

Repair of Multiple Descriptions on Distributed Storage

Anders Høst-Madsen

Department of EE
University of Hawaii
ahm@hawaii.edu

Heechoel Yang

School of Electronic Engineering
Kumoh National Institute of Technology, Gumi
hc.yang@kumoh.ac.kr

Minchul Kim, Jungwoo Lee

Department of ECE
Seoul National University
kmc1222@cml.snu.ac.kr, junglee@snu.ac.kr

Abstract—In multiple descriptions on distributed storage, a source is stored in a shared fashion on multiple servers. When a subset of servers are contacted, the source should be estimated with a certain maximum distortion depending on the number of servers. The problem considered in this paper is how to restore the system operation when one of the servers fail and a new server replaces it, that is, repair. The requirement is that the distortions in the restored system should be no more than in the original system. The question is how many extra bits are needed for repair. We find the optimum solution for a two server problem in the Gaussian case, and an achievable rate for general n nodes. One conclusion is that it is necessary to design the multiple description codes with repair in mind.

I. INTRODUCTION

A common problem in lossy source coding is to enable decoding of a source with a distortion that depends on the quality of the connection a user has: users with good connections can decode with little distortion, while users with poor connection can still obtain a low-quality replica of the source. One variation of this problem is multiple-description coding [1], [2]. In that case, the encoder produces multiple descriptions of the source. The more of these descriptions the decoder has access to, the lower the distortion of the reconstruction is. Traditionally this has been aimed at packet transmission networks, where some packet may be lost. In this paper we consider a scenario where each description is stored on a separate server. A user has access to one or multiple servers, for example depending on network connection, physical location, delay, or cost. The more servers a user has access to, the less distortion in reconstruction. Thus, the descriptions are stored on a distributed storage system [3]. Clearly, this does not change the multiple description problem as such. However, a central issue in distributed storage is how to repair the system when one or more of the servers fail or become unavailable [3] and is replaced by new servers. In this paper we consider repair specifically for multiple-description servers.

To simplify the problem we consider a symmetric multiple description problem as in [4]. A multiple description coding system is specified as follows: when a subset $J \subset \{1, \dots, n\}$ of servers are contacted, a source X should be restored with a distortion at most D_J . If one (or multiple) of the servers fail, we should be able to set up a replacement server with enough

information so that the whole region $D_J, J \subset \{1, \dots, n\}$ is restored. We consider two scenarios

- 1) There is special (highly reliable) repair server that does not participate in the usual operation of the system, but only comes into action if another server fails. The repair server can contact all other (non-failed) servers and use their information combined with its own information to restore the failed server (collaborative repair).
- 2) The repair information is stored in a distributed fashion among the n servers (distributed repair).

For simplicity, in this paper we only consider failure of a single server.

A straightforward solution is to separate the source coding problem (multiple descriptions) and the repair problem. Any existing code for multiple descriptions can then be used, and repair can be done using minimum distance separable (MDS) erasure codes as in traditional distributed storage [3]. We will use this as baseline. For case 1 above, the repair server can simply store the xor (sum modulo 2) of the bits on the operational servers. When one server fails, the xor together with the bits from the surviving servers can restore the failed server. Thus, if each operational server stores lR bits, the repair server also needs to store lR bits. For distributed repair, the xor can instead be stored with an $(n, n-1)$ erasure code. Therefore in addition to the lR bits for operation, each server needs to store $\frac{lR}{n-1}$ bits for repair. It should be clear that these rates are also optimal with separation: even if the system knows in advance which server will fail, it cannot store less information. We can consider this as a separate source channel coding solution, with multiple descriptions being source coding and the repair being channel coding. It is known that in many information theory problems, joint source-channel coding is superior to separation. This is then the question we consider here: can we find a better joint source channel coding solution that can beat the above rates?

II. PROBLEM DESCRIPTION

We consider a symmetric multiple description problem as in [5], and use their notation. We have an i.i.d. (independent identically distributed) source X that takes values in a finite alphabet $\mathcal{X}$ and needs to be restored in the alphabet $\hat{\mathcal{X}}$, with generalizations to a Gaussian source through usual quantization arguments [2]. We will first define the distributed

repair problem. For a source sequence x^l of length l each node stores lR_t bits. For many achievable schemes, these can be split into lR bits for normal operation and lR_r additional bits used only for repair. There are n encoding functions $f_i : \mathcal{X}^l \rightarrow \{1, \dots, 2^{lR_t}\}$, 2^{n-1} decoding function $g_J : \{1, \dots, 2^{lR_t}\}^{|J|} \rightarrow \mathcal{X}^l$, $J \subset I_n$, and n repair functions $h_i : \{1, \dots, 2^{lR_t}\}^{n-1} \rightarrow \{1, \dots, 2^{nR_t}\}$. We define the error probability of repair as

$$P_r^{(n)} = \max_{i=1, \dots, n} P(h_i(f_{I_n - \{i\}}(x^l)) \neq f_i(x^l))$$

We now say that a tuple $(R_t, D_1, \dots, D_{n-1})$ is achievable if there exists a sequence of $(2^{lR_t}, l)$ codes with

$$\lim_{n \rightarrow \infty} \max_{J: |J|=m} E[d_{|J|}(x^l, g_J(f_J(x^l)))] \leq D_m \quad \lim_{n \rightarrow \infty} P_r^{(n)} = 0 \quad (1)$$

with the distortions $d_{|J|}(x^l, \hat{x}^l) = \frac{1}{l} \sum_{i=1}^l \tilde{d}_{|J|}(x_i, \hat{x}_i)$, $\tilde{d}_{|J|}(x_i, \hat{x}_i) \geq 0$. We call this *strong repair*. The repaired node is required to be an exact copy of the failed node, except that we allow a certain, vanishing, error rate. Notice that the randomness in the system is purely due to the source x^l . Thus, for a given sequence x^l either all failures can be repaired exactly, and if they can be repaired once, they can be repaired infinitely many times; or, some failures can never be repaired. The probability of the source sequences that are not repairable should be vanishing small.

For a dedicated repair node, each node stores lR bits and the repair node lR_r bits. The non-collaborative repair functions are instead functions $h_i : \{1, \dots, 2^{lR_r}\} \rightarrow \{1, \dots, 2^{lR}\}$, and the collaborative repair functions are $h_i : \{1, \dots, 2^{lR_r}\} \times \{1, \dots, 2^{lR}\}^{n-1} \rightarrow \{1, \dots, 2^{lR}\}$, with the other definitions similar.

III. TWO NODES

We at first consider a problem with $n = 2$ nodes as this is one of the only cases where the optimum rate distortion region is known, in the Gaussian case [1] with mean-squared distortion. If there is no repair node, the problem is trivial: each node has to be able to achieve the distortion D_2 by itself, and they can therefore be copies of each other. We therefore assume that there is a special repair node, and consider the case when this has access to the surviving operational nodes for repair, collaborative repair. The question is: what is the minimum information the repair node can store, so that (D_1, D_2) can be achieved *without any increase in storage rate of the operational nodes*. Here D_i is the required distortion when i nodes are available.

The problem is most well posed in the Gaussian case, as we know the exact rate distortion region (R, D_1, D_2) . We then want to find the minimum repair rate R_r for every point on the boundary of the rate distortion region (R, D_1, D_2) . We also know that the El-Gamal Cover (EC) coding scheme achieves the optimum rate-distortion region. The idea in the EC scheme is that each node stores an index for use when only that node is accessed, in addition to half the bits of an index

with refinement information that is only used when both nodes are accessed. However, EC is clearly sub-optimum for repair. Consider the point $D_2 = D_1$; in this point it is clear what is the optimum solution. Each node has to be able to restore the source by itself with distortion D_2 , and they can therefore be copies of each other. Repair then is done simply by copying from the surviving node and $R_r = 0$. Now if D_2 is close to D_1 one would expect R_r to be small. On the other hand, in EC the two nodes store independently generated codewords – even if the joint distribution is not independent [1]. Therefore to restore the EC code exactly, $R_r = R$ is needed.

We therefore instead consider the Zhang-Berger (ZB) scheme [6], [2]. In addition to the individual and refinement information stored in the EC scheme, the nodes in the ZB scheme store a common codeword. While this cannot decrease rate in the Gaussian case, a common codeword is great for repair, as it can be simply copied from the surviving node without additional information from the repair node.

Instead of the original characterization of the ZB scheme, we will describe it in the language of PPR [5], both to be consistent with the general problem later, and because the PPR more explicitly characterizes the information stored on nodes in terms of auxiliary random variable, which is essential to calculate repair rate.

Theorem 1 (Zhang-Berger). *A rate R is achievable if*

$$R > I(X; U_1) + H(Y_{12}|U_1) + \frac{1}{2}H(Y_2|Y_{12}, Y_{11}, U_1) - \frac{1}{2}H(Y_{12}, Y_{11}|X, U_1) - \frac{1}{2}H(Y_2|Y_{12}, Y_{11}, X, U_1)$$

for some conditional pdf $p(u_1, y_{11}, y_{12}, y_2|x)$ such that $E[d_1(X, g_{11}(U_1, Y_{11}))] \leq D_1$, $E[d_2(X, g_2(U_1, Y_{11}))] \leq D_2$.

Corollary 2. *A repair rate R_r is achievable if*

$$R_r > H(Y_{12}|Y_{11}, U_1) - \frac{1}{2}H(Y_{12}, Y_{11}|X, U_1) + \frac{1}{2}H(Y_2|Y_{12}, Y_{11}, U_1) - \frac{1}{2}H(Y_2|Y_{12}, Y_{11}, X, U_1) \quad (2)$$

We omit the proof, as it is a special case of Theorem 5 later.

It turns out the ZB is exactly optimum in the Gaussian case

Theorem 3. *Consider a Gaussian source with $E[X^2] = 1$. The optimal repair rate is*

$$R_r = \begin{cases} \frac{1}{4} \log \left(\frac{1}{D_2} \right) & D_2 \leq 2D_1 - 1 \\ \frac{1}{2} \log \left(\frac{2\sqrt{(1-D_1)(D_1-D_2)}}{(D_2-1)\sqrt{D_2}} \right) & 2D_1 - 1 \leq D_2 \leq \frac{D_1}{2-D_1} \\ \frac{1}{2} \log \left(\frac{D_1}{D_2} \right) & \frac{D_1}{2-D_1} \leq D_2 \end{cases}$$

Proof: For achievable rate we let $U_1 = X + Q_{u1}$, $Y_{1i} = X + Q_{1i}$, $Y_2 = X + Q_2$ with $Q_{\dots}$ zero-mean Gaussian, $E[Q_{u1}^2] = \sigma_{u1}^2$, $E[Q_{1i}^2] = \sigma_{q1}^2$, $E[Q_2^2] = \sigma_{q2}^2$, $E[Q_{11}Q_{12}] = \rho_1\sigma_{q1}^2$, and all other noise variables uncorrelated. We first

calculate the distortions,

$$D_1 = \frac{\sigma_{q1}^2 \sigma_{u1}^2}{\sigma_{q1}^2 \sigma_{u1}^2 + \sigma_{q1}^2 + \sigma_{u1}^2}$$

$$D_2 = \frac{(\rho_1 + 1) \sigma_{q1}^2 \sigma_{q2}^2 \sigma_{u1}^2}{(\rho_1 + 1) \sigma_{q1}^2 (\sigma_{q2}^2 \sigma_{u1}^2 + \sigma_{q2}^2 + \sigma_{u1}^2) + 2 \sigma_{q2}^2 \sigma_{u1}^2} \quad (3)$$

The D_1 distortion constraint is always satisfied with equality, and therefore

$$\sigma_{q1}^2 = \frac{D_1 \sigma_{u1}^2}{\sigma_{u1}^2 - D_1 \sigma_{u1}^2 - D_1} \quad (4)$$

Using standard Gaussian calculations of differential entropy, we get

$$R = \frac{1}{2} \log \left(1 + \frac{1}{\sigma_{u1}^2} \right) + \frac{1}{2} \log \left(\frac{\sigma_{u1}^2}{\sqrt{1 - \rho_1^2} D_1 (\sigma_{u1}^2 + 1)} \right)$$

$$+ \frac{1}{4} \log \left(\frac{D_1 ((\rho_1 - 1) \sigma_{q2}^2 (\sigma_{u1}^2 + 1) + (\rho_1 + 1) \sigma_{u1}^2) + 2 \sigma_{q2}^2 \sigma_{u1}^2}{\sigma_{q2}^2 (D_1 (\rho_1 - 1) (\sigma_{u1}^2 + 1) + 2 \sigma_{u1}^2)} \right) \quad (5)$$

and

$$R_r = \frac{1}{2} \log \left(\frac{1 - \rho_1}{\sqrt{1 - \rho_1^2} \sigma_{q2}^2 \sigma_{u1}^2} \right) + \frac{1}{2} \log (2 \sigma_{q2}^2 \sigma_{u1}^2)$$

$$D_1 ((\rho_1 - 1) \sigma_{q2}^2 (\sigma_{u1}^2 + 1) + (\rho_1 + 1) \sigma_{u1}^2) \quad (6)$$

Following [2, Theorem 13.2], there are three regions for D_2 to consider. If $D_2 \geq 2D_1 - 1$, the optimum solution can be achieved without transmitting resolution information, i.e., $\sigma_{q2}^2 = \infty$. From (5) we get

$$R = \frac{1}{2} \log \left(\frac{1}{D_1 \sqrt{1 - \rho_1^2}} \right) \quad (7)$$

independent of σ_{u1}^2 . This region is again split into two. If $D_2 > \frac{D_1}{2-D_1}$ we can achieve $R = \frac{1}{2} \log \left(\frac{1}{D_1} \right)$, which is achieved (and only achieved) for $\rho_1 = 0$. What happens in this region is that the two nodes have independent messages, and the combination results in a distortion less than D_2 . But independent messages are poor for repair. We cannot change ρ_1 because of (7), but we can use the common message in the ZB scheme. We choose the power σ_{u1}^2 so that the combination of the two nodes' information gives exactly a distortion D_2 , which gives $\sigma_{u1}^2 = \frac{D_1 D_2}{2D_2 - D_1 D_2 - D_1}$. This solution is valid for $D_2 > \frac{D_1}{2-D_1}$. We then get from (6) that

$$R_r = \frac{1}{2} \log \left(\frac{D_1}{D_2} \right)$$

For the case $D_2 \leq \frac{D_1}{2-D_1}$ we need to decrease ρ_1 from zero. We store no common message. Then, solving (3) with respect to ρ_1 (for $\sigma_{q2}^2 = \infty$ and $\sigma_{u1}^2 = \infty$) gives $\rho_1 = \frac{D_1 D_2 + D_1 - 2D_2}{D_1 (D_2 - 1)}$

and

$$R = \frac{1}{2} \log \left(\frac{D_2 - 1}{2 \sqrt{(D_1 - 1) D_2 (D_2 - D_1)}} \right)$$

$$R_r = \frac{1}{2} \log \left(\frac{2 \sqrt{(D_1 - 1) (D_2 - D_1)}}{(D_2 - 1) \sqrt{D_2}} \right)$$

In the region $D_2 \leq 2D_1 - 1$ the optimum solution requires storage of resolution information; we use no common message. We get

$$R = \frac{1}{2} \log \left(\sqrt{\frac{1}{D_1 D_2 (1 - \rho_1) (D_1 (\rho_1 - 1) + 2)}} \right)$$

We minimize this with respect to ρ_1 and get $\rho_1 = \frac{D_1 - 1}{D_1}$. Inserting this we get

$$R = R_r = \frac{1}{2} \log \left(\frac{1}{\sqrt{D_2}} \right) \quad (8)$$

For the converse, we can think of the problem as follows. When the repair node has restored the failing operational node, the two operational nodes should be able to estimate X with a distortion (less than or equal to) D_2 . But that also means that the surviving node and the repair node when they cooperate must be able to estimate X with a distortion D_2 . From standard rate-distortion theory we then must have $R + R_r \geq \frac{1}{2} \log \left(\frac{1}{D_2} \right)$. Now it is easy to see that in all three regions above, we have $R + R_r = \frac{1}{2} \log \left(\frac{1}{D_2} \right)$. ■

We notice that in the low distortion case, $D_2 \leq 2D_1 - 1$, separation is exactly optimal as seen from (8), but in the other cases, joint coding for multiple descriptions and repair performs better than separation.

IV. GENERAL n NODES

For more than two nodes the optimum rate distortion region is not known, not even in the Gaussian case. There are therefore many different schemes for multiple description coding and we have to design repair for each specific method. In this paper we will consider the PPR scheme [4], [5], as this is specifically aimed at the symmetric case and is well-suited to repair. The following is our repairable PPR scheme.

Theorem 4 (Distributed repair). *For any symmetric probability distribution [5] $p(\mathbf{y}_{I_{n-2}, I_n}, \mathbf{u}_{I_{n-2}}, y_{n-1} | x)$ and decoding functions g_J the lower convex closure of $(R + R_r, D_1, \dots, D_{n-1})$ is achievable, where $E[d_{|J|}(X, g_J(\mathbf{Y}_{I_{|J|}, J}, \mathbf{U}_{I_{|J|}}))] \leq D_{|J|}$, $|J| \leq n - 1$ and the information needed to encode operational information is*

$$R > I(X; U_1) + H(Y_{1n} | U_1) + \sum_{k=2}^{n-2} \frac{1}{k} H(\mathbf{Y}_{kI_k} | \mathbf{Y}_{I_{k-1}, I_k} \mathbf{U}_{I_k})$$

$$+ \frac{1}{n-1} I(Y_{n-1}; X | \mathbf{Y}_{I_{n-2}, I_{n-1}}, \mathbf{U}_{I_{n-2}})$$

$$- \frac{1}{n} H(\mathbf{Y}_{I_{n-2}, I_n} | X, \mathbf{U}_{I_{n-2}})$$

$$+ \sum_{k=1}^{n-2} \frac{1}{k} (H(U_k | \mathbf{Y}_{I_{k-1}, I_k}, \mathbf{U}_{I_{k-1}}) - H(U_k | X, \mathbf{Y}_{I_{k-1}, I_n}, \mathbf{U}_{I_{k-1}}))$$

with additional information needed to encode repair information

$$R_r > \frac{1}{n-1} \sum_{k=1}^{n-2} \left[H(Y_{kn} | \mathbf{U}_{I_k}, \mathbf{Y}_{kI_{n-1}} \mathbf{Y}_{I_{k-1}I_n}) - \frac{1}{n} H(\mathbf{Y}_{kI_n} | X, \mathbf{Y}_{k-1I_n}, \mathbf{U}_{I_k}) \right]^+$$

with $[x]^+ = \max\{0, x\}$

Proof: Space permits us only to outline how the coding changes compared to [4], [5], and reading the proof therefore requires familiarity with those two papers. A formal proof can be found in [?]. Consider at first layer 1. We generate a codebook $\mathcal{C}_{u1}$ by picking $2^{lR'_{u1}}$ elements uniformly randomly with replacement from the typical set according to the distribution $p_{U_1}(u_1)$. We also generate n independent random codebooks $\mathcal{C}_{1I_n}$ drawn from the typical set according to $p_{Y_{11}}(y_{11})$ with $2^{lR'_{11}}$ codewords. We need to be able to find a codeword in $\mathcal{C}_{u1}$ that is jointly typical with x^l with high probability, which, from standard rate distortion, is the case if

$$R_{u1} = R'_{u1} > H(U_1) - H(U_1 | X) = I(X; U_1)$$

This codeword is stored in all the nodes. We now need to be able to find n codewords from $\mathcal{C}_{1I_n}$ that are jointly typical with x^l and the chosen codeword $u_1^l \in \mathcal{C}_{u1}$. There are about $2^{nlH(Y_{11})}$ (marginally) typical sequences, and about $2^{lH(Y_{11}, \dots, Y_{1n} | U_1, X)}$ that are jointly typical with a given x^l and u_1^l (see, e.g., [7, Section 15.2]); the probability that a given codeword combination in $\mathcal{C}_{1I_n}$ is jointly typical therefore is about $2^{l(H(Y_{11}, \dots, Y_{1n} | U_1, X) - nH(Y_{11}))}$. The probability that no codeword is jointly typical then is about [7], [2]

$$\left(1 - 2^{l(H(Y_{11}, \dots, Y_{1n} | U_1, X) - nH(Y_{11}))} \right)^{2^{nlR'_{11}}} \leq \exp \left(-2^{l(nR'_{11} - (nH(Y_{11}) - H(Y_{11}, \dots, Y_{1n} | U_1, X)))} \right) \quad (9)$$

Thus, if

$$nR'_{11} > nH(Y_{11}) - H(Y_{11}, \dots, Y_{1n} | U_1, X) \quad (10)$$

there is a high probability that at least one of the $2^{nlR'_{11}}$ codeword combinations is jointly typical.

The codewords in $\mathcal{C}_{1j}$ are randomly binned into 2^{lR_1} bins. At the time of decoding, the common codeword $u_1^l \in \mathcal{C}_{u1}$ is available as well as the bin number i for the codeword $y_{1j}^l \in \mathcal{C}_{1j}$. The decoder looks for a codeword in bin i that is typical with u_1^l . There is always one, the actual codeword, but if there is more than one, the decoding results in error. The probability that a random codeword in $\mathcal{C}_{1j}$ is jointly typical with u_1^l is about $2^{l(H(Y_{11} | U_1) - H(Y_{11}))}$ as above, while there are about $2^{l(R'_1 - R_1)}$ codewords in each bin. By the union bound, the probability that there is at least one random codeword in the bin jointly typical is approximately upper bounded by $2^{l(R'_1 - R_1)} 2^{-l(H(Y_{11}) - H(Y_{11} | U_1))}$. Thus, if

$$R'_1 - R_1 < H(Y_{11}) - H(Y_{11} | U_1) \quad (11)$$

there is only one such codeword with high probability. Combining (10) and (11) we get

$$R_1 > H(Y_{11} | U_1) - H(Y_{11}, \dots, Y_{1n} | U_1, X)$$

At layer $k < n - 1$ we similarly generate a random codebook $\mathcal{C}_{uk}$ with $2^{lR'_{uk}}$ typical elements according to the marginal distribution $p_{U_k}(u_k)$ and n independent random codebooks $\mathcal{C}_{kI_n}$ according to the distribution $p_{Y_{k1}}(y_{k1})$ with $2^{lR'_{k1}}$ codewords. We need to be able to find a codeword in $\mathcal{C}_{uk}$ that is jointly typical with x^l and all the codewords chosen in the previous layers. This is possible if

$$R'_{uk} > H(U_k) - H(U_k | X, \mathbf{Y}_{I_{k-1}I_n}, \mathbf{U}_{I_{k-1}})$$

with the same argument as for (10). We also need to be able to find an n -tuple of codewords from $\mathcal{C}_{kI_n}$ that are jointly typical with all prior codewords and x^l , which is possible with high probability if (again as in (10))

$$nR'_{k1} > nH(Y_{k1}) - H(\mathbf{Y}_{kI_n} | X, \mathbf{Y}_{k-1I_n}, \mathbf{U}_{I_k})$$

For $\mathcal{C}_{uk}$ we generate n independent binning partitions each with $2^{lR_{uk}}$ elements. The bin number in the i -th partition is stored in the i -th node. When the decoder has access to k nodes, say nodes $1, \dots, k$ it needs to be able to find a *unique* codeword in the k bins jointly typical with codewords from previous layers. The probability that a random selected codeword is jointly typical is about $2^{l(H(U_k | \mathbf{Y}_{I_{k-1}I_k}, \mathbf{U}_{I_{k-1}}) - H(U_k))}$, as above. There are about $2^{lR'_{uk}} 2^{-lkR_{uk}}$ in each combined bin. Therefore, if

$$kR_{uk} > R'_{uk} + H(U_k | \mathbf{Y}_{I_{k-1}I_k}, \mathbf{U}_{I_{k-1}}) - H(U_k)$$

or

$$R_{uk} > \frac{1}{k} (H(U_k | \mathbf{Y}_{I_{k-1}I_k}, \mathbf{U}_{I_{k-1}}) - H(U_k | X, \mathbf{Y}_{I_{k-1}I_n}, \mathbf{U}_{I_{k-1}})) \quad (12)$$

with high probability there is only one jointly typical codeword in the combined bin. It also needs to find a single codeword in the k bins for $\mathcal{C}_{kI_k}$ s that are jointly typical with $(\mathbf{U}_{I_k}, \mathbf{Y}_{I_{k-1}I_k})$. The probability that a random codeword is jointly typical is about $2^{l(H(\mathbf{Y}_{kI_k} | \mathbf{U}_{I_k}, \mathbf{Y}_{I_{k-1}I_k}) - kH(Y_{k1}))}$, while the number of codewords in the k joint bins is about $2^{lkR'_{k1}} 2^{-lkR_{k1}}$. With high probability there is only one such if

$$k(R'_k - R_k) < kH(Y_{k1}) - H(\mathbf{Y}_{kI_k} | \mathbf{U}_{I_k}, \mathbf{Y}_{I_{k-1}I_k})$$

or

$$R_k > \frac{1}{k} H(\mathbf{Y}_{kI_k} | \mathbf{U}_{I_k}, \mathbf{Y}_{I_{k-1}I_k}) - \frac{1}{n} H(\mathbf{Y}_{kI_n} | X, \mathbf{U}_{I_k}, \mathbf{Y}_{I_{k-1}I_n})$$

(as in [4] this can be repeated for any collection of k nodes).

At layer $n - 1$ only a single codebook is generated, and this is binned into n independent partitions. Upon receipt, in analogy with (12), this can be found uniquely with high probability if

$$R_{n-1} > \frac{1}{n-1} H(Y_{n-1} | \mathbf{Y}_{I_{n-2}I_{n-1}}, \mathbf{U}_{I_{n-2}}) - \frac{1}{n-1} H(Y_{n-1} | X, \mathbf{Y}_{I_{n-2}I_n}, \mathbf{U}_{I_{n-2}})$$

For repair, the joint $2^{nR'_k}$ codewords in $\mathcal{C}_{k1} \times \dots \times \mathcal{C}_{kn}$ at layer $k < n-1$ are binned into $2^{lR_{rk}}$ bins. The single bin number of the n chosen codewords is encoded with an $(n, n-1)$ MDS erasure code.

Now suppose node n is lost, and needs to be recovered. The repair node works from the bottom up. So, suppose the previous $k-1$ layers have been recovered, that is, $\mathbf{y}_{I_{k-1}I_b}^l, \mathbf{u}_{I_{k-1}}^l$ are known without error. First u_k^l is recovered, which can be done since $n-1 \geq k$ nodes are used. It can also decode the codewords in $\mathcal{C}_{kI_{n-1}}$. It restores the bin number of the repair codeword from the erasure code. There are approximately $2^{l(nR'_k - R_{rk})}$ codewords in the bin, but since it knows the codewords in $\mathcal{C}_{kI_{n-1}}$ there are only about $2^{l(R'_k - R_{rk})}$ valid ones. It searches in the bin for valid codewords jointly typical with $\mathbf{y}_{kI_{n-1}}^l, \mathbf{y}_{I_{k-1}I_n}^l, \mathbf{u}_{I_k}^l$. With high probability there is only one such if

$$R'_k - R_{rk} < H(Y_{kn}) - H(Y_{kn}|\mathbf{U}_{I_k}, \mathbf{Y}_{kI_{n-1}}, \mathbf{Y}_{I_{k-1}I_n})$$

$$R_{rk} > H(Y_{kn}|\mathbf{U}_{I_k}, \mathbf{Y}_{kI_{n-1}}, \mathbf{Y}_{I_{k-1}I_n}) - \frac{1}{n}H(\mathbf{Y}_{kI_n}|X, \mathbf{Y}_{k-1I_n}, \mathbf{U}_{I_k}) \quad (13)$$

There is at least one codeword in the bin, namely the correct one. Thus, if there is no error (more than one codeword), the repair is exact, as required from the exact repairability condition in Section II. ■

Theorem 5 (Collaborative repair node). *For any symmetric probability distribution $p(\mathbf{y}_{I_{n-1}I_n}, \mathbf{u}_{I_{n-1}}, y_n|x)$ and decoding functions g_J the lower convex closure of $(R, D_1, \dots, D_n)$ is achievable, where $E[d_{|J|}(X, g_J(\mathbf{Y}_{I_{|J|}J}, \mathbf{U}_{I_{|J|}}))] \leq D_{|J|}, |J| \leq n$ and*

$$R > I(X; U_1) + H(Y_{1n}|U_1) + \sum_{k=2}^{n-1} \frac{1}{k} H(\mathbf{Y}_{kI_k}|\mathbf{Y}_{I_{k-1}I_k}, \mathbf{U}_{I_k}) + \frac{1}{n} H(Y_n|\mathbf{Y}_{I_{n-1}I_n}, \mathbf{U}_{I_{n-1}}) - \frac{1}{n} H(\mathbf{Y}_{I_{n-1}I_n}|X, \mathbf{U}_{I_{n-1}}) + \sum_{k=1}^{n-1} \frac{1}{k} (H(U_k|\mathbf{Y}_{I_{k-1}I_k}, \mathbf{U}_{I_{k-1}}) - H(U_k|X, \mathbf{Y}_{I_{k-1}I_n}, \mathbf{U}_{I_{k-1}}))$$

The additional information the repair node has to store is

$$R_r > \sum_{k=1}^{n-1} \left[H(Y_{kn}|\mathbf{U}_{I_k}, \mathbf{Y}_{kI_{n-1}}, \mathbf{Y}_{I_{k-1}I_n}) - \frac{1}{n} H(\mathbf{Y}_{kI_n}|X, \mathbf{Y}_{k-1I_n}, \mathbf{U}_{I_k}) \right]^+ + \frac{1}{n} H(Y_n|\mathbf{Y}_{I_{n-1}I_n}, \mathbf{U}_{I_{n-1}}) - \frac{1}{n} H(\mathbf{Y}_{I_{n-1}I_n}|X, \mathbf{U}_{I_{n-1}})$$

A. Example Gaussian case

We consider a three node Gaussian case with mean-squared distortion and with distributed repair. This is characterized by $(R+R_r, D_1, D_2)$. We put $U = X + Q_{U1}, Y_{1i} = X + Q_{1i}, Y_2 = X + Q_2$ with $Q_{\dots}$ zero-mean Gaussian, $E[Q_{u1}^2] = \sigma_{u1}^2$, $E[Q_{1i}^2] = \sigma_{q1}^2$, $E[Q_2^2] = \sigma_{q2}^2$, $E[Q_{1i}Q_{1j}] = \rho_{1i}\sigma_{q1}^2$, and all other noise variables uncorrelated.

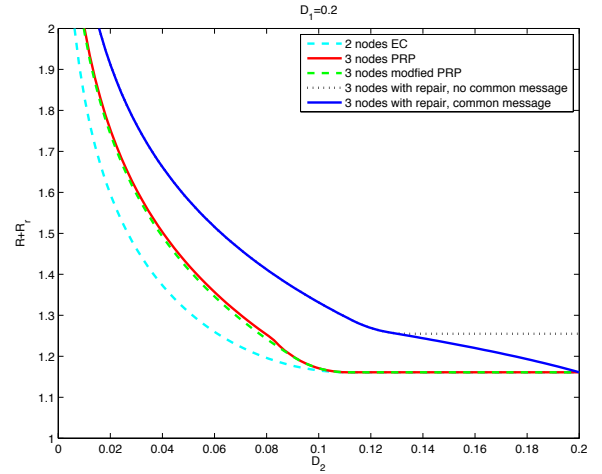


Figure 1. Plots of R or $R + R_r$ for (a) two nodes according to [1] (b) Three nodes with at most two used, without repair according to PPR [5] or Theorem 4 (modified PPR) and (c) Three nodes with distributed repair without or without common message.

Figure 1 shows typical numerical results. First, from the bottom, we have the curve for the optimum region for the two node problem according to EC [1], [2]. Second, we have the curves for the three node problem, but where we use at most two nodes for reconstruction, either using [5, Section V] directly (ignoring the D_3 constraint), or using Theorem 4 without repair. Finally, we have the curves for repair. We see that a common message gives a clear improvement.

V. CONCLUSION AND FUTURE WORK

The results show that multiple descriptions codes have to be designed with repair in mind. We have shown optimality in the two node case with a repair node. This can be generalized to the n -node two-level case considered in [8], see [?].

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