DEFINING KEY DEVELOPMENTAL UNDERSTANDINGS IN CONGRUENCE PROOFS FROM A TRANSFORMATION APPROACH

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Previous work by the authors (St. Goar et al., 2019) identified two potential key developmental understandings (KDUs) (Simon, 2006) in the construction of congruence proofs from a transformation perspective for pre-service secondary teachers in an undergraduate geometry course. We hypothesized the independence of the potential KDUs in previous work, meaning that students may have one potential KDU but not the other, and vice versa. We tested this hypothesis with analysis of an expanded data set and found that this hypothesis did not hold in general. We report on the expanded analysis and discuss implications for the scope and limitation of the potential KDUs.

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A change has come to K-12 geometry instruction, and as a result changes to preparation of future teachers must follow. Many guidelines (Catalyzing Change in High School Mathematics: Initiating Critical Conversations [NCTM], 2018) and standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) now recommend or require the teaching of geometry from a transformation perspective instead of the more traditional approach originating from Euclid's *Elements* (Sinclair, 2008). The concepts and proofs involving congruence and similarity now appeal to rigid motions: reflections, rotations, and translations. That is, two figures are said to be congruent if and only if there exists a sequence of rigid motions carrying one figure onto another. This definition is notably different from those in Euclid's *Elements,* where the criteria for congruence differs for each type of shape. Thus, the reader will note that the differences in mathematical structure between the transformation and *Elements* contexts are substantial.

The resulting danger is that some future teachers may lack the content knowledge to handle the new approach. Without sufficient content knowledge, they may struggle to know what can be proved in this new context and how these proofs may be structured. This lack may affect how they write lesson plans and course materials, adapt or modify materials for the context of their class, and evaluate student thinking and alternate approaches. Future teachers may need support in the transformation context to allow them to thrive in the teaching of geometry.

Relationship to Prior Literature

To answer the call, some undergraduate instructors are beginning to incorporate transformation geometry into their geometry courses for future teachers. Because transformation geometry is becoming a more prominent feature of geometry in post-secondary contexts, research on how pre-service teachers learn these topics are particularly salient. However, at this point research on how pre-service teachers learn transformation geometry is just beginning. Jones and Tzekaki (2016) noted the "limited research explicitly on the topics of congruency and similarity, and little on transformation geometry" (p. 139).

Some key results informing our work are the following. Edwards (2003) explained that students in middle school through undergraduate contexts tend to view transformations from a motion view, as opposed to a map view of transformations. A motion view is characterized by conceptualizing transformations as physical movements, such as picking up a figure and shoving it to where it needs to go. A map view is characterized in terms of inputs and outputs of transformation, and distinguishes the preimage from the image. For instance, a person with a motion view may think of an image and preimage of a figure as being the same object, simply with a different location. But a person with a map view can hold the idea that the image and preimage as different objects, and hence compare them. Research conducted after Edwards' (2003) study with middle school students corroborate her results, even for high school teachers (Hegg et al., 2018; Portnoy et al., 2006; Yanik, 2011). These results also note the difficulty that a motion view may present to generating congruence proofs from a transformation approach.

Based on analysis of future teachers' work on two congruence proofs on a midterm examination, we previously highlighted the importance of supporting pre-service teachers in understanding *both* directions of the "if and only if" in the definition of congruence. Further, we identified two potential key developmental understandings (KDUs; Simon, 2006), stated below:

"Potential KDU 1: Understanding that applying the definition of congruence to prove congruence of two figures means establishing a sequence of rigid motions mapping one *entire* figure to the other *entire* figure" (St. Goar et al., 2019).

"Potential KDU 2: Understanding that using a sequence of transformations to prove that two figures are congruent means justifying deductively that the image of one figure under the sequence of transformations is exactly the other figure" (St. Goar et al., 2019).

As the results by St. Goar et al. (2019) were based on analysis of teachers' work from a single, timed assessment, more work is needed to interrogate the accuracy of these potential KDUs.

Further, *we previously hypothesized the independence of these potential KDUs, meaning that teachers might hold KDU 1 but not KDU 2, or hold KDU 2 but not KDU 1.* We generated this hypothesis empirically from examples of teachers' work in our previous analysis. In considering the literature, we might also support and refine this hypothesis as follows. First, potential KDU 1 pertains to constructing a sequence of rigid motions, and not explicit deductive reasoning about images and preimages, which is the scope of potential KDU 2. Second, constructing a sequence of rigid motions can be consistent with either a motion view or a map view. However, deductive reasoning as needed for congruence proofs might require distinguishing between overlapping figures. Although this could be done under a motion view, it seemed plausible to us that conceiving transformations as maps was more likely to support a teacher in careful work with images and preimages – particularly if the figure is disconnected. It seemed plausible that it is more difficult to conceive of "moving" a disconnected figure than "moving" a connected one. In lieu of the literature, although it is possible for these potential KDUs to be independently held, *the following is a better hypothesis: Teachers hold neither potential KDU (if neither motion or map view is developed), potential KDU 1 but not potential KDU 2 (representing a motion view), or both potential KDUs (representing a map view).*

Objectives

Hence, we proceeded with the following research questions, with the same teachers' work on different congruence proofs than previously analyzed: (1) Do we continue to see evidence of the previously identified potential KDUs? (2) What are the scope and limitations, including the independence, of these potential KDUs?

Conceptual Perspective

Based on Usiskin and Coxford (1972), a *transformation approach* assumes without proof that rigid motions (e.g., reflections, rotations, and translations) are bijections of the plane that preserve both distance and angle measure. Additionally, under such an approach, two subsets of the plane are considered to be congruent if and only if there exist a sequence of rigid motions mapping one subset to the other. Similarity is treated analogously, incorporating dilations.

Key developmental understandings (KDUs) are described by Simon (2006). A key developmental understanding has two primary aspects: (1) Achieving a KDU represents a conceptual advance by the student. A conceptual advance is "a change in a students' ability to think about and/or perceive particular mathematical relationships" (Simon, 2006, p. 362) and (2) Acquiring KDUs does not tend to happen "as the result of an explanation or demonstration. That is, the transition requires a building up of the understanding through students' activity and reflection and usually comes about over multiple experiences" (Simon, 2006, p. 362).

As Simon noted, KDUs generally cannot be found by a mathematician examining their own understanding of a topic, but rather through observing students' mathematical work. As a result, our first steps in identifying these potential KDUs have been through the analysis of future teachers' work. Simon noted also that KDUs may be identified with varying amounts of detail and that "the level of detail specified for a key developmental understanding is adequate if it serves to guide the effort for which it is needed (e.g. curriculum design, further research)" (Simon, 2006 p. 364). Hence our analysis here is meant to achieve this necessary detail so that the potential KDUs can be used to improve undergraduate geometry curricula and research.

We use the term "potential KDU" rather than "KDU" because we see our understanding of teachers' understanding as a work in progress that is only based on analysis of written work as opposed to cognitive interviews, which would be ideal and needed to substantiate a claim of being a KDU. We return to this critical piece in the discussion and questions to the audience.

Methods

We collected the coursework of twenty teachers in an undergraduate geometry course taught by Lai. We examined homework assignments and midterm exams from throughout the semester for tasks where teachers specifically worked on congruence proofs. Here we report analysis of four tasks. This resulted in 69 total proof submissions included in the analysis.

We coded teachers' work on tasks based on evidence of potential KDU 1 and KDU 2. During the course of this analysis, if some criteria had to be changed, then codes were reworked to reflect these updated criteria, consistent with constant comparison (Strauss & Corbin, 1994).

Results

Addressing the first research question, the basic statements of the potential KDUs remained intact after analysis of teachers' work on additional tasks. Addressing the second research question, this analysis provided possible disconfirming evidence for the independence of the potential KDUs. We begin this section by reviewing the scope and limitations of the potential KDUs, and then compare evidence of each potential KDU.

Scope and Limitations of Potential KDUs

Potential KDU 1 is primarily focused on the *construction* of the sequences of rigid motions. That is, in order to have this potential KDU, teachers must construct a sequence of rigid motions from one entire figure to another entire figure. This means that aside from the creation of the rigid motions themselves, the rest of the deductive logic in a transformation proof is not a part of this potential KDU.

Potential KDU 2 focuses on the *deductive reasoning* used in the proof. Specifically, teachers need to attempt to deductively show that their transformation extends to the entire figure. Note that a teachers' work need not show entirely correct logic in order to show evidence of this potential KDU so long as they are attempting to extend arguments about the image of a transformation to entire figures and are using deductive logic to do so.

(Non) Independence of Potential KDUs

We hypothesized previously the independence of potential KDU 1 and potential KDU 2, meaning that, teachers' capacity to engage in deductive reasoning about the correctness of a proof may not depend on their capacity to construct sequences of rigid motions. We refined our view in lieu of the literature to hypothesize that it is most likely that teachers may hold neither potential KDU 1 nor potential KDU 2, hold potential KDU 1 and not potential KDU 2, or hold both. Our analysis suggests that our initial hypothesis is not well-supported, but our new hypothesis is. For brevity, we limit discussion of this to a visual summary of the results of this analysis, shown in Figure 1.

Homework Task 1	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	12	2
No Evidence of KDU 2	2	0

Homework Task 2	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	9	0
No Evidence of KDU 2	1	6

Midterm Task 1	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	4	0
No Evidence of KDU 2	12	1

Midterm Task 2	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	7	3
No Evidence of KDU 2	5	5

Across all Tasks	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	32	5
No Evidence of KDU 2	20	12

Figure 1: The above is a summary of evidence of potential KDU 1 and potential KDU 2 across two homework tasks and two midterm examination tasks.

Discussion and Conclusion

In this report, we expanded on the research by St. Goar et al. (2019) by analyzing future teachers' work on transformation congruence from an undergraduate geometry course. The results confirm the viability of potential KDU 1 and potential KDU 2 as codes for teachers' written work on congruence proofs from a transformation approach. Moreover, the results do corroborate the authors' revised hypothesis that that teachers may hold neither potential KDU 1 nor potential KDU 2, hold potential KDU 1 and not potential KDU 2, or hold both. In other words, the *least likely scenario* is that teachers hold potential KDU 2 but not potential KDU 1. Indeed, across the tasks, there are only 5 out of 69 instances (7%) where teachers' work shows evidence of potential KDU 2 but not potential KDU 1.

While our work was able to corroborate part of our revised hypothesis described above, the revised hypothesis was based on the construct of map view and motion view. We were not able to deduce from the available written work which type of view a teacher might hold, and as a result further research is needed to investigate this possible role of motion view and map view.

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