Semiotic Conflicts in Students' Collective Proof Comprehension Activity

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Making sense of proofs and statements is a fundamental part of advanced mathematics classes; however, researchers have established that students have limited approaches to reading proofs and may struggle to comprehend them. Converting between representation systems can play an essential role in comprehending formal mathematics including proofs and statements. While navigating representation systems, students are likely to evoke an array of personal meanings that can lead to semiotic conflicts in communication. In this study, we examine what conflicts arose as a group of students collectively worked to comprehend the Fundamental Homomorphism Theorem. Our results show that the students had conflicts related to functions and quotient groups that arose when converting between the formal and other representation systems. Although these conflicts can be problematic, we believe that with a productive discussion and instructor intervention (when necessary) these conflicts can be resolved.

Keywords: proof comprehension, representation systems

Proof comprehension is a fundamental activity in advanced mathematics classrooms where students encounter proofs in lecture and textbooks. However, research points to students' passive consumption of proof presentations as inadequate for students' understanding of them (e.g., Lew, et al., 2016). In Mejia-Ramos and Inglis' (2009) overview of mathematics education literature in proof-based settings, they pointed to a lack of studies related to proof comprehension. Since this review, researchers have developed an assessment framework for proof comprehension (Mejia-Ramos, et al., 2012) and suggested and studied productive strategies for proof comprehension (Samkof & Weber, 2015; Weber & Mejia-Ramos, 2013). These approaches share the commonality of comprehension as multidimensional – focusing both on holistic and local line-by-line understanding.

These studies suggest that proof comprehension involves transforming all or some aspects of a formally represented proof into other representation systems (such as informally summarizing key ideas or illustrating a portion of the proof through an example.) In fact, Mejia-Ramos et al. (2012) point to the use of examples (and diagrams) as a key component of their comprehension framework. Other studies have pointed to the role of examples and diagrams in developing better comprehension of proofs for both students and mathematicians (e.g., Weber & Mejia-Ramos, 2013, Samkoff & Weber, 2015, Weber, 2015) at both the level of a high-level generic proof (Lew, et al., 2020), to make sense of inferences (Weber & Mejia-Ramos, 2013), and even the theorem itself (Samkoff & Weber, 2015).

Because of the significant role of representation systems (and their constituent signs) in proof comprehension, theories informed by semiotics can provide a useful lens to analyze proof comprehension. In this study, we focus on proof comprehension as a collective activity – analyzing a group of four students making sense of the *Fundamental Homomorphism Theorem* (*FHT*) statement and proof. We identify *semiotic conflicts* (Godino, et al., 2007) apparent in their discourse as they both *convert* objects between the formal representation system and other mathematical registers and *treat* objects within registers (Duval, 2006). From analyzing these episodes, we share both challenges that may be common to understanding a syntactically and symbolically dense proof, and ways that students' disparate meanings may be resolved.

Theoretical Orientation and Analytic Framing

In the context of abstract algebra, focal mathematical objects are abstract and general. As noted by Presmeg, et al. (2016), engaging with and communicating about such mathematical objects requires the use of *signs* that "are not the mathematical objects themselves but stand for them in some way" (p. 9). As a result, theories of semiotics, that is, theories related to study of signs, have become a prevalent way to analyze mathematical activity and discourse. In this study, we take a view of semiotics influenced by Duval's (2006) *theory of registers of semiotic representation* and Godino et al.'s (2007) *onto semiotics*—two theories that can serve a complementary role (Pino-Fan, et al., 2017). We focus specifically on Duval's (2006) notion of representation systems (registers). Mathematical meaning is inherently constrained and shaped by the available signs and rules of a representation system (natural language, symbolic, figures/drawings, and diagrams/graphs.) In our context, we add the representational system of proof, which we label *formal*: "generalized symbolic statements which can be combined into permitted configurations via the rules of, for example, predicate calculus, propositional logic and acceptable proof frameworks" (Alcock & Inglis, 2008, p. 114). We then consider the activities of *conversion* and *treatment,* "to substitute one semiotic representation for another, only by changing the semiotic system mobilized; and to substitute two semiotics representations within the same semiotic system" (Pino-Fan, et al., 2017, p. 101), respectively.

We then use *onto semiotic* (Godino, et al., 2007) notions of primary objects – focusing on the subset of objects that are not directly observable (concepts/definitions, propositions and procedures, Font, et al., 2013). Mathematical objects contain dualities that reflect that there is not "one 'same' object with different representations." (Font, et al., p. 7). Rather, meaning and representation of an object are subject to dualities including: the *expression* (the sign) and *content* (the meaning referent) and correspondingly what is *ostensive* (observable) and *non-ostensive* (imagined). Objects are also understood both *personally* (to the individual) and *institutionally* (which can refer to the local community of students or a larger community). An object can also be treated as *unitary* (a single thing) or *systematic* (something to be decomposed into a system). Finally, an object can be *intensive* (general) or *extensive* (a specific example). While these dualities are presented as dichotomies, we argue for a spectrum where something like a *generic example* (Font & Contreras, 2008) can serve as a bridge. These dichotomies can lead to *semiotic conflicts,* "disparity or difference of interpretation between the meanings ascribed to an expression by two subjects, being either persons or institutions" (Godino, et al., 2007, p. 133) as students discuss mathematical objects in different representational systems. Such conflicts, while observable by a researcher, are often unnoticed by students. These conflicts can serve to limit student communication or progression but can also serve as a space for students to negotiate new meaning and make mathematical progress.

Background on the Fundamental Homomorphism Theorem

The Fundamental Homomorphism Theorem (FHT) describes an essential relationship between complex group theory ideas including quotient groups, homomorphisms, kernels, and isomorphisms. Each of these topics individually can be quite challenging for students to develop robust and normative meanings (see Dubinsky, et al., 1994; Leron, et al., 1995; Melhuish, 2019; Melhuish, et al., 2020). As Asiala, et al. (1997) documented, the algorithm for creating cosets often supersedes other conceptions. Further, to understand the FHT, students need to conceive of cosets as both sets and objects themselves that can be elements within a group structure. Indeed, Nardi's (2000) study of a tutor and student engaging with the FHT points to the struggle associated with decontextualized nature of learning quotient groups. This study also suggests a fundamental challenge in "co-ordination and understanding of the link between and, as well as the clarification about the definition of" (p. 184) the relevant functions: isomorphisms and homomorphisms. Research related to isomorphism and homomorphism point to reliance on algorithmic processes and the need for a sophisticated understanding of function (Leron, et. al, 1995). Melhuish, et al. (2020) have further documented that students' treatment of homomorphisms is tied to their coordination with the concept of function and that the meaning associated with function can serve as a support or hindrance in productive engagement with homomorphisms, kernels, and the FHT. In sum, the literature points to complexity involved in understanding the FHT and its constituent parts. In this study, we focus on:

(1) What meanings do student evoke from different representation systems as they engage with the FHT and its proof? (2) What semiotic conflicts arise as students engage in comprehension tasks related to the FHT and to what extent are they resolved?

Theorem 2 (The First Isomorphism Theorem). *If $\phi : G \to H$ is a group homomorphism, then*

$$\frac{G}{\ker \phi} \simeq \phi(G).$$

Proof. First, we note that the kernel, $K = \ker \phi$ is normal in G.

Define $\beta : \frac{G}{K} \to \phi(G)$ by $\beta(gK) = \phi(g)$. We first show that β is a well-defined map. If $g_1K = g_2K$, then for some $k \in K, g_1k = g_2$; consequently,

Figure 1. The theorem and beginning lines of the proof.

Methods

As part of a larger project, we have conducted a series of task-based interviews centered on proof in abstract algebra in a large, public university in the United States. In this paper, we focus on a group of four undergraduate mathematics majors who had recently completed an abstract algebra course. Students were first provided the FHT statement and asked to dissect the important terms within the statement. Next, they were given example groups and homomorphisms and asked to identify parts of the example that correlate with the statement of the FHT. Lastly, the students were given the proof of the FHT and prompted to make sense of the theorem globally and locally in conjunction with their prior statement dissection and examples.

The focal transcript (along with video and student work) was analyzed through several passes. First, the three members of the research team independently read through the data and created a set of memos broadly identifying student activity and representational systems at play. From this initial pass, the team arrived at the set of relevant representation systems and initial objects. After this holistic pass, the lead researcher chunked the transcript into a series of 16 episodes. For each episode, the lead researcher created a narrative, first identifying mathematical objects, representation systems, and student meanings ascribed to objects. This narrative was then expanded to describe conversion, treatments, relevant object dualities, semiotic conflicts, and resolution paths for these conflicts. For each episode, other team members read each narrative and compared it to the existing transcript with the aim of challenging initial interpretations. Disagreements were resolved through discussion. Here, we provide a brief overview of pervasive conflicts, and share two illustrative episodes.

Results

An essential part of understanding in the FHT and its proof is understanding the meaning of function and well-defined. Students' personal meanings for function appeared consistent when referencing a *procedure* from the graphical system (vertical line test), but their meanings were in conflict in the diagram representation system (with a subset of students producing 1-1 rather than well-defined), and in verbal descriptions where two of the four students focused on everywhere-defined ("mapped each element to something"). Throughout the tasks, students would bring different personal meanings for the properties and often reach resolution (such as coming to an agreement over the contradicting diagrams), only for the conflict to reemerge in a different representation system (such as identifying well-defined within the proof).

We also documented consistent conflicts related to quotient groups. Of note, when describing what a quotient group is, some students attended to a "list of cosets" whereas others attended to a group structure. A second conflict emerged in relation to the meaning of "factor group" with students in disagreement as to whether the normal subgroup would be "factored out" evoking a meaning of factor related to removal rather than a meaning consistent with factoring as partitioning. A third conflict emerged in relation to the meaning of the elements in the quotient group where some students treated the elements as sets (H as the identity in G/H) whereas others treated the elements as singleton ("e" is the identity G/H). Finally, a related conflict emerged in relation to the representative symbolic notation for cosets (aH) and symbolic set notation ($\{a_1, a_2, a_3, \ldots\}$) where the representative notation was personally meaningful to some students (and connected with coset formation) but did not appear to have meaning beyond the symbols for other students.

Converting the FHT between the formal and diagrammatic representation system

The two pairs of students were each given a homomorphism example (ϕ: $\mathbb{Z}$ to $\mathbb{Z}_4$ where $\phi(x)=x \bmod 4$ and ϕ: $\mathbb{Z}_{12}$ to $\mathbb{Z}_3$ where $\phi(x)=x \bmod 3$, respectively) and tasked with creating a function diagram and identifying where the FHT can be seen in the diagram (a task to convert from the formal representation system to the diagrammatic system). Moreover, this task involved the duality between intensive (general statement) and extensive (diagram example). Both pairs converted between the formal statement and informal symbolic by identifying G and H, respectively. However, at this point, several of the students voiced uncertainty about the quotient group connection and where the isomorphism can be seen in the diagram.

In order to address this issue, the teacher-researcher asked the students to put their diagrams on the board. Student C explains that they began by listing "dozens of elements" but simplified to "just the four unique cases" (a treatment). Student A and D agreed that they had a similar process. Student C noted, "at least I'm starting to really see the coset groups forming individual elements." We interpret this comment as reflecting some resolution around the elements of quotient group conflict. Although, we acknowledge there is still some language inconsistency with the cosets being referenced as groups.

The teacher-researcher then had the students concretely connect the parts of the formal statement with the diagrammatic representation on the board - focusing on the finite example.

The students converted between the systems, and navigated between the exemplar-type duality to identify the kernel, *G*, and *H*. They are then prompted, “where is our isomorphism? Where is our quotient group?” There are two viable ways to create the cosets of the quotient group. The cosets can be built by identifying the kernel and using the coset formation procedure or through creating pre-image subsets. In this case, one pair of students appeared to have used the latter version, but voiced uncertainty about whether these created the quotient groups, “We did that right? […] We don’t know. We think we could have possibly started” identifying the kernel as the identity (further evidencing resolution around the element-set coset conflict.) In contrast, Student C used the coset procedure sharing, “So, I just started with what we have here, kernel of ϕ […] from the left I added the next operation of our H.”

The teacher-researcher had previously requested representative notation by asking for a “name” for the cosets (a more standard notation in the formal representation system). At this point, Student A recognized they have the same cosets, but states “we just didn’t know how to name them.” However, when the teacher-research prompted about the + operation, Student A and D identify the operation from $\mathbb{Z}_3$ (the operation in the image) rather than from $\mathbb{Z}_{12}$ (the operation in the coset) further evidencing that there is a conflict in the personal meanings attribute to the coset formation process. After some discussion, the teacher-researcher advocated that the representative elements are from $\mathbb{Z}_{12}$, but it is unclear whether this conflict was resolved for all the students. However, we do have evidence that the students were all seeing the cosets (regardless of formation process) as the quotient group elements as they are easily able to answer how many elements are in the quotient group itself (3).

At this point, the teacher-researcher focuses on where the isomorphism is. Student B notices “the order of the image of phi is the same as your quotient group.” With this comment, the teacher-researcher prompts the students if the cosets are clear from the original function diagram (Figure 2a). To which Student B and D indicate that “If you got all the purple lines” (B) and “They would regroup it like that” (D). With the teacher-researcher as the scribe, the students produced the following function diagram, a treatment in the diagrammatic representation system (Figure 2b - without the red markings).

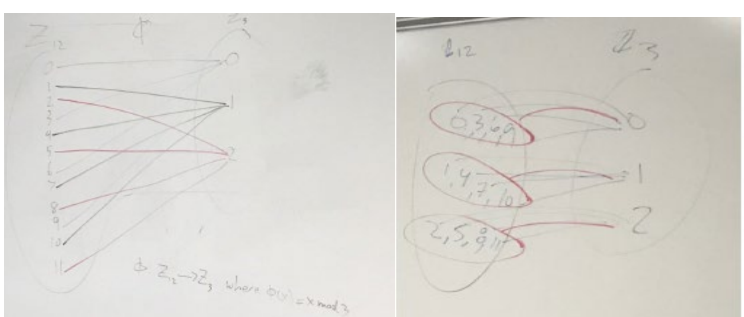

Figure 2. (a) Function diagram of the homomorphism phi from ***Z****12 to* ***Z****3. (b) Reorganized function diagram*

After the creation of this diagram, Student B makes the comment that “you see it's one-to-one and onto” evidencing conversion between the formal meaning of isomorphism and the function diagram instantiation. However, Student B took back their suggestion when asked for further explain indicating a potential conflict between what they see (ostensive) and corresponding non-ostensive mathematical concepts. Student C continues, “we have that grouping, if we change the mapping...” suggesting creating “$\mathbb{Z}_{12}$ so you divide by zero three six nine slash kernel” with Student D suggesting, “bracketed off and that becomes the one element” with the teacher-researcher using their suggestions to add the red circles/lines to the diagram. We see evidence of resolution around both the quotient groups and isomorphism through the use of the diagram instantiation. Student B asked, “so now the red is your factor group?” and student D agreed. Student B also commented, “and now it’s 1-1 and onto” with Student D elaborating, “and there’s your isomorphism.” At this point, the students seemed to have successfully converted from the formal representation to a diagram.

Converting Between a Line in the Proof and a Symbolic Example

In this episode, the students were tasked with making sense of the line if $g_1K = g_2K$, then for some k in K, $g_1k= g_2$ in the FHT proof. The teacher-researcher prompted the students to convert this line to the symbolic/ diagrammatically illustrated example (Figure 2). Recognizing that these objects are cosets, Student B suggested “kernel of phi [as] one of em' and then four plus kernel of phi is another one?” Here we see the student using distinct cosets rather than distinct elements from the same coset. This conflict is recognized and resolved by Student D who suggested changing the second coset, “It’d be 1+ *ker* ϕ.” to which Student B agreed, “Oh, I'm talking about the wrong one. Okay, yeah.” The students continued converting from the formal representation to the informal symbolic example with Student C suggesting another coset equivalence, “so kernel phi would be equal to three plus kernel phi.”

The teacher-researcher returned focus to the formal representation asking, “But how do we know that $g_1k= g_2$ for some k?” Student B made two suggestions: “because it’s normal” and “I thought it was because it was not nonempty, so you have identity in there.” As only one student is suggesting, the teacher-researcher prompted for students to explain what “$g_1K = g_2K$” means. Student B focused on the entire set: “two cosets that are the same” and while Student D attends to the elements, “These sets repeat down in a row?” reflecting uncertainty in their personal meaning. The teacher-researcher then introduced the use of a symbolic example prompting students to create g_1K and g_2K. The teacher-researcher and students constructed the expanded versions (Figure 3) converting between the condensed version of cosets typical of formal proof representation into an expanded symbolic representation. While it was not clear that the meaning shared for the equivalence was in conflict for Student B and D originally, the conflict is explicated in this new representation. Student B suggested the cosets being the same means “all the elements.. They're matching” further explaining “g_2, g_1 are the same g_2k_1, g_1k_1.” Student D responds, “Not necessarily those... all the elements in g_2K there's an element that matches them somewhere in g_1K.” Student B’s personal meaning focused on the element level, while Student D’s personal meaning was at the set level (aligned with the normative meaning for set equality).

Figure 3a and 3b. The cosets of K with representative g1 and g2.

To see if other students understand Student D’s reasoning, the teacher-researcher asked someone to revoice what Student D said. In response, Student A stated, “that's like continuing? So somewhere along the line there would be g_2k in the g_1k function” Here we see that Student A shared their meaning with Student D. However, Student C returned to the formal proof line and asked, “But there's only like g_2 so what does just g_2 look like?” This question may reflect a desire to know the exact match. Student D then explains the idea that “g_2 has to be somewhere in here” because the cosets are equivalent and introduces a “k_m” such that “$g_1k_m = g_2$.” Student C responded that they are finding a “specific one” and Student A suggested “arbitrary.” In the expanded cosets, specific k_i's are listed and Student D introduces k_m. In the formal proof, the line has just “k.” Student C’s personal meaning attributed k_m as more specific than k while Student D and A are seeing k_m as representing an equally arbitrary element.

After some continued conversation, we see an additional layer of complexity and some resolution. Student B commented, “I've been thinking that g_2 is some like a group.” With Student C agreeing, “That's what I was thinking, too.” This may evidence a conflict around the meaning for representation notation: when an element is the referent and when a set is the referent. Student D commented that “g_2 is like another element” with Student B explaining they went from “what is that?” and identified “the whole revelation I had was when you put k_m.” At this point, the conflict seemed to be resolved for the students, although this claim relies on weak evidence for Student C as they had stopped voicing questions.

Discussion

In this paper, a variety of conflicts arose as students conveyed meaning for different objects as they converted and treated them across and within representation systems. The students in the task had difficulty with functions and converting functions from a formal representation to a diagrammatic representation. They had conflicts with quotient groups such as, notational representation, set vs groups, and elements as cosets vs elements as singletons. We saw that students had conflicts in converting a formal representation of the isomorphism in the FHT theorem to a diagrammatic representation of an isomorphism in a given example. We also saw students having conflicts when converting between the line of the formal proof and symbolic concrete and generic representations.

From a research and teaching perspective, we found the identification and, in some cases, resolution of these conflicts to be a useful lens. The conflicting means students evoke may account for some of the disconnect between instructor intentions and students’ comprehension. We found students brought different meanings for functions and quotient groups, and different notation and representation systems often shaped the meanings involved. Prompts to convert between representation systems appeared productive to terms of explicating conflicting meaning and allowing for discussion space to resolve conflicts. Further, instructors may find it to useful to identify the representation systems in which students’ personal meanings may diverge and strengthen the connections between objects symbolized in different systems.

Acknowledgments

This material is based upon work supported by the National Science Foundation under grant (DUE-1836559). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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