OPERATIONALIZING AUTHENTIC AND DISCIPLINARY ACTIVITY FOR THE UNDERGRADUATE CONTEXT

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Many educators and researchers advocate for student engagement in disciplinary activity. This is especially the case in advanced undergraduate courses taken by mathematics majors. In our respective design-based research projects, we found a need to better operationalize the activity of mathematicians in order to both plan for and document student engagement in disciplinary activity. In this report, we share our literature-based efforts to identify the tools and objects used by pure mathematicians in their work. We share the overarching framework we developed, Authentic Mathematical Proof Activities (AMPA), and illustrate the ways we have used this framework to analyze teacher-student activity using an activity theory lens. We conclude with reflections on how tensions between authenticity-to-the-discipline and authenticity-to-the-students shape the teacher-student activity system.

Keywords: Undergraduate Education, Advanced Mathematical Thinking, Design Experiments

What does it mean to document participatory learning in a proof-based classroom? Analysis of classroom activity at this level often focuses on cognitive analogs (such as documenting taken-as-if-shared practices) and analyzing argumentation through lenses such as Toulmin's argumentation scheme (e.g., Rasmussen & Stephan, 2008). While such analyses can provide important insights into student activity, we have found them insufficient for analyzing student activity in relation to our *authentic mathematical activity* design principles. That is, such analyses enable documentation of students' progressing related to content and arriving at mathematical argumentation goals but may oversimplify the nuances involved in the disciplinary mathematical activity we hope to engender.

In mathematics education, the term authentic mathematical activity or practice is often used in conjunction with two simultaneous, but sometimes competing goals: (1) Staying authentic to the disciplinary activity of mathematics and mathematicians; (2) Staying authentic to student communication, activity, and thinking. Lampert (1992) noted that authenticity needs to go back and forth between "being authentic (that is, meaningful and important) to the immediate participants and being authentic in its reflection of a wider mathematical culture" (p. 310). Herbst (2002) has referred to the tension between students having the opportunity to engage in authentic activity and need for proofs to progress in normative ways as a *double bind* when teaching proof. Dawkins et al. (2019) has elaborated on this in the undergraduate proof setting in which such a bind was felt between "supporting success for all students and authentic mathematical activity" (p. 331). As design-based researchers, we have observed a similar tension in our work, resulting in a need to better operationalize authentic activity at this level in order to plan for and then analyze such activity. In this report, we share our efforts using Activity Theory (Engeström, 2000) to better articulate the authentic activities from the discipline and how such activities may or may not be authentically observed in student activity.

Theoretical and Analytic Framing

Activity occurs within larger systems informed by cultural history and norms. Both research mathematicians and students operate within *activity systems* (Engeström, 2000) which account for individuals' goal-driven actions and the way a community works together when they share a common object. These systems consist of: the acting subject, objects (where the action is focused; the motive will be embedded within the object), tools (the means by which the subject acts in relation to the object), the community, the norms and rules of the community, and the division of labor between members of this community as they work towards a goal. We focus heavily on *tools* as the culturally-situated ways that a subject can transform an object toward a desired outcome. Further, we focus on *objective* as a means to capture the compound notion of both a focal object and the embedded motive consistent with Kaptelinin et al. (1995) and Engeström's (2000) treatment of objects.

Researchers in science education have pointed to the key role of tools and usage of tools towards disciplinary objectives in both engendering and analyzing students' disciplinary activity (Nolen et al., 2020). Classroom activity systems may differ substantially from mathematician activity systems in terms of the community, norms, and division of labor; however, tools and objectives can theoretically exist across systems. Instructors, members of both communities, often play a boundary crossing (Akkerman & Bakker, 2011) role connecting between the disparate settings. While work in other contexts has focused primarily on material tools, we argue that conceptual and procedural tools play a more substantial role in the activity of pure mathematicians due to the abstract nature of the discipline. This leads to the natural question:

- What tools towards what objectives do mathematicians use in their discipline that have the potential to be used by undergraduate students in proof-based contexts?

Authenticity to the Discipline: Tools and Objectives

In order to answer this question, we conducted a thorough literature review of both mathematics education research journals and journals that publish mathematician research activity. We created categories of tools and objectives found in student activity from research projects that shared our basic assumptions around desiring student engagement in authentic disciplinary activity (such as inquiry-oriented and anthropological theory of didactic studies). In alignment with concerns voiced in a recent issue of *ZDM* (Hanna & Larvor, 2020; Weber et al., 2020), we verified that such tools and objectives have been documented in empirical studies of mathematicians. We worked reflexively from the two literature bases to arrive at a three-dimensional framework focused on tools and objectives decomposed into motives and objects. Motives include *understanding*, *testing*, and *constructing* (cf. Selden & Selden, 2017) which exist in relation to mathematical objects: *proofs*, *concepts* (including definitions), and (propositional) *statements* (cf. Dawkins, 2015). For example, the mathematical activity of conjecturing would link to the objective: constructing a statement. In terms of tools, we identified nine categories described in Table 1.

Table 1: Mediating Tools in Mathematician Activity

Tool	Description
Analyzing/ Refining	A process of analyzing and/or refining a proof, statement, or definition via attention to the strength and consequence of assumptions.
Formalizing/ Symbolizing	A process of translating informal ideas into symbolic or formal rhetoric form.

Warranting	A process of inferring why a particular claim is true based on the provided premises.
Analogizing/ Transferring	A process of importing a proof, statement, or concept across domains and adapting to the new setting.
Examples	A specific, concrete instantiation of a mathematical statement, concept, or proof representing a class of objects.
Diagrams	A visual representation of a mathematical object (statement, concept, or proof) that captures structural features.
Logic	The rules of logic which allow for precisely quantified statements and deductive arguments.
Structures/ Frameworks	A top-level structure for a proof (or modular section of a proof) which is determined by statements to be proven.
Existent PSC Objects	Proofs, statements, and concepts (definitions) that are accepted as valid in the community.

Detailed case studies of mathematician's work (e.g, Fang & Chapman, 2020, Fernández-León, et al., 2020; Martín-Molina, et al., 2018) reflect several other themes in the ways in which mathematicians use tools towards objectives. Notably, their activity involves coordination of tool use (both in tandem and succession and the use of tools within and outside of the formal-rhetoric system) and transition of objects to tools for continued mathematical activity. Consider an example from Fernández-León et al.'s (2020) study of a mathematician's conjecturing and proving activity. The mathematician (and their colleagues) began with an *existent statement*: "all complete CAT(0) spaces satisfy the (Q4) condition" (p. 7). They then *analyzed/refined* the statement through the process of exploring *examples* to arrive at a new, stronger statement *formalized* as: "every CAT(0) spaces satisfy the (Q4) condition" (p. 12). This statement is *tested* with additional *examples* aided by a *diagram* which rejection of the statement, and a new *refined* statement was *constructed*: "any CAT(0) space with constant curvature satisfies the (Q4) condition" (p. 12). The mathematicians then *tested* this statement with a new *example* producing a proof of the Q4 condition being met (using *logic/framework*) and *analogizing* the proof process in this context. This proof then served as a generic example for *constructing the proof* of the statement (and thus a final *testing* of the statement). Such illustrations help to bolster the claim that authentic mathematical activity is nonlinear, or "zig-zagging" as argued by proponents of authentic activity (e.g., Lampert, 1992) in which a multitude of tools are used to meet objectives and prior objects become tools for continued mathematization.

Authenticity to the Student: Division of Labor, Norms, and Community

A set of mathematician tools and objectives provide a means to document some engagement in disciplinary activity; however, they need to be paired with exploration of other components of the activity system to reflect authenticity to students as well. A traditional undergraduate course tends to contain a division of labor in which students are responsible for taking notes and answering largely closed-form questions, while the instructor presents definitions, theorems, and formal proofs accompanied by verbal informal explanations (Artemeva & Fox, 2011; Paoletti et al., 2018; Weber, 2004). Thurston (1994) and others have questioned the authenticity of the focus on formal products rather than the informal, nonlinear processes involved in the creation of such products. Further, advocates for students to engage in more authentic activity have suggested images of instruction with differing norms and division of labor focusing on student activity driving the mathematical agenda (e.g., Laursen & Rasmussen, 2019).

From an observable standpoint, the division of labor can evidence whether activity is more or less authentic to students. For example, Herbst (2001) illustrated the division of labor in a geometry class in which the community was working collaboratively to produce a proof, but ultimately the teacher introduced the key idea. Thus, this activity may become less authentic to students in order to meet goals of staying authentic to disciplinary aims. Division of labor can provide insight into *agency* and *authority* (e.g., David & Tomaz, 2012; González & DeJarnette, 2012). We operationalize agency as the freedom to make decisions and create tools in the activity system such as who prompts the use of a tool or who creates a tool. Whereas authority reflects how mathematical tools and objects are determined to be valid. The more division of labor reflects students taking on these roles, the more authentic the activity is to them.

Setting and Analysis Process

The driving force behind this theoretical exploration was a need to better analyze the activity of undergraduate students participating in our design-based research studies. Our framework affords analyses of how a tool is introduced by who, and the degree to which student and teacher contributions shape the overall activity (working towards a particular objective). In order to illustrate the potential of our framework to document student activity and provide a means to make claims about authenticity, we share an episode from one of our projects. This episode stems from a larger project aimed at adapting instructional practices from the K-12 literature base (e.g., Stein et al., 2008) to an undergraduate proof-based setting (introduction to abstract algebra) in order to promote student engagement in more authentic proving activity. The project consists of cyclic task development and instructional supports through a series of task-based design experiments starting with small groups of students and then tested in a classroom setting. For the scope of this paper, we share an episode from the second cycle in which we engaged four undergraduate students in a series of task-based interviews. This episode stems from a task (Melhuish, et al., 2020) in which students compare proof approaches and use these proofs as a springboard to refine and test versions of the theorem establishing that the Abelian property is structural (that is, if two groups are isomorphic, and one is commutative, the other is as well).

Our analysis process was as follows: First, we focused on the degree to which the activity was authentic to the discipline. To do so, we coded all tools and associated objectives from the AMPA framework. We then considered how this tool use approximated the complexity of mathematicians (e.g., Are multiple tools being used in conjunction and in succession towards an objective? Do prior objects become tools for new objectives?). To analyze the degree to which the activity was authentic to the student, we analyzed the division of labor in goal-directed actions (e.g., Who introduces the tools? Who uses the tools? Who connects the tools to the objectives?). As a result of analyzing the two facets of authenticity, we further identified shifts in authenticity and provided robust descriptions of these changes to better understand the activity system. The analysis proceeded in several passes – first focusing on the tools and objectives, then focusing on the division of labor and shifts in authenticity. In each stage, at least two researchers analyzed the data with one researcher serving as a first reader and second researcher serving to challenge interpretations. Disagreements were resolved through discussion.

An Illustration of Analyzing Student-Teacher Activity

The following episode describes the student-teacher activity as they used disciplinary tools toward the objective: *construct* a *statement*. This episode occurred after students spent time *understanding* a theorem *statement* (including subdividing the theorem into a set of assumptions and conclusion) and then *understanding* two student *proof* approaches and comparing across them. During this activity the students identified several differences between the proofs including the use of *warrants*. One proof did not appear to use the fact that an isomorphism is 1-1 and onto to warrant any claims whereas the other approach did. The episode began with the teacher researcher prompting students to use the *analyzing/refining* tool in conjunction with the *existing* theorem and proofs to decide, “So, the big question is, did we actually need all of the assumptions in this statement?”

The teacher-researcher positioned the statement and proof as existent objects in which students used their understanding developed in the prior episodes to move forward. The students responded:

> StudentC: Yeah, we definitely, if the final proof is *H* is abelian, for sure *G* is abelian.
> StudentA: Because that's the property that we use, and we also used isomorph- [cross talk]
> StudentD: You would need everything for isomorphic, because you need to know that it is isomorphic.
> StudentC: I mean, couldn't we prove it with homomorphism?
> [continued cross talk]
> StudentC: Our proof worked [proof that did not use onto or 1-1]
> StudentD: All you need to know is that *G* and *H* are homomorphic.

The teacher-researcher asked the students to explicitly construct a revised statement based on the homomorphism-only suggestion. One of the students suggested replacing “isomorphic” with “homomorphic.” The teacher-researcher slightly altered the statement and wrote the conjectured theorem on the board: *Suppose there exists a homomorphism from G to H. Then if G is Abelian, H is Abelian.*

At the beginning of this episode, we can see the students *warranting* by referencing the necessity of abelian in the proof and disagreeing over the necessity of 1-1 and onto warrants. After some cross talk, the students arrived at the antecedents needed to *construct* a new *statement*. The division of labor at this point included the teacher-researcher prompting a tool to use (*analyze/refine*) and the students using this tool in conjunction with their understanding of the statement and proof from their prior activity. Further, the students engaged in debate, reflecting authority in determining what is mathematically true. At the end of the episode, the teacher-researcher rephrased the student suggestion to align with convention reflecting the teacher-researcher using the *formalizing* tool. This was a place in which authenticity to the discipline and authenticity to student objects converged, with the teacher-researcher preserving some aspects of the student object while also acting to bring it closer to the mathematical community standards.

After the new statement was written, the teacher-researcher again prompted for a specific tool use, *testing* the *statement* with *examples,* or producing a counter*example*. This was a consistent role the teacher-researcher assumed. The students began trying to generate *examples* while explaining their strategies such as, “I'm trying to think of groups under specific operations that wouldn't map correctly to other groups. So, maybe a different operation, but I don't know exactly how a homomorphism looks in that sense,” and “... so, since we lost one-to-one and onto, maybe think of some element in *H* that doesn't have a pre-image.” We can see in these contributions that the students were linking the objective (*testing the statement*) with the tool (*examples*) drawing upon *existent* concepts such as the meaning of onto and relationship with the pre-image. The students began suggesting different groups, but also voiced their uncertainty about whether they were creating counterexamples. At this point, the division of labor shifted as the teacher-researcher began scaffolding the tool generation by asking questions about what would need to be true about G, *phi*, and H with students suggesting, "abelian," "homomorphism," and "non-abelian," respectively. While the students confidently answered abelian and homomorphism, their "non-abelian" response conveyed hesitancy, to which the teacher-researcher took on authority to endorse the correctness of non-abelian.

The teacher-researcher next asked for example groups to meet the abelian and non-abelian requirements. The students suggested a number of examples which the teacher-researcher would challenge with questions by asking if the examples met the requirements to be abelian or a group. For example, a student suggested the *example*, "Integers under subtraction, they don't have the abelian property." When the teacher-researcher asked if this was a group, the students disagreed. To resolve this disagreement, the teacher-researcher prompted for the use of an *existent definition*, "So, what properties are you checking right now to decide if it's a group or not?" The students could list the properties of a group, and the teacher-researcher began asking about them one-by-one. The students suggested "0" as the identity to which the teacher-researcher prompted for the definition of identity. Several students made suggestions including, "any other element yields that element," to which another student responded, "So, a minus zero would still be a." The teacher-researcher then asked, "What about zero minus a?" A student shared, "negative a" with two students voicing that was not a group structure. We can see that students used their *definition* for group and the various group properties. We can also see that the division of labor reflected the teacher using the *definition* of group to ask a series of questions to check the properties. As such, the students' agency was more limited. This was another instance in which the teacher-researcher provided scaffolding questions to implicitly challenge a student *tool* that was not conventionally accurate. In terms of authenticity to the students, this episode reflects a shared distribution of labor in which the teacher-researcher never explicitly stated the structure was not a group, but asked questions that implicitly alerted students that more needed to be explored. Through asking these questions, the teacher-researcher changed the *objective* from *testing* the revised *statement* to *testing* the implied *statement: The integers with subtraction is a group.* The students did appear to link the tool and the objective agreeing ultimately that the failure of the identity property (using the *definition*) meant that statement was untrue.

At this point, the teacher-researcher resolicited for a non-abelian group with students making some suggestions and the teacher-researcher taking up the suggestion of the dihedral group *example*. Unlike the first instance, the students engaged in using the *existent definition* to *test* the implicit *statement* that the dihedral group was non-abelian, and came to an agreement using the *example* elements r and s ("$rs \neq sr$"). The spontaneous use of the definition and example reflected a different distribution of labor with students taking on more agency. The students then suggested an abelian group ({-1,1}), but voiced confusion about creating the homomorphism map. At this point, the teacher-researcher interjected to introduce the *diagram* tool and drew a function diagram with the co-domain and domain group. The teacher-researcher further asked the leading question, "If we have a homomorphism, where do we know this identity has to go?" with a student stating to the "identity." Another student asked, "[can] we just pick another one for the negative one to go to?" The teacher-researcher challenged, "do we need to pick a different one?" with a student returning to the assumptions in the statement to say, "it's not 1-1 or onto." Ultimately, the teacher-researcher introduced the map of sending all elements to the identity. While the counterexample was co-constructed by the teacher-researcher and students, there were a number of places in which the authenticity to the students was limited due to the teacher-researcher introducing the tool or providing guiding questions that resulted in students having less degrees of freedom.

The teacher-researcher then asked if this was in fact a counterexample and prompted for the students to explain. A student shared, "Because we have an abelian group that maps to non-abelian group, therefore H does not always have to be abelian." This student's contribution evidenced that they were seeing the *example* as a tool to meet the objective of *testing* the statement, which also provides some indication that limited authenticity to the students' contributions may still be authentic to their activity. This was further evidenced in the next episode when the teacher-researcher had students return to proofs and examples to determine whether 1-1 and/or onto was needed in the statement. The labor shifted from the teacher-researcher providing a specific tool for students to use to allowing students the agency to use whatever they wanted to *test the statement.* For the sake of space, we do not share a detailed analysis of this next portion, but we do note that students used the prior tool for further reasoning. They repurposed the counter*example* and *diagram* (Student C: If you just say one-to-one …) as a means for continued *analysis* and *testing* the onto assumption of the statement, noting that altering the map to make it 1-1 did not fix the issue (Student D: So, yeah it would still be wrong; Student A: Yeah, it contradicts.). The remainder of this task session involved the students and teacher-researcher using both the proofs and examples to arrive at a final statement.

Discussion

We selected the above episode because it provided nuance to authenticity and illustrated a time in our design experiment in which the teacher-researcher shifted the division of labor. We would conjecture that some researchers may read the exchange and feel it was inauthentic because the teacher-researcher engaged in much of the labor, including focusing student objectives and suggesting the type of tools for students to use. Further, at some points, the students themselves did not generate the tool without the teacher-researcher scaffolding. However, it is likely that other researchers would see this episode as illustrative of authentic activity because students engaged with tools of the discipline towards disciplinary objectives and students' contributions were an expected part of the labor throughout. These differing, yet viable, interpretations lend credence to the notion that authenticity is not a binary construct.

Some of this nuance may be attributed to differing types of authenticities and the tensions involved between maintaining authenticity to the discipline and authenticity to the students (Lampert, 1992; Ball, 1993, Dawkins et al., 2019). Weiss et al. (2012) further identified two distinct types of authenticity to the discipline which they deem authentic to practice and authentic to discipline. We conceptualize this divide as a practice and content distinction. For example, this distinction can be seen in Chazan and Ball's (1999) discussion of convention for "testing ideas, for establishing the validity of a proposition, for challenging an assertion" (p. 7) and "definitions, language, concepts, and assumptions" (p. 7), respectively. From our framing, practice is reflected in types of tools and motives, and content is reflected in types of objects. Although unproblematized in the literature, we would argue that student authenticity can similarly be subdivided into content and practice. Practice can be thought of as authentic to students if students have agency to create tools and use them to meet objectives related to knowledge generation. From a content perspective, we can also consider the authenticity in terms of how student-generated tools and objects are positioned in the activity. That is, are student contributions legitimized, and are objects (proofs, statements, and definitions) that students generate used for further mathematical activity?

Because of these differing aspects of authenticity, a student-teacher activity system is likely to be rife with tensions stemming from differences in traditional classroom activity systems and mathematician activity systems. For example, the division of labor in the teacher-student activity system likely necessitates teachers providing tasks and thus setting some of the objectives in the classroom setting. Further, the student-teacher activity system has competing goals related to apprenticing students in mathematician activity while also developing proficiency with conventual mathematical content. Engeström (2001) points to the role of "historically accumulating structural tensions within and between activity systems" (p. 137) as propagating system change – a notion we see reflected in our own work in which the teacher serves a unique role as a member of both the mathematician and undergraduate community, and thus serves as a boundary crosser for the respective activity systems. Teacher scaffolding serves both the role of "help[ing] learners use cultural tools" (Belland, 2016, p. 32) and managing the tensions involved in authentic activity (Williams & Baxter, 1996). We point to these tensions, and the role of the teacher, to emphasize that authenticity is not a binary construct, rather, activity can be authentic in different ways and to different degrees when analyzing an activity system.

The particular episode we selected illustrates a situation in which students were prompted to use certain tools for a certain objective, but had agency in the specifics involved while generating those tools, leading to the construction of a statement that would not be valid in the mathematician community. In terms of content, the student objects stayed centered (although formalized by the teacher-researcher). However, there was variation in how much agency students had in creating the examples to test the statement. In particular, the students seemed at an impasse around generating a counterexample, and so the division of labor shifted to the teacher-researcher. Although, there was a dip in authenticity for the students in trade for authenticity to discipline and practice, we would argue that this ebb in authenticity opened a space for students to engage more authentically in the next portion of the task as students both had increased agency in what tools to use and repurposed the co-constructed counterexample.

We developed the AMPA framework to provide a concrete way to analyze and evidence student engagement in authentic activity. The framework contains operationalizations of mathematician tools and objectives that had the potential for use in the undergraduate setting. As an analytic tool, we complemented the tool and objective analysis with considerations of complexity and division of labor. Student activity more closely approximates disciplinary activity when tools are used in more complex ways and prior objects become tools to meet new disciplinary objectives. This activity is likely to be authentic to student practice if students play a substantial role in the division of labor, have agency to generate tools towards an objective, and authority to evaluate the validity of objects. While our initial attention was to focus on elements of practice, we note that the tension in terms of student-generated and disciplinary content also played a larger role in the activity system. Future research could develop additional analytic tools to further parse the ways that the four types of authenticity (authentic to disciplinary content and practice / authentic to student practice and content) shape school activity systems in both research and classroom settings.

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