Cup products in the étale cohomology of number fields

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ABSTRACT. This paper concerns cup product pairings in étale cohomology related to work of M. Kim and of W. McCallum and R. Sharifi. We will show that by considering Ext groups rather than cohomology groups, one arrives at a pairing which combines invariants defined by Kim with a pairing defined by McCallum and Sharifi. We also prove a formula for Kim's invariant in terms of Artin maps in the case of cyclic unramified Kummer extensions. One consequence is that for all integers $n > 1$, there are infinitely many number fields over which there are both trivial and non-trivial Kim invariants associated to cyclic groups of order n.

CONTENTS

1. Introduction

This paper concerns cup product pairings in étale cohomology which underlie an important case of the arithmetic Chern-Simons theory introduced by M. Kim in [4] as well as a pairing in Galois cohomology studied by McCallum and Sharifi in [6]. Our interest in these pairings arises from the search for new numerical invariants of number fields which pertain to the higher codimension behavior of Iwasawa modules (see [1]).

Suppose F is a number field and O_F is its ring of integers. Let $X = \mathrm{Spec}(O_F)$ and let μ_n be the sheaf of n-th roots of unity in the étale topology on X. The pairing connected with Kim's work is the natural cup product pairing

$$(1.1) \qquad \mathrm{H}^1(X, \mathbb{Z}/n) \times \mathrm{H}^2(X, \mu_n) \longrightarrow \mathrm{H}^3(X, \mu_n) \overset{\mathrm{inv}_n}{=\!=\!=} \mathbb{Z}/n$$

in étale cohomology when inv_n is the invariant map isomorphism (see [5, p. 538]).

Suppose F contains the multiplicative group $\tilde{\mu}_n$ generated by a primitive n-th root of unity, and let G be an abstract finite group acting trivially on $\tilde{\mu}_n = \mu_n(X)$. Let $\pi_1(X, \eta)$ be the étale fundamental group of X relative to a fixed base point η. Then $\pi_1(X, \eta)$ is the Galois group of a maximal everywhere unramified extension of F. Suppose c is a class in $\mathrm{H}^3(G, \tilde{\mu}_n)$, and let $f : \pi_1(X, \eta) \to G$ be a fixed homomorphism. Then $f^*c \in \mathrm{H}^3(\pi_1(X, \eta), \tilde{\mu}_n)$ defines via Čech cohomology a class $f_X^*c \in \mathrm{H}^3(X, \mu_n)$. Kim's invariant in [4] in the unramified case is

$$(1.2) \qquad\qquad S(f, c) = \mathrm{inv}_n(f_X^*c) \in \mathbb{Z}/n.$$

In the ramified case, one replaces X by the complement X' of a non-empty finite set of closed points of X. One must then take a different approach, since $\mathrm{H}^3(X', \mu_n) = \{0\}$; see [4]. We will return to the ramified case in a later paper.

One way to compute (1.2) is to employ the pairing (1.1). Namely, consider the diagram of pairings

$$(1.3) \qquad
\begin{array}{ccccc}
\mathrm{H}^1(G, \mathbb{Z}/n) & \times & \mathrm{H}^2(G, \tilde{\mu}_n) & \longrightarrow & \mathrm{H}^3(G, \tilde{\mu}_n) \\
\big\downarrow {\scriptstyle f_X^*} & & \big\downarrow {\scriptstyle f_X^*} & & \big\downarrow {\scriptstyle f_X^*} \\
\mathrm{H}^1(X, \mathbb{Z}/n) & \times & \mathrm{H}^2(X, \mu_n) & \longrightarrow & \mathrm{H}^3(X, \mu_n)
\end{array}$$

in which the vertical homomorphisms are induced by f. Picking classes $c_1 \in \mathrm{H}^1(G, \mathbb{Z}/n)$ and $c_2 \in \mathrm{H}^2(G, \tilde{\mu}_n)$ such that $c_1 \cup c_2 = c$, the pairing (1.1) leads to a way to compute

$$(1.4) \qquad\qquad S(f, c) = f_X^*(c_1) \cup f_X^*(c_2).$$

The McCallum-Sharifi pairing, on the other hand, is defined using Galois cohomology. It was defined in [6] using the cup product pairing

$$(1.5) \qquad \mathrm{H}^1(G_{F,S}, \tilde{\mu}_n) \times \mathrm{H}^1(G_{F,S}, \tilde{\mu}_n) \to \mathrm{H}^2(G_{F,S}, \tilde{\mu}_n^{\otimes 2})$$

when S is a finite set of places of F containing all the places above n and all real archimedean places, and $G_{F,S}$ is the Galois group of the maximal unramified outside S extension of F.

A pairing which incorporates both Kim's invariant for $G = \mathbb{Z}/n$ and the McCallum-Sharifi pairing is the cup product Ext pairing

$$(1.6) \qquad \mathrm{Ext}_X^1(\mathbb{Z}/n, \mu_n) \times \mathrm{Ext}_X^2(\mathbb{Z}/n, \mu_n) \to \mathrm{Ext}_X^3(\mathbb{Z}/n, \mu_n^{\otimes 2}).$$

To explain this, consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

induced by multiplication by n. The long exact Ext sequence associated to this sequence leads to a diagram

(1.7)

in which the vertical sequences are exact and the pairings in the second and third rows are given by cup products. Note that we have natural isomorphisms

$$(1.8) \qquad\qquad \mathrm{H}^i(X, \mu_n^{\otimes j}) = \mathrm{H}^i(X, \mathbb{Z}/n) \otimes \tilde{\mu}_n^{\otimes j}$$

for all $i, j \geq 0$ since $\tilde{\mu}_n = \mathrm{H}^0(X, \mu_n)$ has order n by assumption.

We show the following result in §2.

Theorem 1.1. *The cup product in the bottom row of* (1.7) *can be used to compute Kim's invariant via* (1.3), (1.4) *and* (1.8). *This pairing is compatible with pushing forward the cup product in the middle row of* (1.7). *The cup product pairing*

$$(1.9) \qquad \mathrm{H}^1(X, \mu_n) \times \mathrm{H}^1(X, \mu_n) \to \mathrm{H}^2(X, \mu_n^{\otimes 2})$$

is compatible with the McCallum-Sharifi pairing, which results from (1.5), *via the natural inflation maps* $\mathrm{H}^i(X, \mu_n) \to \mathrm{H}^i(G_{F,S}, \tilde{\mu}_n)$. *The pairing* (1.9) *arises from the pairing in the second row of* (1.7) *by the natural pull back and push forward procedure. Namely, suppose* $\alpha \in \mathrm{H}^1(X, \mu_n)$ *pulls back to* $\tilde{\alpha} \in \mathrm{Ext}^1_X(\mathbb{Z}/n, \mu_n)$ *in the first column of* (1.7), *and that* $\beta \in \mathrm{H}^1(X, \mu_n)$ *has boundary* $\partial\beta \in \mathrm{Ext}^2_X(\mathbb{Z}/n, \mu_n)$ *under the first vertical map in the second column of* (1.7). *Then*

$$(1.10) \qquad \partial(\alpha \cup \beta) = -(\tilde{\alpha} \cup \partial\beta)$$

where on the left ∂ *is the boundary map* $\mathrm{H}^2(X, \mu_n^{\otimes 2}) \to \mathrm{Ext}^3_X(\mathbb{Z}/n, \mu_n^{\otimes 2})$ *in the third column of* (1.7).

Note that the minus sign on the right side of (1.10) comes from the definition of the differential of the total complex of the tensor product of two complexes.

Another pairing in Galois cohomology that is related to Kim's invariants and different from the McCallum-Sharifi pairing is described in Theorem 1.15 below.

In [3], H. Chung, D. Kim, M. Kim, J. Park and H. Yoo showed how to compute Kim's invariant by comparing local and global trivializations of Galois three cocycles. Using this method they construct infinitely many examples in which the invariant is non-trivial and the finite group involved is either $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$ or the symmetric group S_4.

Our next results use a different approach than [3] in the unramified case. When G is cyclic we prove in Theorem 1.3 below a formula that determines the invariant using Artin maps. One consequence of Theorem 1.3 is the following result. This shows that there are infinitely many number fields F over which there are both trivial and non-trivial Kim invariants associated to cyclic groups of order n. The methods of this paper carry over *mutatis mutandis* to the case of global function fields provided n is prime to the characteristic of the field.

Theorem 1.2. *Suppose* $n > 1$ *is an integer,* $G = \mathbb{Z}/n$ *and that* c *is a fixed generator of* $\mathrm{H}^3(G, \tilde{\mu}_n)$. *Then there are infinitely many totally complex number fields* F *for which there are cyclic unramified Kummer extensions* K_1/F *and* K_2/F *with the following property. Let* $f_i : \pi_1(X, \eta) \to G$ *for* $i = 1, 2$ *be the inflation of an isomorphism* $\mathrm{Gal}(K_i/F) \to G$. *Then*

$$(1.11) \qquad S(f_1, c) = 0 \quad and \quad S(f_2, c) \neq 0.$$

To state our formula for Kim's invariant in terms of Artin maps, let $f : \pi_1(X, \eta) \to G = \mathbb{Z}/n$ be a fixed surjection. Let $c_1 \in \mathrm{H}^1(G, \mathbb{Z}/n) = \mathrm{Hom}(G, \mathbb{Z}/n)$ be the identity map, and let c_2 generate $\mathrm{H}^2(G, \tilde{\mu}_n)$. Then $c = c_1 \cup c_2$ generates the cyclic group $\mathrm{H}^3(G, \tilde{\mu}_n)$ of order n. We wish to use the diagram (1.3) to calculate $S(f, c) = f_X^*(c_1) \cup f_X^*(c_2)$.

The element $f_X^*(c_1) \in \mathrm{H}^1(X, \mathbb{Z}/n)$ factors through an isomorphism

$$\mathrm{Gal}(K/F) \to G = \mathbb{Z}/n$$

for a cyclic unramified extension K/F of degree n which we will use to identify $\mathrm{Gal}(K/F)$ with $G = \mathbb{Z}/n$.

Using the exact sequence of multiplicative groups

(1.12) $$1 \to \tilde{\mu}_n \to K^* \to K^* \to K^*/(K^*)^n \to 1$$

associated to exponentiation by n on K^* we will show that there is an exact sequence

(1.13) $$F^* \to (K^*/(K^*)^n)^{\mathrm{Gal}(K/F)} \to \mathrm{H}^2(\mathrm{Gal}(K/F), \tilde{\mu}_n) \to 1.$$

Let $\gamma \in K^*$ be such that $\gamma(K^*)^n \in (K^*/(K^*)^n)^G$ has image c_2 in

$$\mathrm{H}^2(\mathrm{Gal}(K/F), \tilde{\mu}_n) = \mathrm{H}^2(G, \tilde{\mu}_n)$$

under the homomorphism in (1.13).

Theorem 1.3. *The O_F ideal $\mathrm{Norm}_{K/F}(\gamma)O_F$ is the n-th power I^n of a fractional ideal I of F. The ideal class $[I]$ of I in the ideal class group $\mathrm{Cl}(O_F)$ of O_F depends only on c_2 and is n-torsion. Let $\mathrm{Art} : \mathrm{Cl}(O_F) \to \mathrm{Gal}(K/F) = G = \mathbb{Z}/n$ be the Artin map associated to K/F. Then Kim's invariant of the class $c = c_1 \cup c_2 \in \mathrm{H}^3(G, \tilde{\mu}_n)$ is*

(1.14) $$S(f, c) = f_X^*(c_1) \cup f_X^*(c_2) = \mathrm{Art}([I]) \in G = \mathbb{Z}/n.$$

Note that in this result, the input is f and c_2, from which one determines K and γ. Conversely, we now show how one can start with a cyclic unramified degree n Kummer extension K/F and then use this to determine an f and c_2 for which (1.14) holds.

For the remainder of the paper we fix the following choices.

Definition 1.4. Let ζ_n be a primitive n-th root of unity in F. If m is a divisor of n, we let $\zeta_m = \zeta_n^{n/m}$.

Theorem 1.5. *Suppose K/F is an everywhere unramified cyclic degree n Kummer extension of number fields. By the Hasse norm theorem, $\zeta_n = \mathrm{Norm}_{K/F}(x)$ for some $x \in K$. By Hilbert's Theorem 90, $x^n = \sigma(y)/y$ for some $y \in K$ and a generator σ for $G = \mathrm{Gal}(K/F)$. For all such y, there is a fractional O_F-ideal J such that $\mathrm{Norm}_{K/F}(y)O_F = J^n$. Let $c_1 : G \to \mathbb{Z}/n$ be the isomorphism sending σ to $1 \mod n$. Let $\gamma = y$ in Theorem 1.3, and let $c_2 \in \mathrm{H}^2(G, \tilde{\mu}_n)$ be the image of $\gamma(K^*)^n$ under (1.13). Then c_2 generates $\mathrm{H}^2(G, \tilde{\mu}_n)$, J is the ideal I of Theorem 1.3 and $S(f, c)$ is given by (1.14) when $c = c_1 \cup c_2$.*

This theorem leads to the following result concerning the functorality of Kim's invariant under base extensions.

Corollary 1.6. *Suppose F' is a finite extension of F which is disjoint from K, and let $K' = F'K$ be the compositum of F' and K. The ideal I' associated to K'/F' by Theorem 1.5 may be taken to be $IO_{F'}$. Kim's invariant for K'/F' is the image of the invariant for K/F under the transfer map*

$$\mathrm{Ver}_{K'/K} : \mathrm{Gal}(K/F) \to \mathrm{Gal}(K'/F')$$

when we identify both of these Galois groups with $\mathbb{Z}/n$.

Theorem 1.5 gives the following criterion for the non-triviality of Kim's invariant for cyclic unramified Kummer extensions.

Corollary 1.7. *With the notations of Theorem 1.5, the following are equivalent:*

(i) *The invariant $S(f,c)$ is trivial for all $f : \pi_1(X,\eta) \to G = \mathbb{Z}/n$ factoring through $\mathrm{Gal}(K/F)$ and all $c \in \mathrm{H}^3(G, \tilde{\mu}_n)$.*

(ii) *$[J]$ is contained in $\mathrm{Norm}_{K/F}(\mathrm{Cl}(O_K))$.*

(iii) *The image of $[J]$ under the Artin map $\mathrm{Art} : \mathrm{Cl}(O_F) \to \mathrm{Gal}(K/F)$ is trivial.*

We now describe another way to find an element $\gamma \in K$ with the properties in Theorem 1.3. This method will be used to show Theorem 1.2.

Theorem 1.8. *Let $f : \pi_1(X,\eta) \to G = \mathbb{Z}/n = \mathrm{Gal}(K/F)$ be as above with $c_1 : G \to \mathbb{Z}/n$ the identity map.*

(i) *There is a cyclic degree n^2 extension L/F such that $K \subset L$. This extension is unique up to twisting by a cyclic degree n extension of F, in the following sense. Write $L = K(\gamma^{1/n})$ for some Kummer generator γ. If L' is any other cyclic degree n^2 extension of F which contains K, then $L' = K(\gamma'^{1/n})$ for some $\gamma' \in \gamma \cdot (K^*)^n \cdot F^*$, and conversely all such γ' give rise to such L'.*

(ii) *The coset $\gamma(K^*)^n$ of $K^*/(K^*)^n$ is fixed by the action of $\mathrm{Gal}(K/F)$. Let $c_2 \in \mathrm{H}^2(\mathrm{Gal}(K/F), \tilde{\mu}_n) = \mathrm{H}^2(G, \tilde{\mu}_n)$ be the image of $\gamma(K^*)^n$ under the boundary map in (1.13). The formula in (1.14) determines $S(f,c)$ when c is the generator $c_1 \cup c_2$ of $\mathrm{H}^3(G, \mu_n)$.*

(iii) *Suppose the ideal I of Theorem 1.3 has the form $I = I' \cdot J'$ for some fractional ideals I' and J' of O_F such that any prime in the support of J' is either split in K or unramified in L/K. Then*

(1.15) $S(f,c) = f_X^*(c_1) \cup f_X^*(c_2) = \mathrm{Art}([I']) \in G = \mathbb{Z}/n.$

This description leads to the following corollaries, which we will show lead to a proof of Theorem 1.2.

Corollary 1.9. *Suppose K/F is contained in a cyclic degree n^2 extension L/F such that every prime $\mathcal{P}$ of O_F which ramifies in L splits completely*

in K. Then $S(f, c) = 0$ for all surjections $f : \pi_1(X, \eta) \to \mathrm{Gal}(K/F) = G = \mathbb{Z}/n$ and all $c \in \mathrm{H}^3(G, \tilde{\mu}_n)$.

Corollary 1.10. *Suppose K/F is contained in a cyclic degree n^2 extension L/F with the following properties. There is a unique prime ideal $\mathcal{P}$ of O_F which ramifies in L/F, $\mathcal{P}$ is undecomposed in L and the inertia group of $\mathcal{P}$ in $\mathrm{Gal}(L/F)$ is $\mathrm{Gal}(L/K)$. Furthermore, the residue characteristic of $\mathcal{P}$ is prime to n. Then $S(f, c)$ is of order n for all surjections $f : \pi_1(X, \eta) \to \mathrm{Gal}(K/F) = G = \mathbb{Z}/n$ and all generators c of $\mathrm{H}^3(G, \tilde{\mu}_n)$.*

Remark 1.11. These corollaries explain the examples of [3, §5.5] in the following way. Let $n = 2$, and let $F = \mathbb{Q}(\sqrt{-pt})$ where p is a prime such that $p \equiv 1 \bmod 4$ and t is a positive square-free integer prime to p. Let K be $F(\sqrt{p})$. Then K is contained in the unique cyclic degree 4 extension L of F contained in $F(\tilde{\mu}_p)$. The unique prime $\mathcal{P}$ over p in F is the unique prime which ramifies in L. The examples in [3, §5.5] arise from Corollaries 1.9 and 1.10 because $\mathcal{P}$ splits in K if and only if t is a square mod p since -1 is a square mod p.

The following two results give examples in which our results show that Kim's invariants are trivial, where $\zeta_n \in F$ is fixed as in Definition 1.4.

Theorem 1.12. *Suppose n is a properly irregular prime in the sense that n divides $\#\mathrm{Cl}(\mathbb{Z}[\zeta_n])$ but not $\#\mathrm{Cl}(\mathbb{Z}[\zeta_n + \zeta_n^{-1}])$. If K is any cyclic unramified extension of $F = \mathbb{Q}(\zeta_n)$ then $S(f, c) = 0$ for all surjections $f : \pi_1(X, \eta) \to \mathrm{Gal}(K/F) = G = \mathbb{Z}/n$ and all $c \in \mathrm{H}^3(G, \tilde{\mu}_n)$.*

Theorem 1.13. *Suppose that $n > 2$ is prime and K/F is a cyclic unramified Kummer extension of degree n such that both K and F are Galois over $\mathbb{Q}$. Then $S(f, c) = 0$ for all surjections $f : \pi_1(X, \eta) \to \mathrm{Gal}(K/F) = G = \mathbb{Z}/n$ and all $c \in \mathrm{H}^3(G, \tilde{\mu}_n)$.*

Remark 1.14. Note that in Theorem 1.12, n does not divide $[F : \mathbb{Q}]$. One can also construct many examples of Theorem 1.13 in which $[F : \mathbb{Q}]$ is prime to n. However, the examples we will construct in Theorem 1.2 in which Kim's invariant is non-trivial all have $n | [F : \mathbb{Q}]$. It would be interesting to find examples in which Kim's invariant is non-trivial when n is prime and $[F : \mathbb{Q}]$ is not divisible by n.

We now describe a pairing in Galois cohomology that is different from the McCallum-Sharifi pairing and that gives rise to Kim's invariants.
    Define

$$(1.16) \qquad T(F) = \{a \in F^* : F(a^{1/n})/F \text{ is unramified}\}.$$

Suppose $a \in T(F)$ and $b \in O_F^*$. The field $K = F(a^{1/n})$ is a cyclic Kummer extension of degree m dividing n. Since K/F is unramified, and $b \in O_F^*$, we have $b = \mathrm{Norm}_{K/F}(x)$ for some $x \in K^*$. Let $\sigma \in \mathrm{Gal}(K/F)$ be the unique generator such that $\sigma(a^{1/n})/a^{1/n} = \zeta_m = \zeta_n^{n/m}$, where $\zeta_n \in F$ is as

in Definition 1.4. Since $\text{Norm}_{K/F}(x^m/b) = 1$, there is an element $\nu \in K^*$ such that $x^m/b = \sigma(\nu)/\nu$. Since K/F is unramified and $b \in O_F^*$, the ideal $\text{Norm}_{K/F}(\nu a^{1/n})O_F$ equals I^m for some fractional ideal I of O_F. We define

$$(1.17) \qquad (a,b)_n = [I] \otimes \zeta_m \in \text{Cl}(O_F) \otimes_{\mathbb{Z}} \tilde{\mu}_n$$

where $[I]$ is the ideal class of I in $\text{Cl}(O_F)$. The value $(a,b)_n$ does not depend on the choice of ζ_n in Definition 1.4.

Theorem 1.15. *Suppose $K = F(a^{1/n})$ has degree $n = m$ over F for some $a \in T(F)$. Let $\sigma \in G$ be the generator such that $\sigma(a^{1/n})/a^{1/n} = \zeta_n$ and set $b = \zeta_n$. Fix an isomorphism $c_1 : G = \text{Gal}(K/F) \to \mathbb{Z}/n$ by letting $\sigma \in G$ correspond to $1 \in \mathbb{Z}/n$. Then as above, $b = \text{Norm}_{K/F}(x)$ for some $x \in K^*$ and $x^n/b = x^n/\zeta_n = \sigma(\nu)/\nu$ for some $\nu \in K^*$. When $y = \nu a^{1/n}$, the coset $y(K^*)^n$ lies in $(K^*/(K^*)^n)^G$, and its image under the homomorphism in (1.13) is a generator $c_2 \in \text{H}^2(G, \tilde{\mu}_n)$. Let $c = c_1 \cup c_2 \in \text{H}^3(G, \tilde{\mu}_n)$. There is a unique homomorphism*

$$\kappa : \text{Cl}(O_F) \otimes_{\mathbb{Z}} \tilde{\mu}_n \to \mathbb{Z}/n$$

which sends $[J] \otimes \zeta_n$ to $\text{Art}([J]) \in G = \mathbb{Z}/n$ for all fractional ideals J of O_F. Kim's invariant $S(f,c)$ is given by

$$(1.18) \qquad S(f,c) = \kappa((a,\zeta_n)_n)$$

when $(a,\zeta_n)_n$ is the pairing defined by (1.17).

By contrast, the McCallum-Sharifi pairing is defined in the following way. Let S be the union of set of places of F which have residue characteristics dividing n with the real places. Let $C_{F,S}$ be the S-class group of F. In [6, §2] McCallum and Sharifi define a pairing

$$(1.19) \qquad \langle \, , \, \rangle_S : T(F) \times O_F^* \to (C_{F,S}/nC_{F,S}) \otimes \tilde{\mu}_n.$$

See also [8] for further discussion.

Remark 1.16. Here is an example for which the following three statements hold:

   (i) Kim's invariant $S(f,c)$ in (1.18) is not trivial.
   (ii) The McCallum-Sharifi pairing value $\langle a, \zeta_n \rangle_S$ in (1.19) is trivial.
   (iii) The homomorphism $\text{Cl}(O_F) \to C_{F,S}$ is an isomorphism.

Let $n = 2$, $G = \mathbb{Z}/2$ and $\zeta_n = -1$. Define $F = \mathbb{Q}(\sqrt{-pt})$ for some prime $p \equiv 1 \bmod 4$ and some square-free $t > 0$ such that t is not a square mod p and $-pt \equiv 5 \bmod 8$. When $K = F(\sqrt{p})$ and $a = p$, Remark 1.11 shows (i). Since 2 is inert to F, (iii) holds when S is the set of places of F over 2. Finally (ii) follows from the formula in [6, Thm. 2.4] since $\zeta_2 = -1 = \text{Norm}_{K/F}(\epsilon)$ when ϵ is a fundamental unit of $\mathbb{Q}(\sqrt{p}) \subset K$.

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2. Proof of Theorem 1.1

We assume in this section the notations of Theorem 1.1. We will use the results of Swan in [9] concerning cup products. In [9, §3], Swan considers cup products of covariant left exact functors. This can be used to define the cup product pairings in the middle and bottom rows of (1.7). Namely, in the category of sheaves in the étale topology on X, let

$$0 \to U \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \mathcal{I}^3 \to \cdots$$

and

$$0 \to V \to \mathcal{J}^0 \to \mathcal{J}^1 \to \mathcal{J}^2 \to \mathcal{J}^3 \to \cdots$$

be pure injective resolutions. Then the total complex $\mathcal{I}^\bullet \otimes \mathcal{J}^\bullet$ is a pure, but not necessarily injective, resolution of $U \otimes V$. Let

$$0 \to U \otimes V \to \mathcal{K}^0 \to \mathcal{K}^1 \to \mathcal{K}^2 \to \mathcal{K}^3 \to \cdots$$

be a pure injective resolution, and choose a morphism of resolutions

$$\mathcal{I}^\bullet \otimes \mathcal{J}^\bullet \to \mathcal{K}^\bullet$$

over $U \otimes V$. For étale sheaves A, B on X, the composition of the morphisms

$$\begin{aligned}
\mathrm{Hom}_X(A, \mathcal{I}^\bullet) \otimes \mathrm{Hom}_X(B, \mathcal{J}^\bullet) &\to \mathrm{Hom}_X(A \otimes B, \mathcal{I}^\bullet \otimes \mathcal{J}^\bullet) \\
&\to \mathrm{Hom}_X(A \otimes B, \mathcal{K}^\bullet)
\end{aligned}$$

then induces a cup product pairing

$$\mathrm{Ext}_X^i(A, U) \times \mathrm{Ext}_X^j(B, V) \to \mathrm{Ext}_X^{i+j}(A \otimes B, U \otimes V)$$

(see [9, Thm. 3.4, Lemma 3.6 and §7]). Given morphisms of étale sheaves $C \to A \otimes B$ and $U \otimes V \to T$, we get

(2.1) $$\mathrm{Ext}_X^i(A, U) \times \mathrm{Ext}_X^j(B, V) \to \mathrm{Ext}_X^{i+j}(C, T).$$

The first statement in Theorem 1.1 is that the cup products in the middle and bottom rows of (1.7) are compatible with the vertical homomorphisms from the terms of the middle row to the terms of the bottom row. The latter homomorphisms are those associated to the natural morphism $\mathbb{Z} \to \mathbb{Z}/n$ of étale sheaves on X, since $\mathrm{H}^i(X, \mu_n) = \mathrm{Ext}_X^i(\mathbb{Z}, \mu_n)$ and the terms of the middle row have the form $\mathrm{Ext}_X^i(\mathbb{Z}/n, \mu_n)$. So the above compatibility of the middle and bottom rows follows from the naturality of the cup product (2.1) with respect to morphisms of the arguments. Note that in showing this, we have not used any compatibility of cup products with boundary maps; the latter requires more hypotheses.

We now turn to analyzing the connection of the cup product pairing (1.9)

$$\mathrm{H}^1(X, \mu_n) \times \mathrm{H}^1(X, \mu_n) \to \mathrm{H}^2(X, \mu_n^{\otimes 2})$$

with the diagram (1.7). We are to prove that this is compatible with pulling back and pushing forward arguments to the second row of (1.7).

By the naturality of cup product pairings with respect to either argument, we have a commuting diagram of pairings

$$(2.2) \qquad \mathrm{Ext}^1_X(\mathbb{Z}/n, \mu_n) \;\times\; \mathrm{H}^1(X, \mu_n) \;\longrightarrow\; \mathrm{Ext}^2_X(\mathbb{Z}/n, \mu_n^{\otimes 2})$$

$$\mathrm{H}^1(X, \mu_n) \quad\times\quad \mathrm{H}^1(X,\mu_n) \;\longrightarrow\; \mathrm{H}^2(X, \mu_n^{\otimes 2})$$

in which the left and right vertical homomorphisms are induced by the canonical surjection $\mathbb{Z} \to \mathbb{Z}/n$. We claim that the top row of this diagram fits into a diagram of pairings

$$(2.3) \qquad \mathrm{Ext}^1_X(\mathbb{Z}/n, \mu_n) \;\times\; \mathrm{H}^1(X, \mu_n) \;\longrightarrow\; \mathrm{Ext}^2_X(\mathbb{Z}/n, \mu_n^{\otimes 2})$$

$$\mathrm{Ext}^1_X(\mathbb{Z}/n, \mu_n) \;\times\; \mathrm{Ext}^2_X(\mathbb{Z}/n, \mu_n) \;\longrightarrow\; \mathrm{Ext}^3_X(\mathbb{Z}/n, \mu_n^{\otimes 2})$$

with the right vertical map λ.

that commutes up to the sign (-1) and in which the middle vertical map is the boundary map resulting from the sequence

$$(2.4) \qquad B_\bullet = (0 \to B'' \to B \to B' \to 0) = (0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0)$$

and the right vertical map is the boundary map associated with the Bockstein sequence

$$(2.5) \quad C_\bullet = (0 \to C'' \to C \to C' \to 0) = (0 \to \mathbb{Z}/n \to \mathbb{Z}/n^2 \to \mathbb{Z}/n \to 0).$$

Let $A = \mathbb{Z}/n$. We have a morphism $C_\bullet \to A \otimes B_\bullet$ fitting into a commutative diagram

$$(2.6)$$

$$\mathbb{Z}/n \xrightarrow{\;0\;} \mathbb{Z}/n \xrightarrow{\;1\;} \mathbb{Z}/n \longrightarrow 0$$

$$A \otimes B'' \longrightarrow A \otimes B \longrightarrow A \otimes B' \longrightarrow 0$$

$$0 \longrightarrow C'' \longrightarrow C \longrightarrow C' \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Choosing pure injective resolutions $\mu_n \to \mathcal{I}^\bullet$ and $\mu_n^{\otimes 2} \to \mathcal{K}^\bullet$ and a morphism of resolutions $\mathcal{I}^\bullet \otimes \mathcal{I}^\bullet \to \mathcal{K}^\bullet$ over $\mu_n^{\otimes 2}$, we can apply the respective Hom functors over X to the diagram (2.6) to obtain the commutative diagram in Figure 2.1. It follows from [9, Lemma 3.2] that the diagram (2.3) commutes up to the sign (-1).

FIGURE 2.1. The diagram resulting from applying appropriate Hom functors over X to the diagram (2.6).

In view of diagrams (2.2) and (2.6), the last assertion (1.10) of Theorem 1.1 concerning the relation of (1.9) to the pairing in the middle row of (2.3) will hold if we can show the following assertion. We claim that the rightmost vertical homomorphism

$$\lambda : \operatorname{Ext}^2_X(\mathbb{Z}/n, \mu_n^{\otimes 2}) \to \operatorname{Ext}^3_X(\mathbb{Z}/n, \mu_n^{\otimes 2})$$

in (2.3), which is induced by the boundary map of the Bockstein sequence $C_\bullet$ in (2.5), is the composition of the pullback map

$$\tau : \operatorname{Ext}^2_X(\mathbb{Z}/n, \mu_n^{\otimes 2}) \to \operatorname{H}^2(X, \mu_n^{\otimes 2})$$

associated to $\mathbb{Z} \to \mathbb{Z}/n$ with the boundary map

$$\nu : \operatorname{H}^2(X, \mu_n^{\otimes 2}) \to \operatorname{Ext}^3_X(\mathbb{Z}/n, \mu_n^{\otimes 2})$$

associated to the sequence $B_\bullet$ in (2.4).

This assertion (and the more general fact, which holds in all degrees) can be proved by calculating λ and $\nu \circ \tau$ using a pure injective resolution of the second argument, which in this case is $\mu_n^{\otimes 2}$. To be explicit, let

$$0 \to \mu_n^{\otimes 2} \to \mathcal{K}^0 \to \mathcal{K}^1 \to \mathcal{K}^2 \to \mathcal{K}^3 \to \cdots$$

be a pure injective resolution. The boundary map λ results from taking elements of $\operatorname{Hom}_X(\mathbb{Z}/n, \mathcal{K}^2)$ which go to zero in $\mathcal{K}^3$, lifting these to elements of $\operatorname{Hom}_X(\mathbb{Z}/n^2, \mathcal{K}^2)$ by the injectivity of $\mathcal{K}^2$, and then pushing this lift forward by $\mathcal{K}^2 \to \mathcal{K}^3$ to produce an element of $\operatorname{Hom}_X(\mathbb{Z}/n, \mathcal{K}^3)$. The map τ results from simply inflating a homomorphism in $\operatorname{Hom}_X(\mathbb{Z}/n, \mathcal{K}^2)$ to one in $\operatorname{Hom}_X(\mathbb{Z}, \mathcal{K}^2)$ via the natural surjection $\mathbb{Z} \to \mathbb{Z}/n$. The map ν results from lifting maps from $\operatorname{Hom}_X(\mathbb{Z}, \mathcal{K}^2)$ to $\operatorname{Hom}_X(\mathbb{Z}, \mathcal{K}^2)$ through the multiplication by n homomorphism $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ and then pushing the lift forward by $\mathcal{K}^2 \to \mathcal{K}^3$ to produce an element of $\operatorname{Hom}_X(\mathbb{Z}/n, \mathcal{K}^3)$. Since we can use the lifts involved in calculating λ to do the calculations to find ν on maps which come from the inflation map τ, we see that $\lambda = \nu \circ \tau$.

3. A reformulation of the approach via Artin maps

We describe in this section our approach to proving Theorem 1.3. Instead of the diagram of pairings (1.3), we consider the diagram of pairings (3.1)

$$
\begin{array}{ccccccc}
\operatorname{H}^1(G, \tilde{\mu}_n) & \times & \operatorname{H}^2(G, \tilde{\mu}_n) & \longrightarrow & \operatorname{H}^3(G, \tilde{\mu}_n^{\otimes 2}) & = & \operatorname{H}^3(G, \tilde{\mu}_n) \otimes \tilde{\mu}_n \\
\downarrow{\scriptstyle f_X^*} & & \downarrow{\scriptstyle f_X^*} & & & & \downarrow{\scriptstyle f_X^*} \\
\operatorname{H}^1(X, \mu_n) & \times & \operatorname{H}^2(X, \mu_n) & \longrightarrow & \operatorname{H}^3(X, \mu_n^{\otimes 2}) & = & \operatorname{H}^3(X, \mu_n) \otimes \tilde{\mu}_n = \tilde{\mu}_n
\end{array}
$$

in which the vertical homomorphisms are induced by f. Let $\phi : \mathbb{Z}/n \to \tilde{\mu}_n$ be the isomorphism taking $1 \bmod n$ to ζ_n, where $\zeta_n \in F$ is as in Definition

1.4. Then ϕ takes the generator c_1 of $\mathrm{H}^1(G, \mathbb{Z}/n)$ to a generator $d_1 = \phi(c_1)$ of $\mathrm{H}^1(G, \tilde{\mu}_n)$. We have

$$
\begin{aligned}
\phi(f_X^*(c)) &= \phi(f_X^*(c_1) \cup f_X^*(c_2)) = f_X^*(\phi(c_1)) \cup f_X^*(c_2) \\
&= f_X^*(d_1) \cup f_X^*(c_2).
\end{aligned}
$$

We will show (1.14) of Theorem 1.3 by calculating the cup product of $f_X^*(d_1)$ and $f_X^*(c_2)$ using Mazur's description in [5] of the bottom row of (3.1).

4. Analysis of $f_X^*(d_1)$

Lemma 4.1. *There is a canonical isomorphism*

$$
\mathrm{H}^1(X, \mu_n) = \mathrm{Hom}(\mathrm{Pic}(X), \tilde{\mu}_n).
$$

The restriction of a class $d \in \mathrm{H}^1(X, \mu_n)$ to $\mathrm{H}^1(\mathrm{Spec}(F), \mu_n)$ defines a torsor $Y(d)$ for the group scheme $\pmb{\mu}_n$ over $\mathrm{Spec}(F)$. The scheme $Y(d)$ is isomorphic to $\mathrm{Spec}\left(\frac{F[w]}{(w^n - \xi)}\right)$ as a $\pmb{\mu}_n$-torsor for an element $\xi \in F^*$ which is unique up to multiplication by an element of $(F^*)^n$.

Proof. Our choice of a primitive n-th root of unity ζ_n in F gives an isomorphism of étale sheaves from $\mathbb{Z}/n$ to μ_n. This induces an isomorphism from $\mathrm{H}^1(X, \mu_n)$ to $\mathrm{H}^1(X, \mathbb{Z}/n)$. The group $\mathrm{H}^1(X, \mathbb{Z}/n)$ classifies torsors for the constant group scheme $\mathbb{Z}/n$. Therefore

$$
\mathrm{H}^1(X, \mathbb{Z}/n) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}/n) = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}/n)
$$

where the last isomorphism results from class field theory. Thus

$$
\begin{aligned}
\mathrm{H}^1(X, \mu_n) &= \mathrm{H}^1(X, \mathbb{Z}/n) \otimes_{\mathbb{Z}} \tilde{\mu}_n \\
&= \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}/n) \otimes_{\mathbb{Z}} \tilde{\mu}_n \\
&= \mathrm{Hom}(\mathrm{Pic}(X), \tilde{\mu}_n)
\end{aligned}
$$

and the isomorphism between the far left and far right terms does not depend on the choice of ζ_n. The last statement is clear from Kummer theory over fields of characteristic 0; see [7, p. 125, Thm. 3.9]. $\qquad\square$

Remark 4.2. Suppose the class $d \in \mathrm{H}^1(X, \mu_n)$ has order n. Then $Y(d) = \mathrm{Spec}(K)$ for an everywhere unramified $\mathbb{Z}/n$ extension $K = F(\xi^{1/n})$ of F for an element $\xi \in F^*$ as in Lemma 4.1. Associating d canonically to a homomorphism $d : \mathrm{Pic}(X) \to \tilde{\mu}_n$ as in Lemma 4.1, the element ξ has the property that

$$
(4.1) \qquad \frac{\mathrm{Art}(a)(\xi^{1/n})}{\xi^{1/n}} = d(a) \quad \text{for all } a \in \mathrm{Pic}(X)
$$

where $\mathrm{Art}(a) \in \mathrm{Gal}(K/F)$ is the image of $a \in \mathrm{Pic}(X)$ under the Artin map. The equality (4.1) does not depend on the choice of n-th root $\xi^{1/n}$ of ξ in K. It specifies the class of ξ uniquely in the quotient group $F^*/(F^*)^n$.

Lemma 4.3. *The Pontryagin dual* $\mathrm{H}^2(X, \mathbb{Z}/n)^\star = \mathrm{Hom}(\mathrm{H}^2(X, \mathbb{Z}/n), \mathbb{Q}/\mathbb{Z})$ *of* $\mathrm{H}^2(X, \mathbb{Z}/n)$ *lies in an exact sequence*

$$(4.2) \qquad 1 \to O_F^*/(O_F^*)^n \xrightarrow{\tau} \mathrm{H}^2(X, \mathbb{Z}/n)^\star \xrightarrow{\delta} \mathrm{Pic}(X)[n] \to 0$$

in which $\mathrm{Pic}(X)[n]$ *is the n-torsion in* $\mathrm{Pic}(X)$. *Define* T *to be the subgroup of* $\gamma \in F^*$ *such that* γO_F *is the n-th power of some fractional ideal* $I(\gamma)$. *Then there is a canonical isomorphism*

$$(4.3) \qquad\qquad\qquad T/(F^*)^n = \mathrm{H}^2(X, \mathbb{Z}/n)^\star$$

with the following properties.

(i) *The homomorphisms* τ *and* δ *in* (4.2) *are induced by the inclusion* $O_F^* \subset T$ *and the map which sends* $\gamma \in T$ *to the ideal class* $[I(\gamma)]$ *of* $I(\gamma)$.

(ii) *The homomorphism* $h : \mathrm{H}^1(X, \mu_n) \to \mathrm{H}^2(X, \mathbb{Z}/n)^\star$ *induced by the cup product pairing*

$$\mathrm{H}^1(X, \mu_n) \times \mathrm{H}^2(X, \mathbb{Z}/n) \to \mathrm{H}^3(X, \mu_n) = \mathbb{Z}/n\mathbb{Z}$$

has the following description. Suppose $d \in \mathrm{H}^1(X, \mu_n)$ *gives a* $\pmb{\mu}_n$ *torsor* $Y(d)$ *over* $\mathrm{Spec}(F)$ *as in Lemma 4.1. Let* $\xi \in F^*$ *be associated to* $Y(d)$ *as in Lemma 4.1, so that* ξ *is unique up to multiplication by an element of* $(F^*)^n$. *Then* $\xi \in T$, *and* $h(d)$ *is the coset* $\xi(F^*)^n$ *in* $T/(F^*)^n = \mathrm{H}^2(X, \mathbb{Z}/n)^\star$.

Proof. The exact sequence (4.2) is shown in [5, p. 539]. This utilizes Artin-Verdier duality (c.f. [5, p. 538]), which gives a canonical isomorphism $\mathrm{H}^2(X, \mathbb{Z}/n)^\star = \mathrm{Ext}^1_X(\mathbb{Z}/n, G_{m,X})$. The more precise description in (4.3), together with properties in (i) and (ii) of this description, results from the analysis of $\mathrm{Ext}^1_X(\mathbb{Z}/n, G_{m,X})$ and the computation of duality pairings by Hilbert symbols in [5, p. 540-541]. $\square$

Corollary 4.4. *Suppose* d_1 *is a generator of* $\mathrm{H}^1(G, \tilde{\mu}_n)$. *The class* $d = f_X^*(d_1) \in \mathrm{H}^1(X, \mu_n)$ *corresponds to a* $\pmb{\mu}_n$*-torsor* $Y(d) = \mathrm{Spec}(K)$ *over* $\mathrm{Spec}(F)$ *such that* $K = F(\xi^{1/n})$ *of* F *for an element* $\xi \in F^*$ *with the following properties.*

(i) *The extension* K/F *is everywhere unramified and cyclic of degree* n. *Fixing an embedding of* K *into the maximal unramified extension* F^{un} *of* F *determines a surjection* $\rho : \mathrm{Gal}(F^{\mathrm{un}}/F) = \pi_1(X, \eta) \to \mathrm{Gal}(K/F)$.

(ii) *There is a unique isomorphism* $\lambda : \mathrm{Gal}(K/F) \to G = \mathbb{Z}/n$ *such that* $\lambda \circ \rho : \pi_1(X, \eta) \to G$ *is the homomorphism* $f : \pi_1(X, \eta) \to G$ *used to construct Kim's invariant.*

(iii) *The element* $\xi \in F^*$ *is uniquely determined mod* $(F^*)^n$ *by the requirement that* (4.1) *hold when we identify* d *with an element of* $\mathrm{Hom}(\mathrm{Pic}(X), \tilde{\mu}_n)$ *as in Lemma 4.1.*

(iv) *The image of* $d = f_X^*(d_1)$ *under the homomorphism* $h : \mathrm{H}^1(G, \tilde{\mu}_n) \to \mathrm{H}^2(G, \mathbb{Z}/n)^\star = T/(F^*)^n$ *of Lemma 4.3 is the coset* $\xi(F^*)^n$.

5. Hilbert pairings, Artin maps and $\mathrm{H}^2(X, \mu_n)$

With the notations of §3, our goal is to compute the cup product

(5.1) $$f_X^*(d_1) \cup f_X^*(c_2) = h(f_X^*(d_1))(f_X^*(c_2)) \in \tilde{\mu}_n$$

when c_2 is a generator of $\mathrm{H}^2(G, \tilde{\mu}_n)$, $f_X^*(c_2)$ is the pullback of c_2 to $\mathrm{H}^2(X, \mu_n)$ and $h(f_X^*(d_1))$ is the element of the Pontryagin dual $\mathrm{H}^2(X, \mathbb{Z}/n)^\star$ determined in Corollary 4.4. To do this, we first develop in this section a description of $\mathrm{H}^2(X, \mu_n) = \mathrm{H}^2(X, \mathbb{Z}/n) \otimes \tilde{\mu}_n$ using ideles of F.

Let $j : \mathrm{Spec}(F) \to X$ be the inclusion of the generic point of X into X. Then $j_* \mu_{n,F} = \mu_{n,X}$ since F contains a primitive n-th root of unity. There is a spectral sequence

(5.2) $$\mathrm{H}^p(X, R^q j_*(\mu_{n,X})) \to \mathrm{H}^{p+q}(F, \mu_{n,F}).$$

Consider the $(p, q) = (2, 0)$ term. This is associated to the restriction homomorphism

(5.3) $$\mathrm{H}^2(X, R^0 j_*(\mu_{n,X})) = \mathrm{H}^2(X, \mu_{n,X}) \to \mathrm{H}^2(F, \mu_{n,F}).$$

By the Kummer sequence

(5.4) $$0 \to \mu_{n,F} \to G_{m,F} \to G_{m,F} \to 0$$

and Hilbert's Theorem 90, the homomorphism $\mathrm{H}^2(F, \mu_{n,F}) \to \mathrm{H}^2(F, G_{m,F})$ is injective. The composition of $\mathrm{H}^2(X, \mu_{n,X}) \to \mathrm{H}^2(F, \mu_{n,F})$ with this homomorphism factors through the homomorphism $\mathrm{H}^2(X, \mu_{n,X}) \to \mathrm{H}^2(X, G_{m,X})$. However, elements of $\mathrm{H}^2(X, G_{m,X})$ are elements of the Brauer group of F with trivial local invariants everywhere since F is totally complex, and such elements must be trivial. Thus $\mathrm{H}^2(X, G_{m,X}) = \{0\}$ and it follows that (5.3) is the zero homorphism. Hence the spectral sequence (5.2) gives an exact sequence

(5.5) $$\mathrm{H}^1(F, \mu_{n,F}) \to \mathrm{H}^0(X, R^1 j_* \mu_n) \xrightarrow{\omega} \mathrm{H}^2(X, \mu_{n,X}) \to 0.$$

The homomorphism ω can be realized in the following way (up to possibly multiplying by -1, depending on one's conventions for boundary maps in spectral sequences). Taking the long exact sequence associated to the functor j_* applied to (5.4) gives an exact sequence

(5.6) $$0 \to \mu_{n,X} \to j_* G_{m,F} \to j_* G_{m,F} \to R^1 j_* \mu_{n,F} \to 0$$

since $R^1 j_* G_{m,F} = 0$ by Hilbert's Theorem 90. Splitting (5.6) into two short exact sequences and then taking boundary maps in the associated long exact cohomology sequences over X produces the transgression map ω in (5.5) up to possibly multiplying by -1.

We now recall from [7, p. 36-39] some definitions.

Definition 5.1. Let x be a point of X with residue field $k(x)$. Define $O_x = O_{X,x}$ to be the local ring of x on X. Let $\overline{x}$ be a geometric point of X over x, so that $k(\overline{x})$ is a separable closure of $k(x)$. The Henselization $O_{x,h}$ of O_x (resp. the strict Henselization $O_{x,sh}$ of O_x) is the direct limit of all of

all local rings D (resp. D') which are étale O_x-algebras having residue field $k(x)$ (resp. having residue field inside $\overline{k(x)}$). Let $\hat{O}_x$ be the completion of O_x and let $\hat{O}_{\overline{x}}$ be the direct limit of all finite étale local $\hat{O}_x$ algebras having residue field in $\overline{k(x)}$.

The following result is implicit in [5], but we will recall the argument since the details of the computation enter into some later calculations.

Lemma 5.2. *Let x be a point of X, and let $\overline{x}$ be a geometric point over x. The stalk $(R^1 j_* \mu_n)|_{\overline{x}}$ of $R^1 j_* \mu_n$ at $\overline{x}$ is the cohomology group $\mathrm{H}^1(F_{x,sh}, \mu_n)$, where $F_{x,sh} = F \otimes_{O_F} O_{x,sh}$. The Kummer sequence*

$$1 \to \mu_n \to G_m \to G_m \to 1$$

over $F_{x,sh}$ is exact. The $\mathrm{Gal}(\overline{F}_{x,sh}/F_{x,sh})$ cohomology of this sequence gives an isomorphism

$$(5.7) \qquad F_{x,sh}^*/(F_{x,sh}^*)^n = \mathrm{H}^1(F_{x,sh}, \mu_n) = (R^1 j_* \mu_n)|_{\overline{x}}.$$

This group is trivial if x is the generic point of X. Suppose now that x is a closed point, with residue field $k(x)$. We then have natural isomorphisms $\mathrm{Gal}(F_{x,sh}/F_{x,h}) = \mathrm{Gal}(\overline{k(x)}/k(x)) = \hat{\mathbb{Z}}$ where $F_{x,h} = F \otimes_{O_F} O_{x,sh}$. One has

$$(5.8) \qquad \mathrm{H}^0(\mathrm{Gal}(F_{x,sh}/F_{x,h}), (R^1 j_* \mu_n)|_{\overline{x}}) = \hat{F}_x^*/T_x$$

where $\hat{F}_x = \mathrm{Frac}(\hat{O}_x)$ is the completion of F with respect to the discrete absolute value at x and $T_x \supset (\hat{F}_x^*)^n$ is the subgroup of $\gamma \in \hat{F}_x^*$ such that $\hat{F}_x(\gamma^{1/n})$ is unramified over $\hat{F}_x$. Here $T_x/((\hat{F}_x)^*)^n$ is cyclic of order n. Finally, we have

$$(5.9)$$
$$\mathrm{H}^0(X, R^1 j_* \mu_{n,F}) = \bigoplus_{x \in X^0} \mathrm{H}^0(\mathrm{Gal}(F_{x,sh}/F_{x,h}), (R^1 j_* \mu_n)|_{\overline{x}}) = \bigoplus_{x \in X^0} \hat{F}_x^*/T_x$$

where X^0 is the set of closed points of X.

Proof. The isomorphism (5.7) results from the description of stalks of higher direct images in [7, Thm 1.15] together with the long exact cohomology sequence of the Kummer sequence over $F_{x,sh}$. If x is the generic point of X, then $F_{x,sh}$ is an algebraic closure of F and the groups in (5.7) are trivial. Suppose now that x is a closed point. We then have two exact sequences

$$(5.10) \qquad 1 \to \tilde{\mu}_n \to F_{x,sh}^* \to (F_{x,sh}^*)^n \to 1$$

and

$$(5.11) \qquad 1 \to (F_{x,sh}^*)^n \to F_{x,sh}^* \to F_{x,sh}^*/(F_{x,sh}^*)^n \to 1.$$

Taking the cohomology of the second exact sequence (5.11) with respect to $\Gamma = \mathrm{Gal}(F_{x,sh}/F_{x,h}) = \mathrm{Gal}(\overline{k(x)}/k(x))$ and then taking completions gives

an exact sequence

$$(5.12) \qquad 0 \to ((\hat{F}_x^{\mathrm{un}})^*)^n)^\Gamma \to \hat{F}_x^* \to (F_{x,sh}^*/(F_{x,sh}^*)^n)^\Gamma$$
$$\to \mathrm{H}^1(\Gamma, (F_{x,sh}^*)^n) \to \mathrm{H}^1(\Gamma, F_{x,sh}^*) = 0$$

where $\hat{F}_x^{\mathrm{un}}$ is the maximal unramified extension of the complete local field $\hat{F}_x$. The Γ-cohomology of the first exact sequence (5.10) gives

$$(5.13) \qquad 0 = \mathrm{H}^1(\Gamma, F_{x,sh}^*) \to \mathrm{H}^1(\Gamma, (F_{x,sh}^*)^n) \to \mathrm{H}^2(\Gamma, \tilde{\mu}_n).$$

The cohomology of finite modules for $\Gamma = \hat{\mathbb{Z}}$ is trivial above dimension 1. So (5.13) shows $\mathrm{H}^1(\Gamma, (F_{x,sh}^*)^n) = 0$. In (5.12), the group $((\hat{F}_x^{\mathrm{un}})^*)^n)^\Gamma$ consists of those $\gamma \in \hat{F}_x^*$ such that $\hat{F}_x(\gamma^{1/n})$ is unramified over $\hat{F}_x$, so $((\hat{F}_x^{\mathrm{un}})^*)^n)^\Gamma = T_x$. Hence (5.12) now shows (5.8).

Now $R^1 j_* \mu_n$ has trivial stalk over the generic point of X, and units are n-th powers locally in the étale topology over all closed points $x \in X^0$ having residue fields prime to n. We conclude from (5.6) that $R^1 j_* \mu_{n,F}$ is the sheaf resulting from the direct sum of the stalks $(R^1 j_* \mu_n)|_{\bar{x}}$ as x ranges over X^0, from which (5.9) follows. $\square$

Corollary 5.3. *The exact sequence* (5.5) *is identified with*

$$(5.14) \qquad F^*/(F^*)^n \xrightarrow{r} \bigoplus_{x \in X^0} \hat{F}_x^*/T_x \xrightarrow{\omega} \mathrm{H}^2(X, \mu_n) \to 0.$$

Proof. By the Kummer sequence over F we have $\mathrm{H}^1(F, \mu_n) = F^*/(F^*)^n$. If $\beta \in F^*$, then $F(\beta^{1/n})$ is unramified at almost all places of F, so $\beta \in T_x$ for all but finitely many $x \in X^0$. Thus the natural homomorphisms $F^* \to \hat{F}_x^*/T_x$ give rise to a homomorphism r as in (5.14), and the constructions in Lemma 5.2 identify r with the first map in (5.5). $\square$

Lemma 5.4. *Suppose that in the description* $\mathrm{H}^2(X, \mathbb{Z}/n)^\star = T/(F^*)^n$ *of Lemma 4.3 we are given an element* $\eta \in T$ *describing a class* $\eta(F^*)^n \in T/(F^*)^n$. *Let* $j \in J(F)$ *be an idele of* F *such that the component* j_x *of* j *at almost all* $x \in X^0$ *lies in* T_x, *so that* j *defines an element* $z(j)$ *of* $\oplus_{x \in X^0}(\hat{F}_x^*/T_x)$. *Corollary 5.3 produces an element* $\omega(z(j))$ *of* $\mathrm{H}^2(X, \mu_n)$. *We have* $\mathrm{H}^2(X, \mu_n) = \mathrm{H}^2(X, \mathbb{Z}/n) \otimes_{\mathbb{Z}} \tilde{\mu}_n$ *and thus a natural non-degenerate pairing*

$$(5.15) \qquad \langle \, , \, \rangle : \mathrm{H}^2(X, \mathbb{Z}/n)^\star \times \mathrm{H}^2(X, \mu_n) \to \tilde{\mu}_n$$

resulting from the Pontryagin duality pairing

$$\mathrm{H}^2(X, \mathbb{Z}/n)^\star \times \mathrm{H}^2(X, \mathbb{Z}/n) \to \mathbb{Z}/n.$$

The value of the pairing in (5.15) *on the pair* $\eta(F^*)^n$ *and* $\omega(z(j))$ *is*

$$(5.16) \qquad \langle \eta(F^*)^n, \omega(z(j)) \rangle = \mathrm{Art}(j)(\eta^{1/n})/\eta^{1/n}$$

where $\mathrm{Art}(j)$ *is the image of* j *under the Artin map* $J(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ *when* $F^{\mathrm{ab}} \supset F(\eta^{1/n})$ *is the maximal abelian extension of* F.

Proof. This follows from reducing the computation of duality pairings to the computation of Hilbert symbols, as in [5, §2.4-2.6]. Here is one way to carry this out explicitly.

We have a long exact relative cohomology sequence

(5.17) $\mathrm{H}^1(X, \mathbb{Z}/n) \to \mathrm{H}^1(X - V, \mathbb{Z}/n) \to \mathrm{H}^2_V(X, \mathbb{Z}/n) \to \mathrm{H}^2(X, \mathbb{Z}/n)$

$$\xrightarrow{e} \mathrm{H}^2(X - V, \mathbb{Z}/n) \to \mathrm{H}^3_V(X, \mathbb{Z}/n) \xrightarrow{b} \mathrm{H}^3(X, \mathbb{Z}/n)$$

associated to a choice of a finite non-empty set V of closed points of X which is discussed in [5, §2.5].

Suppose we take V large enough so that $\mathrm{Pic}(X - V) = 0$ and all of the residue characteristics of points of $X - V$ are relatively prime to n. Then the Kummer sequence

$$1 \to \mu_{n,X-V} \to G_{m,X-V} \to G_{m,X-V} \to 1$$

is exact. So

$$\mathrm{H}^1(X - V, G_{m,X-V}) = \mathrm{Pic}(X - V) = 0$$

implies $\mathrm{H}^2(X - V, \mu_{n,X-V})$ equals the n-torsion in the Brauer group $\mathrm{H}^2(X - V, G_{m,X-V})$. This n-torsion has order $n^{\#V-1}$ by the usual theory of elements of the Brauer group of F which are unramified outside of V. By local duality (c.f. [5, p. 540, 538]),

$$\mathrm{H}^3_V(X, \mathbb{Z}/n) = \prod_{P \in V} \tilde{\mu}_n^\star.$$

Global duality gives

$$\mathrm{H}^3(X, \mathbb{Z}/n) = \mathrm{Ext}^0_X(\mathbb{Z}/n, G_m) = \tilde{\mu}_n^\star.$$

By considering the orders of these groups, we see that the map b in (5.17) has kernel exactly $\mathrm{H}^2(X - V, \mathbb{Z}/n)$, so the map e is trivial.

By local duality (*op. cit.*) we have

$$\mathrm{H}^2_V(X, \mathbb{Z}/n) = \prod_{P \in V} (\hat{O}_P^* / (\hat{O}_P^*)^n)^\star.$$

Using these isomorphisms in (5.17) and taking Pontryagin duals gives an exact sequence

(5.18)
$$0 \to \mathrm{H}^2(X, \mathbb{Z}/n)^\star \to \prod_{P \in V} \hat{O}_P^* / (\hat{O}_P^*)^n \to \mathrm{H}^1(X - V, \mathbb{Z}/n)^\star \to \mathrm{H}^1(X, \mathbb{Z}/n)^\star.$$

By class field theory,

$$\mathrm{H}^1(X, \mathbb{Z}/n) = \mathrm{Hom}(\mathrm{Cl}(O_F), \mathbb{Z}/n)$$

and

$$\mathrm{H}^1(X - V, \mathbb{Z}/n) = \mathrm{Hom}(\mathrm{Cl}_{m_V}(O_F), \mathbb{Z}/n)$$

when $\mathrm{Cl}(O_F) = \mathrm{Pic}(O_F)$ is the ideal class group of O_F and $\mathrm{Cl}_{m_V}(O_F)$ is the ray class group of conductor m_V for m_V a sufficiently high power of the product of the prime ideals of O_F corresponding to $P \in V$.

Thus (5.18) becomes

$$(5.19) \quad 0 \to \mathrm{H}^2(X, \mathbb{Z}/n)^\star \to \prod_{P \in V} \hat{O}_P^* / (\hat{O}_P^*)^n \to \frac{\mathrm{Cl}_{m_V}(O_F)}{n\mathrm{Cl}_{m_V}(O_F)} \to \frac{\mathrm{Cl}(O_F)}{n\mathrm{Cl}(O_F)}$$

where the right hand homomorphism is induced by the canonical surjection $\mathrm{Cl}_{m_V}(O_F) \to \mathrm{Cl}(O_F)$.

Now in (5.14), since $\mathrm{H}^2(X, \mu_n)$ is finite, we can take V as above sufficiently large so that there is a surjection

$$(5.20) \qquad \bigoplus_{P \in V \subset X^0} \hat{F}_P^* / T_P \to \mathrm{H}^2(X, \mu_n) \to 0.$$

The compatibility of local and global duality pairings shows that pairing

$$\mathrm{H}^2(X, \mathbb{Z}/n)^\star \times \mathrm{H}^2(X, \mu_n) \to \tilde{\mu}_n$$

in (5.15) results from (5.19), (5.20) and the pairings

$$(5.21) \qquad \frac{\hat{O}_P^*}{(\hat{O}_P^*)^n} \times \frac{\hat{F}_P^*}{T_P} \to \tilde{\mu}_n$$

induced by the Hilbert pairings

$$(5.22) \qquad \frac{\hat{F}_P^*}{(\hat{F}_P^*)^n} \times \frac{\hat{F}_P^*}{(\hat{F}_P^*)^n} \to \tilde{\mu}_n.$$

Note here that (5.21) is non-degenerate since (5.22) is non-degenerate and $T_P/(\hat{F}_P^*)^n$ corresponds by class field theory to the unique cyclic unramified extension of degree n of $\hat{F}_P$.

This description of (5.15) leads to (5.16) by the compatibility of the Artin map with Hilbert pairings. $\qquad \square$

6. Analysis of $f_X^*(c_2)$

Our goal now is to compute the cup product in (5.1) using Lemma 5.4. We have a reasonable description of $h(f_X^*(d_1)) \in \mathrm{H}^2(X, \mathbb{Z}/n)^\star$ from Corollary 4.4 in terms of a Kummer generator $\xi \in F$ for the $\pmb{\mu}_n$-torsor $Y(f_X^*(d_1))$ produced by the generator $d_1 \in \mathrm{H}^1(G, \tilde{\mu}_n)$ and the homomorphism $f : \pi_1(X) \to G$. Recall that we assumed f to be surjective, and we know $K = F(\xi^{1/n})$ is a cyclic degree n Kummer extension which is everywhere unramified over F. In this section we must develop an expression for $f_X^*(c_2) \in \mathrm{H}^2(X, \mu_n)$ when c_2 is a generator for $\mathrm{H}^2(G, \tilde{\mu}_n)$. This will then be used in Lemma 5.4.

Consider the exact sequences of $G = \mathrm{Gal}(K/F)$-modules

$$(6.1) \qquad 1 \to \tilde{\mu}_n \to K^* \to (K^*)^n \to 1$$

and

$$(6.2) \qquad 1 \to (K^*)^n \to K^* \to K^*/(K^*)^n \to 1.$$

Lemma 6.1. *The composition of the boundary maps in the long exact G-cohomology sequences associated to* (6.1) *and* (6.2) *gives an exact sequence*

$$(6.3) \qquad\qquad F^* \to (K^*/(K^*)^n)^G \to \mathrm{H}^2(G, \tilde{\mu}_n) \to 0.$$

Here $\mathrm{H}^2(G, \tilde{\mu}_n)$ is cyclic of order n. So there is an element $\gamma \in K^*$ such that the coset $\gamma(K^*)^n$ is in $(K^*/(K^*)^n)^G$, and the image of this coset in $\mathrm{H}^2(G, \tilde{\mu}_n)$ equals the generator c_2.

Proof. Since G is cyclic, the map $\mathrm{H}^2(G, \tilde{\mu}_n) \to \mathrm{H}^2(G, K^*)$ is the cup product with a generator of $\mathrm{H}^2(G, \mathbb{Z})$ of the map $\hat{H}^0(G, \tilde{\mu}_n) \to \hat{H}^0(G, K^*)$ of Tate cohomology groups. Since K/F is everywhere unramified, every element of $\tilde{\mu}_n$ is a local norm. Hence every element of $\tilde{\mu}_n$ is a global norm from K to F because K/F is cyclic. Therefore $\hat{H}^0(G, \tilde{\mu}_n) \to \hat{H}^0(G, K^*)$ is the trivial map, so $\mathrm{H}^2(G, \tilde{\mu}_n) \to \mathrm{H}^2(G, K^*)$ is the trivial map. Because $\mathrm{H}^1(G, K^*) = 0$, the G cohomology of the exact sequences (6.1) and (6.2) gives (6.3). $\qquad\square$

Lemma 6.2. *With the above notations, the extension $K(\gamma^{1/n})$ is a cyclic degree n^2 extension of F which contains K. There is an idele $j = (j_v)_v$ of $J(F)$ with the following properties. If v is an infinite place of F, $j_v = 1$. Suppose v is finite and that v corresponds to the closed point x of X. Then for all places w of K above v, the images of $j_v \in F_v^* = \hat{F}_x^*$ and $\gamma \in K^*$ in*

$$(\hat{F}_{x,sh}^*/(\hat{F}_{x,sh}^*)^n)^{G_x} = \hat{F}_x^*/T_x$$

agree for any embedding of K_w into $\hat{F}_{x,sh}$ over F_v. Let $\mathrm{ord}_v : F_v^* \to \mathbb{Z}$ be the discrete valuation associated to v. Then $\mathrm{ord}_v(\mathrm{Norm}_{K/F}(\gamma))$ lies in $n\mathbb{Z}$, and there is a congruence of integers

$$(6.4) \qquad \frac{\mathrm{ord}_v(\mathrm{Norm}_{K/F}(\gamma))}{n} \equiv \mathrm{ord}_v(j_v) \equiv \mathrm{ord}_w(\gamma) \mod n_v\mathbb{Z}$$

when n_v is the order the decomposition group $G_w \subset G = \mathrm{Gal}(K/F)$ of any place w over v in K and $\mathrm{ord}_w : K_w^* \to \mathbb{Z}$ is the discrete valuation associated to w. Finally, in the notation of Lemma 5.4, for all such j the element $\omega(z(j)) \in \mathrm{H}^2(X, \mu_n)$ equals $f_X^*(c_2)$.

Proof. Since $\gamma(K^*)^n$ lies in the invariants $(K^*/(K^*)^n)^G$ and $G = \mathrm{Gal}(K/F)$ is cyclic, the extension $K(\gamma^{1/n})$ is abelian over F. Since $\gamma(K^*)^n$ has image of order n in $\mathrm{H}^2(G, \tilde{\mu}_n)$, it must define an element of $K^*/(K^*)^n$ of order n. So $K(\gamma^{1/n})$ is a cyclic degree n extension of K.

By Kummer theory,

$$K^*/(K^*)^n = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{(n)}/K), \tilde{\mu}_n)$$

and

$$F^*/(F^*)^n = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(F^{(n)}/F), \tilde{\mu}_n)$$

when $K^{(n)}$ is the maximal abelian exponent n extension of K and $F^{(n)}$ is defined similarly for F. The natural homomorphism $F^*/(F^*)^n \to K^*/(K^*)^n$

corresponds to the map

$$\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(F^{(n)}/F), \tilde{\mu}_n) \to \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{(n)}/K), \tilde{\mu}_n)$$

which results from restricting continuous homomorphisms from $\mathrm{Gal}(F^{(n)}/F)$ to $\mathrm{Gal}(F^{(n)}/K)$ and then inflating them to $\mathrm{Gal}(K^{(n)}/K)$. The image of

$$F^*/(F^*)^n \to K^*/(K^*)^n$$

is thus contained in the set $\mathcal{H}$ of those elements of $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{(n)}/K), \tilde{\mu}_n)$ which are inflated from elements of the group $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(F^{(n)}/K), \tilde{\mu}_n)$. Let us show that this image is precisely $\mathcal{H}$. It is enough to show that any continuous homomorphism $\mathrm{Gal}(F^{(n)}/K) \to \tilde{\mu}_n$ can be extended to a continuous homomorphism $\mathrm{Gal}(F^{(n)}/F) \to \tilde{\mu}_n$. This is so because the sequence

$$1 \to \mathrm{Gal}(F^{(n)}/K) \to \mathrm{Gal}(F^{(n)}/F) \to \mathrm{Gal}(K/F) \to 1$$

splits owing to the fact that $\mathrm{Gal}(F^{(n)}/F)$ is an exponent n abelian group and $\mathrm{Gal}(K/F)$ is isomorphic to $\mathbb{Z}/n$.

In view of (6.3) and the above discussion of Kummer theory, $\gamma(K^*)^n$ corresponds to a homomorphism $h : \mathrm{Gal}(K^{(n)}/K) \to \tilde{\mu}_n$ such that the smallest power of h which is in $\mathcal{H}$ is the n-th power of h. This means that the compositum $F^{(n)}K(\gamma^{1/n})$ is a cyclic degree n extension of $F^{(n)}$, where $K \subset F^{(n)}$. Now $K(\gamma^{1/n})$ is an abelian extension of F of some exponent m with $n|m|n^2$. If $m \neq n^2$, then $\Gamma = \mathrm{Gal}(K(\gamma^{1/n})/F)$ would have a subgroup H such that Γ/H has exponent n and H has exponent $m/n < n$. Now $K(\gamma^{1/n})^H$ is an exponent n abelian extension of F, so $K(\gamma^{1/n})^H \subset F^{(n)}$. But then

$$[F^{(n)}K(\gamma^{1/n}) : F^{(n)}] \leq [K(\gamma^{1/n}) : K(\gamma^{1/n})^H] = H < n$$

contradicting the fact that $F^{(n)}K(\gamma^{1/n})$ is cyclic of degree n over $F^{(n)}$. Thus $K(\gamma^{1/n})/F$ must be an exponent n^2 abelian extension of F, so in fact it is a cyclic extension of degree n^2 of F.

Suppose now that x is a closed point of X corresponding to a finite place v of F. As in the proof of Lemma 5.2, let $O_{x,sh}$ be the strict Henselization of the local ring O_x of x on X. Define $F_{x,sh} = F \otimes_{O_F} O_{x,sh}$. We know that Y is etale over x so the local ring O_y of each such y lies inside $O_{x,sh}$. Thus the completion K_w of K at each place w over v lies inside the completion $\hat{F}_{x,sh}$ of $F_{x,sh}$. Fix a place $w(v)$ of K over v and choose any embedding of $K_{w(v)}$ into $\hat{F}_{x,sh}$ over $F_v = \hat{F}_x$. The fact that G permutes the places w of K over v leads to a sequence of homomorphisms

(6.5) $(K^*/(K^*)^n)^G \longrightarrow (\oplus_{w|v}(K_w^*/(K_w^*)^n))^G$

$$\|$$

$$(K_{w(v)}^*/(K_{w(v)}^*)^n)^{G_{w(v)}} \longrightarrow (\hat{F}_{x,sh}^*/(\hat{F}_{x,sh}^*)^n)^{G_x}$$

when $G_x = \mathrm{Gal}(F_{x,sh}/F_{x,h})$. However, in Lemma 5.2 we showed that the right hand side of (6.5) is just $\hat{F}_x^*/T_x = F_v^*/T_x$. So we can choose the local component j_v associated to v to come from $\gamma \in K$ in the way described in Lemma 6.2. At infinite v we can certainly choose j_v to be trivial. Now the fact that $\gamma(K^*)^n$ has image $f_X^*(c_2)$ in $\mathrm{H}^2(X, \mu_n)$ together with the construction of Corollary 5.3 shows $\omega(z(j)) = f_X^*(c_2)$.

It remains to show the congruence (6.4) for each finite place v of F. A uniformizer of F_v is one for $\hat{F}_{x,sh}$ and for K_w. Hence we have

$$(6.6) \qquad\qquad \mathrm{ord}_v(j_v) \equiv \mathrm{ord}_w(\gamma) \mod n\mathbb{Z}$$

from the above construction of j_v from γ.

We now fix a place $w(v)$ over v in K. Since $\gamma(K^*)^n$ lies in $(K^*/(K^*)^n)^G$ we know that for each place w over v in K there is an integer $t(w)$ such that $\mathrm{ord}_w(\gamma) = \mathrm{ord}_{w(v)}(\gamma) + nt(w)$. Since each K_w is an unramified cyclic extension of F_v of degree n_v, we have

$$\mathrm{ord}_v(\mathrm{Norm}_{K_w/F_v}(\gamma)) = n_v(\mathrm{ord}_{w(v)}(\gamma) + nt(w)).$$

Hence

$$
\begin{aligned}
\mathrm{ord}_v(\mathrm{Norm}_{K/F}(\gamma)) &= \sum_{w|v} \mathrm{ord}_v(\mathrm{Norm}_{K_w/F_v}(\gamma)) \\
&= \sum_{w|v} n_v(\mathrm{ord}_{w(v)}(\gamma) + nt(w)) \\
&= \Big(\sum_{w|v} n_v\Big) \cdot \mathrm{ord}_{w(v)}(\gamma) + n \cdot n_v \sum_{w|v} t(w) \\
(6.7) \qquad &= n \cdot \mathrm{ord}_{w(v)}(\gamma) + n \cdot n_v \sum_{w|v} t(w).
\end{aligned}
$$

Since $w(v)$ was an arbitrary place of K over v, dividing (6.7) by n and using (6.6) completes the proof of the congruence (6.4). $\qquad\square$

7. Proof of Theorem 1.3

We will adopt the notations of Theorem 1.3. Thus $G = \mathbb{Z}/n$ and c_1 is a generator of $\mathrm{H}^1(G, \mathbb{Z}/n) = \mathrm{Hom}(G, \mathbb{Z}/n)$ given by the identity map. The element c_2 generates $\mathrm{H}^2(G, \tilde{\mu}_n)$ and $c = c_1 \cup c_2$. In Definition 1.4 we picked a particular primitive n-th root of unity ζ_n. Let $\phi : \mathbb{Z}/n \to \tilde{\mu}_n$ be the isomorphism sending $1 \bmod n$ to ζ_n. Then $d_1 = \phi \circ c_1$ is a generator of $\mathrm{H}^1(G, \tilde{\mu}_n)$. Write $K = F(\xi^{1/n})$ as in Corollary 4.4 for an element $\xi \in F^*$ which is determined mod $(F^*)^n$ by $f_X^*(d_1) \in \mathrm{H}^1(X, \mu_n) = \mathrm{Hom}(\mathrm{Pic}(X), \tilde{\mu}_n)$. We have an isomorphism $\mathrm{Gal}(K/F) \to G = \mathbb{Z}/n$ determined by $f^*(c_1) : \pi_1(X, \eta) \to \mathbb{Z}/n$. We will use this isomorphism to identify $\mathrm{Gal}(K/F)$ with $G = \mathbb{Z}/n$ in what follows. The element $\xi \in F$ is the Kummer generator for

K as an everywhere unramified cyclic extension of F for which

$$\text{(7.1)} \qquad \frac{\text{Art}(a)(\xi^{1/n})}{\xi^{1/n}} = \phi(\text{Art}(a)) = f_X^*(d_1)(a)$$

for $a \in \text{Cl}(O_F) = \text{Pic}(X)$, where $\text{Art} : \text{Cl}(O_F) \to \text{Gal}(K/F)$ is the Artin map for K/F.

The element of $\text{H}^2(G, \mathbb{Z}/n)^\star = T/(F^*)^n$ associated to d_1 by Corollary 4.4 is the coset $\xi(F^*)^n$. Let j be an idele of F associated to c_2 as in Lemma 6.2. Then $\omega(z(j)) = f_X^*(c_2) \in \text{H}^2(X, \mu_n)$.

By Lemma 5.4, the value of the pairing

$$\langle \, , \, \rangle : \text{H}^2(X, \mathbb{Z}/n)^\star \times \text{H}^2(X, \mu_n) \to \tilde{\mu}_n$$

in (5.15) on the pair $\xi(F^*)^n$ and $\omega(z(j)) = f_X^*(c_2)$ is

$$\text{(7.2)} \qquad \langle \xi(F^*)^n, \omega(z(j)) \rangle = \text{Art}(j)(\xi^{1/n})/\xi^{1/n}.$$

Here j is the idele constructed in Lemma 5.4, and we are also using Art to denote the Artin map from the ideles $J(F)$ of F to $\text{Gal}(K/F)$. Since $d_1 = \phi \circ c_1$, this is $\phi(n \cdot (f_X^*(c_1) \cup f_X^*(c_2)))$ when

$$S(f, c) = f_X^*(c_1) \cup f_X^*(c_2) \in \text{H}^3(X, \mu_n) = \mathbb{Z}/n\mathbb{Z}$$

is Kim's invariant for f and c.

Combining this with the normalization of ξ in Corollary 4.4 and (7.1) gives

$$\phi(S(f, c)) = \phi(\text{Art}(j)).$$

Thus $\text{Art}(j) = S(f, c)$. Hence the proof of the formula (1.14) is reduced to showing

$$\text{(7.3)} \qquad \text{Art}(j) = \text{Art}([I])$$

for a fractional ideal I of O_F having the properties in Theorem 1.3, where $[I]$ is the ideal class of I in $\text{Cl}(O_F)$.

The first property of I is that it should be an n-th root of $\text{Norm}_{K/F}(\gamma)O_F$ when $\gamma \in K$ is as in Theorem 1.3. The fact that I exists is shown by (6.4) of Lemma 6.2, which showed $\text{ord}_v(\text{Norm}_{K/F}(\gamma))$ is divisible by n for all finite places v of F. Let j_v be the component of j at v. The congruence in (6.4) also shows that $\text{ord}_v(j_v)$ is congruent to $\text{ord}_v(I)$ modulo the order n_v of the decomposition group of a place w over v in K. Since K/F is an unramified extension, this is enough to show the equality (7.3), which completes the proof.

8. Proof of Theorem 1.8 and of Corollaries 1.9 and 1.10

Parts (i) and (ii) of Theorem 1.8 follow from the arguments used in Lemmas 6.1 and 6.2 together with Theorem 1.3.

To show part (iii) of Theorem 1.8, it will suffice to show the following for each place v of F. Let j_v be the v component of an idele j of F with the properties in Lemma 6.2. Let n_v be the local degree of v in K/F, i.e.

the order of the decomposition group in $G = \mathrm{Gal}(K/F)$ of a place over v in K. In view of the equality (7.3), Theorem 1.3 and the congruence (6.4), it will suffice to show n_v divides $\mathrm{ord}_v(j_v)$ if L/K is unramified over v or if v splits in K. Here $L = K(\gamma^{1/n})$ for some γ as in Lemma 6.2. If L/K is not ramified over the place w of K over v, then $\mathrm{ord}_w(\gamma)$ must be divisible by n. But $\mathrm{ord}_v(j_v) \equiv \mathrm{ord}_w(\gamma) \bmod n_v\mathbb{Z}$ by (6.4), and $n_v \big| n$, so we get $n_v \big| \mathrm{ord}_v(j_v)$ in this case. If v splits in K, then $n_v = 1$ so $n_v \big| \mathrm{ord}_v(j_v)$ is trivial. This finishes the proof of Theorem 1.8.

Corollary 1.9 follows directly from Theorem 1.8, since we can take I' to be O_F in this case.

Suppose now that the hypotheses of Corollary 1.10 hold. Let v be the place of F determined by the prime $\mathcal{P}$ in the statement of Corollary 1.10. Then there is a unique place w over v in K, w totally ramifies in L, and v and w have residue characteristic prime to n. Thus $L = K(\gamma^{1/n})$ implies $\mathrm{ord}_w(\gamma)$ is relatively prime to n, and $n_v = n$ since v is undecomposed in K. Since

$$\mathrm{ord}_{\mathcal{P}}(I) \equiv \mathrm{ord}_v(j_v) \equiv \mathrm{ord}_w(\gamma) \mod n_v\mathbb{Z}$$

as above, we conclude $\mathrm{ord}_{\mathcal{P}}(I)$ is relatively prime to $n_v = n$. By part (iii) of Theorem 1.8, we can take $I' = \mathcal{P}^{\mathrm{ord}_{\mathcal{P}}(I)}$ since v is the only place of F over which L/K ramifies. Hence $\mathrm{Art}([I'])$ is a generator of $\mathrm{Gal}(K/F)$ since $\mathrm{Art}([\mathcal{P}])$ is, so Corollary 1.10 follows from (7.3).

9. Proof of Theorem 1.2

By assumption, $n > 1$. It will suffice to construct infinitely many totally complex fields F which have cyclic degree n^2 extensions L_1/F and L_2/F having the properties in Corollary 1.9 and 1.10, respectively. We use a base change argument to do this.

The field $\mathbb{Q}(\zeta_{n^2})$ is totally complex. We start with an initial choice of a field F_1 containing $\mathbb{Q}(\zeta_{n^2})$ together with a cyclic degree n^2 extension N_1 of F_1. Let S_1 be the set of places of F_1 which ramify in N_1. Let F be a number field containing F_1 which is linearly disjoint from N_1 such that for each place w of F over a place v in S_1, the completion $(F_1)_w$ contains the completions of N_1 at places over v. Then $L_1 = FN_1$ will be a cyclic unramified degree n^2 extension of F as required in Corollary 1.9. For simplicity we now replace F_1 by F to be able to assume that N_1/F_1 is a cyclic degree n^2 unramified extension. Any base change of N_1/F_1 by a field extension F of F_1 which is disjoint from N_1 will preserve this property.

We now focus on finding an extension F of F_1 which is disjoint from N_1 for which we can construct an extension L_2/F with the properties in Corollary 1.10.

Let $\mathcal{M}$ be a sufficiently high power of the ideal nO_{F_1} in O_{F_1} such that if $\alpha \in O_{F_1}$ and $\alpha \equiv 1 \bmod \mathcal{M}$, then α is in $((F_1)_v^*)^{n^2}$ for all places v of F_1 dividing n. Choose a prime $\mathcal{Q}$ of O_{F_1} which splits in the ray class field over

F_1 of conductor $\mathcal{M}$. Then by definition of the ray class group of F_1 mod $\mathcal{M}$, there is a generator α for $\mathcal{Q}$ such that $\alpha \equiv 1 \mod \mathcal{M}$. Since F_1 contains a root of unity of order n^2, the extension $N_2 = F_1(\alpha^{1/n^2})$ is an abelian Kummer extension of F_1. It is cyclic of degree n^2 and totally ramified over $\mathcal{Q}$ since α has valuation 1 at $\mathcal{Q}$. Now N_2/F_1 splits over all places of F_1 which divide n, since by construction α is an n^2 power at these places. Finally, at each place v of F_1 which does not divide n and which is not the place $v_{\mathcal{Q}}$ assocated to $\mathcal{Q}$, α has valuation 0 at v, so v is unramified in N_2. Thus N_2/F_1 is a cyclic degree n^2 extension unramified outside of $v_{\mathcal{Q}}$ and totally ramified over $v_{\mathcal{Q}}$.

Let $w_{\mathcal{Q}}$ be the unique place over $v_{\mathcal{Q}}$ in N_2. For simplicity, we define E to be the completion of F_1 at $v_{\mathcal{Q}}$, and we let Y be the completion of N_2 at $w_{\mathcal{Q}}$. Now Y/E is a cyclic degree n^2 totally ramified extension of local fields. There is a unique cyclic unramified extension E' of E of degree n. Consider the compositum $E'Y$. We have $\mathrm{Gal}(E'Y/E) = \mathrm{Gal}(E'Y/E') \times \mathrm{Gal}(E'Y/Y) = J_1 \times J_2$ where $J_1 = \mathrm{Gal}(E'Y/E') \cong \mathbb{Z}/n^2$ is the inertia subgroup of $\mathrm{Gal}(E'Y/E)$ and $J_2 = \mathrm{Gal}(E'Y/Y) \cong \mathrm{Gal}(E'/E) = \mathbb{Z}/n$ is cyclic of order n. Let j_1 be a generator of J_1 and let j_2 be a generator of J_2. The element $(j_1, j_2) \in J_1 \times J_2$ then generates a cyclic subgroup Γ of order n^2 in $J_1 \times J_2$, and $\Gamma \cap (J_1 \times \{0\}) = \Gamma \cap \mathrm{Gal}(E'Y/E')$ has order n. Thus the subfield $E'' = (E'Y)^\Gamma$ of $E'Y$ has the property that $E'Y/E''$ is cyclic of order n^2, $\mathrm{Gal}(E'Y/E'') = \Gamma$ has inertia group $\Gamma \cap (J_1 \times \{0\})$ of order n, and E''/E is cyclic and totally ramified of degree n. Thus E'' can be obtained from E by adjoining the root of an Eisenstein polynomial of degree n in $O_E[x]$. Note that $E'Y = E''Y$ since $\Gamma = \mathrm{Gal}(E'Y/E'')$ and $J_2 = \mathrm{Gal}(E'Y/Y)$ intersect only in the identity element.

We now choose F to be any degree n extension of F_1 which is totally ramified over $v_{\mathcal{Q}}$ such that the completion F_w of F at the unique place w over $v_{\mathcal{Q}}$ is isomorphic to E'' as an extension of $E = (F_1)_{v_{\mathcal{Q}}}$. Such an F can be constructed by finding a monic polynomial of degree n in $O_{F_1}[x]$ which is Eisenstein at $v_{\mathcal{Q}}$ and which locally at $v_{\mathcal{Q}}$ has a root in E''. Because F/F_1 is totally ramified over $v_{\mathcal{Q}}$, it is disjoint from the cyclic unramified degree n^2 extension N_1/F_1 we constructed at the beginning of the proof. Hence FN_1 is a cyclic degree n^2 unramified extension L_1 of F of the kind required in Corollary 1.9.

Consider now the compositum $L_2 = N_2F$ over F_1. We know there are unique places w and $w_{\mathcal{Q}}$ over $v_{\mathcal{Q}}$ in F and N_2, respectively, and $F_w = E''$ while $(N_2)_{w_{\mathcal{Q}}} = Y$. Since $E''Y = E'Y$ has degree n^3 over $F_v = E$, and $[L_2 : F_1] = [L_2 : F] \cdot [F : F_1] \leq n^2 \cdot n$, we see $[L_2 : F] = n^2$ and there is a unique place $\tilde{w}$ over $v_{\mathcal{Q}}$ in $L_2 = N_2F$. Thus L_2/F is a cyclic degree n^2 extension since it is the base change by $F_1 \subset F$ of N_2/F_1. The only place of F which can ramify in L_2 is the unique place w over $v_{\mathcal{Q}}$, since N_2/F_1 is unramified outside of $v_{\mathcal{Q}}$. Further, $\tilde{w}$ is the unique place of L_2 over w, and $(L_2)_{\tilde{w}}/F_w$ is the extension $E''Y/E''$. We showed that this local extension is

cyclic of order n^2 with inertia group of order n. Thus if we let $\mathcal{P}$ be the prime of F determined by w, the extension L_2/F will now have all of the properties required in Corollary 1.10. Theorem 1.2 now follows from Corollaries 1.9 and 1.10. Note that we can vary the above construction in many ways, e.g. by choosing different primes $\mathcal{Q}$, so we can construct infinitely many F with the properties in Theorem 1.2.

10. Proof of Theorems 1.5 and 1.15

We will use the notations of Theorems 1.5 and 1.3. Since $\sigma(y)/y = x^n \in (K^*)^n$, the coset $y(K^*)^n$ lies in $(K^*/(K^*)^n)^G$. Recall that we have exact sequences

$$1 \to \tilde{\mu}_n \to K^* \to (K^*)^n \to 1 \quad \text{and} \quad 1 \to (K^*)^n \to K^* \to K^*/(K^*)^n \to 1.$$

By the construction of the boundary map

$$\delta_0 : \hat{H}^0(G, K^*/(K^*)^n)) \to \mathrm{H}^1(G, (K^*)^n)$$

the class $\delta_0(y(K^*)^n)$ is represented by the one cocycle which sends σ^i to $\sigma^i(y)/y = (x^n)\sigma(x^n)\cdots\sigma^{i-1}(x^n)$ for $i \geq 0$. Thus $\delta_0(y(K^*)^n)$ is the cup product $[x^n] \cup t$, where $[x^n]$ is the class in $\mathrm{H}^{-1}(G, (K^*)^n)$ represented by the element $x^n \in K^*$ of norm 1 to F, and t is an appropriate generator of $\mathrm{H}^2(G, \mathbb{Z})$. The image of $[x^n]$ under the boundary map $\mathrm{H}^{-1}(G, (K^*)^n) \to \hat{H}^0(G, \tilde{\mu}_n) = \tilde{\mu}_n$ is the class represented by $\zeta_n = \mathrm{Norm}_{K/F}(x)$. Since boundary maps respect cup products with t, we find that $\delta_0(y(K^*))$ maps to an element of order n under the boundary map $\mathrm{H}^1(G, (K^*)^n) \to \mathrm{H}^2(G, \tilde{\mu}_n)$. This proves that $y(K^*)^n$ has image of order n under the map $(K^*/(K^*)^n)^G \to \mathrm{H}^2(G, \tilde{\mu}_n)$ which was used in (1.13) just prior to Theorem 1.3. Hence (1.13) shows that if we take $\gamma = y$ in Theorem 1.3, Theorem 1.5 now follows from Theorem 1.3. Theorem 1.15 is proved similarly.

11. Proof of Theorem 1.12

The hypotheses of Theorem 1.12 are that n is a properly irregular prime, so that n divides $\#\mathrm{Cl}(\mathbb{Z}[\zeta_n])$ but not $\#\mathrm{Cl}(\mathbb{Z}[\zeta_n + \zeta_n^{-1}])$, and K is a cyclic unramified extension of $F = \mathbb{Q}(\zeta_n)$ of degree n. We are to show that $S(f, c) = 0$ for all surjections $f : \pi_1(X, \eta) \to \mathrm{Gal}(K/F) = G = \mathbb{Z}/n$ and all $c \in \mathrm{H}^3(G, \tilde{\mu}_n)$.

Lemma 11.1. *There is a $\mathbb{Z}_n$ extension of F which contains K and which is unramified outside n.*

Before proving this lemma, we note how it implies Theorem 1.12. The lemma shows that there is a cyclic degree n^2 extension L of F which is unramified outside of n and contains K. The unique prime $\mathcal{P}$ over n in F is principal, so it splits in K. Hence Corollary 1.9 shows the conclusion of Theorem 1.12.

Proof of Lemma 11.1. Let $E = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ be the real subfield of F. Write $\Gamma = \mathrm{Gal}(F/E) = \{e, \sigma\}$. Then σ acts by inversion on the Sylow n-subgroup of $\mathrm{Cl}(O_F)$ since n does not divide $\#\mathrm{Cl}(O_E)$. Therefore K is contained in the maximal n-elementary extension N of F which is unramified outside of p, Galois over E and for which σ acts by inversion on $\mathrm{Gal}(N/F)$.

The Kummer pairing gives a Γ-equivariant isomorphism $\mathrm{Gal}(N/F) = \mathrm{Hom}(T/(F^*)^n, \tilde{\mu}_n)$ when $T = \{\xi \in F^* : \xi^{1/n} \in N\}$. Since σ acts by inversion on both $\mathrm{Gal}(N/F)$ and $\tilde{\mu}_n$, we conclude that it acts trivially on $T/(F^*)^n$. Because n is odd, this implies that the inclusion $E^* \to F^*$ induces a surjection $s : T' \to T/(F^*)^n$ when we let $T' = (E^*)^n \cdot \mathrm{Norm}_{F/E}(T)$.

Since E has class number prime to n, and N/F is unramified outside of n, we now see that $s(T'') = T/(F^*)^n$ when T'' is the subgroup of n-units in T'. The subgroup of n-units in E has no n-torsion and rank $d = (n-1)/2$. We conclude that $\mathrm{Gal}(N/F)$ is an elementary abelian n-group of dimension at most d over $\mathbb{Z}/n$.

By class field theory there is a $\mathbb{Z}_n^d$ extension $\tilde{F}$ of F which is unramified outside of n such that $\sigma \in \Gamma$ acts by inversion on $\mathrm{Gal}(\tilde{F}/F)$. The maximal n-elementary subextension N' of F in $\tilde{F}$ then has $\mathrm{Gal}(N/F) = (\mathbb{Z}/n)^d$, so in fact $N' = N$. This implies Lemma 11.1 since $K \subset N = N'$. $\qquad\square$

12. Proof of Theorem 1.13

The hypotheses of Theorem 1.13 are that $n > 2$ is prime and K/F is a cyclic unramified Kummer extension of degree n such that both K and F are Galois over $\mathbb{Q}$. The action of $\Delta = \mathrm{Gal}(F/\mathbb{Q})$ on $G = \mathrm{Gal}(K/F)$ is then via a character $\chi : \Delta \to \mathrm{Aut}(G)$. If we fix an isomorphism $\iota : G \to \mathbb{Z}/n$ we get an isomorphism between $\mathrm{Aut}(G)$ and $(\mathbb{Z}/n)^*$ that is independent of the choice of ι. In this way we can identify χ with a character $\chi : \Delta \to (\mathbb{Z}/n)^*$.

Theorem 1.3 gives a Δ-equivariant homomorphism

$$(12.1) \qquad \mathrm{H}^2(G, \tilde{\mu}_n) = (K^*/(K^*)^n)^G/\mathrm{Image}(F^*) \to G = \mathrm{Gal}(K/F)$$

sending the class of $\gamma(K^*)^n$ to $\mathrm{Art}([I])$ in the notation of Theorem 1.3. The action of Δ on $G = \mathrm{Gal}(K/F)$ is given by the character χ. To determine the action of Δ on $\mathrm{H}^2(G, \tilde{\mu}_n)$, we use the exact sequence

$$0 \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Q}/\mathbb{Z} \to 0$$

of modules with trivial G-action produced by multiplication by n on $\mathbb{Q}/\mathbb{Z}$. The boundary map in the long exact cohomology sequence of this sequence produces Δ-equivariant isomorphisms

$$(12.2) \qquad \mathrm{H}^2(G, \tilde{\mu}_n) = \mathrm{H}^2(G, \mathbb{Z}/n) \otimes \tilde{\mu}_n = \mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) \otimes \tilde{\mu}_n.$$

Since $\mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z})$ is $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$, we see from (12.2) that the action of Δ on $\mathrm{H}^2(G, \tilde{\mu}_n)$ is via the character $\chi^{-1} \cdot \omega$ where $\omega : \Delta \to (\mathbb{Z}/n)^*$ is the Teichmüller character giving the action of Δ on $\tilde{\mu}_n$. If (12.1) is not the trivial homomorphism, it must be an isomorphism between cyclic groups of order n,

and since it is Δ-equivariant we would have to have $\chi^{-1} \cdot \omega = \chi$. This would force $\omega = \chi^2$, which is impossible since ω has even order $n - 1 = \#(\mathbb{Z}/n)^*$. Thus (12.1) must be trivial under the hypotheses of Theorem 1.13, which completes the proof.

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