the covering complex, by the following observation which follows from Lemmas 1 and 2.

Lemma 3 *If \mathcal{O}_E is a principal ideal domain, for any two E -rational points $p, q \in \partial_\infty X$ and $\gamma \in \text{U}(H, \mathcal{O}_E)$, $\text{lev}(\gamma p, \gamma q) = \text{lev}(p, q)$.*

More importantly, levels allow us to find the optimal height u of a horosphere $H_u = \partial B_u$ based at a preferred E -rational point $\infty \in \partial_\infty X$ such that the orbit ΓB_u covers X . This relies on the following result, which is part of Corollary 1 in Section 3:

Lemma 4 *There exists a decreasing function $u : \mathbb{N} \rightarrow \mathbb{R}$ such that, for any E -rational point $p \in \partial_\infty X$ with depth n , and any (integral) $A_p \in \Gamma$ satisfying $A_p(\infty) = p$, the set $H_u \cap A_p(H_u)$ is empty if and only if $u > u(n)$.*

In fact we will see in Corollary 1 that the function u is given by $u(n) = \frac{2}{\sqrt{n}}$.

Definition 2 *The covering depth of Γ is the unique $n \in \mathbb{N}$ such that $\frac{2}{\sqrt{n+1}} < u^{\text{cov}} \leq \frac{2}{\sqrt{n}}$, where u^{cov} denotes the maximal height such that $\Gamma B_{u^{\text{cov}}}$ covers X .*

In practice, for the purpose of finding a presentation of Γ , we will not need to explicitly determine the covering depth or u^{cov} . It will suffice to bound the covering depth from above, and use the covering of X at the corresponding height to apply Macbeath's theorem.

2.3 Reduction modulo the vertex stabilizer Γ_∞

We choose a preferred cusp point $\infty \in \partial_\infty X$ of Γ (in general we will take $\infty = \pi([1, 0, \dots, 0]^T)$ in the Siegel model, see section 3), and consider the cusp stabilizer $\Gamma_\infty = \text{Stab}_\Gamma(\infty)$. Since Γ is a lattice, it is well known that Γ_∞ acts cocompactly on all horospheres based at ∞ . Let n denote the covering depth of Γ and u^{cov} the corresponding covering height, so that the Γ -translates of the horoball B_u cover X , and let $D_\infty \subset H_u$ be a compact fundamental domain for the action of Γ_∞ on $H_u \simeq \partial_\infty X \setminus \{\infty\}$. In practice we will choose D_∞ to be an affinely convex polytope in Heisenberg coordinates (see section 3).

Assume that we are given a finite presentation $\Gamma_\infty = \langle S_\infty | R_\infty \rangle$. Then we may reduce the procedure in Macbeath's theorem to finitely many additional generators and relations as follows. Let $\{p_1, \dots, p_k\}$ denote the E -rational points with depth at most n in D_∞ , and assume for simplicity that they are ordered in such a way that the first r of them form a system of representatives under the action of Γ_∞ . Assume moreover that we have found for each $i = 1, \dots, r$ an element $A_i \in \Gamma$ such that $A_i(\infty) = p_i$ (this is possible in principle since Γ is assumed to have a single cusp).

Generators: The group Γ is generated by $\{S_\infty, A_1, \dots, A_r\}$. This follows easily from part (1) of Macbeath's theorem and Lemma 4, as any E -rational point of $\partial_\infty X$ with depth at most n is in the Γ_∞ -orbit of one of p_1, \dots, p_r . Note that with the notation from the Theorem, we are using the open set $V = B_u = B$ to cover X , and $E(B) = \{\gamma \in \Gamma \mid B \cap \gamma B \neq \emptyset\} = \Gamma_\infty \{A_1, \dots, A_r\} \Gamma_\infty = \{\gamma_\infty^1 A_i \gamma_\infty^2 \mid \gamma_\infty^1, \gamma_\infty^2 \in \Gamma_\infty, i = 1, \dots, r\}$. Indeed, by Lemma 4, $B \cap \gamma B \neq \emptyset$ if and only if $\gamma \infty$ is either ∞ or an E -rational point of depth at most n , which is a Γ_∞ -translate of one of p_1, \dots, p_r .

Relations: We now rephrase part (2) of Macbeath's theorem in this context. Let $\gamma, \gamma' \in E(B)$ satisfy $B \cap \gamma B \cap \gamma \gamma' B \neq \emptyset$, and first assume that $\gamma \infty \neq \infty$ and $\gamma \gamma' \infty \neq \infty$. After conjugating by an element of Γ_∞ we may assume that $\gamma = A_a$, $\gamma' = \gamma_\infty^1 A_b \gamma_\infty^2$, $\gamma \gamma' = \gamma_\infty^3 A_c \gamma_\infty^4$ for some $a, b, c \in \{1, \dots, r\}$ and $\gamma_\infty^1, \dots, \gamma_\infty^4 \in \Gamma_\infty$. The corresponding relation $\gamma \gamma' = \gamma \cdot \gamma'$ is then: $A_a \gamma_\infty^1 A_b \gamma_\infty^2 = \gamma_\infty^3 A_c \gamma_\infty^4$. Taking the image of ∞ under both sides of this relation gives: $A_a(\gamma_\infty^1 p_b) = \gamma_\infty^3 p_c$.

In practice this is how we will detect the relations, finding which points of depth at most n are sent to points of depth at most n by the generators A_a . One then recovers the relation as follows. For each triple (a, b, c) for which there exist $\gamma_\infty^1, \gamma_\infty^3 \in \Gamma_\infty$ such that $A_a(\gamma_\infty^1 p_b) = \gamma_\infty^3 p_c$, we obtain a relation $R_{a,b,c}$ by identifying the element $A_c^{-1}(\gamma_\infty^3)^{-1} A_a \gamma_\infty^1 A_b \in \Gamma_\infty$ as a word in the generators S_∞ . The key point is that there are only finitely many such triples (a, b, c) to be checked, by Lemma 6.

Now assume that one of $\gamma \infty$, $\gamma' \infty$ is ∞ but not both (as the relations in Γ_∞ have already been considered). The corresponding relation can be obtained as above, using the point $p_\infty = \infty$ with corresponding group element $A_\infty = \text{Id}$.

Summarizing the above discussion gives:

Lemma 5 *With the above notation, Γ admits the presentation $\Gamma = \langle S_\infty, A_1, \dots, A_r | R_\infty, R_{a,b,c} \rangle$.*

2.4 The method in practice

We now give an outline of the method we use to apply Macbeath's theorem:)

- (1) Find an explicit (affine) fundamental domain $D_\infty \subset \partial_\infty X \simeq H_u$ for the action of $\Gamma_\infty = \text{Stab}_\Gamma(\infty)$, and a presentation $\Gamma_\infty = \langle S_\infty | R_\infty \rangle$.
- (2) Find the covering depth n of Γ . Consider the corresponding covering complex $\mathcal{C}(\Gamma B_{u^{cov}}) \simeq \mathcal{C}_n(\Gamma)$.
- (3) Find all E -rational points $\{p_1, \dots, p_d\}$ in D_∞ with depth at most n (and denote $p_\infty = \infty$).
- (4) For each of the r Γ_∞ -orbits of points p_a , find an explicit $A_a \in \Gamma$ such that $A_a(\infty) = p_a$.

(5) For each triple (a, b, c) for which there exist $\gamma_1^1, \gamma_3^3 \in \Gamma_\infty$ such that $A_a(\gamma_1^1 p_b) = \gamma_3^3 p_c$ we obtain a relation $R_{a,b,c}$ by identifying the element $A_c^{-1}(\gamma_3^3)^{-1} A_a \gamma_1^1 A_b \in \Gamma_\infty$ as a word in the generators S_∞ .

To find all such triples, we seek triple intersections of the form described above. A necessary condition for the triple intersection $B \cap A_a B \cap \gamma_3^3 A_c B$ to be nonempty is for all three pairwise intersections to be nonempty. We get two of them, $B \cap \gamma_3^3 A_c B \neq \emptyset$ and $B \cap A_a B \neq \emptyset$, for free since A_a and A_c are generators. For each of the finitely many A_a 's and A_c 's, we want to find all γ_3^3 such that the third pairwise intersection is nonempty.

The third intersection $A_a B \cap \gamma_3^3 A_c B$ will be nonempty if and only if the level between the corresponding points p_a and $\gamma_3^3 p_c$ is at most the covering depth n . By Lemma 6, this translates to a bound on the Cygan distance between these points, which by the triangle inequality gives a bound on the Cygan distance of each point to the reference point $(0, 0)$. As there are finitely many points $\{p_1, \dots, p_r\}$, we obtain an upper bound on these distances, and cover a ball centered at $(0, 0)$ of the corresponding radius by translates of D_∞ . The finite collection of translates required to do this is our finite list of candidates for γ_3^3 . For each such γ_3^3 , we iterate over the A_a 's and A_c 's, checking the level between p_a and $\gamma_3^3 p_c$ to find all intersections.

Finally, we take all the pairs (a, c) for which the three pairwise intersections are nonempty, and compute coordinates for the point $A_a^{-1} \gamma_3^3 A_c p_\infty$. If the depth of this point is greater than n , it will not give us a relation so we discard it. If it is less than n , then it is of the form $\gamma_1^1 p_b$. We can easily identify γ_1^1 and b by figuring out which translate of D_∞ the point lives in. The last step is to identify the group element $A_b^{-1}(\gamma_1^1)^{-1} A_a^{-1} \gamma_3^3 A_c$ as a word in the generators of Γ_∞ .

Then by Macbeath's theorem and Lemma 5, $\Gamma = \langle S_\infty, A_1, \dots, A_r | R_\infty, R_{a,b,c} \rangle$. Note that in step (5) we don't in fact check which triples of horoballs have nonempty triple intersection (which is harder to check); rather we write a relation for each triple satisfying the necessary condition arising from Lemma 6. In principle this may yield redundant relations which we later eliminate.

Step (5) was largely carried out by a computer. Using the bound on the level coming from the Cygan distance (see Lemma 6), we found a Cygan ball containing all centers of horoballs that could possibly have nonempty intersections with any of $\{A_1 B, \dots, A_r B\}$. We then covered this ball by Γ_∞ -translates of D_∞ , and iterated over the list of centers to generate a list of cycles. We then compute the matrix representation of $A_c^{-1}(\gamma_3^3)^{-1} A_a \gamma_1^1 A_b$ and attempt to identify it as a word in Γ_∞ . For $d = 1, 3, 7$ we did this very inefficiently, by iterating over words in Γ_∞ of increasing length until we found it. This way of searching would have taken unreasonably long for the quaternions, so instead we devised a way of guessing the element based on distance, and then making small corrections to get it exactly right. See [MCode](#) for the details of the computations.

In order to avoid tedious repetition of similar arguments or straightforward computations, we will only give one detailed proof for each step for the Picard modular groups; we will give detailed arguments for the quaternionic Hurwitz lattice. We will usually choose the most difficult case, or the most instructive if the various cases are of similar difficulty. Step (1) is routine and we just state the results, except for the Hurwitz lattice, where we find a presentation for Γ_∞ in Lemma 15 and use Philippe's fundamental domain found in [\[Ph\]](#) (see Proposition 4). We give a detailed argument and proof for step (2) for the Picard modular group $\Gamma(7)$ in Lemma 13, and for step (3) for the same group and depth 2 in Lemma 14.

There seems to be no general strategy for step (4); we find all relevant matrices in this paper by combining two tricks, which luckily cover all the cases we need. The first trick is to use stabilizers of vertical complex lines in the Heisenberg group: it is easy to find such a matrix when it stabilizes the vertical axis, then we carry over to other vertical lines by conjugating by a horizontal translation. The second trick is to hit all relevant integral points by the group elements that we already know, and see if we land in the Γ_∞ -orbit of the point we are trying to reach.

2.5 A toy example: $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$

In order to illustrate the method, we now go through its steps for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ exactly as we will for the more complicated Picard and Hurwitz modular groups. The results are either well-known or elementary and we state them without proof.

Presentation and fundamental domain for the cusp stabilizer Γ_∞ : The cusp stabilizer Γ_∞ has presentation $\langle T \rangle$; a fundamental domain for its action on $\partial \mathbb{H}_\mathbb{R}^2 \setminus \{\infty\} \cong \mathbb{R}$ is $D_\infty = [0, 1]$. Concretely we use the following generator for Γ_∞ :

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Covering depth and \mathbb{Q} -rational points in D_∞ : The covering depth of $\mathrm{PSL}(2, \mathbb{Z})$ is 1. The \mathbb{Q} -rational points of depth 1 in D_∞ are 0 and 1, both in the same Γ_∞ -orbit. An integral lift of 0 is $p_0 = [0, 1]^T$; we denote $p_1 = Tp_0$.

In the Picard modular and Hurwitz cases this is a more difficult estimation, owing to the fact that we cannot simply look at a 2-dimensional picture. In the Picard modular case we can look at a 3-dimensional picture to guess at the covering depth which we may then verify, but in the Hurwitz case it is much more challenging to use a picture for help; see Figure 5 for some visual intuition in that case.

Generators: The following element $A_0 \in \Gamma$ maps the point $\infty = [1, 0]^T$ to p_0 :

$$A_0 = I_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

In the Picard modular and Hurwitz cases we must find generators mapping ∞ to points of depth greater than 1. While there is no general method for doing this, in practice we were always able to do it by combining the two tricks described at the end of the previous section.

Relations: We list in Table 1 the relations obtained for $\mathrm{PSL}(2, \mathbb{Z})$ by applying generators to points of depth at most 1 as described in part (5) of section 2.4. The second relation is obtained by following the corresponding cycle of points, which gives $I_0 T I_0 T I_0 \in \Gamma_\infty$. The latter element is computed to be T^{-1} , giving the relation $(I_0 T)^3 = \mathrm{Id}$.

For each matrix A and point p for which $A.p$ has depth less than or equal to the covering depth, we obtain a relation from the following cycle of points

$$\infty \xrightarrow{A_p} p \xrightarrow{W^{-1}A} p' \xrightarrow{A_{p'}^{-1}} \infty$$

We write $A.p = W.p'$ where W is an element of Γ_∞ written as a word in its generators. Then $A_{p'}^{-1}W^{-1}AA_p$ is an element of Γ_∞ , where A_p and $A_{p'}$ are our chosen generators taking ∞ to p and p' respectively. We write that element as a word W' in the generators of Γ_∞ , and the relation we obtain from the cycle is $A_{p'}WAA_p = W'$

$A.p$	p'	$A_{p'}^{-1}W^{-1}AA_p$	W'
$I_0 p_0$	∞	I_0^2	Id
$I_0 p_1$	p_0	$(TI_0)^{-1}(TI_0)^{-1}I_0$	T

Table 1: Action of generators on vertices for $\mathrm{PSL}(2, \mathbb{Z})$

This step works in exactly the same way in the Picard modular and Hurwitz cases, but with more points and more cycles.

3 Horosphere intersections

Our main reference for this section is [KP]. We will use the *Siegel model* of hyperbolic space $\mathrm{H}_{\mathbb{K}}^n$ (with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$), which is the projective model (as described in Section 2.2) associated to the Hermitian form on \mathbb{K}^{n+1} given by $\langle Z, W \rangle = W^* J Z$ with:

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then hyperbolic space $\mathrm{H}_{\mathbb{K}}^n$ can be parametrized by $\mathbb{K}^{n-1} \times \mathrm{Im} \mathbb{K} \times \mathbb{R}^+$ as follows, denoting as before by π the projectivization map: $\mathrm{H}_{\mathbb{K}}^n = \{\pi(\psi(\zeta, v, u)) \mid \zeta \in \mathbb{K}^{n-1}, v \in \mathrm{Im} \mathbb{K}, u \in \mathbb{R}^+\}$, where:

$$\psi(\zeta, v, u) = \begin{pmatrix} (-|\zeta|^2 - u + v)/2 \\ \zeta \\ 1 \end{pmatrix} \quad (2)$$

(For consistency between the cases $\mathbb{K} = \mathbb{C}, \mathbb{H}$, we use the convention that elements of $\text{Im } \mathbb{C}$ are of the form it with $t \in \mathbb{R}$). With this parametrization the boundary at infinity $\partial_\infty \text{H}_{\mathbb{K}}^n$ corresponds to the one-point compactification:

$$\{\pi(\psi(\zeta, v, 0)) \mid \zeta \in \mathbb{K}^{n-1}, v \in \text{Im } \mathbb{K}\} \cup \{\infty\}$$

where $\infty = \pi((1, 0, \dots, 0)^T)$. The coordinates $(\zeta, v, u) \in \mathbb{K}^{n-1} \times \text{Im } \mathbb{K} \times \mathbb{R}^+$ are called the *horospherical coordinates* of the point $\pi(\psi(\zeta, v, u)) \in \text{H}_{\mathbb{K}}^n$.

Definition 3 For a fixed $u_0 \in \mathbb{R}^+$, the level set $H_{u_0} = \{\pi(\psi(\zeta, v, u_0)) \mid \zeta \in \mathbb{K}^{n-1}, v \in \text{Im } \mathbb{K}\}$ is called the *horosphere* at height u_0 based at ∞ , and $B_{u_0} = \{\pi(\psi(\zeta, v, u)) \mid \zeta \in \mathbb{K}^{n-1}, v \in \text{Im } \mathbb{K}, u > u_0\}$ is called the *horoball* at height u_0 based at ∞ .

The punctured boundary $\partial_\infty \text{H}_{\mathbb{K}}^n \setminus \{\infty\}$ is then naturally identified to the *generalized Heisenberg group* $\text{Heis}(\mathbb{K}, n)$, defined as the set $\mathbb{K}^{n-1} \times \text{Im } \mathbb{K}$ equipped with the group law:

$$(\zeta_1, v_1)(\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + 2\text{Im}(\bar{\zeta}_2 \cdot \zeta_1))$$

where \cdot denotes the usual Euclidean dot-product on \mathbb{K}^{n-1} . This is the classical 3-dimensional Heisenberg group when $\mathbb{K} = \mathbb{C}$ and $n = 2$. The identification of $\partial_\infty \text{H}_{\mathbb{K}}^n \setminus \{\infty\}$ with $\text{Heis}(\mathbb{K}, n)$ is given by the simply-transitive action of $\text{Heis}(\mathbb{K}, n)$ on $\partial_\infty \text{H}_{\mathbb{K}}^n \setminus \{\infty\}$, where the element $(\zeta_1, v_1) \in \text{Heis}(\mathbb{K}, n)$ acts on the vector $\psi(\zeta_2, v_2, 0)$ by left-multiplication by the following *Heisenberg translation* matrix in $\text{U}(n, 1, \mathbb{K})$:

$$T_{(\zeta_1, v_1)} = \begin{pmatrix} 1 & -\zeta_1^* & (-|\zeta_1|^2 + v_1)/2 \\ 0 & \text{I}_{n-1} & \zeta_1 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

Given an element $U \in \text{U}(n, \mathbb{K})$, the *Heisenberg rotation* by U is given by the following matrix:

$$R_U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4)$$

There is an additional class of isometries fixing ∞ when $\mathbb{K} = \mathbb{H}$, coming from the action of diagonal matrices which is non-trivial in the non-commutative case. Recall that our convention is that matrices act on vectors on the left, and scalars act on vectors on the right. Then, for any unit quaternion $q \in \mathbb{H}$, the diagonal matrix $C_q = q \text{Id}$ acts by the isometry of hyperbolic space given by conjugating horospherical coordinates (the result of multiplying the vector form (2) by q on the left, then normalizing by q^{-1} on the right):

$$C_q : (\zeta, v, u) \mapsto (q\zeta q^{-1}, qvq^{-1}, u) \quad (5)$$

For this reason, when $\mathbb{K} = \mathbb{H}$ the relevant projectivization of $\text{U}(n, 1, \mathbb{H})$ acting on $\text{H}_{\mathbb{H}}^n$ is $\text{PU}(n, 1, \mathbb{H}) = \text{U}(n, 1, \mathbb{H})/\{\pm \text{Id}\}$ rather than $\text{U}(n, 1, \mathbb{H})/\text{U}(1)$.

Heisenberg translations and rotations, as well as conjugation by unit quaternions, preserve the following distance function on $\text{Heis}(\mathbb{K}, n)$, called the *Cygan metric*, defined for $(\zeta_1, v_1), (\zeta_2, v_2) \in \text{Heis}(\mathbb{K}, n)$ by:

$$d_C((\zeta_1, v_1), (\zeta_2, v_2)) = \left| |\zeta_1 - \zeta_2|^4 + |v_1 - v_2 + 2\text{Im}(\bar{\zeta}_2 \cdot \zeta_1)|^2 \right|^{1/4} \quad (6)$$

$$= |2\langle \psi(\zeta_1, v_1, 0), \psi(\zeta_2, v_2, 0) \rangle|^{1/2} \quad (7)$$

This is in fact the restriction to $\partial_\infty \text{H}_{\mathbb{K}}^n \setminus \{\infty\}$ of an incomplete distance function on $\overline{\text{H}_{\mathbb{K}}^n} \setminus \{\infty\}$ called the *extended Cygan metric* (see [KP]), defined for $(\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2) \in \mathbb{K}^{n-1} \times \text{Im } \mathbb{K} \times \mathbb{R}^{\geq 0} \simeq \overline{\text{H}_{\mathbb{K}}^n} \setminus \{\infty\}$ by:

$$d_{XC}((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)) = \left| (|\zeta_1 - \zeta_2|^2 + |u_1 - u_2|)^2 + |v_1 - v_2 + 2\text{Im}(\bar{\zeta}_2 \cdot \zeta_1)|^2 \right|^{1/4} \quad (8)$$

$$= |2\langle \psi(\zeta_1, v_1, u_1), \psi(\zeta_2, v_2, u_2) \rangle|^{1/2} \quad (9)$$

We define *Cygan spheres*, *Cygan balls*, *extended Cygan spheres* and *extended Cygan balls* in the usual way relative to these distance functions.

When $\Gamma < \mathrm{U}(n, 1, \mathcal{O}_E)$ is an integral lattice with \mathcal{O}_E a principal ideal domain as in section 2.2, the Cygan distance relates to levels and depths of integral boundary points as follows:

Lemma 6 *Let $g = (g_{i,j})$ and $h = (h_{i,j}) \in \mathrm{U}(n, 1, \mathcal{O}_E)$ satisfy $g\infty \neq \infty, h\infty \neq \infty$, and denote $(\zeta_g, v_g), (\zeta_h, v_h)$ the horospherical coordinates of $g\infty, h\infty$ respectively. Then:*

$$d_C((\zeta_g, v_g), (\zeta_h, v_h)) = \left(\frac{4\mathrm{lev}(g\infty, h\infty)}{\mathrm{depth}(g\infty)\mathrm{depth}(h\infty)} \right)^{1/4}.$$

In particular, given $n \geq 1$:

$$\mathrm{lev}(g\infty, h\infty) \leq n \iff d_C((\zeta_g, v_g), (\zeta_h, v_h)) \leq \left(\frac{4n}{\mathrm{depth}(g\infty)\mathrm{depth}(h\infty)} \right)^{1/4}.$$

Proof. By Lemma 2, the first column vector of g (resp. h) is a primitive integral lift of $g\infty$ (resp. $h\infty$), and it can be written in terms of horospherical coordinates as $g_{n+1,1}\psi(\zeta_g, v_g)$ (resp. $h_{n+1,1}\psi(\zeta_h, v_h)$). Therefore, using (6) we have:

$$\begin{aligned} \mathrm{lev}(g\infty, h\infty) &= \langle g_{n+1,1}\psi(\zeta_g, v_g), h_{n+1,1}\psi(\zeta_h, v_h) \rangle \\ &= |g_{n+1,1}|^2|h_{n+1,1}|^2 \langle \psi(\zeta_g, v_g), \psi(\zeta_h, v_h) \rangle \\ &= \mathrm{depth}(g\infty)\mathrm{depth}(h\infty) \frac{d_C((\zeta_g, v_g), (\zeta_h, v_h))^4}{4}. \end{aligned}$$

□

When we apply Macbeath's theorem we argue that the images under Γ of the horoball B_u based at ∞ at a certain height $u > 0$ cover X , or equivalently cover the horosphere $H_u = \partial B_u$. The following result allows us to control the traces on H_u of these images in terms of Cygan spheres depending only on arithmetic data. It involves *Ford isometric spheres*, whose definition we first recall.

Definition 4 *The Ford isometric sphere I_g of an isometry $g \in \mathrm{U}(n, 1, \mathbb{K})$ is defined as*

$$I_g = \{z = (\zeta, v, u) \in \mathrm{H}_{\mathbb{K}}^n \mid |\langle \psi(z), \psi(\infty) \rangle| = |\langle \psi(z), g^{-1}\psi(\infty) \rangle|\}$$

By Proposition 4.3 of [KP], the Ford isometric sphere I_g of $g = (g_{i,j}) \in \mathrm{U}(n, 1, \mathbb{K})$ is in fact the extended Cygan sphere S with center $g^{-1}(\infty)$ and radius $\sqrt{2/|g_{n+1,1}|}$.

Proposition 1 *Let $g = (g_{i,j}) \in \mathrm{U}(n, 1, \mathbb{K})$ satisfy $g(\infty) \neq \infty$, $S = I_{g^{-1}}$ the extended Cygan sphere with center $g(\infty)$ and radius $\sqrt{2/|g_{n+1,1}|}$, and H_{u_0} the horosphere based at ∞ at height $u_0 > 0$. Then $H_{u_0} \cap g(H_{u_0}) = H_{u_0} \cap S$.*

Proof. Using the following standard form for g and g^{-1} , equation (1.2) of [KP], where $a, b, c, d \in \mathbb{K}$, $\alpha, \beta, \gamma, \delta \in \mathbb{K}^{n-1}$ and $A \in \mathrm{M}_{n-1}(K)$:

$$g = \begin{pmatrix} a & \gamma^* & b \\ \alpha & A & \beta \\ c & \delta^* & d \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} \bar{d} & \beta^* & \bar{b} \\ \delta & A^* & \gamma \\ \bar{c} & \alpha^* & \bar{a} \end{pmatrix}, \quad (10)$$

from Equation (2) and Definition 4 we get that: $z = (\zeta, v, u) \in I_{g^{-1}} = S \iff |\frac{c}{2}(-|\zeta|^2 - u + v) + \alpha^*\zeta + \bar{a}|^2 = 1$.

Now fix $u_0 > 0$ and let $z = (\zeta, v, u_0) \in H_{u_0}$. We claim that: $g^{-1}(z) \in H_{u_0} \iff z \in S$.

Indeed, using the above forms for g^{-1} and $\psi(z)$ we have:

$$g^{-1}\psi(z) = \begin{pmatrix} \frac{\bar{d}}{2}(-|\zeta|^2 - u_0 + v) + \beta^* \zeta + \bar{b} \\ \frac{\bar{\delta}}{2}(-|\zeta|^2 - u_0 + v) + A^* \zeta + \gamma \\ \frac{\bar{\epsilon}}{2}(-|\zeta|^2 - u_0 + v) + \alpha^* \zeta + \bar{a} \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}.$$

The corresponding point is in the horosphere H_{u_0} if and only if its u -coordinate in horospherical coordinates equals u_0 . Now the u -coordinate of a point is recovered from any lift $(\zeta_1, \zeta_2, \zeta_3)^T$ by:

$$u = -\zeta_1 \zeta_3^{-1} - \overline{\zeta_1 \zeta_3^{-1}} - |\zeta_2 \zeta_3^{-1}|^2 = -\zeta_3^{-1} \zeta_1 - \zeta_3^{-1} \overline{\zeta_1 \zeta_3^{-1}} \zeta_3 - |\zeta_2 \zeta_3^{-1}|^2 = -|\zeta_3|^{-2} (\overline{\zeta_3} \zeta_1 + \overline{\zeta_1} \zeta_3 + |\zeta_2|^2).$$

(Note that we conjugated by ζ_3 in the second step). Therefore: $g^{-1}(z) \in H_{u_0} \iff \overline{\zeta_3} \zeta_1 + \overline{\zeta_1} \zeta_3 + |\zeta_2|^2 = -u_0 |\zeta_3|^2$. Expanding the left-hand side gives:

$$\begin{aligned} & \left((-|\zeta|^2 - u_0 - v) \frac{c}{2} + \zeta^* \alpha + a \right) \left(\frac{\bar{d}}{2} (-|\zeta|^2 - u_0 + v) + \beta^* \zeta + \bar{b} \right) \\ & + \left((-|\zeta|^2 - u_0 - v) \frac{d}{2} + \zeta^* \beta + b \right) \left(\frac{\bar{c}}{2} (-|\zeta|^2 - u_0 + v) + \alpha^* \zeta + \bar{a} \right) \\ & + \left((-|\zeta|^2 - u_0 - v) \frac{\delta^*}{2} + \zeta^* A + \gamma^* \right) \left(\frac{\bar{\delta}}{2} (-|\zeta|^2 - u_0 + v) + A^* \zeta + \gamma \right) \end{aligned}$$

If we further expand by distributing and collecting the terms in $(-|\zeta|^2 - u_0)$, ζ , v (and their conjugates and products), all terms vanish by the relations $gg^{-1} = g^{-1}g = \text{Id}$ applied to the standard forms of g, g^{-1} from equation (10), except for the term in $(-|\zeta|^2 - u_0)/2$ which has coefficient $(a\bar{d} + b\bar{c} + \gamma^*\delta) + (cb + d\bar{a} + \delta^*\gamma) = 2$ and the term in $\zeta\zeta^*$ which has coefficient $\alpha\beta^* + \beta\alpha^* + AA^* = \text{Id}$. Therefore the left-hand side simplifies to $-u_0$, whence:

$$g^{-1}(z) \in H_{u_0} \iff -u_0 = -u_0 |\zeta_3|^2 \iff |\zeta_3|^2 = 1 \iff z \in S.$$

This proves the claim and hence the Lemma. \square

Corollary 1 *Let E be a number field such that \mathcal{O}_E is a principal ideal domain, $p \in \partial_\infty X$ an E -rational point with depth $n \geq 1$ and $g_p \in \text{U}(H, \mathcal{O}_E)$ satisfying $g_p(\infty) = p$. Then $H_{u_0} \cap g_p(H_{u_0}) = H_{u_0} \cap S$, with S the extended Cygan sphere centered at p with radius $(\frac{4}{n})^{1/4}$. In particular: $H_{u_0} \cap g_p(H_{u_0}) = \emptyset \iff u_0 > u(n) = \frac{2}{\sqrt{n}}$.*

Proof. Since $g_p(\infty) = p$ and $e_1 = (1, 0, \dots, 0)^T$ is a lift of ∞ , the first column vector of g_p is a lift P_0 of p , and since $g_p \in \text{U}(H, \mathcal{O}_E)$ it is an integral lift. In fact by Lemma 2 it is a primitive lift, therefore the depth of p is $|\langle P_0, e_1 \rangle|^2 = |g_{n+1,1}|^2$, denoting as above $g_p = (g_{i,j})$, and the result follows from Proposition 1. The second part of the statement follows by using this radius in the formula (8) for the extended Cygan metric. \square

We will also use the following observation, which is Lemma 1 of [FPP], in our covering arguments:

Lemma 7 *Extended Cygan balls are affinely convex in horospherical coordinates.*

Finally, when considering the action of a discrete subgroup Γ_∞ of $\text{Isom}(\partial_\infty X)$ (relative to the Cygan metric) it is convenient to consider its vertical and horizontal components defined as follows (see [FP] for the case $\mathbb{K} = \mathbb{C}$ and $n = 2$). The homomorphism $\Pi : \text{Heis}(\mathbb{K}, n) \rightarrow \mathbb{K}^{n-1}$ given by projection to the first factor in the decomposition of the set $\text{Heis}(\mathbb{K}, n)$ as $\mathbb{K}^{n-1} \times \text{Im } \mathbb{K}$ induces a short exact sequence:

$$1 \longrightarrow \text{Im } \mathbb{K} \longrightarrow \text{Isom}(\text{Heis}(\mathbb{K}, n)) \xrightarrow{\Pi^*} \text{Isom}(\mathbb{K}^{n-1}) \longrightarrow 1, \quad (11)$$

where the isometries of \mathbb{K}^{n-1} are relative to the Euclidean metric and $\text{Im } \mathbb{K}$ acts by ("vertical") translations. Denoting $\Gamma_\infty^v = \Gamma_\infty \cap \text{Im } \mathbb{K}$ and $\Gamma_\infty^h = \Pi^*(\Gamma_\infty)$ this gives the short exact sequence:

$$1 \longrightarrow \Gamma_\infty^v \longrightarrow \Gamma_\infty \xrightarrow{\Pi^*} \Gamma_\infty^h \longrightarrow 1. \quad (12)$$

4 Picard modular groups

In this section we use the method described in Section 2.4 to compute presentations for the Picard modular groups $\Gamma(d) = \mathrm{PU}(2, 1, \mathcal{O}_d)$ with $d = 1, 3, 7$. The following propositions summarize the results in this section. Recall that presentations for $\Gamma(d) = \mathrm{PU}(2, 1, \mathcal{O}_d)$ with $d = 1, 3$ were obtained in [FP] and [FFP] respectively. We only include these cases as test cases for our method; it turns out that the presentation we obtain when $d = 1$ simplifies nicely (thanks to Magma [Mag]) so we include it below, whereas the presentation we obtain when $d = 3$ is much more complicated than the Falbel-Parker presentation, so we only include it in the appendix. We note the abelianization in each case as a corollary of the presentation.

Corollary 2 ([FP]) *The abelianization of $\Gamma(3)$ is $\mathbb{Z}/6\mathbb{Z}$.*

Proposition 2 *The Picard modular group $\Gamma(1) = \mathrm{PU}(2, 1, \mathcal{O}_1)$ admits the presentation $\langle I, A \mid \mathcal{R}_1 \rangle$, where \mathcal{R}_1 is the following set of 6 relations:*

$$\begin{aligned} I^2 &= \mathrm{Id} \\ A^8 &= \mathrm{Id} \\ IA^{-2}IA^2IA^2IA^{-2} &= \mathrm{Id} \\ (IA^3IA^{-3})^3 &= \mathrm{Id} \\ (A^{-1}IA^{-2}IA^{-1}IA^3IA^{-1}IA^{-2}I)^3 &= \mathrm{Id} \\ IA^{-2}IAIA^3IAIA^{-2}IA^{-1}IAIA^2IA^{-1}IAIA^{-2} &= \\ IA^{-1}IAIA^{-2}IA^{-1}IAIA^{-2}IA^{-1}IA^3IAIA^3IA^{-1}IA^3IA^3IA &= \mathrm{Id} \end{aligned}$$

Corollary 3 ([FFP]) *The abelianization of $\Gamma(1)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.*

Proposition 3 *The Picard modular group $\Gamma(7) = \mathrm{PU}(2, 1, \mathcal{O}_7)$ admits the presentation $\langle T_1, R, I \mid \mathcal{R}_7 \rangle$, where \mathcal{R}_7 is the following set of 13 relations:*

$$\begin{aligned} R^2 &= \mathrm{Id} \\ I^2 &= \mathrm{Id} \\ (RI)^2 &= \mathrm{Id} \\ RT_1RT_1 &= T_1RT_1R \\ (T_1IT_1^{-1}R)^4 &= \mathrm{Id} \\ (T_1^{-1}IT_1R)^4 &= \mathrm{Id} \\ T_1^{-1}IT_1^{-1}IT_1IT_1IT_1^{-3}IT_1IT_1IT_1^{-1}IT_1^{-1} &= \mathrm{Id} \\ (T_1^{-1}IT_1IT_1IT_1^{-1}IT_1^{-1}I)^2 &= \mathrm{Id} \\ (IT_1^{-1}R)^7 &= \mathrm{Id} \\ T_1^{-1}IT_1IT_1IT_1^{-2}IT_1^{-1}IT_1IT_1^2IT_1^{-1}IT_1^{-1}IT_1I &= \mathrm{Id} \\ T_1^{-1}IT_1IT_1IRT_1IT_1IT_1^{-1}IT_1^{-1}IT_1RT_1^{-1}IRT_1^{-1}I &= \mathrm{Id} \\ RT_1IRT_1IT_1IT_1^{-1}IT_1^{-1}IRT_1^{-1}IRT_1^{-1}IT_1^{-1}IT_1IT_1IT_1^{-1} &= \mathrm{Id} \\ RT_1IRT_1RT_1^{-1}IT_1IT_1IRT_1IT_1IT_1^{-1}RT_1RIT_1RT_1^{-1}IT_1IT_1IT_1IT_1^{-1} &= \mathrm{Id} \end{aligned}$$

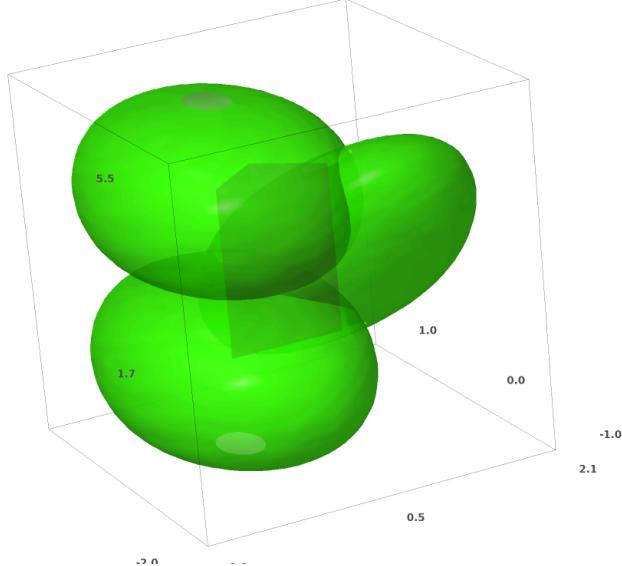
Corollary 4 *The abelianization of $\Gamma(7)$ is $\mathbb{Z}/2\mathbb{Z}$.*

The action of $\Gamma_\infty(d) = \mathrm{Stab}_{\Gamma(d)}(\infty)$ on $\partial\mathbb{H}_\mathbb{C}^2$ is well understood for all d , see [FP] for $d = 3$, [FFP] for $d = 1$ and Section 5.3 of [PW] for all other values (using unpublished notes of Falbel-Francsics-Parker). We will refer to these papers for presentations and fundamental domains for $\Gamma_\infty(d)$ which we state in Lemmas 8, 10 and 12. We will denote $\tau = \frac{1+i\sqrt{d}}{2}$ when $d \equiv 3 \pmod{4}$, so that $\mathcal{O}_d = \mathbb{Z}[\tau]$.

4.1 The Eisenstein-Picard modular group $\Gamma(3) = \mathrm{PU}(2, 1, \mathcal{O}_3)$

Presentation and fundamental domain for the cusp stabilizer $\Gamma_\infty(3)$:

Figure 1: Covering the prism $D_\infty(3)$ by Cygan balls of depth 1



Lemma 8 1. The cusp stabilizer $\Gamma_\infty(3)$ admits the following presentation:

$$\Gamma_\infty(3) = \left\langle T_1, T_\tau, R \mid \begin{array}{l} [[T_1, T_\tau], T_1], [[T_1, T_\tau], T_\tau], [[T_1, T_\tau], R], \\ R^{-1}T_\tau R = T_1, R^{-1}T_1 R = T_1 T_\tau^{-1}, R^6 \end{array} \right\rangle.$$

2. Let $D_\infty(3) \subset \partial H_{\mathbb{C}}^2$ be the affine convex hull of the points with horospherical coordinates $(0,0)$, $(1,0)$, $(\frac{\tau+1}{3}, 0)$, $(0, 2\sqrt{3})$, $(1, 2\sqrt{3})$, $(\frac{\tau+1}{3}, 2\sqrt{3})$. Then $D_\infty(3)$ is a fundamental domain for $\Gamma_\infty(3)$ acting on $\partial H_{\mathbb{C}}^2 \setminus \{\infty\}$.

Concretely, we use the following generators for $\Gamma_\infty(3)$ (recall that $\tau = \frac{1+i\sqrt{3}}{2}$)

$$T_1 = T_{(1,\sqrt{3})} = \begin{bmatrix} 1 & -1 & \tau^2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad T_\tau = T_{(\tau,\sqrt{3})} = \begin{bmatrix} 1 & -\bar{\tau} & \tau^2 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Covering depth and $\mathbb{Q}[i\sqrt{3}]$ -rational points in $D_\infty(3)$:

We denote $B((z,t), r)$ the open extended Cygan ball centered at $p = (z,t) \in \partial_\infty H_{\mathbb{C}}^2$ with radius r (see Equation 8 for the definition of the extended Cygan metric). Recall that $u(n) = \frac{2}{\sqrt{n}}$ is the height at which balls of depth n appear, in the sense of Corollary 1.

Lemma 9 Let $u = u(5) + \varepsilon = 0.895$ and H_u the horosphere of height u based at ∞ . Then the prism $D_\infty(3) \times \{u\}$ is covered by the intersections with H_u of the following extended Cygan balls of depth 1: $B((0,0), \sqrt{2})$, $B((0, 2\sqrt{3}), \sqrt{2})$ and $B((1, \sqrt{3}), \sqrt{2})$.

We omit the proof, which is similar to the proof of Lemma 13 but much simpler; see Figure 4.1.

Note that in order to cover $D_\infty(3) \times \{u\}$ we only need balls of depth 1, in particular none of depths 3 or 4 even though they are present at the height $u = u(5) + \varepsilon$ which we consider. It seems however necessary to pass to this height, as we have observed experimentally that at height $u = u(4) + \varepsilon$ balls of depth at most 3 do not cover $D_\infty(3) \times \{u\}$ (there are no $\mathbb{Q}[i\sqrt{3}]$ -rational points of depth 2), i.e. that the covering depth is more than 3.

Corollary 5 *The covering depth of $\Gamma(3)$ is at most 4.*

By inspection, we see that the $\mathbb{Q}[i\sqrt{3}]$ -rational points in $D_\infty(3)$ with depth at most 4 are, in horospherical coordinates:

- Depth 1: $(0, 0)$, $(0, 2\sqrt{3})$ and $(1, \sqrt{3})$, all in the same $\Gamma_\infty(3)$ -orbit;
- Depth 3: $(0, \frac{2}{3}\sqrt{3})$, $(1, \frac{5}{3}\sqrt{3})$ in one $\Gamma_\infty(3)$ -orbit, and $(0, \frac{4}{3}\sqrt{3})$, $(1, \frac{1}{3}\sqrt{3})$ in the other;
- Depth 4: $(0, \sqrt{3})$, $(1, 0)$ and $(1, 2\sqrt{3})$, all in the same $\Gamma_\infty(3)$ -orbit.

Integral lifts of representatives of $\Gamma_\infty(3)$ -orbits of these points are:

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad p_{3,1} = \begin{bmatrix} -1 \\ 0 \\ i\sqrt{3} \end{bmatrix} \quad p_{3,2} = \begin{bmatrix} -2 \\ 0 \\ i\sqrt{3} \end{bmatrix} \quad p_4 = \begin{bmatrix} i\sqrt{3} \\ 0 \\ 2 \end{bmatrix}$$

Generators: The following elements $A_\alpha \in \Gamma(3)$ map the point $\infty = [1, 0, 0]^T$ to the corresponding p_α as above (for $\alpha = 0; 3, 1; 3, 2; 4$):

$$A_0 = I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_{3,1} = A_3 = \begin{bmatrix} -1 & 0 & i\sqrt{3} \\ 0 & 1 & 0 \\ i\sqrt{3} & 0 & 2 \end{bmatrix}$$

$$A_{3,2} = A_3^{-1} \quad A_4 = \begin{bmatrix} -i\sqrt{3} & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & i\sqrt{3} \end{bmatrix}$$

Relations: We obtain a complete set of relations between these generators by applying generators to points of depth at most 4 as described in part (5) of section 2.4. The detailed steps of the computation can be found on the companion Sagemath Jupyter notebook in [\[MCODE\]](#). The direct output is a presentation with 8 generators and 583 relations, which simplifies, thanks to Magma [\[Mag\]](#), to the presentation in the appendix. More specifically, we obtain that particular simplification by specifying a subset of the generators which must be preserved, using in this case the command `Simplify (G: Preserve:=[1,4,5]);`.

To illustrate the steps involved we compute by hand the cycles and relations corresponding to triples (a, b, c) for which the point $A_a(p_b)$ has depth at most 4, hence is of the form $\gamma_\infty(p_c)$ for some $\gamma_\infty \in \Gamma_\infty$ and $p_c \in D_\infty$ with depth at most 4. In the notation of part (5) of section 2.4, this corresponds to $\gamma_\infty^1 = \text{Id}$ and $\gamma_\infty^3 = \gamma_\infty$. The results are listed in Table 2, with the same notation as in the example of section 2.5.

4.2 The Gauss-Picard modular group $\Gamma(1) = \text{PU}(2, 1, \mathcal{O}_1)$

Presentation and fundamental domain for the cusp stabilizer $\Gamma_\infty(1)$:

Lemma 10 1. *The cusp stabilizer $\Gamma_\infty(1)$ admits the following presentation:*

$$\Gamma_\infty(1) = \left\langle T_2, T_\tau, T_v, R \mid \begin{array}{l} [T_\tau, T_2] = T_v^4, [T_v, T_2], [T_v, T_\tau], [T_v, R], R^4, \\ RT_2R^{-1} = T_\tau^2 T_2^{-1} T_v^4, RT_\tau R^{-1} = T_\tau T_2^{-1} T_v^2 \end{array} \right\rangle.$$

2. *Let $D_\infty(1) \subset \partial H_\mathbb{C}^2$ be the affine convex hull of the points with horospherical coordinates $(0, 0)$, $(1, 0)$, $(\tau, 0)$, $(0, 2)$, $(1, 2)$, $(\tau, 2)$. Then $D_\infty(1)$ is a fundamental domain for $\Gamma_\infty(1)$ acting on $\partial H_\mathbb{C}^2 \setminus \{\infty\}$.*

Concretely, we use the following generators for $\Gamma_\infty(1)$ (denoting $\tau = 1 + i$):

$\mathbf{A} \cdot \mathbf{p}$	\mathbf{p}'	$\mathbf{A}_{\mathbf{p}'}^{-1} \mathbf{W}^{-1} \mathbf{A} \mathbf{A}_{\mathbf{p}}$	\mathbf{W}'
$I_0 p_0$	∞	I_0^2	Id
$A_4 p_4$	∞	A_4^2	Id
$R p_0$	p_0	$I_0 R I_0$	R
$R p_{3,1}$	$p_{3,1}$	$A_3^{-1} R A_3$	R
$R p_{3,2}$	$p_{3,2}$	$A_3 R A_3^{-1}$	R
$I_0(0, 2\sqrt{3})$	$p_{3,2}$	$A_3 T_v I_0 T_v I_0$	R^3
$I_0(1, \sqrt{3})$	p_0	$I_0 T_\tau^{-1} T_v I_0 T_1 I_0$	$T_v^{-1} T_1 T_\tau^{-1}$
$I_0 p_{3,1}$	p_0	$I_0 T_v I_0 A_3$	$R^3 T_v^{-1}$
$I_0 p_{3,2}$	p_4	$A_4^{-1} T_v I_0 A_3^{-1}$	R^3
$I_0 p_4$	$p_{3,1}$	$A_3^{-1} T_v I_0 A_4$	R^3
$A_3 p_0$	p_4	$A_4^{-1} A_3 I_0$	$R^3 T_v$
$A_3 p_4$	p_0	$I_0 T_v^{-1} A_3 A_4$	R^3
$A_3 p_{3,1}$	$p_{3,2}$	A_3^3	R^3
$A_4 p_0$	$p_{3,2}$	$A_3 A_4 I_0$	$R^3 T_v$
$A_4 p_{3,1}$	p_0	$I_0 T_v^{-1} A_4 A_3$	R^3

Table 2: Action of generators on vertices for $d = 3$

$$T_2 = T_{(2,0)} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad T_\tau = T_{(\tau,0)} = \begin{bmatrix} 1 & -\bar{\tau} & -1 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_v = T_{(0,2)} = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Covering depth and $\mathbb{Q}[i]$ -rational points in $D_\infty(1)$:

We denote $B((z,t),r)$ the open extended Cygan ball centered at $p = (z,t) \in \partial_\infty H_\mathbb{C}^2$ with radius r (see Equation 8 for the definition of the extended Cygan metric). Recall that $u(n) = \frac{2}{\sqrt{n}}$ is the height at which balls of depth n appear, in the sense of Corollary 1.

Lemma 11 *Let $u = u(5) + \varepsilon = 0.895$ and H_u the horosphere of height u based at ∞ . Then the prism $D_\infty(1) \times \{u\}$ is covered by the intersections with H_u of the following extended Cygan balls:*

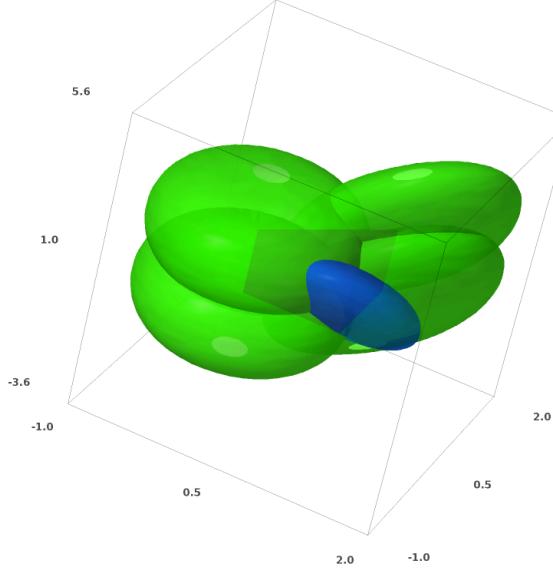
- (depth 1) $B((0,0), \sqrt{2})$, $B((0,2), \sqrt{2})$, $B((\tau,0), \sqrt{2})$, $B((\tau,2), \sqrt{2})$, $B((1, \sqrt{7}), \sqrt{2})$,
- (depth 2) $B((1,1), \sqrt[4]{2})$.

We omit the proof, which is similar to the proof of Lemma 13 but simpler; see Figure 4.2.

Note that in order to cover $D_\infty(1) \times \{u\}$ we only need balls of depth at most 2, in particular none of depth 4 even though they are present at the height $u = u(5) + \varepsilon$ which we consider. It is however necessary to pass to this height, as we have observed experimentally that at height $u = u(4) + \varepsilon$ balls of depth at most 2 do not cover $D_\infty(1) \times \{u\}$ (there are no $\mathbb{Q}[i]$ -rational points of depth 3).

Corollary 6 *The covering depth of $\Gamma(1)$ is at most 4.*

Figure 2: Covering the prism $D_\infty(1)$ by Cygan balls of depth 1 and 2



By inspection, we see that the $\mathbb{Q}[i]$ -rational points in $D_\infty(1)$ with depth at most 4 are, in horospherical coordinates:

- Depth 1: $(0, 0)$, $(0, 2)$, $(\tau, 0)$ and $(\tau, 2)$, all in the same $\Gamma_\infty(1)$ -orbit;
- Depth 2: $(1, 1)$
- Depth 4: $(0, 1)$, $(\tau, 1)$ in one $\Gamma_\infty(1)$ -orbit and $(1, 0)$, $(1, 2)$ in the other.

Integral lifts of representatives of $\Gamma_\infty(1)$ -orbits of these points are:

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad p_2 = \begin{bmatrix} -1 \\ 1+i \\ 1+i \end{bmatrix} \quad p_{4,1} = \begin{bmatrix} i \\ 0 \\ 2 \end{bmatrix} \quad p_{4,2} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

Generators: The following elements $A_\alpha \in \Gamma(1)$ map the point $\infty = [1, 0, 0]^T$ to the corresponding p_α as above (for $\alpha = 0; 2; 4, 1; 4, 2$) :

$$A_0 = I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 & 2+i \\ 1+i & 2+i & -1-i \\ 1+i & 1+i & -i \end{bmatrix}$$

$$A_{4,1} = \begin{bmatrix} i & 0 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & -i \end{bmatrix} \quad A_{4,2} = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{bmatrix}$$

Relations: We obtain a complete set of relations between these generators by applying generators to points of depth at most 4 as described in part (5) of section 2.4. The detailed steps of the computation can be found on the companion Sagemath Jupyter notebook in [MCode](#). The direct output is a presentation with 8 generators and 247 relations, which simplifies, thanks to Magma [Mag](#), to the presentation in Proposition 2. More specifically, we obtain that particular simplification by specifying a subset of the generators which must be preserved, using in this case the command `Simplify (G: Preserve:=[5,6]);`.

To illustrate the steps involved we compute by hand the cycles and relations corresponding to triples (a, b, c) for which the point $A_a(p_b)$ has depth at most 4, hence is of the form $\gamma_\infty(p_c)$ for some $\gamma_\infty \in \Gamma_\infty$ and $p_c \in D_\infty$ with depth at most 4. In the notation of part (5) of section 2.4, this corresponds to $\gamma_\infty^1 = \text{Id}$ and $\gamma_\infty^3 = \gamma_\infty$. The results are listed in Table 3, with the same notation as in the example of section 2.5.

$\mathbf{A} \cdot \mathbf{p}$	\mathbf{p}'	$\mathbf{A}_{\mathbf{p}'}^{-1} \mathbf{W}^{-1} \mathbf{A} \mathbf{A}_{\mathbf{p}}$	\mathbf{W}'
$I_0 p_0$	∞	I_0^2	Id
$A_{4,1} p_{4,1}$	∞	$A_{4,1}^2$	Id
$A_{4,2} p_{4,2}$	∞	$A_{4,2}^2$	Id
$R p_0$	p_0	$I_0 R I_0$	R
$R p_{4,1}$	$p_{4,1}$	$A_{4,1}^{-1} R A_{4,1}$	R
$I_0(0, 2)$	p_0	$I_0 T_v I_0 T_v I_0$	$R T_v^{-1}$
$I_0(\tau, 0)$	$(\tau, 0)$	$(T_\tau I_0)^{-1} I_0 T_\tau I_0$	$R^3 T_\tau$
$I_0(\tau, 2)$	p_2	$(T_v T_\tau I_0)^{-1} I_0 T_\tau T_2^{-1} A_2$	R^2
$I_0 p_2$	p_0	$I_0 T_\tau^{-1} T_v I_0 A_2$	$R T_2^{-1} T_v^{-1}$
$I_0 p_{4,1}$	p_0	$I_0 T_v^2 I_0 A_{4,1}$	$R T_v^{-1}$
$I_0 p_{4,2}$	p_0	$I_0 T_2^{-1} I_0 A_{4,2}$	T_2^{-1}
$A_2 p_0$	p_0	$I_0 T_v^2 I_0 A_{4,1}$	$R T_v^{-1}$
$A_2 p_2$	∞	A_2^2	$R^{-1} T_2^{-1} T_\tau T_v^{-3}$
$A_2 p_{4,1}$	p_2	$A_2^{-1} T_v^{-4} T_2^{-1} T_\tau A_2 A_{4,1}$	$R^2 T_2^{-1}$
$A_2 p_{4,2}$	p_2	$A_2^{-1} T_v^{-1} A_2 A_{4,2}$	$T_2 T_\tau^{-1} R$
$A_{4,1} p_0$	p_0	$I_0 T_v^{-1} A_{4,1} I_0$	$R^{-1} T_v^2$
$A_{4,1} p_2$	p_2	$A_2^{-1} A_{4,1} A_2$	$R^2 T_\tau^{-1} T_2^{-1} T_v^{-2}$
$A_{4,2} p_0$	p_0	$I_0 T_2^{-1} A_{4,2} I_0$	T_2^{-1}
$A_{4,2} p_2$	$(\tau, 0)$	$(T_\tau I_0)^{-1} T_v A_{4,2} A_2$	$R^{-1} T_\tau T_2^{-1}$

Table 3: Action of generators on vertices for $d = 1$

4.3 The Picard modular group $\Gamma(7) = \text{PU}(2, 1, \mathcal{O}_7)$

Presentation and fundamental domain for the cusp stabilizer $\Gamma_\infty(7)$:

Lemma 12 1. The cusp stabilizer $\Gamma_\infty(7)$ admits the following presentation:

$$\Gamma_\infty(7) = \left\langle T_1, T_\tau, T_v, R \mid [T_\tau, T_1] = T_v, [T_v, T_1], [T_v, T_\tau], [T_v, R], (R T_\tau)^2, (R T_1)^2 = T_v, R^2 \right\rangle$$

2. Let $D_\infty(7) \subset \partial \mathbb{H}_\mathbb{C}^2$ be the affine convex hull of the points with horospherical coordinates $(0, 0)$, $(1, 0)$, $(\tau, 0)$, $(0, 2\sqrt{7})$, $(1, 2\sqrt{7})$, $(\tau, 2\sqrt{7})$. Then $D_\infty(7)$ is a fundamental domain for $\Gamma_\infty(7)$ acting on $\partial \mathbb{H}_\mathbb{C}^2 \setminus \{\infty\}$.

Concretely, we use the following generators for $\Gamma_\infty(7)$ (denoting $\tau = \frac{1+i\sqrt{7}}{2}$):

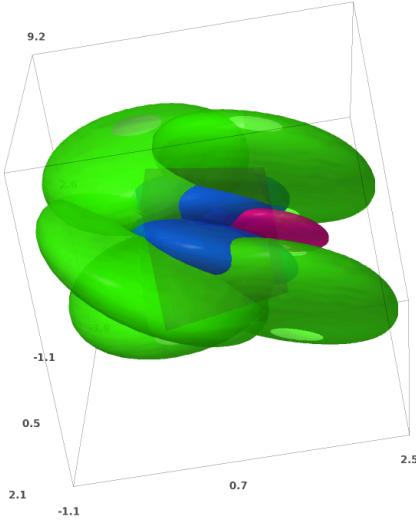
$$T_1 = T_{(1, \sqrt{7})} = \begin{bmatrix} 1 & -1 & \tau - 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad T_\tau = T_{(\tau, 0)} = \begin{bmatrix} 1 & -\bar{\tau} & -1 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_v = T_{(0, 2\sqrt{7})} = \begin{bmatrix} 1 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Covering depth and $\mathbb{Q}[i\sqrt{7}]$ -rational points in $D_\infty(7)$:

We denote $B((z, t), r)$ the open extended Cygan ball centered at $p = (z, t) \in \partial_\infty \mathbb{H}_\mathbb{C}^2$ with radius r (see Equation 8 for the definition of the extended Cygan metric). Recall that $u(n) = \frac{2}{\sqrt{n}}$ is the height at which balls of depth n appear, in the sense of Corollary 1.

Figure 3: Covering the prism $D_\infty(7)$ by Cygan balls of depth 1, 2 and 4



Lemma 13 Let $u = u(8) + \varepsilon = 0.70711$ and H_u the horosphere of height u based at ∞ . Then the prism $D_\infty(7) \times \{u\}$ is covered by the intersections with H_u of the following extended Cygan balls:

- (depth 1) $B((0,0), \sqrt{2})$, $B((0,2\sqrt{7}), \sqrt{2})$, $B((\tau,0), \sqrt{2})$, $B((\tau,2\sqrt{7}), \sqrt{2})$, $B((1,\sqrt{7}), \sqrt{2})$,
- (depth 2) $B((\tau/2, 3\sqrt{7}/2), \sqrt[4]{2})$, $B(((\tau+1)/2, \sqrt{7}), \sqrt[4]{2})$
- (depth 4) $B((\tau, \sqrt{7}), 1)$.

Proof. Figure 4.3 shows the prism $D_\infty(7)$ and the relevant Cygan balls. We prove the result by dissecting the prism $D_\infty(7) \times \{u\}$ into affine polyhedra, each of which lies in one of the extended Cygan balls. This is reminiscent of the proof of Proposition 5.2 of [Zh]. Consider the following points of $\partial_\infty H_{\mathbb{C}}^2$, in horospherical coordinates (see Figure 4.3):

$$\begin{aligned}
 q_1 &= (0.65\tau, 2.8) & q_2 &= (\tau, 2) & q_3 &= (\tau, 3.3) & q_4 &= (0.35 + 0.65\tau, 3.4) & q_5 &= (0.7 + 0.3\tau, 4.2) \\
 q_6 &= (1, 4.3) & q_7 &= ((\tau+1)/2, 1.5) & q_8 &= (\tau, 1.5) & q_9 &= ((\tau+1)/2, 0) & q_{10} &= (1, 1) \\
 q_{11} &= (\tau, 4) & q_{12} &= ((\tau+1)/2, 2\sqrt{7}) & q_{13} &= (\tau/2, 4) & q_{14} &= (\tau/2, 2\sqrt{7}) & q_{15} &= (0, 3.5) \\
 q_{16} &= (0.3\tau, 3) & q_{17} &= (\tau/2, 1) & q_{18} &= (0, 1.7) & q_{19} &= (\tau/2, 0)
 \end{aligned}$$

Denoting $\text{Hull}(S)$ the affine hull (in horospherical coordinates) of a subset $S \subset H_u \simeq \partial_\infty H_{\mathbb{C}}^2 \times \{u\}$, we claim that the following affinely convex pieces of $D_\infty(7) \times \{u\}$ are each contained in the corresponding (open) extended Cygan sphere:

- $D_1 = \text{Hull}((0,0), (1,0), q_9, q_{10}, q_{17}, q_{18}, q_{19}) \subset B((0,0), \sqrt{2})$
- $D_2 = \text{Hull}((0,2\sqrt{7}), (1,2\sqrt{7}), q_6, q_{12}, q_{13}, q_{14}, q_{15}) \subset B((0,2\sqrt{7}), \sqrt{2})$
- $D_3 = \text{Hull}((\tau,0), q_7, q_8, q_9, q_{17}, q_{19}) \subset B((\tau,0), \sqrt{2})$
- $D_4 = \text{Hull}((\tau,2\sqrt{7}), q_{11}, q_{12}, q_{13}, q_{14}) \subset B((\tau,2\sqrt{7}), \sqrt{2})$
- $D_5 = \text{Hull}((1,\sqrt{7}), q_6, q_7, q_{10}, q_{15}, q_{16}, q_{17}, q_{18}) \subset B((1,\sqrt{7}), \sqrt{2})$

- $D_6 = \text{Hull}(q_1, q_3, q_4, q_5, q_{11}, q_{12}, q_{13}, q_{15}, q_{16}) \subset B((\tau/2, 3\sqrt{7}/2), \sqrt[4]{2})$
- $D_7 = \text{Hull}(q_1, q_2, q_4, q_5, q_6, q_7, q_8, q_{16}, q_{17}) \subset B(((\tau+1)/2, \sqrt{7}), \sqrt[4]{2})$
- $D_8 = \text{Hull}(q_1, q_2, q_3, q_4) \subset B((\tau, \sqrt{7}), 1)$

To verify each of these claims, we check numerically that each of the vertices indeed belongs to the ball in question using Equation (8), then extend to the whole affine convex hull by Lemma 7. For example, the point $q_1 = (0.65\tau, 2.8)$ indeed belongs to $B((\tau/2, 3\sqrt{7}/2), \sqrt[4]{2})$, $B(((\tau+1)/2, \sqrt{7}), \sqrt[4]{2})$ and $B((\tau, \sqrt{7}), 1)$ because:

$$\begin{aligned} d_{XC}((0.65\tau, 2.8, u), (\tau/2, 3\sqrt{7}/2, 0)) &\simeq 1.179 < \sqrt[4]{2} \simeq 1.189 \\ d_{XC}((0.65\tau, 2.8, u), ((\tau+1)/2, \sqrt{7}, 0)) &\simeq 1.172 < \sqrt[4]{2} \simeq 1.189 \\ d_{XC}((0.65\tau, 2.8, u), (\tau, \sqrt{7}, 0)) &\simeq 0.982 < 1 \end{aligned}$$

The result then follows as the prism $D_\infty(7) \times \{u\}$ is the union of the affinely convex pieces D_1, \dots, D_8 . Indeed, each of these pieces has all of its vertices on the boundary of the prism, the union of the boundaries of the pieces covers the boundary of the prism, and each piece shares a codimension-1 face with its neighbors, see Figure 4.3. \square

Corollary 7 *The covering depth of $\Gamma(7)$ is at most 7.*

Note that in the above covering argument we have only needed balls of depth at most 4 (in particular none of depth 7) even though they are present at the height $u = u_8 + \varepsilon$ which we consider. It is however necessary to pass to this height, as we have observed experimentally that at height $u = u_7 + \varepsilon$ balls of depth at most 4 do not cover $D_\infty(7) \times \{u\}$ (there are no $\mathbb{Q}[i\sqrt{7}]$ -rational points of depths 5 or 6).

By inspection, we see that the $\mathbb{Q}[i\sqrt{7}]$ -rational points in $D_\infty(7)$ with depth at most 7 are the following, in horospherical coordinates. We give in Lemma 14 below a detailed justification for the points of depth 2.

- Depth 1: $(0, 0)$, $(0, 2\sqrt{7})$, $(\tau, 0)$, $(\tau, 2\sqrt{7})$ and $(1, \sqrt{7})$, all in the same $\Gamma_\infty(7)$ -orbit;
- Depth 2: $(\frac{\tau}{2}, \frac{3}{2}\sqrt{7})$ in one $\Gamma_\infty(7)$ -orbit and $(\frac{\tau+1}{2}, \sqrt{7})$ in the other;
- Depth 4: $(0, \sqrt{7})$, $(\tau, \sqrt{7})$, $(1, 0)$, $(1, 2\sqrt{7})$ in one $\Gamma_\infty(7)$ -orbit, $(\frac{\tau}{2}, \frac{1}{2}\sqrt{7})$ in a second, $(\frac{\tau+1}{2}, 0)$, $(\frac{\tau+1}{2}, 2\sqrt{7})$ in a third, $(\frac{\tau+1}{4}, \frac{5\sqrt{7}}{4})$ in a fourth, and $(\frac{\tau+2}{4}, \frac{3\sqrt{7}}{2})$ in a fifth.
- Depth 7: for each $k = 1, \dots, 6$, there is a $\Gamma_\infty(7)$ -orbit containing the 3 points $(0, \frac{2k}{7}\sqrt{7})$, $(\tau, \frac{2k}{7}\sqrt{7})$, $(1, \frac{2k+7}{7}\sqrt{7})$ (with $2k+7$ taken mod 7).

Integral lifts of representatives of $\Gamma_\infty(7)$ -orbits of these points are:

$$\begin{aligned} p_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & p_{2,1} &= \begin{bmatrix} -\tau - 2 \\ -1 \\ \tau - 1 \end{bmatrix} & p_{2,2} &= \begin{bmatrix} 2 \\ 1 - \tau \\ -\tau \end{bmatrix} \\ p_{4,1} &= \begin{bmatrix} i\sqrt{7} \\ 0 \\ 2 \end{bmatrix} & p_{4,2} &= \begin{bmatrix} \tau - 1 \\ \tau \\ 2 \end{bmatrix} & p_{4,3} &= \begin{bmatrix} -1 \\ \tau + 1 \\ 2 \end{bmatrix} & p_{4,4} &= \begin{bmatrix} -2\tau - 1 \\ -1 \\ \tau - 2 \end{bmatrix} & p_{4,5} &= \begin{bmatrix} 2\tau - 4 \\ \tau \\ \tau + 1 \end{bmatrix} \\ p_{7,1} &= \begin{bmatrix} -1 \\ 0 \\ i\sqrt{7} \end{bmatrix} & p_{7,2} &= \begin{bmatrix} -2 \\ 0 \\ i\sqrt{7} \end{bmatrix} & p_{7,3} &= \begin{bmatrix} -3 \\ 0 \\ i\sqrt{7} \end{bmatrix} & p_{7,4} &= \begin{bmatrix} -4 \\ 0 \\ i\sqrt{7} \end{bmatrix} & p_{7,5} &= \begin{bmatrix} -5 \\ 0 \\ i\sqrt{7} \end{bmatrix} & p_{7,6} &= \begin{bmatrix} -6 \\ 0 \\ i\sqrt{7} \end{bmatrix} \end{aligned}$$

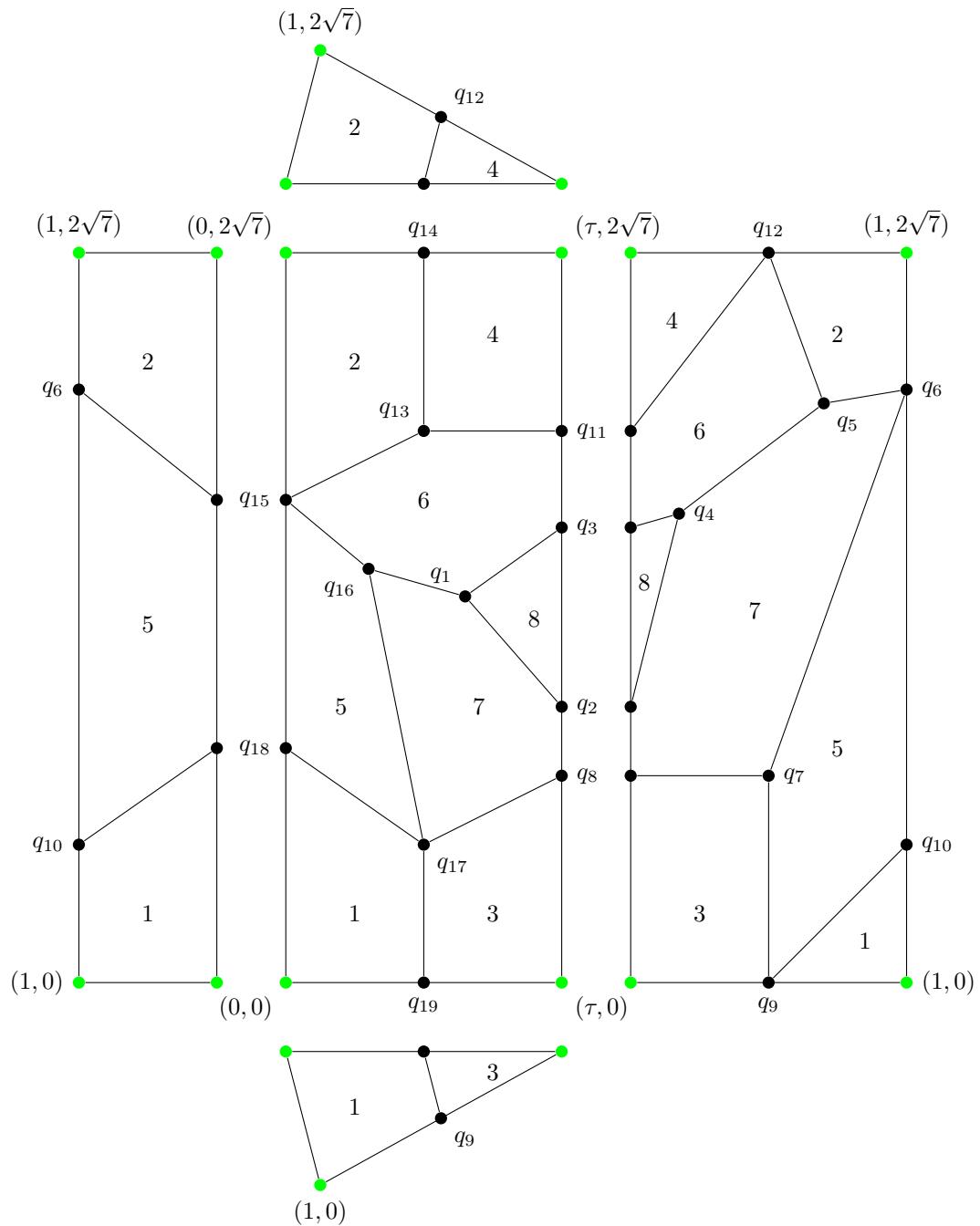


Figure 4: Affine cell decomposition of the prism $D_\infty(7)$

Lemma 14 *The $\mathbb{Q}[i\sqrt{7}]$ -rational points in $D_\infty(7)$ with depth 2 are exactly $(\frac{\tau}{2}, \frac{3}{2}\sqrt{7})$ and $(\frac{\tau+1}{2}, \sqrt{7})$.*

Proof. We illustrate the general procedure for finding points of depth n , then specialize to the present case.

The depths at which there will be E -rational points are the natural numbers n such that $|z|^2 = n$ has a solution $z \in \mathcal{O}_d$. For $d = 7$, the first few n 's are $1, 2, 4, 7, \dots$. Begin by assuming we have found all points of depth less than n . In this case those are just the points at depth 1.

1. Find all $q = x + y\tau \in \mathcal{O}_d$ (up to multiplication by a unit) with $|q|^2 = n$. From the geometry of numbers, we know that there are only finitely many such q . In this case, there are two possibilities: either $q = \tau := \frac{1+i\sqrt{7}}{2}$ or $q = \tau - 1$.
2. Consider the standard lift of a point of $\partial H_{\mathbb{K}}^2 \setminus \{\infty\}$:

$$P = \begin{pmatrix} \frac{-|z|^2+it}{2} \\ z \\ 1 \end{pmatrix}$$

Note that $t = b\sqrt{7}$ for some $b \in \mathbb{Q}$. The calculation is more transparent if we rewrite the first coordinate:

$$\frac{-|z|^2+it}{2} = \frac{-|z|^2-b}{2} + b\tau$$

For P to be the vector form of an E -rational point p of depth 2, it must satisfy the following:

(a) P does not have depth 1 (in general, P does not have depth less than n). In other words, the coordinates of P are not in \mathcal{O}_7 .

(b) Pq 's coordinates must be in \mathcal{O}_7 for some q from step (1).

Next, we do some calculations to make sure (b) is satisfied.

3. Find all z in the projection to \mathbb{C} of D_∞ such that $zq \in \mathcal{O}_7$. If $q = \tau$ we can have $z = 0, 1, \tau$, or $\frac{1+\tau}{2}$. If $q = \tau - 1$, we can have $z = 0, 1, \tau, \frac{\tau}{2}$.
4. For each possible z , find $|z|^2$ and compute $\left(\frac{-|z|^2-b}{2} + b\tau\right)q$. Use this to list all b 's such that $(z, b) \in D_\infty$ and $\left(\frac{-|z|^2-b}{2} + b\tau\right)q \in \mathcal{O}_7$.

z	$ z ^2$	q	$\left(\frac{- z ^2-b}{2} + b\tau\right)q$	b 's	point(s) in horo. coords
0	0	τ	$-2b + \frac{b\tau}{2}$	0, 2	$(0, 0), (0, 2\sqrt{7})$
		$\tau - 1$	$-\frac{3b}{2} - \frac{b}{2}\tau$	0, 2	$(0, 0), (0, 2\sqrt{7})$
1	1	τ	$2b + \frac{b-1}{2}\tau$	1	$(1, \sqrt{7})$
		$\tau - 1$	$\frac{1-3b}{2} - \frac{1+b}{2}\tau$	1	$(1, \sqrt{7})$
τ	2	τ	$-2b - \frac{b-2}{2}$	0, 2	$(0, 0), (0, 2\sqrt{7})$
		$\tau - 1$	$\frac{2-3b}{2} - \frac{2+b}{2}\tau$	0, 2	$(0, 0), (0, 2\sqrt{7})$
$\frac{1+\tau}{2}$	1	τ	$2b + \frac{b-1}{2}\tau$	1	$(\frac{1+\tau}{2}, \sqrt{7})$
$\frac{\tau}{2}$	$\frac{1}{2}$	$\tau - 1$	$\frac{1-6b}{4} - \frac{1+2b}{4}\tau$	$\frac{3}{2}$	$(\frac{\tau}{2}, \frac{3\sqrt{7}}{2})$

5. Get rid of all the ones that are level 1 (or from any previous level). What you are left with are the level 2 points: $\left\{(\frac{1+\tau}{2}, \sqrt{7}), (\frac{\tau}{2}, \frac{3\sqrt{7}}{2})\right\}$.

□

Generators: The following elements $A_\alpha \in \Gamma(7)$ map the point $\infty = [1, 0, 0]^T$ to the corresponding p_α as above (for $\alpha = 0; 2, 1; 2, 2; 4, 1; \dots; 4, 5; 7, 1; \dots; 7, 6$) :

$$\begin{aligned}
A_0 = I_0 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & A_{2,1} &= \begin{bmatrix} -\tau - 2 & -5 & 5 \\ -1 & -1 & 0 \\ \tau - 1 & i\sqrt{7} & -i\sqrt{7} \end{bmatrix} & A_{2,2} &= \begin{bmatrix} 2 & -\tau & \tau - 1 \\ 1 - \tau & -2 & \tau \\ -\tau & \tau - 1 & 2 \end{bmatrix} \\
A_{4,1} &= \begin{bmatrix} i\sqrt{7} & 0 & 4 \\ 0 & 1 & 0 \\ 2 & 0 & -i\sqrt{7} \end{bmatrix} & A_{4,2} &= \begin{bmatrix} \tau - 1 & \tau & 2 \\ \tau & 2 & 1 - \tau \\ 2 & 1 - \tau & -\tau \end{bmatrix} & A_{4,3} &= \begin{bmatrix} -1 & \tau - 2 & 2 \\ \tau + 1 & 3 & -\tau - 1 \\ 2 & 2 - \tau & -1 \end{bmatrix} \\
A_{4,4} &= \begin{bmatrix} -2\tau - 1 & -4 & 4 \\ -1 & -1 & 0 \\ \tau - 2 & i\sqrt{7} & -i\sqrt{7} \end{bmatrix} & A_{4,5} &= \begin{bmatrix} 2\tau - 4 & 3 + 3\tau & 4 - 3\tau \\ \tau & 2 & 1 - \tau \\ 1 + \tau & 3 - 2\tau & -2 - \tau \end{bmatrix} \\
A_{7,1} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ i\sqrt{7} & 0 & -1 \end{bmatrix} & A_{7,2} &= \begin{bmatrix} -2 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ i\sqrt{7} & 0 & 3 \end{bmatrix} & A_{7,3} &= \begin{bmatrix} -3 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ i\sqrt{7} & 0 & 2 \end{bmatrix} \\
A_{7,4} &= \begin{bmatrix} -4 & 0 & 3i\sqrt{7} \\ 0 & 1 & 0 \\ i\sqrt{7} & 0 & 5 \end{bmatrix} & A_{7,5} &= \begin{bmatrix} -5 & 0 & 3i\sqrt{7} \\ 0 & 1 & 0 \\ i\sqrt{7} & 0 & 4 \end{bmatrix} & A_{7,6} &= \begin{bmatrix} -6 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ i\sqrt{7} & 0 & 1 \end{bmatrix}
\end{aligned}$$

Relations: We obtain a complete set of relations between these generators by applying generators to points of depth at most 4 as described in part (5) of section 2.4. The detailed steps of the computation can be found on the companion Sagemath Jupyter notebook in [MCODE](#). The direct output is a presentation with 18 generators and 406 relations, which simplifies, thanks to Magma [Mag](#), to the presentation in Proposition 3. More specifically, we obtain that particular simplification by specifying a subset of the generators which must be preserved, using in this case the command `Simplify (G: Preserve:=[1,4,5]);`.

To illustrate the steps involved we compute by hand the cycles and relations corresponding to triples (a, b, c) for which the point $A_a(p_b)$ has depth at most 7, hence is of the form $\gamma_\infty(p_c)$ for some $\gamma_\infty \in \Gamma_\infty$ and $p_c \in D_\infty$ with depth at most 7. In the notation of part (5) of section 2.4, this corresponds to $\gamma_\infty^1 = \text{Id}$ and $\gamma_\infty^3 = \gamma_\infty$. The results are listed in Table 4, with the same notation as in the example of section 2.5.

5 The Hurwitz quaternion modular group $\text{PU}(2, 1, \mathcal{H})$

In this section we use the method described in Section 2.4 to compute a presentation for the Hurwitz modular group $\Gamma(\mathcal{H}) = \text{PU}(2, 1, \mathcal{H})$ (also denoted $\text{PSp}(2, 1, \mathcal{H})$). Recall that the Hurwitz integer ring is $\mathcal{H} = \mathbb{Z}[i, j, k, \sigma] \subset \mathbb{H}$, with $\sigma = \frac{1+i+j+k}{2}$. The resulting presentation is unfortunately too large for Magma to handle directly (see comments at the end of this section), however it still allows us to obtain the following results.

Theorem 2 *The abelianization of $\Gamma(\mathcal{H})$ is $\mathbb{Z}/3\mathbb{Z}$.*

Theorem 3 *$\Gamma(\mathcal{H})$ is generated by $\Gamma_\infty(\mathcal{H})$ and I_0 .*

Theorem 3 was in fact stated in [\[Ph\]](#) (Th. 4.4.2), but the very short proof given there is inadequate (it would indeed apply to any group containing I_0 and a nontrivial stabilizer of ∞). However, it is shown in [\[Ph\]](#) that $\Gamma_\infty(\mathcal{H})$ can be generated by only 3 elements, which combined with the above shows that $\Gamma(\mathcal{H})$ can be generated by 4 elements.

We will use Theorem 3 in the proof of Theorem 2, so for future reference we note that Theorem 3 follows from the result of computational simplification of a partial presentation by Magma. More specifically, we enter the Magma command `Simplify(G: Preserve:=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17]);` applied to the group G presented by all generators and the first thousand relations in our main presentation (see file [QuaternionsTruncated1000.txt](#) at [MCODE](#)). This returns a presentation of G with 17 generators, corresponding to 16 generators for $\Gamma_\infty(\mathcal{H})$ and I_0 . Theorem 3 follows since $\Gamma(\mathcal{H})$ is a quotient of G .

$\mathbf{A} \cdot \mathbf{p}$	\mathbf{p}'	$\mathbf{A}_{\mathbf{p}'}^{-1} \mathbf{W}^{-1} \mathbf{A} \mathbf{A}_{\mathbf{p}}$	\mathbf{W}'
$I_0 p_0$	∞	I_0^2	Id
$A_{4,1} p_{4,1}$	∞	$A_{4,1}^2$	Id
$A_{4,2} p_{4,2}$	∞	$A_{4,2}^2$	Id
$A_{4,3} p_{4,3}$	∞	$A_{4,3}^2$	Id
$R p_0$	p_0	$I_0 R I_0$	R
$I_0 p_{2,2}$	$p_{4,3}$	$A_{4,3}^{-1} T_1 I_0 A_{2,2}$	T_1
$I_0 p_{4,1}$	$p_{7,5}$	$A_{7,5}^{-1} T_v I_0 A_{4,1}$	R
$I_0 p_{4,2}$	$p_{2,1}$	$A_{2,1}^{-1} T_v T_1 I_0 A_{4,2}$	$R T_v^{-1} T_{\tau}^{-1} T_1^{-2}$
$I_0 p_{4,3}$	p_0	$A_{4,3}^{-1} I_0 T_{\tau} T_1 T_v^{-1} I_0$	$T_1 T_{\tau}$
$I_0 p_{7,1}$	p_0	$A_{7,1}^{-1} I_0 T_v^{-1} I_0$	R
$I_0 p_{7,2}$	$p_{4,1}$	$A_{7,2}^{-1} I_0 T_v^{-1} A_{4,1}$	Id
$I_0(0, 2\sqrt{7})$	$p_{7,6}$	$A_{7,6}^{-1} T_v I_0 T_v I_0$	Id
$I_0(1, \sqrt{7})$	$p_{2,1}$	$A_{2,1}^{-1} T_v I_0 T_1 I_0$	$T_1^2 T_v^{-1}$
$I_0(\tau, 0)$	$(\tau, 0)$	$(T_{\tau} I_0)^{-1} I_0 T_{\tau} I_0$	$R T_{\tau}$
$A_{2,1} p_0$	$p_{7,5}$	$A_{7,5}^{-1} A_{2,1} I_0$	$R T_v T_1^{-1}$
$A_{2,2} p_0$	$p_{4,2}$	$A_{4,2}^{-1} A_{2,2} I_0$	Id
$A_{4,1} p_0$	$p_{7,4}$	$A_{7,4}^{-1} A_{4,1} I_0$	Id
$A_{4,1} p_{4,4}$	$p_{2,2}$	$A_{2,2}^{-1} T_{\tau} A_{4,1} A_{4,4}$	T_1^{-1}
$A_{4,1} p_{7,3}$	p_0	$I_0 T_v^{-1} A_{4,1} A_{7,3}$	R
$A_{4,2} p_0$	$p_{2,2}$	$A_{2,2}^{-1} A_{4,2} I_0$	Id
$A_{4,2} p_{7,2}$	$p_{2,2}$	$A_{2,2}^{-1} T_1 T_{\tau}^{-1} A_{4,2} A_{7,2}$	$T_{\tau} T_1^{-2} T_v$
$A_{4,3} p_0$	p_0	$I_0 T_1^{-1} T_{\tau}^{-1} T_v A_{4,3} I_0$	$T_1^{-1} T_{\tau}^{-1} T_v$
$A_{4,4} p_0$	$p_{7,4}$	$A_{7,4}^{-1} A_{4,4} I_0$	$R T_1^{-1}$
$A_{4,5} p_{2,2}$	$p_{7,5}$	$A_{7,5}^{-1} A_{4,5} A_{2,2}$	$R T_v$
$A_{7,1} p_0$	p_0	$I_0 A_{7,1} I_0$	$R T_v^{-1}$
$A_{7,2} p_{4,1}$	p_0	$I_0 A_{7,2} A_{4,1}$	T_v^{-1}
$A_{7,2} p_{7,3}$	∞	$A_{7,2} A_{7,3}$	R
$A_{7,2} p_{7,4}$	$p_{7,1}$	$A_{7,1}^{-1} A_{7,2} A_{7,4}$	R
$A_{7,3} p_0$	$p_{4,1}$	$A_{4,1}^{-1} A_{7,3} I_0$	$R T_v$
$A_{7,3} p_{4,2}$	$p_{2,1}$	$A_{2,1}^{-1} T_1 A_{7,3} A_{4,2}$	$T_{\tau}^{-1} T_1^2 T_v$
$A_{7,3} p_{7,1}$	$p_{7,4}$	$A_{7,4}^{-1} A_{7,3} A_{7,1}$	Id
$A_{7,3} p_{7,2}$	∞	$A_{7,3} A_{7,2}$	R
$A_{7,4} p_0$	$p_{4,1}$	$A_{4,1}^{-1} A_{7,4} I_0$	Id
$A_{7,5} p_{2,2}$	$p_{4,3}$	$A_{4,3}^{-1} T_1 T_v^{-1} A_{7,5} A_{2,2}$	$R T_{\tau}^{-2} T_1$
$A_{7,5} p_{4,1}$	p_0	$I_0 T_v^{-1} A_{7,5} A_{4,1}$	R
$A_{7,5} p_{4,4}$	p_0	$I_0 T_v^{-1} T_1 A_{7,5} A_{4,4}$	Id
$A_{7,5} p_{7,3}$	$p_{7,6}$	$A_{7,6}^{-1} A_{7,5} A_{7,3}$	Id
$A_{7,5} p_{7,4}$	∞	$A_{7,5} A_{7,4}$	$R T_v$
$A_{7,6} p_0$	p_0	$I_0 T_v^{-1} A_{7,6} I_0$	T_v
$A_{7,6} p_{7,1}$	∞	$A_{7,6} A_{7,1}$	$R T_v$
$A_{7,6} p_{7,2}$	$p_{7,5}$	$A_{7,5}^{-1} A_{7,6} A_{7,2}$	R

Table 4: Action of generators on vertices for $d = 7$

Presentation and fundamental domain for $\Gamma_\infty(\mathcal{H})$:

In this section we study the action of the cusp group $\Gamma_\infty(\mathcal{H}) = \text{Stab}_{\Gamma(\mathcal{H})}(\infty)$ on $\partial_\infty \mathbb{H}^2 \setminus \{\infty\} \simeq \mathbb{H} \times \text{Im } \mathbb{H}$. The referee informed us that a fundamental domain for the action of $\Gamma_\infty(\mathcal{H})$ on $\partial_\infty \mathbb{H}^2 \setminus \{\infty\}$ had been found by Philippe in her thesis [Ph]. In a previous version of this paper we only used a *coarse fundamental domain*, i.e. a domain whose translates cover the space; using her fundamental domain simplified the arguments and computations in this section, so we thank the referee for drawing our attention to it. In fact Philippe gives in [Ph] (Theorem 4.3.11, Proposition 4.3.22 and pp. 103–104) two fundamental domains, related by a sequence of cut-and paste operations. We denote these two domains by $D_\infty^1(\mathcal{H})$ and $D_\infty^2(\mathcal{H})$ and will in fact use both, as they have different geometric/combinatorial advantages. More specifically, $D_\infty^2(\mathcal{H})$ is simpler combinatorially as it is a product of a 4-simplex and a 3-cube, whereas $D_\infty^1(\mathcal{H})$ is a union of two such objects. However each of the two isometric pieces of $D_\infty^1(\mathcal{H})$ is contained in a smaller Cygan ball centered at 0, which makes it easier to use in the covering argument below.

Recall that $\partial_\infty \mathbb{H}^2 \setminus \{\infty\} \simeq \mathbb{H} \times \text{Im } \mathbb{H}$; we will refer to \mathbb{H} as the horizontal direction and $\text{Im } \mathbb{H}$ as the vertical direction. The vertices of the horizontal component of these domains consist in the following points in \mathbb{H} (following the notation in [Ph]):

$$\begin{aligned} p_0 &= 0, & p_3 &= i, & c_1 &= (1 + i/3 + j + k)/2, & c_2 &= (-1 + i/3 - j - k)/2, \\ q_0 &= i/2, & q_{14} &= (i + k)/2, & q_{18} &= (i + j)/2, & q_{16} &= (i - k)/2, & q_{20} &= (i - j)/2. \end{aligned}$$

Proposition 4 ([Ph]) *Consider the subsets $S = \text{Hull}(p_0, p_3, c_1, q_{14}, q_{18})$, $C_1 = \text{Hull}(p_0, q_0, c_1, q_{14}, q_{18})$ and $C_2 = \text{Hull}(p_0, q_0, c_2, q_{16}, q_{20})$ of \mathbb{H} . Then $D_\infty^1(\mathcal{H}) = (C_1 \cup C_2) \times [-1, 1]^3$ and $D_\infty^2(\mathcal{H}) = S \times [-1, 1]^3$ are fundamental domains for the action of $\Gamma_\infty(\mathcal{H})$ on $\partial_\infty \mathbb{H}^2 \setminus \{\infty\} \simeq \mathbb{H} \times \text{Im } \mathbb{H}$.*

We now determine a presentation for $\Gamma_\infty(\mathcal{H})$ by more algebraic methods, understanding this group as a sequence of normal extensions of simpler subgroups. The presentation we give is highly redundant in terms of generators, but we will need this "geometrically complete" set of generators when we are required to algorithmically identify given elements of $\Gamma_\infty(\mathcal{H})$ in terms of the generators. (Also note that [Ph] does not identify a presentation corresponding to the above fundamental domains).

We use the notation of Equations (3), (4) and (5), namely $T_{(\zeta, v)}$, R_w , C_w respectively denote Heisenberg translation by (ζ, v) , Heisenberg rotation by w and conjugation by w . Note that for any unit quaternion $w \in \mathcal{H}$ and purely imaginary $q \in \mathcal{H}$, $\Gamma_\infty(\mathcal{H})$ contains the following Heisenberg translations:

$$T_w = T_{(w, i+j+k)} = \begin{pmatrix} 1 & -\bar{w} & \frac{-1+i+j+k}{2} \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{v_q} = T_{(0, 2q)} = \begin{pmatrix} 1 & 0 & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (13)$$

Lemma 15 $\Gamma_\infty(\mathcal{H})$ admits the following presentation:

$$\Gamma_\infty(\mathcal{H}) = \langle \mathcal{S}_\infty \mid \mathcal{R}_\infty \rangle = \langle R_i, R_\sigma, C_i, C_\sigma, T_1, T_i, T_j, T_k, T_\sigma, T_{v_i}, T_{v_j}, T_{v_k} \mid \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5 \rangle$$

where the \mathcal{A}_n are the following sets of relations:

$$\mathcal{A}_1 = \{R_i^4, R_i^2 R_\sigma^{-3}, (R_i R_\sigma)^3, (R_i^{-1} R_\sigma)^2 R_i^{-1} R_\sigma^{-2}\}$$

$$\mathcal{A}_2 = \{C_i^2, C_\sigma^3, (C_i C_\sigma)^3\}$$

$$\mathcal{A}_3 = \left\{ \begin{array}{l} [T_{v_w}, T], [T_1, T_w] = T_{v_w}^{-2}, [T_1, T_\sigma] = T_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1}, \\ [T_w, T_{\hat{w}}] = T_{v_w \hat{w}}^2, [T_w, T_\sigma] = T_{v_w} T_{v_{\hat{w}}}^{-1} T_{v_{w\hat{w}}}, T_1 T_i T_j T_k = T_\sigma^2 T_{v_i} T_{v_j}^{-1} T_{v_k} \end{array} \right\}$$

where T runs over $T_1, T_i, T_j, T_k, T_\sigma, T_{v_i}, T_{v_j}, T_{v_k}$, w runs over i, j, k , and $\hat{i} = j, \hat{j} = k, \hat{k} = i$.

$$\mathcal{A}_4 = \{C_i R_i C_i = R_i, C_i R_\sigma C_i = R_i R_\sigma^4 R_i, C_\sigma R_i C_\sigma^{-1} = R_\sigma R_i R_\sigma^{-1}, C_\sigma R_\sigma C_\sigma^{-1} = R_\sigma\}$$

$$\mathcal{A}_5 = \{GTG^{-1} = E_{G,T}\},$$

where G runs over $R_i, R_\sigma, C_i, C_\sigma$, T runs over $T_1, T_i, T_j, T_k, T_\sigma, T_{v_i}, T_{v_j}, T_{v_k}$, and $E_{G,T}$ is the entry in the G column and T row of the table below:

	R_i	R_σ	C_i	C_σ
T_1	T_i	T_σ	$T_1 T_{v_j}^{-1} T_{v_k}^{-1}$	T_1
T_i	$T_1^{-1} T_{v_i} T_{v_j} T_{v_k}$	$T_1^{-1} T_k^{-1} T_\sigma T_{v_j} T_{v_k}^2$	$T_i T_{v_j}^{-1} T_{v_k}^{-1}$	T_j
T_j	T_k	$T_1^{-1} T_i^{-1} T_\sigma T_{v_i}^2 T_{v_k}$	$T_j^{-1} T_{v_i}$	T_k
T_k	$T_j^{-1} T_{v_i} T_{v_j} T_{v_k}$	$T_1^{-1} T_j^{-1} T_\sigma T_{v_i} T_{v_j}^2$	$T_k^{-1} T_{v_i}$	T_i
T_σ	$T_\sigma^{-1} T_i T_k T_{v_j} T_{v_k}^{-1}$	$T_1^{-1} T_\sigma$	$T_j^{-1} T_k^{-1} T_\sigma T_{v_j}$	T_σ
T_{v_i}	T_{v_i}	T_{v_i}	T_{v_i}	T_{v_j}
T_{v_j}	T_{v_j}	T_{v_j}	$T_{v_j}^{-1}$	T_{v_k}
T_{v_k}	T_{v_k}	T_{v_k}	$T_{v_k}^{-1}$	T_{v_i}

Proof. To obtain a presentation for $\Gamma_\infty(\mathcal{H})$, we identify 3 of its subgroups, observe that some of them normalize each other, and build up the presentation via a sequence of extensions using the following procedure. Suppose that G is a group with subgroups N and K where N is normal in G and G is an extension of N by K . Suppose also that we know presentations for N and K , $N = \langle S_N \mid R_N \rangle$ and $K = \langle S_K \mid R_K \rangle$. Then G admits the presentation $\langle S_N \cup S_K \mid R_N \cup R_K \cup R \rangle$, where the set R consists of relations of the form $knk^{-1} = n'$ where k runs over all elements of S_K , n runs over the elements of S_N , and $n' \in N$ is expressed as a word in the generators S_N .

The three subgroups of $\Gamma_\infty(\mathcal{H})$ we identify are the rotation, conjugation, and translation subgroups.

The rotation subgroup consists of all Heisenberg rotations R_w where w is a Hurwitz integral unit quaternion. It is isomorphic to the binary tetrahedral group, which has order 24 (see e.g. [CoS]). It admits the presentation

$$\langle R_i, R_\sigma \mid \mathcal{A}_1 \rangle$$

The conjugation subgroup consists of all conjugations by unit quaternions. Elements of this group also correspond to Hurwitz unit integral quaternions, only C_w acts the same as C_{-w} . Thus, this group is isomorphic to the quotient of the binary tetrahedral group by -1 , which is the tetrahedral group (or the alternating group on 4 elements). It admits the presentation

$$\langle C_i, C_\sigma \mid \mathcal{A}_2 \rangle$$

The translation subgroup consists of all Heisenberg translations. It admits the presentation

$$\langle T_1, T_i, T_j, T_k, T_\sigma, T_{v_i}, T_{v_j}, T_{v_k} \mid \mathcal{A}_3 \rangle$$

The rotation subgroup is normalized by the conjugation subgroup. The extension of the rotation subgroup by the conjugation subgroup is a finite group of order 288. We obtain the four relations in \mathcal{A}_4 by conjugating R_i and R_σ by C_i and C_σ .

The translation subgroup is normalized by the rotation-conjugation subgroup. The conjugates of the translation generators by the rotation and conjugation generators are listed in the table. The translations T are along the left side, the rotations/conjugations G are along the top, and the table entry $E_{G,T}$ is the element GTG^{-1} written as a word in the translation generators. \mathcal{A}_5 contains these relations. \square

Corollary 8 *The abelianization of $\Gamma_\infty(\mathcal{H})$ is $(\mathbb{Z}/3\mathbb{Z})^2$, generated by (the images of) R_σ and C_σ .*

Covering depth and $\mathbb{Q}[i, j, k]$ -rational points in $D_\infty(\mathcal{H})$:

We denote as before $B((\zeta, v), r)$ the open extended Cygan ball centered at $p = (\zeta, v) \in \partial_\infty \mathbb{H}^2_{\mathbb{H}}$ with radius r (see Equation 8 for the definition of the extended Cygan metric d_{XC}). Recall that $u(n) = \frac{2}{\sqrt{n}}$ is the height at which balls of depth n appear, in the sense of Corollary 1.

Lemma 16 Let $u = u(5) + \varepsilon = 0.89443$ and H_u the horosphere of height u based at ∞ . Then the prism $D_\infty^1(\mathcal{H}) \times \{u\}$ is covered by the intersections with H_u of the following 17 extended Cygan balls of depth 1: $B((0,0), \sqrt{2})$, $B((\sigma, \pm i \pm j \pm k), \sqrt{2})$, $B((\sigma - 1 - j - k, \pm i \pm j \pm k), \sqrt{2})$.

Proof. Recall from Proposition 4, using the notation from [Ph], that $D_\infty^1(\mathcal{H}) = (C_1 \cup C_2) \times [-1, 1]^3 \subset \mathbb{H} \times \text{Im } \mathbb{H}$, where C_1 and C_2 are the 4-simplices $C_1 = \text{Hull}(p_0, q_0, q_{14}, q_{18}, c_1) \subset \mathbb{H}$ and $C_2 = \text{Hull}(p_0, q_0, q_{16}, q_{20}, c_2) \subset \mathbb{H}$.

Claim: $C_1 \times [0, 1]^3$ is contained in $B_0 \cup B_\sigma$, where $B_0 = B((0,0), \sqrt{2})$ and $B_\sigma = B((\sigma, i+j+k), \sqrt{2})$.

Using coordinates $(x_1 + x_2 i + x_3 j + x_4 k, t_1 i + t_2 j + t_3 k)$ on $\mathbb{H} \times \text{Im } \mathbb{H}$, we separate $C_1 \times [0, 1]^3$ into 2 pieces by the piecewise linear hypersurface $A = \{(x_1 + x_2 i + x_3 j + x_4 k, t_1 i + t_2 j + t_3 k) \in \mathbb{H} \times \text{Im } \mathbb{H} \mid T = L(X)\}$, where we denote $T = t_1 + t_2 + t_3$, $X = x_1 + x_2 + x_3 + x_4$, and L is the piecewise linear function defined by:

$$L(X) = \begin{cases} 2.7 & \text{if } X \in [0, 1/2] \\ -1.4X + 3.4 & \text{if } X \in [1/2, 1] \\ -0.9X + 2.9 & \text{if } X \in [1, 5/3]. \end{cases}$$

Heuristically, the coordinate X (resp. T) measures the (l^1 -) distance from the origin $(0,0)$ in the horizontal (resp. vertical) direction, and the choice of the hypersurface A was inspired by the position of the 40 vertices of $C_1 \times [0, 1]^3$ relative to the Cygan balls $B((0,0), \sqrt{2})$ and $B((\sigma, i+j+k), \sqrt{2})$, see Figure 5 and below.

The claim is then verified by showing that $C_1 \times [0, 1]^3 \cap A^- \subset B((0,0), \sqrt{2})$ and $C_1 \times [0, 1]^3 \cap A^+ \subset B((\sigma, i+j+k), \sqrt{2})$, where A^\pm denote the 2 half-spaces bounded by A , with $(0,0) \in A^-$ and $(\sigma, i+j+k) \in A^+$. This is done as in the proof of Lemma 13, by checking the vertices numerically using Equation (8), then extending the result to their affinely convex hull by Lemma 7.

Since A is only piecewise linear, we first subdivide A^+ and A^- into affinely convex pieces as follows. First we add the base vertices $s_1, \dots, s_5 \in \mathbb{H}$ defined as the respective intersections of the affine segments $[p_0, q_{18}], [p_0, c_1], [p_0, q_{14}]$ with the hyperplane $\{X = 1/2\}$, and $[q_0, c_1], [p_0, c_1]$ with the hyperplane $\{X = 1\}$ (see Figure 5). Explicitly:

$$s_1 = \frac{i+j}{4}, \quad s_2 = \frac{3+i+3j+3k}{20}, \quad s_3 = \frac{i+k}{4}, \quad s_4 = \frac{3+5i+3j+3k}{14}, \quad s_5 = \frac{3+i+3j+3k}{10}.$$

Since $C_1 = \text{Hull}(p_0, q_0, q_{14}, q_{18}, c_1)$ with $X(p_0) = 0$, $X(q_0) = 1/2$, $X(q_{14}) = X(q_{18}) = 1$ and $X(c_1) = 5/3$, this ensures that:

$$\begin{aligned} C_1 \cap \{X \leq 1/2\} &= \text{Hull}(p_0, q_0, s_1, s_2, s_3), \\ C_1 \cap \{1/2 \leq X \leq 1\} &= \text{Hull}(q_0, q_{14}, q_{18}, s_1, s_2, s_3, s_4, s_5), \\ C_1 \cap \{1 \leq X \leq 5/3\} &= \text{Hull}(p_0, q_0, s_1, s_2, s_3). \end{aligned}$$

This produces a total of 10 vertices in the horizontal factor \mathbb{H} , above each of which lies a 3-cube in the vertical factor, spanned by $0, i, j, k, i+j, i+k, j+k, i+j+k$. We check numerically, using Equation (8) that each of these 80 vertices lies in B_0, B_σ , or both as indicated in Figure 5. Finally, we check that each point of intersection of an edge of a cube with the hypersurface A lies in both B_0 and B_σ (of course, the values appearing in the definition of the function L above were chosen to satisfy this property). Explicitly, these intersection points comprise:

- 3 points at height $T = 2.7$ on the top 3 edges of the cube above p_0 ,
- 12 points at height $T = 2.7$ on the top 3 edges of the cubes above q_0, s_1, s_2, s_3 ,
- the 12 vertices at height $T = 2$ (with vertical coordinates $i+j, i+k, j+k$) above q_{18}, s_4, s_5, q_{14} ,
- and 6 points at height $T = 1.4$ on the middle 6 edges of the cube above c_1 .

For example, the relevant point above $c_1 = (1+i/3+j+k)/2$ on the vertical edge $[j, i+j]$ has coordinates $(c_1, 0.4i+j) = (1/2, 1/6, 1/2, 1/2, 0.4, 1, 0)$ and satisfies:

$$d_{XC}((c_1, 0.4i+j, u), (0, 0, 0)) \simeq 1.410 < \sqrt{2},$$

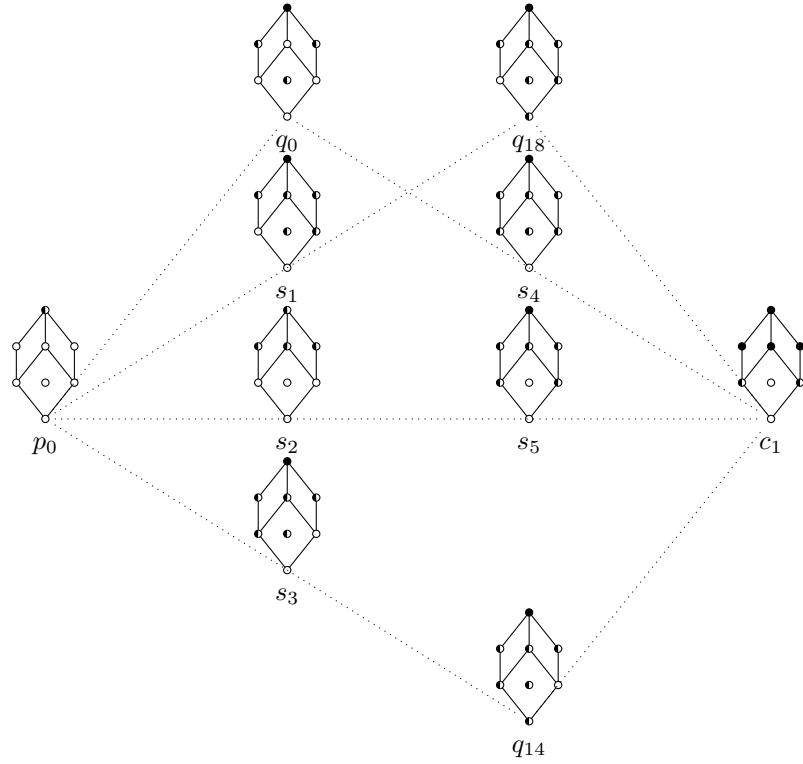


Figure 5: Vertices of the prism $C_1 \times [0, 1]^3$ ($p_0, q_0, q_{14}, q_{18}, c_1$) and its intersections with the level sets $\{X = 1/2\}$ (s_1, s_2, s_3) and $\{X = 1\}$ (s_4, s_5). The vertices of the horizontal base are labelled by name, above each lies a vertical 3-cube spanned by $0, i, j, k, i+j, i+k, j+k, i+j+k$, ordered here from bottom to top and left to right. A vertex is colored white if it belongs to $B((0, 0), \sqrt{2})$, black if it belongs to $B((\sigma, i+j+k), \sqrt{2})$, and black and white if it belongs to both. Cubes in a common column belong to a common level set of $X = x_1 + x_2 + x_3 + x_4$.

$$d_{XC}((c_1, 0.4i + j, u), (\sigma, i + j + k, 0)) \simeq 1.394 < \sqrt{2}.$$

All other computations are similar, and the claim follows.

We now complete the proof of Lemma 16 from the claim. Replacing C_1 by C_2 by negating the first, third and fourth horizontal coordinates gives that the other subprism $C_2 \times [0, 1]^3$ is contained in $B((0, 0), \sqrt{2}) \cup B((\sigma - 1 - j - k, i + j + k), \sqrt{2})$, since that transformation is an (affine) isometry of the Cygan metric.

Finally, by likewise negating some or all of the vertical coordinates we get that the 14 other subprisms $C_1 \times [-1, 0] \times [0, 1]^2, C_2 \times [-1, 0] \times [0, 1]^2, \dots$ are contained in the extended Cygan balls obtained by switching the corresponding signs of the vertical coordinates of the center. This results in the 17 Cygan balls in the statement of Lemma 16. \square

Corollary 9 *The covering depth of $\Gamma(\mathcal{H})$ is at most 4.*

We now use the other fundamental domain to list the points up to depth 4, namely $D_\infty^2(\mathcal{H}) = S \times (-1, 1]^3$, where S is the simplex $S = \text{Hull}(0, \frac{i+j}{2}, \frac{i+k}{2}, \frac{1}{2}(1 + \frac{i}{3} + j + k), i) \subset \mathbb{H}$.

Lemma 17 *The $\mathbb{Q}[i, j, k]$ -rational points in $D_\infty^2(\mathcal{H})$ with depth at most 4 are, in horospherical coordinates:*

- Depth 1: $(0, 0)$;
- Depth 2: $(0, i + j)$;

- *Depth 3:* $(0, \frac{\pm 2i \mp 2j \pm 2k}{3}), (\frac{1+3i+j+k}{6}, \frac{3i-j-k}{3}), (\frac{1+3i+j+k}{6}, \frac{-i+3j+k}{3}), (\frac{1+i+j+3k}{6}, \frac{3i-j-k}{3})$ (5 points total);
- *Depth 4:* $(0, i), (0, i+j+k), (\frac{i+j}{2}, \frac{\pm i \pm j \pm k}{2})$ (10 points total);

and all of their Γ_∞ -translates.

Proof. We use the general procedure outlined in Lemma 14.

Let $\Gamma_\infty^{fin} \subseteq \Gamma_\infty$ be the finite subgroup of order 288 described in Lemma 15. Γ_∞^{fin} describes two different actions by units $u \in \mathcal{H}$ on $\partial_\infty H_\mathbb{H}^2$. One is the action by left multiplication on the first coordinate

$$R_u : p = (z, t) \mapsto (uz, t)$$

The other is by conjugation

$$C_u : p = (z, t) \mapsto (uzu^{-1}, utu^{-1})$$

Consider the standard lift P of a point $p \in \partial H_\mathbb{K}^2 \setminus \{\infty\}$:

$$P = \begin{pmatrix} \frac{-|z|^2 + it_1 + jt_2 + kt_3}{2} \\ z \\ 1 \end{pmatrix}$$

For P to be the vector form of a $\mathbb{Q}(i, j, k)$ -rational point p of depth n , it must satisfy the following:

1. P does not have depth less than n
2. Pq 's coordinates must be in \mathcal{H} for some q with $|q|^2 = n$. Write $z = z_1 + iz_2 + jz_3 + kz_4$ and $q = q_1 + iq_2 + jq_3 + kq_4$. Then we have

$$\begin{aligned} z_1 q_1 - z_2 q_2 - z_3 q_3 - z_4 q_4, \\ z_1 q_2 + z_2 q_1 + z_3 q_4 - z_4 q_3, \\ z_1 q_3 + z_3 q_1 - z_2 q_4 + z_4 q_2, \\ z_1 q_4 + z_4 q_1 + z_2 q_3 - z_3 q_2 \end{aligned} \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2} \quad (14)$$

and

$$\begin{aligned} \frac{1}{2}(-|z|^2 q_1 - t_1 q_2 - t_2 q_3 - t_3 q_4), \\ \frac{1}{2}(-|z|^2 q_2 + t_1 q_1 + t_2 q_4 - t_3 q_3), \\ \frac{1}{2}(-|z|^2 q_3 + t_2 q_1 - t_1 q_4 + t_3 q_2), \\ \frac{1}{2}(-|z|^2 q_4 + t_3 q_1 + t_1 q_3 - t_2 q_2) \end{aligned} \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2} \quad (15)$$

To list the points of depth n , we first assume we have already listed the points of depth less than n , and then we follow these steps:

1. Find all $q \in \mathcal{H}$ with $|q|^2 = n$. From the geometry of numbers, we know that there will only be finitely many q 's. Even better, we only need to consider one representative for every Γ_∞^{fin} orbit of q 's under the Γ_∞^{fin} action on \mathcal{H} . The reason for this is that if u is a unit, then
 - if Pqu has coordinates in \mathcal{H} then so does Pq .
 - if Puq has coordinates in \mathcal{H} then $C_u^{-1}Pq = u^{-1}Puq$ also has coordinates in \mathcal{H} .
 - if $Puqu^{-1}$ has coordinates in \mathcal{H} then $C_u^{-1}Pq$ also has coordinates in \mathcal{H} .

Thus, we will always find a point in the Γ_∞^{fin} -orbit of P using our chosen q .

2. For each q , find all potential first coordinates z , which are solutions to (14). Like in the previous step, we only need one z representing each Γ_∞^{fin} -orbit. We pick z inside the fundamental domain C , and if necessary we modify our choice of q by a unit so that $zq \in \mathcal{H}$.

3. For each z , find all solutions to (15) with $-1 \leq t_1, t_2, t_3 < 1$. Keep only those solutions that are not on the list of points of depth less than n .

In practice, here's how that goes:

Depth 1: In step 1, q must be a unit, and there is a single orbit of q 's represented by 1. In step 2, we must have $z \in \mathcal{H} \cap D_\infty^h(\mathcal{H})$, and so $z = 0$ is the only solution to (14).

It follows that $|z|^2 = 0$, and so in step 3, the solutions to (15) satisfying the inequalities are $(t_1, t_2, t_3) \in \mathbb{Z}^3$ with $0 \leq t_1, t_2, t_3 \leq 2$ and t_1, t_2, t_3 all even. This gives us exactly the depth 1 point listed above.

Depth 2: In step 1, there is one orbit of q 's, represented by $1 + i$. In step 2, we find $z \in C$ by solving (14) which specializes to

$$z_1 - z_2, z_1 + z_2, z_3 + z_4, z_4 - z_3 \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

Using some basic algebra, we conclude that

$$2z_1, 2z_2, 2z_3, 2z_4 \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

The possible values of z in C are then represented by $0, \frac{i}{2}, \frac{i+j}{2}$.

The values of $|z|^2$ corresponding to these z 's are $0, \frac{1}{4}, \frac{1}{2}$ respectively. In step 3, (15) specializes to

$$\frac{1}{2}(-|z|^2 - t_1), \frac{1}{2}(-|z|^2 + t_1), \frac{1}{2}(-t_2 + t_3), \frac{1}{2}(t_3 - t_2) \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

With $z = 0$, we find that taking two of t_1, t_2, t_3 equal to 1 and the other equal to 0 gives a solution. These solutions are all new points of depth 2, and are Γ_∞^{fin} -equivalent (by conjugation). This is the depth 2 point listed above.

For the other two z 's, with $|z|^2 = \frac{1}{2}$ and $\frac{1}{4}$, there are no solutions to (15).

Depth 3: In step 1, there is one orbit of q 's, represented by $1 - j - k$. In step 2, we find z by solving (14) which specializes to

$$z_1 + z_3 + z_4, z_2 - z_3 + z_4, -z_1 + z_3 + z_4, -z_1 + z_4 - z_2 \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

Using some basic algebra, we conclude that

$$3z_1, 3z_2, 3z_3, 3z_4 \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

The possible values of z in C are then represented by

$$0, \frac{i}{3}, \frac{i+j}{3}, \frac{1+3i+j+k}{3}$$

The values of $|z|^2$ corresponding to these z 's are $0, \frac{1}{9}, \frac{2}{9}$, and $\frac{1}{3}$ respectively. In step 3, (15) specializes to

$$\frac{1}{2}(-|z|^2 + t_2 + t_3), \frac{1}{2}(t_1 - t_2 + t_3), \frac{1}{2}(|z|^2 + t_2 + t_1), \frac{1}{2}(|z|^2 + t_3 - t_1) \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

Using some basic algebra, we find that

$$\frac{3|z|^2}{2}, \frac{3t_1}{2}, \frac{3t_2}{2}, \frac{3t_3}{2} \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

meaning that $3|z|^2 \in \mathbb{Z}$ and its parity is the same as that of $3t_1, 3t_2, 3t_3$. Therefore, we see that $|z|^2 = 0$ or $\frac{1}{3}$. We then solve (15), and in addition to finding the depth 1 point we already have, we find five new solutions at depth 3.

Depth 4:

In step 1, there is one orbit of q 's, represented by 2. In step 2, we find z by solving (14) which specializes to

$$2z_1, 2z_2, 2z_3, 2z_4 \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

The possible values of z in C are then represented by

$$0, \frac{1}{2}, \frac{i+j}{2}$$

The values of $|z|^2$ corresponding to these z 's are $0, \frac{1}{4}$, and $\frac{1}{2}$ respectively. In step 3, (15) specializes to

$$|z|^2, t_1, t_2, t_3 \in \mathbb{Z}, \text{ or } \in \mathbb{Z} + \frac{1}{2}$$

Thus, we must have $z = 0$ or $z = \frac{i+j}{2}$. When we solve for t_1, t_2, t_3 , we recover all the depth 1 and 2 solutions we already found, as well as ten new points of depth 4. \square

Integral lifts of representatives of $\Gamma_\infty(\mathcal{H})$ -orbits of these points are:

$$\begin{aligned} p_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & p_2 &= \begin{bmatrix} -1 \\ 0 \\ i+j \end{bmatrix} & p_{3,1} &= \begin{bmatrix} 1 \\ 0 \\ -i+j-k \end{bmatrix} & p_{3,2} &= \begin{bmatrix} 1 \\ 0 \\ i-j+k \end{bmatrix} \\ p_{3,3} &= \begin{bmatrix} 1 \\ \frac{1-i-j+k}{2} \\ \frac{-1-3i+j+k}{2} \end{bmatrix} & p_{3,4} &= \begin{bmatrix} 1 \\ -k \\ \frac{-1-i-3j-k}{2} \end{bmatrix} & p_{3,5} &= \begin{bmatrix} 1 \\ \frac{1-i+j-k}{2} \\ \frac{-1-i-j-3k}{2} \end{bmatrix} & p_{4,1} &= \begin{bmatrix} i \\ 0 \\ 2 \end{bmatrix} \\ p_{4,2} &= \begin{bmatrix} i+j+k \\ 0 \\ 2 \end{bmatrix} & p_{4,3} &= \begin{bmatrix} 1 \\ 1-i \\ -1-i-j-k \end{bmatrix} & p_{4,4} &= \begin{bmatrix} 1 \\ -i-k \\ -1+i-j-k \end{bmatrix} & p_{4,5} &= \begin{bmatrix} 1 \\ -i+k \\ -1-i+j-k \end{bmatrix} \\ p_{4,6} &= \begin{bmatrix} 1 \\ -1-i \\ -1+i+j-k \end{bmatrix} & p_{4,7} &= \begin{bmatrix} 1 \\ 1-j \\ -1-i-j+k \end{bmatrix} & p_{4,8} &= \begin{bmatrix} 1 \\ -j-k \\ -1+i-j+k \end{bmatrix} & p_{4,9} &= \begin{bmatrix} 1 \\ -j+k \\ -1-i+j+k \end{bmatrix} \\ p_{4,10} &= \begin{bmatrix} 1 \\ -1-j \\ -1+i+j+k \end{bmatrix} \end{aligned}$$

Generators: The following elements $A_\alpha \in \Gamma(\mathcal{H})$ map the point $\infty = [1, 0, 0]^T$ to the corresponding p_α as above (for $\alpha = 0; 2; 3, 1; \dots; 3, 5; 4, 1; \dots; 4, 10$) :

$$\begin{aligned}
A_0 = I_0 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & A_2 &= \begin{bmatrix} -1 & 0 & i+j \\ 0 & 1 & 0 \\ i+j & 0 & 1 \end{bmatrix} \\
A_{3,1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i+j-k & 0 & 1 \end{bmatrix} & A_{3,2} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i-j+k & 0 & 1 \end{bmatrix} \\
A_{3,3} &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1-i-j+k}{-1-3i+j+k} & 1 & 0 \\ \frac{-1-3i+j+k}{2} & \frac{-1-i-j+k}{2} & 1 \end{bmatrix} & A_{3,4} &= \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ \frac{-1+i-3j-k}{2} & -k & 1 \end{bmatrix} \\
A_{3,5} &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1-i+j-k}{-1-i-j-3k} & 1 & 0 \\ \frac{-1-i-j-3k}{2} & \frac{-1-i+j-k}{2} & 1 \end{bmatrix} & A_{4,1} &= \begin{bmatrix} i & 0 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & -i \end{bmatrix} \\
A_{4,2} &= \begin{bmatrix} i+j+k & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -i-j-k \end{bmatrix} & A_{4,3} &= \begin{bmatrix} 1 & 0 & 0 \\ 1-i & 1 & 0 \\ -1-i-j-k & -1-i & 1 \end{bmatrix} \\
A_{4,4} &= \begin{bmatrix} 1 & 0 & 0 \\ -i-k & 1 & 0 \\ -1+i-j-k & -i-k & 1 \end{bmatrix} & A_{4,5} &= \begin{bmatrix} 1 & 0 & 0 \\ -i+k & 1 & 0 \\ -1-i+j-k & -i+k & 1 \end{bmatrix} \\
A_{4,6} &= \begin{bmatrix} 1 & 0 & 0 \\ -1-i & 1 & 0 \\ -1+i+j-k & 1-i & 1 \end{bmatrix} & A_{4,7} &= \begin{bmatrix} 1 & 0 & 0 \\ 1-j & 1 & 0 \\ -1-i-j+k & -1-j & 1 \end{bmatrix} \\
A_{4,8} &= \begin{bmatrix} 1 & 0 & 0 \\ -j-k & 1 & 0 \\ -1+i-j+k & -j-k & 1 \end{bmatrix} & A_{4,9} &= \begin{bmatrix} 1 & 0 & 0 \\ -j+k & 1 & 0 \\ -1-i+j+k & -j+k & 1 \end{bmatrix} \\
A_{4,10} &= \begin{bmatrix} 1 & 0 & 0 \\ -1-j & 1 & 0 \\ -1+i+j+k & 1-j & 1 \end{bmatrix}
\end{aligned}$$

Relations: We find a complete set of relations by applying generators to points of depth at most 4 as described in part (5) of section 2.4. The detailed steps of the computation can be found in the companion files in [\[MCode\]](#). The direct output is a presentation with 33 generators and 968,480 relations, which is unfortunately too large for Magma to handle directly (the text file for this presentation, `quaternions.m`, is slightly over 200 MB and can also be found at [\[MCode\]](#)).

To illustrate the steps involved we compute by hand some of the cycles and relations corresponding to triples (a, b, c) for which the point $A_a(p_b)$ has depth at most 4, hence is of the form $\gamma_\infty(p_c)$ for some $\gamma_\infty \in \Gamma_\infty$ and $p_c \in D_\infty$ with depth at most 4. In the notation of part (5) of section 2.4, this corresponds to $\gamma_\infty^1 = \text{Id}$ and $\gamma_\infty^3 = \gamma_\infty$. The results are listed in Tables 5–9, with the same notation as in the example of section 2.5.

We would also like to make explicit certain relations in the group, which essentially follow from these tables.

Proposition 5 *The following relations hold among the generators of $\Gamma_\infty(\mathcal{H})$ and $I = I_0$:*

$$\begin{aligned}
&I^2, [I, R_i], [I, R_\sigma], [I, C_i], [I, C_\sigma], (T_{v_i} I)^3 = R_i C_i, (T_{v_i} T_{v_j} I)^4 = R_i^2, (T_{v_i} T_{v_j} T_{v_k} I)^6, \\
&[IT_{v_i}^{-2} IR_i T_{v_i}^{-1} C_i, R_\sigma], [C_\sigma^{-1} (IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1})^3 R_i^2 C_\sigma, R_i], [C_\sigma^{-1} (IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1})^3 R_i^2 C_\sigma, R_\sigma], \\
&[C_\sigma^{-1} (IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1})^3 R_i^2 C_\sigma, C_\sigma], (IT_{v_i}^{-2} IR_i T_{v_i}^{-1} C_i)^2, IT_{v_i}^{-1} (IT_{v_j})^2 = (T_{v_j} IT_{v_i}^{-1})^2 R_i C_i, \\
&[(T_{v_i} T_{v_j} I)^2, C_\sigma^{-1} (IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1})^3 R_i^2 C_\sigma], IT_1 T_\sigma^{-1} IT_\sigma I = T_1 T_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1} C_\sigma^{-1}
\end{aligned} \tag{16}$$

It is straightforward to check that each of these relations hold. However we find it instructive to illustrate how some of them can be deduced from the cycle relations in the tables. We now give more details on how to obtain the relations $(T_{v_i} T_{v_j} T_{v_k} I)^6$ and $[(T_{v_i} T_{v_j} I)^2, C_\sigma^{-1}(IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1})^3 R_i^2 C_\sigma]$. The rest of the details are either straightforward or similar to these two.

Substituting for A_2 in (2) using (1) (both from Table 6), we obtain

$$A_{4,2} = IT_{v_i}^{-1} T_{v_j}^{-1} IT_{v_i}^{-1} T_{v_k}^{-1} T_{v_i} T_{v_j} IT_{v_i} T_{v_j} IC_\sigma^{-1} R_i C_\sigma T_{v_i} T_{v_j} T_{v_k}^{-1} C_\sigma^{-1} C_i C_\sigma$$

Use the relations $C_\sigma T_{v_w} C_\sigma^{-1} = T_{v_{\hat{w}}}$ and $[R_i, T_{v_w}] = 1$ from $\Gamma_\infty(\mathcal{H})$ and the relations (5) and (6) from Table 9 to get

$$A_{4,2} = C_\sigma^{-1} IT_{v_j}^{-1} T_{v_k}^{-1} IT_{v_i} IT_{v_j} T_{v_k} IT_{v_j} T_{v_k} T_{v_i}^{-1} R_i C_i C_\sigma$$

Use the relation $(IT_{v_i})^3 = R_i C_i$ to make the substitution $IT_{v_i} I = T_{v_i}^{-1} IT_{v_i}^{-1} R_i C_i$.

$$A_{4,2} = C_\sigma^{-1} IT_{v_j}^{-1} T_{v_k}^{-1} T_{v_i}^{-1} IT_{v_i}^{-1} R_i C_i T_{v_j} T_{v_k} IT_{v_j} T_{v_k} T_{v_i}^{-1} R_i C_i C_\sigma$$

Use the relations $[C_i, T_{v_i}] = 1$ and $C_i T_{v_w} C_i^{-1} = T_{v_w}^{-1}$ if $w \neq i$ from $\Gamma_\infty(\mathcal{H})$ to get

$$\begin{aligned} A_{4,2} &= C_\sigma^{-1} IT_{v_j}^{-1} T_{v_k}^{-1} T_{v_i}^{-1} IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1} IT_{v_j}^{-1} T_{v_k}^{-1} T_{v_i}^{-1} R_i^2 C_\sigma \\ &= C_\sigma^{-1} (IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1})^3 R_i^2 C_\sigma \end{aligned}$$

The relation $(T_{v_i} T_{v_j} T_{v_k} I)^6$ comes from substituting this expression for $A_{4,2}$ in the relation (4) from Table 8. The relation $[(T_{v_i} T_{v_j} I)^2, C_\sigma^{-1}(IT_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1})^3 R_i^2 C_\sigma]$ comes from first observing that (3) in Table 8 and (2) in Table 6 are conjugate to each other by A_2 . Since both are equal to $A_{4,2}$, we obtain $[A_2, A_{4,2}] = 1$. Then to get the relation that appears in the presentation, we substitute for A_2 using (1) from Table 6 and for $A_{4,2}$ using the expression obtained above.

Corollary 10 *The abelianization of $\Gamma(\mathcal{H})$ is $\mathbb{Z}/3\mathbb{Z}$, generated by (the image of) R_σ , or trivial.*

Proof. By Theorem 3, $\Gamma = \Gamma(\mathcal{H})$ is generated by $\Gamma_\infty(\mathcal{H})$ and I_0 , hence by \mathcal{S}_∞ and I_0 (where \mathcal{S}_∞ denotes our generating set for $\Gamma_\infty(\mathcal{H})$, see Lemma 15). Therefore Γ is a quotient of the abstract group Δ with generators \mathcal{S}_∞ and I_0 and relations \mathcal{R}_∞ together with the relations listed in Proposition 5 (with \mathcal{S}_∞ the set of relations for $\Gamma_\infty(\mathcal{H})$, see Lemma 15). By Corollary 8, the abelianization of Δ is a quotient of $(\mathbb{Z}/3\mathbb{Z})^2$, generated by (the images of) R_σ and C_σ . By inspecting the relations listed in Proposition 5, we see that the abelianization of Δ is $\mathbb{Z}/3\mathbb{Z}$, generated by (the image of) R_σ . \square

We then obtain Theorem 2 by combining these results with a computational check using the full presentation. More specifically, while the full presentation is too large for Magma to handle at once, we found that it could handle files approximately 10 times smaller in a few hours. To check that the abelianization remains unchanged after adding the roughly 10^6 relations of the mondo presentation, we subdivide the file with all relations into 10 pieces (each with approximately 10^5 relations), see files `mondo1.m`, ..., `mondo10.m` at [MCode](#) (each of these also contains the relations from $\Gamma_\infty(\mathcal{H})$). Recall from Corollary 8 that the abelianization of $\Gamma_\infty(\mathcal{H})$ is $(\mathbb{Z}/3\mathbb{Z})^2$. We add to each file enough relations to bring the abelianization down to $\mathbb{Z}/3\mathbb{Z}$ - in practice we simply add the relations $C_\sigma = 1 = I_0$ (which hold in the abelianization by the above, but of course not in the group). This produces the files `mondo1modified.m`, ..., `mondo10modified.m`, for which Magma is able to compute that all 10 abelianizations are $\mathbb{Z}/3\mathbb{Z}$, proving Theorem 2.

$\mathbf{A} \cdot \mathbf{p}$	\mathbf{p}'	$\mathbf{A}_{\mathbf{p}'}^{-1} \mathbf{W}^{-1} \mathbf{A} \mathbf{A}_{\mathbf{p}}$	\mathbf{W}'
Ip_0	∞	I^2	Id
$I(\sigma, i + j + k)$	p_0	$IT_1 T_{\sigma}^{-1} IT_{\sigma} I$	$T_1 T_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1} C_{\sigma}^{-1}$
Ip_2	p_0	$IT_{v_i} T_{v_j} IA_2$	$T_{v_i}^{-1} T_{v_j}^{-1} R_i^2$
$Ip_{3,1}$	p_0	$IT_{v_i} T_{v_j} T_{V_k} IA_{3,1}$	Id
$Ip_{3,2}$	p_0	$IT_{v_i}^{-1} T_{v_j} T_{V_k} IA_{3,2}$	Id
$Ip_{3,3}$	p_0	$IT_{\sigma} T_i^{-1} T_{v_j} T_{v_k} IA_{3,3}$	Id
$Ip_{4,1}$	p_0	$IT_{v_i}^2 IA_{4,1}$	$T_{v_i}^{-1} R_i C_i$
$Ip_{4,2}$	$p_{3,1}$	$A_{3,1}^{-1} T_{v_i} T_{v_j} T_{v_k} IA_{4,2}$	$R_i^1 T_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1}$
$Ip_{4,3}$	p_0	$IT_{v_i}^2 T_{v_j} T_{v_k}^2 T_k^{-1} T_i^{-1} IA_{4,3}$	Id
$Ip_{4,4}$	p_0	$IT_k^{-1} T_1^{-1} T_{v_j}^2 T_{v_k} IA_{4,4}$	Id

Table 5: Cycles coming from I

$\mathbf{A} \cdot \mathbf{p}$	\mathbf{p}'	$\mathbf{A}_{\mathbf{p}'}^{-1} \mathbf{W}^{-1} \mathbf{A} \mathbf{A}_{\mathbf{p}}$	\mathbf{W}'
$A_2 p_0$	p_0	$I_0 T_{v_j}^{-1} T_{v_i}^{-1} A_2 I_0$	$A_2 = (T_{v_i} T_{v_j} I_0)^2 \quad (1)$
$A_2 p_2$	∞	A_2^2	R_i^2
$A_2 p_{3,1}$	p_0	$I_0 T_{v_i}^{-1} T_{v_j}^{-1} T_{v_k}^{-1} A_2 A_{3,1}$	$R_{\sigma}^{-1} R_i^{-1} R_{\sigma} C_{\sigma}^{-1} C_i C_{\sigma} T_{v_k}$
$A_2 p_{3,3}$	$p_{3,3}$	$A_{3,3}^{-1} A_2^{-1} R_i^{-1} T_{v_i} T_{v_j} C_{\sigma}^{-1} C_i C_{\sigma} A_{3,3}$	$C_{\sigma}^{-1} C_i C_{\sigma} R_i$
$A_2 p_{4,1}$	p_2	$A_2^{-1} T_{v_i}^{-1} A_2 A_{4,1}$	$C_{\sigma} R_i^{-1} C_{\sigma}^{-1} T_{v_i} T_{v_j} C_{\sigma} C_i C_{\sigma}^{-1}$
$A_2 p_{4,2}$	p_2	$A_2^{-1} T_{v_k}^{-1} A_2 A_{4,2}$	$C_{\sigma}^{-1} R_i C_{\sigma} T_{v_i} T_{v_j} T_{v_k}^{-1} C_{\sigma}^{-1} C_i C_{\sigma} \quad (2)$
$A_2 p_{4,3}$	p_2	$A_2^{-1} T_{\sigma}^{-1} A_2 A_{4,3}$	$T_1^{-1} T_{\sigma} T_k^{-1} T_{v_i} C_i$

Table 6: Cycles coming from A_2

$\mathbf{A.p}$	\mathbf{p}'	$\mathbf{A}_{\mathbf{p}'}^{-1} \mathbf{W}^{-1} \mathbf{A} \mathbf{A}_{\mathbf{p}}$	\mathbf{W}'
$A_{3,1}p_0$	p_0	$IA_{3,1}I$	$T_{v_i}^{-1}T_{v_j}^{-1}T_{v_k}^{-1}$
$A_{3,2}p_0$	p_0	$IA_{3,2}I$	$T_{v_i}T_{v_j}^{-1}T_{v_k}^{-1}$
$A_{3,2}p_{4,1}$	$p_{3,1}$	$A_{3,1}^{-1}A_{3,2}A_{4,1}$	$R_iC_iT_{v_i}^{-1}$
$A_{3,3}p_0$	p_0	$IA_{3,3}I$	$T_{v_j}^{-1}T_{v_k}^{-1}T_iT_{\sigma}^{-1}$

Table 7: Cycles coming from $A_{3,1}, A_{3,2}, A_{3,3}$

$\mathbf{A.p}$	\mathbf{p}'	$\mathbf{A}_{\mathbf{p}'}^{-1} \mathbf{W}^{-1} \mathbf{A} \mathbf{A}_{\mathbf{p}}$	\mathbf{W}'
$A_{4,1}p_0$	p_0	$IT_{v_i}^{-1}A_{4,1}I$	$T_{v_i}^2R_i^{-1}C_i$
$A_{4,1}p_2$	p_2	$A_2^{-1}T_{v_j}A_{4,1}A_2$	$T_{v_i}^{-1}T_{v_j}R_iC_i$
$A_{4,1}p_{3,1}$	$p_{3,2}$	$A_{3,2}^{-1}C_iT_{v_i}^{-1}A_{4,1}A_{3,1}$	R_i^{-1}
$A_{4,1}p_{4,1}$	∞	$A_{4,1}^2$	Id
$A_{4,1}p_{4,3}$	$p_{4,4}$	$A_{4,4}^{-1}C_iT_{v_i}^{-1}R_i^{-1}R_{\sigma}^2A_{4,1}A_{4,3}$	R_{σ}^{-1}
$A_{4,2}p_0$	$p_{3,2}$	$A_{3,2}^{-1}C_iT_{v_i}^{-1}T_{v_j}^{-1}T_{v_k}^{-1}A_{4,2}I$	$T_{v_i}T_{v_j}T_{v_k}R_i^2C_i$
$A_{4,2}p_2$	p_2	$A_2^{-1}T_{v_k}^{-1}A_{4,2}A_2$	$T_{v_i}T_{v_j}T_{v_k}C_{\sigma}^{-1}R_iC_iC_{\sigma}$ (3)
$A_{4,2}p_{3,1}$	p_0	$IT_{v_i}^{-1}T_{v_j}^{-1}T_{v_k}^{-1}A_{4,2}A_{3,1}$	$T_{v_i}T_{v_j}T_{v_k}R_i^2$
$A_{4,2}p_{3,3}$	$p_{3,3}$	$A_{3,3}^{-1}C_{\sigma}^{-1}C_iC_{\sigma}T_{v_i}^{-1}T_{v_j}^{-1}R_i^2A_{4,2}A_{3,3}$	$C_iT_j^{-1}T_k^{-1}T_{v_j}T_{v_k}^2$
$A_{4,2}p_{4,2}$	∞	$A_{4,2}^2$	Id (4)
$A_{4,2}p_{4,3}$	$p_{4,4}$	$A_{4,4}^{-1}C_iT_{v_i}^{-1}T_{v_j}^{-1}T_{v_k}^{-1}R_{\sigma}^{-1}R_iA_{4,2}A_{4,3}$	$C_iT_{v_i}^{-1}T_{v_k}^{-1}R_{\sigma}R_iR_{\sigma}^{-1}R_i^{-1}R_{\sigma}^{-1}T_iT_k$
$A_{4,3}p_0$	p_0	$IA_{4,3}I$	$T_iT_kT_{v_i}^{-2}T_{v_j}^{-1}T_{v_k}^{-2}$
$A_{4,4}p_0$	p_0	$IA_{4,4}I$	$T_1T_kT_{v_j}^{-2}T_{v_k}^{-1}$
$A_{4,4}p_{4,1}$	$p_{4,3}$	$A_{4,3}^{-1}R_{\sigma}A_{4,4}A_{4,1}$	$C_iR_{\sigma}R_i^{-1}R_{\sigma}R_i^{-1}R_{\sigma}^{-1}T_{v_i}^{-1}$

Table 8: Cycles coming from $A_{4,1}, A_{4,2}, A_{4,3}, A_{4,4}$

6 Appendix: presentation for the Eisenstein-Picard modular group $\text{PU}(2, 1, \mathcal{O}_3)$

The presentation for the Eisenstein-Picard modular group $\text{PU}(2, 1, \mathcal{O}_3)$ which we obtain is $\langle T_1, R, I \mid \mathcal{R}_3 \rangle$, where \mathcal{R}_3 is the following set of 19 relations:

$$\begin{aligned}
I^2 &= \text{Id} \\
R^{-1}IRI &= \text{Id} \\
R^6 &= \text{Id} \\
RT_1R^{-1}T_1^{-1}R^{-1}T_1R &= \text{Id} \\
IT_1IRT_1IT_1^{-1}IR^{-1}T_1^{-1} &= \text{Id} \\
T_1^{-1}IT_1IRT_1IT_1^{-1}R^{-1}I &= \text{Id} \\
R^{-2}T_1^{-1}IR^2IR^{-1}T_1RT_1^{-1} &= \text{Id} \\
IR^2T_1IR^2T_1R^2IT_1 &= \text{Id} \\
IRT_1^{-1}R^{-1}T_1IT_1^{-1}IT_1R^{-1}T_1^{-1}R &= \text{Id} \\
T_1^{-1}RT_1^2R^{-1}T_1^{-1}RT_1^{-1}R^{-1}T_1R^{-1}T_1R &= \text{Id} \\
IT_1^{-1}RT_1IR^{-3}T_1RIRT_1 &= \text{Id} \\
IR^2T_1^{-1}R^{-1}IR^{-1}T_1^{-1}R^{-2}IT_1^{-1}R^2 &= \text{Id} \\
IR^{-1}T_1IR^{-1}IT_1^{-1}R^{-2}T_1IT_1^{-1}IR^{-2}T_1^{-1} &= \text{Id} \\
IT_1^{-1}R^{-2}T_1IT_1^{-1}R^{-2}T_1IT_1^{-1}IT_1^{-1}R^{-1}IT_1R^{-1} &= \text{Id} \\
T_1IT_1^2IRT_1^{-1}IRT_1IR^{-2}T_1IT_1^{-2}IT_1^{-1}I &= \text{Id} \\
T_1R^{-1}IT_1^{-1}R^2IT_1^{-1}IT_1IR^{-1}T_1IT_1^{-1}IRT_1^{-1}R^{-1}I &= \text{Id} \\
T_1IT_1^{-1}R^2T_1^{-1}R^{-1}T_1R^{-1}T_1IT_1^{-1}R^{-2}T_1IT_1^{-1}RIT_1^{-1}R^2T_1^{-1}R^{-1} &= \text{Id} \\
T_1^{-2}RT_1IT_1IT_1^{-1}R^2T_1RIT_1^{-1}RT_1^2R^{-1}IT_1^{-1}R^{-2}T_1R^{-3}T_1IT_1R &= \text{Id} \\
(T_1^{-1}RT_1R^{-2}T_1IT_1IT_1^{-1}R^{-2}T_1IT_1^2)^2 &= \text{Id}
\end{aligned}$$

Image of a point	Cycle of points	Relation
$R_i p_0 = p_0$	$\infty \xrightarrow{I_0} p_0 \xrightarrow{R_i} p_0 \xrightarrow{I_0} \infty$	$[R_i, I_0] = \text{Id}$ (5)
$R_\sigma p_0 = p_0$	$\infty \xrightarrow{I_0} p_0 \xrightarrow{R_\sigma} p_0 \xrightarrow{I_0} \infty$	$[R_\sigma, I_0] = \text{Id}$
$C_i p_0 = p_0$	$\infty \xrightarrow{I_0} p_0 \xrightarrow{C_i} p_0 \xrightarrow{I_0} \infty$	$[C_i, I_0] = \text{Id}$ (6)
$C_\sigma p_0 = p_0$	$\infty \xrightarrow{I_0} p_0 \xrightarrow{C_\sigma} p_0 \xrightarrow{I_0} \infty$	$[C_\sigma, I_0] = \text{Id}$
$R_i p_2 = p_2$	$\infty \xrightarrow{A_2} p_2 \xrightarrow{R_i} p_2 \xrightarrow{A_2^{-1}} \infty$	$[R_i, A_2] = \text{Id}$
$R_\sigma p_2 = p_2$	$\infty \xrightarrow{A_2} p_2 \xrightarrow{R_\sigma} p_2 \xrightarrow{A_2^{-1}} \infty$	$[R_\sigma, A_2] = \text{Id}$
$R_i p_{4,1} = p_{4,1}$	$\infty \xrightarrow{A_{4,1}} p_{4,1} \xrightarrow{R_i} p_{4,1} \xrightarrow{A_{4,1}^{-1}} \infty$	$[R_i, A_{4,1}] = \text{Id}$
$R_\sigma p_{4,1} = p_{4,1}$	$\infty \xrightarrow{A_{4,1}} p_{4,1} \xrightarrow{R_\sigma} p_{4,1} \xrightarrow{A_{4,1}^{-1}} \infty$	$[R_\sigma, A_{4,1}] = \text{Id}$
$R_i p_{4,2} = p_{4,2}$	$\infty \xrightarrow{A_{4,2}} p_{4,2} \xrightarrow{R_i} p_{4,2} \xrightarrow{A_{4,2}^{-1}} \infty$	$[R_i, A_{4,2}] = \text{Id}$
$R_\sigma p_{4,2} = p_{4,2}$	$\infty \xrightarrow{A_{4,2}} p_{4,2} \xrightarrow{R_\sigma} p_{4,2} \xrightarrow{A_{4,2}^{-1}} \infty$	$[R_\sigma, A_{4,2}] = \text{Id}$
$C_i p_{4,1} = p_{4,1}$	$\infty \xrightarrow{A_{4,1}} p_{4,1} \xrightarrow{C_i} p_{4,1} \xrightarrow{A_{4,1}^{-1}} \infty$	$[C_i, A_{4,1}] = \text{Id}$
$C_\sigma p_{4,2} = p_{4,2}$	$\infty \xrightarrow{A_{4,2}} p_{4,2} \xrightarrow{C_\sigma} p_{4,2} \xrightarrow{A_{4,2}^{-1}} \infty$	$[C_\sigma, A_{4,2}] = \text{Id}$

Table 9: Action of generators on vertices for \mathcal{H} , degenerate cycles coming from R 's and C 's

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ALICE MARK

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY

alice.h.mark@vanderbilt.edu

JULIEN PAUPERT

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, ARIZONA STATE UNIVERSITY

paupert@asu.edu