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On BBW parabolics for simple classical Lie superalgebras

Dimitar Grantcharov [a,1], Nikolay Grantcharov [b,2], Daniel K. Nakano [c,*,3], Jerry Wu [d]

[a] *University of Texas at Arlington, Department of Mathematics, Arlington, TX 76019, United States of America*
[b] *Department of Mathematics, University of Chicago, Chicago, IL 60637, United States of America*
[c] *Department of Mathematics, University of Georgia, Athens, GA 30602, United States of America*
[d] *Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840, United States of America*

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ABSTRACT

In this paper the authors introduce a class of parabolic subalgebras for classical simple Lie superalgebras associated to the detecting subalgebras introduced by Boe, Kujawa and Nakano. These parabolic subalgebras are shown to have good cohomological properties governed by the Bott-Borel-Weil theorem involving the zero component of the Lie superalgebra in conjunction with the odd roots. These results are later used to verify an open conjecture given by Boe, Kujawa and Nakano pertaining to the equality of various support varieties.

* Corresponding author.
E-mail addresses: grandim@uta.edu (D. Grantcharov), nikolayg@uchicago.edu (N. Grantcharov), nakano@uga.edu (D.K. Nakano), jerryxzwu@gmail.com (J. Wu).
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Support varieties
Sheaf cohomology

1. Introduction

1.1. Let $\mathfrak{g}$ be a classical simple Lie superalgebra over $\mathbb{C}$ and G be the corresponding supergroup (scheme) with Lie $G = \mathfrak{g}$. Given a parabolic subgroup scheme P, a major open question has been to compute the higher sheaf cohomology group $R^j\mathrm{ind}_P^G N$ for $j \geq 0$ where N is a finite-dimensional P-module. General theory on this topic can be found in [29], and some computations for Lie superalgebras such as $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, and $\mathfrak{q}(n)$ are presented in [29,12,13,22,23,26,27]. For reductive algebraic groups, when P is a Borel subgroup and N is a one-dimensional module, the answer is given by the classical Bott-Borel-Weil (BBW) theorem.

In this paper we introduce parabolic subsupergroups $P = B$ such that the higher sheaf cohomology $R^j\mathrm{ind}_B^G(-)$ can be computed using data from the BBW theorem. These subgroups are obtained by using the detecting subalgebras via the stable action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$. One striking feature about these subalgebras is the interplay between the even roots and the odd roots with their associated finite reflection groups, and the fact that our approach allows for a uniform treatment of all classical simple Lie superalgebras. In particular as a byproduct of our work, we obtain an important computation of the higher sheaf cohomology groups of G/B for the trivial line bundle: $\mathcal{H}^j(G/B, \mathcal{L}(0)) := R^j\mathrm{ind}_B^G\mathbb{C}$ for $j \geq 0$ (cf. Theorem 4.10.1). For classical simple Lie superalgebras other than $\mathfrak{p}(n)$, it is shown that the polynomial $p_{G,B}(t) = \sum_{i=0}^{\infty} \dim R^i\mathrm{ind}_B^G\mathbb{C}\ t^i$ is equal to a Poincaré polynomial for a finite reflection group $W_{\bar{1}}$ specialized at a power of t. This indicates that the combinatorics of the length function on $W_{\bar{1}}$ plays in important role in this setting, and opens the possibilities for developing a general theory involving these parabolic subsupergroups.

1.2. For finite groups it is well-known that the cohomology is detected on the collection of elementary abelian p-subgroups. Moreover, Quillen [24,25] showed that these subgroups can be used to describe the spectrum of the cohomology ring. Later Avrunin and Scott [1] demonstrated that the support varieties for finite groups consist of taking unions of support varieties for elementary abelian subgroup whose varieties can be described using rank varieties.

In the study of classical simple Lie superalgebras, Boe, Kujawa and Nakano [4] used invariant theory for reductive groups to show that there are natural classes of "subalgebras" that detect the cohomology (see also [2]). These subalgebras come in one of two families: $\mathfrak{f}$ (when $\mathfrak{g}$ is stable) and $\mathfrak{e}$ (when $\mathfrak{g}$ is polar). In all cases, $\mathfrak{g}$ admits a stable action and in most cases $\mathfrak{g}$ admits a polar action (cf. [4, Table 5]).

In this situation, the restriction maps induce isomorphisms[4]:

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C})^N \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}, \mathbb{C})^{W_\mathfrak{e}}$$

where N is a reductive group and $W_\mathfrak{e}$ is a finite pseudoreflection group. These relative cohomology rings may be identified with the invariant ring $S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$, where $S^\bullet$ denotes the symmetric algebra, and so are finitely generated. This property was used to construct support varieties for modules in the category $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ (i.e., finite-dimensional $\mathfrak{g}$-modules that are completely reducible over $\mathfrak{g}_{\bar{0}}$).

The main application of the existence and properties of the BBW type parabolic subalgebras is our verification of the following theorem.

Theorem 1.2.1. *Let $\mathfrak{g}$ be a simple classical Lie superalgebra and let M be in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$.*

(a) *If $\mathfrak{g}$ is stable then the map on support varieties*

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/N \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$$

is an isomorphism.

(b) *If $\mathfrak{g}$ is stable and polar then the maps on support varieties*

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W_\mathfrak{e} \to \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/N \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$$

are isomorphisms, where $W_\mathfrak{e}$ is a pseudoreflection group.

The aforementioned theorem has been a conjecture that was first introduced in [4]. In that paper, the equality of the varieties in Theorem 1.2.1 was shown to hold on the complement of the discriminant locus (i.e., an open dense set). This provided strong evidence for the validity of the conjecture. Later, Lehrer, Nakano and Zhang [20] proved the conjecture for the general linear Lie superalgebra and more generally type I classical simple Lie superalgebra via a cohomological embedding theorem.

Kac and Wakimoto defined a combinatorial invariant called the atypicality of a weight λ when $\mathfrak{g}$ is a basic classical simple Lie superalgebra. The support varieties in Theorem 1.2.1 play a prominent role in the theory because they provide a geometric interpretation of this combinatorial invariant. It is conjectured that for the basic simple Lie superalgebras, the dimension of the support variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L(\lambda))$ equals the atypicality of the finite-dimensional irreducible representation $L(\lambda)$. This has been verified in a number of cases including $\mathfrak{gl}(m|n)$ [5] and $\mathfrak{osp}(m|2n)$ [18].

[4] There are some errors in the statements in [4] and [20]. In these papers "$\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C})^{N/N_{\bar{0}}}$" should be replaced with "$\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C})^N$" and "$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/(N/N_{\bar{0}})$" should be replaced with "$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/N$".

1.3. For the detecting subalgebra $\mathfrak{e}$ one has a realization of the support variety $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$ as a rank variety:

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \cong \mathcal{V}^{\text{rank}}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) := \{x \in \mathfrak{e}_{\bar{1}} : \ M|_{U(\langle x\rangle)} \text{ is not projective}\} \cup \{0\}.$$

The establishment of Theorem 1.2.1 along with this rank variety description (i) provides a concrete realization of $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ and (ii) shows that the assignment $(-) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(-)$ satisfies the properties as stated in [3] for a support datum. These important properties are stated in the following corollary.

Corollary 1.3.1. *Let $\mathfrak{g}$ be a simple classical Lie superalgebra which is both stable and polar, and let M_1, M_2 and M be in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$.*

(a) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) \cong \mathcal{V}^{\text{rank}}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W_{\mathfrak{e}}$;
(b) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1 \otimes M_2) = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_2)$.
(c) *Let X be a conical subvariety of $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$. Then there exists L in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ with $X = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L)$.*
(d) *If M is indecomposable then* $\mathrm{Proj}(\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M))$ *is connected.*

Note that the verification of the corollary above follows by the same line of reasoning as given in [20, Theorem 5.2.1].

1.4. The paper is organized as follows. In the next section, Section 2, the structure theory for the detecting subalgebras and their relationship to various support variety theories is reviewed. Given a classical simple Lie superalgebra, $\mathfrak{g}$, the construction of a parabolic subalgebra, $\mathfrak{b}$, that is generated by the negative Borel subalgebra for $\mathfrak{g}_{\bar{0}}$ and the detecting subalgebra $\mathfrak{f}$ is presented in Section 3. These parabolics are defined via hyperplanes in the span of the roots in a Euclidean space. A comparison theorem is proved between the relative cohomology for $(\mathfrak{b},\mathfrak{b}_{\bar{0}})$ and $(\mathfrak{f},\mathfrak{f}_{\bar{0}})$ (cf. Theorem 3.4.1) and between the relative cohomology for $(\mathfrak{g},\mathfrak{g}_{\bar{0}})$ and $(\mathfrak{b},\mathfrak{b}_{\bar{0}})$ (cf. Theorem 3.5.1). The latter relationship involves a natural grading on the group algebra of a finite reflection group $W_{\bar{1}}$.

In Section 4, we investigate sheaf cohomology for G/B where $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{b} = \text{Lie } B$. In particular, we consider the Poincaré series, $p_{G,B}(t) = \sum_{i=0}^{\infty} \dim R^i\mathrm{ind}_B^G\mathbb{C}\ t^i$ and give a complete computation for all Lie superalgebras except when $\mathfrak{g} = \mathfrak{p}(n)$. It is shown that $p_{G,B}(t)$ is directly related to the standard Poincaré polynomial of $W_{\bar{1}}$ via the natural length function on the finite reflection group $W_{\bar{1}}$ (cf. Table 7.2.1). Our calculations use an intricate and detailed analysis of the (odd) dot action of $W_{\bar{1}}$ on a natural subset, $\Phi_{\bar{1}}$, of odd roots. Section 5 is devoted to investigating the situation for $\mathfrak{g} = \mathfrak{p}(n)$. For $\mathfrak{p}(2)$ and $\mathfrak{p}(3)$ it is shown that $p_{G,B}(t)$ is governed by the BBW theorem. However, for $\mathfrak{p}(4)$ this is not the case and open questions are presented at the end of this section.

Finally, in Section 6, we indicate how our computation fit into a more functorial setting involving natural spectral sequences (see Theorem 6.4.1 and Theorem 6.5.1). For all classical Lie superalgebras with the possible exception of $\mathfrak{g} = \mathfrak{p}(n)$, it is shown that the spectral sequence in Theorem 6.4.1 collapses. This result enables us to prove the conjecture involving the equality of supports stated as Theorem 1.2.1.

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2. Preliminaries

2.1. Notation

We will use and summarize the conventions developed in [4–6]. For more details we refer the reader to [4, Section 2].

Throughout this paper, let $\mathfrak{a}$ be a Lie superalgebra over the complex numbers $\mathbb{C}$. In particular, $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space with a supercommutator $[\ ,\ ] : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}$. A finite-dimensional Lie superalgebra $\mathfrak{a}$ is called *classical* if there is a connected reductive algebraic group $A_{\bar{0}}$ such that $\mathrm{Lie}(A_{\bar{0}}) = \mathfrak{a}_{\bar{0}}$, and the action of $A_{\bar{0}}$ on $\mathfrak{a}_{\bar{1}}$ differentiates to the adjoint action of $\mathfrak{a}_{\bar{0}}$ on $\mathfrak{a}_{\bar{1}}$. The Lie superalgebra $\mathfrak{a}$ is *basic classical* if it is a classical Lie superalgebra with a nondegenerate invariant supersymmetric even bilinear form. In this paper our main focus will be on classical "simple" Lie superalgebras. The algebras of interest are listed in Table 7.2.1. Although some of these Lie superalgebras are not simple in the true sense, they are close enough to being simple and are ones of general interest. With a slight abuse of notation we will let $A(m|n)$ denote the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ for $m \neq n$ and $\mathfrak{sl}(n|n)$ and $\mathfrak{psl}(n|n)$ for $m = n$. For the Lie superalgebras of type Q we use the notation of [23]. Namely, $\mathfrak{q}(n)$ will be the Lie superalgebra with even and odd parts $\mathfrak{gl}_n$, while $\mathfrak{psq}(n)$ is the corresponding simple subquotient of $\mathfrak{q}(n)$. The Lie superalgebras that fall into the family of type P will be denoted by $P(n)$. These algebras include $\mathfrak{p}(n)$ and its enlargement $\widetilde{\mathfrak{p}}(n)$.

Let $U(\mathfrak{a})$ be the universal enveloping superalgebra of $\mathfrak{a}$. We will use the term $\mathfrak{a}$-module to be a unital module for $U(\mathfrak{a})$. If M and N are $\mathfrak{a}$-modules one can use the antipode and coproduct of $U(\mathfrak{a})$ to define a $\mathfrak{a}$-module structure on the dual M^* and the tensor product $M \otimes N$.

Let $\mathfrak{a}$ be an arbitrary Lie superalgebra (not necessary classical). In this paper we will study homological properties of the category of $\mathfrak{a}$-modules where the projective objects are relatively projective $U(\mathfrak{a}_{\bar{0}})$-modules. Given $\mathfrak{a}$-modules, M, N, let $\mathrm{Ext}^n_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M, N)$ denote the n-extension group defined by using a relatively projective $U(\mathfrak{a}_{\bar{0}})$-resolution for M. Under the conditions that either $\mathfrak{a}_{\bar{1}}$ is finitely semisimple over $\mathfrak{a}_{\bar{0}}$ or $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ is a direct sum of $\mathfrak{a}_{\bar{0}}$-modules (cf. [19, 3.1.8 Corollary, 3.1.15 Remark]), there is a concrete realization for these extension groups via the relative Lie superalgebra cohomology for the pair $(\mathfrak{a}, \mathfrak{a}_{\bar{0}})$:

$$\mathrm{Ext}^n_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M, N) \cong \mathrm{H}^n(\mathfrak{a}, \mathfrak{a}_{\bar{0}}; M^* \otimes N).$$

The later cohomology group can be computed using an explicit complex. For a detailed discussion about the complex to compute relative Lie superalgebra cohomology the reader is referred to [4, Section 2.3]. Set

$$p_{\mathfrak{a}}(t) = \sum_{i=0}^{\infty} \dim \mathrm{H}^i(\mathfrak{a}, \mathfrak{a}_{\bar{0}}, \mathbb{C}) t^i. \tag{2.1.1}$$

When $\mathfrak{a}$ is a classical Lie superalgebra, let $\mathcal{F}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}$ be the full subcategory of finite-dimensional $\mathfrak{a}$-modules which are finitely semisimple over $\mathfrak{a}_{\bar{0}}$ (a $\mathfrak{a}_{\bar{0}}$-module is *finitely semisimple* if it decomposes into a direct sum of finite-dimensional simple $\mathfrak{a}_{\bar{0}}$-modules). The projectives in the category $\mathcal{F} := \mathcal{F}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}$ are the finite-dimensional relatively projective $U(\mathfrak{a}_{\bar{0}})$-modules. Moreover, $\mathcal{F}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}$ is a Frobenius category (i.e., where injectivity is equivalent to projectivity) [6]. Given M, N in $\mathcal{F}$, $\mathrm{Ext}^n_{\mathcal{F}}(M, N) \cong \mathrm{Ext}^n_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M, N)$. Let R be the cohomology ring

$$\mathrm{H}^{\bullet}(\mathfrak{a}, \mathfrak{a}_{\bar{0}}; \mathbb{C}) = S^{\bullet}(\mathfrak{a}_{\bar{1}}^*)^{\mathfrak{a}_{\bar{0}}} \cong S^{\bullet}(\mathfrak{a}_{\bar{1}}^*)^{A_{\bar{0}}}.$$

The last isomorphism holds because $A_{\bar{0}}$ is reductive and acts semisimply on the symmetric algebra. Moreover, since $A_{\bar{0}}$ is reductive it follows that R is finitely generated.

2.2. Support varieties

We recall the definition of the support variety of a finite-dimensional $\mathfrak{a}$-supermodule M (cf. [4, Section 6.1]). Let $\mathfrak{a}$ be a classical Lie superalgebra, $R := \mathrm{H}^{\bullet}(\mathfrak{a}, \mathfrak{a}_{\bar{0}}; \mathbb{C})$, and M_1, M_2 be in $\mathcal{F} := \mathcal{F}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}$. According to [4, Theorem 2.5.3], $\mathrm{Ext}^{\bullet}_{\mathcal{F}}(M_1, M_2)$ is a finitely generated R-module. Set $J_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M_1, M_2) = \mathrm{Ann}_R(\mathrm{Ext}^{\bullet}_{\mathcal{F}}(M_1, M_2))$ (i.e., the annihilator ideal of this module). The *relative support variety of the pair* (M, N) is

$$\mathcal{V}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M, N) = \mathrm{MaxSpec}(R/J_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M, N)) \tag{2.2.1}$$

In the case when $M = M_1 = M_2$, set $J_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M) = J_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M, M)$, and

$$\mathcal{V}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M) := \mathcal{V}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M, M).$$

The variety $\mathcal{V}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M)$ is called the *support variety* of M. In this situation, $J_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M) = \mathrm{Ann}_R \ \mathrm{Id}$ where Id is the identity morphism in $\mathrm{Ext}^0_{\mathcal{F}}(M, M)$.

2.3. Structure theory for the detecting subalgebras

The main ideas used in constructing the detecting subalgebras $\mathfrak{f}$ and $\mathfrak{e}$ for classical simple Lie superalgebras are summarized below.

Let $\mathfrak{g}$ be a classical simple Lie superalgebra as described in [4, Section 8]. It was shown that the action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ admits a *stable* action. The reader is referred to [4, Section 3.2] [21] for a detailed exposition on stable actions.

Fix a generic element $x_0 \in \mathfrak{g}_{\bar{1}}$ (cf. [4, Section 8.9] for an explicit construction). Set

$$H = \mathrm{Stab}_{G_{\bar{0}}} x_0 := G_{\bar{0},x_0},$$

and

$$\mathfrak{f}_{\bar{1}} = \mathfrak{g}_{\bar{1}}^H = \{z \in \mathfrak{g}_{\bar{1}} : \ h.z = z \text{ for all } h \in H\}.$$

Note that the roots of $\mathfrak{f}_{\bar{1}}$ are listed in Table 7.1.1. One can construct the detecting subalgebra $\mathfrak{f}$ by letting $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$ with $\mathfrak{f} := \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$.

Now let $N = N_{G_{\bar{0}}}(H)$ and $N_{\bar{0}}$ be the connected component of the identity. Since x_0 is semisimple, H is reductive as well as N. Set

$$W_{\bar{1}} = W_{\mathfrak{f}} := N_{G_{\bar{0}}}(H)/N_{\bar{0}}.$$

The finite group $W_{\bar{1}}$ is a pseudo-reflection group.

The action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is a *polar* representation (as in [10]). In particular,

$$\dim \mathfrak{e}_{x_0} = \text{Kr. dim } S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$$

where

$$\mathfrak{e}_{x_0} := \{x \in \mathfrak{g}_{\bar{1}} : \ [\mathfrak{g}_{\bar{0}}, x] \subseteq [\mathfrak{g}_{\bar{0}}.x_0]\}.$$

Set $\mathfrak{e}_{\bar{1}} = \mathfrak{e}_{x_0}$, $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ with $\mathfrak{e}_{\bar{0}} = [\mathfrak{e}_{\bar{1}}, \mathfrak{e}_{\bar{1}}]$.

One can obtain a finite reflection group $W_{\mathfrak{e}}$ by setting

$$W_{\mathfrak{e}} = N_{G_{\bar{0}}}(\mathfrak{e}_{\bar{1}})/\mathrm{Stab}_{G_{\bar{0}}}(\mathfrak{e}_{\bar{1}}).$$

2.4. In this section, we compare the support varieties for the classical Lie superalgebras $\mathfrak{g}$, $\mathfrak{f}$, and $\mathfrak{e}$ under the restriction maps. Assume that $\mathfrak{g}$ is both stable and polar. Without the assumption that $\mathfrak{g}$ is polar, the statements concerning cohomology and support varieties for $\mathfrak{g}$ and $\mathfrak{f}$ remain true. We recall the exposition given in [4, Section 6.1].

First there are natural maps of rings given by restriction,

$$\text{res} : \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \to \mathrm{H}^{\bullet}(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \to \mathrm{H}^{\bullet}(\mathfrak{e}, \mathfrak{e}_{\bar{0}}, \mathbb{C}),$$

which induce isomorphisms

$$\mathrm{res}: \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})^N \to \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}, \mathbb{C})^{W_{\mathfrak{e}}}. \tag{2.4.1}$$

The map on cohomology above induces a morphism of varieties:

$$\mathrm{res}^*: \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C}) \longrightarrow \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C})$$

and isomorphisms (by passing to quotient spaces)

$$\mathrm{res}^*: \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})/W_{\mathfrak{e}} \to \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C})/N \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}). \tag{2.4.2}$$

Let M be a finite-dimensional $\mathfrak{g}$-module. Then res^* induces maps between support varieties:

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \to \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M).$$

Since M is a $\mathfrak{g}_{\bar{0}}$-module, the first two varieties are stable under the action of $W_{\mathfrak{e}}$ and N respectively. Consequently, we obtain the following induced maps of varieties using (2.4.2):

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W_{\mathfrak{e}} \hookrightarrow \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/N \hookrightarrow \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M).$$

These maps are embeddings because if $x \in R$ annihilates the identity in $\mathrm{H}^0(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, M^* \otimes M)$ then it must annihilate the identity elements in $\mathrm{H}^0(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, M^* \otimes M)$ and $\mathrm{H}^0(\mathfrak{e}, \mathfrak{e}_{\bar{0}}, M^* \otimes M)$, and the restriction maps induce isomorphisms on the cohomology given in (2.4.1).

2.5. Support varieties for stable and polar detecting subalgebras

We record the result proved in [20, Theorem 4.5.1] that shows that the support varieties for $\mathfrak{e}$ and $\mathfrak{f}$ coincide after taking the geometric quotient.

Theorem 2.5.1. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra which is stable and polar. If $M \in \mathcal{F}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$ then we have the following isomorphism of varieties:*

$$\mathrm{res}^*: \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W_{\mathfrak{e}} \to \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/N.$$

3. Construction of $\mathfrak{b}$

3.1. Generalities on parabolic subalgebras

Let $\mathfrak{g}$ be a classical simple Lie superalgebra with a fixed Cartan subalgebra $\mathfrak{h}$ and root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$. For the definitions of $\epsilon_i, \delta_j, \epsilon$ we follow the convention of [17] with the exception of the superalgebras $D(2,1,\alpha)$, $F(4)$, and $G(3)$. For the latter we use the following notation: $(\epsilon,0,0)$, $(0,\epsilon,0)$, $(0,0,\epsilon)$ for $\epsilon_1,\epsilon_2,\epsilon_3$, respectively, if $\mathfrak{g}=D(2,1,\alpha)$; $(0,\epsilon)$ for δ if $\mathfrak{g}=G(3)$ or $\mathfrak{g}=F(4)$.

In what follows we use the terminology and setting of [11]. A *parabolic subalgebra* of $\mathfrak{g}$ is a subalgebra that contains a Borel subalgebra of $\mathfrak{g}$. We will consider only parabolic subalgebras that contain $\mathfrak{h}$. Every such parabolic subalgebra corresponds to a parabolic set of roots, as explained below.

Assume first that Φ is symmetric, i.e., $\Phi=-\Phi$. This is true for all classical Lie superalgebras $\mathfrak{g}$ except for those of type P. We call a proper subset S of Φ a *parabolic set in* Φ if

$$\Phi=S\cup(-S),\quad \text{and}\ \ \alpha,\beta\in S\ \ \text{with}\ \ \alpha+\beta\in\Phi\quad\text{implies}\quad \alpha+\beta\in S.$$

In the case when $\Phi\neq-\Phi$, we call $S\subsetneq\Phi$ a *parabolic subset* if $S=\widetilde{S}\cap\Phi$ for some parabolic subset $\widetilde{S}$ of $\Phi\cup(-\Phi)$.

To assign a parabolic set of roots to a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we use the correspondence $\mathfrak{p}\mapsto\Phi_{\mathfrak{p}}$, where $\Phi_{\mathfrak{p}}$ are the roots of $\mathfrak{p}$ relative to $(\mathfrak{g},\mathfrak{h})$. For the reverse direction we proceed as follows.

For a parabolic subset of roots S, we call $S^0:=S\cap(-S)$ the *Levi component* of S, $S^-:=S\backslash(-S)$ the *nilpotent component* of S, and $S=S^0\sqcup S^-$ the *Levi decomposition of* S. Then $\mathfrak{p}_S=\mathfrak{h}\oplus\left(\bigoplus_{\mu\in S}\mathfrak{g}^\mu\right)$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, and $\mathfrak{l}_S=\mathfrak{h}\oplus\left(\bigoplus_{\mu\in S^0}\mathfrak{g}^\mu\right)$ and $\mathfrak{n}_S^-=\bigoplus_{\mu\in S^-}\mathfrak{g}^\mu$ are called the *Levi subalgebra*, and the *nilradical* of $\mathfrak{p}_S$, respectively.

Let V_Φ be a real vector space such that $\Phi\subset V_\Phi\setminus\{0\}$. An element $\mathcal{H}$ in V_Φ^* defines a parabolic subset of roots $S=S(\mathcal{H})$ as follows. We define S^0 (respectively, S^-) to be the subset of Φ consisting of all roots α such that $\alpha(h)=0$ (respectively, $\alpha(h)<0$) for all $h\in\mathcal{H}$. Note that we identify the elements of $(V_\Phi^*)^*$ and V_Φ. A parabolic subset of roots S that is of the form $S(\mathcal{H})$ for some $\mathcal{H}$ is called *principal parabolic subset.* Note that $\ker\mathcal{H}$ is a hyperplane in V_Φ, and the roots in S^0 (respectively, S^-) can be treated as those that are on (respectively, "below") the hyperplane $\ker\mathcal{H}$.

3.2. A parabolic subalgebra, $\mathfrak{b}$, that arises from taking a principal parabolic subset $S=S(\mathcal{H})=S^0\sqcup S^-$, where $\mathcal{H}$ is listed in Table 7.1.2, will be called a *BBW parabolic subalgebra*. Later, in Theorem 4.10.1, it will be shown that these subalgebras have very special cohomological properties involving equality of various Poincaré series. There exists a natural triangular decomposition of $\mathfrak{g}=\mathfrak{u}^+\oplus\mathfrak{f}\oplus\mathfrak{u}$ where the roots in $\mathfrak{u}_1^+$ (resp. $\mathfrak{u}$) coincide with $-(S^-)$ (resp. S^-). The BBW parabolic subalgebra identifies with $\mathfrak{b}=\mathfrak{f}\oplus\mathfrak{u}$. Even though $\mathfrak{b}$ is a parabolic subalgebra and technically is not a Borel subalgebra, we will view $\mathfrak{b}$ as being analogous to a Borel subalgebra for a complex simple Lie algebra, and the detecting subalgebra $\mathfrak{f}$ like a maximal torus. In the cases when $\mathfrak{g}=\mathfrak{gl}(n|n)$ or $\mathfrak{q}(n)$, $\mathfrak{b}$ can be realized as matrices of the form:

$$\mathfrak{b} = \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathfrak{g} : A,\ B,\ C,\ D \in L_n(\mathbb{C}) \right\}$$

where $L_n(\mathbb{C})$ are the set of $n \times n$ lower triangular matrices. We add that there exists a supergroup scheme B with Lie $B = \mathfrak{b}$ that corresponds to the (super) Hopf algebra $U(\mathfrak{b}) \cong \operatorname{Dist}(B)$.

For this paper, let $\Phi_{\bar{1}}^{-}$ (resp. $\Phi_{\bar{1}}^{+}$) correspond with the roots in $\mathfrak{u}_{\bar{1}}$ (resp. $\mathfrak{u}_{\bar{1}}^{+}$). One has $\Phi_{\bar{1}} = \Phi_{\bar{1}}^{+} \cup \Phi_{\bar{1}}^{-}$ and in the case when $\mathfrak{g} \neq \mathfrak{p}(n)$, $\Phi_{\bar{1}}^{-} = -(\Phi_{\bar{1}}^{+})$. In particular, we will take the liberty of calling $\Phi_{\bar{1}}^{-}$ the negative roots of $\mathfrak{f}$. The authors realize that this convention is not the standard practice in the literature. However, in Section 4.3, we will demonstrate that the dot action of $W_{\bar{1}}$ on $\Phi_{\bar{1}}^{+}$ is compatible with the dot action of the Weyl group of $G_{\bar{0}}$ on $\Phi_{\bar{0}}^{+}$. This key observation entailing the compatibility of these even and odd roots allows us to successfully complete the computations in the paper.

In Table 7.1.3 we describe the odd negative roots of the principal parabolic subsets $S = S^0 \sqcup S^-$ corresponding to the parabolic subalgebras $\mathfrak{b} = \mathfrak{f} \oplus \mathfrak{u}$. The elements $\mathcal{H}$ defining P are listed in Table 7.1.2. For $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{sl}(m|n), \mathfrak{osp}(2m|2n), \mathfrak{osp}(2m+1|2n)$, we let $V_\Phi = \operatorname{Span}\{\epsilon_i, \delta_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and fix E_i and D_j to be the basis vectors of V_Φ^* that are dual to ϵ_i and δ_j, respectively. Also, for these superalgebras, we let $E_i = 0$ and $D_j = 0$ whenever $i > m$ and $j > n$. For all exceptional Lie superalgebras we choose $V_\Phi = \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z}\Phi)$. For $\mathfrak{g} = D(2,1,\alpha)$ we let E_1, E_2, E_3 to be the dual to $(\epsilon, 0, 0)$, $(0, \epsilon, 0)$, $(0, 0, \epsilon)$, respectively. Lastly, if $\mathfrak{g} = G(3), F(4)$ we use L_i for the vectors in V_Φ^* dual to the fundamental weights ω_i of G_2 $(i = 1, 2)$, $\mathfrak{so}(7)$ $(i = 1, 2, 3)$, respectively, and E for the dual of $(0, \epsilon)$.

Note that x_i are arbitrary real numbers subject to the conditions listed in the table. In all cases $\Phi_{\bar{1}}^{-}$ corresponds to the odd part of S^-.

3.3. For each classical simple Lie superalgebra $\mathfrak{g}$ we can define a parabolic subalgebra $\mathfrak{b}$ via the decomposition of odd roots given in Table 7.1.3 and in Section 5.2 for $\mathfrak{g} = \mathfrak{p}(n)$ that satisfies the following properties:

(a) $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ where $\mathfrak{b}_{\bar{0}}$ is a (negative) Borel subalgebra of $\mathfrak{g}_{\bar{0}}$ with maximal torus $\mathfrak{t}_{\bar{0}}$.
(b) $\mathfrak{t} = \mathfrak{t}_{\bar{0}} \oplus \mathfrak{t}_{\bar{1}}$ where $\mathfrak{t}_{\bar{1}} = \mathfrak{f}_{\bar{1}}$ where $\mathfrak{f}$ is the (stable) detecting subalgebra.
(c) $\mathfrak{f}$ is a subalgebra of $\mathfrak{t}$.
(d) $\mathfrak{f}_{\bar{1}}$ is $T_{\bar{0}}$-stable where Lie $T_{\bar{0}} = \mathfrak{t}_{\bar{0}}$.
(e) $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ where $\mathfrak{u}$ is a nilpotent Lie superalgebra.
(f) $\mathfrak{u} = \mathfrak{u}_{\bar{0}} \oplus \mathfrak{u}_{\bar{1}}$ where $\mathfrak{u}_{\bar{0}}$ is the unipotent radical of $\mathfrak{b}_{\bar{0}}$.

In this setting one has a weight space decomposition $\mathfrak{u}_{\bar{1}} = \oplus_{\lambda \in \mathfrak{t}_{\bar{0}}^*} (\mathfrak{u}_{\bar{1}})_\lambda$ where $(\mathfrak{u}_{\bar{1}})_\lambda$ is a $\mathfrak{t}_{\bar{0}}$-module with composition factors of the form λ.

3.4. Comparison of cohomology

We first compare the relative cohomology for $(\mathfrak{b}, \mathfrak{b}_{\bar{0}})$ and $(\mathfrak{f}, \mathfrak{f}_{\bar{0}})$.

Theorem 3.4.1. *Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ be the parabolic subalgebra as defined in Section 3.3. Then*

(a) $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C}) \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)$.
(b) *The restriction map*

$$\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C})^{T_{\bar{0}}}$$

is an isomorphism. Moreover, $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C})^{T_{\bar{0}}} \cong \mathrm{H}^\bullet(\mathfrak{t}, \mathfrak{t}_{\bar{0}}, \mathbb{C})$.

Proof. (a) Since $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$ it follows that

$$\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C}) \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^{F_{\bar{0}}} \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*). \tag{3.4.1}$$

(b) Next observe that

$$S^n(\mathfrak{b}_{\bar{1}}^*)^{F_{\bar{0}}} \cong S^n(\mathfrak{f}_{\bar{1}}^* \oplus \mathfrak{u}_{\bar{1}}^*)^{F_{\bar{0}}} \cong \bigoplus_{i+j=n} S^i(\mathfrak{f}_{\bar{1}}^*) \otimes S^j(\mathfrak{u}_{\bar{1}}^*)^{F_{\bar{0}}} \cong S^n(\mathfrak{f}_{\bar{1}}^*).$$

The last isomorphism holds since (i) $S^\bullet(\mathfrak{u}_{\bar{1}}^*)^{F_{\bar{0}}} \subseteq S^\bullet(\mathfrak{u}_{\bar{1}}^*)^{T_{\bar{0}}}$ and (ii) the duals of roots in $\mathfrak{u}_{\bar{1}}^*$ under the $T_{\bar{0}}$-grading are positive (see Section 3.1). It follows that

$$S^n(\mathfrak{b}_{\bar{1}}^*)^{T_{\bar{0}}} \cong S^n(\mathfrak{f}_{\bar{1}}^*)^{T_{\bar{0}}}$$

and $\dim S^n(\mathfrak{b}_{\bar{1}}^*)^{B_{\bar{0}}} \leq \dim S^n(\mathfrak{f}_{\bar{1}}^*)^{T_{\bar{0}}}$ for $n \geq 0$.

Since $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong S^\bullet(\mathfrak{b}_{\bar{1}}^*)^{B_{\bar{0}}}$, the restriction map $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{t}, \mathfrak{t}_{\bar{0}}, \mathbb{C})$ is given by the restriction map on functions:

$$S^\bullet(\mathfrak{b}_{\bar{1}}^*)^{B_{\bar{0}}} \to S^\bullet(\mathfrak{t}_{\bar{1}}^*)^{T_{\bar{0}}}. \tag{3.4.2}$$

Finally, observe that as $B_{\bar{0}}$-module, one has a short exact sequence

$$0 \to \mathfrak{u}_{\bar{1}} \to \mathfrak{b}_{\bar{1}} \to \mathfrak{t}_{\bar{1}} \to 0.$$

Therefore,

$$0 \to \mathfrak{t}_{\bar{1}}^* \to \mathfrak{b}_{\bar{1}}^* \to \mathfrak{u}_{\bar{1}}^* \to 0$$

with $B_{\bar{0}}$-acting trivially on $\mathfrak{t}_{\bar{1}}^*$. This shows there exists a subring $S \subseteq S^\bullet(\mathfrak{b}_{\bar{1}}^*)^{B_{\bar{0}}}$ such that the restriction map induces an isomorphism of $S \cong S^\bullet(\mathfrak{t}_{\bar{1}}^*)^{T_{\bar{0}}} = S^\bullet(\mathfrak{f}_{\bar{1}}^*)^{T_{\bar{0}}}$. The statement of (b) now follows because $\dim S^n(\mathfrak{b}_{\bar{1}}^*)^{B_{\bar{0}}} \leq \dim S^n(\mathfrak{t}_{\bar{1}}^*)^{T_{\bar{0}}}$ for $n \geq 0$. $\square$

3.5. We can now demonstrate how the relative cohomology for $\mathfrak{b}$ is related to the relative cohomology for $\mathfrak{g}$ and the dual of the group algebra of $W_{\bar{1}}$. One can view this result as a functorial interpretation of the harmonic decomposition for $S^\bullet(\mathfrak{f}_{\bar{1}}^*)$.

Theorem 3.5.1. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra. There exists a detecting subalgebra $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$ obtained by using the stable action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ and a proper parabolic subalgebra $\mathfrak{b}$ with the following properties*

(a) $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ *where* $\mathfrak{b}_{\bar{1}} \cong \mathfrak{f}_{\bar{1}} \oplus \mathfrak{u}_{\bar{1}}$ *and* $\mathfrak{b}_{\bar{0}}$ *is a Borel subalgebra for* $\mathfrak{g}_{\bar{0}}$.
(b) *There exists a finite reflection group $W_{\bar{1}}$ isomorphic to $N/N_{\bar{0}}$ and a grading on the coordinate algebra, $\mathbb{C}[W_{\bar{1}}]$, such that as graded vector spaces,*

$$\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \otimes \mathbb{C}[W_{\bar{1}}]_\bullet.$$

Proof. Let $\mathfrak{b}$ be as in Section 3.3. One has the harmonic decomposition (cf. [4, Theorem 3.5]):

$$S^\bullet(\mathfrak{f}_{\bar{1}}^*) \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N \otimes [\mathrm{ind}_H^N \mathbb{C}]_\bullet, \tag{3.5.1}$$

as graded $S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N$-modules. Applying $T_{\bar{0}}$ fixed points and using the fact that $T_{\bar{0}} \leq N$, one has

$$S^\bullet(\mathfrak{f}_{\bar{1}}^*)^{T_{\bar{0}}} \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N \otimes [\mathrm{ind}_H^N \mathbb{C}]_\bullet^{T_{\bar{0}}}. \tag{3.5.2}$$

From the definition of the induced module, one has

$$\begin{aligned}\mathrm{ind}_H^N \mathbb{C} &\cong [\mathbb{C}[N] \otimes \mathbb{C}]^H \\ &\cong \mathrm{Hom}_H(\mathbb{C}, \mathbb{C}[N])\end{aligned}$$

Now by applying $T_{\bar{0}}$ fixed points and using the fact that N_0 is generated by $T_{\bar{0}}$ and H:

$$[\mathrm{ind}_H^N \mathbb{C}]^{T_{\bar{0}}} \cong [\mathrm{Hom}_H(\mathbb{C}, \mathbb{C}[N])]^{T_{\bar{0}}} \cong \mathrm{Hom}_{N_0}(\mathbb{C}, \mathbb{C}[N]) \cong \mathbb{C}[W_{\bar{1}}].$$

Here $\mathbb{C}[W_{\bar{1}}]$ is the coordinate algebra of $W_{\bar{1}}$ which is dual to the group algebra of $W_{\bar{1}}$.

Next one can use the isomorphisms: $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^{T_{\bar{0}}}$ by Theorem 3.4.1(b), and $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N$ [4, Theorem 4.1]. One can now reinterpret (3.5.2) as

$$\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \otimes \mathbb{C}[W_{\bar{1}}]_\bullet. \quad \square \tag{3.5.3}$$

The reader should be made aware that the grading on $\mathbb{C}[W_{\bar{1}}]_\bullet$ is not always given by the Poincaré series for the finite reflection group $W_{\bar{1}}$. We will explore this important issue in the upcoming sections.

3.6. Let W be a finite reflection group and consider the Poincaré polynomial (cf. [15, Section 1.11])

$$p_W(t) = \sum_{w \in W} t^{l(w)}. \tag{3.6.1}$$

Note that the coefficient of t^j is precisely $|\{w \in W : \ l(w) = j\}|$. In general one has the identity

$$p_W(t) = \prod_{i=1}^{n} (1 + t + \cdots + t^{e_i}),$$

where e_i are the exponents of W. Set

$$z_{\mathfrak{b},\mathfrak{g}}(t) = p_{\mathfrak{b}}(t)/p_{\mathfrak{g}}(t) \tag{3.6.2}$$

We now provide some examples that show how to compute $z_{\mathfrak{b},\mathfrak{g}}(t)$.

Example 3.6.1 *($\mathfrak{g} = \mathfrak{q}(n)$ and $\mathfrak{gl}(m|n)$).* Assume that $m \geq n$. One has $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^{T_{\bar{0}}}$. This implies that

$$\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong \mathbb{C}[z_1, z_2, \ldots, z_n]$$

where the degree of z_j ($j = 1, 2, \ldots, n$) is 1 for $\mathfrak{q}(n)$ and 2 for $\mathfrak{gl}(m|n)$. Furthermore, by [4, Table 1],

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \cong \mathbb{C}[z_1, z_2, \ldots, z_n]^{\Sigma_n}.$$

Hence, $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C})$ is a polynomial algebra generated in degrees $1, 2, \ldots, n$. Therefore, $z_{\mathfrak{b},\mathfrak{g}}(t) = p_{\Sigma_n}(t^r)$ where $r = 1$ for $\mathfrak{q}(n)$ and $r = 2$ for $\mathfrak{gl}(m|n)$.

Example 3.6.2 *($\mathfrak{g} = D(2, 1, \alpha)$, $G(3)$, $F(4)$).* A direct computation shows that $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong \mathbb{C}[z]$ where z is of degree 2. From [4, Table 1], $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C})$ is a polynomial algebra generated in degree 4. Therefore,

$$z_{\mathfrak{b},\mathfrak{g}}(t) = \frac{1 - t^4}{1 - t^2} = 1 + t^2 = p_{\Sigma_2}(t^2).$$

One can compute $z_{\mathfrak{b},\mathfrak{g}}(t)$ for the other classical simple Lie superalgebras by using the ideas presented in the preceding examples. Table 7.2.1 provides the relationship between $z_{\mathfrak{b},\mathfrak{g}}(t)$ and the Poincaré polynomial for $W_{\bar{1}}$ for other classical simple Lie superalgebras. Note that the x's, y's, and z's have degree one. We can summarize these results in the following theorem.

Theorem 3.6.3. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra. Assume that $\mathfrak{g}$ is not isomorphic to $P(n)$. There exists a detecting subalgebra $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$ obtained by using the stable action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ and a parabolic subalgebra $\mathfrak{b}$ such that $z_{\mathfrak{b},\mathfrak{g}}(t) = p_{W_{\bar{1}}}(s)$, where $s = t$ for Lie algebras $\mathfrak{g}$ of type Q, and $s = t^2$ otherwise.*

4. Connections with the geometry of G/B

4.1. Supergroups and the induction functor

Let G be an affine supergroup scheme over $\mathbb{C}$ and $\mathrm{Mod}(G)$ be the category of rational modules for G. For a general overview and details about supergroup schemes, the reader is referred to work of Brundan and Kleshchev [8, Sections 2,4,5] [9, Section 2].

In the case when $\mathfrak{g}$ is a classical Lie superalgebra and $\mathfrak{g} = \mathrm{Lie}\, G$, the category $\mathrm{Mod}(G)$ is equivalent to locally finite integral modules for $\mathrm{Dist}(G) = U(\mathfrak{g})$ (cf. [8, Corollary 5.7]). In particular, if $\mathfrak{g}$ is a classical Lie superalgebra, then $\mathrm{Mod}(G)$ is equivalent to $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ (i.e., the category of $\mathfrak{g}$-supermodules that are completely reducible over $\mathfrak{g}_{\bar{0}}$).

Let H be a closed subgroup scheme of G and $R^j\mathrm{ind}_H^G(-)$ be the higher right derived functors of the induction functor $\mathrm{ind}_H^G(-)$. In the case when $\mathfrak{g} = \mathrm{Lie}\, G$ is a classical simple Lie superalgebra and $H = P$ where P is a parabolic subgroup, the following two propositions provide information about $R^\bullet \mathrm{ind}_P^G M$ when restricted to $G_{\bar{0}}$.

Proposition 4.1.1. *Let $\mathfrak{g} = \mathrm{Lie}\, G$ be a classical simple Lie superalgebra and P be a parabolic subgroup with M a P-module.*

(a) *Assume that $R^n \mathrm{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^i((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)] = 0$ for n-odd, and n-even when $n \neq i$. Then*

$$(R^n \mathrm{ind}_P^G M)|_{G_{\bar{0}}} \cong R^n \mathrm{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)]$$

for $n \geq 0$.

(b) *Assume that $M \cong \mathbb{C}$ and $R^n \mathrm{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[\Lambda^i((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)] = 0$ for $i \neq n$. Then*

$$(R^n \mathrm{ind}_P^G \mathbb{C})|_{G_{\bar{0}}} \cong R^n \mathrm{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)]$$

for $n \geq 0$.

Proof. We will employ results provided in the exposition given in [9, Section 2]. Let $X = G/P$ and M be a P-module, with $\mathcal{L}(M)$ being the associated quasi-coherent $\mathcal{O}_X G$-(super)module. First note that $H^n(G/P, \mathcal{L}(M)) \cong R^n\mathrm{ind}_P^G M$ for all $n \geq 0$. Now according to [9, (6)], one has

$$R^n\mathrm{ind}_P^G M|_{G_{\bar{0}}} \cong H^n(G/P, \mathcal{L}(M))|_{G_{\bar{0}}} \cong H^n(G/P, \mathrm{res}_{G_{\bar{0}}}^G(\mathcal{L}(M))) \tag{4.1.1}$$

for all $n \geq 0$. Next observe that by [9, (2), Theorem 2.7] there exists a canonical filtration of $\mathcal{L}(M)$:

$$\mathcal{J}^0 = \mathcal{L}(M) \supseteq \mathcal{J}^1 \supseteq \mathcal{J}^2 \supseteq \dots \mathcal{J}^{t-1} \supseteq \mathcal{J}^t = \{0\}$$

with

$$\mathcal{J}^i/\mathcal{J}^{i+1} \cong \mathcal{L}_{ev}(M \otimes \Lambda^i((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)) \tag{4.1.2}$$

Finally, by [9, equation after (2)] one has the following isomorphisms:

$$H^n(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) \cong H^j(G_{\bar{0}}/P_{\bar{0}}, \mathcal{L}_{ev}(M \otimes \Lambda^i((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)) \cong R^n\text{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^i((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)]. \tag{4.1.3}$$

(a) The filtration described above yields the short exact sequence:

$$0 \to \mathcal{J}^{i+1} \to \mathcal{J}^i \to \mathcal{J}^i/\mathcal{J}^{i+1} \to 0.$$

Next apply the long exact sequence in cohomology, and use the fact that $H^n(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) = 0$ for n-odd and $i \geq 0$ to obtain a five term exact sequences for $(n-1)$-even:

$$\begin{aligned} 0 &\to H^{n-1}(G/P, \mathcal{J}^{i+1}) \to H^{n-1}(G/P, \mathcal{J}^i) \to H^{n-1}(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) \\ &\to H^n(G/P, \mathcal{J}^{i+1}) \to H^n(G/P, \mathcal{J}^i) \to 0. \end{aligned}$$

Using these five term sequences, we first show that $H^n(G/P, \mathcal{J}^i) = 0$ for n-odd and all $i \geq 0$. First consider the case when $i+1 = t$. Then $H^n(G/P, \mathcal{J}^{i+1}) = 0$ for all $n \geq 0$ and from the sequences above $H^n(G/P, \mathcal{J}^{t-1}) = 0$ for n-odd. Now apply the process again for $i+1 = t-1$, and the prior result to show that $H^n(G/P, \mathcal{J}^{t-2}) = 0$ for n-odd. Continuing this process proves that $H^n(G/P, \mathcal{J}^i) = 0$ for n-odd and $i \geq 0$ and the statement of part (a) of the theorem in the case when n is odd.

Next we finish off the statement of part (a) when n is even. From the results in the prior paragraph, the five term exact sequences become short exact sequences of the form:

$$0 \to H^n(G/P, \mathcal{J}^{i+1}) \to H^n(G/P, \mathcal{J}^i) \to H^n(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) \to 0, \tag{4.1.4}$$

for $n \geq 0$ (here n can be either even or odd). Using these short exact sequences, we can conclude that for $i \leq n$

$$H^n(G/P, \mathcal{J}^0) = H^n(G/P, \mathcal{J}^i) \tag{4.1.5}$$

and for $n < i$,

$$H^n(G/P, \mathcal{J}^i) = 0. \tag{4.1.6}$$

Combining these equations and using the assumptions in the theorem, it follows that as $G_{\bar{0}}$-module,

$$H^n(G/P, \mathcal{J}^0) = H^n(G/P, \mathcal{J}^n/\mathcal{J}^{n+1}) = H^n(G/P, \oplus_{i\geq 0}\mathcal{J}^i/\mathcal{J}^{i+1}).$$

The result now follows by applying the identifications provided in the first paragraph.

(b) In order to prove the statement we need to consider the π-graded category of G-(super)modules where $\pi = \mathbb{Z}_2$. In this category, the simple modules consist of the simple G-modules with their images under the parity change functor Π. In particular, one has the trivial module $\mathbb{C}$ with trivial π-action and the module $\Pi\mathbb{C}$ with trivial G-action and the non-trivial element in π acting as (-1). Denote the graded category by π-$(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$.

The short exact sequence

$$0 \to \mathcal{J}^{i+1} \to \mathcal{J}^i \to \mathcal{J}^i/\mathcal{J}^{i+1} \to 0,$$

along with the long exact sequence in cohomology and the fact that $H^k(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) = 0$ for $i \neq k$ yields the five term exact sequence:

$$\begin{aligned} 0 &\to H^i(G/P, \mathcal{J}^{i+1}) \to H^i(G/P, \mathcal{J}^i) \to H^i(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) \\ &\to H^{i+1}(G/P, \mathcal{J}^{i+1}) \to H^{i+1}(G/P, \mathcal{J}^i) \to 0. \end{aligned}$$

Moreover, one obtains the following isomorphisms:

$$H^k(G/P, \mathcal{J}^{i+1}) \cong H^k(G/P, \mathcal{J}^i) \quad \text{for } k < i \text{ and } k > i+1. \tag{4.1.7}$$

Now fix $n \geq 0$. From the isomorphisms in (4.1.7),

$$H^n(G/P, \mathcal{J}^0) \cong H^n(G/P, \mathcal{J}^1) \cong \ldots \cong H^n(G/P, \mathcal{J}^{n-1}) \cong H^n(G/P, \mathcal{J}^n), \tag{4.1.8}$$

$$0 = H^n(G/P, \mathcal{J}^t) \cong H^n(G/P, \mathcal{J}^{t-1}) \cong \ldots \cong H^n(G/P, \mathcal{J}^{n+2}) \cong H^n(G/P, \mathcal{J}^{n+1}). \tag{4.1.9}$$

One can use the five term sequence above along with (4.1.8) and (4.1.9) to obtain a four term exact sequence

$$\begin{aligned} 0 &\to H^n(G/P, \mathcal{J}^0) \to H^n(G/P, \mathcal{J}^n/\mathcal{J}^{n+1}) \to H^{n+1}(G/P, \mathcal{J}^{n+1}) \\ &\to H^{n+1}(G/P, \mathcal{J}^n) \to 0. \end{aligned}$$

From the exact sequence and the isomorphism $R^n\mathrm{ind}_P^G\mathbb{C}|_{G_{\bar{0}}} \cong H^n(G/P, \mathcal{J}^0)$, one has an injection:

$$f_n : R^n\mathrm{ind}_P^G\mathbb{C}|_{G_{\bar{0}}} \hookrightarrow H^n(G/P, \mathcal{J}^n/\mathcal{J}^{n+1}).$$

The statement of part (b) will now follow if we show that f_n is an isomorphism for all $n \geq 0$. Using the hypothesis and [9, Corollary 2.8], one has

$$\sum_{n\geq 0}(-1)^n R^n \mathrm{ind}_P^G \mathbb{C} = \sum_{n\geq 0}(-1)^n H^n(G/P, \mathcal{J}^n/\mathcal{J}^{n+1}).$$

Here the sum is taken in the Grothendieck group of π-graded $G_{\bar{0}}$-modules. Using the fact that f_n is an injection, the simple modules appearing in $H^n(G/P, \mathcal{J}^n/\mathcal{J}^{n+1})$ and $R^n \mathrm{ind}_P^G \mathbb{C}$ have the same parity (depending on the parity of n, see [9, Lemma 4.4]).

It follows that

$$\sum_{n\geq 0} R^{2n} \mathrm{ind}_P^G \mathbb{C} = \sum_{n\geq 0} H^{2n}(G/P, \mathcal{J}^{2n}/\mathcal{J}^{2n+1}),$$

$$\sum_{n\geq 0} R^{2n+1} \mathrm{ind}_P^G \mathbb{C} = \sum_{n\geq 0} H^{2n+1}(G/P, \mathcal{J}^{2n+1}/\mathcal{J}^{2n+2}).$$

Hence,

$$\sum_{n\geq 0} \dim R^n \mathrm{ind}_P^G \mathbb{C} = \sum_{n\geq 0} \dim H^n(G/P, \mathcal{J}^n/\mathcal{J}^{n+1}).$$

This proves that f_n is an isomorphism for all n. $\square$

Proposition 4.1.2. *Let $\mathfrak{g} = \mathrm{Lie}\, G$ be a classical simple Lie superalgebra and P be a parabolic subgroup with M a P-module. Assume that $R^j \operatorname{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)] = 0$ for $j > 0$. Then*

$$(R^j \operatorname{ind}_P^G M)|_{G_{\bar{0}}} \cong R^j \operatorname{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)]$$

for $j \geq 0$.

Proof. We use the setting as described in Proposition 4.1.1. For $j > 0$, $H^j(G/P, \mathcal{J}^i/J^{i+1}) = 0$ for all i. It follows that $H^j(G/P, \mathcal{J}^i) = 0$ for all i, thus $R^j \operatorname{ind}_P^G M = 0$ for $j > 0$.

Now consider the case when $j = 0$. For each i, one has the short exact sequence

$$0 \to \mathcal{J}^{i+1} \to \mathcal{J}^i \to \mathcal{J}^i/\mathcal{J}^{i+1} \to 0.$$

Applying the long exact sequence in cohomology and using the fact that $H^1(G/P, \mathcal{J}^i) = 0$ yields a short exact sequence:

$$0 \to H^0(G/P, \mathcal{J}^{i+1}) \to H^0(G/P, \mathcal{J}^i) \to H^0(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) \to 0.$$

For each i, this short exact sequence splits over $G_{\bar{0}}$ and one can deduce that

$$\begin{aligned}(R^0 \operatorname{ind}_P^G M)|_{G_{\bar{0}}} &\cong H^0(G/P, \mathcal{J}^0) \cong \oplus_i \mathrm{H}^0(G/P, \mathcal{J}^i/\mathcal{J}^{i+1}) \\ &\cong R^0 \operatorname{ind}_{P_{\bar{0}}}^{G_{\bar{0}}}[M \otimes \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*]. \quad \square\end{aligned}$$

The theorem above justifies the statement of [7, Proposition 6.1.1] when the additional hypothesis is added. The results in [7, Proposition 6.5.3] can be justified by using Theorem 4.1.2.

Let P be a parabolic subgroup with $P \subseteq G$ and let

$$p_{G,P}(t) = \sum_{i=0}^{\infty} \dim R^i \mathrm{ind}_P^G \mathbb{C}\ t^i. \tag{4.1.10}$$

The following proposition will be useful in making the transition from computing $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\ \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ to computing $p_{G,B}(t)$ where B is the parabolic defined in Section 3.2.

Proposition 4.1.3. *Let $j \geq 0$.*

(a) *If $\mathfrak{g} \neq \mathfrak{q}(n)$ and $(R^j \mathrm{ind}_B^G \mathbb{C})|_{G_{\bar{0}}} \cong \mathbb{C}^{\oplus t}$, then $R^j \mathrm{ind}_B^G \mathbb{C} \cong \mathbb{C}^{\oplus t}$ as a G-module.*
(b) *If $\mathfrak{g} = \mathfrak{q}(n)$ with $(R^j \mathrm{ind}_B^G \mathbb{C})|_{G_{\bar{0}}} \cong \mathbb{C}^{\oplus t}$ and $(R^j \mathrm{ind}_B^G \mathbb{C})|_{G_{\bar{0}}} \cong R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^k((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ for some $k \geq 0$, then $R^j \mathrm{ind}_B^G \mathbb{C} \cong \mathbb{C}^{\oplus t}$ as a G-module.*

Proof. (a) The statement follows immediately if there are no self-extensions of the trivial module, that is, $\mathrm{Ext}^1_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C},\mathbb{C}) = 0$. This space identifies with $S^1(\mathfrak{g}_{\bar{1}})^{G_{\bar{0}}}$. For all types other than $\mathfrak{g} = \mathfrak{q}(n)$ this is always equal to zero (cf. [4, Table 1]).

(b) Let $\mathfrak{g} = \mathfrak{q}(n)$. For $j > 0$, the simple modules appearing as G-composition factors in the π-G-module, $R^j \mathrm{ind}_B^G \mathbb{C}$, all have the same parity. (i.e., they are either all $\mathbb{C}$ or all $\Pi\mathbb{C}$). See [9, Lemma 4.4.].

Since

$$\mathrm{Ext}^1_{\pi\text{-}(\mathfrak{g},\,\mathfrak{g}_{\bar{0}})}(\Pi\mathbb{C},\Pi\mathbb{C}) \cong \mathrm{Ext}^1_{\pi\text{-}(\mathfrak{g},\,\mathfrak{g}_{\bar{0}})}(\mathbb{C},\mathbb{C}) \cong S^1(\mathfrak{g}_{\bar{1}}^*)^{\pi\text{-}G_{\bar{0}}} \subseteq (\mathfrak{g}_{\bar{1}})^{\pi} = 0,$$

it follows by using the hypothesis that $R^j \mathrm{ind}_B^G \mathbb{C}$ as a π-G-module is isomorphic to $\mathbb{C}^{\oplus t}$ if j is even and $\Pi\mathbb{C}^{\oplus t}$ if j is odd. Hence, as G-module (disregarding the grading), $R^j \mathrm{ind}_B^G \mathbb{C} \cong \mathbb{C}^{\oplus t}$. $\square$

We remark that Proposition 4.1.3 is stated for the ungraded situation where one simply considers rational G-modules. One can also formulate a statement via the graded category π-G-modules where one distinguishes between $\mathbb{C}$ where π acts trivially and $\Pi\mathbb{C}$ where a non-trivial element of π acts as (-1). As rational G-modules, the grading is ignored and $\mathbb{C} \cong \Pi\mathbb{C}$.

4.2. Poincare series for exceptional Lie superalgebras

In the following theorem, we compute $p_{G,B}(t)$ for exceptional Lie superalgebras. Although $p_{G,B}(t)$ is a polynomial of degree 2, the verification extensively uses the representation theory of $\mathfrak{sl}_2$, G_2 and $\mathfrak{so}_7$ along with the classical Bott-Borel-Weil (BBW) theorem.

Theorem 4.2.1. *Let $\mathfrak{g} = D(2,1,\alpha)$, $G(3)$ or $F(4)$ and $\mathfrak{b}$ be the parabolic subalgebra described in Table 7.1.3. Then*

$$p_{G,B}(t) = 1 + t^2 = z_{\mathfrak{b},\mathfrak{g}}(t) = p_{W_{\bar{1}}}(t^2).$$

Proof. The last two equalities follow from Theorem 3.6.3. It remains to show that $p_{G,B}(t) = 1 + t^2$.

First consider $\mathfrak{g} = D(2,1,\alpha)$. One has $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $\mathfrak{g}_{\bar{1}} \cong V \boxtimes V \boxtimes V$ where V is the 2-dimensional natural representation of $\mathfrak{sl}_2$. Let $G_{\bar{0}} = G_{\bar{0},(1)} \times G_{\bar{0},(2)} \times G_{\bar{0},(3)}$ denote the product of three copies of SL_2 with Borel subgroup $B_{\bar{0}} = B_{\bar{0},(1)} \times B_{\bar{0},(2)} \times B_{\bar{0},(3)}$ (corresponding to the negative roots). For a given one-dimensional $B_{\bar{0}}$-module, $\mu = (\mu_1, \mu_2, \mu_3)$, one has

$$R^n \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \mu = \bigoplus_{n_1+n_2+n_3=n} R^{n_1} \mathrm{ind}_{B_{\bar{0},(1)}}^{G_{\bar{0},(1)}} \mu_1 \boxtimes R^{n_2} \mathrm{ind}_{B_{\bar{0},(2)}}^{G_{\bar{0},(2)}} \mu_2 \boxtimes R^{n_3} \mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}} \mu_3 \quad (4.2.1)$$

by the Künneth Theorem. It follows that if any of the components vanish then $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \mu = 0$.

The weights of $\Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ are $\{(-\epsilon, -\epsilon, -\epsilon), (-\epsilon, -\epsilon, \epsilon), (\epsilon, -\epsilon, -\epsilon)\}$. Let $X(T_{\bar{0}})$ be the integral weights of $G_{\bar{0}}$ and $\overline{C}_{\mathbb{Z}}$ be the closure of the bottom alcove in $X(T_{\bar{0}})$. Moreover, let $X(T_{\bar{0}})_+$ be the set of dominant integral weights. See [16, p. 571-572] for precise definitions.

By the BBW theorem, since all the weights of $\Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ are in $\overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+$, it follows that at least one component in the decomposition (4.2.1) vanishes, so $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$. Similarly, $\Lambda^3((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ is one-dimensional spanned by a vector of weight $\mu = (\epsilon, -3\epsilon, -\epsilon)$. The last component vanishes in (4.2.1), thus $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^3((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$. Also, note that $\Lambda^0((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong \mathbb{C}$, so $R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^0((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$ for $j > 0$ and is isomorphic to $\mathbb{C}$ for $j = 0$.

We need to analyze $\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$. This will entail using two-dimensional $B_{\bar{0}}$-modules. Similar methods will also be employed for the $G(3)$ and $F(4)$-cases. A direct computation shows that as a $B_{\bar{0}}$-module, $\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ has head isomorphic to $(0, -2\epsilon, 0)$ and two-dimensional socle $(0, -2\epsilon, -2\epsilon) \oplus (-2\epsilon, -2\epsilon, 0)$. Therefore, one has a short exact sequence:

$$0 \to (-2\epsilon, -2\epsilon, 0) \to \Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \to N \to 0 \quad (4.2.2)$$

where N is a two-dimensional $B_{\bar{0}}$-module isomorphic as $(B_{\bar{0},(1)} \times B_{\bar{0},(2)}) \times B_{\bar{0},(3)}$-module to $(0, -2\epsilon) \boxtimes N'$ where N' is a two-dimensional $B_{\bar{0},(3)}$-module with socle -2ϵ and head $\mathbb{C}$.

As a $B_{\bar{0},(3)}$-module, one has

$$0 \to N' \to L(2\epsilon) \to 2\epsilon \to 0 \quad (4.2.3)$$

where $L(2\epsilon)$ is the three-dimensional adjoint representation for $G_{\bar{0},(3)}$. Now by the tensor identity, $R^j\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}L(2\epsilon) \cong [R^j\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}\mathbb{C}] \otimes L(2\epsilon)$. This is zero for $j > 0$. Applying the long exact sequence in cohomology to (4.2.3) and the fact that $R^j\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}\ 2\epsilon = 0$ for $j > 0$ shows that $R^j\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}N' = 0$ for $j > 0$. It remains to look at the remaining part of the long exact sequence:

$$0 \to \mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}N' \to L(2\epsilon) \to L(2\epsilon) \to R^1\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}N' \to 0.$$

The only dominant weight of N' is 0 so $\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}N'$ is either 0 or $\mathbb{C}$. This proves the arrow from $L(2\epsilon)$ to $L(2\epsilon)$ must be an isomorphism, thus $R^\bullet\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}N' = 0$.

Now apply the long exact sequence in cohomology (4.2.2) and use the fact that $R^\bullet\mathrm{ind}_{B_{\bar{0},(3)}}^{G_{\bar{0},(3)}}N' = 0$. This yields

$$R^j\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}(-2\epsilon, -2\epsilon, 0) \cong R^j\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$$

for all $j \geq 0$. Applying the Künneth theorem and the BBW theorem shows that $R^j\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$ for $j \neq 2$ and $R^2\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong \mathbb{C}$. Consequently, $p_{G,B}(t) = 1 + t^2$.

For $G(3)$ and $F(4)$ the calculations are much more lengthy and involved to show that $p_{G,B}(t) = 1 + t^2$. First, $G_{\bar{0}} \cong G_{\bar{0},(1)} \times G_{\bar{0},(2)}$ has two components and for any one-dimensional $B_{\bar{0}} = B_{\bar{0},(1)} \times B_{\bar{0},(2)}$-module $\mu = (\mu_1, \mu_2)$ one has

$$R^n\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\ \mu = \bigoplus_{n_1+n_2=n} R^{n_1}\mathrm{ind}_{B_{\bar{0},(1)}}^{G_{\bar{0},(1)}}\ \mu_1 \boxtimes R^{n_2}\mathrm{ind}_{B_{\bar{0},(2)}}^{G_{\bar{0},(2)}}\ \mu_2 \tag{4.2.4}$$

In these cases the last component $G_{\bar{0},(2)}$ is isomorphic to SL_2, so to prove the vanishing, the focus will be more on the first component $G_{\bar{0},(1)}$ which is G_2 (resp. $\mathfrak{so}_7$) for $G(3)$ (resp. $F(4)$).

We will outline the ideas to handle $G(3)$. The ideas are similar for $F(4)$ and involve more verifications. In the case of $G(3)$, one has $\dim \mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}} = 6$.

(1) Show that if $k \neq 0, 2$ then $R^\bullet\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^k((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$.

One of the main ideas to analyze $\Lambda^k((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ for $k \neq 0, 2$ is to find a filtration of $B_{\bar{0}}$-modules whose subquotients are either one-dimensional or two-dimensional modules N_j such that $R^\bullet\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}N_j = 0$. For the one-dimensional modules, one shows that the weights are in $\overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+$. For the two dimensional modules, one uses the argument as given in $D(2,1,\alpha)$ so that these modules are submodules of the adjoint modules for a parabolic subgroup in $G_{\bar{0},(1)}$ corresponding to an SL_2. For example, in $G(3)$, the weights for $\Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ are

$$\{(-\omega_1 + \omega_2, -\epsilon), (2\omega_1 - \omega_2, -\epsilon), (0, -\epsilon), (\omega_1 - \omega_2, -\epsilon), (-2\omega_1 + \omega_2, -\epsilon), (-\omega_1, -\epsilon)\}.$$

By the BBW theorem, all the weights except for $(0, -\epsilon)$ and $(-2\omega_1 + \omega_2, -\epsilon) = -\alpha_1$ yield no cohomology. One can see the vectors of these weights form a subquotient with the desired properties.

(2) For $k = 0$, analyze $R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^0((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \mathbb{C}$.

From Kempf's vanishing theorem, $R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \mathbb{C} = 0$ for $j > 0$, and one has $\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \mathbb{C} = \mathbb{C}$.

(3) For $k = 2$, show that $R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$ for $j \neq 2$ and $R^2 \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = \mathbb{C}$.

A technique that is used in the verification of (3) is the existence of an embedding of $\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ into $\Lambda^2(L) \boxtimes (-2\epsilon)$ where L is an irreducible $G_{\bar{0},(1)}$-module. For example, when $\mathfrak{g} = G(3)$, one has

$$0 \to \Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \to \Lambda^2(L(\omega_1)) \boxtimes (-2\epsilon) \to M \to 0. \tag{4.2.5}$$

From the tensor identity, one has that $R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^2(L(\omega_1)) \boxtimes (-2\epsilon) = 0$ for $j \geq 0$. This allows one to dimension shift via the long exact sequence in cohomology to concentrate on calculating $R^j \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} M$. In the case M is a 6-dimensional module. This makes the computations tractable to verify (3). A similar short exact sequence to (4.2.5) exists for $F(4)$ via the spin representation $L(\omega_3)$ for $\mathfrak{so}_7$ and the same technique can be utilized in this case. $\square$

4.3. Consider $R^n \mathrm{ind}_B^G \mathbb{C} \cong R^n \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ as a $G_{\bar{0}}$-module for $n \geq 0$. The weights of $\Lambda^n((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ are $-\rho(J)$ where $J \subseteq \Phi_{\bar{1}}^+$ and $|J| = n$. Here $\rho(J) = \sum_{\alpha \in J} \alpha$. Set $\rho_{\bar{1}} = \frac{1}{2} \sum_{\alpha \in \Phi_{\bar{1}}^+} \alpha$ and let

$$w \cdot \lambda = w(\lambda + \rho_{\bar{1}}) - \rho_{\bar{1}}$$

for $w \in W_{\bar{1}}$ and λ in the $\mathbb{Q}$-span of $\Phi_{\bar{1}}$. This will be referred to as the *odd* dot action of $W_{\bar{1}}$.

One has

$$\rho_{\bar{1}} - \rho(J) = \frac{1}{2} \sum_{\gamma \in J'} \gamma - \frac{1}{2} \sum_{\alpha \in J} \alpha \tag{4.3.1}$$

where $J' = \Phi_{\bar{1}} - J$. Now $w \in W_{\bar{1}}$ permutes the set of odd roots $\Phi_{\bar{1}}$. Under the condition that $\Phi_{\bar{1}}^- = -\Phi_{\bar{1}}^+$, it follows that $w(\rho_{\bar{1}} - \rho(J)) = \rho_{\bar{1}} - \rho(J_1)$ where $J_1 \subseteq \Phi_{\bar{1}}^+$, and consequently

$$w \cdot (-\rho(J)) = -\rho(J_1) \tag{4.3.2}$$

Let $G_{\bar{0}}$ be a reductive algebraic group, Δ be the simple roots in $\Phi_{\bar{0}}^+$ and $W_{\bar{0}}$ be the Weyl group for the corresponding root system, $\Phi_{\bar{0}}$, for $G_{\bar{0}}$. Let $\rho^{\Phi_{\bar{0}}} := \rho_{\bar{0}} = \frac{1}{2} \sum_{\alpha \in \Phi_{\bar{0}}^+} \alpha$ and denote the *even* dot action by $w \circ \lambda = w(\lambda + \rho_{\bar{0}}) - \rho_{\bar{0}}$ where $w \in W_{\bar{0}}$ and $\lambda \in X(T_{\bar{0}})$.

Table 4.3.1 provides the relationship between $\rho_{\bar{1}}$ and $\rho_{\bar{0}}$. Observe that in the case when $\mathfrak{g}$ is $A(n|n)$ or $\mathfrak{osp}(2n+1|2n)$, $G_{\bar{0}} \cong G_{\bar{0},(1)} \times G_{\bar{0},(2)}$. In these cases $\rho_{\bar{0}}$ is a sum $\rho_{\bar{0},(1)} + \rho_{\bar{0},(2)}$ where $\rho_{\bar{0},(j)}$ is the half sum of positive roots arising from $G_{\bar{0},(j)}$ where $j = 1, 2$.

Table 4.3.1
Sums of even and odd roots.

$\mathfrak{g}$	
$\mathfrak{q}(n)$	$\rho_{\bar{1}} = \rho_{\bar{0}}$
$\mathfrak{psq}(n)$	$\rho_{\bar{1}} = \rho_{\bar{0}}$
$A(n\|n)$	$\rho_{\bar{1}} = \rho_{\bar{0}} = \rho^{A_{n-1}}_{\bar{0},(1)} + \rho^{A_{n-1}}_{\bar{0},(2)}$
$\mathfrak{osp}(2n+1\|2n)$	$\rho_{\bar{1}} = \rho_{\bar{0}} = \rho^{B_n}_{\bar{0},(1)} + \rho^{C_n}_{\bar{0},(2)}$

A key idea to calculate $R^\bullet \mathrm{ind}_B^G \mathbb{C}$ entails connecting the even and odd dot actions on weights of $\Lambda^\bullet(\mathfrak{u}_{\bar{1}})$ as shown in the next example.

Example 4.3.1. Let $\mathfrak{g} = \mathfrak{q}(n)$ or $\mathfrak{psq}(n)$. There exists a $B_{\bar{0}}$-isomorphism $\Lambda^\bullet(\mathfrak{u}_{\bar{0}}^-) \cong \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$. Furthermore, $\rho_{\bar{1}} = \rho_{\bar{0}}$ and the even and odd dot actions coincide.

One can now directly apply [16, II 6.18, Proposition] to conclude that $R^n \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong \mathbb{C}^{\oplus t_n}$ where $t_n = |\{w \in \Sigma_n : \ l(w) = n\}|$. The contributions in this cohomology group are given by weights in $\Lambda^n((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$, so one can apply Proposition 4.1.3(b) to conclude that $p_{G,B}(t) = p_{W_{\bar{1}}}(t)$. This result generalizes the $\mathfrak{q}(2)$ example computed by Brundan [9, Lemma 4.4] for all $\mathfrak{q}(n)$, $n \geq 1$.

4.4. Combinatorics with odd roots

We will start by focusing on the cases when $\mathfrak{g}$ is of type $A(n|n)$ or $\mathfrak{osp}(2n+1|2n)$.

In this setting $G_{\bar{0}}$ is a product of two reductive algebraic groups which is unlike the case for type Q. The dot action of the group $W_{\bar{1}}$ on $\Phi_{\bar{1}}$ is more complicated in this setting, yet one still has a beautiful connection between $w \cdot 0$ with natural subsets of roots in $\Phi_{\bar{1}}^+$. We will consider the following set of even simple roots for $\mathfrak{g}_{\bar{0}}$ given in Table 4.4.1 (cf. [14, 12.1]).

Set $I = \{1, 2, \dots, n-1\}$ for $A(n|n)$ (resp. $I = \{1, 2, \dots, n\}$ for $\mathfrak{osp}(2n+1|2n)$). Let $s_{j,\bar{0},(1)}$ (resp. $s_{j,\bar{0},(2)}$) be the reflection corresponding to the jth root in $\Delta_{\bar{0},(1)}$ (resp. $\Delta_{\bar{0},(2)}$). For $j \in I$, set

$$s_j := s_{j,\bar{0},(1)} s_{j,\bar{0},(2)}.$$

Then s_j is a simple reflection in $W_{\bar{1}}$ and $W_{\bar{1}}$ is generated by $\{s_j : \ j \in I\}$.

Example 4.4.1. Let $\mathfrak{g} = A(n|n)$. Observe that $s_j(\epsilon_j - \delta_{j+1}) = \delta_{j+1} - \epsilon_j \in \Phi_{\bar{1}}^-$ and $s_j(\delta_j - \epsilon_{j+1}) = \epsilon_{j+1} - \delta_j \in \Phi_{\bar{1}}^-$. Furthermore,

$$s_j(\Phi_{\bar{1}}^+ - \{\epsilon_j - \delta_{j+1}, \delta_j - \epsilon_{j+1}\}) \subset \Phi_{\bar{1}}^+.$$

Table 4.4.1
Even simple roots.

$\mathfrak{g}$	$\bar{\Delta}_{\bar{0},(1)}$	$\bar{\Delta}_{\bar{0},(2)}$
$A(n\|n)$	$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n\}$	$\{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n\}$
$\mathfrak{osp}(2n+1\|2n)$	$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\}$	$\{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\}$

That is, the only roots in $\Phi^+_{\bar{1}}$ that are sent to $\Phi^-_{\bar{1}}$ are $\epsilon_j - \delta_{j+1}$ and $\delta_j - \epsilon_{j+1}$.

From direct computations, one can show that Example 4.4.1 extends to the other algebras listed in Table 4.4.1. There are two sets of roots in $\Phi^+_{\bar{1}}$, $\bar{\Delta}_{\bar{1},(1)} = \{\beta_j : j \in I\}$ and $\bar{\Delta}_{\bar{1},(2)} = \{\gamma_j : j \in I\}$ with the property that

- $s_j(\{\beta_j, \gamma_j\}) \subset \Phi^-_{\bar{1}}$ with $s_j(\beta_j) = -\beta_j$ and $s_j(\gamma_j) = -\gamma_j$
- $s_j(\Phi^+_{\bar{1}} - \{\beta_j, \gamma_j\}) \subset \Phi^+_{\bar{1}}$

The following table gives this correspondence.

$\mathfrak{g}$	$\bar{\Delta}_{\bar{1},(1)}$	$\bar{\Delta}_{\bar{1},(2)}$
$A(n\|n)$	$\{\epsilon_1 - \delta_2, \dots, \epsilon_{n-1} - \delta_n\}$	$\{\delta_1 - \epsilon_2, \dots, \delta_{n-1} - \epsilon_n\}$
$\mathfrak{osp}(2n+1\|2n)$	$\{\epsilon_1 - \delta_2, \dots, \epsilon_{n-1} - \delta_n, \delta_n\}$	$\{\delta_1 - \epsilon_2, \dots, \delta_{n-1} - \epsilon_n, \epsilon_n + \delta_n\}$

For $w \in W_{\bar{1}}$ set

$$\Phi(w) = -(w\Phi^+_{\bar{1}} \cap \Phi^-_{\bar{1}}) = w\Phi^-_{\bar{1}} \cap \Phi^+_{\bar{1}} \subset \Phi^+_{\bar{1}}. \tag{4.4.1}$$

The following results establish some basic facts about $\Phi(w)$.

Proposition 4.4.2. *Let $w \in W_{\bar{1}}$.*

(a) $|\Phi(w)| = 2 \cdot l(w)$.
(b) $w \cdot 0 = -\rho(\Phi(w))$.
(c) *If $w = s_{j_1} \dots s_{j_t}$ is a reduced expression, then*

$$\Phi(w) = \{\beta_{j_1}, s_{j_1}\beta_{j_2}, s_{j_1}s_{j_2}\beta_{j_3}, \dots, s_{j_1}s_{j_2}\dots s_{j_{t-1}}\beta_{j_t}\}$$
$$\cup \{\gamma_{j_1}, s_{j_1}\gamma_{j_2}, s_{j_1}s_{j_2}\gamma_{j_3}, \dots, s_{j_1}s_{j_2}\dots s_{j_{t-1}}\gamma_{j_t}\}.$$

(d) *If* $w \cdot 0 = -\rho(J)$ *for some* $J \subset \Phi_{\bar{1}}^{+}$ *then* $J = \Phi(w)$.

Proof. (a) (b) and (c): One proves these statements using induction on $l(w)$. When $w = \mathrm{id}$ (i.e., $l(w) = 0$), these statements are clear. Now suppose that $w \in W_{\bar{1}}$ and $w = s_j w'$ where $l(w) = l(w') + 1$. One has $\beta_j \notin \Phi(w')$ and $\gamma_j \notin \Phi(w')$ due to the minimality of the expression $w = s_j w'$. Since s_j sends all roots in $\Phi_{\bar{1}}^{+}$ other than β_j and γ_j to $\Phi_{\bar{1}}^{+}$, one can express

$$\Phi(w) = s_j(\Phi(w')) \cup \{\beta_j, \gamma_j\}. \tag{4.4.2}$$

This is a disjoint union of sets. This proves (a) and (c).

For (b), one observes that

$$\begin{aligned} w \cdot 0 &= s_j \cdot (w' \cdot 0) = s_j \cdot (-\rho(\Phi(w'))) = -s_j\rho(\Phi(w')) + s_j \cdot 0 \\ &= -s_j\rho(\Phi(w')) - \beta_j - \gamma_j = -\rho(\Phi(w)) \end{aligned}$$

by using (4.4.2).

(d) We adapt the line of reasoning given in [28, Lemma 3.1.2(b)]. Statement (d) will be proved by induction on $l(w)$. If $w = \mathrm{id}$ or equivalently $l(w) = 0$ then $w \cdot 0 = 0$, so $J = \varnothing = \Phi(\mathrm{id})$.

Let $w \in W_{\bar{1}}$ with $l(w) > 0$. One can write $w = s_j w'$ with $l(w) = l(w') + 1$. We have $\beta_j, \gamma_j \in \Phi(w)$ and these elements are not in $\Phi(w')$.

Let $w \cdot 0 = -\rho(J)$ where $J = \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \in \Phi_{\bar{1}}^{+}$. Then

$$w' \cdot 0 = s_j \cdot (w \cdot 0) = s_j(w \cdot 0) + s_j\rho - \rho = -(s_j\sigma_1 + \cdots + s_j\sigma_m + \beta_j + \gamma_j).$$

There are two cases.

Case 1: $\sigma_i \neq \beta_j$ or γ_j for all i. Without loss of generality we may assume that $i = m$ when there is equality. In this case, each of the three sets (that we will denote by J')

- $\{s_j\sigma_1, \ldots, s_j\sigma_m, \beta_j, \gamma_j\}$
- $\{s_j\sigma_1, \ldots, s_j\sigma_{m-1}, \beta_j\}$
- $\{s_j\sigma_1, \ldots, s_j\sigma_{m-1}, \gamma_j\}$

yields distinct elements in $\Phi_{\bar{1}}^{+}$ whose sum equals $-w' \cdot 0$. Now by induction, $-\rho(J') = \Phi(w')$. This is a contradiction because $\beta_j \notin \Phi(w')$ and $\gamma_j \notin \Phi(w')$.

Case 2: $\sigma_i = \beta_j$ and $\sigma_k = \gamma_j$ for some i, k. We may assume that $i = m$ and $k = m-1$, so $w' \cdot 0 = -(s_j\sigma_1 + \cdots + s_j\sigma_{m-2})$. By induction, $\Phi(w') = \{s_j\sigma_1, \ldots, s_j\sigma_{m-2}\}$. Consequently, $\Phi(w) = \Phi(w') \cup \{\beta_j, \gamma_j\} = \{\sigma_1, \ldots, \sigma_m\}$. □

4.5. In this section we compute $p_{G,B}(t)$ for the algebras listed in Table 4.4.1. This will be accomplished in a series of steps. Recall that $G_{\bar{0}} = G_{\bar{0},(1)} \times G_{\bar{0},(2)}$. It will be convenient to view a weight of $G_{\bar{0},(1)} \times G_{\bar{0},(2)}$ as a pair (σ_1, σ_2) that is expressed as $\sigma_1 + \sigma_2$ when considered as a weight of $\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$.

When $\rho_{\bar{0}} = \rho_{\bar{1}}$, the even dot action is compatible with the odd action for $w \in W_{\bar{1}}$. That is,

$$(w, w) \circ \mu = w \cdot \mu$$

for $w \in W_{\bar{1}}$ and for any weight μ (i.e., in the span of the ϵ's and δ's). This observation about the compatibility of the actions is central to making the computations in the paper.

(1) *If* $\sigma = (\sigma_1, \sigma_2)$ *is a weight of* $\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ *and* $R^n \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}(\sigma_1, \sigma_2) \neq 0$ *then* $\sigma = (w_1 \circ 0, w_2 \circ 0)$ *where* $w_1, w_2 \in W_{\bar{1}}$. Here $\circ$ denotes the dot action of the Weyl group of $G_{\bar{0},(1)} \times G_{\bar{0},(2)}$.

First consider $\mathfrak{g} = A(n|n)$. Then $W_{\bar{1}}$ can be identified as the diagonal embedding of $\Delta : \Sigma_m \hookrightarrow \Sigma_m \times \Sigma_m$ with $w \in W_{\bar{1}}$ represented as (w, w). There exists $\lambda_j \in X(T_{\bar{0},(j)})_+$ and $w_j \in \Sigma_m$ for $j = 1, 2$ such that $\sigma_1 = w_1 \circ \lambda_1$ and $\sigma_2 = w_2 \circ \lambda_2$. We have

$$w_{\bar{1}} \circ \lambda_1 + w_2 \circ \lambda_2 = -\rho(J) \tag{4.5.1}$$

for some $J \subseteq \Phi_{\bar{1}}^+$. Since $\rho_{\bar{0}} = \rho_{\bar{0},(1)} + \rho_{\bar{0},(2)} = \rho_{\bar{1}}$, applying $(w_1^{-1}, w_1^{-1}) \in W_{\bar{1}}$ to this equation yields

$$\lambda_1 + (w_1^{-1} w_2) \circ \lambda_2 = w_1^{-1} \cdot (-\rho(J)) = -\rho(J_1) \tag{4.5.2}$$

for some $J_1 \subseteq \Phi_{\bar{1}}^+$.

We claim that the dominance condition on λ_1 forces $\lambda_1 = 0$. One has

$$-\rho(J_1) = \sum_{i>j} m_{i,j}(\epsilon_i - \delta_j) + \sum_{i>j} n_{i,j}(\delta_i - \epsilon_j) \tag{4.5.3}$$

with $1 \leq i, j \leq n$ and $m_{i,j}, n_{i,j} \geq 0$. In (4.5.2), the term $(w_1^{-1} w_2) \circ \lambda_2$ only involves δ_j's. The term involving ϵ_1 in (4.5.3) is less than or equal to zero, whereas the term involving ϵ_n is greater than or equal to zero. Since λ_1 is dominant it follows that $\lambda_1 = 0$. Therefore, $w_1 \circ 0 + w_2 \circ \lambda_2 = -\rho(J)$. Apply w_2^{-1} to both sides and repeat the argument above to get that $\lambda_2 = 0$.

Next consider $\mathfrak{g} = \mathfrak{osp}(2n+1|2n)$. Then $W_{\bar{1}} = \Delta(\Sigma_m \ltimes (\mathbb{Z}_2)^m)$. Given (4.5.1), one can use the same line of reasoning as in the preceding paragraph with a few modifications. One needs to add the additional term to (4.5.3): $\sum_{i,j} q_{i,j}(\epsilon_i + \delta_j) + \sum_j r_j \delta_j$ with $q_{i,j}, r_j \leq 0$. The dominance condition for $\mathfrak{so}(2n+1)$ (resp. $\mathfrak{sp}(2n)$) entails that the coefficient involving ϵ_n (resp. δ_n) is greater than or equal to zero. This allows us to show that $\lambda_1 = 0$ and $\lambda_2 = 0$.

(2) *If* $R^n \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}(w_1 \circ 0, w_2 \circ 0) \neq 0$ *for* $w_1, w_2 \in W_{\bar{1}}$ *then* $w_1 = w_2$.

Suppose that $w_1 \circ 0 + w_2 \circ 0 = -\rho(J)$ for some $J \subseteq \Phi_{\bar{1}}^{+}$. Then $(w_1^{-1}w_2) \circ 0 = -\rho(J_1)$ for some $J_1 \subseteq \Phi_{\bar{1}}^{+}$. Set $w = w_1^{-1}w_2$, and note that $-\rho(J_1)$ consists of a negative sum of roots for $G_{\bar{0},(2)}$ (i.e., roots involving δ's).

Let $l(w) > 0$ and $w = s_{i,\bar{0},(2)}w'$ be a reduced expression. Then

$$s_{i,\bar{0},(1)} \circ 0 + w' \circ 0 = s_i \cdot (-\rho(J_1)) = s_i(-\rho(J_1)) + s_i \cdot 0.$$

Now $s_{i,\bar{0},(1)} \cdot 0 = -(\epsilon_i - \epsilon_{i+1})$ and $s_i \cdot 0 = -(\epsilon_i - \epsilon_{i+1}) - (\delta_i - \delta_{i+1})$. Set $\bar{\alpha} = \delta_i - \delta_{i+1}$. It follows that

$$\begin{aligned} w' \cdot 0 &= s_i(-\rho(J_1)) - \bar{\alpha} \\ &= -\rho(J_1) + \langle -\rho(J_1), \bar{\alpha}^{\vee} \rangle \bar{\alpha} - \bar{\alpha} \\ &= w \cdot 0 + [\langle -\rho(J_1), \bar{\alpha}^{\vee} \rangle - 1]\bar{\alpha}. \end{aligned}$$

Therefore,

$$w' \cdot 0 - w \cdot 0 = -[\langle -\rho(J_1), \bar{\alpha}^{\vee} \rangle + 1]\bar{\alpha}. \tag{4.5.4}$$

From the explicit descriptions of the negative roots summing to $w' \cdot 0$ and $w \cdot 0$, one can conclude that $w' \cdot 0 - w \cdot 0 = -(w')^{-1}(\bar{\alpha})$. The equation (4.5.4) shows that $(w')^{-1}(\bar{\alpha}) = \bar{\alpha}$ and $\langle -\rho(J_1), \bar{\alpha}^{\vee} \rangle = 0$. Therefore,

$$0 = \langle -\rho(J_1), \bar{\alpha}^{\vee} \rangle = 0 = \langle w \cdot 0, \bar{\alpha}^{\vee} \rangle = \langle w\rho_{\bar{0}}^{(2)} - \rho_{\bar{0}}^{(2)}, \bar{\alpha}^{\vee} \rangle.$$

Consequently, $\langle w\rho_{\bar{0}}^{(2)}, \bar{\alpha}^{\vee} \rangle = 1$. On the other hand,

$$\langle w\rho_{\bar{0}}^{(2)}, \bar{\alpha}^{\vee} \rangle = \langle s_{i,\bar{0},(2)} w' \rho_{\bar{0}}^{(2)}, \bar{\alpha}^{\vee} \rangle = \langle w' \rho_{\bar{0}}^{(2)}, -\bar{\alpha}^{\vee} \rangle = \langle \rho_{\bar{0}}^{(2)}, -\bar{\alpha}^{\vee} \rangle = -1.$$

This is a contradiction, so $l(w) = 0$ and $w_1 = w_2$.

(3) $\dim \Lambda^{\bullet}((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)_{(w \circ 0, w \circ 0)} = 1$ *for all* $w \in W_{\bar{1}}$.

The statement (3) follows from Proposition 4.4.2.

Let $n \geq 0$. According to the Künneth formula

$$R^n \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} (w \circ 0, w \circ 0) = \bigoplus_{n_1 + n_2 = n} R^{n_1} \operatorname{ind}_{B_{\bar{0},(1)}}^{G_{\bar{0},(1)}} w \circ 0 \boxtimes R^{n_2} \operatorname{ind}_{B_{\bar{0},(2)}}^{G_{\bar{0},(2)}} w \circ 0. \tag{4.5.5}$$

This shows that

$$R^n \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}(w \cdot 0, w \cdot 0) \cong \begin{cases} \mathbb{C} & n = 2l(w) \\ 0 & \text{otherwise.} \end{cases}$$

From (1), (2) and (3), one can conclude that $R^n \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong \mathbb{C}^{\oplus t_n}$ where $t_n = |\{w \in W_{\bar{1}} : \frac{n}{2} = l(w)\}|$ for n even and zero when n is odd. Consequently, by Proposition 4.1.3(a), one has that $p_{G,B}(t) = p_{W_{\bar{1}}}(t^2)$.

4.6. Computing Poincaré series via spectral sequences

Let P be a parabolic subgroup such that $B \subseteq P \subseteq G$. The following result enables one to compute $p_{G,B}(t)$ from $p_{G,P}(t)$ and $p_{P,B}(t)$.

Proposition 4.6.1. *Let P be a parabolic subgroup such that $B \subseteq P \subseteq G$. Suppose that*

(a) $R^{2\bullet} \mathrm{ind}_B^P \mathbb{C} \cong \mathbb{C}^{\oplus s}$ *and* $R^{2\bullet+1} \mathrm{ind}_B^P \mathbb{C} = 0$;
(b) $R^{2\bullet+1} \mathrm{ind}_P^G \mathbb{C} = 0$.

Then $p_{G,B}(t) = p_{G,P}(t) \cdot p_{P,B}(t)$.

Proof. There exists a first quadrant spectral sequence

$$E_2^{i,j} = R^i \mathrm{ind}_P^G \, R^j \mathrm{ind}_B^P \mathbb{C} \Rightarrow R^{i+j} \mathrm{ind}_B^G \mathbb{C}.$$

From (a), since the P-modules, $R^\bullet \mathrm{ind}_B^P \mathbb{C}$ are either 0 or a direct sum of trivial modules, one can regard these modules as G-modules (i.e., the P-module structure lifts to G). Therefore, by the tensor identity, the E_2-page can be expressed as a tensor product

$$E_2^{i,j} = R^i \mathrm{ind}_P^G \mathbb{C} \otimes R^j \mathrm{ind}_B^P \mathbb{C}.$$

According to (b), $E_2^{i,j} = 0$ has non-zero terms only if i and j are both even. The differentials in the spectral sequence have bidegree $(r, 1-r)$. Therefore, the spectral sequence must collapse and yields $R^\bullet \mathrm{ind}_B^G \mathbb{C} \cong R^\bullet \mathrm{ind}_P^G \mathbb{C} \otimes R^\bullet \mathrm{ind}_B^P \mathbb{C}$. This proves the statement of the proposition. $\square$

4.7. $\mathfrak{g} = \mathfrak{osp}(2n|2n)$ for $n \geq 1$

We begin by comparing the even and odd roots for $\mathfrak{g} = \mathfrak{osp}(2n|2n)$ through the information below. (See Table 4.7.1.)

Table 4.7.1
Even simple roots.

$\mathfrak{g}$	$\bar{\Delta}_{\bar{0},(1)}$	$\bar{\Delta}_{\bar{0},(2)}$
$\mathfrak{osp}(2n\|2n)$	$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}$	$\{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\}$

In the case when $\mathfrak{g} = \mathfrak{osp}(2n|2n)$ one has $\rho_{\bar{0}} \neq \rho_{\bar{1}}$. Instead,

$$\rho_{\bar{1}} = \rho^{D_n} + 2[\epsilon_1 + \cdots + \epsilon_n] + \rho^{C_n}$$

This necessitates the use of different techniques than the ones used for $A(n|n)$ and $\mathfrak{osp}(2n+1|2n)$ (when $\rho_{\bar{0}} = \rho_{\bar{1}}$). The root system for type D_n embeds in the root system for type C_n and one also has the relationship $\rho^{C_n} = \rho^{D_n} + [\epsilon_1 + \cdots + \epsilon_n]$.

Consider the subalgebra $\mathfrak{gl}(n|n)$ in $\mathfrak{osp}(2n|2n)$ and the parabolic subalgebra generated by $\mathfrak{gl}(n|n)$ and the root vectors of weights $\{-\epsilon_i - \delta_j : \ 1 \leq i, j \leq n\}$, and let P be the corresponding parabolic subgroup scheme. From our prior section, $R^{\bullet}\mathrm{ind}_B^P \mathbb{C}$ is isomorphic to a direct sum of trivial modules and

$$\sum_{j=0}^{\infty} \dim R^j \mathrm{ind}_B^P \mathbb{C}\ t^j = p_{\Sigma_n}(t^2).$$

It suffices to show that

$$\sum_{i=0}^{\infty} \dim R^i \mathrm{ind}_P^G \mathbb{C}\ t^i = p_{W_{\bar{1}}/\Sigma_n}(t^2) = (1+t^2)(1+t^4)\dots(1+t^{2n}) \tag{4.7.1}$$

If this holds, then by Proposition 4.6.1, $p_{G,B}(t) = p_{W_{\bar{1}}}(t^2)$.

For the case $\mathfrak{g} = \mathfrak{osp}(2n|2n)$, one has $W_{\bar{1}} \cong \Sigma_n \ltimes (\mathbb{Z}_2)^{n-1}$. First observe that

$$R^n \mathrm{ind}_P^G \mathbb{C}|_{G_{\bar{0}}} \cong R^n \mathrm{ind}_{P_{\bar{0}}}^{G_{\bar{0}}} \Lambda^{\bullet}((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*) \cong R^n \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^{\bullet}((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*).$$

The weights of $(\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*$ are the roots $-\Phi_{\mathfrak{p}}^+ := \{-\epsilon_i - \delta_j : \ 1 \leq i, j \leq n\}$. Suppose that $w_1 \circ \lambda + w_2 \circ \mu = -\rho(J)$ where $J \subseteq \Phi_{\mathfrak{p}}^+$. Using the argument in Section 4.5(1), we can deduce that

$$1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \tag{4.7.2}$$

and

$$1 \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n \geq 0. \tag{4.7.3}$$

From (4.7.3), $\mu = \delta_1 + \delta_2 + \cdots + \delta_s$. Then

$$w_2 \circ \mu = w_2(\mu + \rho^{C_n}) - \rho^{C_n}.$$

If $s \geq 1$, then the first term in $\mu + \rho^{C_n}$ is $(n+1)\delta_1$ and must be sign changed to obtain a summand involving $-\delta_j$'s in $-\rho(J)$. However, if this term is sign changed to $-(n+1)$ and possibly permuted, then the corresponding term in $w_2 \circ \mu$ is at most $-n-2$ which less than $-n$. This leads to a contradiction, thus $s = 0$ and $\mu = 0$. Consider $w_2 \circ 0 = w_2(\rho^{C_n}) - \rho^{C_n}$ with $\rho^{C_n} = n\delta_1 + (n-1)\delta_2 + \cdots + \delta_n = (n, n-1, \ldots, 1)$. Using (4.7.4) shows that w_2 must fix $n\delta_1$. This proves that $w_1 \circ \lambda + w_2 \circ 0 = -\rho(J)$ where $J \subseteq \{-\epsilon_i - \delta_j : \ 1 \leq i \leq n, \ 2 \leq j \leq n\}$.

Next from (4.7.2), $\lambda = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_s \pm \epsilon_n$ and consider $w_1 \circ \lambda = w_1(\lambda + \rho^{D_n}) - \rho^{D_n}$. Now if $s \geq 1$ then the term $n\epsilon_1$ in $\lambda + \rho^{D_n}$ must change sign so there is an ϵ-coefficient in $w_1 \circ \lambda$ less than or equal to $-n$. However, from the preceding paragraph, in $-\rho(J)$ the coefficient of ϵ_i is greater than $-(n-1)$. Therefore, $s = 0$ and by the dominance condition, $\lambda = 0$.

We will prove (4.7.1) by induction n. Assume for $n-1$,

$$\sum_{i=0}^{\infty} \dim R^i \mathrm{ind}_P^G \mathbb{C} \ t^i = (1+t^2)(1+t^4)\ldots(1+t^{2(n-1)})$$

given via $2^{(n-1)}$ solutions of $w_1 \circ 0 + w_2 \circ 0 = -\rho(J)$, $J \subseteq \{\epsilon_i - \delta_j : \ 2 \leq i \leq n, \ 2 \leq j \leq n\}$ with $l(w_1) = l(w_2)$. For $\mathfrak{osp}(2|2)$ (i.e., $n-1=1$), this can be verified directly.

Suppose for $\mathfrak{g} = \mathfrak{osp}(2n|2n)$, one has

$$w_1 \circ 0 + w_2 \circ 0 = -\rho(J) \tag{4.7.4}$$

with $J \subseteq \{\epsilon_i - \delta_j : \ 1 \leq i \leq n, \ 1 \leq j \leq n\}$. First consider $w_2 \circ 0 = w_2(\rho^{C_n}) - \rho^{C_n}$ with $\rho^{C_n} = n\delta_1 + (n-1)\delta_2 + \cdots + \delta_n = (n, n-1, \ldots, 1)$. From (4.7.4) one can deduce that w_2 must fix $n\delta_1$ and either (i) fix $(n-1)\delta_2$ or (ii) permute and sign change $(n-1)\delta_2$ to $-(n-1)\delta_n$. In the first case (i),

$$w_2 \circ 0 = (0, 0, *, *, \ldots, *).$$

In the case (i) consider $w_1 \circ 0$ where $\rho^{D_n} = (n-1)n\epsilon_1 + (n-2)\delta_2 + \cdots + \epsilon_{n-1} = (n-1, n-2, \ldots, 1, 0)$. Under w_1, either $(n-1)$ is fixed or gets sign changed to $-(n-1)$ and is permuted in the ϵ_n-position. The latter is not possible because $w_2 \circ 0 = (0, 0, *, *, \ldots, *)$ (i.e., only $\epsilon_n + \delta_j$ can occur for $j = 3, \ldots, n$). Hence, in case (i), $w_1 \circ 0 = (0, *, *, \ldots, *)$. The conclusion is that in case (i), we are reduced to the $2^{(n-1)}$ solutions of $w_1 \circ 0 + w_2 \circ 0 = -\rho(J)$ in $\mathfrak{osp}(2(n-1)|2(n-1))$.

Next we handle case (ii). We can reduce to the solutions of (4.7.4) in $\mathfrak{osp}(2(n-1)|2(n-1))$ by multiplying by an element with length $2n$. In case (ii), $w_2 \circ 0 = (0, b_1, b_2, \ldots, b_{n-2}, -n)$ and $w_1 \circ 0 = (a_1, a_2, \ldots, a_{n-1}, -(n-1))$ with

$$(a_1, a_2, \ldots, a_{n-1}, -(n-1)) + (0, b_1, b_2, \ldots, b_{n-2}, -n) = -\rho(J).$$

Note that $a_i, b_j \leq 0$.

Set $\tau_1 = s_{\epsilon_n} s_{n-1,\bar{0},(1)} s_{n-2,0,(1)} \ldots s_{1,\bar{0},(1)}$ where $s_{\epsilon_n}(\epsilon_j) = \epsilon_j$ for $j = 1, 2, \ldots, n-1$ and $s_{\epsilon_n}(\epsilon_n) = -\epsilon_n$. Even though τ_1 is not in the Weyl group for type D_n, it can be shown that $\tau_1^{-1} w_1 \circ 0 = w_1' \circ 0$ for some $w_1' \in \Sigma_n \ltimes (\mathbb{Z}_2)^{n-1}$. By direct computation,

$$\tau_1^{-1} w_1 \circ 0 = (0, 1 + a_1, 1 + a_2, \ldots, 1 + a_{n-1}).$$

On the other hand, let $\tau_2 = s_{n,\bar{0},(2)} s_{n-1,\bar{0},(2)} \ldots s_{2,\bar{0},(2)} \in \Sigma_n \ltimes (\mathbb{Z}_2)^n$. One can verify that

$$\tau_2^{-1} w_2 \circ 0 = (0, 0, 1 + b_1, 1 + b_2, \ldots, 1 + b_{n-2}).$$

Next we need to show that

$$\begin{aligned}\tau_1^{-1} w_1 \circ 0 + \tau_2^{-1} w_2 \circ 0 &= (0, 1 + a_1, 1 + a_2, \ldots, 1 + a_{n-1}) \\ &\quad + (0, 0, 1 + b_1, 1 + b_2, \ldots, 1 + b_{n-2}) = -\rho(J_2),\end{aligned}$$

for some $J_2 \subseteq \{\epsilon_i - \delta_j : \ 2 \leq i \leq n, \ 2 \leq j \leq n\}$. From our assumption,

$$(a_1, a_2, \ldots, a_{n-1}, -(n-1)) + (0, b_1, b_2, \ldots, b_{n-2}, -n) = -\rho(J).$$

This implies that $\{\epsilon_i + \delta_n : \ 1 \leq i \leq n\} \cup \{\epsilon_n + \delta_j : \ 2 \leq j \leq n-1\} \subseteq J$, and

$$(a_1 + 1, a_2 + 1, \ldots, a_{n-1} + 1, 0) + (0, b_1 + 1, b_2 + 1, \ldots, b_{n-2} + 1, 0) = -\rho(J_1),$$

for some $J_1 \subseteq \{\epsilon_i - \delta_j : \ 1 \leq i \leq n-1, \ 1 \leq j \leq n-1\}$. This claim now follows by applying a permutation of the coordinates. We can now conclude by the induction hypothesis that

$$\tau_1^{-1} w_1 \circ 0 + \tau_2^{-1} w_2 \circ 0 = \tilde{w}_1 \circ 0 + \tilde{w}_2 \circ 0 = -\rho(J_2),$$

for unique $\tilde{w}_1, \tilde{w}_2 \in \Sigma_{n-1} \ltimes (\mathbb{Z}_2)^{n-2}$. Moreover, one can verify that $l(\tau_j \tilde{w}_j) = n + l(\tilde{w}_j)$ for $j = 1, 2$. Consequently, in case (ii), we are reduced to the $2^{(n-1)}$ solutions of $w_1 \circ 0 + w_2 \circ 0 = -\rho(J)$ in $\mathfrak{osp}(2(n-1)|2(n-1))$ by multiplying by $\tau^{-1} = \tau_1^{-1}\tau_2^{-1}$ whose total length is $2n$. This proves (4.7.1).

4.8. $\mathfrak{g} = \mathfrak{osp}(2(n+1)|2n)$ *for* $n \geq 1$

Consider the embedding $\mathfrak{osp}(2n|2n) \hookrightarrow \mathfrak{osp}(2(n+1)|2n)$, and let $\mathfrak{p}$ be the parabolic subalgebra generated by $\mathfrak{osp}(2n|2n)$ and the root vectors with weights of the form $-\epsilon_1 \pm \delta_j$ for $j = 1, 2, \ldots, n$. Let P be the parabolic subgroup scheme with Lie $P = \mathfrak{p}$. One has $B \subseteq P \subseteq G$.

Table 4.9.1
Embeddings.

$\mathfrak{g}'$	$\mathfrak{g}$	$\Phi^+_{\bar{1},P}$ $[n \leq m-1]$
$A(n\|m-1)$	$A(n\|m)$	$\{\epsilon_i - \delta_m : 1 \leq i \leq n\}$
$\mathfrak{osp}(2n+1\|2(m-1))$	$\mathfrak{osp}(2n+1\|2m)$	$\{-\epsilon_i + \delta_1 : 2 \leq i \leq n\} \cup \{\epsilon_i + \delta_1 : 1 \leq i \leq n\} \cup \{\delta_1\}$
$\mathfrak{osp}(2(m-1)+1\|2n)$	$\mathfrak{osp}(2m+1\|2n)$	$\{\epsilon_1 - \delta_i : 1 \leq i \leq n\} \cup \{\epsilon_1 + \delta_i : 1 \leq i \leq n\}$
$\mathfrak{osp}(2n\|2(m-1))$	$\mathfrak{osp}(2n\|2m)$	$\{-\epsilon_i + \delta_1 : 1 \leq i \leq n\} \cup \{\epsilon_i + \delta_1 : 1 \leq i \leq n\}$
$\mathfrak{osp}(2(m-1)\|2n)$	$\mathfrak{osp}(2m\|2n)$	$\{\epsilon_1 - \delta_i : 2 \leq i \leq n\} \cup \{\epsilon_1 + \delta_i : 1 \leq i \leq n\}$

In this case, we have $W_{\bar{1}} \cong \Sigma_n \ltimes (\mathbb{Z}_2)^n$ for $\mathfrak{g} = \mathfrak{osp}(2(n+1)|2n)$. In order to show that $p_{G,B}(t) = p_{W_{\bar{1}}}(t^2)$, we use Proposition 4.6.1 to reduce our computation to proving that $p_{G,P}(t) = 1 + t^{2n}$. Here we are using information about the Poincaré series for $\mathfrak{osp}(2n|2n)$.

The weights of $(\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*$ are $-\epsilon_1 \pm \delta_j$ for $j = 1, 2, \dots, n$. Suppose we have a weight of the form $w_1 \circ \lambda + w_2 \circ \mu = -\rho(J)$ where $J \subseteq \{-\epsilon_1 \pm \delta_j : j = 1, 2, \dots, n\}$. Then $w_1 \circ \lambda = -k\epsilon_1 = -k\omega_1$. Therefore, $\langle w_1 \circ \lambda, \alpha^\vee \rangle = 0$ for $\alpha \in \Delta_{\bar{0},(1)}$. It follows that $\langle \lambda + \rho_{\bar{0}}^{(1)}, w_1^{-1}\alpha^\vee \rangle = 1$. Consequently, $w_1^{-1}\alpha \geq 0$ and $w_1^{-1}\alpha \in \Delta_{\bar{0},(1)}$. We have $w_1^{-1}\{\alpha_2, \dots, \alpha_{n+1}\} \subseteq \{\alpha_2, \dots, \alpha_{n+1}\}$, thus $\langle \lambda, \alpha_j \rangle = 0$ for $j = 2, 3, \dots, n+1$.

Now consider $w_1 \circ s\omega_1 = -k\omega_1$ or equivalently,

$$w_1 \circ s\epsilon_1 = -k\epsilon_1.$$

A direct computation using the dot action for D_{n+1} shows that there are two solutions: (i) $w_1 = 1$, $s = 0$, $k = 0$ and (ii) $s = 0$, $k = 2n$ and $l(w_1) = 2n$. This proves the assertion.

4.9. We now extend our computation for $p_{G,B}(t)$ when $\mathfrak{g} = A(p|q)$ and when $\mathfrak{g} = \mathfrak{osp}(p|q)$. We consider the following embeddings of Lie superalgebras $\mathfrak{g}' \subseteq \mathfrak{g}$ and set of positive roots $\Phi^+_{\bar{1},P}$. Set $n \leq m-1$.

Let $\mathfrak{p}$ be the subalgebra generated by $\mathfrak{b}$ and $\mathfrak{g}'$ and P be the corresponding parabolic subgroup scheme with $\mathfrak{p} = \operatorname{Lie} P$.

Theorem 4.9.1. *Let $\mathfrak{g}' = \operatorname{Lie} G'$ and $\mathfrak{g} = \operatorname{Lie} G$ be as in Table 4.9.1. Then $p_{G,B}(t) = p_{G',B'}(t)$.*

Proof. One has $B \subseteq P \subseteq G$. We prove the theorem by induction on m. One has $R^\bullet \operatorname{ind}_B^P \mathbb{C}$ as a G'-module identifies with $R^\bullet \operatorname{ind}_{B'}^{G'} \mathbb{C}$ (cf. [16, I. 6.14(1)]). The cohomology in odd degree vanishes and cohomology in even degree is isomorphic to a direct sum of trivial modules. Therefore, by Proposition 4.6.1 it suffices to show that $p_{G,P}(t) = 1$ to prove that $p_{G,B}(t) = p_{G',B'}(t)$.

One has

$$R^j \operatorname{ind}_P^G \mathbb{C}|_{G_{\bar{0}}} \cong R^j \operatorname{ind}_{P_{\bar{0}}}^{G_{\bar{0}}} \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*) \cong R^j \operatorname{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*).$$

The last isomorphism follows by using the spectral sequence relating the composition of induction functors [16, I. 4.5(c) Proposition] and the fact that $R^t\mathrm{ind}_{B_{\bar{0}}}^{P_{\bar{0}}}\mathbb{C} = 0$ for $t > 0$.

The weights of $\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}}$ coincide with $\Phi^+_{\bar{1},P}$. Let σ be a weight of $\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}})^*)$ with $R^j\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\sigma \neq 0$ for some $j \geq 0$.

If $\mathfrak{g} = A(n|m)$ then by the argument given in Section 4.5(1), $\sigma = w_1 \circ \lambda_1 + w_2 \circ 0$. It follows that $w_2 \circ 0 = c\ \delta_m$ where $c \geq 0$. This can only happen if $c = 0$, which implies that $\sigma = 0$. This proves the statement of the theorem for $A(n|m)$.

In the other cases, the arguments given in Sections 4.5(1), and 4.7 show that $\sigma = w_1 \circ 0 + w_2 \circ 0$. Consider the second case in Table 4.9.1. Then $w_2 \circ 0 = -c\delta_1 = -c\omega_1$. In the root system C_m this means that $c = 0$ or $c = 2m$. However, $c \leq 2n \leq 2(m-1) < 2m$ which implies that $c = 0$, thus $\sigma = 0$. The other three cases in the table are handled with a similar argument. $\square$

4.10. The computation of $R^\bullet\mathrm{ind}_B^G\mathbb{C}$ and $p_{G,B}(t)$

The following theorem relates the sheaf theoretic Poincaré polynomial with the Poincaré polynomial for $W_{\bar{1}}$ when $\mathfrak{g}$ is not of type P.

Theorem 4.10.1. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra with $\mathfrak{g} = \mathrm{Lie}\,G$. Assume that $\mathfrak{g}$ is not isomorphic to $P(n)$. Let B be the parabolic subgroup such that $\mathfrak{b} = \mathrm{Lie}\,B$ where $\mathfrak{b}$ is the parabolic subalgebra defined in Table 7.1.3. Then*

(a) *$R^\bullet\,\mathrm{ind}_B^G\,\mathbb{C}$ is a direct sum of trivial modules.*
(b) *The number of trivial modules in $R^n\,\mathrm{ind}_B^G\,\mathbb{C}$ is given by*

$$p_{G,B}(t) = z_{\mathfrak{b},\mathfrak{g}}(t) = p_{W_{\bar{1}}}(s)$$

where s is the parameter defined in Table 7.2.1.

Proof. Parts (a) and (b) were proved for the various classical Lie superalgebras in the following way. First, it was established that $R^\bullet\,\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\,\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ is a direct sum of trivial modules. Now from Propositions 4.1.1 and 4.1.3, it follows that $R^\bullet\,\mathrm{ind}_B^G\,\mathbb{C}$ is a direct sum of trivial modules. Part (b) was verified along the way via the calculation of $R^\bullet\,\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\,\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$.

For the statements of (a) and (b) the cases when $\mathfrak{g} = D(2,1,\alpha)$, $G(3)$ and $F(4)$ were proved in Theorem 4.2.1. For type Q the statements was verified in Example 4.3.1, and for the type A families and orthosymplectic Lie superalgebras in Section 4.9. $\square$

4.11. The preceding theorem motivates the following definition.

Definition 4.11.1. Let G be an algebraic supergroup where $\mathfrak{g} = \mathrm{Lie}\,G$ is a classical simple Lie superalgebra and B be a parabolic subgroup with $\mathfrak{b} = \mathrm{Lie}\,B$ such that

(a) $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ where $\mathfrak{b}_{\bar{1}} \cong \mathfrak{f}_{\bar{1}} \oplus \mathfrak{u}_{\bar{1}}$ where $\mathfrak{b}_{\bar{0}}$ is a Borel subalgebra for $\mathfrak{g}_{\bar{0}}$.
(b) There exists a finite reflection group $W_{\bar{1}}$ such that as graded vector spaces,

$$\mathrm{H}^{\bullet}(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \otimes \mathbb{C}[W_{\bar{1}}]_{\bullet}.$$

Then $\mathfrak{b}$ is called an *BBW parabolic subalgebra* if and only if

$$p_{G,B}(t) = z_{\mathfrak{b},\mathfrak{g}}(t) = p_{W_{\bar{1}}}(s)$$

where $s = t^r$ for some $r \geq 1$.

5. Results for the Lie superalgebra $\mathfrak{p}(n)$

5.1. In this section we will present results for the Lie superalgebra $\mathfrak{p}(n)$ and explain how the theory differs from the other classical simple Lie superalgebras. Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{p}(n)$ where $n \geq 2$. This Lie superalgebra embeds into $\mathfrak{gl}(n|n)$ as $2n \times 2n$ matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array}\right), \tag{5.1.1}$$

where A, B and C are $n \times n$ matrices over $\mathbb{C}$ with $A \in \mathfrak{sl}_n(\mathbb{C})$, B symmetric, and C skew-symmetric.

Let V be the n-dimensional natural representation for $\mathfrak{sl}_n(\mathbb{C})$ with weights ϵ_j, $j = 1, 2, \ldots, n$. One has

$$\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}_n(\mathbb{C}) \text{ and } \mathfrak{g}_{\bar{1}} \cong S^2(V) \oplus \Lambda^2(V^*).$$

The weights of $\mathfrak{g}_{\bar{1}}$ are given by

$$\Phi_{\bar{1}} = \{\epsilon_i + \epsilon_j : \ 1 \leq i \leq j \leq n\} \cup \{-\epsilon_i - \epsilon_j : \ 1 \leq i < j \leq n\}.$$

The Lie superalgebra $\widetilde{\mathfrak{p}}(n)$, which is an enlargement of $\mathfrak{p}(n)$, is constructed by taking $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}_n(\mathbb{C})$.

5.2. Cohomology and Hilbert series

For the sake of convenience, we will redefine the detecting subalgebra $\mathfrak{f}$ as follows. The vector space $\mathfrak{f}_{\bar{1}}$ is the span of the root vectors in $\mathfrak{g}_{\bar{1}}$ whose weights are of the form

$$\Phi_{\mathfrak{f}_{\bar{1}}} = \begin{cases} \{\pm(\epsilon_{1+j} + \epsilon_{2l-j})\} & \text{for } j = 0, 1, \ldots, l-1, \ n = 2l \\ \{\pm(\epsilon_{1+j} + \epsilon_{2l+1-j}), 2\epsilon_{l+1}\} & \text{for } j = 0, 1, \ldots, l-1, \ n = 2l+1. \end{cases}$$

Set $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$ and $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$.

In both cases when n is even or odd, H is a torus of dimension l and $N/N_{\bar{0}} \cong \Sigma_l \ltimes (\mathbb{Z}_2)^l$. One can define a parabolic subalgebra $\mathfrak{b}$ as follows. We have

$$\Phi_{\bar{1}} = \{\epsilon_i + \epsilon_j : \ 1 \leq i, j \leq n\} \cup \{-\epsilon_i - \epsilon_j : \ 1 \leq i < j \leq n\}.$$

Set

$$\Phi_{\bar{1}}^- = \{\epsilon_i + \epsilon_j : \ n+1 < i+j\} \cup \{-\epsilon_i - \epsilon_j : \ i < j, \ i+j < n+1\}$$

and $\mathfrak{b}$ be the parabolic subalgebra generated by the root vectors with roots in $\Phi_{\bar{0}}^- \cup \Phi_{\mathfrak{f}_{\bar{1}}} \cup \Phi_{\bar{1}}^-$ and $\mathfrak{t}_{\bar{0}}$. The defining hyperplanes for the parabolic are given by

$$\mathcal{H} = \begin{cases} \sum_{i=1}^{l} x_i(E_i - E_{2l+1-i}), \ x_1 > x_2 > \cdots > x_l > 0 & n = 2l \\ \sum_{i=1}^{l} x_i(E_i - E_{2l+2-i}), \ x_1 > x_2 > \cdots > x_l > 0 & n = 2l+1. \end{cases}$$

The computation of $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) = S^\bullet(\mathfrak{f}_{\bar{1}}^*)^{T_{\bar{0}}}$ is given in Table 5.2.1.

Table 5.2.1
Cohomology and Hilbert series.

$\mathfrak{g}$	$W_{\bar{1}}$	$\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$
$\mathfrak{p}(n)$, $n = 2l$	$\Sigma_l \ltimes (\mathbb{Z}_2)^l$	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ly_l, x_1x_2 \ldots x_l, y_1y_2 \ldots y_l]$
$\mathfrak{p}(n)$, $n = 2l+1$	$\Sigma_l \ltimes (\mathbb{Z}_2)^l$	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ly_l, x_1^2x_2^2 \ldots x_l^2 x_{l+1}]$

The goal for the remainder of this section is to compute $z_{\mathfrak{b},\mathfrak{g}}(t)$ when n is even and when n is odd.

In the case when $n = 2l$ is even, set

$$S = \mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ly_l, x_1x_2 \ldots x_l, y_1y_2 \ldots y_l]$$

and

$$T = \mathbb{C}[f_1, f_2, \ldots, f_{l-1}, x_1x_2 \ldots x_l, y_1y_2 \ldots y_l]$$

where f_j is the jth symmetric polynomial in $\{x_1y_1, x_2y_2, \ldots, x_ly_l\}$. Then as in the case for $\mathfrak{sl}(l|l)$, S is free T-module of rank $|\Sigma_l|$. Furthermore, if $p_S(t)$ (resp. $p_T(t)$) are the Poincaré polynomials of S (resp. T) then

$$p_{\Sigma_l}(t^2) = p_S(t)/p_T(t) = p_{\mathfrak{b}}(t)/p_T(t). \tag{5.2.1}$$

Now we use the fact that T is a polynomial algebra generated in degrees $2, 4, \ldots, 2l-2$, l and l. Therefore,

$$p_{\mathfrak{b}}(t) = p_{\Sigma_l}(t^2) \cdot p_T(t) = \frac{(1-t^{2l})}{(1-t^2)^l(1-t^l)^2}. \tag{5.2.2}$$

From [4, Table 1] $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}},\mathbb{C})$ is a polynomial algebra generated in degrees $4, 8, \dots$, $4(l-1)$, l, and n. Consequently,

$$z_{\mathfrak{b},\mathfrak{g}}(t) = \frac{(1-t^4)(1-t^8)\dots(1-t^{4(l-1)})(1-t^n)(1+t^l)}{(1-t^2)^l} = p_{W'_{\bar{1}}}(t^2)p_{\mathbb{Z}_2}(t^l) \qquad (5.2.3)$$

where $W'_{\bar{1}} = \Sigma_l \ltimes (\mathbb{Z}_2)^{l-1}$.

In the case when $n = 2l+1$ is odd, $\mathrm{H}^\bullet(\mathfrak{b},\mathfrak{b}_{\bar{0}},\mathbb{C})$ is a polynomial algebra with l generators in degree 2 and one generator in degree $n = 2l+1$. On the other hand, $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}},\mathbb{C})$ is a polynomial algebra with generators in degrees $4, 8, \dots, 4l$ and n (cf. [4, Table 1]). Therefore, for n odd,

$$z_{\mathfrak{b},\mathfrak{g}}(t) = \frac{(1-t^4)(1-t^8)\dots(1-t^{4l})}{(1-t^2)^l} = p_{W_{\bar{1}}}(t^2). \qquad (5.2.4)$$

Note that after cancellation by the factors $(1-t^2)^l$ in (5.2.3) and (5.2.4), one obtains that

$$z_{\mathfrak{b},\mathfrak{g}}(1) = 2^l \cdot (l)! = |W_{\bar{1}}|.$$

5.3. $\mathfrak{p}(2)$ *and* $\mathfrak{p}(3)$

First let $\mathfrak{g} = \mathfrak{p}(2)$. Then $\Phi_{\bar{1}} = \{2\epsilon_2\} = \{-\alpha\}$ where α is the positive root in $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_2$. Therefore, one sees that

$$R^j\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^\bullet((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = \begin{cases} \mathbb{C} & j = 0, 1 \\ 0 & \text{else.} \end{cases}$$

It follows that

$$z_{\mathfrak{b},\mathfrak{g}}(t) = \frac{(1-t^2)(1+t)}{(1-t^2)} = 1 + t = p_{W_{\bar{1}}}(t) = p_{G,B}(t) \qquad (5.3.1)$$

and $\mathfrak{b}$ is a BBW parabolic subalgebra.

Next, let $\mathfrak{g} = \mathfrak{p}(3)$. It will be convenient to use the root basis and the fundamental weight basis for our calculations for $\Phi_{\bar{0}} = A_2$. One has

$$\Phi_{\bar{1}}^- = \{2\epsilon_3, \epsilon_2+\epsilon_3, -\epsilon_1-\epsilon_2\} = \{2\omega_2, -\omega_1, -\omega_2\}.$$

Now $-\omega_1, -\omega_2 \in \overline{C}_{\mathbb{Z}}$, and

$$s_{\alpha_2}\cdot(-2\omega_2) = -\omega_1 \in \overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+.$$

Therefore, $R^\bullet\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$. Similarly,

$$s_{\alpha_1} \cdot (\omega_2 - 3\omega_1) = \omega_1 - \omega_2 \in \overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+,$$

thus, $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^3((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$.

The weights of $\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ are $\{2\omega_2 - \omega_1, -3\omega_2, -\omega_1 - \omega_2\}$. The weight $-2\omega_2 - \omega_1$ is conjugate to $-\omega_2$ by $s_{\alpha_1}s_{\alpha_2}$, and $-\omega_1 - \omega_2 \in \overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+$. So these weights do not contribute to give any cohomology. On the other hand,

$$(s_{\alpha_2}s_{\alpha_1}) \cdot 0 = -3\omega_2. \tag{5.3.2}$$

Consequently, $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = \mathbb{C}$.

In summary, one has

$$z_{\mathfrak{b},\mathfrak{g}}(t) = 1 + t^2 = p_{W_{\bar{1}}}(t^2) = p_{G,B}(t) \tag{5.3.3}$$

and $\mathfrak{b}$ is again a BBW parabolic subalgebra.

5.4. $\mathfrak{p}(4)$

Next consider the Lie superalgebra $\mathfrak{g} = \mathfrak{p}(4)$. One has

$$\begin{aligned}\Phi_{\bar{1}}^- &= \{2\epsilon_4, 2\epsilon_3, \epsilon_3 + \epsilon_4, \epsilon_2 + \epsilon_4, -\epsilon_1 - \epsilon_2, -\epsilon_1 - \epsilon_2\} \\ &= \{-2\omega_3, -2\omega_2 + 2\omega_3, -\omega_2, -\omega_1 + \omega_2 - \omega_3, -\omega_2, -\omega_1 + \omega_2 - \omega_3\}.\end{aligned}$$

It is useful to express the elements in $\Phi_{\bar{1}}^-$ in terms of fundamental weights of $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_n$. Note that $-\omega_2$ and $-\omega_1 + \omega_2 - \omega_3$ occur with multiplicity two.

The weights of $\Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ are precisely the ones in $\Phi_{\bar{1}}^-$. All of these weights are conjugate to a weight in $\overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+$, thus $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$. Next observe that $\Lambda^6((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ is one-dimensional and spanned by a vector of weight $-2\rho_{\bar{0}}$. Therefore, $\Lambda^5((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong \Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)^* \otimes (-2\rho_{\bar{0}})$. Let $\mu = -\lambda - 2\rho_{\bar{0}}$ be a weight of $\Lambda^5((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ where λ is a weight of $\Lambda^1((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$. Then

$$w_0 \cdot \mu = w_0(-\lambda) = -w_0\lambda.$$

The possible weights of the form $w_0 \cdot \mu$ are $\{-2\omega_1, 2\omega_1 - 2\omega_2, -\omega_2, -\omega_1 + \omega_2 + \omega_3\}$ which are all conjugate to a weight in $\overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+$. Consequently, $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^5((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$.

The distinct weights of $\Lambda^3((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ are

$$\begin{aligned}&\{-3\omega_2, -\omega_1 - \omega_2 - \omega_3, -\omega_1 - 3\omega_3, -2\omega_2 - 2\omega_3, -2\omega_1 + 2\omega_2 - -4\omega_3, \\ &-\omega_1 - 2\omega_2 + \omega_1, -2\omega_1 - 2\omega_1 + \omega_2 - 2\omega_3\}.\end{aligned}$$

A lengthy verification shows that all of the weights above are conjugate to a weight in $\overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+$, thus $R^\bullet \mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} \Lambda^3((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$.

The distinct weights in $\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$ that are conjugate to a weight in $\overline{C}_{\mathbb{Z}} - X(T_{\bar{0}})_+$ are

$$\{-2\omega_2, -\omega_2 - 2\omega_3, -\omega_1 + \omega_2 - 3\omega_3, -\omega_1 - \omega_2 + \omega_3, -\omega_1 - \omega_3\}.$$

For the other two weights: $-3\omega_2+2\omega_3$ (multiplicity 2), and $-2\omega_1+2\omega_2-2\omega_3$ (multiplicity 1), one has

$$(s_{\alpha_1}s_{\alpha_2}) \cdot (-\omega_2 - 2\omega_3) = 0, \tag{5.4.1}$$

$$(s_{\alpha_1}s_{\alpha_3}) \cdot (-2\omega_1 + 2\omega_2 - 2\omega_3) = 0. \tag{5.4.2}$$

Consequently, $R^j\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$ for $j \neq 2$ and $R^2\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^2((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong \mathbb{C}^{\oplus 3}$. By using duality this also holds for $\Lambda^4((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*)$.

Finally, $w_0(-2\rho_{\bar{0}}) = 0$ and $l(w_0) = 6$, thus $R^j\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^6((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) = 0$ for $j \neq 6$ and $R^6\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}}\Lambda^6((\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})^*) \cong \mathbb{C}$. By gathering all this information, one can now conclude that

$$p_{G,B}(t) = 1 + 3t^2 + 3t^4 + t^6 = (1+t^2)^3 = z_{\mathfrak{b},\mathfrak{g}}(t).$$

For $\mathfrak{g} = \mathfrak{p}(4)$, one has $W_{\bar{1}} = \Sigma_2 \ltimes (\mathbb{Z}_2)^2$. The Poincaré polynomial

$$p_{W_{\bar{1}}}(t) = \frac{(1-t^2)(1-t^4)}{(1-t)^2} = (1+t)(1+t+t^2+t^3).$$

From this, it is clear that $p_{G,B}(t) \neq p_{W_{\bar{1}}}(t^r)$ for any $r \geq 1$, and $\mathfrak{b}$ is not a BBW parabolic.

5.5. Given our computations for $\mathfrak{g} = \mathfrak{p}(n)$, we conclude this section with two open questions about the parabolic subalgebra $\mathfrak{b}$.

(5.5.1) Does $p_{G,B}(t) = z_{\mathfrak{b},\mathfrak{g}}(t)$?

(5.5.2) Is there a natural subset of elements in Σ_n that describes the grading on $\mathbb{C}[W_{\bar{1}}]_\bullet$ given by $z_{\mathfrak{b},\mathfrak{g}}(t)$?

6. Comparing cohomology and supports for $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$, $(\mathfrak{b}, \mathfrak{b}_{\bar{0}})$ and $(\mathfrak{f}, \mathfrak{f}_{\bar{0}})$

In the section assume that $\mathfrak{g}$ is a classical Lie superalgebra, $\mathfrak{b}$ is the BBW parabolic subalgebra and $\mathfrak{f}$ is the detecting subalgebra as defined in Section 3.3.

6.1. By using the finite generation of the cohomology ring $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$, one can define two types of support varieties. Let $\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M)$ be the variety associated to the annihilator of $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$ on $\mathrm{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, M)$. One has an injection of $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \hookrightarrow \mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$ such that $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C})$. Set $\widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M)$ to be the variety associated to the annihilator of $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C})$ on $\mathrm{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, M)$.

The following theorem compares the support varieties for cohomology in $(\mathfrak{b}, \mathfrak{b}_{\bar{0}})$, $(\mathfrak{t}, \mathfrak{t}_{\bar{0}})$ and $(\mathfrak{f}, \mathfrak{f}_{\bar{0}})$.

Theorem 6.1.1. *Let M be a finite-dimensional $\mathfrak{b}$-module.*

(a) $\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \cong \mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M)$.
(b) $\widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \cong \mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M)/N$.
(c) $\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M) \cong \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/T_{\bar{0}}$.
(d) $\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M)/N \cong \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/N$.

Proof. (a) First observe that by Theorem 3.4.1(b), the restriction map $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{t}, \mathfrak{t}_{\bar{0}}, \mathbb{C})$ is an isomorphism. Therefore, $\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M) \subseteq \mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M)$ (cf. argument in [4, Section 6.1]).

Let $M = \oplus_{\lambda\in\mathfrak{t}_{\bar{0}}^*} M_\lambda$ be a weight space decomposition of M. Note that each M_λ is a $\mathfrak{t}$-module. Next observe one can construct a $\mathfrak{b}$-stable filtration of M:

$$M := M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_s \supseteq \{0\}$$

such that $M_i/M_{i+1} \cong M_{\lambda_i}$ for some $\lambda_i \in \mathfrak{t}_{\bar{0}}^*$.

The filtration above provides a short exact sequence $0 \to M_s \to M \to M/M_s \to 0$. One can then use the long exact sequence in cohomology to show that

$$\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, M') \subseteq \mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M_s, M') \cup \mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M/M_s, M')$$

for all finite-dimensional $\mathfrak{b}$-modules N. Specializing $M = M'$, one obtains

$$\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \subseteq \mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M_s) \cup \mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M/M_s).$$

Applying this procedure inductively yields

$$\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \subseteq \bigcup_{\lambda\in t_{\bar{0}}^*} \mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M_\lambda). \tag{6.1.1}$$

Here M_λ is regarded as $\mathfrak{b}$-module with trivial $\mathfrak{u}$-action.

Next apply the LHS spectral sequence for M_λ:

$$E_2^{i,j} = \mathrm{Ext}^i_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(\mathbb{C}, \mathrm{Ext}^j_{(\mathfrak{u},\mathfrak{u}_{\bar{0}})}(\mathbb{C}, \mathbb{C}) \otimes M_\lambda^* \otimes M_\lambda) \Rightarrow \mathrm{Ext}^{i+j}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M_\lambda, M_\lambda).$$

By using the identification of

$$R := \mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{t}, \mathfrak{t}_{\bar{0}}, \mathbb{C}) \cong S^\bullet(\mathfrak{t}_{\bar{1}}^*)^{T_{\bar{0}}}$$

one has that R acts on the rows of E_2 and the abutment. It follows that

$$\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M_\lambda) \subseteq \mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M_\lambda).$$

Since $\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M) = \cup_{\lambda\in\mathfrak{t}_{\bar{0}}^*}\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M_\lambda)$, one has $\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \subseteq \mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M)$.

(b) The result can be obtained using the argument given in (a) and replacing (i) $\mathrm{H}^\bullet(\mathfrak{b},\mathfrak{b}_{\bar{0}},\mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{t},\mathfrak{t}_{\bar{0}},\mathbb{C})$ by $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}},\mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{t},\mathfrak{t}_{\bar{0}},\mathbb{C})^N$, (ii) $\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(-)$ by $\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(-)/N$, and (iii) $\mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(-)$ by $\widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(-)$.

(c) We have $\mathfrak{f} \trianglelefteq \mathfrak{t}$, so one can apply the Lyndon-Hochschild-Serre spectral sequence for relative cohomology

$$E_2^{i,j} = \mathrm{H}^i(\mathfrak{t}/\mathfrak{f},\mathfrak{t}_{\bar{0}}/\mathfrak{f}_{\bar{0}},\mathrm{H}^j(\mathfrak{f},\mathfrak{f}_{\bar{0}},M')) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{t},\mathfrak{t}_{\bar{0}},M')$$

for any $\mathfrak{t}$-module M'. The spectral sequence collapses ($\mathfrak{t}/\mathfrak{t}_{\bar{0}} \cong \mathfrak{f}/\mathfrak{f}_{\bar{0}}$) and yields:

$$\mathrm{H}^\bullet(\mathfrak{t},\mathfrak{t}_{\bar{0}},M') \cong \mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar{0}},M')^{T_{\bar{0}}}. \tag{6.1.2}$$

This proves that the restriction map: $\mathrm{H}^\bullet(\mathfrak{t},\mathfrak{t}_{\bar{0}},M') \hookrightarrow \mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar{0}},M')$ is an injective map, so by [20, Theorem 4.4.1], $\mathcal{V}_{(\mathfrak{t},\mathfrak{t}_{\bar{0}})}(M) \cong \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/T_{\bar{0}}$.

(d) One can obtain this part by using (c) and taking quotients with N. $\square$

6.2. Geometric induction and spectral sequences

Let G (resp. B) be the supergroup (scheme) such that Lie $G = \mathfrak{g}$ (resp. Lie $B = \mathfrak{b}$). If M is a G-module (resp. B-module) then one can consider M as a $\mathfrak{g}$-module (resp. $\mathfrak{b}$-module) by differentiation. The following result provides a spectral sequence that relates the relative cohomology for $\mathfrak{g}$ and $\mathfrak{b}$ via the higher right derived functors of $\mathrm{ind}_B^G(-)$.

Proposition 6.2.1. *Let M_1 be a G-module and M_2 be a B-module. Then there exists a first quadrant spectral sequence.*

$$E_2^{i,j} = \mathrm{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, R^j\,\mathrm{ind}_B^G\,M_2) \Rightarrow \mathrm{Ext}^{i+j}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M_1,M_2).$$

Proof. The spectral sequence is constructed via a composition of functors. Let $\mathcal{F}_1(-) = \mathrm{Hom}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M,-)$ and $\mathcal{F}_2(-) = \mathrm{ind}_B^G(-)$. We are regarding $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) on the relative category $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ (resp. $\mathcal{C}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}$) where the injective objects are relatively projective over $U(\mathfrak{g}_{\bar{0}})$ (resp. $U(\mathfrak{b}_{\bar{0}})$).

The functors $\mathcal{F}_1$ and $\mathcal{F}_2$ are left exact. Furthermore, an injective object in $\mathcal{C}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}$ is a direct summand of $\mathrm{ind}_{B_{\bar{0}}}^B N$ for some $B_{\bar{0}}$-module N. Observe that

$$\mathcal{F}_2(\mathrm{ind}_{B_{\bar{0}}}^B N) \cong \mathrm{ind}_B^G[\mathrm{ind}_{B_{\bar{0}}}^B N] \cong \mathrm{ind}_{B_{\bar{0}}}^G N = \mathrm{ind}_{G_{\bar{0}}}^G[\mathrm{ind}_{B_{\bar{0}}}^{G_{\bar{0}}} N].$$

Therefore, $\mathcal{F}_2(\mathrm{ind}_{B_{\bar{0}}}^B N)$ is an injective module in $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. It follows that injective objects in $\mathcal{C}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}$ are taken to objects acyclic for $\mathcal{F}_1$. Finally, observe that

$$\mathcal{F}_1 \circ \mathcal{F}_2(-) = \mathrm{Hom}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, \mathrm{ind}_B^G(-)) \cong \mathrm{Hom}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, -).$$

The existence of the spectral sequence now follows by [16, I. 4.1 Proposition]. □

6.3. Restricting relative $U(\mathfrak{g}_{\bar{0}})$-injectives to $U(\mathfrak{b})$

We can use the spectral sequence to investigate what happens when an relative injective $U(\mathfrak{g}_{\bar{0}})$-module restricts to $\mathfrak{b}$.

Theorem 6.3.1. *Let I be a $\mathfrak{g}$-module that is a relatively injective $U(\mathfrak{g}_{\bar{0}})$-module and M be any finite-dimensional $\mathfrak{g}$-module. Then*

(a) $\mathrm{Ext}^j_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, I) \cong \mathrm{Hom}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, [R^j \mathrm{ind}_B^G \mathbb{C}] \otimes I)$
(b) $\mathrm{Ext}^j_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, I) = 0$ *for* $j > \dim G_{\bar{0}}/B_{\bar{0}}$.

Proof. One can apply the spectral sequence given in Proposition 6.2.1:

$$E_2^{i,j} = \mathrm{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, [R^j \mathrm{ind}_B^G \mathbb{C}] \otimes I) \Rightarrow \mathrm{Ext}^{i+j}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, I). \tag{6.3.1}$$

Since I is injective the spectral sequence collapses and yields (a). For part (b), one has $R^j \mathrm{ind}_B^G \mathbb{C} = 0$ for $j \geq \dim G_{\bar{0}}/B_{\bar{0}}$ by Proposition 4.1.1. □

The result above shows that I restricted to $\mathfrak{b}$ need not be a relatively injective $U(\mathfrak{b}_{\bar{0}})$-module. However, the result does show that if I is a relatively injective $U(\mathfrak{g}_{\bar{0}})$-module then $\{0\} = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(I) = \mathcal{V}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(I)$.

6.4. Collapsing of the spectral sequence

The next result shows that the spectral sequence given in Proposition 6.2.1 collapses when $M = \mathbb{C}$ and $\mathfrak{b}$ is a BBW parabolic subalgebra.

Theorem 6.4.1. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra with $\mathfrak{g} \neq P(n)$. Then the following spectral sequence collapses:*

$$E_2^{i,j} = \mathrm{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}, [R^j \mathrm{ind}_B^G \mathbb{C}]) \Rightarrow \mathrm{Ext}^{i+j}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(\mathbb{C}, \mathbb{C}). \tag{6.4.1}$$

Proof. It suffices to show that

$$\sum_{i+j=n} \dim E_2^{i,j} = \dim \mathrm{H}^n(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \tag{6.4.2}$$

for all $n \geq 0$. This will insure that the differentials d_r are zero for $r \geq 2$.

Since $R^j \text{ind}_B^G \mathbb{C} \cong \mathbb{C}^{\oplus m_j}$ by Theorem 4.10.1(a), one has

$$\bigoplus_{i+j=n} E_2^{i,j} \cong \bigoplus_{i+j=n} \mathrm{H}^i(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \otimes R^j \text{ind}_B^G \mathbb{C} \tag{6.4.3}$$

for all $n \geq 0$.

Now by Theorem 4.10.1(b), $p_{\mathfrak{b}}(t) = p_{\mathfrak{g}}(t) \cdot p_{G,B}(t)$. Therefore, by comparing coefficients of t^n, one can conclude that (6.4.2) holds. □

6.5. We can now give conditions via the collapsing of the spectral sequence in Proposition 6.2.1 for $M_1 \cong \mathbb{C}$ and $M_2 \cong \mathbb{C}$ to insure that $\widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \cong \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$.

Theorem 6.5.1. *Let M a finite-dimensional $\mathfrak{g}$-module. Suppose that*

(a) $R^j \,\text{ind}_B^G \mathbb{C} \cong \mathbb{C}^{\oplus m_j}$ *for* $j > 0$.
(b) *The spectral sequence*

$$E_2^{i,j} = \text{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}, [R^j \,\text{ind}_B^G \mathbb{C}]) \Rightarrow \text{Ext}^{i+j}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(\mathbb{C}, \mathbb{C}) \tag{6.5.1}$$

collapses and yields an isomorphism of $R = \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) = S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$-modules.

Then $\text{res}^* : \widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ *is an isomorphism.*

Proof. Let M be a finite-dimensional $\mathfrak{g}$-module. By assumption, $R^j \,\text{ind}_B^G \mathbb{C} \cong \mathbb{C}^{\oplus m_j}$ for $j > 0$. Using the tensor identity, $R^j \,\text{ind}_B^G M \cong [R^j \,\text{ind}_B^G \mathbb{C}] \otimes M$, one has two spectral sequences:

$$E_2^{i,j} = \text{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}, \mathbb{C}^{\oplus m_j}) \Rightarrow \text{Ext}^{i+j}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(\mathbb{C}, \mathbb{C}), \tag{6.5.2}$$

$$\bar{E}_2^{i,j} = \text{Ext}^i_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(\mathbb{C}, [M^* \otimes M]^{\oplus m_j}) \Rightarrow \text{Ext}^{i+j}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, M). \tag{6.5.3}$$

The spectral sequence (6.5.2) acts on (6.5.3) in the following way. There exists a natural map of $\mathbb{C}$-algebras $\rho : \mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C}) \to \text{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, M)$ that is defined by taking an extension class in $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$ and tensoring the class by M. Set $\widehat{J}_{\mathfrak{b},M} = \text{Ann}_R \text{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, M)$. Then one has an injective ring homomorphism

$$\rho : \mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})/\widehat{J}_{\mathfrak{b},M} \hookrightarrow \text{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M, M). \tag{6.5.4}$$

For $j \geq 0$, there also exist maps on the direct sum of algebras:

$$\rho_j : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C})^{\oplus m_j} \to \text{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, M)^{\oplus m_j} \tag{6.5.5}$$

with

$$\rho_j : [R/J_M]^{\oplus m_j} \hookrightarrow \operatorname{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M,M)^{\oplus m_j}. \tag{6.5.6}$$

Furthermore, there is a compatibility of differentials:

$$\rho_j(d_r(x)) = \bar{d}_r(\rho_j(x)). \tag{6.5.7}$$

Since (6.5.2) collapses, $d_r(x) = 0$ for $r \geq 2$, thus $\bar{d}_r(\rho_j(x)) = 0$ for $r \geq 2$, $j \geq 0$. Therefore, the differentials on $[R/J_M]^{\oplus m_j}$ in (6.5.3) are zero, and $\operatorname{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M,M)$ contains a copy of the module $\oplus_{j\geq 0}[R/J_M]^{\oplus m_j}$.

Now suppose that $y \in R$ annihilates $\operatorname{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M,M)$. Then y annihilates R/J_M so $y \in J_M$. Consequently, $\operatorname{Ann}_R \operatorname{Ext}^\bullet_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M,M) \subseteq J_M$, and $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) \subseteq \widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M)$. The other inclusion holds by looking at the action of R on the spectral sequence (6.5.3) [e.g., if R annihilates $\bar{E}_2$, then it annihilates the abutment]. Hence, $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M)$. □

6.6. Proof of Theorem 1.2.1

For $\mathfrak{g} = \mathfrak{p}(n)$, the first isomorphism in Theorem 1.2.1(b) can be deduced from [20, Theorem 5.1.1(a)] since $P(n)$ is type I. Now assume that $\mathfrak{g} \neq P(n)$, then the first isomorphism in Theorem 1.2.1(b) follows from Theorem 2.5.1. Therefore, it suffices to prove that $\operatorname{res}^* : \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/N \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ is an isomorphism. From Theorem 6.5.1 $\operatorname{res}^* : \widehat{\mathcal{V}}_{(\mathfrak{b},\mathfrak{b}_{\bar{0}})}(M) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ is an isomorphism. The statement of the theorem now follows by applying Theorem 6.1.1(b)(d).

7. Tables for BBW parabolics and Poincaré series

7.1. BBW parabolics

The following tables provide a reference for the construction of BBW parabolic subalgebras. In these tables, the roots for the detecting subalgebras and the BBW parabolics are given as well as the defining hyperplanes. One should note that although $\Phi_{\mathfrak{f}_{\bar{1}}} = \emptyset$ for $\mathfrak{g} = \mathfrak{q}(n), \mathfrak{psq}(n)$ in Table 7.1.1, the algebra $\mathfrak{f}_{\bar{1}}$ is not trivial, and it equals the odd part of the Cartan subalgebra of $\mathfrak{g}$. Also, for convenience, the obvious restrictions for the indexes are not included in some cases. For example, in Table 7.1.3, the restrictions of

Table 7.1.1
Roots for the detecting subalgebras.

$\mathfrak{g}$	$\Phi_{\mathfrak{f}_{\bar{1}}}$
$\mathfrak{gl}(m\|n), \mathfrak{sl}(m\|n)$ $[m \leq n]$	$\{\pm\,(\epsilon_i - \delta_i) \mid 1 \leq i \leq m\}$
$\mathfrak{osp}(2m\|2n)$	$\{\pm\,(\epsilon_i - \delta_i) \mid 1 \leq i \leq \min(m,n)\}$
$\mathfrak{osp}(2m+1\|2n)$ $[m \geq n]$	$\{\pm\,(\epsilon_i - \delta_i) \mid 1 \leq i \leq n\}$
$\mathfrak{osp}(2m+1\|2n)$ $[m < n]$	$\{\pm\,(\epsilon_i - \delta_i) \mid 1 \leq i \leq m\}$
$\mathfrak{q}(n), \mathfrak{psq}(n)$	$\emptyset$
$D(2,1,\alpha)$	$\{\pm\,(\epsilon, -\epsilon, \epsilon)\}$
$G(3)$	$\{\pm\,(\omega_1, -\epsilon)\}$
$F(4)$	$\{\pm\,(\omega_3, -\epsilon)\}$

Table 7.1.2
Hyperplanes of BBW parabolics.

$\mathfrak{g}$	$\mathcal{H}$
$\mathfrak{gl}(m\|n), \mathfrak{sl}(m\|n)$ $[m \leq n]$	$\sum_{i=1}^{n} x_i(E_i + D_i),\ x_1 > x_2 > \cdots > x_n$
$\mathfrak{osp}(2m\|2n)$	$\sum_{i=1}^{r} x_i(E_i + D_i),\ x_1 > x_2 > \cdots > x_r > 0$ $[r = \max(m,n)]$
$\mathfrak{osp}(2m+1\|2n)$	$\sum_{i=1}^{r} x_i(E_i + D_i),\ x_1 > x_2 > \cdots > x_r > 0$ $[r = \max(m,n)]$
$\mathfrak{q}(n), \mathfrak{psq}(n)$	$\sum_{i=1}^{n} x_i E_i,\ x_1 > x_2 > \cdots > x_n$
$D(2,1,\alpha)$	$x_1E_1 + (x_1 + x_3)E_2 + x_3E_3,\ x_1 > x_3 > 0$
$G(3)$	$x_1L_1 + x_2L_2 + x_1E,\ 2x_1 > x_2 > x_1 > 0$
$F(4)$	$x_1L_1 + x_2L_2 + x_3L_3 + x_3E,\ 2x_1 > x_3 > x_2 > x_1 > 0$

Table 7.1.3
Roots of BBW parabolics.

$\mathfrak{g}$	$\Phi_{\bar{1}}^{-}$
$\mathfrak{gl}(m\|n), \mathfrak{sl}(m\|n)$ $[m \leq n]$	$\{-\epsilon_i + \delta_j, -\delta_i + \epsilon_j \mid i < j\}$
$\mathfrak{osp}(2m\|2n)$	$\{-\epsilon_i + \delta_j, -\delta_i + \epsilon_j, -\epsilon_k - \delta_\ell, \mid i < j\}$
$\mathfrak{osp}(2m+1\|2n)$ $[m \geq n]$	$\{-\epsilon_i + \delta_j, -\delta_i + \epsilon_j, -\epsilon_k - \delta_\ell, -\delta_t \mid i < j\}$
$\mathfrak{osp}(2m+1\|2n)$ $[m < n]$	$\{-\epsilon_i + \delta_j, -\delta_i + \epsilon_j, -\epsilon_k - \delta_\ell, -\delta_t \mid i < j, t \leq m\}$
$\mathfrak{q}(n), \mathfrak{psq}(n)$	$\{-\epsilon_i + \epsilon_j \| i < j\}$
$D(2,1,\alpha)$	$\{(-\epsilon,-\epsilon,-\epsilon), (-\epsilon,-\epsilon,\epsilon), (\epsilon,-\epsilon,-\epsilon)\}$
$G(3)$	$\{(-\omega_1 + \omega_2, -\epsilon), (2\omega_1 - \omega_2, -\epsilon), (0,-\epsilon), (\omega_1 - \omega_2, -\epsilon),$ $(-2\omega_1 + \omega_2, -\epsilon), (-\omega_1, -\epsilon)\}$
$F(4)$	$\{(\omega_2 - \omega_3, -\epsilon), (\omega_1 - \omega_2 + \omega_3, -\epsilon), (\omega_1 - \omega_3, -\epsilon),$ $(-\omega_2 + \omega_3, -\epsilon), (-\omega_1 + \omega_2 - \omega_3, -\epsilon), (-\omega_1 + \omega_3, -\epsilon), (-\omega_3, -\epsilon)\}$

the indexes for $\Phi_{\mathfrak{f}_{\bar{1}}}$ when $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ should be $i < j$, $1 \leq i,k \leq m$, $1 \leq j,\ell \leq n$, but we just write $i < j$.

7.2. Poincaré series

For the parabolic subalgebras $\mathfrak{b}$ given in Section 7.1, the table below provides gives a description of the cohomology $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$ and relationship between $z_{\mathfrak{b},\mathfrak{g}}(t)$ with $p_{W_{\bar{1}}}(t)$.

Table 7.2.1
Cohomology and Hilbert series.

$\mathfrak{g}$	$W_{\bar{1}}$	$\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{b}_{\bar{0}}, \mathbb{C})$	$z_{\mathfrak{b},\mathfrak{g}}(t)$
$\mathfrak{gl}(m\|n)$ $[m \geq n]$	Σ_n	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ny_n]$	$p_{W_{\bar{1}}}(t^2)$
$\mathfrak{sl}(m\|n)$ $[m > n]$	Σ_n	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ny_n]$	$p_{W_{\bar{1}}}(t^2)$
$\mathfrak{sl}(n\|n)$	Σ_n	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ny_n, x_1x_2\ldots x_n, y_1y_2\cdots y_n]$	$p_{W_{\bar{1}}}(t^2)$
$\mathfrak{psl}(n\|n)$	Σ_n	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ny_n, x_1x_2\ldots x_n, y_1y_2\cdots y_n]$	$p_{W_{\bar{1}}}(t^2)$
$\mathfrak{q}(n)$	Σ_n	$\mathbb{C}[z_1, z_2, \ldots, z_n]$	$p_{W_{\bar{1}}}(t)$
$\mathfrak{psq}(n)$	Σ_n	$\mathbb{C}[z_1, z_2, \ldots, z_n, z_1z_2\ldots z_n]/(z_1 + z_2 + \cdots + z_n)$	$p_{W_{\bar{1}}}(t)$
$\mathfrak{osp}(2m+1\|2n)$	$\Sigma_r \ltimes (\mathbb{Z}_2)^r$	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ry_r]$ $[r = \min(m,n)]$	$p_{W_{\bar{1}}}(t^2)$
$\mathfrak{osp}(2m\|2n)$ $[m > n]$	$\Sigma_n \ltimes (\mathbb{Z}_2)^n$	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_ny_n]$	$p_{W_{\bar{1}}}(t^2)$
$\mathfrak{osp}(2m\|2n)$ $[m \leq n]$	$\Sigma_m \ltimes (\mathbb{Z}_2)^{m-1}$	$\mathbb{C}[x_1y_1, x_2y_2, \ldots, x_my_m]$	$p_{W_{\bar{1}}}(t^2)$
$D(2,1,\alpha)$	Σ_2	$\mathbb{C}[xy]$	$p_{W_{\bar{1}}}(t^2)$
$G(3)$	Σ_2	$\mathbb{C}[xy]$	$p_{W_{\bar{1}}}(t^2)$
$F(4)$	Σ_2	$\mathbb{C}[xy]$	$p_{W_{\bar{1}}}(t^2)$

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