Adapting K-12 Teaching Routines to the Advanced Mathematics Classroom

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In recent years, there has a been a push for undergraduate mathematics classrooms to move away from purely lecture to a model where students are more actively engaged in their own learning. Such a transition is hardly a trivial task and requires robust instructional supports. Our recent work endeavors to adapt research-based supports from the K-12 level to the undergraduate abstract algebra classroom. In this report, we share preliminary results from a design-based research project directly aimed at adapting best practices to this new setting. We share several illustrations of how particular teaching routines (Melhuish & Thanheiser, 2017; Teachers Development Group, 2013) can productively unfold in a proof-based setting.

Keywords: instructional practices, proof

The teaching of undergraduate mathematics is a largely understudied practice (e.g., Speer, Smith, & Horvath, 2010). This is even more so the case for advanced mathematics classrooms with proof as their focal practice (see Rasmussen & Wawro, 2017.) From a practical standpoint, we have little knowledge of what may be productive instructional supports for this level classroom. This is not the case in the K-12 setting where there is a set of research-based instructional strategies (colloquially referred to as *best practices*) that include focusing on student thinking, facilitating student discourse, and creating a classroom where students engage authentically with mathematics (e.g., Jacobs & Spangler, 2017). Many instructional supports have been developed and studied in this setting in order to provide instructors with the tools to keep a classroom both student-centered and mathematically productive (e.g., Melhuish & Thanheiser, 2017; Stein, Engle, Smith, & Hughes, 2008).

As we move towards undergraduate classrooms being more student-centered (as recommended by Saxe & Braddy, 2015), we are in a position to develop similar instructional supports for the undergraduate level. In this preliminary report, we share some initial analysis of a design-based research experiment aimed at adapting K-12 supports to the needs of a proof-based abstract algebra classroom.

Background and Framing

If we want students to engage in authentic mathematical activity during classroom lessons, the first step is providing opportunities for them to do so (Cai, et al., 2017). As a mathematics education field, we have a general consensus that we value student-centered classrooms and that noticing student thinking and facilitating meaningful discussion are key instructional practices that support such classrooms (Jacobs & Spangler, 2017). At the K-12 level, there are a number of instructional routines that can serve to support students in engaging in authentic mathematics while continuing to move the mathematics forward. These include teaching routines such as selecting and sequencing student ideas (Stein, et al., 2008) comparing across student strategies (e.g., Durkin, Star, & Rittle-Johnson, 2017), leveraging visual representations (e.g., Arcavi, 2003), and providing students the time and space to make sense of tasks (e.g., Kelemanick, Lucenta, & Creighton, 2016). Through these and other best practices, classrooms can become a place where students do not just engage with open tasks, but their thinking can move the mathematical agenda forward.

Pedagogy at the Undergraduate Level

At the undergraduate level, we know substantially less about how this type of instruction may play out. We have some evidence that active learning (Freeman, et al., 2014) and inquiry-based learning (Laursen, et al., 2014) may support student learning gains, confidence, and more equitable outcomes. However, such analysis has been large-scale, and gives little insight into the nature of these classrooms. In fact, we can find other narratives countering that more student-centered approaches are positively aligned with student outcomes (Sonnert, Sadler, Sadler, & Bressoud, 2015) and are more equitable (Johnson, et al., 2019). Yet, another research-line, research on inquiry-oriented curricula, provides a series of results indicating with careful pedagogy, this type of classroom can be associated with student learning (e.g., Rasmussen & Kwon, 2007). Furthermore, beyond course outcomes, if we value students engaging in mathematical activity, it is propitious to develop classroom settings that are student-focused. This literature raises questions about not whether student-centered classrooms are better, but rather under what conditions they work best.

Answering that question can serve a substantial need for instructors wishing to move away from a traditional lecture model. Johnson, Keller, and Fukawa-Connelly (2018) recently surveyed instructors of abstract algebra classes to discover the nature of their instruction and why they choose to orchestrate their classrooms in particular ways. One important result is that many instructors do not think lecture is better for students; however, they identified a lack of resources and support in implementing ambitious pedagogy.

The Math Habits Framework

Mathematically Productive Teaching Routines	**Catalytic Teaching Habits**	
Generates student engagement in mathematically productive discourse by: (**Structure**) Structuring mathematically worthwhile talk (**Sequence**) Working with selected & sequenced student math ideas (**Public Record**) Working with public records of student mathematical thinking (**Confer**) Conferring to understand students' mathematical thinking & reasoning (**Represent)** Eliciting reasoning about visual representations (**Meaning**) Making meaning of tasks, contexts, and/or language	*Uses questions/actions to elicit students':* • Private reasoning time • Perceptions of the meanings of specific math concepts or properties • Mathematical noticings, wonderings, or conjectures • Metacognition or reflection • Mathematical reasoning on a problem or argument	*Orients students to ideas by:* • Prompting students to analyze contradictions or misconceptions • Prompting students to analyze a strategy or argument • Prompting students to compare or connect across students' reasoning • Exposing mathematical content in a non-verbal mode • Revoicing or recapping student ideas

The underlying hypothesis of our work is that best instructional practices at the K-12 level can be adapted to the abstract algebra setting. As such, we developed a series of tasks and instructional supports focused on key aspects of mathematically productive classrooms. In this study, we are focusing on the Math Habits framework (Melhuish & Thanheiser, 2017), a research-informed framework that captures teaching routines (extended routines that can support student engagement in authentic mathematical activity), catalytic teaching habits (the individual moves that teachers make to engage students in mathematical activity), and the resulting student activity. This framework operationalizes the triangle that occurs between math content, teachers, and student interactions (Cohen, Raudenbush, & Ball, 2003). For the scope of this report, we focus on the teaching portion to illustrate how particular routines may play out in the abstract algebra setting.

Methods

The data presented in this report stems from a design-based research project focused on developing instructional supports for the abstract algebra classroom. We developed three central tasks related to fundamental proof activities: validating, constructing, and comprehending. In this report we focus on the validating task (which also includes proof analysis to modify and test proofs and statements, c.f. Lakatos, 2015). The task centers on the proof that the Abelian property of a group is structural (preserved by isomorphism). We conducted two iterations with undergraduates who had recently completed introductory abstract algebra (three and four undergraduates, respectively). In the spirit of design-based research, we developed a set of conjectures of how students would respond to various instructional moves, then refined and altered tasks based on a close analysis of transcript and video data between implementations. We share data from the second implementation that incorporates substantial analysis-informed changes from the first. This lesson was roughly two hours in duration.

The focal lesson we share in this preliminary report was videoed and transcribed. This data was then analyzed using the Math Habits framework to identify the nature of the instructional moves and resulting student activity. Two researchers, who were not involved in the planning or implementing the lesson, coded the data independently and reconciled differences through discussion.

Sample Results

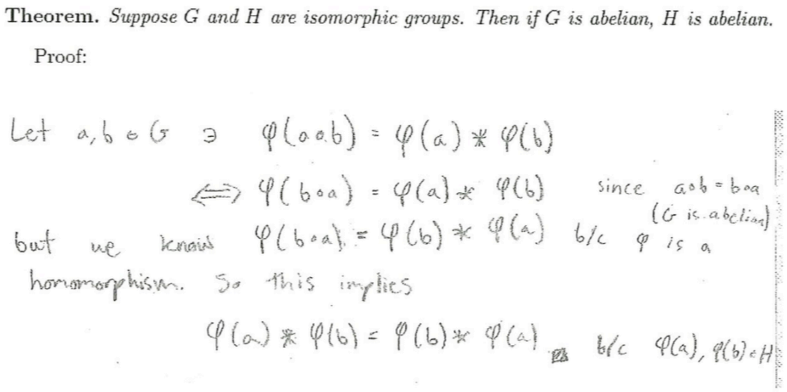
During the course of our focal lesson, 55% of the time was spent with the instructor(s) engaged in at least one teaching routine with all six appearing. Additionally, 62 catalytic teaching habits occurred covering all ten habits at some point in the lesson.

Teaching Routine Vignette 1: Making meaning of tasks, contexts, and/or language.

The first teaching routine that occurred was focused on making meaning of the theorem statement. Students were prompted to read the theorem and "sketch out how you might go about proving it." This incorporated the CTH: private reasoning time, as the students were intentionally given time to write out their thoughts before engaging with each other. The instructor-researcher then asked, "So what are the types of things that we think about when we're going to prove something?" (CTH: metacognition/reflection). From this prompt, the students suggested the "givens" and "what we want to prove." The instructor-researcher then created a public record on the board of the students' suggestions (CTH: exposing content in a non-verbal mode) [Record: G and H are isomorphic. G is abelian. G, H groups; What we want to prove: H abelian.]

At this point we unpacked various vocabulary terms asking students to provide their definitions for abelian and isomorphic (CTH: press for perceptions of the meaning of mathematical ideas). This discussion served both to remind the students of the meaning of terms, but also provide a scaffold for the eventual proof by creating a record that included the definition of isomorphism (existence of an alpha that's 1-1, onto, and a homomorphism) and abelian (for every $a,b \in H$, $ab=ba$.)

Teaching Routine Vignette 2: Working with public records of students' thinking.


Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

Proof:

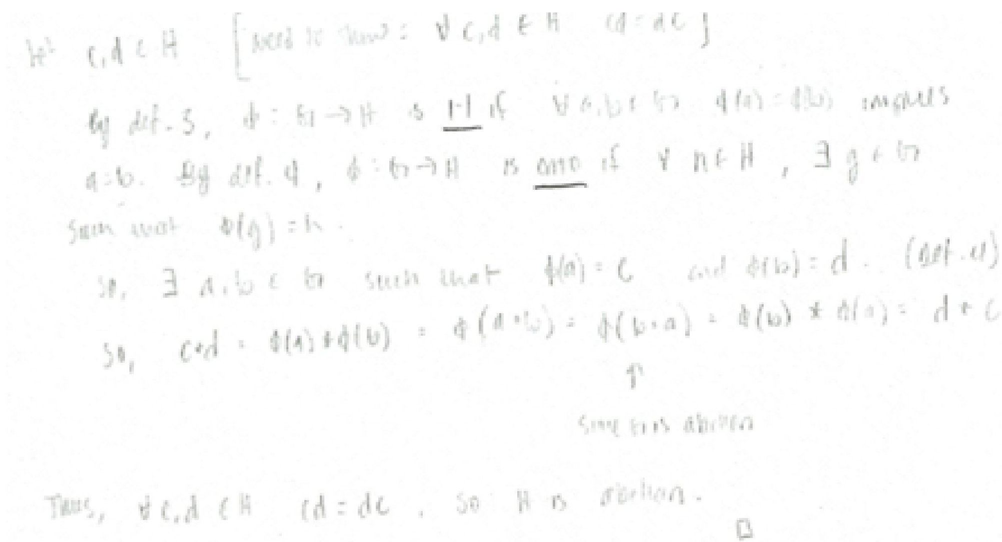
Let $a, b \in G \ni \varphi(a \circ b) = \varphi(a) * \varphi(b)$

$\Longleftrightarrow \varphi(b \circ a) = \varphi(a) * \varphi(b)$ since $a \circ b = b \circ a$ (G is abelian)

but we know $\varphi(b \circ a) = \varphi(b) * \varphi(a)$ b/c φ is a homomorphism. So this implies

$\varphi(a) * \varphi(b) = \varphi(b) * \varphi(a)$ b/c $\varphi(a), \varphi(b) \in H$

Figure 1. An invalid (or incomplete) G-first approach.


Let $c, d \in H$ [need to show: $\forall c, d \in H$ $cd = dc$]

By def. 3, $\phi: G \to H$ is <u>1-1</u> if $\forall a, b \in G$ $\phi(a) = \phi(b)$ implies $a = b$. By def. 4, $\phi: G \to H$ is <u>onto</u> if $\forall h \in H$, $\exists g \in G$ such that $\phi(g) = h$.

So, $\exists a, b \in G$ such that $\phi(a) = c$ and $\phi(b) = d$. (def. 4)

So, $cd = \phi(a) * \phi(b) = \phi(a \circ b) = \phi(b \circ a) = \phi(b) * \phi(a) = d * c$

since G is abelian

Thus, $\forall c, d \in H$ $cd = dc$. So H is abelian. □

Figure 2. A valid H-first approach

The next vignette we share is focused on the teaching routine of working with public records of student thinking. We note the routine is not focused only on students sharing thinking, but also having students engage with each other's mathematical ideas. In this iteration, we provided the students with two common approaches[1] to this proof (one that argued that the images of arbitrary elements of the domain group, *G*, commute, and one that argued that arbitrary elements of the co-domain group, *H*, commute) as shown in Figures 1 and 2. The two sets of partners were provided one of each of the approaches and the following directions as presenter and listener roles:

Be prepared to explain this proof approach to your classmates. This explanation should include a function diagram that connects to the proof approaches. (CTH: non-verbal representation)

What is one thing about this proof approach that makes sense to you? What is something that you have a question about? (CTH: make sense of a strategy or argument)

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[1] See Melhuish, Larsen, & Cook, 2019 for the frequency of these approaches and some of the underlying proof issues.

These prompts positioned the students to make sense of a proof approach, including through using a visual representation, and to engage meaningfully with each other's approaches. This task was particularly productive as the students successfully outlined the proofs including identifying important warrants, asked robust questions (such as identifying when elements are in the domain and codomain), and provided revoicings of each others' ideas.

The next prompts focused on connecting across ideas: "…the next thing that we're going to have you think about... kind of like a series of activities going through... is thinking about what's the same and what's different about these approaches, and you're welcome to chat again with your partner about this" (CTH: compare and connect across strategies). Through this prompt the students identified a number of commonalities including shared warrants (homomorphism property and abelian) and differences (beginning with elements in G rather than H, the use of 1-1 and onto). The instructors again kept a record of the similarities and differences on the board. By comparing and contrasting, the students (who found both arguments compelling) were positioned to notice the important differences across the proofs. Exploring these differences eventually lead to determining which warrants were essential (onto), which were not needed (1-1), and a discussion on how one approach better aligned with the statement via attending to the givens and what we want to prove.

Discussion

The early iterations of this experiment illustrated substantial promise in adapting teaching routines from the K-12 setting to the proof-based setting. First, we provide an existence proof that the types of teaching moves and routines from this setting can be adapted to the formal proof setting. Furthermore, these moves and routines seemed to be productive in the sense that students engaged in authentic mathematical activity (which we defined as proof analysis, construction, and comprehension) as a result of the teaching prompts and tasks. We also found that this iteration was productive in terms of meeting the underlying instructional goals which included: developing an appreciation the role of proof frameworks (Selden & Selden, 1995), deeply exploring the impact of a function being 1-1, onto, and a homomorphism, and arriving at important statement modifications. Comparing student strategies seemed particularly crucial to identifying the important differences between approaches which otherwise may have remained hidden. Furthermore, unpacking the statement in terms of givens and want-to-proves placed focus on the alignment of proof and the statement to be proven. In particular, the students arrived at noting the importance of starting with arbitrary elements from the codomain group, an approach rarely taken by students in introductory abstract algebra classes (Melhuish, et al., 2019).

Currently, this project is in early stages of analysis and implementation. As such, we acknowledge limitations of the generalizability of this work. Currently, we see a great deal of promising in adapting instructional supports from the K-12 level. If we want undergraduate student-centered classrooms to be productive, we need to develop such supports at this level, and then study their impact. Future research will include scaling from small-group lab settings to full classroom implementations to further adapt and refine this work.

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