The Role of Lines and Points in the Construction of Emergent Shape Thinking

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In this study, we focus on a student's meanings for lines and points in the context of graphing covarying quantities. Specifically, we illustrate a student conceiving a line as representing a direction of movement of a dot on a coordinate plane. Consequently, the student did not conceive a dot moving in the coordinate plane as leaving a trace of infinitely many points; similarly, points on a line did not exist until they were physically and visually plotted. We conclude that the student's meanings for lines and points had a significant impact on his graphing activities, in particular, on his construction of emergent shape thinking.

Keywords: Graphical Shape Thinking, Quantitative and Covariational Reasoning

Graphing is critical for understanding various ideas in mathematics (Kaput, 2008; Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Despite its importance, students experience a number of challenges in interpreting and making sense of graphs that may affect their learning of many topics in algebra and calculus (e.g., Clement, 1989; Leinhardt et al., 1990; Moore & Thompson, 2015). In addressing those challenges and difficulties, a number of researchers have suggested that quantitative and covariational reasoning is productive for students' construction of graphing meanings (e.g., Carlson, 1998; Frank, 2017; Johnson, McClintock, & Hornbein, 2017; Saldanha & Thompson, 1998; Thompson, Hatfield, Yoon, Joshua & Byerley, 2017). Specifically, some researchers (e.g., Frank, 2017, 2018; Moore & Thompson, 2015) have emphasized that a productive meaning for a graph is conceiving it as an emergent trace of points that unite two covarying quantities' magnitudes and/or values (i.e., *emergent shape thinking*). In this paper, we contribute to extant theorization of emergent shape thinking by demonstrating how a student's meanings for lines and points influenced his construction of emergent shape thinking. In particular, the student's meaning for a line as indicating a direction of movement of a dot was a contraindication of him conceiving of a graph as an emergent trace of a point, and an implication of his meaning for a line is that it constrained him from thinking of a line as consisting of infinitely many points.

It is worth noting that, in this study, we situated students' learning about quantities' relationships within contexts emphasizing the quantities' *magnitudes* independent of numerical *values*. By a quantity's magnitude, we mean the general sense of the quantitative size of an object's measurable attribute (e.g., length), whereas, by quantity's value, we mean the result of measuring that attribute. Researchers have argued that reasoning with quantities' magnitudes supports students in understanding quantities' covariational relationships (Liang, Stevens, Tasova, & Moore, 2018; Thompson, Carlson, Byerley, & Hatfield, 2014).

Background

Emergent Shape Thinking

Moore and Thompson (2015) introduced the notion of *emergent shape thinking* to describe a person who envisions a graph "*simultaneously* as what is made (a trace) and how it is made (covariation)" (p. 785). This conception involves (1) representing two inter-dependent quantities' magnitudes and/or values varying on each axis of a coordinate system, (2) forming a multiplicative object (Saldanha & Thompson, 1998; Thompson, 2011; Thompson et al., 2017) by uniting those two quantities' magnitudes or values as a single object, and (3) assimilating the process of a multiplicative object moving within the plane in ways invariant with the two covarying quantities as generating a graph, or conceiving a given graph as an emergent record of all instantiated moments of the simultaneous coordination of two covarying quantities. In this study, we report on a student who demonstrated compelling evidence of the first two elements of emergent shape thinking that are listed above; however, demonstrated a contraindication of the third element of emergent shape thinking due to his meanings for points and lines.

Students' understanding of points on a line

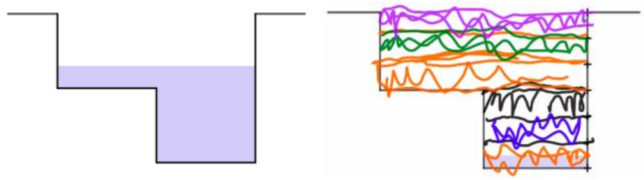
Researchers (e.g., Kerslake, 1981; Mansfield, 1985) have investigated how students conceived points and lines in the context of graphing. They have revealed that students (especially secondary level students) tend to "see" points on a line only if they are visually marked on a graph. For example, Manfield (1985) reported that some secondary and undergraduate students did not perceive points in between two marked points on a line, and some students only perceived endpoints of a straight line on a paper or points on the vertices of a zigzag line. These authors also reported on students who believed no line has points until they are placed on the line. Similarly, Kerslake (1981) reported that about 89% of secondary students (*N*=1798) did not conceive of infinitely many points on a line. When asked how many points are on a straight line that goes through three points, several students answered "three" or the number of places where the line and a coordinate grid intersect. Although there were some students being aware that there were many points (e.g., "lots" or "hundreds" p. 123) on a line, Kerslake reported that their conception was still constrained "by the physical constraints of actually drawing the points" (p. 123). For example, a student said there are points "as many as there is room for" on the line in between two points plotted (p. 123). In this study, we found a student whose meanings for points on a line was similar to that of the students reported by these aforementioned researchers. We contribute to these findings by discussing how such meanings constrained his construction of emergent shape thinking.

Method

We conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with four secondary students. In the teaching experiment, we aimed to investigate the mental actions involved in the students' conceiving situations quantitatively and representing particular quantitative relationships on number lines and coordinate systems. In this paper, we focus on one of the four students, Zane, since his meanings for the lines and points were consistent and clearly described by him throughout the teaching experiment. We believe it is important to document his ways of thinking in order to add nuances to our models of students' thinking in a graphing activity in terms of emergent shape thinking.

Zane participated in 16 one-hour long videotaped teaching experiment sessions over the course of seven weeks. The first author was the teacher-researcher (TR), and the second author served as the witness-researcher (WR). Before conducting the teaching experiment, the TR developed an initial sequence of tasks by considering particular design principles focused on graphing covarying quantities (e.g., Frank, 2017; Moore & Thompson, 2015; Thompson & Carlson, 2017). The TR revised and implemented those tasks based on on-going inferences and analysis of Zane's thinking. Each task was designed with a dynamic geometry software and displayed on a tablet device. We recorded all sessions using two video cameras to capture Zane's work and his gestures and a screen recorder to capture Zane's activities on the tablet device. We transcribed the video and digitized Zane's written work for both on-going and retrospective conceptual analyses (Thompson, 2008). Our analysis relied on the generative and axial methods (Corbin & Strauss, 2008), and it was guided by an attempt to developing working models of Zane's thinking based on his observable and audible behaviors.

In this paper, we report data from Zane's activity in the Swimming Pool Task adapted from Swan (1985). We presented Zane a dynamic diagram of a pool (see Figure 1a), where he could fill or drain the pool by dragging a point on a given slider. We designed the task to support Zane in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool.



(a) (b)

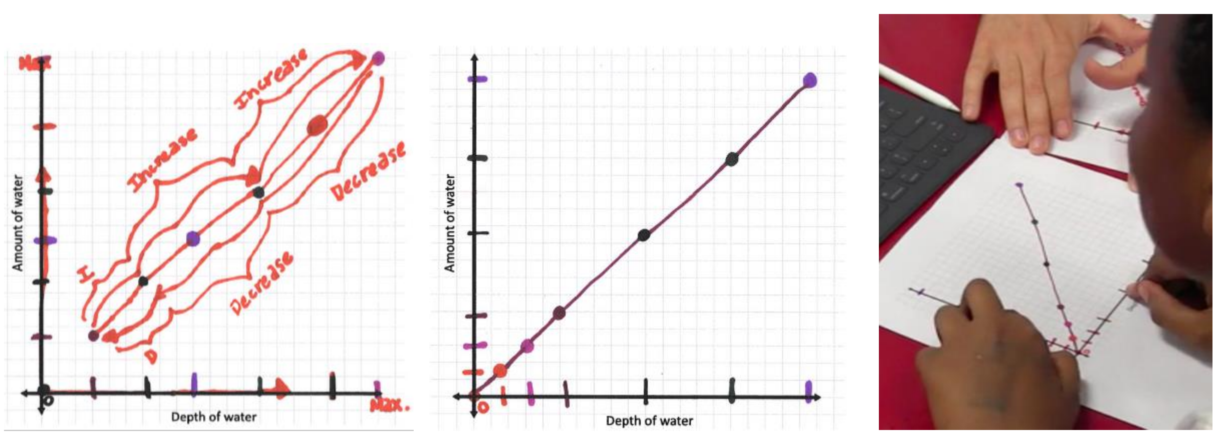
Figure 1.(a) A diagram of the pool (b) illustration of Zane's partitioning activity.

Analysis and Findings

In this section, we illustrate Zane's meanings for his graphs, including the tick marks, points, and lines as he perceived on the graphs. Then, we discuss how these meanings influence his assimilation of what we perceive to be an emergent trace of a point.

Zane's construction and interpretation of his displayed graphs

We asked Zane to sketch a graph that shows the relationship between AoW and DoW as the pool fills up. Zane started with drawing tick marks on each axis and plotting points corresponding to two related tick marks (see his color-coded points and tick marks in Figure 2a and 2b), then he connected those points with line segments. He initially constructed Figure 2a and adjusted his graph to Figure 2b to represent bigger increments at the top half of the pool. He also drew arrows to show "increase" and "decrease" in both quantities (Figure 2a).



(a) (b) (c)

Figure 2. (a) Zane's first draft, (b) Zane's second draft, and (c) Zane moving his fingers on axes.

Meanings for tick marks. When questioned about his tick marks, Zane referred to the quantity's magnitude by drawing a line segment from the origin to the tick mark on the axis. He also used his fingers to simulate the quantities' variation as the TR played the animation to fill the empty pool. He initially placed his left and right index fingers at the origin saying "I started from zero" and then moved his left index finger up along the vertical and his right index finger to the right along the horizontal axis (Figure 2c). While he was moving his fingers, we inferred that he wanted to make sure both fingers hit each corresponding tick marks at the same time so as to match AoW and DoW as he perceived in the animation.

In order to determine if Zane perceived quantities' magnitudes *in between* his tick marks, the TR asked him if he moved his fingers by jumping from one tick mark to another. He responded that he moved his fingers continuously and described an intermediate state:

> Because, I mean, on the thing [*pointing to the pool in Figure 1b*], it is not like very jumping up [*moving up his finger very fast from the bottom of the pool*]. It is really just, like, because the water can be [*pointing to the orange shaded area at the bottom of the pool, Figure 1b*] also a half of it too [*pointing to the water level in Figure 1b*].

In summary, we infer that Zane could simultaneously coordinate both quantities' variations on the Cartesian coordinate system. He conceived of the distance from each finger to the origin as representing the magnitude of AoW or DoW, and he could keep track of the two quantities' variations simultaneously and continuously, including intermediate states between tick marks.

Meanings for points. As the conversation continued, the TR tried to gain insights into the extent to which he coordinated those tick marks on the axes to construct meaning of points on the graph. The TR asked Zane to show the point on his graph that shows the AoW and DoW when the pool is full. Zane first pointed to the far right and top purple tick marks on each axis (see Figure 2b, also see his gesture illustrated in Figure 3a), and then, he pointed to the corresponding purple point on the plane (see Figure 2b). His actions showed that he could associate these two tick marks on each axis to the point on the plane. Then the TR asked him to move his fingers correspondingly on each axis as we played the animation. The following excerpt demonstrates his activity:

TR: I am gonna take out water. You are gonna

Zane: Go down [*moving his right and left index finger to the left and down, along the horizontal and vertical axis respectively. Then, he put his finger back in their original position at the very end tick marks on each axis, see Figure 3a*].

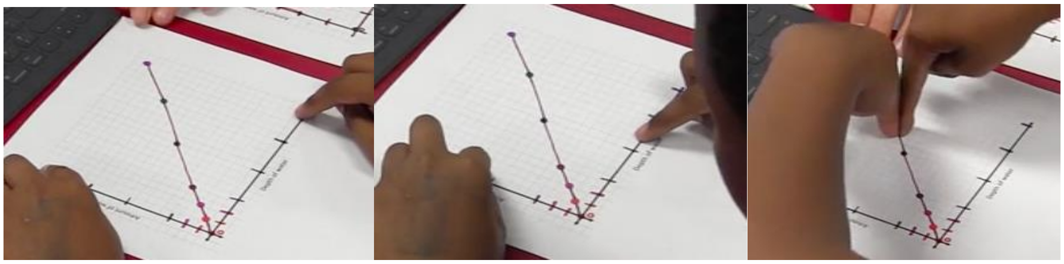
TR: Yes. But, when you do this, you gotta imagine what happens to this point [*pointing to the corresponding point*] ... when I start changing, you are gonna move your fingers and imagine what happens to the corresponding point.

Zane: [*We played the animation and Zane moved his left and right index fingers smoothly on each axis to the left and down, respectively. The TR stopped the animation where the water level is within the area that are shaded in green [see Figure 1b], and he immediately stopped moving his both fingers, see Figure 3b*].

TR: Okay. Where is the corresponding point?

Zane: [He *simultaneously moved his left index finger to the right horizontally and right index finger up vertically, and stopped when the two fingers met; see Figure 3c*] like right here. [*Then he plotted a point on his graph and added two corresponding tick marks on each axis*].

We interpreted that Zane was able to conceive of a point on the plane as a multiplicative object that simultaneously unites the two quantities' variations. Later, he described a point moving up and down along the line as representing *both* quantities' increases and decreases at the same time, saying "the dot represents both." That is, he held in mind two quantities associated with a point and imagined variation of the two quantities as the point moved.



(a) (b) (c)

Figure 3. (a) the location of Zane's fingers before the animation started, (b) the place where Zane stopped, and (c) Zane pointing to the corresponding point where both fingers met.

Meanings for lines. When questioned why he connected the points with lines, Zane responded that the line shows "where the dots go." Additionally, he said, "it also helps to person who comes in, they will understand that the line is probably moving down and up." We infer that Zane conceived of a line on the coordinate plane as showing a path of movement of a dot in either direction. We hypothesized that this meaning for a line might be related to his meaning constructed outside a graphing context prior to the study. Therefore, the TR drew a straight line on a blank paper and asked him how he would define a line. He responded, "a point, hmm, something that goes and never stops." The TR followed up, "What is that goes and never stops?" After an eight-second pause, he said "hmm, from a start point [*placed his right index finger on the left end side of the line*] to an end point [*moves his finger to the right end side of the line*]. We inferred that Zane conceived a line as describing how an object moves from one location to another, and this was compatible with his understanding of lines in the graphical context.

Based on this inference, the WR hypothesized that he might not conceive of a line as consisting of infinitely many points. To test this hypothesis, the WR asked him, "how many points do we need to plot in order to fully describe what is going on here?" Zane added three additional points on each segment between the points that he originally plotted on his graph (see Figure 2b) and said "24". The WR then asked him to compare two graphs, a graph with 24 dots plotted and another graph that includes a line, and discuss how they were similar or different. He replied "wait, did the person who has the line also have 24 points too?" We inferred from his activity that, for him, a line with dots and a line without dots are different graphs and the dots are critical components that convey additional information to a line.

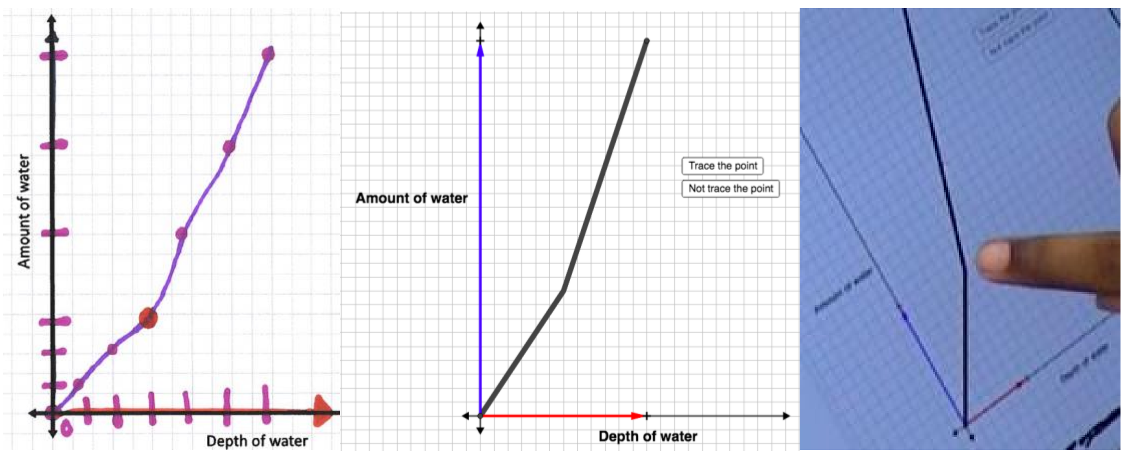
As an additional evidence, when asked whether his graph (see Figure 4a) shows every single moment of how the two quantities vary in the situation, Zane said no because in order to show it, you need to plot an additional point. We infer that, for Zane, lines do not have points until they are visually plotted. He needed to physically plot additional points to represent moments in between two available points, even if there is a line connecting them. That is, he did not conceive of his line as showing these extra moments. In the next section, we demonstrate that such meaning for a line played a role in his construction and interpretation of what we perceived to be in-progress trace.

Zane's Interpretations of an Emergent Trace

Given these interpretations of Zane's meanings for tick marks, lines, and points, the TR hypothesized that he likely did not interpret his prior finger activity (Figure 2c and Figure 3) as generating infinitely many coordinate points. To test this hypothesis, the TR showed him an animation on an tablet device (see https://youtu.be/97EOENUM_co) and asked: "is this trace [*Figure 4b*] showing us the relationship between depth of water and amount of water for this pool?" He replied "no" and struggled to make sense of what the animation was showing, which suggested that he did not perceive the animation as a simulation of his prior graphing activity on paper.

The WR asked Zane whether those dots[1] on his paper (see Figure 4a) are "part of the line on the computer." Zane replied, "there is only one dot," pointing to the animating dot that produced the trace (see his gesture in Figure 4c). When asked "is there any other dots on this graph?" he shook his head. Moreover, he interpreted his graph (Figure 4a) as having more dots than the one produced in the animation (Figure 4b), commenting that mine is better because "mine have more dots".

Zane also claimed that he could not construct his graph in the same way as the animation did due to physical constraints of human, saying, "well, I cannot do that, because, like, can you do dots and dots [*tapping his right index finger very fast along his graph shown in Figure 4a*] and trace it?" This is an additional contraindication that he conceived of graphing a line as a way to represent infinitely many points.



(a) (b) (c)

Figure 4. (a) Zane's last draft and (b) an instance of the animation, and (c) Zane pointing to the "only" point on the trace.

In summary, despite his success in the finger activity and being able to conceive of a point as a multiplicative object, Zane assimilated his activity as well as the animation as one dot moving along a line path instead of one dot generating infinitely many points by leaving a trace. We claimed that his meaning for a line as describing a direction of movement (as opposed to consisting of infinitely many points) played a critical role in his construction and constrained him from conceiving a graph as an emergent, in-progress trace (i.e., the third component of emergent shape thinking).

Discussion

Moore and Thompson (2015) and Frank (2017, 2018) emphasized the important role of forming a multiplicative object in students' development of emergent shape thinking. What we have found in this study is that multiplicative object is a necessary but not sufficient condition

1 We use the word *dot* instead of *point* when we have evidence of Zane referring to a visual circular object that is plotted on the plane, but do not have explicit evidence of him holding the two quantities *in mind* (i.e., conceiving a multiplicative objective) at the moment. By using the word "dot", we are also genuine to Zane's language in this activity.

for emergent shape thinking. We found that Zane constructed a meaning for individual points as multiplicative objects, and he could do the finger tool activity (i.e., moving index fingers on each axis) smoothly to represent continuous, simultaneous co-variation of two quantities. However, he interpreted his activity of connecting points on his graph as representing a direction of movement of one point from one location to another; hence he did not imagine a segment or a line as including infinitely many points and representing every moment of the quantities' covariation. In turn, he did not perceive an emergent trace generated by a continuously moving point as a line. Thus, we claim that Zane's meanings for lines and points constrained him from developing a meaning of graphs consistent with emergent shape thinking (i.e., a graph as an emergent trace of points). Given the results provided by this study and other researchers (e.g., Manfield, 1985; Kerslake, 1981) regarding students' understanding of lines and points, it is important that we take into account students' meanings for lines and points constructed in other contexts and consider how those meanings may influence students' construction of graphing meanings.

Although we report on findings from a secondary student, the implication of this study is important for both the secondary and the undergraduate mathematics education community. We are drawing on the construct, emergent shape thinking, that was initially developed from undergraduate students. The results of this study regarding a secondary student's meanings afford us a better understanding of students' graphing activity, including their construction of emergent shape thinking. Prior researchers (e.g., Frank, 2017) have reported on students' difficulty with constructing an emergent meaning for graphs, and the case of Zane suggests that one source of such difficulty can be students' meanings for lines and points in general. If students conceived of a line as showing *one* point's movement, it would be difficult for them to assimilate a line graph as an emergent trace that includes *infinitely many* points, despite being able to conceiving each point as a representation of two quantities' magnitudes (and/or values) at a particular moment. Consequently, they would need to physically plot points to show quantities' magnitudes (and/or values) at any specified moments (i.e., a pointwise meaning for a graph). Thus, we conjecture that meanings for lines and points consistent with Zane's might help explain students' difficulties with constructing and representing smooth and continuous images of covariation in graphical contexts (Castillo-Garsow, Johnson, & Moore, 2013).

Our analysis supports the affordance of Castillo-Garsow's (2012) distinction between "a problem situation, the method used to solve it, and the reasoning that derives or selects that method" (p. 56) when characterizing students' covariational reasoning. Zane conceptualized two quantities' covariation continuously regarding the pool situation; however, he used a discrete method (i.e., plotting points) when constructing his graph due to a constrain implied by his meanings of lines. We argue that it is important for researchers to be aware of such a discrepancy and be aware that a pointwise graph activity (i.e., plotting the points first, and then connecting them) does not necessarily imply that their images of quantities' covariation is discrete, and vice versa. When characterizing students' covariational reasoning, it is important for us to take into account students' activities and reasoning in various context (e.g., situations, graphs in different coordinate systems, and number lines) before we make claims about their covariational reasoning. We believe identifying this inconsistency and its causes can allow us to better advance students' understandings of graphs.

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