A Quantitative Reasoning Framing of Concept Construction

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Researchers are producing a growing number of studies that illustrate the importance of quantitative and covariational reasoning for students' mathematical development. These researchers' contributions often are in the context of learning of specific topics or developing particular reasoning processes. In both contexts, researchers are detailed in their descriptions of the intended topics or reasoning processes. There is, however, a lack of specificity relative to generalized criteria for the construction of a concept. We address this lack of specificity by introducing the construct of an abstracted quantitative structure. We discuss the construct, ideas informing its development and criteria, and empirical examples of student actions that illustrate its use. We also discuss potential implications for research and teaching.

Keywords: Quantitative reasoning, Covariational reasoning, Abstraction, Concept.

Steffe and Thompson enacted and sustained research programs that have characterized students' (and teachers') mathematical development in terms of their conceiving and reasoning about measurable or countable attributes (see Steffe & Olive, 2010; Thompson & Carlson, 2017). Thompson (1990, 2011) formalized such reasoning into a system of mental operations he termed *quantitative reasoning.* Over the past few decades, other researchers have adopted quantitative reasoning to investigate students' and teachers' meanings in various ways. Some researchers have adopted quantitative reasoning to characterize individuals' meanings within specific topical or representational areas including exponential relationships (Castillo-Garsow, 2010; Ellis, Özgür, Kulow, Williams, & Amidon, 2015), graphs or coordinate systems (Frank, 2017; Lee, 2017; Lee, Moore, & Tasova, 2019), and function (Oehrtman, Carlson, & Thompson, 2008; Paoletti & Moore, 2018). Other researchers have adopted quantitative reasoning to characterize types of individuals' reasoning. A predominant example is reasoning about quantities changing in tandem, or *covarying* (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012, 2015b; Saldanha & Thompson, 1998; Stalvey & Vidakovic, 2015).

We introduce the construct *abstracted quantitative structure* that marries and extends these two research themes by offering framing criteria for concept construction. Defined generally, an abstracted quantitative structure is a system of quantitative operations a person has interiorized to the extent they can operate *as if* it is independent of specific figurative material. That person can thus re-present this structure in several ways including to accommodate to novel experiences permitting the associated quantitative operations. As we illustrate in this paper, an abstracted quantitative structure is a type of quantitative reasoning that has implications for an individual's meanings within specific topical or representational areas, and her or his engagement in other related forms of reasoning. In what follows, we first discuss background information on quantitative (and covariational) reasoning that underpins the abstracted quantitative structure construct. We then provide a more formal definition for an abstracted quantitative structure and data to illustrate both indications and contraindications of individuals having constructed such a structure. We close with its implications for research and teaching.

Background

Quantitative Reasoning

Thompson (2011) defined quantitative reasoning as the mental operations involved in conceiving a context as entailing measurable attributes (i.e., quantities) and relationships between those attributes (i.e., quantitative relationships). A premise of quantitative reasoning is that quantities and their relationships are idiosyncratic constructions that occur and develop over time (e.g., hours, weeks, or even years). A researcher or a teacher cannot take quantities or their relationships as a given when working with students or teachers (Izsák, 2003; Moore, 2013; Thompson, 2011). Furthermore, and reflecting one implication of the present work, a researcher or teacher should not assume a student has constructed a system of quantities and their relationships based on actions within only one context (e.g., situation, graph, or formula).

Thompson (Smith III & Thompson, 2008; Thompson, 1990) distinguished between *quantitative operations/magnitudes* and *arithmetic operations/measures* to differentiate between the mental actions involved in constructing a quantity via a quantitative relationship and the actions used to determine a quantity's numerical measure. Following Thompson (1990), we illustrate these distinctions using a comparison between two heights. Thompson (1990) described that an additive comparison requires one to construct an image of the measurable attribute that indicates by how much one height exceeds the other height (*Figure 1*). Constructing such a quantity through the *quantitative operation* of comparing two other quantities additively does not depend on having numerical measures, nor does it require executing a calculation; an important aspect of Thompson's quantitative reasoning is that it foregrounds constructing and operating on magnitudes (i.e., amount-ness) of quantities in the context of figurative material (e.g., coordinate systems and phenomena) that permit those operations. *Arithmetic operations*, on the other hand, are those operations between numerical measures such as addition, subtraction, multiplication, etc. that one uses to determine a quantity's measure, and are often in the context of inscriptions or glyphs that signify quantities but do not provide the perceptual material to operate on quantitatively (Moore, Stevens, Paoletti, Hobson, & Liang, online).

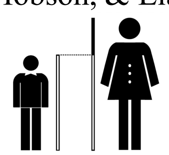

Figure 1. An image of an additive comparison based in magnitudes

Covariational Reasoning

A form of quantitative reasoning involves constructing relationships between two quantities that vary in tandem, or covariational reasoning (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Researchers have conveyed that covariational reasoning is critical for key concepts of K–16 mathematics including function (Carlson, 1998; Oehrtman et al., 2008), modeling dynamic situations (Carlson et al., 2002; Johnson, 2012, 2015b; Paoletti & Moore, 2017), and calculus (Johnson, 2015a; Thompson, 1994; Thompson & Silverman, 2007). Researchers have also illustrated that covariational reasoning is critical to constructing function classes (Ellis, 2007; Lobato & Siebert, 2002; Moore, 2014).

Carlson et al. (2002), Confrey and Smith (1995), Ellis (2011), Johnson (2015a, 2015b), and Thompson and Carlson (2017) are researchers who have detailed covariation frameworks and mental actions. Due to space constraints and the empirical examples we use below, we narrow our focus to a mental action (or operation) identified by Carlson et al. (2002). A critical mental action, especially for differentiating between various function classes, is to compare amounts of change (Figure 2, MA3). MA3 is also important for understanding and justifying that a graph and its curvature appropriately model covarying quantities of a situation (Figure 3) (Stevens & Moore, 2016). Furthermore, and as we illustrate in more detail below, such reasoning enables understanding invariance among different representations of quantities' covariation (Moore, Paoletti, & Musgrave, 2013), which is the foundation for an abstracted quantitative structure.

Mental Action	Descriptions of Mental Actions
MA1	Coordinating the value of one variable with changes in the other
MA2	Coordinating direction of change of one variable with changes in the other variable
MA3	Coordinating amount of change of one variable with changes in the other variable
MA4	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable
MA5	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

Figure 2. Carlson et al. (2002, p. 357) covariational reasoning mental actions.

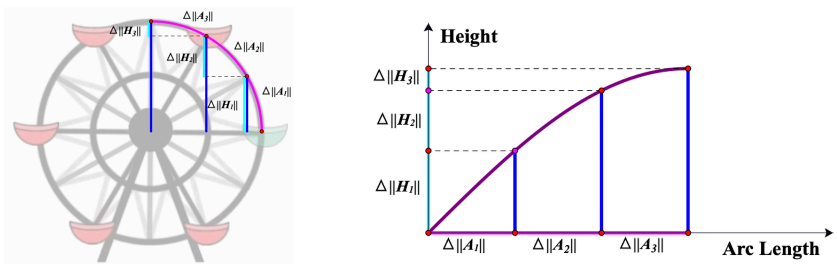

Figure 3. For equal increases in arc length from the 3 o'clock position, height increases by decreasing amounts.

Figurative and Operative Thought

Piagetian notions of *figurative* and *operative* thought (Piaget, 2001; Steffe, 1991; Thompson, 1985) also inform our characterization of an abstracted quantitative structure. These two constructs enable differentiating between thought based in and constrained to figurative material (e.g., perceptual objects and sensorimotor actions)—termed *figurative thought*—and thought in which figurative material is subordinate to logico-mathematical operations, their re-presentation, and possibly their transformations—termed *operative thought*. Quantitative and covariational reasoning are examples of operative thought due to their basis in logico-mathematical operations (Steffe & Olive, 2010). To illustrate the figurative and operative distinction, Steffe (1991) characterized a child's counting scheme as figurative if his counting required re-presenting particular sensorimotor actions and operative if it entailed unitized records of counting that did not require the child to re-present particular perceptual material or sensorimotor experience. As another example, Moore et al. (online) illustrated figurative graphing meanings in which prospective secondary teachers' graphing actions were constrained to particular perceptual features (e.g., drawing a graph solely left-to-right) even when they perceived those features as constraining their graphing of a conceived relationship. In contrast, Moore et al. (online) described that a prospective secondary teacher's graphing meaning is operative in the event that the perceptual and sensorimotor features of their graphing actions are persistently dominated by the mental operations associated with re-presenting quantitative and covariational operations across various attempts to construct graphical re-presentations.

Abstracted Quantitative Structure

Our notion of an abstracted quantitative structure draws on the aforementioned constructs to apply and extend von Glasersfeld's (1982) definition of concept to the area of quantitative and covariational reasoning. von Glaserfeld defined a *concept* as, "any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual 'input'" (p. 194). In the introduction, we defined an abstracted quantitative structure as a system of quantitative (including covariational) operations a person has interiorized to the extent he or she can operate *as if* it is independent of specific figurative material. Using von Glaserfeld's framing, an abstracted quantitative structure is a system of quantitative operations that an individual has interiorized so that it:

1. is recurrently usable beyond its initial experiential construction;
2. can be re-presented in the absence of available perceptual material including that in which it was initially constructed;
3. can be transformed to accommodate to novel contexts permitting the associated quantitative operations, see generalizing assimilation (Steffe & Thompson, 2000);
4. is anticipated as re-presentable in any figurative material that permits the associated quantitative operations.

Clarifying 2., an individual having constructed an abstracted quantitative structure can re-present it in thought and through the regeneration of previous experiences. Clarifying 3., a feature of an abstracted quantitative structure is that it can accommodate novel contexts through additional processes of experiential construction within the context of figurative material in which such construction has not previously occurred. This action is a hallmark of operative thought because it entails an individual transforming and using operations of their quantitative structure to accommodate to novel quantities and associated figurative material, as opposed to having fragments of figurative activity dominate their thought (Thompson, 1985). This action is also a hallmark of quantitative reasoning because it enables conceiving mathematical equivalence in a context differing figuratively from that in which a quantitative structure has been previously constructed (Moore et al., 2013). Clarifying 4., an abstracted quantitative structure's mathematical properties (e.g., quantities' covariation) are anticipated independent of any particular instantiation of them, thus understood as not tied to any particular quantities and associated figurative material. It is in this way that the quantitative operations of an abstracted quantitative structure are abstract; the individual not only understands that the operations are re-presentable in previous experiences, but she also anticipates that the operations *could be* relevant to novel but not yet had experiences (e.g., some coordinate system not yet experienced).

We next use empirical examples to illustrate the extent students have constructed an abstracted quantitative structure. Each example is drawn from a study that either used clinical interview (Ginsburg, 1997) or teaching experiment (Steffe & Thompson, 2000) methodologies to build second-order models of student thinking (Ulrich, Tillema, Hackenberg, & Norton, 2014). It was in our reflecting on these second-order models (and those developed during other studies) that we identified themes in their reasoning, one of which is the notion of an abstracted quantitative structure. We acknowledge the way we have defined abstracted quantitative structure presents an inherent problem in attempts to characterize a student as having or having not constructed such. First, it is impossible to investigate a student's reasoning in every context in which an abstracted quantitative structure could be relevant. Second, to characterize a students' quantitative reasoning necessarily involves focusing on their enactment of operations in the context of particular figurative material. No conceptual structure is truly representation free, as “operations have to operate on something and that something is the figurative material contained in the operations, figurative material that has its origin in the construction of the operations” (L. P. Steffe, personal communication, July 24, 2019). For this reason, we find it necessary to use the criteria above to discuss a student’s actions in terms of *indications* and *contraindications* of her or him having constructed an abstracted quantitative structure.

A Contraindication of Re-Presentation

Critical criteria of an abstracted quantitative structure are the ability to re-present that structure in the absence of available perceptual material and the ability to transform its operations to accommodate to novel contexts. As a contraindication of these criteria, consider Lydia’s actions during a teaching experiment focused on trigonometric relationships and re-presentation (Liang & Moore, 2018). Prior to the actions presented here, Lydia had constructed incremental changes compatible with those presented in Figure 3 (left). We took her actions to indicate her reasoning quantitatively and subsequently presented her the *Which One?* task. The task (Figure 4, left) presented numerous red segments that varied in tandem as the user varied a horizontal (blue) segment, which represented the rider’s arc length traveled along the circle. We asked her to choose the red segment that covaried with the blue segment in a way compatible with the vertical height and arc length of the rider. We conjectured this would help determine the extent she could re-present her previous actions in a similar context with less perceptual material available (i.e., the circle) and novel material (i.e., the red and blue segments).

Lydia became perturbed as to whether or not the horizontal red segment should vary at a changing rate with respect to the horizontal blue segment. After much effort, she abandoned considering the segments in the horizontal orientation and re-oriented them vertically. She chose the correct segment by checking whether the heights matched pointwise within the displayed circle (Figure 4, middle). Both her questioning how the red segments should vary with respect to the blue segment and her requiring re-orienting the red segments were a contraindication of her having constructed an abstracted quantitative structure. We thus returned her to the question of whether the chosen red segment and blue segment entailed the same quantitative relationship as she identified in her previous activity (see Figure 3, left; from Liang and Moore (2018)):

> *Lydia*: Not really…Um, I don’t know. [*laughs*] Because that was just like something that I had seen for the first time, so I don’t know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.
>
> *TR.*: So is what we’re looking at right now different than what we were looking at with the Ferris wheel?
>
> *Lydia*: No. It’s – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don’t see it here, and so –
>
> *TR.*: And by you “don’t see it here,” you mean you don’t see it in that red segment?
>
> *Lydia*: Yes.

As the interaction continued, Lydia expressed uncertainty as to how to determine if the blue segment and her chosen red segment entailed the same relationship as she had illustrated in her previous activity, although she knew the segments were correct pointwise. As a further contraindication of her having constructed an abstracted quantitative structure, it was only after much subsequent teacher-researcher guiding and their introducing perceptual material using their pens (Figure 4, right) that she was able to conceive the red and blue segments’ covariation as compatible with the relationship she had constructed in the Ferris wheel situation.

Liang and Moore (2018) illustrated Lydia's repeated engagement in quantitative and covariational reasoning eventually led to her re-presenting quantitative operations including transforming those operations in novel contexts. This enabled her to conceive mathematical equivalence across numerous contexts including situations, oriented segments, and Cartesian graphs. As indication of having constructed abstracted quantitative structures, Lydia re-presented particular quantitative operations in contexts without given perceptual material (Lee et al., 2019).

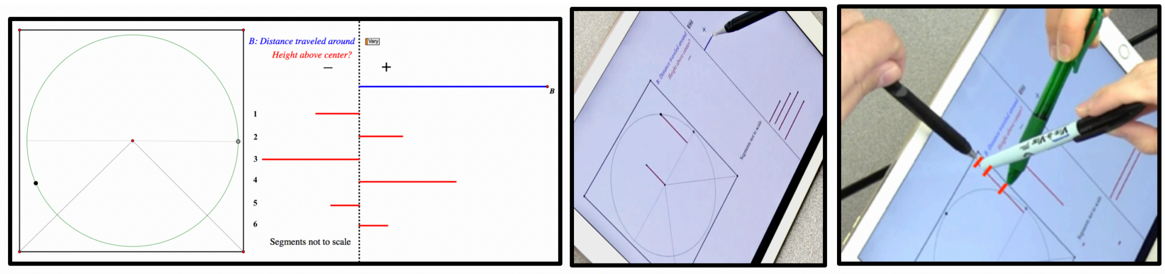

Figure 4. (left) Which One?, (middle) Lydia checking segment pointwise, and (right) Lydia attempting to re-present a quantitative structure with assistance.

An Indication of Re-Presentation and Accommodation

We turn to two prospective secondary teachers'—Kate and Jack—actions when asked to determine a formula for an unnamed polar coordinate system graph (Figure 5, which is $r = \sin(\theta)$; see Moore et al. (2013) for the detailed study). After investigating a few points, Kate and Jack conjectured that $r = \sin(\theta)$ is the appropriate formula and drew a Cartesian sine graph to explore their conjecture (from Moore et al., 2013, p. 468). Important to note, Kate and Jack were not familiar with graphing the sine relationship in the polar coordinate system.

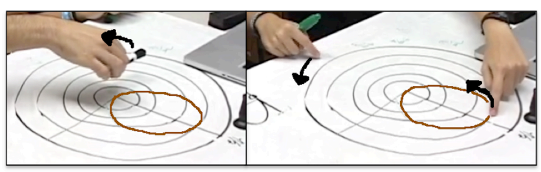

Figure 5. Kate and Jack covary quantities with respect to the given graph (from Moore et al., 2013, p. 467).

Kate: This gets us from zero to right here is zero again [*tracing along Cartesian horizontal axis from 0 to π*]. So, we start here [*pointing to the pole in the polar coordinate system*].

Jack: Ya, and you're sweeping around because [*making circular motion with pen*], theta's increasing, distance from the origin increases and then decreases again [*Jack traces along Cartesian graph from 0 to π as Kate traces along corresponding part of the polar graph*].

TR.: OK, so you're saying as theta increases the distance from the origin does what?

Jack: It increases until pi over 2 [*Kate traces along polar graph*] and then it starts decreasing [*Kate traces along polar graph as Jack traces along Cartesian graph*].

TR.: And then what happens from like pi to two pi.

Kate: It's the same.

Jack: Um, same idea except your, the radius is going to be negative, so it gets more in the negative direction of the angle we're sweeping out [*using marker to sweep out a ray from π to 3π/2 radians – see Figure 5*] until three pi over two where it's negative one away and then it gets closer to zero [*continuing to rotate marker*].

TR.: OK, so from three pi over two to two pi, can you show me where on this graph [*pointing to polar graph*] we would start from and end at?

Kate: This is the biggest in magnitude, so it's the furthest away [*placing a finger on a ray defining 3π/2 and a finger at (1, π/2)*], and then [the distance from the pole] gets smaller in magnitude [*simultaneously tracing one index finger along an arc from 3π/2 to 2π and the other index finger along the graph – see Figure 5*].

Kate and Jack's actions indicate their having constructed (or constructing) an abstracted quantitative structure associated with the sine relationship. They transformed and re-presented the quantitative and covariational operations they associated with a Cartesian graph to accommodate to a polar coordinate system displayed graph. This re-presentation enabled them to conceive two graphs as representing equivalent quantitative structures despite their perceptual differences, which is a contraindication of their reasoning being dominated by figurative aspects of thought. We note that Kate and Jack did not provide evidence related to criteria 4. Such evidence would involve their identifying the potential of not yet experienced coordinate systems and associated graphs that enable re-presenting the same quantitative structure.

Discussion and Implications

We envision the construct of an abstract quantitative structure as useful in several ways. First, it provides criteria to research and distinguish between students' meanings in terms of their foregrounding figurative material and activity and their foregrounding logico-mathematical operations such as a quantitative structure. In our description of Lydia's activity, we underscore that she did not encounter much difficulty assimilating the figurative material; she was able to assimilate the segments and their variation to quantitative operations. Rather, Lydia struggled to accommodate the relationship she constructed with previous figurative material in a way that she could re-present it with novel figurative material. Seeing how difficult it was for Lydia to re-present a relationship within a circular context further demonstrates how powerful Kate and Jack's reasoning was because not only did they re-present a quantitative structure in a novel context, but they abstracted the associated operations such that they could identify the same relationship within a perceptually different representational system. We, therefore, call for researchers and educators to attend not only to students' meanings for various representations (e.g., Cartesian coordinate system, polar coordinate system, formulas, tables, etc.), but also to the quantitative structures students construct and the extent they can re-present (and potentially transform) those structures. In doing so, we can obtain more detailed insights to the extent students construct mental operations in which figurative material is a consequence of those operations including how those operations enable accommodating to novel contexts.

Second, we hypothesize that students' abstracted quantitative structures play an important role in their productive generalization (Ellis, 2007) and transfer (Lobato & Siebert, 2002). Researchers have recently characterized the role of different forms of abstraction in generalization (Ellis, Tillema, Lockwood, & Moore, 2017). Researchers have also recently characterized different forms of transfer including how a student's novel activity can result in cognitive reorganizations regarding their previous activity (Hohensee, 2014; Lobato, Rhodehamel, & Hohensee, 2012). We envision students' construction of abstracted quantitative structures to be a province of each, and we argue that future research should explore these potential relationships as it relates to students' mathematical development.

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