On Notions of Detectability and Observers for Hybrid Systems

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***Abstract*— Notions of observer and detectability are well established for continuous-time and discrete-time systems, and are known to be linked, since a system must be detectable to admit an observer. Unfortunately, defining such notions for a hybrid system is not straightforward because solutions do not share the same hybrid time domain. In this paper, we propose to define observers and detectability for hybrid systems, such that detectability is still necessary for the existence of an observer and such that standard definitions are recovered for continuous-time and discrete-time systems, when seen as special cases of hybrid systems. We rely on a recent definition of hybrid systems with hybrid inputs and use jump reparametrizations to define convergence and equality of outputs.**

I. INTRODUCTION

A. Context

For continuous-time (CT) and discrete-time (DT) systems, asymptotic observers are commonly defined as dynamical systems taking the plant's output as input and whose state asymptotically converges to the plant's state. The existence of such an object then requires some intrinsic properties of the plant, in particular that the plant be asymptotically detectable: the trajectories giving the same output must asymptotically converge to one another [1]. However, those notions are not straightforward to define for a hybrid plant, since they involve comparisons of hybrid solutions defined on different hybrid time domains.

When the jump times of the plant are assumed to be known, the difficulties due to a possible mismatch of the trajectories' domains disappear since the observer can be synchronized with the plant and observability notions also reduce to comparing outputs with the same time domain (see [2] and literature therein). However, when the plant's jump times are unknown, the observer must be a hybrid system which does not necessarily jump at the same time as the plant. This difficulty is avoided in [3] thanks to a change of coordinates transforming the jump map into the identity map and thus somehow making the jumps disappear in the observer. As for [4], an extended system containing both the plant and the observer is directly analyzed. In the particular setting of switched systems, the problem has been handled by estimating the switching signal ([5], [6] and literature therein.)

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As far as we know, no general notions of (incremental) detectability and observers exist in the literature for general hybrid systems. This paper thus proposes to define such notions, building from recent definitions of hybrid systems with hybrid inputs [7], [8] and the literature of hybrid reference tracking [9] and hybrid incremental stability [10], [11], where methods for the comparison of hybrid solutions were also introduced.

B. Detectability and Observers for CT Systems

Consider a CT system

$$\dot{x} = f(x) \quad , \qquad y = h(x) \tag{1}$$

initialized in a set of interest $\mathcal{X}_0$. We usually define an observer as a dynamical system of the form

$$\dot{z} = F(z, y) \quad , \qquad \hat{x} = H(z, y) \tag{2}$$

initialized in a set $\mathcal{Z}_0$ and whose *complete solutions*, i.e. solutions defined on $\mathbb{R}_{\geq 0}$, are asked to verify stability and convergence properties. If z lives in the same space as x, we may directly take $\hat{x} = z$, but that is not necessarily the case for nonlinear systems. The existence of such an observer then intrinsically requires some detectability properties of the system (1): the output y should somehow contain enough information to uniquely determine the plant's state.

Definition 1.1: The system (2) is an *asymptotic observer* of (1) on $\mathcal{X}_0$ if there exists a (known) set of initial conditions $\mathcal{Z}_0$ such that for any complete solution x of (1) initialized in $\mathcal{X}_0$, any maximal solution z of (2) initialized in $\mathcal{Z}_0$ with input $y = h(x)$ is also complete and verifies $\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0$.

Definition 1.2: The system (1) is *asymptotically detectable* on $\mathcal{X}_0$ if any pair of complete solutions x_a and x_b of (1) initialized in $\mathcal{X}_0$ such that

$$h(x_a(t)) = h(x_b(t)) \quad \forall t \in \mathbb{R}_{\geq 0} \tag{3}$$

verifies $\lim_{t\to+\infty} |x_a(t) - x_b(t)| = 0$.

As shown for instance in [1], this detectability property is necessary for the existence of an observer.

Theorem 1.3: If system (1) *admits an* asymptotic observer *on* $\mathcal{X}_0$*, then system* (1) *is* asymptotically detectable *on* $\mathcal{X}_0$.

Proof: Consider complete solutions x_a and x_b of (1) such that $h(x_a(t)) = h(x_b(t))$ for all $t \geq 0$. Take a maximal solution z of (2) initialized in $\mathcal{Z}_0$ with input $y = h(x_a)$. Then, by definition, z is complete and $\lim_{t\to+\infty} |x_a(t) - \hat{x}(t)| = 0$ with $\hat{x} = H(z, h(x_a))$. But since $h(x_a) = h(x_b)$, z is also solution of (2) with input

$y = h(x_b)$, and thus $\lim_{t\to+\infty} |x_b(t) - \hat{x}(t)| = 0$ with $\hat{x} = H(z, h(x_a)) = H(z, h(x_b))$. It thus follows by triangle inequality that $\lim_{t\to+\infty} |x_a(t) - x_b(t)| = 0$. ■

C. Towards Hybrid Systems

Consider now a general hybrid system

$$\mathcal{H}\left\{\begin{array}{ll} \dot{x} \in f(x) & x \in C \\ x^+ \in g(x) & x \in D \end{array}\right. , \quad y = h(x) \tag{4}$$

with state $x \in \mathbb{R}^{d_x}$ and output $y \in \mathbb{R}^{d_y}$. The solutions are now hybrid arcs $(t, j) \mapsto x(t, j)$ defined on a hybrid time domain $\operatorname{dom} x \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N}$ according to [12], with both continuous-time evolution in C and discrete events in D (flows and jumps).

If we want to properly define notions of observers and detectability as it has been done for CT/DT systems, we need to think about both definitions together in a way that ensures that detectability is a necessary condition for the existence of an observer. Besides, both CT and DT detectability/observers should be recovered as particular cases when $D = \emptyset$ and $C = \emptyset$ respectively. The main difficulties are as follows:

- The observer should be a hybrid system taking the (hybrid) output of $\mathcal{H}$ as input, but the latter has its own hybrid time domain that may differ from the hybrid time domain of the observer's solution. Hence, a more general notion of solution must be used;
- Trajectories do not share a common time domain. This makes the comparison between x_a and x_b for detectability, or between x and $\hat{x}$ for observers, not straightforward;
- Completeness can happen either in the time-horizon (if t goes to $+\infty$) or the jump horizon (if j goes to $+\infty$) and at different times;
- Asking for exact convergence of $\hat{x}$ to x may be too restrictive around the jump times where an arbitrarily small mismatch of jump times between $\hat{x}$ and x leads to a significant error if $g \neq \mathrm{Id}$, i.e., g is not the identity map.

A notion of solutions to hybrid systems with hybrid inputs was proposed in [7], [8], relying on a so-called *jump reparametrization*. This process, recalled in Section II, enables to compare $\hat{x}$ and x on a common domain and thus to formulate a definition of asymptotic observers that resembles Definition 1.1. Note that more general notions of convergence will be allowed, with $(\hat{x}, x)$ required to converge to a set $\mathcal{A} \subset \mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$ possibly larger than

$$\mathcal{A} = \left\{(x, \hat{x}) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_x} \ : \ x = \hat{x}\right\} . \tag{5}$$

Then, in order to determine whether two hybrid outputs are "equal" as in Definition 1.2, we propose in Section III an algorithm that enables to write two hybrid arcs on a common time domain and thus define *asymptotic detectability relative to* $\mathcal{A}$. Regarding the completeness condition, we will see that only complete solutions sharing the same time horizon need to be compared.

The link between these definitions and more intuitive extended systems is investigated in Section IV.

Finally, in Section V, we show that the proposed definitions preserve the tight link between asymptotic detectability and observers exhibited for CT systems in Theorem 1.3, namely, we prove the following main result.

Theorem 1.4: Let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$. If $\mathcal{H}$ admits an asymptotic observer on $\mathcal{X}_0$ relative to $\mathcal{A}$, then $\mathcal{H}$ is asymptotically detectable on $\mathcal{X}_0$ relative to $\mathcal{A}$.

D. Notations and Preliminaries

We denote by $\mathbb{R}$ (resp. $\mathbb{N}$) the set of real (resp. natural) numbers, and $\mathbb{R}_{\geq 0} := [0, +\infty)$, $\mathbb{R}_{>0} := (0, +\infty)$, and $\mathbb{N}_{>0} := \mathbb{N}\backslash\{0\}$. For a set $\mathcal{S}$, $\mathrm{cl}(\mathcal{S})$ denotes its closure, $\mathrm{int}(\mathcal{S})$ its interior, and $\operatorname{card} \mathcal{S}$ its cardinality (possibly infinite).

The set of maximal solutions to a hybrid system $\mathcal{H}$ initialized in $\mathcal{X}_0$ is denoted $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$, or $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0; u)$ if $\mathcal{H}$ takes u as input. For a hybrid arc $(t, j) \mapsto \phi(t, j)$ defined on a hybrid time domain $\operatorname{dom} \phi$, we denote $\operatorname{dom}_t \phi$ (resp. $\operatorname{dom}_j \phi$) its projection on the time (resp. jump) axis, and for a positive integer j, $t_j(\phi)$ the time stamp associated to the jth jump (i.e., the only time satisfying $(t_j(\phi), j) \in \operatorname{dom} \phi$ and $(t_j(\phi), j-1) \in \operatorname{dom} \phi$), and $\mathcal{I}_j(\phi)$ the largest interval such that $\mathcal{I}_j(\phi) \times \{j\} \subseteq \operatorname{dom} \phi$. We define also $\mathcal{T}(\phi) = \{t_j(\phi) \ : \ j \in \operatorname{dom}_j \phi \cap \mathbb{N}_{>0}\}$ as the set of jump times of ϕ, $T(\phi) = \sup \operatorname{dom}_t \phi \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ the maximal time of the domain, $J(\phi) = \sup \operatorname{dom}_j \phi \in \mathbb{N} \cup \{+\infty\}$ the total number of jumps, and, for a time t in $\mathbb{R}_{\geq 0}$, $\mathcal{J}_t(\phi) = \{j \in \mathbb{N}_{>0} \ : \ t_j(\phi) = t\}$ the set of jump counters associated to the jumps occurring at time t. It follows that $\operatorname{card} \mathcal{J}_t(\phi)$ is the number of jumps of ϕ occurring at time t. A hybrid arc ϕ is said to be t-complete (resp. j-complete) if $\operatorname{dom}_t \phi$ (resp. $\operatorname{dom}_j \phi$) is unbounded, complete if $\operatorname{dom} \phi$ is unbounded, and Zeno if it is complete with $\sup \operatorname{dom}_t \phi < \infty$.

Finally, we will need to consider convergence to a subset $\mathcal{A}$ of $\mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$. For that, a map $d_{\mathcal{A}} : \mathbb{R}^{d_x} \times \mathbb{R}^{d_x} \to \mathbb{R}_{\geq 0}$ defines a distance relative to $\mathcal{A}$ if for all x_a, x_b, x_c in $\mathbb{R}^{d_x}$,

$$d_{\mathcal{A}}(x_a, x_b) = 0 \iff (x_a, x_b) \in \mathcal{A} \tag{6a}$$

$$d_{\mathcal{A}}(x_a, x_b) = d_{\mathcal{A}}(x_b, x_a) \tag{6b}$$

$$d_{\mathcal{A}}(x_a, x_b) \leq d_{\mathcal{A}}(x_a, x_c) + d_{\mathcal{A}}(x_c, x_b) \tag{6c}$$

II. Hybrid Asymptotic Observers

Inspired from (2), we define an observer as a hybrid system taking the plant's output y as input and producing an estimate $\hat{x}$ of the plant's state as output, namely

$$\hat{\mathcal{H}}\left\{\begin{array}{ll} \dot{z} \in F(z, y) & (z, y) \in \hat{C} \\ z^+ \in G(z, y) & (z, y) \in \hat{D} \end{array}\right. , \quad \hat{x} = H(z, y) \tag{7}$$

with state $z \in \mathbb{R}^{d_z}$, such that "$\hat{x}$ converges to x" in some sense. First, solutions to (7) must be defined with care because the hybrid input y coming from the plant $\mathcal{H}$ has its own time domain and its jumps have no reason to happen when (z, y) is in the jump set $\hat{D}$. Therefore, their jumps are not necessarily synchronized. Appropriate definitions have been given in [8] which we briefly recall next.

A. Reparametrization and Definition of Solutions

Definition 2.1 ([7], [8]): Given a hybrid arc ϕ, a hybrid arc ϕ^{r} is a *j-reparametrization* of ϕ if there exists a function

$\rho : \mathbb{N} \to \mathbb{N}$ such that

$$\rho(0) = 0 \quad , \qquad \rho(j+1) - \rho(j) \in \{0, 1\} \quad \forall j \in \mathbb{N} \quad (8)$$

$$\phi^{\mathrm{r}}(t, j) = \phi(t, \rho(j)) \qquad \forall (t, j) \in \operatorname{dom} \phi^{\mathrm{r}} \ . \quad (9)$$

The hybrid arc ϕ^{r} is a *full j-reparametrization* of ϕ if

$$\operatorname{dom} \phi = \bigcup_{(t,j) \in \operatorname{dom} \phi^{\mathrm{r}}} (t, \rho(j)) \ . \quad (10)$$

The map ρ is called *j-reparametrization map* from ϕ to ϕ^{r}.

In other words, ϕ^{r} takes at each time t the same values as ϕ, but maybe associated to a different jump index: initially $\phi^{\mathrm{r}}(t, 0) = \phi(t, 0)$ for all $t \in \mathcal{I}_0(\phi^r)$, and when ϕ^r jumps,

- either $\rho(1) = 1$ and $\phi^{\mathrm{r}}(t, 1) = \phi(t, 1)$ for all $t \in \mathcal{I}_1(\phi^r)$,
- or $\rho(1) = 0$ and $\phi^{\mathrm{r}}(t, 1) = \phi(t, 0)$ for all $t \in \mathcal{I}_1(\phi^r)$,

and so on. This means that if $\rho(j+1) = \rho(j) + 1$, the jth jump of ϕ^{r} corresponds to an actual jump in the domain of ϕ, and if $\rho(j+1) = \rho(j)$, ϕ^{r} exhibits a jump that ϕ does not exhibit and, necessarily,

$$\begin{aligned} \phi^{\mathrm{r}}(t_{j+1}, j+1) &= \phi(t_{j+1}, \rho(j+1)) = \phi(t_{j+1}, \rho(j)) \\ &= \phi^{\mathrm{r}}(t_{j+1}, j) \end{aligned}$$

namely, the jump is trivial. Therefore, as long as it is defined, ϕ^{r} is the "same" as ϕ, modulo additional trivial jumps. If the whole hybrid arc ϕ is "contained" in ϕ^{r}, then the parametrization is "full".

[8, Definition 4] defines solutions to hybrid systems with hybrid inputs like $\hat{\mathcal{H}}$ as pairs $\phi = (z, y^{\mathrm{r}})$ where y^{r} is a j-reparametrization of y that is defined on the same domain as z. An algorithm to build ϕ is provided in [8] and can be summed up as the following. As long as the input y does not jump, z evolves like in a standard hybrid system, z flowing according to F if ϕ is in $\hat{C}$, and jumping according to G if ϕ is in $\hat{D}$. In this case, a trivial jump is added to y^{r}. On the other hand, when y jumps, z can either jump according to G or be reset identically, depending on whether ϕ is in $\hat{C}, \hat{D}$ or both. The precise jump logic is recalled in Appendix.

B. Definition of Asymptotic Observer for $\mathcal{H}$

Definition 2.2: Let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$. The hybrid system $\hat{\mathcal{H}}$ is an *asymptotic observer* for $\mathcal{H}$ on $\mathcal{X}_0 \subseteq \mathbb{R}^{d_x}$ relative to $\mathcal{A}$ if there exist a distance function $d_{\mathcal{A}}$ relative to $\mathcal{A}$ and a subset $\mathcal{Z}_0$ of $\mathbb{R}^{d_z}$ such that for any complete plant solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ with output y and any observer solution $\phi = (z, y^{\mathrm{r}}) \in \mathcal{S}_{\hat{\mathcal{H}}}(\mathcal{Z}_0; y)$ with output $\hat{x}$ and j-reparametrization map ρ from y to y^{r}:

(a) ϕ is complete with $\operatorname{dom}_t \phi = \operatorname{dom}_t x$;

(b) denoting x^{r} the full j-reparametrization of x on the domain of ϕ defined by

$$x^{\mathrm{r}}(t, j) = x(t, \rho(j)) \qquad \forall (t, j) \in \operatorname{dom} \phi \ ,$$

we have

$$\lim_{t+j \to +\infty} d_{\mathcal{A}}\Big(\hat{x}(t, j), x^{\mathrm{r}}(t, j)\Big) = 0 \ . \quad (11)$$

Condition (a) ensures that the observer solution exists as long as the underlying plant solution x does. In particular, the extra condition $\operatorname{dom}_t x = \operatorname{dom}_t \phi$ means that they both "achieve their completeness" at the same time:

- either they are both t-complete;
- or they are both Zeno with same dom_t.

As for Condition (b), it translates the intuitive idea of "$\hat{x}$ converges to x" (relative to $\mathcal{A}$), even if $\hat{x}$ and x do not share the same domain. This is done by *reparametrizing* x into x^{r}, which is defined on the domain of $\hat{x}$.

When convergence of $\hat{x}$ to x is required, $\mathcal{A}$ can be chosen as the diagonal set (5). However, the generic set $\mathcal{A}$ in Definition 2.2 allows to consider more general notions of convergence of $(x, \hat{x})$. This is important because exact convergence of $\hat{x}$ to x is in general difficult to obtain unless $g = \mathrm{Id}$ or unless the jumps of the observer become perfectly synchronized with those of the plant after some time. Indeed, if $\hat{x}$ and x do not jump exactly at the same time and $g \neq \mathrm{Id}$, it may not be possible to make the estimation error $\hat{x} - x$ small: if $x = \hat{x}$ before the jump, then $\hat{x} \in g(x) \cup g^{-1}(x)$ after one jump of either x or $\hat{x}$. This is the so-called *peaking* phenomenon. In that case, denoting

$$\underline{g}(x) = \begin{cases} g(x) & \text{if } x \in D \\ \emptyset & \text{otherwise} \end{cases} \ , \quad (12)$$

we can only expect $(x, \hat{x})$ to converge to

$$\mathcal{A} = \Big\{ (x, \hat{x}) \in (C \cup D \cup g(D))^2 \ : \\ x = \hat{x} \ \text{ or } \ x \in \underline{g}(\hat{x}) \ \text{ or } \ \hat{x} \in \underline{g}(x) \Big\} \ , \quad (13)$$

as in [4], or even to a larger set when consecutive jumps are possible [9]. More generally, we might be interested in estimating only part of the state x, which can be captured by a proper choice of $\mathcal{A}$.

III. Hybrid Asymptotic Detectability

In order to define detectability in a way that extends Definition 1.2, we need to compare the outputs of two hybrid solutions and decide whether they are "equal" in some sense.

A. Motivation

Methods to compare hybrid arcs have been developed in the literature. In [12, Definition 5.23], notions of ε and (ε, τ)-closeness were first introduced. This distance is related to the graphical distance of the graphs of the hybrid arcs, namely hybrid arcs are compared at the same jump index and "close in time." It was used in the context of incremental stability [13], but was then observed to be too restrictive [10] and was consequently relaxed in [10, Definition 4] or in [11, Definition 1] by allowing to compare solutions "close in time" but maybe at different jump indexes.

In the context of detectability, we must decide under which conditions on the outputs we want to require trajectories to converge to each other. Our approach is to think of detectability as a necessary condition for the existence of an observer, similarly to CT and DT systems. Therefore, the question becomes: which pairs of outputs would the observer in Definition 2.2 not distinguish? The observer would then produce asymptotically the same estimate and the corresponding plant trajectories would thus have to converge to each other, along a similar reasoning as in Theorem 1.3.

First, we notice that Definition 2.2 concerns only complete trajectories and that the estimate $\hat{x}$ is also complete with the same time horizon. Therefore, in comparing pairs of complete solutions (x_a, x_b) of $\mathcal{H}$ with the "same output", the only way we can exploit the observer asymptotic convergence is if $\mathrm{dom}_t\, x_a = \mathrm{dom}_t\, x_b$.

The meaning of "same output" should then be clarified. In the spirit of graphical distance [12], equality of outputs would require equality of the time domains. This is however restrictive because the observer may not either be able to distinguish outputs that are the same up to trivial jumps added to their domains. On the other hand, the spirit of [10, Definition 4] would consider two outputs y_a and y_b "equal" if for all $(t, j) \in \mathrm{dom}\, y_a$,

$$\exists j' \in \mathbb{N} \;:\; (t, j') \in \mathrm{dom}\, y_b \;,\; y_a(t, j) = y_b(t, j')$$

and vice-versa. This time, this definition would be too broad since it does not respect the causality/order of the jumps which indeed is seen by the observer. In particular, this definition would not apply to DT systems.

All in all, we propose an intermediate definition based on an algorithm that *reparametrizes* two hybrid arcs onto a common time domain in order to compare them pointwisely, while preserving the order and simultaneity of the jumps.

B. Algorithm $\mathcal{R}_c$

Two hybrid arcs x_a and x_b can be reparametrized onto a common hybrid time domain, constructed by either

- preserving the time stamp and, as time evolves, adding jumps whenever either x_a or x_b jumps. When only one hybrid arc jumps, a trivial jump is added to the other; when both jump, their jumps are recorded simultaneously;
- or preserving the jump numbering and letting both hybrid arcs flow until they can both jump at the same time. When one arc flows for a longer time than the other, the other arc is kept constant while waiting for the other's jump time.

Given the role of time in applications, we explore the first process as formalized in Algorithm 3.1. By preserving the order and simultaneity of the jumps, this process also applies to discrete hybrid arcs.

Definition 3.1: Given two hybrid arcs x_a and x_b, we define the reparametrized hybrid arcs $(x_a^{\mathrm{r}}, x_b^{\mathrm{r}}) := \mathcal{R}_c(x_a, x_b)$ by Algorithm 3.1.

Algorithm 3.1 preserves the time stamp, but changes the jump numbering, and it stops when one of the arcs has reached the end of its domain. It thus gives pairs $(x_a^{\mathrm{r}}, x_b^{\mathrm{r}})$ defined on a common time domain which are j-reparametrizations of x_a and x_b, at least on the "common" part of their domains. However, it can happen that x_a ends earlier and "blocks" x_b so that x_b^{r} does not contain all the information about x_b, i.e., it is not a full j-reparametrization.

Lemma 3.2: Consider two complete hybrid arcs x_a and x_b such that $\mathrm{dom}_t\, x_a = \mathrm{dom}_t\, x_b$. Then, the hybrid arc

Algorithm 3.1 $(x_a^{\mathrm{r}}, x_b^{\mathrm{r}}) = \mathcal{R}_c(x_a, x_b)$

$j \leftarrow 0 \;,\; t_j \leftarrow 0 \;,\; j_a \leftarrow 0 \;,\; j_b \leftarrow 0$
$\mathcal{I}_a \leftarrow \{t \in \mathbb{R}_{\geq 0} \;:\; (t, j_a) \in \mathrm{dom}\, x_a\}$
$\mathcal{I}_b \leftarrow \{t \in \mathbb{R}_{\geq 0} \;:\; (t, j_b) \in \mathrm{dom}\, x_b\}$
while $\mathcal{I}_a \neq \emptyset$ and $\mathcal{I}_b \neq \emptyset$ **do**
  $T_{m,a} \leftarrow \sup \mathcal{I}_a \;,\; T_{m,b} \leftarrow \sup \mathcal{I}_b$
  $T_m \leftarrow \min\{T_{m,a}, T_{m,b}\}$ ▷ $\min\{+\infty, +\infty\} = +\infty$
  if $(T_m, j_a) \notin \mathrm{dom}\, x_a$ or $(T_m, j_b) \notin \mathrm{dom}\, x_b$ **then**
    $x_a^{\mathrm{r}}(t, j) \leftarrow x_a(t, j_a) \quad \forall t \in [t_j, T_m)$
    $x_b^{\mathrm{r}}(t, j) \leftarrow x_b(t, j_b) \quad \forall t \in [t_j, T_m)$
  else
    $x_a^{\mathrm{r}}(t, j) \leftarrow x_a(t, j_a) \quad \forall t \in [t_j, T_m]$
    $x_b^{\mathrm{r}}(t, j) \leftarrow x_b(t, j_b) \quad \forall t \in [t_j, T_m]$
  end if
  if $T_m = T_{m,a}$ **then**
    $j_a \leftarrow j_a + 1$
  end if
  if $T_m = T_{m,b}$ **then**
    $j_b \leftarrow j_b + 1$
  end if
  $j \leftarrow j + 1 \;,\; t_j \leftarrow T_m$
  $\mathcal{I}_a \leftarrow \{t \in \mathbb{R}_{\geq 0} \;:\; (t, j_a) \in \mathrm{dom}\, x_a\}$
  $\mathcal{I}_b \leftarrow \{t \in \mathbb{R}_{\geq 0} \;:\; (t, j_b) \in \mathrm{dom}\, x_b\}$
end while
return $(x_a^{\mathrm{r}}, x_b^{\mathrm{r}})$

$(x_a^{\mathrm{r}}, x_b^{\mathrm{r}}) = \mathcal{R}_c(x_a, x_b)$ is such that both x_a^{r} and x_b^{r} are full j-reparametrizations of x_a and x_b, respectively.

Proof: See the report version [14]. ■

C. A Definition of Asymptotic Detectability

The following definition extends Definition 1.2.

Definition 3.3: Let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$. The hybrid plant $\mathcal{H}$ is said to be *asymptotically detectable* on $\mathcal{X}_0$ relative to $\mathcal{A}$ if there exists a distance function $d_{\mathcal{A}}$ relative to $\mathcal{A}$ such that any pair of complete solutions $x_a, x_b \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ verifying $\mathrm{dom}_t\, x_a = \mathrm{dom}_t\, x_b$ and

$$h(x_a^{\mathrm{r}}(t, j)) = h(x_b^{\mathrm{r}}(t, j)) \quad \forall (t, j) \in \mathrm{dom}\, \phi^{\mathrm{r}} \tag{14}$$

where $(x_a^{\mathrm{r}}, x_b^{\mathrm{r}}) := \mathcal{R}_c(x_a, x_b)$, verify

$$\lim_{t+j \to +\infty} d_{\mathcal{A}}(x_a^{\mathrm{r}}(t, j), x_b^{\mathrm{r}}(t, j)) = 0 \;. \tag{15}$$

IV. Observers and Detectability via Extended Systems

Before proceeding to the proof of Theorem 1.4, we exhibit the link between our definitions and alternative definitions via extended systems, which will be useful for the proof.

A. Observers

Instead of defining an observer as $\hat{\mathcal{H}}$ in (7), a first idea could have been to define an observer directly through an extended system of the form

$$\hat{\mathcal{H}}_{\rm ext}\left\{\begin{array}{ll}\begin{pmatrix}\dot{x}\\ \dot{z}\end{pmatrix}\in\begin{pmatrix}f(x)\\ F(z,h(x))\end{pmatrix} & (x,z)\in C\times\hat{C}\\ \begin{pmatrix}x^+\\ z^+\end{pmatrix}\in\ G_{\rm ext}(x,z) & (x,z)\in\ \hat{D}_{\rm ext}\\ \hat{x}=H(z,h(x)) & \end{array}\right. \tag{16}$$

with a jump map $G_{\rm ext}$ and a jump set $\hat{D}_{\rm ext}$ to be defined. In doing that, we are facing three main difficulties. First, a jump logic has to be designed in a way that does not assume synchronous jumps of z and x since the jump times of the plant $\mathcal{H}$ are not necessarily known. Second, it is not straightforward to deduce from $\hat{\mathcal{H}}_{\rm ext}$ the hybrid dynamics of z to be implemented as an observer algorithm with input y and output $\hat{x}$. Third, without any assumption about the domain of solutions to $\mathcal{H}$, we would require something like: any complete solution of $\hat{\mathcal{H}}_{\rm ext}$ initialized in $\mathcal{X}_0\times\mathcal{Z}_0$ verifies $\lim_{t+j\to+\infty} d_{\mathcal{A}}\left(x(t,j),\hat{x}(t,j)\right)=0$. But a solution to $\hat{\mathcal{H}}_{\rm ext}$ may be complete without browsing the whole underlying maximal solution of $\mathcal{H}$, for instance if the z-component induced Zeno or finite time escape earlier than x. Therefore, $\hat{x}$ would not provide any estimate of x after a certain time, which is not acceptable.

This being said, an extended system of the form (16) may be handy for design since it allows for Lyapunov analysis. Actually, in [2, Section 4.1], solutions $(z,y^{\rm r})$ to $\hat{\mathcal{H}}$ are shown to be such that $(x^{\rm r},z)$ is solution to $\hat{\mathcal{H}}_{\rm ext}$ with jump set

$$\hat{D}_{\rm ext}=\Big\{(x,z)\in\mathbb{R}^{d_x}\times\mathbb{R}^{d_z}:x\in D\ ,\ (z,h(x))\in{\rm cl}(\hat{C})\cup\hat{D}\Big\}\\ \cup\Big\{(x,z)\in\mathbb{R}^{d_x}\times\mathbb{R}^{d_z}:x\in{\rm cl}(C)\cup D\ ,\ (z,h(x))\in\hat{D}\Big\}$$

and jump map

$$G_{\rm ext}(x,z)=\begin{pmatrix}\underline{g}(x)\\ \underline{\rm Id}_z(z,h(x))\end{pmatrix}\cup\begin{pmatrix}\underline{\rm Id}_x(x)\\ \underline{G}(z,h(x))\end{pmatrix}\cup\begin{pmatrix}\underline{g}(x)\\ \underline{G}(z,h(x))\end{pmatrix}$$

where $\underline{g}$ is defined in (12), and in the same spirit

$$\underline{G}(z,h(x))=\left\{\begin{array}{ll}G(z,h(x)) & \text{if } (z,h(x))\in\hat{D}\\ \emptyset & \text{otherwise}\end{array}\right.,$$

$$\underline{\rm Id}_x(x)=\left\{\begin{array}{ll}x & \text{if } x\in{\rm cl}(C)\\ \emptyset & \text{otherwise}\end{array}\right.,\tag{17}$$

$$\underline{\rm Id}_z(z,h(x))=\left\{\begin{array}{ll}z & \text{if } (z,h(x))\in{\rm cl}(\hat{C})\\ \emptyset & \text{otherwise}\end{array}\right..$$

Therefore, any analysis made on $\hat{\mathcal{H}}_{\rm ext}$ may hold for solutions of $\hat{\mathcal{H}}$. However, the reverse is not true because $\hat{\mathcal{H}}_{\rm ext}$ has a larger set of solutions, see [8] for more details.

Lemma 4.1: Let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}^{d_x}\times\mathbb{R}^{d_x}$ and $d_{\mathcal{A}}$ a distance relative to $\mathcal{A}$. Assume any $x\in\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ is t-complete. If each $(x,z)\in\mathcal{S}_{\hat{\mathcal{H}}_{\rm ext}}(\mathcal{X}_0\times\mathcal{Z}_0)$ is t-complete and verifies $\lim_{t+j\to+\infty} d_{\mathcal{A}}\left(x(t,j),\hat{x}(t,j)\right)=0$, then $\hat{\mathcal{H}}$ is an asymptotic observer for $\mathcal{H}$ on $\mathcal{X}_0$ relative to $\mathcal{A}$.

Proof: See the report version [14]. ■

Remark 4.2: We allow here both x and z to jump simultaneously whereas this kind of jump is decomposed into two successive jumps in [9], [10]. Our goal is indeed to cover the framework of DT systems when $C=\hat{C}=\emptyset$.

B. Detectability

Similarly to $\hat{\mathcal{H}}_{\rm ext}$, it can be proved that given $x_a,x_b\in\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$, $(x_a^{\rm r},x_b^{\rm r}):=\mathcal{R}_c(x_a,x_b)$ is a solution to the extended hybrid system

$$\mathcal{H}^{\rm r}\left\{\begin{array}{ll}\begin{pmatrix}\dot{x}_a\\ \dot{x}_b\end{pmatrix}\in\begin{pmatrix}f(x_a)\\ f(x_b)\end{pmatrix} & (x_a,x_b)\in C\times C\\ \begin{pmatrix}x_a^+\\ x_b^+\end{pmatrix}\in\ g^{\rm r}(x_a,x_b) & (x_a,x_b)\in D^{\rm r}\end{array}\right.\tag{18}$$

where

$$D^{\rm r}=\Big\{(x_a,x_b)\in\mathbb{R}^{d_x}\times\mathbb{R}^{d_x}\ :x_a\in D\ ,\ x_b\in{\rm cl}(C)\cup D\Big\}\\ \cup\Big\{(x_a,x_b)\in\mathbb{R}^{d_x}\times\mathbb{R}^{d_x}:x_a\in{\rm cl}(C)\cup D\ ,\ x_b\in D\Big\}$$

$$g^{\rm r}(x_a,x_b)=\begin{pmatrix}\underline{g}(x_a)\\ \underline{\rm Id}_x(x_b))\end{pmatrix}\cup\begin{pmatrix}\underline{\rm Id}_x(x_a)\\ \underline{g}(x_b)\end{pmatrix}\cup\begin{pmatrix}\underline{g}(x_a)\\ \underline{g}(x_b))\end{pmatrix}$$

with $\underline{g}$ and $\underline{\rm Id}$ defined in (12) and (17).

Lemma 4.3: Let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}^{d_x}\times\mathbb{R}^{d_x}$ and $d_{\mathcal{A}}$ a distance relative to $\mathcal{A}$. If each complete solution $\phi=(x_a,x_b)\in\mathcal{S}_{\mathcal{H}^{\rm r}}(\mathcal{X}_0\times\mathcal{X}_0)$ such that

$$h(x_a(t,j))=h(x_b(t,j))\quad\forall(t,j)\in{\rm dom}\,\phi$$

verifies $\lim_{t+j\to+\infty} d_{\mathcal{A}}\left(x_a(t,j),x_b(t,j)\right)=0$, then $\mathcal{H}$ is asymptotically detectable on $\mathcal{X}_0$ relative to $\mathcal{A}$.

Note that this handy condition for detectability is sufficient but not necessary. Indeed, unless trajectories are t-complete, a complete trajectory of $\mathcal{H}^{\rm r}$ could correspond to $\mathcal{R}_c(x_a,x_b)$ with x_a and x_b not verifying ${\rm dom}_t\, x_a={\rm dom}_t\, x_b$, which are not required to converge to each other in Definition 3.3, nor for the existence of an observer.

More generally, even if the trajectories are t-complete, the jump map $g^{\rm r}$ allows x_a and x_b to jump consecutively using first $\begin{pmatrix}g(x_a)\\ x_b\end{pmatrix}$ and then $\begin{pmatrix}x_a\\ g(x_b)\end{pmatrix}$ whenever x_a and x_b are in $D\cap{\rm cl}(C)$, whereas x_a and x_b solutions to $\mathcal{H}$ are forced to jump from $D\cap{\rm cl}(C)$ if no flow is possible from there. In that case, this jump would be recorded simultaneously in $\mathcal{R}_c(x_a,x_b)$. Therefore, $\mathcal{H}$ could be detectable relative to $\mathcal{A}$ defined in (5) without $\mathcal{H}^{\rm r}$ verifying the assumption of Lemma 4.3.

V. Proof of Theorem 1.4

The proof follows the same ideas as those in the proof of Theorem 1.3, but requires extra technical steps to take care of the different hybrid time domains. We give here only a sketch of the proof which is fully available in [14].

Consider complete solutions $(x_a,x_b)\in\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)\times\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ with $\mathcal{H}$ in (4) such that ${\rm dom}_t\, x_a={\rm dom}_t\, x_b$ and $\phi^{\rm r}:=(x_a^{\rm r},x_b^{\rm r})=\mathcal{R}_c(x_a,x_b)$ verifies (14). According to Lemma 3.2, $x_a^{\rm r}$ and $x_b^{\rm r}$ are full j-reparametrizations of x_a and x_b.

Consider $\phi_b = (z_b, y_{b,\text{cl}}) \in \mathcal{S}_{\hat{\mathcal{H}}}(\mathcal{Z}_0; y_b)$ with $y_b = h(x_b)$. Define the corresponding full j-reparametrization of x_b with

$$x_{b,\text{cl}}(t,j) = x_b(t, \rho_b(j)) \quad \forall (t,j) \in \operatorname{dom} \phi_b \ . \tag{19}$$

From [8, Lemma 1], $\phi_{b,\text{cl}} = (x_{b,\text{cl}}, z_b) \in \mathcal{S}_{\hat{\mathcal{H}}_{\text{ext}}}(\mathcal{X}_0 \times \mathcal{Z}_0)$ and, by definition of asymptotic observer, $d_{\mathcal{A}}(\hat{x}_b, x_{b,\text{cl}})$ with $\hat{x}_b = H(z_b, h(x_{b,\text{cl}}))$ asymptotically converges to 0.

Put x_a and $\phi_{b,\text{cl}}$ on a same domain, namely consider $\bar{\phi} = (\bar{x}_a, (\bar{x}_b, \bar{z})) = \mathcal{R}_c(x_a, (x_{b,\text{cl}}, z_b))$. According to Lemma 3.2, $\bar{x}_a$ and $(\bar{x}_b, \bar{z})$ are full j-reparametrizations of x_a and $(x_{b,\text{cl}}, z_b)$ respectively, which means that

$$\lim_{t+j \to +\infty} d_{\mathcal{A}}\Big(\bar{\hat{x}}(t,j), \bar{x}_b(t,j)\Big) = 0 \tag{20}$$

with $\bar{\hat{x}} = H(\bar{z}, h(\bar{x}_b))$.

Prove that $(\bar{x}_a, \bar{x}_b)$ is a full j-parametrization of $\phi^{\text{r}} = (x_a^{\text{r}}, x_b^{\text{r}}) = \mathcal{R}_c(x_a, x_b)$ and thus

$$h(\bar{x}_a(t,j)) = h(\bar{x}_b(t,j)) \quad \forall (t,j) \in \operatorname{dom} \bar{\phi} \ . \tag{21}$$

Prove that $\bar{\phi} = (\bar{x}_a, (\bar{x}_b, \bar{z}))$ is solution to an extended system $\mathcal{R}_c(\mathcal{H}, \hat{\mathcal{H}}_{\text{ext}})$, and from (21), deduce that $(\bar{x}_b, (\bar{x}_a, \bar{z}))$ is also solution to $\mathcal{R}_c(\mathcal{H}, \hat{\mathcal{H}}_{\text{ext}})$. Then, extract from $(\bar{x}_a, \bar{z})$ a solution $\phi_{a,\text{cl}} = (x_{a,\text{cl}}, z_a)$ to $\hat{\mathcal{H}}_{\text{ext}}$ such that $(\bar{x}_a, \bar{z})$ is a full j-reparametrization of $(x_{a,\text{cl}}, z_a)$.

Prove that there exists x'_a solution to $\mathcal{H}$ initialized in $\mathcal{X}_0$ such that $(z_a, y_{a,\text{cl}})$ is solution to $\hat{\mathcal{H}}$ with input $y'_a = h(x'_a)$ where $y_{a,\text{cl}} = h(x_{a,\text{cl}})$ and $x_{a,\text{cl}}$ is a full j-reparametrization of x'_a. Deduce that $d_{\mathcal{A}}(\hat{x}_a, x_{a,\text{cl}})$ with $\hat{x}_a = H(z_a, h(x_{a,\text{cl}}))$ vanishes asymptotically.

Since $(\bar{x}_a, \bar{z})$ is a full j-reparametrization of $(x_{a,\text{cl}}, z_a)$, deduce that $d_{\mathcal{A}}\left(H(\bar{z}, h(\bar{x}_a)), \bar{x}_a\right)$ asymptotically converges to 0, and so does $d_{\mathcal{A}}(\bar{\hat{x}}, \bar{x}_a)$ with (21). By triangle inequality, using (6) and (20), $d_{\mathcal{A}}(\bar{x}_a, \bar{x}_b)$ converges to 0.

Conclude that $d_{\mathcal{A}}(x_a^{\text{r}}, x_b^{\text{r}})$ also converges to 0, which ensures asymptotic detectability of $\mathcal{H}$ along Definition 3.3.

Remark 5.1: This proof, similarly to the proof of Theorem 1.3, heavily relies on a triangle inequality, so property (6c) is crucial. This differs from [10] where the distance is only required to be definite (6a) and symmetric (6b).

APPENDIX

Definition 1.1: Consider a hybrid arc y. A pair $\phi = (z, y^{\text{r}})$ is a solution to $\hat{\mathcal{H}}$ in (7) with input y and output $\hat{x}$ if

1) $\operatorname{dom} z = \operatorname{dom} y^{\text{r}} (= \operatorname{dom} \phi)$
2) y^{r} is a j-reparametrization of y with reparametrization map ρ_y, and with also $\operatorname{card} \mathcal{J}_{T(y)}(\phi) = \operatorname{card} \mathcal{J}_{T(y)}(y)$ if this reparametrization is full.
3) for all $j \in \mathbb{N}$ such that $\mathcal{I}_j(\phi)$ has nonempty interior,

$$(z(t,j), y^{\text{r}}(t,j)) \in \hat{C} \quad \forall t \in \text{int } \mathcal{I}_j(\phi)$$
$$\dot{z}(t,j) \in F(z(t,j), y^{\text{r}}(t,j)) \quad \text{for almost all } t \in \mathcal{I}_j(\phi)$$

4) for all $t \in \mathcal{T}(\phi)$, denoting $j_0 = \min \mathcal{J}_t(\phi)$ and $n_y = \operatorname{card} \mathcal{J}_t(y)$, we have
   a) for all $j \in \mathcal{J}_t(\phi)$ such that $j < j_0 + n_y$, we have $\rho_y(j) = \rho_y(j-1) + 1$, and:
   if $j = j_0$ and $t > 0$,
   - $(z(t,j_0-1), y^{\text{r}}(t,j_0-1)) \in \hat{C} \cup \hat{D}$
   - $z(t,j_0) \in G_e^0(z(t,j_0-1), y^{\text{r}}(t,j_0-1))$

   else
   - $(z(t,j-1), y^{\text{r}}(t,j-1)) \in \text{cl}(\hat{C}) \cup \hat{D}$
   - $z(t,j) \in G_e(z(t,j-1), y^{\text{r}}(t,j-1))$

   with

$$G_e^0(z,y) = \begin{cases} z & \text{if } (z,y) \in \hat{C} \setminus \hat{D} \\ G(z,y) & \text{if } (z,y) \in \hat{D} \setminus \hat{C} \\ \{z, G(z,y)\} & \text{if } (z,y) \in \hat{D} \cap \hat{C} \end{cases}$$

$$G_e(z,y) = \begin{cases} z & \text{if } (z,y) \in \text{cl}(\hat{C}) \setminus \hat{D} \\ G(z,y) & \text{if } (z,y) \in \hat{D} \setminus \text{cl}(\hat{C}) \\ \{z, G(z,y)\} & \text{if } (z,y) \in \hat{D} \cap \text{cl}(\hat{C}) \end{cases}$$

   b) for all $j \in \mathcal{J}_t(\phi)$ such that $j \geq j_0 + n_y$, we have $\rho_y(j) = \rho_y(j-1)$ and
   - $(z(t,j-1), y^{\text{r}}(t,j-1)) \in \hat{D}$
   - $z(t,j) \in G(z(t,j-1), y^{\text{r}}(t,j-1))$
5) for all $(t,j) \in \operatorname{dom} \phi$, $\hat{x}(t,j) = H(z(t,j), y^{\text{r}}(t,j))$.

See [14] and more generally [8] for more information about the construction of solutions.

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