# Upper bounds and Cost Evaluation in Dynamic Two-player Zero-sum Games

Santiago J. Leudo, Francesco Ferrante, and Ricardo G. Sanfelice

***Abstract*— Optimality in a min-max sense for constrained difference equations in the presence of disturbances is studied as a two-player zero-sum game. Sufficient conditions that permit the evaluation and to upper bound the cost of solutions to such systems are presented. Cost evaluation results are also presented for the case in which a control input aims to minimize a cost functional while the objective of disturbance is to maximize it. Sufficient conditions in the form of Hamilton-Jacobi-Isaacs equations are provided to certify closed-loop saddle point optimality. The results are illustrated in an example featuring a linearized and discretized model of an inverted pendulum.**

## I. Introduction

The use of game theoretical tools to model conflicting behaviors in natural and man-made systems is a well-established approach; see [1], [2]. Loosely speaking, a game is an optimization problem with multiple decision makers (players), a set of constraints that enforces the "rules" of the game, and a set of payoff functions to be optimized. Constraints on the actions of the players formulated as dynamic relationships with the state variables lead to *dynamic games*; see [1], [3] and the references therein. In this setting, an interesting situation occurs when one players aims to minimize a given performance measure, while another aims to maximize it. This problem is commonly called a dynamic two-player zero-sum noncooperative game. This type of dynamic game is studied in [4] from an *optimal control* standpoint.

An interesting application of dynamic two-player zero-sum noncooperative games consists of tackling the effect of disturbances in closed-loop control systems. In particular, this scenario can be modeled as a zero-sum two-player game in which one player determines the control input, and attempts to minimize a cost, while the other player selects the disturbance and aims to maximize it; see, e.g., [1, Chapter 6.6]. Optimality is attained in a min-max sense, via the so-called saddle point conditions. For such a problem, it is challenging to find the exact, or even an upper bound on the cost, or to characterize the family of controllers that yields it. One is always interested in finding conditions to assess the cost without explicitly computing it, as in general, it requires computing system trajectories.

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In [5], connections between Lyapunov theory, optimal control, and cost evaluation are presented. The results therein are presented in a tutorial way for nonquadratic cost functionals for continuous-time systems, yet without considering disturbances or constraints. It is shown in [5] that, if the solution to the steady-state Hamilton-Jacobi-Bellman equation is a Lyapunov function, besides optimality, asymptotic stability of the origin is guaranteed. The results in [5] are extended in [6] to two-player zero-sum differential games, while maintaining the applicability to continuous-time systems that exhibit nonlinear dynamics with a nonquadratic stage cost. The conditions therein are presented in the form of Hamilton-Jacobi-Isaacs (HJI) equations. Connections with Lyapunov functions, guaranteeing optimality in the min-max sense and asymptotic stability of closed sets, are included therein. In addition, the author extends an inverse optimal control approach to derive a cost functional that guarantees the existance of a saddle point for a given family of controllers. More recently, [7] proposed an extension of the results in [5] to a broad class of discrete-time systems: difference inclusions with constraints on the state and on the input. In particular, in [7], the authors present results in terms of Lyapunov-like inequalities to upper-bound and evaluate the cost. A unique feature of the setting in [7] is that completeness of feasible maximal solutions is not required, i.e., the results therein can be applied to systems with solutions that have a bounded domain of definition.

The main contribution of this paper is to derive upper bounds and to exactly evaluate the cost in an optimal nonquadratic control problem for constrained nonlinear difference equations with disturbances. To this end, we formulate it as a zero-sum game. We show in Proposition 3.1 that for discrete-time systems modeled by constrained difference equations, under pointwise sufficient conditions, the cost is upper bounded. More precisely, we find the value of the bound in terms of a Lyapunov-like function $V$. Moreover, when $V$ vanishes along solutions to the system, this bound is solution-independent, as stated in Corollary 3.2. Under stronger conditions, presented in Corollary 4.1, it is possible to evaluate the total cost in an exact form. In Section V, these results are used as a basis to express players actions as state-feedback laws, and provide sufficient conditions in terms of the function $V$ to bound the cost. For the case of solutions that may have bounded domain, further conditions on the final value are presented in Theorem 5.1. This result allows to characterize the strategy of the disturbance player that lower-bounds the cost, along with the control player strategy that upper-bounds the cost.

When $V$ vanishes along solutions, sufficient conditions are presented in Corollary 5.4, allowing to evaluate in an exact form the cost, which is the *saddle point value*. In Corollary 5.7, the case when the function $V$ is a Lyapunov function is considered. Sufficient conditions over the players feedback laws are presented, allowing to evaluate exactly the cost while guaranteeing asymptotic stability of a desired set for the closed-loop system. This cost is thus attained at the solution of the zero-sum game and is optimal in the min-max sense. Finally, the problem of robustly and optimally stabilizing the inverted pendulum is studied in Section 6. Due to space constraints, proofs and other details are not included and will be published elsewhere.

**Notation.** Given vectors $x, y$, we use the equivalent notation $(x, y) = [x^\top y^\top]^\top$. The symbol $\mathbb{N}$ denotes the set of natural numbers including zero. The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative reals. Given a vector $x$ and a nonempty set $\mathcal{A}$, the distance from $x$ to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$. Given a matrix $A \in \mathbb{R}^{n \times n}$, we say that $A$ is Schur if its eigenvalues are contained in the open unit circle. When $A$ is symmetric, the scalars $\underline{\lambda}(A)$ and $\bar{\lambda}(A)$ denote the minimum and largest eigenvalue of $A$, respectively. In addition, we denote with $\mathbb{S}^n_+$ the set of real positive definite matrices of dimension $n$ and with $I$, the identity matrix.

## II. Problem Statement

We consider optimal control problems for discrete-time systems described by constrained difference equations, under the presence of disturbances. Following [7] and inspired by [8], we consider constrained difference equations of the form

$$x^+ = g(x, u, w) \qquad (x, u, w) \in D, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$, and the constraint $D \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, which is the playable set. The symbol $x^+$ represents the value of the state after a discrete step. Let $\mathcal{X}$ be the set of functions $\phi : \text{dom}\phi \to \mathbb{R}^n$, $\mathcal{U}$ the set of functions $u_\phi : \text{dom}u_\phi \to \mathbb{R}^m$, and $\mathcal{W}$ the set of functions $w_\phi : \text{dom}w_\phi \to \mathbb{R}^p$, with domains that are subsets of $\mathbb{N}$. A triple $(\phi, u_\phi, w_\phi)$ is a solution to (1) if $\phi \in \mathcal{X}$, $u_\phi \in \mathcal{U}$, $w_\phi \in \mathcal{W}$, $(\phi(0), u_\phi(0), w_\phi(0)) \in D$, $\text{dom}\phi = \text{dom}u_\phi = \text{dom}w_\phi = \mathbb{N} \cap \{0, 1, ..., \bar{J}\}$ for some $\bar{J} \in \mathbb{N} \cup \{\infty\}$, and for all $j \in \text{dom}\,\phi$ such that $j + 1 \in \text{dom}\,\phi$,

$$\begin{aligned} (\phi(j), u_\phi(j), w_\phi(j)) &\in D \\ \phi(j+1) &= g(\phi(j), u_\phi(j), w_\phi(j)) \end{aligned}$$

A solution is said to be *maximal* if it is not the truncation of any other solution. We denote by $\mathcal{S}_{u,w}(M)$ the set of all maximal solutions $(\phi, u_\phi, w_\phi)$ to (1) such that $\phi(0) \in M$. We define the projection of $D$ onto $\mathbb{R}^n$ as

$$\Pi(D) = \{\xi \in \mathbb{R}^n : \exists (u, w) \in \mathbb{R}^m \times \mathbb{R}^p \text{ s.t. } (\xi, u, w) \in D\}$$

Consider the following cost associated to (1): Given $\xi \in \mathbb{R}^n$, the unique solution $(\phi, u_\phi, w_\phi)$ to (1) with $\phi(0) = \xi$, and $q_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$, define the functional

$$J(\xi, u_\phi, w_\phi) := \lim_{j \to \sup \text{dom}\phi} \sum_{i=0}^{j-1} q_d(\phi(i), u_\phi(i), w_\phi(i)) \tag{2}$$

When the limit in (2) exists, we say that $J(\xi, u_\phi, w_\phi)$ exists. For the discrete dynamics described by (1), the disturbance action is determined by the player that aims to maximize (2), while the control action is assigned by the player that wants to minimize it. This setting is modeled as a two-player zero-sum game as follows:

$$\begin{aligned} \min_{u_\phi} \max_{w_\phi} \quad & J(\xi, u_\phi, w_\phi) \\ \text{subject to} \quad & (\phi, u_\phi, w_\phi) \in \mathcal{S}_{u,w}(\xi) \end{aligned} \tag{3}$$

## III. Cost Upper Bounds based on Lyapunov Functions

In this section, we provide upper bounds on the cost of solutions to (1) from $\xi$ when applying a specific pair of actions $u_\phi, w_\phi$, as follows.

*Proposition 3.1:* (Solution Dependent Upper Bound) *Given $q_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ and $(g, D)$ defining (1), suppose there exists $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$V(g(x, u, w)) - V(x) + q_d(x, u, w) \leq 0 \qquad \forall (x, u, w) \in D \tag{4}$$

*Let $(\phi, u_\phi, w_\phi)$ be a solution to (1) from $\xi \in \Pi(D)$ and assume $V \circ \phi$ is bounded. Then, $J(\xi, u_\phi, w_\phi)$ in (2) is finite and satisfies*

$$J(\xi, u_\phi, w_\phi) + \limsup_{j \to \sup \text{dom}\,\phi} V(\phi(j)) \leq V(\xi) \tag{5}$$

Notice that (4) is a pointwise condition that holds on the playable set $D$. Since $q_d$ maps to nonnegative reals, and $V \circ \phi$ is bounded, the sum in (2) converges. Thus, the cost is upper bounded and the bound is expressed in terms of $V$, the initial condition $\xi$, and the solution $\phi$. This result is an extension of Proposition 1 in [7] to the case of constrained difference equations in the presence of disturbances.

To derive an upper bound on the cost $J(\xi, u_\phi, w_\phi)$ that is solution independent, we present the following result.

*Corollary 3.2:* (Solution Independent Upper Bound) *Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, $q_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$, and $(g, D)$ defining (1), suppose that there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is uniformly continuous on a neighborhood of $\mathcal{A}$ and such that $V(\mathcal{A}) = \{0\}$ and (4) holds. Let $(\phi, u_\phi, w_\phi)$ be a maximal solution to (1) with $\phi(0) = \xi \in \Pi(D)$ such that*

$$\lim_{j \to \sup \text{dom}\,\phi} |\phi(j)|_{\mathcal{A}} = 0 \tag{6}$$

*Then,*

$$J(\xi, u_\phi, w_\phi) \leq V(\xi) \tag{7}$$

*Remark 3.3:* The input pair $(u_\phi, w_\phi)$ renders the maximal solution to (1) convergent to $\mathcal{A}$. Uniform continuity of $V$ is required for $V \circ \phi$ to vanish when $\phi$ converges to the set $\mathcal{A}$.

The bound in (7) is expressed independently of the solution. Indeed, it only depends on the function $V$ evaluated at the initial state $\xi$.

## IV. Exact Cost Evaluation based on Lyapunov Functions

By strengthening assumptions in Corollary 3.2, we show that the exact value of the cost is given by $V(\xi)$.

*Corollary 4.1:* (Exact Cost Evaluation) *Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, $q_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$, and $(g, D)$ defining (1), suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$V(g(x,u,w)) - V(x) + q_d(x,u,w) = 0 \qquad \forall (x,u,w) \in D \tag{8}$$

*Let $(\phi, u_\phi, w_\phi)$ be a solution to (1) from $\xi \in \Pi(D)$ such that $V \circ \phi$ is bounded. Then, $J(\xi, u_\phi, w_\phi)$ is finite and satisfies*

$$J(\xi, u_\phi, w_\phi) \;+ \lim_{j \to \sup \operatorname{dom} \phi} V(\phi(j)) \;=\; V(\xi) \tag{9}$$

*Moreover, if $V$ is uniformly continuous on a neighborhood of $\mathcal{A}$, $V(\mathcal{A}) = \{0\}$, and*

$$\lim_{j \to \sup \operatorname{dom} \phi} |\phi(j)|_{\mathcal{A}} = 0 \tag{10}$$

*Then*

$$J(\xi, u_\phi, w_\phi) = V(\xi) \tag{11}$$

## V. Upper Bounds and Exact Cost Evaluation for Closed-Loop Systems

Building upon the results in the previous section, we are interested in finding the control action that minimizes the cost in the presence of a disturbance action. The disturbance action is determined by the player that aims to maximize the cost, while the control action is determined by the player that seeks to minimize it. The solution of the two-player zero-sum game in (3) is given by the Nash-equilibrium or saddle point equilibrium, and such strategies are modeled as feedback laws. We present sufficient conditions, as those employed in Sections III and IV, to find bounds over the cost. Likewise, when solutions converge to the closed set $\mathcal{A}$ where $V$ vanishes, existence and exact evaluation of the cost at the saddle point are studied.

Functions $\theta : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^n \to \mathbb{R}^p$ are the feedback laws that define the actions of the control player and the disturbance player, respectively. Now, given such a pair of feedback laws, consider the closed-loop system

$$x^+ = g(x, \theta(x), \psi(x)) \qquad (x, \theta(x), \psi(x)) \in D. \tag{12}$$

A function $\phi$ is a solution to (12) if $\phi \in \mathcal{X}$, $(\phi(0), \theta(\phi(0)), \psi(\phi(0)) \in D$, and for all $j \in \operatorname{dom} \phi$ such that $j+1 \in \operatorname{dom} \phi$:

$$(\phi(j), \theta(\phi(j)), \psi(\phi(j)) \in D$$

$$\phi(j+1) = g(\phi(j), \theta(\phi(j)), \psi(\phi(j)))$$

Given the feedback laws $\theta$ and $\psi$, we denote by $\mathcal{S}(M)$ the set of maximal solutions $\phi$ to (12) with $\phi(0) \in M$. We denote by $\mathcal{S}_u^\psi(M)$ the set of maximal solutions $(\phi_\psi, u_\phi)$ to

$$x^+ = g(x, u, \psi(x)) \qquad (x, u, \psi(x)) \in D \tag{13}$$

with $\phi_\psi(0) \in M$ and $u_\phi \in \mathcal{U}$. Additionally, we denote by $\mathcal{S}_w^\theta(M)$ the set of maximal solutions $(\phi_\theta, w_\phi)$ to

$$x^+ = g(x, \theta(x), w) \qquad (x, \theta(x), w) \in D \tag{14}$$

with $\phi_\theta(0) \in M$ and $w_\phi \in \mathcal{W}$. Likewise, $J(\xi)$ is the cost of the solution $\phi$ to (12) from $\xi$, $J^\psi(\xi, u_\phi)$ is the cost of the solution $\phi_\psi$ to (13) with input $u_\phi$, and $J^\theta(\xi, w_\phi)$ is the cost of the solution $\phi_\theta$ to (14) with input $w_\phi$.

In this section, we present sufficient conditions to find bounds on the cost when applying the feedback pair $(\theta, \psi)$. When solutions converge to $\mathcal{A}$ (a closed set where $V$ vanishes), these conditions allow to evaluate the cost at the saddle point given by such a feedback pair. This evaluation is achieved by applying the results in the previous section.

### A. Bounds based on Cost Evaluation

*Theorem 5.1:* (Cost bounds for not necessarily complete solutions) *Consider system (1) with cost functional defined by (2). Suppose there exist functions $\theta : \mathbb{R}^n \to \mathbb{R}^m$, $\psi : \mathbb{R}^n \to \mathbb{R}^p$, $V : \mathbb{R}^n \to \mathbb{R}$, and $q_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ such that*

$$V(g(x, \theta(x), \psi(x))) - V(x) + q_d(x, \theta(x), \psi(x)) = 0 \qquad \forall x : (x, \theta(x), \psi(x)) \in D, \tag{15}$$

$$V(g(x, u, \psi(x))) - V(x) + q_d(x, u, \psi(x)) \geq 0 \qquad \forall (x,u) : (x, u, \psi(x)) \in D, \tag{16}$$

$$V(g(x, \theta(x), w)) - V(x) + q_d(x, \theta(x), w) \leq 0 \qquad \forall (x,w) : (x, \theta(x), w) \in D. \tag{17}$$

*Then, the following hold:*

*i) Let $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ be a solution to (12) from $\xi \in \Pi(D)$. Suppose that $V \circ \phi$ is bounded. Then, $J(\xi)$ is finite and satisfies*

$$J(\xi) \;+ \lim_{j \to \sup \operatorname{dom} \phi} V(\phi(j)) \;=\; V(\xi) \tag{18}$$

*ii) Let $(\phi_\psi, u_\phi)$ and $(\phi_\theta, w_\phi)$ be solutions to (13) and to (14), respectively, both from $\xi$. Assume the following:*

*a) $V \circ \phi$ is bounded,*

*b) $V \circ \phi_\psi$ and $V \circ \phi_\theta$ are bounded,*

*c) and*

$$\limsup_{j \to \sup \operatorname{dom} \phi_\psi} V(\phi_\psi(j)) \leq \lim_{j \to \sup \operatorname{dom} \phi} V(\phi(j)) \leq \liminf_{j \to \sup \operatorname{dom} \phi_\theta} V(\phi_\theta(j)) \tag{19}$$

*Then*

$$J^\theta(\xi, w_\phi) \leq J(\xi) \leq J^\psi(\xi, u_\phi) \tag{20}$$

## B. Exact Cost Evaluation with Convergent Solutions

Theorem 5.1 presents sufficient conditions to establish bounds on the cost for solutions to (12) that are not necessarily complete. This leads to solution-dependent bounds, i.e., (20). Condition (15) is the discrete-time version of the steady-state Hamilton-Jacobi-Isaacs equation [4], along with (16), (17), which imply that the feedback pair is in equilibrium with respect to the cost functional. Now, we show that when maximal solutions to (12), (13), and (14) are complete and converge to the set $\mathcal{A}$, the feedback pair $(\theta, \psi)$ allows to attain an upper bound for the cost associated to (14) and a lower bound for the cost associated to (13). Thus, the feedback pair becomes a saddle point equilibrium and it is formally introduced via the following definitions.

*Definition 5.2:* (Set of maximal responses) *Given $\xi \in \mathbb{R}^n$, $\psi : \mathbb{R}^n \to \mathbb{R}^p$, and $u_\phi \in \mathcal{U}$, we denote the set of maximal responses to (13) from $\xi$ as*

$$\mathcal{R}^{\psi}(\xi, u_\phi) := \{\phi_\psi \in \mathcal{X} : (\phi_\psi, u_\phi) \in \mathcal{S}_u^{\psi}(\xi)\}.$$

*In addition, given $\xi \in \mathbb{R}^n$, $\theta : \mathbb{R}^n \to \mathbb{R}^p$, and $w_\phi \in \mathcal{W}$, we denote the set of maximal responses to (14) from $\xi$ as*

$$\mathcal{R}^{\theta}(\xi, w_\phi) := \{\phi_\theta \in \mathcal{X} : (\phi_\theta, w_\phi) \in \mathcal{S}_w^{\theta}(\xi)\}$$

Furthermore, let us define the sets $\mathcal{X}_\mathcal{A} := \{\phi \in \mathcal{X} : \lim_{j \to \sup \operatorname{dom} \phi} |\phi(j)|_\mathcal{A} = 0\}$, $\mathcal{U}_\psi(\xi) := \{u_\phi \in \mathcal{U} : \exists \phi \in \mathcal{R}^{\psi}(\xi, u_\phi) \cap \mathcal{X}_\mathcal{A}\}$, and $\mathcal{W}_\theta(\xi) := \{w_\phi \in \mathcal{W} : \exists \phi \in \mathcal{R}^{\theta}(\xi, w_\phi) \cap \mathcal{X}_\mathcal{A}\}$ [1]. The former is the set of convergent trajectories. The latter two sets define the sets of actions yielding (13) and (14), respectively, to responses from $\xi$ that converge to $\mathcal{A}$ under the respective feedback laws. The following definition is inspired by [1].

*Definition 5.3:* (Saddle Point) *The pair $(\theta, \psi)$ is a saddle point for the game (3) with cost functional $J(\xi, u_\phi, w_\phi)$ on $\mathcal{U}_\psi(\xi) \times \mathcal{W}_\theta(\xi)$ if $J(\xi) \le J^{\psi}(\xi, u_\phi)$ for every $u_\phi \in \mathcal{U}_\psi(\xi)$ and $J^{\theta}(\xi, w_\phi) \le J(\xi)$ for every $w_\phi \in \mathcal{W}_\theta(\xi)$.*

By applying the results presented in Section IV, we show that the saddle point value is $V(\xi)$.

*Corollary 5.4:* (Cost evaluation for convergent maximal solutions) *Consider a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a given $(g, D)$ defining (12). Suppose there exist functions $q_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$, $\theta : \mathbb{R}^n \to \mathbb{R}^m$, $\psi : \mathbb{R}^n \to \mathbb{R}^p$, and a function $V : \mathbb{R}^n \to \mathbb{R}$ that is uniformly continuous on a neighborhood of $\mathcal{A}$ such that $V(\mathcal{A}) = \{0\}$, and (15), (16) and (17) hold. Furthermore, suppose $\theta$ and $\psi$ are such that $\mathcal{U}_\psi(\xi)$ and $\mathcal{W}_\theta(\xi)$ are nonempty. If the unique maximal solution $\phi$ to (12) from $\xi \in \Pi(D)$ converges to $\mathcal{A}$, then*

$$J(\xi) = V(\xi). \tag{21}$$

*In addition,*

$$J(\xi) = \min_{u_\phi \in \mathcal{U}_\psi(\xi)} \max_{w_\phi \in \mathcal{W}_\theta(\xi)} J(\xi, u_\phi, w_\phi) \tag{22}$$

[1]Maximal solutions to (13) and (14) are unique. Hence, the existence statements in these sets imply convergence of such responses to $\mathcal{A}$.

Notice that (22) establishes the existence of a saddle point, according to Definition 5.3 and the definition of a saddle point presented in [1].

Convergence of solutions to (12) to the set $\mathcal{A}$ can be guaranteed in a general way under the existence of a Lyapunov function ensuring asymptotic stability of $\mathcal{A}$. We present an extension of Corollary 5.4 to provide sufficient conditions for $(\theta, \psi)$ to be a saddle point under the existence of such a Lyapunov function. In this way, both optimality in the min-max sense of $(\theta, \psi)$, and stability of $\mathcal{A}$ for the closed-loop system are attained. With that aim, we first introduce the following definitions.

*Definition 5.5:* (Class-$\mathcal{K}_\infty$ functions) *A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a class $\mathcal{K}_\infty$ function, also written as $\gamma \in \mathcal{K}_\infty$, if $\gamma$ is zero at zero, continuous, strictly increasing, and unbounded.*

*Definition 5.6:* (Positive definite functions with respect to a set $\mathcal{A}$) *Let $\theta : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^n \to \mathbb{R}^p$. We say that a function $\rho : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{\ge 0}$ is positive definite with respect to a set $\mathcal{A} \subset \mathbb{R}^n$, in composition with $\theta$ and $\psi$, also written as $\rho \in \mathcal{PD}_{\theta,\psi}(\mathcal{A})$, if $\rho(s, \theta(s), \psi(s)) > 0$ for all $s \notin \mathcal{A}$ and $\rho(\mathcal{A}, \theta(\mathcal{A}), \psi(\mathcal{A})) = \{0\}$.*

*Corollary 5.7:* (Cost evaluation under the existence of a Lyapunov function) *Let $\mathcal{A} \subset \mathbb{R}^n$ be closed, $q_d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$, and suppose there exist functions $\theta : \mathbb{R}^n \to \mathbb{R}^m$, and $\psi : \mathbb{R}^n \to \mathbb{R}^p$ such that the following hold:*

*i) $q_d \in \mathcal{PD}_{\theta,\psi}(\mathcal{A})$.*

*ii) For each $\xi \in \Pi(D)$ the maximal solution $\phi \in \mathcal{S}(\xi)$ is complete.*

*iii) For each $\xi \in \Pi(D)$, every $\phi_\psi \in \mathcal{R}^{\psi}(\xi, u_\phi)$ with $u_\phi \in \mathcal{U}_\psi(\xi)$ satisfies $\phi_\psi \in \mathcal{X}_\mathcal{A}$.*

*iv) For each $\xi \in \Pi(D)$, every $\phi_\theta \in \mathcal{R}^{\theta}(\xi, w_\phi)$ with $w_\phi \in \mathcal{W}_\theta(\xi)$ satisfies $\phi_\theta \in \mathcal{X}_\mathcal{A}$.*

*In addition, assume that there exist a function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(\mathcal{A}) = \{0\}$, (15), (16), and (17) hold and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_\mathcal{A}) \le V(x) \le \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in \Pi(D) \tag{23}$$

*Then, for each $\xi \in \Pi(D)$*

$$J(\xi) = V(\xi) \tag{24}$$

*Moreover*

$$J(\xi) = \min_{u_\phi \in \mathcal{U}_\psi(\xi)} \max_{w_\phi \in \mathcal{W}_\theta(\xi)} J(\xi, u_\phi, w_\phi) \tag{25}$$

In this case, continuity of $V$ is not required. This is due to (23). For further details on Lyapunov functions in these settings, see [9] and [10].

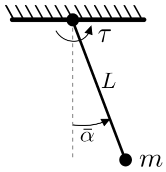


Fig. 1. Pendulum setting and variables notation.

## VI. Application to Stabilization of the Inverted Pendulum

We present an application of the results to the inverted pendulum system, solving it as a zero-sum game in which the controller minimizes the cost to take the pendulum to its upper equilibrium point in the presence of disturbances that are assumed to be seeking to maximize that cost.

### A. Disturbance-free scenario

We can model the dynamics of the friction-less inverted pendulum with no disturbances of Fig. 1 as

$$\dot{\bar{\alpha}} = \omega \tag{26}$$

$$\dot{\omega} = \frac{1}{mL^2}\tau - \frac{\gamma}{L}\sin\bar{\alpha} \tag{27}$$

Here, $\bar{\alpha}$ is the counter-clockwise angle of the pendulum with respect to the lowest equilibrium point, $mL^2$ is the moment of inertia, $L$ is the length of the pendulum, $m$ is the mass of the the pendulum and $\tau$ is the input torque.

A linearization of this system around $\bar{\alpha} = \pi$, and the change of variables $\bar{\alpha} = \pi + \alpha$, yield

$$\begin{bmatrix}\dot{\alpha}\\ \dot{\omega}\end{bmatrix} = F\begin{bmatrix}\alpha\\ \omega\end{bmatrix} + \Gamma\tau = \begin{bmatrix}0 & 1\\ \frac{\gamma}{L} & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \omega\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{mL^2}\end{bmatrix}\tau \tag{28}$$

Thus, to allow to apply the results presented herein, we obtain the ZOH discretization of (28) with sampling time $T_s$, as

$$x^+ = Ax + B_1u = e^{FT_s}x + \int_0^{T_s} e^{Ft}\Gamma dtu \tag{29}$$

with $x = (\alpha, \omega)$, $u = \tau$, $w = 0$. Let $\mathcal{A} = \{0\}$ and

$$q_d(x, u, w) := x^\top Qx + u^2. \tag{30}$$

with $Q \in \mathbb{S}^2$. We pursue to design a static linear feedback law $\theta(x) = -K_1x$. Since $(A, B_1)$ is controllable, there exists a $K_1$ such that $A - B_1K_1$ is Schur. Furthermore, let $V(x) := x^\top Px$, with $P \in \mathbb{S}^2$, and let us consider first the unconstrained case, i.e., $D = \mathbb{R}^4$. Thus, system (12) specializes into

$$x^+ = (A - B_1K_1)x \qquad (x, \theta(x), \star) \in D. \tag{31}$$

Given the feedback law $\theta$ and the Lyapunov-like function $V$, conditions (15) and (16) turn into

$$A_1^\top PA_1 - P + Q + K_1^\top K_1 = 0 \tag{32}$$

$$\begin{bmatrix}A^\top PA - P + Q & A^\top PB_1\\ B_1^\top PA & 1 + B_1^\top PB_1\end{bmatrix} \succeq 0 \tag{33}$$

where $A_1 = (A - B_1K_1)$. Without disturbances, it is clear that (15) implies (17). Therefore, we conclude that if there exist matrices $K_1$, $P$, and $Q$ such that (32) and (33) hold, and $V$ is bounded along the solution $\phi$ to the closed loop system (31) from $\xi \in \Pi(D)$, then item 1) of Theorem 5.1 implies that (18) holds. Thus,

$$J(\xi) = \xi^\top P\xi - \lim_{j\to\sup\text{dom}\,\phi} \phi(j)^\top P\phi(j) \tag{34}$$

is the optimal cost of the solution to system (29) from $\xi$, and it is attained by setting $u = \theta(x) = -K_1x$. This result can be shown to hold for $T_s = 0.1s$, and $m = 0.2\text{kg}, L = 0.3\text{m}, \gamma = 9.81\text{m/s}^2, Q = 10I, P = \left[\begin{smallmatrix}114.7824 & 5.2293\\ 5.2293 & 10.2898\end{smallmatrix}\right], K_1 = \begin{bmatrix}0.7533 & 0.2068\end{bmatrix}$. Matrix $P$ is found by solving LMI (33) and it is used to solve for $K_1$ in (32). In addition, given that $\mathcal{A}$ is closed, $V$ is uniformly continuous and $K_1$ renders $A_1$ Schur, for $\xi \in \Pi(D)$, Corollary 5.4 implies that $V(\xi)$ is the optimal cost and it is attained under the feedback gain $K_1$. See Figure 2.

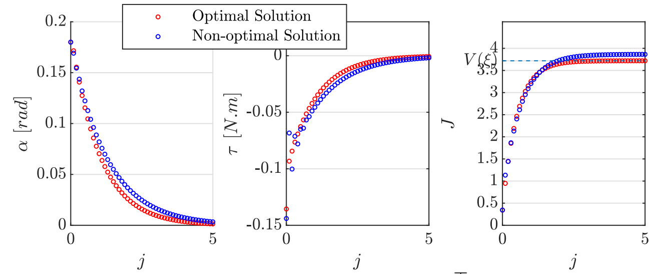


Fig. 2. Solutions to system (29) from $\xi = [0.18\ 0]^\top$. Trajectories, control actions and costs associated when applying $u = \theta(x) = -K_1x$ (red) and $u = -\hat{K}_1x$ with $\hat{K}_1 \neq K_1$ (blue). The dashed line displays the value of $V(\xi)$ that is attained as the optimal cost.

*Constrained case (Solution-Dependent Analysis):* Now, consider the playable set $D = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \setminus \tilde{D}$ with $\tilde{D} = \{(x, u, w) : \underline{\alpha} \leq \alpha \leq \breve{\alpha},\ \underline{\omega} \leq \omega \leq \breve{\omega}\}$ and notice that these state constraints may yield incomplete solutions, i.e., solutions exiting $D$ (entering $\tilde{D}$). Let us study this behavior by analyzing specific solutions. In particular, let us compare the trajectory of (29) with a given control action $u = \hat{K}_1x$, (s.t. $\hat{K}_1 \neq K_1$), and the solution to (31), from the same initial condition. Let $\tilde{D}$ be defined by $\underline{\alpha} = 0.04, \breve{\alpha} = 0.05, \underline{\omega} = -0.05, \breve{\omega} = 0$ and let us choose $\xi = [0.18\ 0]^\top$. When setting $\hat{K}_1 = [0.8\ \ 0.3]$ (non-optimal), one has $V \circ \phi_\psi$ bounded for the solution $(\phi_\psi, u_\phi)$ to (29), and particularly

$$\begin{aligned}&\limsup_{j\to\sup\text{dom}\,\phi_\psi} V(\phi_\psi(j)) = 0.249\\ &< 0.2574 = \lim_{j\to\sup\text{dom}\,\phi} V(\phi(j))\end{aligned} \tag{35}$$

Then, item 2) of Theorem 5.1 implies that for the feedback pair $(\theta, \psi) = (-K_1x, \star)$, (20) holds, which can be verified as

$$J(\xi) = 3.4615 < 3.6113 = J^\psi(\xi, u_\phi) \tag{36}$$

This illustrates that, when (19) holds over $V$ evaluated at the last value of the solutions to (29) and (31) from the same $\xi$, the cost of the solution to (31) provides a lower bound. Specific solutions are analyzed to exploit the applicability of this preliminary result to the case of incomplete maximal solutions with bounded final values as shown in Figure 3.

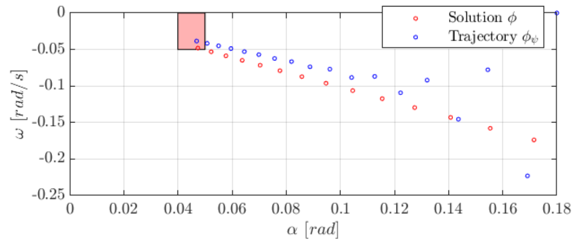


Fig. 3. Non-complete maximal solutions due to state constraints. In blue, we have the trajectory $\phi_\psi$ of (29) with $u = \hat{K}_1 x$, (s.t. $\hat{K}_1 \neq K_1$), and in red, the solution $\phi$ to (31), both from $\xi = [0.18 \quad 0]^\top$ when they cannot jump anymore at $\tilde{D} = \{(x, u, w) : \underline{\alpha} \leq \alpha \leq \breve{\alpha},\ \underline{\omega} \leq \omega \leq \breve{\omega}\}$ (shaded region).

### B. Scenario under disturbances

If we include the effect of additive disturbances to the control action, the system dynamics are described by

$$x^+ = Ax + B_1\tilde{u} = e^{FT_s}x + \int_0^{T_s} e^{Ft}\Gamma dt\tilde{u} \quad (37)$$

with $x = (\alpha, \omega)$, $\tilde{u} = u + w$ and $u = \tau$. We maintain the same parameters but now pick

$$q_d(x, u, w) := x^\top Qx + u^2 + R_2 w^2. \quad (38)$$

In this case, let us set $V(x) = x^\top Px$ and $D = \{(x, u, w) : |u| \leq \bar{u}, x^\top Px \leq c_1\}$. In the previous section, we illustrated that there exist matrices $K_1$, $K_2$ that render $A - B_1(K_1 + K_2)$ Schur. Using such laws, (12) becomes

$$x^+ = (A - B_1K_1 - B_1K_2)x \qquad (x, \theta(x), \psi(x)) \in D \quad (39)$$

If there exist parameters $Q$, $R_2$, $K_1, K_2, P$, $\bar{u}$ and $c_1$ such that conditions (15), (16) and (17) are satisfied, and if $V \circ \phi$ is bounded for the solution $\phi$ to the closed loop system (39) from $\xi \in \Pi(D)$, then item 1) of Theorem 5.1 implies that (18) holds with $q_d$ as defined in (38), $\theta = -K_1x, \psi = -K_2x$, and $V(x) = x^\top Px$. This is the case when $R_2 = -2134$, $K_2 = \begin{bmatrix}-35.316 & -9.694\end{bmatrix} \times 10^{-5}$, $\bar{u} = 1$ and $c_1 = 126$, and the other parameters have the values specified in the previous subsection. On this scenario, $K_1$ and $K_2$ are matrices such that $q_d \in \mathcal{PD}_{\theta,\psi}(\{0\})$ and for each $\xi \in \Pi(D)$, $(\theta, \psi)$ yields (39) to a maximal solution $\phi$, that is complete and converges to $\mathcal{A}$. By taking $\alpha_1(|x|_\mathcal{A}) = \underline{\lambda}(P)|x|^2$ and $\alpha_2(|x|_\mathcal{A}) = \bar{\lambda}(P)|x|^2$, from Corollary 5.7, we have that $V(\xi)$ is the saddle point value and $(\theta, \psi)$ is the saddle point equilibrium for the game (3). Notice that this problem is solved over the set of actions $\mathcal{U}_\psi(\xi)$ and $\mathcal{W}_\theta(\xi)$, which render trajectories from $\xi$ convergent to $\mathcal{A}$. These sets are not required to be explicitly characterized. This behavior is illustrated in Figure 4.

## VII. Conclusions and Future Work

In this paper we address the optimal control problem for constrained difference equations with disturbances as a zero-sum game. The results are presented based on cost evaluation approaches, laying on conditions over Lyapunov-like functions. This allows to characterize the saddle point equilibrium for the game and, under additional conditions, to guarantee asymptotic stability of a closed set. The proposed approach does not require the dynamics to be linear, nor the running cost to be quadratic, which is the common case treated in the literature of optimal control. An application is presented for the optimal stabilization of an inverted pendulum under the presence of disturbances.
Future work includes the extension of these results to solve optimal control problems for hybrid inclusions (in the framework of [11]) in the presence of disturbances, based on the approach in [12].

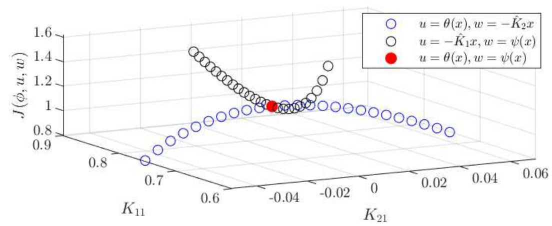


Fig. 4. Saddle point behavior in cost of solutions to the difference equation (37) from $\xi = [-0.1, 0.1]^\top$ when modifying the feedback gains around the optimal value. In the optimal case, $u = -K_1x$ and $w = -K_2x$ (red). Variations are considered in $u = -\hat{K}_1x = -[K_{11}, K_{12}]x$ with $K_{11} \in [0.6536, 0.8936]$ (black) and in $w = -\hat{K}_2x = -[K_{21}, K_{22}]x$ with $K_{21} \in [-0.0504, 0.0696]$ (blue), while keeping the rest of the parameters fixed.

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