Semicontinuity Properties of Solutions and Reachable Sets of Nominally Well-Posed Hybrid Dynamical Systems

Berk Altın and Ricardo G. Sanfelice

***Abstract*— Nominally well-posed hybrid inclusions are a class of hybrid dynamical systems characterized by outer/upper semicontinuous dependence of solutions on initial conditions. In the context of reachable sets of hybrid systems, this property guarantees outer/upper semicontinuous dependence with respect to both initial conditions and time. This article defines a counterpart to the notion of nominal well-posedness, referred to as nominal *inner* well-posedness, which ensures inner/lower semicontinuous dependence of solutions on initial conditions. Consequently, it is shown that reachable sets of nominally inner well-posed hybrid systems depend inner/lower semicontinuously on initial conditions and time under appropriate assumptions. Sufficient conditions guaranteeing nominal inner well-posedness are provided and demonstrated with an example.**

I. Introduction

For dynamical systems, the notion of well-posedness is the cornerstone of robustness of asymptotic stability. In simple terms, (nominal) well-posedness is the property that the limit of a convergent sequence of solutions is a solution to the system, with convergence defined in an appropriate sense. Over the past decade, this property has been shown to be instrumental in the development of a robust stability theory for hybrid dynamical systems [1], [2]. When such a property holds, stability theory results such as converse Lyapunov theorems and invariance principles are possible. Remarkably, although well-posedness is a property of solutions to a system, it can be certified by only verifying certain properties of the data defining the system; see [2, Ch. 6]. For hybrid dynamical systems modeled as hybrid inclusions (Section II), these properties are known as the *hybrid basic conditions* and require the maps defining the dynamics to be "mildly continuous" and the corresponding constraint sets to be closed. The robustness afforded to such systems has found use in various application areas; e.g., event-triggered control and vehicle platooning [3].

A fundamental consequence of well-posedness is that solutions depend on initial conditions upper semicontinuously. That is, every solution starting nearby a given point is close to a solution from that point [4]. This relationship between the set of solutions and the initial conditions is the key point in establishing robustness of pre-asymptotic stability [2] and of numerical algorithms simulating hybrid systems [5]. Although well-posedness is useful for robustness related properties and results, it does not allow one to conclude the existence of a solution starting nearby a given point that is close to a solution from that point. A version of well-posedness, which is introduced as nominal *inner* well-posedness in Section III of this article, provides this constructive relationship. Unlike the existing notion of well-posedness, referred to as of nominal *outer* well-posedness in this article, inner well-posedness guarantees lower semicontinuous dependence with respect to initial conditions. That is, every solution starting *from* a given point is close to a solution starting *nearby* that point. A similar property has found use in the formulation of relaxation theorems for hybrid systems [6]. Importantly, given a hybrid system and its discretization [5], while "outer" notions of well-posedness ensure that the reachable sets are underapproximated as the discretization step size is decreased, it does not guarantee that they are overapproximated.[1] In other words, being a property aimed at robustness, "outer" well-posedness implies upper semicontinuous dependence of reachable sets on the step size, but not necessarily lower semicontinuous dependence. With inner well-posedness, we aim to fill this gap and therefore ensure continuous dependence. To our knowledge, such results are not available even in the context of continuous-time systems, with continuity properties restricted solely to the step size; see [7]. Our motivation for the pursuit of this endeavor comes primarily from trajectory planning and optimal control problems, especially those arising in the context of model predictive control [8], [9], [10].

Being a natural counterpart of the (nominal outer) well-posedness notion in [2], we show that the convergence property required by nominal inner well-posedness is equivalent to closeness over a bounded horizon. Furthermore, we establish a link between lower semicontinuous dependence on initial conditions and nominal inner well-posedness. With inner well-posedness revealing such properties, in Section IV, we establish semicontinuity properties of reachable sets under the said well-posedness properties, with respect to both the initial conditions and time. In particular, we show that under appropriate assumptions, reachable sets over a compact hybrid horizon are locally bounded, and depend outer and inner semicontinuously with respect to both the initial conditions and hybrid time. Sufficient conditions for nominal inner well-posedness presented in the Appendix solidify the findings of the article.

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Berk Altın and Ricardo G. Sanfelice are with the Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA; berkaltin@ucsc.edu, ricardo@ucsc.edu.

[1] That is, the reachable set of the discretized system converges to a subset of the true reachable set. This subset need to be equal to the true reachable set.

II. Preliminaries

This section introduces the style of notation followed throughout the paper and presents the relevant background from hybrid systems theory and set-valued analysis.

A. Notation

We use $\mathbb{R}$ to represent real numbers and $\mathbb{R}_{\geq 0}$ its nonnegative subset. The set of nonnegative integers is denoted $\mathbb{N}$. The Euclidean norm is denoted $|.|$. The notation $S_1 \subset S_2$ indicates that S_1 is a subset of S_2, not necessarily proper. The distance of a vector $x \in \mathbb{R}^n$ to a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is $|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$. The closed unit ball in $\mathbb{R}^n$ is denoted $\mathbb{B}$. Given a set $S \subset \mathbb{R}^n$, its closure, interior, and boundary are denoted $\operatorname{cl} S$, $\operatorname{int} S$, and ∂S, respectively. The domain and graph of a set-valued mapping $M : S \rightrightarrows \mathbb{R}^m$ is denoted $\operatorname{dom} M$ and $\operatorname{gph} M$, respectively. The restriction of M to a set $\widehat{S} \subset S$ is denoted $M|_{\widehat{S}}$.

B. Hybrid Inclusions

We consider hybrid systems in the framework of [2], where a hybrid system $\mathcal{H}$ is described by the combination of a constrained differential and difference inclusion:

$$\mathcal{H} \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{1}$$

Above, the *flow map* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defines the continuous-time behavior (*flows*) of the state $x \in \mathbb{R}^n$ on the *flow set* $C \subset \operatorname{dom} F$. Similarly, the *jump map* $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defines the discrete transitions (*jumps*) of x on the *jump set* $D \subset \operatorname{dom} G$. The 4-tuple (C, F, D, G) is referred to as the data of $\mathcal{H}$. The notation $\mathcal{H} = (C, F, D, G)$ is used to refer to the hybrid system in (1) and define its data.

Solutions of the hybrid system $\mathcal{H}$ belong to a class of functions called *hybrid arcs* and are parametrized by hybrid time (t, j), where $t \in \mathbb{R}_{\geq 0}$ denotes the ordinary time elapsed during flows and $j \in \mathbb{N}$ denotes the number of jumps that have occurred. A function x mapping a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ to $\mathbb{R}^n$ is a hybrid arc if a) its domain, denoted $\operatorname{dom} x$, is a *hybrid time domain*, and b) it is locally absolutely continuous on each connected component of $\operatorname{dom} x$. Formally, a set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in E$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ with $t_0 = 0$ such that

$$E \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}).$$

Then, a function $x : \operatorname{dom} x \to \mathbb{R}^n$ is said to be a hybrid arc if $\operatorname{dom} x$ is a hybrid time domain and for every $j \geq 0$, the function $t \mapsto x(t, j)$ is locally absolutely continuous on the interval $I^j := \{t : (t, j) \in \operatorname{dom} x\}$.

A hybrid arc x is bounded if its range is bounded. It is called complete if its domain is unbounded. It is said to escape to infinity at hybrid time (T, J) if $x(t, J)$ tends to infinity as t tends to T. If x is a solution, it is said to be maximal if it cannot be extended to another solution. The notation $\mathcal{S}_{\mathcal{H}}(S)$ refers to the set of all maximal solutions x of $\mathcal{H}$ originating from S; i.e. $x(0, 0) \in S$ for every $x \in \mathcal{S}_{\mathcal{H}}(S)$. If every $x \in \mathcal{S}_{\mathcal{H}}(S)$ is bounded or complete, we say that $\mathcal{H}$ is *pre-forward complete from* S. Throughout the paper, we say that t is a jump time of x if there exists j such that $(t, j), (t, j + 1) \in \operatorname{dom} x$. We say that $(T, J) \in \operatorname{dom} x$ is the terminal (hybrid) time of x if $t \leq T$ and $j \leq J$ for all $(t, j) \in \operatorname{dom} x$. Similarly, T is referred to as the terminal ordinary time of x.

Definition 2.1: A hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$ is a solution of the hybrid system $\mathcal{H}$ in (1) if $x(0, 0) \in \operatorname{cl}(C) \cup D$, and

- for every $j \in \mathbb{N}$, $x(t, j) \in C$ for all $t \in \operatorname{int} I^J$ and
$$\dot{x}(t, j) \in F(x(t, j)) \text{ for almost all } t \in I^j,$$
where $I^j := \{t : (t, j) \in \operatorname{dom} x\}$;
- for every $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,
$$x(t, j) \in D \text{ and } x(t, j + 1) \in G(x(t, j)).$$

In this paper, we establish regularity properties for reachable set mappings defined below. Given x_0, T, and J, the reachable set mappings collect the values of all solutions originating from x_0 at time (T, J). Motivated by converse safety/invariance problems, an alternative formulation is provided in [11], wherein the reachable set mappings collect the values of all solutions originating from x_0 *until* time (T, J).

Definition 2.2 (Reachable Set Mapping): The reachable set mapping $\mathcal{R}_{\mathcal{H}} : (\operatorname{cl}(C) \cup D) \times \mathbb{R}_{\geq 0} \times \mathbb{N} \rightrightarrows \mathbb{R}^n$ of the hybrid system $\mathcal{H}$ in (1) is the set-valued mapping that associates with every x_0, T, and J, the reachable set of $\mathcal{H}$ from x_0 at time (T, J), i.e.,

$$\mathcal{R}_{\mathcal{H}}(x_0, T, J) := \{x(T, J) : x \in \mathcal{S}_{\mathcal{H}}(x_0), (T, J) \in \operatorname{dom} x\}.$$

C. Notions of Convergence and Regularity from Set-Valued Analysis

Consider a sequence of sets $\{S_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$. The *inner limit* of the sequence, denoted $\liminf_{i \to \infty} S_i$, is the set of all x for which there exists $\imath \in \mathbb{N}$ and a sequence $\{x_i\}_{i=\imath}^{\infty}$ convergent to x such that $x_i \in S_i$ for every $i \geq \imath$. The *outer limit* of the sequence, denoted $\limsup_{i \to \infty} S_i$, is the union of the inner limits of all subsequences of $\{S_i\}_{i=0}^{\infty}$. When the inner and outer limits are equal to each other, the *limit* of the sequence, denoted $\lim_{i \to \infty} S_i$, is defined to be equal to the inner and outer limits.

Let $S \subset \mathbb{R}^n$, $x \in S$. A set-valued mapping $M : S \rightrightarrows \mathbb{R}^m$ is said to be *locally bounded* at x if there exists $\varepsilon > 0$ such that the set $M((x + \varepsilon\mathbb{B}) \cap S)$ is bounded. It is said to be *outer semicontinuous* at x if for any sequence $\{x_i\}_{i=0}^{\infty} \in S$ convergent to x and any convergent sequence $\{y_i\}_{i=0}^{\infty}$ such that $y_i \in M(x_i)$ for all $i \geq 0$, $\lim_{i \to \infty} y_i \in M(x)$. It is said to be *inner semicontinuous* at x if for any sequence $\{x_i\}_{i=0}^{\infty} \in S$ convergent to x and any $y \in M(x)$, there exists $\imath \in \mathbb{N}$ and a sequence $\{y_i\}_{i=\imath}^{\infty}$ convergent to y such that $y_i \in M(x_i)$ for all $i \geq \imath$. Given a set $\widehat{S} \subset S$, M is said to be locally bounded, outer semicontinuous, or inner semicontinuous *relative to* $\widehat{S}$ if the respective properties hold for $M|_{\widehat{S}}$ (the restriction of M to $\widehat{S}$) for all $x \in \widehat{S}$.

The definitions of set convergence, local boundedness, and semicontinuity here follow their counterparts in [12], in particular, [12, Definitions 4.1,5.4, and 5.14]. For locally bounded set-valued maps with closed values, outer semicontinuity is equivalent to the property commonly known as *upper semicontinuity* [13, Definition 1.4.1]. Inner semicontinuity coincides with the property commonly known as *lower semicontinuity* [13, Definition 1.4.2].

III. Structure of the Set of Solutions

A sequence $\{x_i\}_{i=0}^{\infty}$ of hybrid arcs is said to be *locally eventually bounded* if for any $\tau \geq 0$, there exist $\imath \in \mathbb{N}$ and a compact set K such that $x_i(t, j) \in K$ for every $i \geq \imath$ and $(t, j) \in \operatorname{dom} x_i$ with $t + j \leq \tau$. It is said to *converge graphically* to a mapping $M : \mathbb{R}_{\geq 0} \times \mathbb{N} \rightrightarrows \mathbb{R}^n$ if the sequence $\{\operatorname{gph} x_i\}_{i=0}^{\infty}$ converges to $\operatorname{gph} M$ (in the set convergence sense), where

graph gph denotes the graph

. The mapping M is called the *graphical limit* of $\{x_i\}_{i=0}^{\infty}$. See [2] for details.

Graphical convergence offers a convenient way of establishing structurally advantageous properties (e.g. upper and lower semicontinuous dependence on initial conditions) for hybrid systems. It is motivated by the fact that two solutions of a hybrid system need not have the same time domain, which renders the uniform norm insufficient to quantify closeness of solutions. To measure closeness in the hybrid setting, we rely on the notion of (τ, ε)-closeness ([2, Definition 5.23]), which is closely related to graphical convergence.

Definition 3.1: Given $\tau \geq 0$ and $\varepsilon > 0$, two hybrid arcs x and x' are said to be (τ, ε)-close if the following hold:

- for every $(t, j) \in \operatorname{dom} x$ satisfying $t + j \leq \tau$, there exists $(t', j) \in \operatorname{dom} x'$ such that $|t - t'| < \varepsilon$ and $|x(t, j) - x'(t', j)| < \varepsilon$;
- for every $(t', j') \in \operatorname{dom} x'$ satisfying $t' + j' \leq \tau$, there exists $(t, j') \in \operatorname{dom} x$ such that $|t' - t| < \varepsilon$ and $|x'(t', j') - x(t, j')| < \varepsilon$.

Next, using the notion of graphical convergence, we define two complementary properties for the set of solutions of hybrid inclusions. These are later used to establish semicontinuous dependence of solutions on initial conditions, as well as regularity of the reachable set mapping in Definition 2.2.

A. Sequential Properties of Solutions

In [2] and the related literature, a *nominally well-posed* hybrid system [2, Definition 6.2] has a graphical convergence property that can roughly be interpreted as outer semicontinuous dependence of solutions on initial conditions. In this paper, we establish the counterpart of this definition by considering hybrid systems wherein solutions depend on initial conditions in an inner semicontinuously. To differentiate the two, we revise the terminology and refer to the notion in [2, Definition 6.2] as nominal *outer* well-posedness, as seen below.

Definition 3.2: A hybrid system $\mathcal{H}$ is said to be *nominally outer well-posed* on a set $S \subset \mathbb{R}^n$ if for every graphically convergent sequence of solutions $\{x_i\}_{i=0}^{\infty}$ of $\mathcal{H}$ satisfying $\lim_{i\to\infty} x_i(0, 0) =: x_0 \in S$, the following holds:

- if the sequence $\{x_i\}_{i=0}^{\infty}$ is locally eventually bounded, then the graphical limit x is a solution of $\mathcal{H}$ originating from x_0;
- if the sequence $\{x_i\}_{i=0}^{\infty}$ is not locally eventually bounded, then there exists $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $x = M|_{\operatorname{dom} M \cap ([0,T) \times \{0,1,\dots,J\})}$ is a solution of $\mathcal{H}$ originating from x_0 that escapes to infinity at time (T, J), where M is the graphical limit of $\{x_i\}_{i=0}^{\infty}$.

If $S = \mathbb{R}^n$, then $\mathcal{H}$ is simply said to be *nominally outer well-posed.*

Remark 3.3: The definition above is equivalent to [2, Definition 6.2], except for the fact that nominal outer well-posedness is defined as a local property.

Lemma 3.4: Suppose that a hybrid system $\mathcal{H}$ is nominally outer well-posed on a compact set of initial conditions K, and suppose that $\mathcal{H}$ is pre-forward complete from K. Then, for every $\tau > 0$, there exists $\delta > 0$ and a compact set K' such that for every $x \in \mathcal{S}_{\mathcal{H}}(K + \delta\mathbb{B})$ and every $(t, j) \in \operatorname{dom} x$ with $t + j \leq \tau$, $x(t, j) \in K'$.

Proof Follows the same steps as the proof of [2, Proposition 6.13]. ■

Although nominal outer well-posedness of a hybrid system can be difficult to check, a set of mild regularity conditions called the hybrid basic conditions [2, Assumption 6.5] turn out to be sufficient for this property [2, Th. 6.8].

Theorem 3.5: A hybrid system $\mathcal{H} = (C, F, D, G)$ is nominally outer well-posed if the following hold.

(A1) The sets C and D are closed.

(A2) The flow map F is locally bounded and outer semicontinuous relative to C, and $C \subset \operatorname{dom} F$. Furthermore, for every $x \in C$, the set $F(x)$ is convex.

(A3) The jump map G is locally bounded and outer semicontinuous relative to D, and $D \subset \operatorname{dom} G$.

Remark 3.6: A broad class of systems satisfying the conditions of Theorem 3.5 are the Krasovskii regularizations [2, Definition 4.13] of systems for which the flow and jump maps are locally bounded [2, Example 6.6]. For systems with explicit discrete modes (e.g. switched systems and hybrid automata), (A1)-(A3) can be equivalently expressed in terms of regularity requirements on the data of each mode. For example, the convexity requirement in (A2) can be replaced with the requirement that for each mode q and the associated vector field F_q, the set $F_q(z)$ is convex for all z belonging to the domain of F_q; see [2, Example 6.7].

Definition 3.2 refers to two mutually exclusive cases. In the first case, the graphical limit of the sequence is a solution x whose domain is closed (due to the set limit always being a closed set [2, p.99]). Thus, x is either bounded or complete. In the second case, the graphical limit leads to a (maximal) solution that escapes to infinity at (T, J). In both cases, the graph of the solution is closed. Consequently, the notion of nominal inner well-posedness is defined as follows.

Definition 3.7: A hybrid system $\mathcal{H} = (C, F, D, G)$ is said to be nominally inner well-posed on a set $S \subset \mathbb{R}^n$ if for every solution x of $\mathcal{H}$ originating from S, the following holds:

($\star$) given any sequence $\{\xi_i\}_{i=0}^{\infty} \in \mathrm{cl}(C) \cup D$ convergent to $x(0,0)$, for every $i \geq 0$, there exists a solution x_i of $\mathcal{H}$ originating from ξ_i such that

(a) if x is bounded or complete and $\operatorname{dom} x$ is closed, then the sequence $\{x_i\}_{i=0}^{\infty}$ is locally eventually bounded and graphically convergent to x;

(b) if x escapes to infinity at hybrid time (T, J), then the sequence $\{x_i\}_{i=0}^{\infty}$ is not locally eventually bounded but graphically convergent to a mapping M such that $x = M|_{\operatorname{dom} M \cap ([0,T) \times \{0,1,\ldots,J\})}$.

If $S = \mathbb{R}^n$, then $\mathcal{H}$ is simply said to be nominally inner well-posed.

Sufficient conditions for nominal inner well-posedness relying on tangent cones are provided in the Appendix. They are similar to some of the conditions used for the hybrid relaxation results in [6]. In the Appendix, these conditions are utilized to show that a hybrid model of a sampled-data control system is nominally inner well-posed, provided the control law is continuous and the plant dynamics are locally Lipschitz.

The next two results show the relationship between the graphical convergence property described in ($\star$) and an alternative formulation using (τ, ε)-closeness. In particular, for a bounded or complete solution x, these two properties are equivalent.[2]

Proposition 3.8: Let x be a solution of a hybrid system $\mathcal{H} = (C, F, D, G)$. Suppose that the graph of x is closed, and for every $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ such that the following holds: for any $x_0' \in (x(0,0) + \delta\mathbb{B}) \cap (\mathrm{cl}(C) \cup D)$, there exists a solution x' of $\mathcal{H}$ originating from x_0' such that x and x' are (τ, ε)-close. Then, ($\star$) holds.

Proof Consider the sequence $\{(k, 1/k)\}_{k=1}^{\infty}$ and let $\{\delta_k\}_{k=1}^{\infty}$ be a strictly decreasing positive sequence such that for every $k \geq 1$ the following holds: for all $x_0' \in (x(0,0) + \delta_k\mathbb{B}) \cap (\mathrm{cl}(C) \cup D)$, there exists a solution x' of $\mathcal{H}$ originating from x_0' such that x and x' are $(k, 1/k)$-close. Let $\{i_k'\}_{k=1}^{\infty}$ be a strictly increasing positive sequence such that given any $k \geq 1$, $|\xi_i - x_0| \leq \delta_k$ for all $i \geq i_k'$. For every $k \geq 1$ and every $i \in \{i_k', i_k' + 1, \ldots, i_{k+1}' - 1\}$, pick a solution x_i' of $\mathcal{H}$ originating from ξ_i such that x_i' and x are $(k, 1/k)$-close. By [2, Theorem 5.25], the sequence $\{x_i'\}_{i=1}^{\infty}$ is graphically convergent to x. In the case of a bounded or complete x, local eventual boundedness of $\{x_i'\}_{i=1}^{\infty}$ follows from the fact that for any $\tau \geq 0$, the set of all $x(t,j)$ with $t + j \leq \tau$ is bounded. Otherwise, if x escapes to infinity at hybrid time (T, J), it is obvious that $\{x_i'\}_{i=1}^{\infty}$ is not locally eventually bounded. ■

[2]The closed graph requirement in Proposition 3.8 excludes bounded solutions whose domains are not closed. However, this is possible for a solution x only when it has a finite number of jumps, say J, and the interval $I^J := \{t : (t, J) \in \operatorname{dom} x\}$ is open to the right. If the system is nominally outer well-posed, x cannot be maximal, so its graph can be closed by including the "terminal point" $\lim_{t \to \sup I^J} x(t, J)$.

Proposition 3.9: Let x be a bounded or complete solution of a hybrid system $\mathcal{H} = (C, F, D, G)$. Suppose that the domain of x is closed and ($\star$) holds. Then, for every $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ such that the following holds: for any $x_0' \in (x(0,0) + \delta\mathbb{B}) \cap (\mathrm{cl}(C) \cup D)$, there exists a solution x' of $\mathcal{H}$ originating from x_0' such that x and x' are (τ, ε)-close.

Proof If the conclusion of the proposition were false, there would exist a sequence $\{\xi_i\}_{i=1}^{\infty} \in \mathrm{cl}(C) \cup D$ such that for every $i \geq 1$, $\xi_i \in x(0,0) + (1/i)\mathbb{B}$, and no solution x_i of $\mathcal{H}$ satisfying $x_i(0,0) = \xi_i$ is such that x and x_i are (τ, ε)-close. Since x is bounded or complete and the domain of x is closed, the graph of x is closed. For each $i \geq 1$, pick a solution x_i of $\mathcal{H}$ with initial condition $x_i(0,0) = \xi_i$ such that the sequence $\{x_i\}_{i=1}^{\infty}$ is locally eventually bounded and graphically convergent to x. Then, by [2, Theorem 5.25], there exists $\imath \in \mathbb{N}$ such that for every $i \geq \imath$, x and x_i are (τ, ε)-close, which is a contradiction. ■

B. Semicontinuous Dependence on Initial Conditions

Following the definitions of upper and lower semicontinuity of set-valued mappings in [13, Definitions 1.4.1 and 1.4.2], semicontinuous dependence of solutions on initial conditions are defined next. This definition is similar to the one in [6] up to a change in terminology (upper/lower versus outer/inner semicontinuity).

Definition 3.10: Given a set $S \subset \mathbb{R}^n$, solutions of a hybrid system $\mathcal{H} = (C, F, D, G)$ are said to depend

- upper semicontinuously on initial conditions on S if for every $x_0 \in S$, $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ such that for any $x' \in \mathcal{S}_{\mathcal{H}}(x_0 + \delta\mathbb{B})$, there exists a solution x of $\mathcal{H}$ originating from x_0 such that x' and x are (τ, ε)-close;
- lower semicontinuously on initial conditions on S if for every $x_0 \in S$, $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(x_0)$ and any $x_0' \in (x_0 + \delta\mathbb{B}) \cap (\mathrm{cl}(C) \cup D)$, there exists a solution x' of $\mathcal{H}$ originating from x_0' such that x and x' are (τ, ε)-close.

Next, we establish a relationship between lower semicontinuous dependence on initial conditions and nominal inner well-posedness. In particular, for nominally outer well-posed systems that are pre-forward complete, these two notions are the same, with nominal outer well-posedness ensuring that the δ in Proposition 3.9 holds uniformly for all solutions from the same point, given τ and ε .

Theorem 3.11: Suppose that the solutions of a hybrid system $\mathcal{H} = (C, F, D, G)$ depend lower semicontinuously on initial conditions at x_0. Then, $\mathcal{H}$ is nominally inner well-posed at x_0 if maximal solutions of $\mathcal{H}$ originating from x_0 have closed graphs.

Proof For any maximal solution x, the graphical convergence property follows directly from Proposition 3.8. If x is not maximal, consider a maximal extension, say $\hat{x}$. For any sequence $\{\xi_i\}_{i=0}^{\infty} \in \mathrm{cl}(C) \cup D$ convergent to x_0, construct a sequence $\{\hat{x}_i\}_{i=0}^{\infty}$ of solutions that is graphically convergent to $\hat{x}$, where, for each $i \geq 0$, $\hat{x}_i$ originates from ξ_i. For each $i \geq 0$, let x_i be the truncation of $\hat{x}_i$ up to some time (T_i, J_i) such that the sequence $\{(T_i, J_i)\}_{i=0}^{\infty}$ converges to the terminal time (T, J) of x. Then, it is straightforward to show that $\{x_i\}_{i=0}^{\infty}$ converges graphically to x. ■

Theorem 3.12: Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system. Given an initial condition x_0, suppose that $\mathcal{H}$ is nominally outer and inner well-posed at x_0 and pre-forward complete from x_0. Then, solutions of $\mathcal{H}$ depend lower semicontinuously on initial conditions at x_0.

Proof If the conclusion of the theorem were false, there would exist $\tau, \varepsilon > 0$ and sequences $\{z_i\}_{i=1}^{\infty} \in \mathcal{S}_{\mathcal{H}}(x_0)$ and $\{\xi_i\}_{i=1}^{\infty} \in \mathrm{cl}(C) \cup D$ such that for every $i \geq 1$, $\xi_i \in x_0 + (1/i)\mathbb{B}$, and no solution z'_i of $\mathcal{H}$ satisfying $z'_i(0,0) = \xi_i$ is such that z_i and z'_i are (τ, ε)-close. Due to pre-forward completeness from x_0 and nominal outer well-posedness at x_0, the sequence $\{z_i\}_{i=1}^{\infty}$ is locally eventually bounded by Lemma 3.4, so as a result of [2, Theorem 6.1], without relabeling, one can extract graphically convergent subsequence, where the limits of the sequence, say z, is a solution of $\mathcal{H}$, with $z(0,0) = x_0$. Note that since $\operatorname{dom} x$ is the limit of $\{\operatorname{dom} z_i\}_{i=0}^{\infty}$, it is closed, and as such, the graph of z is closed and z is bounded or complete. Pick a locally eventually bounded sequence of solutions $\{z'_i\}_{i=1}^{\infty}$ of $\mathcal{H}$ such that for every $i \geq 1$, $z'_i = \xi_i$ and $\{z'_i\}_{i=1}^{\infty}$ is graphically convergent to z. Fix $c > 2$ such that $\varepsilon/c \leq \tau$. Now, take $\imath \geq 1$ such that for every $i \geq \imath$, the pair z_i, z, and the pair z'_i, z are $(2\tau, \varepsilon/c)$-close (existence of such $\imath$ is guaranteed by [2, Theorem 5.25]). Then, for every $i \geq \imath$ and any $(t_i, j) \in \operatorname{dom} z_i$ with $t_i + j \leq \tau$, there exists $(t, j) \in \operatorname{dom} z$ such that $|t_i - t| < \varepsilon/c$ and $|z_i(t_i, j) - z(t, j)| < \varepsilon/c$. Note that since $\varepsilon/c \leq \tau$, $t + j \leq 2\tau$, hence, there exists $(t'_i, j) \in \operatorname{dom} z'_i$ such that $|t - t'_i| < \varepsilon/c$ and $|z'(t, j) - z_i(t_i, j)| < \varepsilon/c$. Consequently, for every $i \geq \imath$ and any $(t_i, j) \in \operatorname{dom} z_i$ with $t_i + j \leq \tau$, there exists $(t'_i, j) \in \operatorname{dom} z'_i$ such that $|t_i - t'_i| < 2\varepsilon/c < \varepsilon$ and $|z_i(t_i, j) - z'_i(t'_i, j)| < 2\varepsilon/c < \varepsilon$. Similarly, for every $i \geq \imath$ and any $(t'_i, j) \in \operatorname{dom} z'_i$ with $t'_i + j \leq \tau$, there exists $(t_i, j) \in \operatorname{dom} z_i$ such that $|t'_i - t_i| < 2\varepsilon/c < \varepsilon$ and $|z'_i(t'_i, j) - z_i(t_i, j)| < 2\varepsilon/c < \varepsilon$. In other words, for every $i \geq \imath$, z_i and z'_i are (τ, ε)-close, which is a contradiction. ■

Moreover, for nominally outer well-posed systems, semicontinuous dependence of the solution set is uniform over compact sets from where the system is pre-forward complete.

Proposition 3.13: Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system. Given a compact set K, suppose that $\mathcal{H}$ is nominally outer well-posed on K and pre-forward complete from K. Then, for all $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ such that the following holds:

- *for every $x' \in \mathcal{S}_{\mathcal{H}}(K + \delta\mathbb{B})$, there exists a solution x of $\mathcal{H}$ originating from K such that x' and x are (τ, ε)-close.*

If, in addition, $\mathcal{H}$ is nominally inner well-posed on K, then for all $\varepsilon > 0$ and $\tau \geq 0$, there exists $\delta > 0$ such that the following holds:

- *for every $x \in \mathcal{S}_{\mathcal{H}}(K)$ and $x'_0 \in (x(0,0) + \delta\mathbb{B}) \cap (\mathrm{cl}(C) \cup D)$, there exists a solution x' of $\mathcal{H}$ originating from x'_0 such that x and x' are (τ, ε)-close.*

Proof The first item restates the conclusion of [2, Proposition 6.14], with the proof proceeding with the same exact steps. The proof of the second item follows the arguments in the proof of Theorem 3.12. It relies on the existence of sequences $\{z_i\}_{i=1}^{\infty} \in \mathcal{S}_{\mathcal{H}}(K)$ (as opposed to $\{z_i\}_{i=1}^{\infty} \in \mathcal{S}_{\mathcal{H}}(x_0)$ before) and $\{z'_i\}_{i=1}^{\infty}$. As in the proof of Theorem 3.12, the former can be assumed to be graphically convergent with limit z, which is a solution of $\mathcal{H}$, with $z(0,0) \in K$ (as opposed to $z(0,0) = x_0$ before). The latter sequence, $\{z'_i\}_{i=1}^{\infty}$, is chosen to be graphically convergent to z. ■

IV. Regularity Properties of Reachable Set Mappings

In this section, we establish semicontinuity properties of reachable set mappings under nominal inner and outer well-posedness, along with compactness of reachable sets.

Theorem 4.1: Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system. Given an initial condition $x_0 \in \mathrm{cl}(C) \cup D$, suppose that $\mathcal{H}$ is nominally outer well-posed at x_0 and pre-forward complete from x_0. Then, for every $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, the reachability mapping $\mathcal{R}_{\mathcal{H}}$ is locally bounded and outer semicontinuous at (x_0, T, J).

Proof Local boundedness is a straightforward consequence of Lemma 3.4. For outer semicontinuity, take any sequence $\{\xi_i\}_{i=0}^{\infty}$ convergent to x_0 and any sequence $\{T_i\}_{i=0}^{\infty}$ convergent to T. For every $i \geq 0$, let x_i be a solution of $\mathcal{H}$ originating from ξ_i with terminal time (T_i, J), and suppose that $\{x(T_i, J)\}_{i=0}^{\infty}$ is convergent. By Lemma 3.4, the sequence $\{x_i\}_{i=0}^{\infty}$ is locally eventually bounded, so by [2, Theorem 6.1] and nominal well-posedness, it has a graphically convergent subsequence, whose limit, say x, is a solution of $\mathcal{H}$ originating from x_0. Without relabeling, pass to this subsequence. Then, by definition of graphical convergence, $\{x(T_i, J)\}_{i=0}^{\infty}$ must converge to $x(T, J)$. ■

A straightforward consequence of Theorem 4.1 is that the reachable set from x_0 at time (T, J) is compact. More generally, the following result can be observed.

Proposition 4.2: Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system. Given a compact set of initial conditions $K \subset \mathrm{cl}(C) \cup D$, suppose that $\mathcal{H}$ is nominally outer well-posed on K and pre-forward complete from K. Then, for every compact set $\mathcal{C} \subset K \times \mathbb{R}_{\geq 0} \times \mathbb{N}$, the set $\mathcal{R}_{\mathcal{H}}(\mathcal{C})$ is compact.

Proof By Theorem 4.1, the reachability mapping $\mathcal{R}_{\mathcal{H}}$ is locally bounded and outer semicontinuous on $\mathcal{C}$. Due to compactness of $\mathcal{C}$ and local boundedness, by considering a finite subcover, it follows that $\mathcal{R}_{\mathcal{H}}(\mathcal{C})$ is bounded. Moreover, by [2, Lemma 5.10], the graph of $\mathcal{R}_{\mathcal{H}}|_{\mathcal{C}}$ is closed. Since $\mathcal{C}$ is compact, the projection of $\operatorname{gph} \mathcal{R}_{\mathcal{H}} \cap (\mathcal{C} \times \mathbb{R}^n)$ onto $\mathbb{R}^n$ (which is precisely $\mathcal{R}_{\mathcal{H}}(\mathcal{C})$) yields a closed set. Hence, $\mathcal{R}_{\mathcal{H}}(\mathcal{C})$ is compact. ■

For a nominally inner well-posed system, inner semicontinuity of the reachable set from x_0 at time (T, J) does not depend on pre-forward completeness. Instead, it can be observed if no maximal solution originating from x_0 jumps or terminates at ordinary time T, as shown below. Due to space constraints, the proof of this result will be published in another venue.

Theorem 4.3: Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system that is nominally inner well-posed at an initial condition x_0. Given a hybrid time $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, suppose that there exists no $x \in \mathcal{S}_{\mathcal{H}}(x_0)$ such that T is a jump time or the terminal ordinary time of x. Then, the reachability mapping $\mathcal{R}_{\mathcal{H}}$ is inner semicontinuous at (x_0, T, J).

V. Concluding Remarks

In our upcoming work, we define a stronger notion of inner well-posedness that guarantees lower semicontinuous dependence of solutions on initial conditions as well as the magnitude of perturbations. With this stronger notion, we provide results that allow inner semicontinuous approximations of reachable sets without any knowledge of solutions. Along with the accompanying generalization of Theorem 1.1, these results will form the foundation for a concrete theory linking solutions and reachable sets of hybrid systems to their discretizations.

Appendix

Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, denote by $T_S(x)$ the Bouligand tangent cone (also called the contingent cone) to S at x [2, Definition 5.12]; that is, the set of all v such that $v = \lim_{i\to\infty}(x_i - x)/\tau_i$ for a sequence $\{x_i\}_{i=0}^{\infty} \in S$ convergent to x and a positive sequence $\{\tau_i\}_{i=0}^{\infty}$ convergent to zero. Similarly, denote by $M_S(x)$ the Dubovitsky-Miliutin tangent cone to S at x [14, Definition 4.3.1]: $v \in M_S(x)$ if and only if there exists $r, \bar{\delta} > 0$ such that $x + \delta w \in S$ for all $\delta \in (0, \bar{\delta}]$ and $w \in v + r\mathbb{B}$. If $x \in \partial S$, then $M_S(x) = \mathbb{R}^n \backslash T_{\mathbb{R}^n \backslash S}(x)$; [14, Lemma 4.3.2]. Also, a set valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be Lipschitz on S if there exists $L \geq 0$ such that $M(x) \subset M(x') + L|x - x'|\mathbb{B}$ for all $x, x' \in S$.

Sufficient conditions for nominal inner well-posedness are given below. The set $\widetilde{C}$ appearing in (H6) of these conditions is precisely the set of all points where the hybrid system $\mathcal{H}$ has solutions that flow. The proof will be published elsewhere.

Theorem 1.1: Given a hybrid system $\mathcal{H} = (C, F, D, G)$, suppose that the flow set C is closed and (A2) holds. Then, $\mathcal{H}$ is nominally inner well-posed if the following hold.

(H1) For every $x \in C$, there exists an extension of $F|_C$ that is closed valued and Lipschitz on a neighborhood of x.

(H2) For every $x \in \partial C$ such that $F(x) \cap T_C(x)$ is nonempty, there exists $r > 0$ such that $F(x') \subset M_{\operatorname{int} C}(x')$ for all $x' \in (x + r\mathbb{B}) \cap \partial C$, and $(x + r\mathbb{B}) \cap D \subset C$.

(H3) For every $x \in \operatorname{int} C \cap \partial D$, $F(x) \cap M_{\operatorname{int} D}(x)$ is nonempty.

(H4) For every $x \in \partial C \cap \partial D$, either of the following hold:

- *there exists $r > 0$ such that $(x + r\mathbb{B}) \cap C \subset D$;*
- *$F(x) \cap M_{\operatorname{int} C}(x) \cap M_{\operatorname{int} D}(x)$ is nonempty;*
- *$F(x) \cap T_C(x)$ is empty and there exists $r > 0$ such that $(x + r\mathbb{B}) \cap \partial C \subset D$.*

(H5) The jump map G is inner semicontinuous relative to D.

(H6) The mapping $\widetilde{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, where

$$\widetilde{G}(x) := G(x) \cap (\widetilde{C} \cup D) \quad \forall x \in \mathbb{R}^n,$$
$$\widetilde{C} := \operatorname{int}(C) \cup \{x \in \partial C : F(x) \cap T_C(x) \neq \varnothing\},$$

is inner semicontinuous relative to D.

Example 1.2: Consider the sample-and-hold control of a continuous-time system $\dot{x}_p = f(x_p, u)$. Denoting by T_s the sampling period and by κ the control law, the closed loop can be modeled as in (1) with state $x = (x_p, \tau_s, u)$ and data

$$C = \{x : \tau_s \in [0, T_s]\}, \quad F(x) = (f(x_p, u), 1, 0) \quad \forall x \in C,$$
$$D = \{x : \tau_s = T_s\}, \qquad G(x) = (x_p, 0, \kappa(x_p) \quad \forall x \in D,$$

where τ_s is a timer variable regulating sampling times. For this system, the flow set C is closed, and (H3) and the second condition of (H2) holds since $D \subset \partial C$. In addition, (H4) and the first condition of (H2) holds as given $x \in \partial C$, $F(x) \cap T_C(x)$ is nonempty if and only if $\tau_s = 0$. For (A2) to hold, it is necessary and sufficient for f to be continuous, while strengthening this to f being locally Lipschitz guarantees (H1). Similarly, for (H5) to hold, it necessary and sufficient for κ to be continuous. Continuity of κ also guarantees (H6), as the image of G is a subset of $\{x \in \partial C : \tau_s = 0\}$. Hence, this system is nominally inner well-posed.

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