Two Students' Conceptions of Solutions to a System of Linear Equations

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Systems of linear equations (SLE) comprise a fundamental concept in linear algebra, but there is little research regarding the teaching and learning of SLE, especially students' conceptions of solutions. In this study, we examine students' understanding of solutions to SLE in the context of an experientially real task sequence. We interviewed two undergraduate mathematics majors, who were also preservice teachers, to see how they thought about solutions to SLE in $\mathbb{R}^3$*, especially linear systems with multiple solutions. We found participants used their knowledge of SLE in* $\mathbb{R}^2$ *to think about systems in higher dimensions, sometimes ran into algebraic complications, and initially did not find the third dimension intuitive to think about geometrically. Our findings highlight students' ways of reasoning with infinite solution sets, such as moving toward the notion of parametrization.*

Keywords: linear algebra, systems of linear equations, student thinking, solution

Systems of linear equations (SLE) are a core concept in linear algebra. Having a deep understanding of this concept helps in understanding many applications, as well as other topics in linear algebra. Instruction often foregrounds solving methods that focus first on unique solution cases, before moving to the much more common cases of no solution or infinitely many solutions. There is evidence that such procedural instructional approaches to SLE are often inadequate for helping students make sense of SLE in these more common cases (Litke, 2020; Vaiyavutjamai & Clements, 2006). Procedural instruction links to memorization of rules for knowing when a system has no solution or infinitely many solutions and what that means for the geometry of the SLE (Huntley et al., 2007).

In the current study, we examine two students' understanding of solutions to SLE, from both algebraic and geometric interpretations. We used two research questions to guide our analysis: *(1) How did students reason about solutions in the context of the system of equations task sequence? (2) What did students wonder about in this same context?* We discuss the ways participants consider solutions to SLE and questions they voice as they work through a task sequence designed as part of a research-based curriculum design project in linear algebra.

Literature

Research regarding SLE is limited but growing. In this small pool of research, some regard solutions to SLE, whether through task development (e.g., Possani et al., 2010; Sandoval & Possani, 2016) or examining students' understanding of solutions algebraically and geometrically (e.g., Harel, 2017; Huntley et al., 2007; Oktaç, 2018; Zandieh & Andrews-Larson, 2019). For SLE instruction, Possani et al. (2010) developed a task about a set of streets, asking students which streets could be closed for road work without disrupting the flow of traffic. Major parts of this task were modeling the context, using variables to represent important quantities, and then writing a system based on those understandings. Possani and colleagues argue that their task helped students create a solid foundation for understanding SLE and their solutions, and that students were able to transfer what they learned in this task to other problems regarding SLE. Sandoval and Possani (2016) developed a task to see when moving between representations in $\mathbb{R}^3$ (e.g., graphs versus equations of vectors or planes) makes sense for students. They found students could naturally interpret geometric representations of solutions to SLE but that connecting geometric representations to algebraic representations to be less natural.

In regard to student thinking about SLE, several authors have found single solutions to SLE to be more intuitive for students to interpret than systems with no or infinitely many solutions (Harel, 2017; Huntley et al., 2007; Oktaç, 2018). Oktaç (2018) and Huntley et al. (2007) noted that when students worked to interpret results of algebraic manipulation like $8x = 8x$ or $2 = 4$, students often rely on memorized rules to decide which had infinitely many solutions or no solution. Huntley and colleagues also noted these rules were unhelpful to students in deciding whether parallel lines or the same line in a graph meant no solution or infinitely many solutions. Zandieh and Andrews-Larson (2019) found that, when solving SLE, students tended to row reduce on matrices or manipulate the given systems. Students who used the row reduced echelon form were generally successful in solving, but connections between row reduced forms and their geometric representations were not straightforward for them to interpret. Thus, the research points to a need for more tasks that provide opportunities for students to algebraically and geometrically examine SLE with infinitely many solutions, especially those in $\mathbb{R}^3$.

Theoretical Framework

We drew from Realistic Mathematics Education (RME) when designing the task sequence. RME is a theory used to design mathematics tasks and involves Freudenthal's (1973, 1991) framing of mathematics as a human activity. Three core heuristics of RME are guided reinvention, emergent models, and didactical phenomenology (Larsen, 2018). The first refers to which aspects of a concept students are to reinvent along with an instructor's guidance. The second refers to models students develop to organize their mathematical activity. The third refers to the selection and use of phenomena in a task sequence to build toward a mathematics concept. We wanted students to develop reasoning about what the *collection* of solutions to an SLE looks like and how those solutions relate to one another (both numerically and geometrically). We hoped students would be able to build imagery for that collection of things (solutions) by working through our task sequence. Our goal is to understand our participants' reasoning related to this reinvention of infinite solution sets.

Another important heuristic in RME is the use of experientially real starting points (Gravemeijer, 1999), an aspect of didactical phenomenology. These starting points are intended to help students engage in the development of a mathematical idea, rather than have them simply apply prescribed procedures to a situation. In our task context, students find valid combinations of meals in a meal plan. We assume most students are either familiar with having a meal plan that requires a limited number of meals, or, at least, familiar with experiences less specific, like using a limited amount of money to make purchases. Historically, SLE have been used in agricultural and economic situations (Andrews-Larson, 2015), which present contexts for reasoning about SLE (i.e., didactical phenomenology). In terms of the mathematics in the tasks, we assume students have prior experience writing equations that represent a situation, listing solutions (e.g., to at least one or two equation equations), graphing (more in 2d than in 3d), and manipulating equations using substitution and elimination.

Methods

This study is part of a broader NSF-funded project (1914793, 1914841, 1915156) focused on the development of curricular materials in inquiry-oriented linear algebra. This particular task sequence was designed to help students create or reinvent ways of thinking about solutions to SLE, especially infinite solutions sets. Participants were two undergraduate students at a large public university in the Southeastern United States. Both were mathematics majors and preservice secondary mathematics teachers. Both participants were white; one was a woman ("R"), and one was a man ("L"). The students were recruited by asking faculty in a local secondary teacher preparation program to identify two math majors preparing to be secondary math teachers who had taken Calculus I but not linear algebra, were neither atypically "strong" nor "weak" in terms of their mathematical preparation and would be willing to explain their reasoning. Two of the authors interviewed the participants across four consecutive days on Zoom, working through as much of the designed task sequence as possible. We used a paired teaching experiment (PTE) to see how students reasoned with the tasks (Steffe &Thompson, 2000) and to allow for discussion between the participants and the interviewer. When working through the task sequence, participants briefly worked individually before discussing their reasoning. The interviewer prompted participants to think about a problem and explain their ideas, asking questions to further their thinking at some points.

Task Sequence

The tasks posed to participants in the PTE begins with an experientially real situation focused on a university meal plan (see Figure 1). During the first session, both participants worked to identify numbers of breakfasts, lunches, and dinners satisfying the given constraints, as well as estimate the number of options. During the second session, the pair of participants worked to represent the set of solutions they identified in a three-dimensional space using the corner of a cardboard box. During the third session, they explored the geometry of linear systems with three unknowns using Geogebra. During the fourth and final session, they generated examples of systems of equations with specified numbers of solutions and developed generalizations based on this work. In this proposal, we focus on participants' work during days 1 and 2 of the PTE.

PTE Day 1. Participants were first prompted to consider a constraint regarding the number of meals that can be purchased in the context of the meal plans (Figure 1, Part 1). They were asked to list a few different choices for the 210-meal plan and estimate the total number of possible 210-meal plans. The goal for this part of the task was to have the participants reason about and organize a large solution set (not to treat it as a combinatorics problem).

Participants were then asked to consider an additional constraint related to the cost of meals (Figure 1, Part 2). Participants were asked to write equations corresponding to each constraint, and to identify a solution that satisfied the constraints for the number of meals but not the cost (and vice versa) as well as a solution that simultaneously satisfied both constraints. This pressed participants to consider solutions that would simultaneously satisfy two different equations.

PTE Day 2. Participants were first asked to predict what it would look like if all of the solutions to the number of meals constraint ($b+l+d=210$) were graphed. (Note: here b, l, and d are the number of breakfasts, lunches, and dinners, respectively.) They were then asked to find a way to represent (algebraically and geometrically) all of the "no dinner" meal plans (ignoring the 105-meal maximum from day 1), and to make similar predictions corresponding to the cost constraint equation ($5b+7l+10d=1500$). Finally, they were asked to consider the graph of both sets of solutions simultaneously using the corner of a cardboard box to represent $\mathbb{R}^3$.

Part 1: Number of Meals. A university meal plan called the "210-meal plan" requires that a student purchase exactly 210 meals for a 15–week semester (i.e., on average 2 per day for 15 x 7=105 days).
Part 2: Costs of Meals. You just made an estimate of how many different choices would fit the requirements of the 210-meal plan. As you read the brochure more carefully, you noticed that the cost of your 210 meals must add up to exactly $1500. Breakfasts cost $5 each, lunches cost $7 each, and dinners cost $10 each.

Figure 1: The Meal Plans Context.

Data Sources, Analysis, and Limitations

Data sources consisted of video recordings of the interviews, field notes, and participant work. The four authors watched the video recordings of the interviews together, taking notes regarding what we each noticed regarding participants' reasoning about solutions to linear systems as we watched the participants work through the task sequence. Two of the authors used an emergent coding method (Glaser & Strauss, 2017) along with researcher notes to locate moments that offered insight into participants' conceptions of solutions. These instances were selectively transcribed and recordings were reviewed for additional details. Due to space constraints, we limit our analysis to the first days of the PTE. Though our interpretations were triangulated across data sources and study authors, we do not make broad claims about the generalizability of these findings. Rather, our findings document possible reasoning paths and productive starting points of students engaging in the described task sequence.

Findings

In the first day, participants first with the SLE algebraically, one equation at a time. In connecting finding solutions to counting solutions of the first equation, participants wondered about how solutions might be ordered. Participants then considered solutions to the entire SLE and started down a path toward parametrization. On day 2, participants began connecting their algebraic work to the graph of the SLE.

"Are we thinking about (105, 105, 0) being different from (105, 0, 105)?"

In this part of the task, participants looked for solutions to a linear equation ($b + l + d = 210$) and estimated the number of solutions to that equation in the context of the given "real world" constraints. In trying to estimate the number of solutions, Student R asked, *"Are we thinking about (105, 105, 0) being different from (105, 0, 105)?"* Student L replied, *"Does it matter?"*

Here, we saw participants reasoning with and wondering about order in solutions. When adding three numbers, order does not matter, but when counting solutions, order does matter. We saw both participants wonder how solutions with the same three values might count as the same or different solutions. This question prompted the development of a shared notation, that breakfast is the first value, lunch the second, and dinner the third.

"Get all the variables in terms of one variable"

In the second part of the task, an additional constraint was added., and participants began to use algebraic manipulation they learned in secondary school, primarily substitution. We found that, because the system had many solutions, both participants ended up in some circulation of algebra, seemingly looking for a single value (as evidenced by Student L's comment that the result was *“probably going to be a positive number”*). Both ended up plugging different things into equations but found *“the variables will cancel”* (Student L) or *“everything zeroed out”* (Student R), as shown in Figures 2 and 3. Student L articulated a goal for plugging things in: *“I’m just trying to get all the variables in terms of one variable like* [R] *was doing, but every time I try it I guess they’ll either cancel or like I’ll run into another issue where I just can’t. Pretty much what I’m trying to do is try to relate all the quantities to like one specific variable.”*

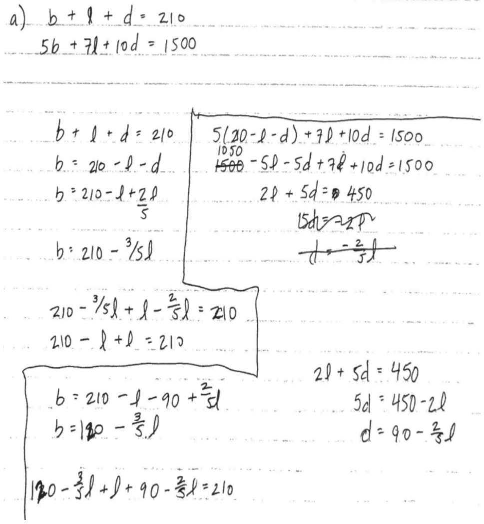

Figure 2. Student R’s algebra.

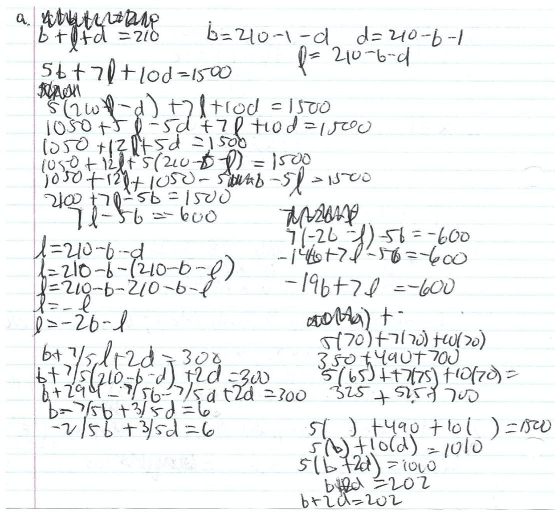

Figure 3. Student L’s algebra.

After both participants used specific solutions they had found to the first equation to see if they work for the second equation, the interviewer prompted Student R to try using an equation she found in conjunction with her previously identified solutions to the first equation. Student R had found dinner in terms of lunch, d = 90 – 2/5 l, shown at the bottom right of Figure 2.

I: Okay, so talk me through. How did you put those two ideas together?

R: [Shows work] Okay, so I have the d = 90 – 2/5l and I plugged in 70 and I got d = 62. So then I plugged that into the price equation with my 70 times 7 and I got 5b + 1110 = 1500. And then I got b = 78, which is how many breakfasts there are. So then I added 78 + 62+70 and that equals 210.

I: And then did it satisfy the 1500? Or I guess you plugged into the 1500 one, didn’t you?

R: Right.

I: Okay, so you found one that works.

R: Yes. The ordered triple is (78, 70, 62). Which is a very random number. So I’m like how are we supposed to find more. So does it work like that, well obviously every number doesn’t work. So do you have to like get lucky with the number you choose to plug in?

I: What do you mean every number doesn’t work?

R: Because we know that some, like (70, 70, 70) doesn’t make both equations true but then when lunch is 70, there is a solution. So like, if I just like plugged in 75 for lunches like, how are we supposed to know which number works or is it just guess and check?

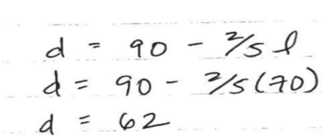

Figure 4. Student R finds d in terms of l.

Student R used information from her solution to the first equation, (70, 70, 70), that she found earlier in the interview to find a solution to the SLE. By plugging 70 in for lunch into an equation for d in terms of l (Figure 4) and using the second equation to find b, she found that (78, 70, 62) was a solution for both equations. Student R had also previously stated that *"because* [the equation] *has to equal exactly 1500, and lunch is $7, we have to make sure that whatever number, however many lunches we have, is a multiple of 5."* She planned to plug 75 in for l in her next iteration because she knew all values of l in the solutions needed to be a multiple of 5. Through this process, Student R was able to find many solutions to the SLE but said she did not *"know why they worked."* Here, both participants were heading toward the notion of parameter and using that parameter to find solutions. Student R wondered why plugging in values for one variable led to finding multiple solutions to the SLE.

"They would intersect because we found… numbers that will satisfy both equations"

On day two, participants were asked to think about the graph of each equation of the SLE, especially considering each line corresponding to skipping a certain meal throughout the semester. In thinking about how to graph a 'no dinners' line, both participants drew on their prior knowledge of graphing a line in the xy-plane:

R: I first started listing out some of the triples with d equaling 0 but then I was like wait, I think there is an easier way to do this… b+l is 210 … and then I just graphed that which was a line with the y and x intercept both being 210 and connected them.

L: I kind of did the same thing where I just plugged in 0 in the equation and I got b+l=210 but then from there I wasn't really sure what to do. Obviously, b or l can be any number between 0 and 210 including 0 and 210. ... But ... I'm not really sure how to even display that geometrically either.

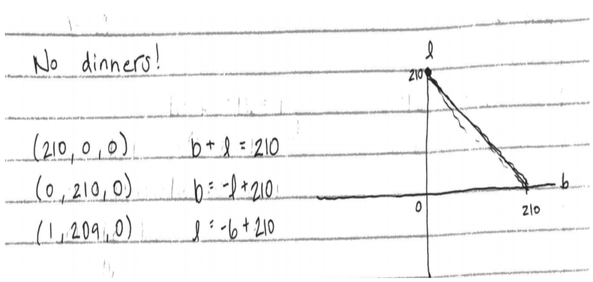

Figure 5. Student R's work in the plane.

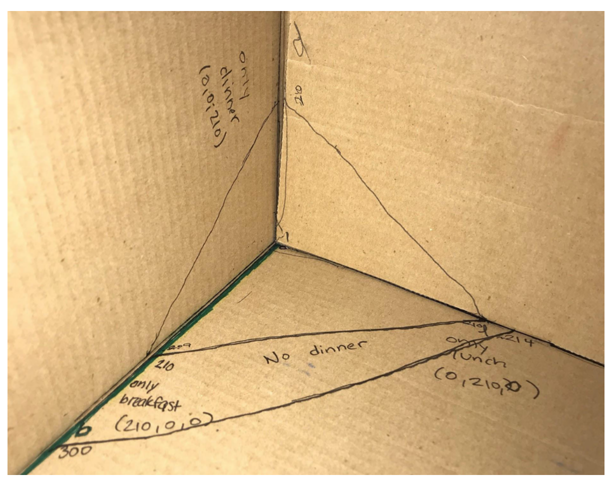

Figure 6. Student R's work on the box.

Student R showed L her graph (see Figure 5) to which he replied, *"Ok, so just a line."* To this point neither participant had tried to illustrate the "no dinners" meal plans or any other solutions in the box. After some help from the interviewer in setting up a coordinate system in the corner of a box, the participants started using their knowledge of lines in 2-dimensions, to create the "no dinners," "no lunches," and "no breakfasts" lines in the box (see Figure 6). The participants then wondered what all the solutions to the b+l+d=210 equation would be. They considered a pyramid or a triangle and whether it would be hollow. L suggested that *"the d would get higher, but the b and the l would get closer in"* and R recognized that if we considered all real numbers (not just integers) then *"it would be flat and all the points would be connected."*

The interviewer next prompted the participants to consider the second equation starting with the "no dinner" line for that equation. Student R added this to her box (see Figure 6) and noted that the triangle is no longer equilateral. The interviewer asked, *"What do you think it means that the lines don't intersect?"* Student R initially noted *"the first equation is inside the graph of the second equation. So, in that way they do intersect, or they overlap."* When the interviewer asked for clarification, Student L stated a different perspective, then analogized to equations of lines.

L: I would think that eventually they would intersect because we found that there are numbers that will satisfy both equations. [Student R: Hmm. That's true.]

I: Why would that tell you that they would meet?

L: Because if there's two lines that intersect at a certain point, then that point is a solution … then that x value that they hit at will be able to be plugged in and you get the same y for both of them.

The participants made substantial progress in building their geometric understanding of 3D solution sets using the box, but still wondered about the exact shape of the intersection of these two equations. That question would not be answered until Day 3.

Discussion

We focus our discussion on the potential of our task sequence to support participants' reinvention of infinite solution sets. Our task setting functioned as an accessible entry point for participants to anchor their algebraic and graphical approaches. Students' prior knowledge of SLE acted as productive beginnings of students' reasoning toward our goal of guided reinvention of infinite solution sets. Asking participants to count possible solutions pressed them to organize their reasoning and notation, identifying the meaning of the *order* in listing solutions as ordered triples. Participants leveraged their prior knowledge about systems, particularly substitution, when the second equation was introduced. Students began by looking for a single unknown value. When participants linked solutions they had identified to the first equation to their algebraic manipulation working with the pair of equations, Student R identified a type of parameterized solution. Finding values for b and d in terms of l seemed to broaden Student R's prior algebraic reasoning to allow her to generate solutions more easily by varying values of l. Initially, the parameterized solution was not as intuitive for students in understanding *why* it was generating solutions, but this was alleviated through a discussion. Finally, using a box and no-dinner lines, participants extended their understanding of lines in a plane to a three-dimensional context. This linked their prior conceptions of solutions as numbers that satisfy the given equations to points of intersection. They also algebraically reasoned with points of intersection as x-values that will get the same y-values in both equations. Here, we found that students' conceptions of solution and what they wondered about largely related to the context of the task and drew from students' prior knowledge. The task allowed students to extend their reasoning about solutions to less familiar situations (i.e., from a single solution to a large solution set), leading to their reinvention of infinite solution sets.

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