Empirical Re-Conceptualization: Bridging from Empirical Patterns to Insight and Understanding

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Identifying patterns is an important part of mathematical investigation, but many students struggle to explain or justify their pattern-based generalizations or conjectures. These findings have led some researchers to argue for a de-emphasis on pattern-based activities, but others argue that empirical investigation can support the discovery of insight into a problem's structure. We introduce a phenomenon we call empirical re-conceptualization, in which learners identify a conjecture based on an empirical pattern, and then re-interpret that conjecture from a structural perspective. We elaborate this construct by drawing on interview data from undergraduate calculus students and research mathematicians, providing a representative example of empirical re-conceptualization from each participant group. Our findings indicate that developing empirical results can foster subsequent insights, which can in turn lead to justification and proof.

Keywords: Generalization, Patterns, Conjecturing, Justification

Introduction and Motivation

Identifying patterns is a fundamental aspect of mathematical activity, with curricular materials and instructional techniques geared towards supporting students' abilities to leverage empirically-based generalizations and conjectures. However, forming pattern-based conjectures is not sufficient; it is also important for students to understand and justify the patterns they develop. A robust body of research reveals a common phenomenon whereby students are able to leverage patterns in order to develop conjectures, but then struggle to understand, explain, or justify their results (e.g., Čadež & Kolar, 2014; Mason, 1996; Pytlak, 2015; Zazkis & Liljedah, 2002). Indeed, these findings have led some researchers to argue for a de-emphasis on pattern-based activity, dismissing it as unsophisticated (e.g., Carraher et al., 2008; Mhlolo, 2016).

In contrast, we have observed a phenomenon we call *empirical re-conceptualization*, in which participants identify a pattern based on empirical evidence alone to form a conjecture, and then re-interpret their conjecture from a structural perspective. In this paper, we address the following research questions: (a) What characterizes students' and mathematicians' abilities to leverage empirical patterns to develop mathematical insights? (b) What are the potential affordances of engaging in empirical patterning activity? We describe and elaborate this construct across two participant populations, mathematicians and undergraduate students. In this manner, we highlight empirical re-conceptualization as a phenomenon that marks productive mathematical activity at multiple levels and populations. Our findings indicate that developing results from empirical patterns, even those that are poorly understood or unjustified in the moment, can serve as a launching point for subsequent insights, including verification, justification, and proof.

Literature Review and Theoretical Perspectives

There is ample evidence that students are adept at identifying and developing mathematical patterns (e.g., Blanton & Kaput, 2002; Pytlak, 2015; Rivera & Becker, 2008). However, the patterns students identify may not always be those that are mathematically useful. As Carraher et al. (2008) noted, a pattern is not a well-defined concept in mathematics, and there is little agreement on what constitutes a pattern, much less its properties and operations. Students who do identify patterns can then experience difficulties in shifting to algebraic thinking (e.g., Čadež & Kolar, 2015, Moss, Beatty, McNab, & Eisenband, 2006; Mason, 1996). Further, both secondary and undergraduate students who work with patterns struggle to justify them (Hargreaves et al., 1998; Zazkis & Liljedah, 2002). An emphasis on empirical patterning without meaning can promote the learning of routine procedures without understanding (Fou-Lai Lin et al., 2004; MacGregor & Stacey, 1995), or the generalization of a relationship divorced from the structure that produced it (Küchemann, 2010). Further, students' challenges with extending pattern generalization to meaningful learning has been shown to contribute to difficulty in multiple domains, including functions (Ellis & Grinstead, 2008; Zaslavsky, 1997), geometric relationships (Vlahović-Štetić, Pavlin-Bernardić, & Rajter, 2010), and combinatorics (Kavousian, 2008; Lockwood & Reed, 2016), among others.

Despite these drawbacks, researchers also point out the affordances of empirical investigation and pattern development. The activity of developing empirically-based conjectures can support the discovery of insight into a problem's underlying structure, which can, in turn, foster proof construction (Tall et al., 2011; de Villiers, 2010). The degree to which pattern generalization is an effective mode of proof development is an unresolved question, but there is evidence that students can engage in a dynamic interplay between empirical patterning and deductive argumentation (e.g., Schoenfeld, 1986). Similarly, research mathematicians regularly engage in experimentation and deduction as complementary activities (de Villiers, 2010). It may be that students become stuck in a focus on empirical relationships divorced from structure because they lack sufficient experience with this way of thinking. Küchemann (2010) found that with practice, students could learn to glean structure from patterns. Similarly, Tall et al. (2011) argued that attention to the similarities and differences in empirical patterns could support the development of mathematical thinking and proof. These works offer a precedent for positioning empirical patterning as a bridge to insight and deduction.

Structural reasoning. Harel and Soto (2017) introduced five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category, pattern generalization, further distinguishes between two types of generalizing: *Result pattern generalization*, and *process pattern generalization* (Harel, 2001). Result pattern generalization (RPG) is a way of thinking in which one attends solely to regularities in the result. The example Harel gives is observing that 2 is an upper bound for the sequence $\sqrt{2}$,$\sqrt{2 + \sqrt{2}}$, $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$, … because the value checks for the first several terms. RPG is typically the type of pattern generalization observed in studies in which students then struggle to shift from recursive to explicit relationships or justify their patterns (e.g., Čadež & Kolar, 2015; Schliemann, Carraher, & Brizuela, 2007). When we refer to the identification of a pattern based on empirical evidence, we are referring to RPG. In contrast, process pattern generalization (PPG) entails attending to regularity in the process, even if that attention may first be initiated by noticing a regularity in the result (Harel, 2001). To extend the above example, Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We define empirical re-conceptualization as the process of re-interpreting a generalization or conjecture from a pattern (identified by RPG) from a structural perspective. By structural perspective, we mean the five major categories of structural reasoning, with the exception of the RPG sub-category of pattern generalization. Thus, a student could reason about the regularity in process from one term to the next of a sequence (PPG). One could also reduce unfamiliar structures into familiar ones, either by constructing new structures or by forming conceptual entities. One may also carry out structural operations in thought without performing calculations, reason in terms of general structures (either by reasoning with conceptual entities or reasoning with operations on conceptual entities), or engage in epistemological justification. In short, re-interpreting a generalization or conjecture from a structural perspective entails the ability to recognize, explore, and reason with general structures.

Figurative and operative thought. The other construct we draw upon to characterize the phenomenon of empirical re-conceptualization addresses a distinction in mental activity (Piaget, 1976, 2001; Steffe, 1991; Thompson, 1985). When engaged in figurative activity, one attends to similarity in perceptual or sensorimotor characteristics. In contrast, operative mental activity entails attending to similarity in structure or function through the coordination and transformation of mental operations. For instance, a student could associate the sine curve with circular motion through conceiving both as representing an invariant relationship of co-varying quantities (an operative association), or through conceiving both as smooth because the motion is perceived as continuous (a figurative association) (Moore et al., 2019). A shift from RPG to PPG is often accompanied by a shift from figurative to operative mental activity, and we consider operative activity to be a hallmark of the ability to reason structurally.

Methods

We drew on interview data from two participant sources, mathematicians and undergraduate students, both stemming from larger projects investigating participants' use of examples to generalize, conjecture, and prove.

Mathematician data. The mathematician data consisted of two sets of hour-long interviews. Thirteen mathematicians participated in Interview 1, and 10 continued for Interview 2. The participants included 7 professors, 3 postdoctoral researchers, and 3 lecturers. Twelve participants hold a Ph.D. in mathematics, and one holds a Ph.D. in computer science. There were 8 men and 5 women. Each interview presented two novel mathematics tasks, which were chosen to be accessible (i.e., they did not require specialized content knowledge) but not trivial (i.e., a solution was not immediately available). For the purposes of this report, we focus on the *Interesting Numbers Task* (Andreescu, Andrica, & Feng, 2007), which we phrased as follows: "Most positive integers can be expressed with the sum of two or more consecutive integers. For example, 24 = 7 + 8 + 9, and 51 = 25 + 26. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore interesting. What are all the interesting numbers?" Below we highlight an exemplar from one mathematician's work on the interesting numbers task, the work of Dr. Fisher.

Undergraduate data. The undergraduate data consisted of a set of hour-long individual interviews with 10 undergraduate male calculus students. The students solved a set of tasks designed to engender generalizing activity (and ultimately to generalize the binomial theorem). For this paper, we report on the *Passwords Task*, which asked students for the number of passwords of length 3, 4, 5, and eventually length n, which consisted of the characters A or B, where repetition was allowed. We asked the students to create tables to organize passwords with a certain number of As, and we also had them reason about the total number of passwords of a given length. Below we focus on one student's work, Raoul, and his reasoning about the total number of length n AB passwords.

Analysis. All interviews were videoed and transcribed, and were also recorded with a Livescribe pen, which yields both an audio record of the interview and a pdf document of the participant's written work. We used gender-preserving pseudonyms for all participants. Using the constant-comparative method (Strauss & Corbin, 1990), we analyzed the interview data in order to identify the participants' generalizations and conjectures and to characterize the mental activity that fostered them. For the first round of analysis we drew on Ellis et al.'s (2017) *Relating-Forming-Extending* Framework for generalizing, which yielded the emergent category of empirical reconceptualization. In subsequent rounds we further examined the participants' talk, gestures, and task responses to further refine the characteristics of empirical reconceptualization.

Results

In order to characterize the participants' abilities to leverage empirical patterns to develop mathematical insights, we present two exemplar cases of empirical re-conceptualization, one from each participant group. In both cases, the participants began with an empirically-based conjecture that they did not understand and could not justify. That initial conjecture, however, then served as a launch point to engage in empirical reconceptualization.

Mathematician Case: Dr. Fisher

Dr. Fisher is a mathematics professor at a large public university. She initially approached the Interesting Numbers Task algebraically by expressing the non-interesting numbers as $m + (m+1) + \cdots + (m+k-1)$. After trying to simplify that expression, she switched to a different approach, in which she listed all of the interesting numbers by ruling out those that were non-interesting. She checked the first numbers 1 to 7 in her head, noting that 3 = 1+2, 5 = 2+3, 6 = 1+2+3, and 7 = 3+4, so only 1, 2, and 4 were interesting. Then she realized she could list all of the non-interesting numbers as sums in an organized table (Figure 1):

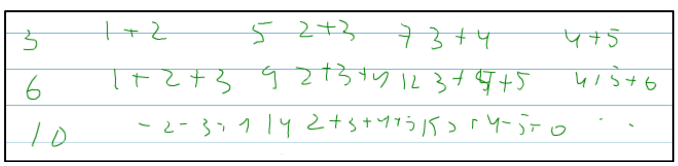
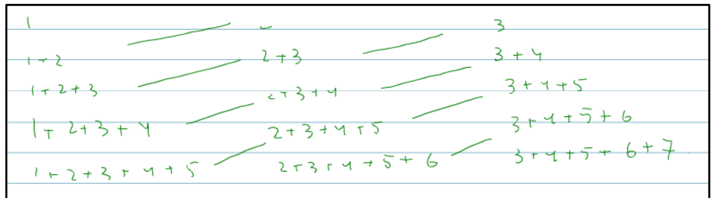


Figure 1. Dr. Fisher's initial table.

Using her table, Dr. Fisher was able to rule out every number up to 24 other than 1, 2, 4, 8, 16, 19, and 23. She recognized the pattern in the first five terms as the powers of 2 and (correctly) conjectured that the interesting numbers were the powers of 2. She made this conjecture even though she had not yet been able to rule out 19 and 23, noting that, "19 and 23 are sort of bothering me from the power of 2 that is showing up in the pattern." Dr. Fisher's conjecture emerged from RPG and figurative activity, in which she noticed a familiar pattern in the (incomplete) data. However, at this time she had no sense of why it should continue.

To determine why the powers of 2 were interesting, Dr. Fisher rewrote her table (Figure 2). She then suddenly noticed a structural pattern in the sums, observing that the sums in the second column could be obtained from the sums in the first column moving diagonally by removing the "+1". Similarly, she saw that the sums in the third column could be obtained from the sums in the second column moving diagonally by removing the "+2". In other words, when moving diagonally up the table, 1+2+3+4 becomes 2+3+4, which becomes 3+4.

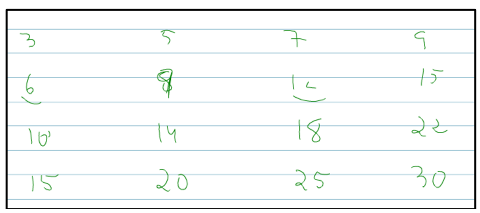


1	2	3
1+2	2+3	3+4
1+2+3	2+3+4	3+4+5
1+2+3+4	2+3+4+5	3+4+5+6
1+2+3+4+5	2+3+4+5+6	3+4+5+6+7

Figure 2. Dr. Fisher's refined table attending to sums.

Dr. Fisher: Oh. I see. I see what's happening. […] So, what I see here is that they differ by 1, right? And then they differ by 2. Right? So, I have these numbers [in the first column] and then automatically get these minus 1. […] And then I automatically get those numbers minus 1 and then I get them minus 2 also. And then I get the minus 3 et cetera. So, this actually should tell me most of them by just subtracting.

This insight led Dr. Fisher to formalize the non-interesting numbers as $\binom{n}{2} - 1 - 2 - \cdots - k$, which she simplified to $\binom{n}{2} - \binom{k}{2}$. However, she still did not understand why the powers of 2 did not have that form. To try to answer that question, she returned again to the table and rewrote a third version, this time listing the values of the sums rather than the sums themselves (Figure 3). She evaluated the first row of sums (1+2, 2+3, 3+4, …), then filled in the rest of the table using the diagonal relationship she observed earlier.



3	5	7	9
6	9	12	15
10	14	18	22
15	20	25	30

Figure 3: Dr. Fisher's third refined table.

During this activity, Dr. Fisher noticed a pattern in the second row, "Now what I'm getting here is that these are multiples of 3. Yes, I got it!" To prove this observation, she represented an arbitrary element of the first row by $2k+1$, then used the diagonal relationship to justify that the next element down the diagonal would be $(2k-1) + (k-2) = 3(k-1)$. She gave a similar proof for the multiples of 5 appearing in the 4th row, then realized that a generalized version of the argument would prove that every multiple of $2k+1$ (above a certain point) will appear in the $2k$ row. Her argument proved that every multiple of an odd number is non-interesting, so every interesting number must have no odd factors (i.e. be a power of 2).

Dr. Fisher initially engaged in RPG, generating a conjecture based solely on the numerical pattern she observed in her list of interesting numbers. Once she had the conjecture, however, she shifted from figurative to operative activity, attending to how the sums changed as the number of summands and starting value changed within her table. Through her analysis of the structural relationships in the data, Dr. Fisher was able to engage in multiple rounds of PPG, forming generalizations that led to a partial proof of her conjecture.

Undergraduate Case: Raoul

Raoul was an undergraduate calculus student at a large public university. In his work on the Passwords Task, Raoul leveraged an empirical pattern to develop a conjecture that for an n-

character password made of As and Bs, there would be 2^n total passwords. In the excerpt below, Raoul explained that he saw the pattern based on his prior work determining 3-character, 4-character, and 5-character passwords. However, he did not understand why the total number of passwords was 2^n:

Raoul: Over here I get, for 3 characters, I get 8 numbers. Four characters, 8 times 2, 16. Five characters, 16 times 2. Oh! 2 power *n*.
Interviewer: Okay. Why did you think that?
Raoul: Well I guess, I just started seeing the pattern. I mean 8 is the 2 cubed. I knew that, and I knew that 5 is…no sorry, 32 is 2 power 5. I knew that too, and here is the same, 2 power 4.
Interviewer: Okay.
Raoul: So, I guess, 2 power *n*.
Interviewer: Cool and can you explain, so, why does that make sense? Does it? Why do you think that's true?
Raoul: Um, it doesn't make sense to me why it has to be 2 power *n*. Two power *n*. I have no idea.

Raoul recognized the familiar numbers 8, 16, and 32 as powers of 2, a recognition based in figurative activity. He then engaged in RPG, conjecturing that an *n*-character password would have length 2^n, but he could not justify it combinatorially. The interviewer then asked Raoul why the number of passwords doubled from 4 to 5 characters:

Interviewer: Why would it make sense, let's say from 4 to 5 it doubles, times 2. Why would there be twice as many possibilities here as there were in the 4 case?
Raoul: That's what I'm trying to think. If I can figure that out, I would be able to find why it makes sense to be 2 power n.
Interviewer: Okay, what would you guess for a 2-character password?
Raoul: Two-character passwords? Four.
Interviewer: Okay, how about for 1-character password?
Raoul: There'd be only 2. A and B, so if it's 2, AB, AA, BA, BB. Hmm. Oh, hold up.

Reflecting on a 1-character and 2-character password, Raoul experienced an insight. He began to relate the doubling phenomenon to the combinatorial context of adding another character: "So, I noticed that this is the pattern that I got with 2 characters, so what I find that, *when I increase it to 3 characters* there will be, another character will be adding up, and that could either be A or B, so the number of passwords would be doubled." Important in Raoul's explanation is his shift of attention to what occurred when he moved from the 2-character case to the 3-character case. This marked a shift to operative activity, in that he was now coordinating the mental operation of imagining what happens when increasing the password length by one character. This also fostered PPG, as Raoul could now attend to a regularity in the process and understand why the number of passwords would double each time the length grew by 1. He was then able to explain his reasoning with the case of moving from a 3-character password to a 4-character password: "One character has to be added up, and that character can either be A or B. So, for this one pattern, for 3 characters, there's going to be 2 [options], there's going to be one more pattern if I make it a 4 character."

Raoul began by making an empirically-based generalization that the total number of passwords would be 2^n, but he could not justify his general statement. However, the development of his initial generalization was still important for Raoul's progress. Before he wrote 2^n, the doubling aspect of the relationship was not foregrounded. Once Raoul had produced a generalization, the interviewer could then ask about doubling, which provided an opportunity for Raoul to shift to operative activity, PPG, and ultimately produce a combinatorial justification for why the number of passwords would double when adding a character.

Discussion and Implications

In this paper we have introduced a new phenomenon, empirical re-conceptualization, in which learners develop an initial conjecture based on empirical evidence or RPG, and then are able to re-conceptualize that generalization or conjecture from a structural perspective. Dr. Fisher began with an empirically-based conjecture, but that conjecture enabled her to then begin an in-depth investigation that supported an attention to how the sums changed as the summands and starting values change. She was able to reduce unfamiliar structures into familiar ones and reason in terms of general structures, both elements of structural reasoning. Raoul was similarly able to leverage his empirically-based generalization, 2^n, by shifting from RPG to PPG, and he did so by carrying out structural operations in thought, imagining what would occur when shifting from a 2-character password to a 3-character password, and again from a 3-character password to a 4-character password. Our findings indicate that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and proof. This confirms de Villiers' (2010) claim that "experimental investigation can also sometimes contribute to the discovery of a hidden clue or underlying structure of a problem, leading eventually to the construction or invention of a proof" (p. 215).

Certainly, students frequently identify patterns that they do not understand or cannot justify; this remains a common problem. A danger remains that students will engage in empirical investigation but then not seek to re-conceive their resulting generalizations or conjectures structurally. Our interest is in understanding why some participants in our study were able to engage in empirical re-conceptualization, while others were not. We note that both Raoul and Dr. Fisher had mechanisms by which they could shift their attention towards structural relationships. At times this ability was spontaneous (in the case of Dr. Fisher) and at other times, it required direction from the interviewer (in the case of Raoul). This suggests that directing students towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. In addition, it suggests that when students engage in empirical patterning activities, it is preferable to have them do so within a particular, concrete context. Ultimately, our findings indicate that the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

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