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Cyclic pairings and derived Poisson structures

Ajay C. Ramadoss and Yining Zhang

Abstract. By a fundamental theorem of D. Quillen, there is a natural duality - an instance of general Koszul duality - between differential graded (DG) Lie algebras and DG cocommutative coalgebras defined over a field k of characteristic 0. A cyclic pairing (i.e., an inner product satisfying a natural cyclicity condition) on the cocommutative coalgebra gives rise to an interesting structure on the universal enveloping algebra $\mathcal{U}\mathfrak{a}$ of the Koszul dual Lie algebra $\mathfrak{a}$ called the derived Poisson bracket. Interesting special cases of the derived Poisson bracket include the Chas-Sullivan bracket on string topology. We study the derived Poisson brackets on universal enveloping algebras $\mathcal{U}\mathfrak{a}$, and their relation to the classical Poisson brackets on the derived moduli spaces $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ of representations of $\mathfrak{a}$ in a finite dimensional reductive Lie algebra $\mathfrak{g}$. More specifically, we show that certain derived character maps of $\mathfrak{a}$ intertwine the derived Poisson bracket with the classical Poisson structure on the representation homology $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ related to $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$.

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1. Introduction

Fix a field k of characteristic 0. In [14], Goldman discovered a symplectic structure on the G-character variety of the fundamental group π of a Riemann surface, where G is a connected Lie group (for example, GL_n). In [15], he further found a Lie bracket on the free k-vector space $k[\hat{\pi}]$ spanned by the conjugacy classes of π. It was shown that the natural trace maps

$$\mathrm{Tr}_n : k[\hat{\pi}] \to \mathcal{O}[\mathrm{Rep}_n(\pi)]^{\mathrm{GL}_n}$$

are Lie algebra homomorphisms, where $\mathrm{Rep}_n(\pi)$ is the affine scheme parametrizing the n-dimensional representations of π. It was later understood that the above structure is a special case of an H_0-Poisson structure on an associative algebra in the sense of [8]: such a structure on an associative algebra A is given by a Lie bracket on $A/[A, A]$ satisfying a certain (noncommuatative) version of the Leibniz rule. If A has an H_0-Poisson structure, there is an induced Poisson structure on the commutative algebra $\mathcal{O}[\mathrm{Rep}_n(A)]^{\mathrm{GL}_n}$ for each n. Further, in this case, the canonical trace map

$$\mathrm{Tr}_n : A/[A, A] \to \mathcal{O}[\mathrm{Rep}_n(A)]^{\mathrm{GL}_n}$$

is a Lie algebra homomorphism for each n. The notion of an H_0-Poisson structure was extended in [1] to arbitrary DG (augmented) associative algebras: a Poisson structure on $R \in \mathtt{DGA}_{k/k}$ is a Lie bracket on $R_\natural := R/(k+[R, R])$ satisfying the noncommutative Leibniz rule referred to above. A *derived* Poisson structure on $A \in \mathtt{DGA}_{k/k}$ is a Poisson structure on some cofibrant resolution of A. In particular, a derived Poisson structure on A gives a graded Lie bracket on the reduced cyclic homology $\overline{\mathrm{HC}}_\bullet(A)$ of A such that the canonical higher character map,

$$\mathrm{Tr}_n : \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_\bullet(A, n)^{\mathrm{GL}_n}$$

is a graded Lie algebra homomorphism for each n (see [1, Thm. 2]), where $\mathrm{HR}_\bullet(A, n)$ stands for the representation homology parametrizing n-dimensional representations of A (see Section 2.1 for the definition).

When $A \in \mathtt{DGA}_{k/k}$ is Koszul dual to a *cyclic* DG coalgebra C, then A acquires a canonical derived Poisson structure (see [1, Thm. 15, Lem. 8]). The general properties of such derived Poisson structures have been studied in [7]. In this sequel to [5], we study the behaviour of certain derived character maps of a (DG) Lie algebra $\mathfrak{a}$ with respect to a canonical derived

Poisson structure on the universal enveloping algebra $\mathcal{U}\mathfrak{a}$ acquired from a cyclic pairing on the Koszul dual coalgebra of $\mathfrak{a}$. In order to state our main results, we recall that a natural direct sum decomposition of the (reduced) cyclic homology $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})$ that is Koszul dual to the Hodge (or λ-) decomposition of cyclic homology of commutative algebras was found in [2] (also see [16]):

$$\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a}) \cong \bigoplus_{p=1}^{\infty} \mathrm{HC}^{(p)}_\bullet(\mathfrak{a}) \,. \tag{1.1}$$

The direct summands of (1.1) appeared in [2] as domains of the *Drinfeld traces*. The Drinfeld trace $\mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a}) : \mathrm{HC}^{(p)}_\bullet(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ associated with an invariant polynomial $P \in I^p(\mathfrak{g}) := \mathrm{Sym}^p(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}$ is a certain derived character map with values in the representation homology of $\mathfrak{a}$ in a finite dimensional Lie algebra $\mathfrak{g}$ (see Section 3.3 for a recapitulation of the construction). A natural interpretation of the strong Macdonald conjecture for a reductive Lie algebra $\mathfrak{g}$ has been given in terms of the Drinfeld traces in [2, Sec. 9]. It was shown in [5] that if $\mathfrak{a}$ is the Quillen model of a simply connected space X, then the summands of (1.1) are the common eigenspaces of Frobenius operations on the S^1-equvariant homology $\mathrm{H}^{S^1}_\bullet(LX;k)$ of the free loop space LX of X.

If $\mathfrak{a}$ is Koszul dual to a *cyclic* cocommutative (DG) coalgebra C, then $\mathcal{U}\mathfrak{a}$ acquires an associated derived Poisson structure. As a result, there is a Lie bracket on $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})$. Such derived Poisson structures arise in topology: it is known that if M is a closed simply connected manifold, there is a derived Poisson structure on $\mathcal{U}\mathfrak{a}_M$ (where $\mathfrak{a}_M$ is the Quillen model of M) that is associated with a cyclic pairing[1] on the Lambrechts-Stanley model of M (see [1, Sec. 5.5]). The induced Lie bracket[2] on $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a}_M) \cong \overline{\mathrm{H}}^{S^1}_\bullet(LM;k)$ corresponds to the Chas-Sullivan bracket on string topology. This Poisson structure on $\mathcal{U}\mathfrak{a}_M$ induces a graded Poisson structure on $\mathrm{H}_\bullet(\mathcal{U}\mathfrak{a}_M) \cong \mathrm{H}_\bullet(\Omega M;k)$ (of degree $2-\dim M$), while the Goldman bracket may be seen as a Poisson structure on $\mathrm{H}_\bullet(\Omega\Sigma;k) \cong k[\pi]$. The above Poisson structure on $\mathcal{U}\mathfrak{a}_M$ may therefore be viewed as an analog of the Goldman bracket. In [5], it was shown that the above cyclic Poisson structure on $\mathcal{U}\mathfrak{a}$ preserves the Hodge filtration

$$F_p\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a}) := \bigoplus_{r\leqslant p+2} \mathrm{HC}^{(r)}_\bullet(\mathfrak{a})\,,$$

thus making $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})$ a filtered Lie algebra. Moreover, in general,

$$\{\mathrm{HC}^{(2)}_\bullet(\mathfrak{a}), \mathrm{HC}^{(p)}_\bullet(\mathfrak{a})\} \subseteq \mathrm{HC}^{(p)}_\bullet(\mathfrak{a})\,,$$

making $\mathrm{HC}^{(p)}_\bullet(\mathfrak{a})$ a graded Lie module over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$. If, in addition, $\mathfrak{g}$ is reductive, there is a derived Poisson structure on the (homotopy commutative

[1]This pairing is of degree $-n$, where $n = \dim M$.

[2]This bracket is of degree $2-n$.

DG algebra) $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ representing the derived scheme parametrizing the representations of $\mathfrak{a}$ in $\mathfrak{g}$. This induces a graded Poisson structure on the representation homology $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$. In the case when $\mathfrak{a}=\mathfrak{a}_M$, this Poisson structure is of degree $2-n$, where $n=\dim M$ (see [1, 5]).

In this context, it is natural to ask how the Drinfeld traces intertwine these structures. The first step in this direction was made in [5] where it was shown that the Drinfeld trace $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{HC}^{(2)}_\bullet(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ corresponding to the Killing form is a graded Lie algebra homomorphism. The map $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})$ therefore equips $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ with the structure of a graded Lie module[3] over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$. The following theorem is our first main result.

Theorem 1.1 (see Theorem 4.1)**.** *For any $P\in I^p(\mathfrak{g})$, the Drinfeld trace $\mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a}) : \mathrm{HC}^{(p)}_\bullet(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ is a homomorphism of graded $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$-modules.*

Recall that

$$\overline{\mathrm{HH}}_\bullet(\mathcal{U}\mathfrak{a}) \cong \mathrm{H}_\bullet(\mathfrak{a};\mathcal{U}\mathfrak{a}) \cong \bigoplus_{p=0}^{\infty} \mathrm{H}_\bullet(\mathfrak{a};\mathrm{Sym}^p(\mathfrak{a}))\ ,$$

where $\mathfrak{a}$ acts on $\mathcal{U}\mathfrak{a}$ and the $\mathrm{Sym}^p(\mathfrak{a})$ via the adjoint action (see [17, Thm. 3.3.2]). Let $\mathrm{HH}^{(p)}_\bullet(\mathfrak{a}) := \mathrm{H}_\bullet(\mathfrak{a};\mathrm{Sym}^p(\mathfrak{a}))$. It was shown in [5] that the Connes differential $B : \overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a}) \to \overline{\mathrm{HH}}_{\bullet+1}(\mathcal{U}\mathfrak{a})$ restricts to a map $B : \mathrm{HC}^{(p)}_\bullet(\mathfrak{a}) \to \mathrm{HH}^{(p-1)}_{\bullet+1}(\mathfrak{a})$ for all $p\geqslant 1$. In this paper, we extend the construction of the Drinfeld traces to give a map $\mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a}) : \mathrm{HH}^{(p)}_{\bullet+1}(\mathfrak{a}) \to \mathrm{H}_\bullet[\Omega^1(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}))]$ for any $P\in I^{p+1}(\mathfrak{g})$, where $\Omega^1(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}))$ stands for the DG module of Kähler differentials of any (cofibrant) commutative DG algebra representing the derived affine scheme $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$. Assume that $\mathfrak{a}$ is Koszul dual to a cyclic cocommutative DG coalgebra C. In this case, it was shown in [5] that $\mathrm{HH}^{(p)}_\bullet(\mathfrak{a})$ is a graded Lie module over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$ for all p, and that $B : \mathrm{HC}^{(p)}_\bullet(\mathfrak{a}) \to \mathrm{HH}^{(p-1)}_{\bullet+1}(\mathfrak{a})$ is a graded $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$-module homomorphism. Our next result extends Theorem 1.1 as follows.

Theorem 1.2 (see Theorem 4.2)**.** *For any $P\in I^{p+1}(\mathfrak{g})$, there is a commuting diagram of graded Lie modules over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$*

$$\begin{array}{ccc} \mathrm{HC}^{(p+1)}_\bullet(\mathfrak{a}) & \xrightarrow{B} & \mathrm{HH}^{(p)}_{\bullet+1}(\mathfrak{a}) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a})} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a})} \\ \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g}) & \xrightarrow{d} & \mathrm{H}_\bullet[\Omega^1(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}))] \end{array}\ ,$$

[3]In what follows, all Lie brackets as well as Lie module structures are of homological degree $n+2$, where n is the degree of the cyclic pairing on the Koszul dual coalgebra.

where the horizontal arrow in the bottom of the above diagram is induced by the universal derivation.

Next, we consider (augmented) associative (DG) algebras that are Koszul dual to cyclic coassociative (conilpotent, DG) coalgebras. In this case, $\overline{\mathrm{HC}}_\bullet(A)$ is a graded Lie algebra, over which $\overline{\mathrm{HH}}_{\bullet+1}(A)$ is a graded Lie module. In line with the Kontsevich-Rosenberg principle in noncommutative geometry, $\overline{\mathrm{HC}}_\bullet(A)$ should be seen as a *derived space of functions* on 'Spec A'. Similarly, $\overline{\mathrm{HH}}_{\bullet+1}(A)$ should be viewed as a *derived space of* 1*-forms* on 'Spec A' (see [3, Sec. 5]). The Connes differential B is an analog of the de Rham differential. Given that the the module of 1-forms of a (commutative) Poisson algebra is a Lie module over that algebra itself, with the universal derivation being a Lie module homomorphism, it is natural to expect that $\overline{\mathrm{HH}}_{\bullet+1}(A)$ to be a graded Lie module over $\overline{\mathrm{HC}}_\bullet(A)$, with B being a Lie module homomorphism. Indeed, by [7, Thm. 1.2], $B : \overline{\mathrm{HC}}_\bullet(A) \to \overline{\mathrm{HH}}_{\bullet+1}(A)$ is a homomorphism of graded Lie modules over $\overline{\mathrm{HC}}_\bullet(A)$. The Kontsevich-Rosenberg principle also leads one to expect the trace maps Tr_n to induce the classical Poisson structures on $\mathrm{DRep}_n(A)$ and its space of 1-forms. Confirming this expectation, we prove the following associative analog of Theorem 1.2, which was stated in [7] (see *loc. cit.*, Theorem 1.3) without proof.

Theorem 1.3 (see Theorem 4.4)**.** *There is a commutative diagram of* $\overline{\mathrm{HC}}_\bullet(A)$*-module homomorphisms*

$$\begin{array}{ccc} \overline{\mathrm{HC}}_\bullet(A) & \xrightarrow{B} & \overline{\mathrm{HH}}_{\bullet+1}(A) \\ \downarrow{\scriptstyle \mathrm{Tr}_n} & & \downarrow{\scriptstyle \mathrm{Tr}_n} \\ \mathrm{HR}_\bullet(A,n) & \xrightarrow{d} & \mathrm{H}_\bullet[\Omega^1(\mathrm{DRep}_n(A))] \end{array} .$$

Our final result is a common generalization of [1, Thm. 2] (for derived Poisson structures induced by cyclic pairings) and [5, Thm. 5.1]. Following [1, 8, 13], we define the notion of a Poisson structure for an algebra over a (finitely generated) cyclic binary quadratic operad $\mathcal{P}$ and show that if A is a $\mathcal{P}$-algebra that is Koszul dual to a cyclic coalgebra C over the (quadratic) Koszul dual operad $\mathcal{Q}$, then A acquires a derived Poisson structure. Further, if $\mathcal{S}$ is a finite dimensional $\mathcal{P}$-algebra with a nondegenerate cyclic pairing, then the representation homology $\mathrm{HR}_\bullet(A,\mathcal{S})$ acquires a graded Poisson structure such that a certain canonical trace map from the $\mathcal{P}$-cyclic homology $\mathrm{HC}_\bullet(\mathcal{P},A)$ of A to $\mathrm{HR}_\bullet(A,\mathcal{S})$ is a graded Lie algebra homomorphism (see Theorem 5.1).

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2. Preliminaries

In this section we review derived representation schemes and derived Poisson structures. The material in this section is primarily drawn from [1, 3, 2, 4, 5].

2.1. Derived representation schemes. We begin by reviewing derived representation schemes of associative and Lie algebras. Let $\mathtt{DGA}_k$ (resp., $\mathtt{DGCA}_k$) denote the category of associative (resp., commutative) DG k-algebras. Let $\mathcal{M}_n(k)$ denote the algebra of $n \times n$ matrices with entries in k.

2.1.1. Associative algebras. Consider the functor

$$(-)_n \,:\, \mathtt{DGA}_{k/k} \to \mathtt{DGCA}_{k/k}\,, \quad A \mapsto [(A *_k \mathcal{M}_n(k))^{\mathcal{M}_n(k)}]_{\natural\natural}\,, \tag{2.1}$$

where $(A *_k \mathcal{M}_n(k))^{\mathcal{M}_n(k)}$ denotes the subalgebra of elements in the free product $A *_k \mathcal{M}_n(k)$ that commute with every element of $\mathcal{M}_n(k)$ and where $(-)_{\natural\natural}$ denotes abelianization. Note that if A is augmented, then A_n has a natural augmentation coming from (2.1) applied to the augmentation map of A. This defines a functor $\mathtt{DGA}_{k/k} \to \mathtt{DGCA}_{k/k}$ from the category of augmented associative DG algebras to the category of augmented commutative DG algebras, which we again denote by $(-)_n$.

Recall that $\mathtt{DGA}_{k/k}$ and $\mathtt{DGCA}_{k/k}$ are model categories where the weak equivalences are the quasi-isomorphisms and the fibrations are the degree-wise surjections. Let $\mathcal{M}'_n(-) : \mathtt{DGCA}_{k/k} \to \mathtt{DGA}_{k/k}$ denote the functor $B \mapsto k \oplus \mathcal{M}_n(\bar{B})$. The functors $(-)_n : \mathtt{DGA}_{k/k} \rightleftarrows \mathtt{DGCA}_{k/k} : \mathcal{M}'_n(-)$ form a (Quillen) adjoint pair.

Thus, A_n is the commutative (DG) algebra corresponding to the (DG) scheme $\mathrm{Rep}_n(A)$ parametrizing the n-dimensional representations of A. Since the functor $(-)_n$ is left Quillen, it has a well behaved left derived functor

$$\mathbb{L}(-)_n \,:\, \mathtt{Ho}(\mathtt{DGA}_{k/k}) \to \mathtt{Ho}(\mathtt{DGCA}_{k/k})\,.$$

Like for any left derived functor, we have $\mathbb{L}(A)_n \cong R_n$ in $\mathtt{Ho}(\mathtt{DGCA}_{k/k})$, where $R \xrightarrow{\sim} A$ is any cofibrant resolution in $\mathtt{DGA}_{k/k}$. We define

$$\mathrm{DRep}_n(A) := \mathbb{L}(A)_n \text{ in } \mathtt{Ho}(\mathtt{DGCA}_{k/k})\,, \quad \mathrm{HR}_\bullet(A,n) := \mathrm{H}_\bullet[\mathbb{L}(A)_n]\,.$$

$\mathrm{DRep}_n(A)$ is called the *derived representation algebra* for n-dimensional representations of A. The homology $\mathrm{HR}_\bullet(A,n)$ is called the *representation homology* parametrizing n-dimensional representations of A. It is easy to verify that $\mathrm{GL}_n(k)$ acts naturally by automorphisms on the graded (commutative) algebra $\mathrm{HR}_\bullet(A,n)$. We denote the corresponding (graded) subalgebra of $\mathrm{GL}_n(k)$-invariants by $\mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$.

Let $R \xrightarrow{\sim} A$ be a cofibrant resolution. The unit of the adjunction $(-)_n : \mathtt{DGA}_{k/k} \rightleftarrows \mathtt{DGCA}_k : \mathcal{M}'_n(-)$ is the universal representation

$$\pi_n : R \to \mathcal{M}'_n(R_n) \hookrightarrow \mathcal{M}_n(R_n)\,.$$

It is not difficult to verify that the composite map

$$R \xrightarrow{\pi_n} \mathcal{M}_n(R_n) \xrightarrow{\mathrm{Id}\otimes \mathrm{Tr}_n} R_n$$

vanishes on $[R,R]$ and that the image of the above composite map is contained in R_n^{GL}. The above composite map therefore induces a map of complexes

$$\mathrm{Tr}_n : R/(k+[R,R]) \to R_n^{\mathrm{GL}}\,,$$

which on homologies gives the *derived character map*

$$\mathrm{Tr}_n : \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}\,.$$

2.1.2. Lie algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Consider the functor

$$(-)_\mathfrak{g} : \mathtt{DGLA}_k \to \mathtt{DGCA}_{k/k}\,,\quad \mathfrak{a} \mapsto \mathfrak{a}_\mathfrak{g}\,,$$

where

$$\mathfrak{a}_\mathfrak{g} := \frac{\mathrm{Sym}_k(\mathfrak{a}\otimes\mathfrak{g}^*)}{\langle\langle (x\otimes\xi_1).(y\otimes\xi_2) - (-1)^{|x||y|}(y\otimes\xi_1).(x\otimes\xi_2) - [x,y]\otimes\xi\rangle\rangle}\,,$$

$\mathtt{DGLA}_k$ is the category of DG Lie algebras over k, $\mathfrak{g}^*$ is the vector space dual to $\mathfrak{g}$ and where $\xi \mapsto \xi_1\wedge\xi_2$ is the map dual to the Lie bracket on $\mathfrak{g}$. The augmentation on $\mathfrak{a}_\mathfrak{g}$ is the one induced by the map taking the generators $\mathfrak{a}\otimes\mathfrak{g}^*$ to 0. Let $\mathfrak{g}(-) : \mathtt{DGCA}_{k/k} \to \mathtt{DGLA}_k$ denote the functor $B \mapsto \mathfrak{g}(\bar{B}) := \mathfrak{g}\otimes\bar{B}$. Recall that $\mathtt{DGLA}_k$ is a model category where the weak-equivalences are the quasi-isomorphisms and the fibrations are the degree-wise surjections. It is shown in [2, Section 6.3] that the functors $(-)_\mathfrak{g} : \mathtt{DGLA}_k \rightleftarrows \mathtt{DGCA}_{k/k} : \mathfrak{g}(-)$ form a (Quillen) adjoint pair.

Thus, $\mathfrak{a}_\mathfrak{g}$ is the commutative (DG) algebra corresponding to the (DG) scheme $\mathrm{Rep}_\mathfrak{g}(\mathfrak{a})$ parametrizing representations of $\mathfrak{a}$ in $\mathfrak{g}$. Since the functor $(-)_\mathfrak{g}$ is left Quillen, it has a well behaved left derived functor

$$\mathbb{L}(-)_\mathfrak{g} : \mathtt{Ho}(\mathtt{DGLA}_k) \to \mathtt{Ho}(\mathtt{DGCA}_{k/k})\,.$$

Like for any left derived functor, we have $\mathbb{L}(\mathfrak{a})_\mathfrak{g} \cong \mathcal{L}_\mathfrak{g}$ in $\mathtt{Ho}(\mathtt{DGCA}_{k/k})$, where $\mathcal{L} \xrightarrow{\sim} \mathfrak{a}$ is any cofibrant resolution in $\mathtt{DGLA}_k$. We define

$$\mathrm{DRep}_\mathfrak{g}(\mathfrak{a}) := \mathbb{L}(\mathfrak{a})_\mathfrak{g} \text{ in } \mathtt{Ho}(\mathtt{DGCA}_{k/k})\,,\quad \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g}) := \mathrm{H}_\bullet[\mathbb{L}(\mathfrak{a})_\mathfrak{g}]\,.$$

$\mathrm{DRep}_\mathfrak{g}(\mathfrak{a})$ is called the *derived representation algebra* for representations of $\mathfrak{a}$ in $\mathfrak{g}$. The homology $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ is called the *representation homology* of $\mathfrak{a}$ in $\mathfrak{g}$. It is not difficult to check that $\mathfrak{g}$ acts naturally by derivations on the graded (commutative) algebra $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$. We denote the corresponding (graded) subalgebra of $\mathfrak{g}$-invariants by $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}}$.

2.2. Derived Poisson structures. The notion of a derived Poisson algebra was first introduced in [1], as a higher homological extension of the notion of an H_0-Poisson algebra introduced by Crawley-Boevey in [8].

2.2.1. Definitions. Let A be an (augmented) DG algebra. The space $\underline{\mathrm{Der}}(A)$ of graded k-linear derivations of A is naturally a DG Lie algebra with respect to the commutator bracket. Let $\underline{\mathrm{Der}}(A)^\natural$ denote the subcomplex of $\underline{\mathrm{Der}}(A)$ comprising derivations with image in $k + [A, A] \subseteq A$. It is easy to see that $\underline{\mathrm{Der}}(A)^\natural$ is a DG Lie ideal of $\underline{\mathrm{Der}}(A)$, so that $\underline{\mathrm{Der}}(A)_\natural := \underline{\mathrm{Der}}(A)/\underline{\mathrm{Der}}(A)^\natural$ is a DG Lie algebra. The natural action of $\underline{\mathrm{Der}}(A)$ on A induces a Lie algebra action of $\underline{\mathrm{Der}}(A)_\natural$ on the quotient space $A_\natural := A/(k + [A, A])$. We write $\varrho : \underline{\mathrm{Der}}(A)_\natural \to \underline{\mathrm{End}}(A_\natural)$ for the corresponding DG Lie algebra homomorphism.

Now, following [1], we define a *Poisson structure* on A to be a DG Lie algebra structure on $A_\natural$ such that the adjoint representation $\mathrm{ad} : A_\natural \to \underline{\mathrm{End}}(A_\natural)$ factors through ϱ: i. e., there is a morphism of DG Lie algebras $\alpha : A_\natural \to \underline{\mathrm{Der}}(A)_\natural$ such that $\mathrm{ad} = \varrho \circ \alpha$. It is easy to see that if A is a commutative DG algebra, then a Poisson structure on A is the same thing as a (graded) Poisson bracket on A. On the other hand, if A is an ordinary k-algebra (viewed as a DG algebra), then a Poisson structure on A is precisely a H_0-Poisson structure in the sense of [8].

Let A and B be two Poisson DG algebras, i.e. objects of $\mathtt{DGA}_{k/k}$ equipped with Poisson structures. A *morphism* $f : A \to B$ of Poisson algebras is then a morphism $f : A \to B$ in $\mathtt{DGA}_{k/k}$ such that $f_\natural : A_\natural \to B_\natural$ is a morphism of DG Lie algebras. With this notion of morphisms, the Poisson DG algebras form a category which we denote $\mathtt{DGPA}_k$. Note that $\mathtt{DGPA}_k$ comes with two natural functors: the forgetful functor $U : \mathtt{DGPA}_k \to \mathtt{DGA}_{k/k}$ and the cyclic functor $(-)_\natural : \mathtt{DGPA}_k \to \mathtt{DGLA}_k$. We say that a morphism f is a *weak equivalence* in $\mathtt{DGPA}_k$ if Uf is a weak equivalence in $\mathtt{DGA}_{k/k}$ and $f_\natural$ is a weak equivalence in $\mathtt{DGLA}_k$; in other words, a weak equivalence in $\mathtt{DGPA}_k$ is a quasi-isomorphism of DG algebras, $f : A \to B$, such that the induced map $f_\natural : A_\natural \to B_\natural$ is a quasi-isomorphism of DG Lie algebras.

Although we do not know at the moment whether the category $\mathtt{DGPA}_k$ carries a Quillen model structure (with weak equivalences specified above), it has a weaker property of being a saturated homotopical category in the sense of Dwyer-Hirschhorn-Kan-Smith [10] (see [5, Sec. 3.1]). This allows one to define a well-behaved homotopy category of Poisson algebras and consider derived functors on $\mathtt{DGPA}_k$: we define the homotopy category

$$\mathtt{Ho}(\mathtt{DGPA}_k) := \mathtt{DGPA}_k[\mathcal{W}^{-1}],$$

where $\mathcal{W}$ is the class of weak equivalences.

Definition 1. By a *derived Poisson algebra* we mean a cofibrant associative DG algebra A equipped with a Poisson structure (in the sense of Defintion 2.2.1), which is viewed up to weak equivalence, i.e. as an object in $\mathtt{Ho}(\mathtt{DGPA}_k)$.

Since the complex $A_\natural$ computes the (reduced) cyclic homology of a cofibrant DG algebra A, the (reduced) cyclic homology of a derived Poisson algebra A carries a natural structure of a graded Lie algebra (see [5, Prop. 3.3]).

Another important result of [1] that holds for the derived Poisson algebras in $\mathtt{Ho}(\mathtt{DGPA}_k)$ and that motivates our study of these objects is the following

Theorem 2.1 (see [1, Theorem 2])**.** *If A is a derived Poisson DG algebra, then, for any n, there is a unique graded Poisson bracket on the representation homology $\mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$, such that the derived character map $\mathrm{Tr}_n : \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$ is a Lie algebra homomorphism.*

2.2.2. Necklace Lie algebras. The simplest example of a derived Poisson algebra is when $A = T_kV$, the tensor algebra generated by an even dimensional k-vector space V equipped with a symplectic form $\langle -,-\rangle : V\times V \to V$. In this case, A acquires a double Poisson structure in the sense of [20]. The double bracket

$$\{\{-,-\}\} : \bar{A}\otimes\bar{A} \to A\otimes A$$

is given by the formula

$$\{\{(v_1,\dots,v_n),(w_1,\dots,w_m)\}\} = \sum_{\substack{i=1,\dots,n\\ j=1,\dots,m}} \langle v_i,w_j\rangle (w_1,\dots,w_{j-1},v_{i+1},\dots,v_n)\otimes(v_1,\dots,v_{i-1},w_{j+1},\dots,w_m)\,, \tag{2.2}$$

where $(v_1,\dots,v_n)$ denotes the element $v_1\otimes\dots\otimes v_n \in T_kV$ for $v_1,\dots,v_n \in V$. This double bracket can be extended to $A\otimes A$ by setting $\{\{a,1\}\} = \{\{1,a\}\} = 0$. The above double bracket induces a (derived) Poisson structure on A: the corresponding Lie bracket on $A_\natural$ is given by the formula

$$\{\bar\alpha,\bar\beta\} = \overline{\mu\circ\{\{\alpha,\beta\}\}}\,,$$

where $\mu : A\otimes A\to A$ is the product and where $\bar a$ denotes the image of $a\in A$ under the canonical projection $A\to A_\natural$. $A_\natural = T_kV_\natural$ equipped with the above Lie bracket is the well known necklace Lie algebra (see [6, 13]).

3. Koszul, Calabi-Yau algebras

In this section, we recall results about derived Poisson structures on Koszul Calabi-Yau algebras from [7, 5].

3.1. Cyclic coalgebras. We now describe our basic construction of derived Poisson structures associated with cyclic coalgebras. Recall (*cf.* [12]) that a graded associative k-algebra is called *n-cyclic* if it is equipped with a symmetric bilinear pairing $\langle -,-\rangle : A \times A \to k$ of degree n such that

$$\langle ab, c\rangle = \langle a, bc\rangle\,, \quad \forall\ a, b, c \in A\,.$$

Dually, a graded coalgebra C is called *n-cyclic* if it is equipped with a symmetric bilinear pairing $\langle -,-\rangle : C \times C \to k$ of degree n such that

$$\langle v', w\rangle v'' = \pm\langle v, w''\rangle w'\,, \quad \forall\ v, w \in C,$$

where v' and v'' are the components of the coproduct of v written in the Sweedler notation. Note that if A is a finite dimensional graded $-n$-cyclic algebra whose cyclic pairing is non-degenerate, then $C := \mathrm{Hom}_k(A, k)$ is a graded n-cyclic coalgebra. A DG coalgebra C is n-cyclic if it is n-cyclic as a graded coalgebra and

$$\langle du, v\rangle \pm \langle u, dv\rangle = 0\,,$$

for all homogeneous $u, v \in C$, i.e, if $\langle -,-\rangle : C[n] \otimes C[n] \to k[n]$ is a map of complexes. *By convention, we say that $C \in \mathtt{DGC}_{k/k}$ is n-cyclic if $\bar{C}$ is n-cyclic as a non-counital DG coalgebra.*

Assume that $C \in \mathtt{DGC}_{k/k}$ is equipped with a cyclic pairing of degree n and let $R := \mathbf{\Omega}(C)$ denote the (associative) cobar construction of C. Recall that $R \cong T_k(\bar{C}[-1])$ as a graded k-algebra. For $v_1, \ldots, v_n \in \bar{C}[-1]$, let $(v_1, \ldots, v_n)$ denote the element $v_1 \otimes \ldots \otimes v_n$ of R. By [1, Theorem 15], the cyclic pairing on C of degree n induces a double Poisson bracket of degree $n+2$ (in the sense of [20])

$$\{\{-,-\}\} : \bar{R} \otimes \bar{R} \to R \otimes R$$

given by the formula

$$\{\{(v_1, \ldots, v_n), (w_1, \ldots, w_m)\}\} = \sum_{\substack{i=1,\ldots,n\\ j=1,\ldots,m}} \pm\langle v_i, w_j\rangle (w_1, \ldots, w_{j-1}, v_{i+1}, \ldots, v_n) \otimes (v_1, \ldots, v_{i-1}, w_{j+1}, \ldots, w_m)\,. \tag{3.1}$$

The above double bracket can be extended to $R \otimes R$ by setting $\{\{r, 1\}\} = \{\{1, r\}\} = 0$. Let $\{-,-\}$ be the bracket associated to (3.1):

$$\{-,-\} := \mu \circ \{\{-,-\}\} : R \otimes R \to R\,, \tag{3.2}$$

where μ is the multiplication map on R. Let $\natural : R \to R_\natural$ be the canonical projection and let $\{-,-\} : \natural \circ \{-,-\} : R \otimes R \to R_\natural$. We recall that the bimodule $R \otimes R$ (with outer R-bimodule structure) has a double bracket (in the sense of [7, Defn. 3.5]) given by the formula

$$\{\{-,-\}\} : R \times (R \otimes R) \to R \otimes (R \otimes R) \oplus (R \otimes R) \otimes R\,,$$
$$\{\{r, p \otimes q\}\} := \{\{r, p\}\} \otimes q \oplus (-1)^{|p|(|r|+n)} p \otimes \{\{r, q\}\}\,.$$

This double bracket restricts to a double bracket on the sub-bimodule $\Omega^1 R$ of $R \otimes R$ ([7, Corollary 5.2]). Let $\{-,-\} : R \otimes \Omega^1 R \to \Omega^1 R$ be the map $\mu \circ \{\{-,-\}\}$, where μ is the bimodule action map and let $\{-,-\} : R \otimes \Omega^1 R \to \Omega^1 R_\natural$ denote the map $\natural \circ \{-,-\}$.

The bracket $\{-,-\} : R \otimes R \to R_\natural$ descends to a DG $(n+2)$-Poisson structure on R. In particular, it descends to a (DG) Lie bracket $\{-,-\}_\natural$ on $R_\natural$ of degree $n+2$. The restriction of the bracket (3.2) to $\bar{R}$ induces a degree $n+2$ DG Lie module structure over $R_\natural$ on $\bar{R}$ and the bracket $\{-,-\} : R \otimes \Omega^1 R \to \Omega^1 R_\natural$ induces a degree $n+2$ DG Lie module structure over $R_\natural$ on $\Omega^1 R_\natural$ (see [7, Proposition 3.11]). On homologies, we have (see [7], Theorem 1.1 and Theorem 1.2)

Theorem 3.1. *Let $A \in \mathtt{DGA}_{k/k}$ be an augmented associative algebra Koszul dual to $C \in \mathtt{DGC}_{k/k}$. Assume that C is n-cyclic. Then,*
(i) $\overline{\mathrm{HC}}_\bullet(A)$ has the structure of a graded Lie algebra (with Lie bracket of degree $n+2$).
(ii) $\overline{\mathrm{HH}}_\bullet(A)$ has a graded Lie module structure over $\overline{\mathrm{HC}}_\bullet(A)$ of degree $n+2$.
(iii) The maps S, B and I in the Connes periodicity sequence are homomorphisms of degree $n+2$ graded Lie modules over $\overline{\mathrm{HC}}_\bullet(A)$.

The Lie bracket of degree $n+2$ on $\overline{\mathrm{HC}}_\bullet(A)$ that is induced by a $(n+2)$-Poisson structure on $R_\natural$ as above is an example of a derived $(n+2)$-Poisson structure on A.

3.1.1. Convention. Since we work with algebras that are Koszul dual to n-cyclic coalgebras, all Lie algebras that we work with have Lie bracket of degree $n+2$. Similarly, all Lie modules are degree $n+2$ Lie modules. We therefore, drop the prefix "degree n+2" in the sections that follow. Following this convention, we shall refer to (derived) $(n+2)$-Poisson structures as (derived) Poisson structures.

3.2. Dual Hodge decomposition. Given a Lie algebra $\mathfrak{a}$ over k, we consider the symmetric ad-invariant k-multilinear forms on $\mathfrak{a}$ of a (fixed) degree $p \geq 1$. Every such form is induced from the universal one: $\mathfrak{a} \times \mathfrak{a} \times \ldots \times \mathfrak{a} \to \lambda^{(p)}(\mathfrak{a})$, which takes its values in the space $\lambda^{(p)}(\mathfrak{a})$ of coinvariants of the adjoint representation of $\mathfrak{a}$ in $\mathrm{Sym}^p(\mathfrak{a})$. The assignment $\mathfrak{a} \mapsto \lambda^{(p)}(\mathfrak{a})$ defines a (non-additive) functor on the category of Lie algebras that extends in a canonical way to the category of DG Lie algebras:

$$\lambda^{(p)} : \mathtt{DGLA}_k \to \mathtt{Com}_k , \quad \mathfrak{a} \mapsto \mathrm{Sym}^p(\mathfrak{a})/[\mathfrak{a}, \mathrm{Sym}^p(\mathfrak{a})] . \tag{3.3}$$

The category $\mathtt{DGLA}_k$ has a natural model structure (in the sense of Quillen [19]), with weak equivalences being the quasi-isomorphisms of DG Lie algebras. The corresponding homotopy (derived) category $\mathtt{Ho}(\mathtt{DGLA}_k)$ is obtained from $\mathtt{DGLA}_k$ by localizing at the class of weak equivalences, i.e. by formally inverting all the quasi-isomorphisms in $\mathtt{DGLA}_k$. The functor (3.3), however,

does *not* preserve quasi-isomorphisms and hence does not descend to the homotopy category $\mathtt{Ho}(\mathtt{DGLA}_k)$. To remedy this problem, one has to replace $\lambda^{(p)}$ by its (left) derived functor

$$\mathbb{L}\lambda^{(p)} : \mathtt{Ho}(\mathtt{DGLA}_k) \to \mathcal{D}(k) \ , \tag{3.4}$$

which takes its values in the derived category $\mathcal{D}(k)$ of k-complexes. We write $\mathrm{HC}^{(p)}_\bullet(\mathfrak{a})$ for the homology of $\mathbb{L}\lambda^{(p)}(\mathfrak{a})$ and call it the *Lie-Hodge homology* of $\mathfrak{a}$.

For $p = 1$, the functor $\lambda^{(1)}$ is just abelianization of Lie algebras; in this case, the existence of $\mathbb{L}\lambda^{(1)}$ follows from Quillen's general theory (see [19, Chapter II, §5]), and $\mathrm{HC}^{(1)}_\bullet(\mathfrak{a})$ coincides (up to shift in degree) with the classical Chevalley-Eilenberg homology $\mathrm{H}_\bullet(\mathfrak{a}, k)$ of the Lie algebra $\mathfrak{a}$. For $p = 2$, the functor $\lambda^{(2)}$ was introduced by Drinfeld [9]; the existence of $\mathbb{L}\lambda^{(2)}$ was established by Getzler and Kapranov [12] who suggested that $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$ should be viewed as an (operadic) version of cyclic homology for Lie algebras.

Observe that each $\lambda^{(p)}$ comes together with a natural transformation to the composite functor $\mathcal{U}_\natural := (-)_\natural \circ \mathcal{U} : \mathtt{DGLA}_k \to \mathtt{DGA}_{k/k} \to \mathtt{Com}_k$, where $\mathcal{U}$ denotes the universal enveloping algebra functor on the category of (DG) Lie algebras. The natural transformations $\lambda^{(p)} \to \mathcal{U}_\natural$ are induced by the symmetrization maps

$$\mathrm{Sym}^p(\mathfrak{a}) \to \mathcal{U}\mathfrak{a} \ , \quad x_1 x_2 \dots x_p \mapsto \frac{1}{p!} \sum_{\sigma \in \mathbb{S}_p} \pm\, x_{\sigma(1)} \cdot x_{\sigma(2)} \cdot \ldots \cdot x_{\sigma(p)} \ , \tag{3.5}$$

which, by the Poincaré-Birkhoff-Witt Theorem, assemble to an isomorphism of DG $\mathfrak{a}$-modules $\mathrm{Sym}_k(\mathfrak{a}) \cong \mathcal{U}\mathfrak{a}$. From this, it follows that $\lambda^{(p)} \to \mathcal{U}_\natural$ assemble to an isomorphism of functors

$$\bigoplus_{p=1}^{\infty} \lambda^{(p)} \cong \mathcal{U}_\natural \ . \tag{3.6}$$

On the other hand, by a theorem of Feigin and Tsygan [11] (see also [3]), the functor $(-)_\natural$ has a left derived functor $\mathbb{L}(-)_\natural : \mathtt{Ho}(\mathtt{DGA}_{k/k}) \to \mathcal{D}(k)$ that computes the reduced cyclic homology $\overline{\mathrm{HC}}_\bullet(R)$ of an associative algebra $R \in \mathtt{DGA}_{k/k}$. Since $\mathcal{U}$ preserves quasi-isomorphisms and maps cofibrant DG Lie algebras to cofibrant DG associative algebras, the isomorphism (3.6) induces an isomorphism of derived functors from $\mathtt{Ho}(\mathtt{DGLA}_k)$ to $\mathcal{D}(k)$:

$$\bigoplus_{p=1}^{\infty} \mathbb{L}\lambda^{(p)} \cong \mathbb{L}(-)_\natural \circ \mathcal{U} \ . \tag{3.7}$$

At the level of homology, (3.7) yields the direct decomposition (*cf.* [2, Theorem 7.2].

$$\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a}) \cong \bigoplus_{p=1}^{\infty} \mathrm{HC}_\bullet^{(p)}(\mathfrak{a}) \ . \tag{3.8}$$

As explained in [2], the existence of (3.7) is related to the fact that $\mathcal{U}\mathfrak{a}$ is a cocommutative Hopf algebra, and in a sense, the Lie Hodge decomposition (3.8) is Koszul dual to the classical Hodge decomposition of cyclic homology for commutative algebras.

The Lie Hodge decomposition (3.8) also extends to (reduced) Hochschild homology (see [5, Sec. 2.1]):

$$\overline{\mathrm{HH}}_\bullet(\mathcal{U}\mathfrak{a}) \cong \bigoplus_{p=0}^{\infty} \mathrm{HH}_\bullet^{(p)}(\mathfrak{a}) \, .$$

Under Kassel's isomorphism $\overline{\mathrm{HH}}_\bullet(\mathcal{U}\mathfrak{a}) \cong \mathrm{H}_\bullet(\mathfrak{a}; \mathrm{Sym}(\mathfrak{a}))$ (see [17, Theorem 3.3.2]), the summand $\mathrm{HH}^{(p)}(\mathfrak{a})$ is identified with $\mathrm{H}_\bullet(\mathfrak{a}; \mathrm{Sym}^p(\mathfrak{a}))$. The Connes periodity sequence for $\mathcal{U}\mathfrak{a}$ decomposes into a direct sum of Hodge components (see [5, Theorem 2.2]): the summand of Hodge degree p is given by the long exact sequence

$$\ldots \xrightarrow{S} \mathrm{HC}_{n-1}^{(p+1)}(\mathfrak{a}) \xrightarrow{B} \mathrm{HH}_n^{(p)}(\mathfrak{a}) \xrightarrow{I} \mathrm{HC}_n^{(p)}(\mathfrak{a}) \xrightarrow{S} \mathrm{HC}_{n-2}^{(p+1)}(\mathfrak{a}) \longrightarrow \ldots . \tag{3.9}$$

Note that $\mathfrak{a}$ is Koszul dual to a cocommutative (coaugmented, conilpotent) DG coalgebra C (for example, C may be taken to be the Chevalley-Eilenberg coalgebra $\mathcal{C}(\mathfrak{a}; k)$). Thus, $\mathcal{U}\mathfrak{a}$ is Koszul dual to C viewed as a coassociative DG coalgebra. When C carries a cyclic pairing (of degree n), $\mathcal{U}\mathfrak{a}$ acquires a derived Poisson structure, giving a Lie bracket (of degree $n+2$) on $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})$. Further, in this case, $\overline{\mathrm{HH}}_\bullet(\mathcal{U}\mathfrak{a})$ has a graded Lie module structure over $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})$ of degree $n+2$. We have (see [5, Theorems 3.3 and 3.4]):

Theorem 3.2. *For all p, the derved Poisson bracket on $\mathrm{HC}_\bullet(\mathcal{U}\mathfrak{a})$ equips the direct summand $\mathrm{HC}_\bullet^{(p)}(\mathfrak{a})$ with a graded Lie module structure over $\mathrm{HC}_\bullet^{(2)}(\mathfrak{a})$ (of degree $n+2$), i.e.,*

$$\{\mathrm{HC}_\bullet^{(2)}(\mathfrak{a}), \mathrm{HC}_\bullet^{(p)}(\mathfrak{a})\} \subset \mathrm{HC}_\bullet^{(p)}(\mathfrak{a}) \ .$$

Further, $\mathrm{HH}_\bullet^{(p)}(\mathfrak{a})$ is equipped with a graded Lie module structure over $\mathrm{HC}_\bullet^{(2)}(\mathfrak{a})$ (of degree $n+2$).

There exists a cyclic cocommutative DG coalgebra Koszul dual to $\mathfrak{a}$ for a large class of interesting examples: unimodular Lie algebras (of which semisimple Lie algebras are examples), Quillen models of simply connected manifolds, etc.

3.3. Drinfeld traces and Poisson structures on representation algebras.

3.3.1. Drinfeld traces. Let $\mathcal{L} \xrightarrow{\sim} \mathfrak{a}$ be a cofibrant resolution. The unit of the adjunction $(-)_{\mathfrak{g}} : \mathtt{DGLA}_k \rightleftarrows \mathtt{DGCA}_{k/k} : \mathfrak{g}(-)$ is the universal representation

$$\pi_{\mathfrak{g}} : \mathcal{L} \to \mathfrak{g}(\mathcal{L}_{\mathfrak{g}}) .$$

Let $\lambda^{(p)} : \mathtt{DGLA}_k \to \mathtt{Com}_k$ be the functor $\mathfrak{a} \mapsto \mathrm{Sym}^p(\mathfrak{a})/[\mathfrak{a}, \mathrm{Sym}^p(\mathfrak{a})]$. There is a natural map $\lambda^{(p)}[\mathfrak{g}(\mathcal{L}_{\mathfrak{g}})] \to \mathcal{L}_{\mathfrak{g}} \otimes \lambda^{(p)}(\mathfrak{g})$. For $P \in I^p(\mathfrak{g}) := \mathrm{Sym}^p(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}$, evaluation at P gives a linear functional ev_P on $\lambda^{(p)}(\mathfrak{g})$. One thus has the composite map

$$\lambda^{(p)}(\mathcal{L}) \xrightarrow{\lambda^{(p)}(\pi_{\mathfrak{g}})} \lambda^{(p)}[\mathfrak{g}(\mathcal{L}_{\mathfrak{g}})] \longrightarrow \mathcal{L}_{\mathfrak{g}} \otimes \lambda^{(p)}(\mathfrak{g}) \xrightarrow{\mathrm{Id} \otimes \mathrm{ev}_P} \mathcal{L}_{\mathfrak{g}}$$

for $P \in I^p(\mathfrak{g})$. On homologies, this gives the map

$$\mathrm{Tr}_{\mathfrak{g}}(P, \mathfrak{a}) : \mathrm{HC}^{(p)}_{\bullet}(\mathfrak{a}) \to \mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} ,$$

which we call the *Drinfeld trace map* associated to P (see [2, Section 7] for further details regarding this construction). If $\mathfrak{g}$ is semisimple, the Killing form is a canonical element of $I^2(\mathfrak{g})$. We denote the associated Drinfeld trace by

$$\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{HC}^{(2)}_{\bullet}(\mathfrak{a}) \to \mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} .$$

3.3.2. Cyclic Lie coalgebras. Recall from [12, Sec. 4.5] that a cyclic pairing $\langle -, - \rangle$ of degree n on a DG Lie algebra $\mathfrak{a}$ is a symmetric, ad-invariant pairing (of degree n) that is compatible with differential: compatibility with differential is equivalent to the assertion that $\langle -, - \rangle : \mathfrak{a} \otimes \mathfrak{a} \to k[-n]$ is a map of complexes.

Dually, a cyclic pairing of degree n on a DG Lie coalgebra $\mathfrak{G}$ is a symmetric pairing compatible with differential satisfying

$$x^1 \langle x^2, y \rangle = \pm y^2 \langle x, y^1 \rangle$$

for all $x, y \in \mathfrak{G}$, where $]x[= x^1 \otimes x^2$, etc. in the Sweedler notation. It is not difficult to verify that if $\mathfrak{a}$ is a finite dimensional DG Lie algebra with a non-degenerate cyclic pairing, then $\mathfrak{a}^*$ is a DG Lie coalgebra with cyclic pairing (see [22, Prop. 2.1]).

Recall that for a DG Lie coalgebra $\mathfrak{G}$, one has the Chevalley-Eilenberg algebra $\mathcal{C}^c(\mathfrak{G}; k)$ which is the construction formally dual to the Chevalley-Eilenberg coalgebra $\mathcal{C}(\mathfrak{a}; k)$ of a DG Lie algebra. In particular, $\mathcal{C}^c(\mathfrak{G}; k)$ is an augmented, *commutative* DG algebra.

Lemma 3.1 (see [5, Lemma 5.1])**.** *If $\mathfrak{G} \in \mathtt{DGLC}_k$ is equipped with a cyclic pairing of degree n, then the Chevalley-Eilenberg algebra $\mathcal{C}^c(\mathfrak{G}; k)$ acquires a DG Poisson structure of degree $n + 2$.*

Indeed, $\mathcal{C}^c(\mathfrak{G};k) \cong \mathrm{Sym}(\mathfrak{G}[-1])$ as a graded algebra. The symmetric pairing of degree n on $\mathfrak{G}$ gives a skew-symmetric pairing on $\mathfrak{G}[-1]$ of degree $n+2$. This in turn, gives the required graded Poisson structure on $\mathrm{Sym}(\mathfrak{G}[-1])$. Cyclicity ensures that this structure is compatible with the differential on $\mathcal{C}^c(\mathfrak{G};k)$.

3.3.3. Poisson structures. Let $\mathfrak{a} \in \mathtt{DGLA}_k$ be Koszul dual to $C \in \mathtt{DGCC}_{k/k}$. Assume that C is equipped with a cyclic pairing of degree n. By [2, Theorem 6.7],

$$\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathcal{C}^c(\mathfrak{g}^*(\bar{C});k) \ .$$

If $\mathfrak{g}$ is semisimple, tensoring the cyclic pairing on $\bar{C}$ with the paring dual to the Killing form on $\mathfrak{g}^*$ gives a cyclic pairing of degree n on the DG Lie coalgebra $\mathfrak{g}^*(\bar{C}) := \mathfrak{g}^* \otimes \bar{C}$. It follows from Lemma 3.1 that $\mathcal{C}^c(\mathfrak{g}^*(\bar{C});k)$ has a DG Poisson structure of degree $n+2$. Thus, $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ has a graded Poisson structure of degree $n+2$. By Theorem 3.2, $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$ has a Lie bracket of degree $n+2$ arising from the derived Poisson structure on $\mathcal{U}\mathfrak{a}$ corresponding to the cyclic pairing on C.

Theorem 3.3 (see [5, Theorem 5.1])**.** *The Drinfeld trace*

$$\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) \ : \ \mathrm{HC}^{(2)}_\bullet(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$$

corresponding to the Killing form on $\mathfrak{g}$ is a homomorphism of graded Lie algebras.

4. Main results

Throughout this section, unless stated otherwise, let $\mathfrak{a} \in \mathtt{DGLA}_k$ be Koszul dual to $C \in \mathtt{DGCC}_{k/k}$. Further assume that C is equipped with a cyclic pairing of degree n. By Theorem 3.2, the corresponding derived Poisson structure on $\mathcal{U}\mathfrak{a}$ equips $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})$ with the structure of a graded Lie algebra (with the Lie bracket having degree $n+2$) of which $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$ is a graded Lie subalgebra. Moreover, the corresponding Lie bracket equips each Lie Hodge summand $\mathrm{HC}^{(p)}_\bullet(\mathfrak{a})$ with the structure of a graded Lie module over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$.

4.1. The main theorem. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. By Theorem 3.3, there is a Poisson structure on $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ such that the Drinfeld trace $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{HC}^{(2)}_\bullet(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ corresponding to the Killing form on $\mathfrak{g}$ is a homomorphism of graded Lie algebras. This equips $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$ with the structure of a graded Lie module over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$.

Theorem 4.1. *For any $P \in I^p(\mathfrak{g})$, the Drinfeld trace*

$$\mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a}) \ : \ \mathrm{HC}^{(p)}_\bullet(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})$$

is a homomorphism of graded Lie modules over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$.

We recall some technicalities before proving Theorem 4.1. Let $R := \mathbf{\Omega}(C) \in \mathtt{DGA}_{k/k}$. Let $\mathcal{L} := \mathbf{\Omega}_{\mathtt{Comm}}(C) \in \mathtt{DGLA}_k$. Then $R \cong \mathcal{UL}$. Recall (see Section 3.1) that the cyclic pairing on C equips R with a double Poisson bracket, and therefore, a derived Poisson structure. In particular, $R_\natural$ is a DG Lie algebra. By [7, Prop. 3.11], the bracket (3.2) equips $\bar{R}$ with the structure of a DG Lie module over $R_\natural$, with $R_\natural$ acting on $\bar{R}$ by derivations. The isomorphism of functors (3.6) applied to $\mathcal{L}$ gives an isomorphism

$$R_\natural \cong \bigoplus_{p=1}^{\infty} \lambda^{(p)}(\mathcal{L}) \,. \tag{4.1}$$

By [5, Prop. 3.4, Cor. 3.1], $\lambda^{(2)}(\mathcal{L})$ is a Lie subalgebra of $R_\natural$, and each $\lambda^{(p)}(\mathcal{L})$ is a $\lambda^{(2)}(\mathcal{L})$-module. Again by [5, Prop. 3.4, Cor. 3.1], $\mathcal{L}$ is a $\lambda^{(2)}(\mathcal{L})$-submodule of R, and the symmetrization map gives an isomorphism of $\lambda^{(2)}(\mathcal{L})$-modules

$$\mathrm{Sym}(\mathcal{L}) \cong R\,,$$

where the $\lambda^{(2)}(\mathcal{L})$-action on $\mathcal{L}$ is extended to an action on $\mathrm{Sym}(\mathcal{L})$ by derivations. By (the proof of) [2, Thm. 6.7], $\mathcal{L}_\mathfrak{g} = \mathcal{C}^c(\mathfrak{g}^*(\bar{C}); k)$. It follows that $\mathcal{L}_\mathfrak{g}$ has a Poisson structure induced by the cyclic pairing pairing on $\mathfrak{g}^*(\bar{C})$ obtained by tensoring the pairing on $\bar{C}$ with the Killing form on $\mathfrak{g}$ (see Section 3.3.3). By (the proof of) [5, Thm. 5.1], the trace $\mathrm{Tr}_\mathfrak{g}(\mathcal{L}) : \lambda^{(2)}(\mathcal{L}) \to \mathcal{L}_\mathfrak{g}$ is a graded Lie algebra homomorphism. This equips $\mathcal{L}_\mathfrak{g}$ with the structure of a graded Lie module over $\lambda^{(2)}(\mathcal{L})$. Since $\mathcal{L}_\mathfrak{g}$ is freely generated by $\mathfrak{g}^* \otimes V$ as a graded commutative algebra, where $V := \bar{C}[-1]$, $\Omega^1(\mathcal{L}_\mathfrak{g}) \cong \mathcal{L}_\mathfrak{g} \otimes \mathfrak{g}^* \otimes V$ as a graded $\mathcal{L}_\mathfrak{g}$-module. Let $\bar{\partial} : \lambda^{(2)}(\mathcal{L}) \to \mathcal{L} \otimes V$ denote the cyclic derivative (see [5, Lemma 6.2]) and let $d : \mathcal{L}_\mathfrak{g} \to \Omega^1(\mathcal{L}_\mathfrak{g})$ denote the universal derivation. The proof of Theorem 4.1 relies on the following Lemma, whose detailed proof we postpone.

Lemma 4.1 (see [5, Lemma 5.2])**.** *The following diagram commutes:*

$$\begin{array}{ccccc}
\lambda^{(2)}(\mathcal{L}) & \xrightarrow{\bar{\partial}} & \mathcal{L} \otimes V & \xrightarrow{\pi_\mathfrak{g} \otimes \mathrm{Id}} & \mathcal{L}_\mathfrak{g} \otimes \mathfrak{g} \otimes V \\
& \searrow^{\mathrm{Tr}_\mathfrak{g}(\mathcal{L})} & & & \downarrow \cong \\
& & \mathcal{L}_\mathfrak{g} & \xrightarrow{d} & \Omega^1(\mathcal{L}_\mathfrak{g})
\end{array}$$

Here, the vertical isomorphism on the right identifies $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form.

Proposition 4.1. *The universal representation $\pi_\mathfrak{g} : \mathcal{L} \to \mathcal{L}_\mathfrak{g} \otimes \mathfrak{g}$ is a $\lambda^{(2)}(\mathcal{L})$-module homomorphism, where $\lambda^{(2)}(\mathcal{L})$ acts trivially on $\mathfrak{g}$.*

Proof. Since $\lambda^{(2)}(\mathcal{L})$ acts on R (resp., $\mathcal{L}_\mathfrak{g}$) by derivations, it acts on the DG Lie algebras $\mathcal{L}$ (resp., $\mathcal{L}_\mathfrak{g} \otimes \mathfrak{g}$) by Lie derivations. Since $\mathcal{L}$ is freely generated

as a graded Lie algebra by $V := \bar{C}[-1]$, it suffices to verify that for any $\alpha \in \lambda^{(2)}(\mathcal{L})$ and for any $u \in V$,

$$\pi_{\mathfrak{g}}(\{\alpha, u\}) \,=\, \{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha), \pi_{\mathfrak{g}}(u)\}\ . \tag{4.2}$$

It follows from (3.1) and [5, Lemma 6.2] that the restriction of the action of $\lambda^{(2)}(\mathcal{L})$ on $\mathcal{L}$ to the (graded) subspace V of $\mathcal{L}$ is given by the composite map

$$\lambda^{(2)}(\mathcal{L}) \otimes V \xrightarrow{\bar{\partial} \otimes \mathrm{Id}} (\mathcal{L} \otimes V) \otimes V \longrightarrow \mathcal{L} \otimes (V \otimes V) \xrightarrow{\mathrm{Id} \otimes \langle -,\, - \rangle} \mathcal{L}\,,$$

where $\bar{\partial} : \lambda^{(2)}(\mathcal{L}) \to \mathcal{L} \otimes V$ is the cyclic derivative. Since the Poisson structure on $\mathcal{L}_{\mathfrak{g}}$ arises from a skew symmetric pairing on $\mathfrak{g}^* \otimes V$, the restriction of the action of $\mathcal{L}_{\mathfrak{g}}$ on itself (via the Poisson bracket) to the (graded) subspace $\mathfrak{g}^* \otimes V$ is given by the composite map

$$\begin{aligned} &\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^* \otimes V \xrightarrow{d \otimes \mathrm{Id}} \Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes \mathfrak{g}^* \otimes V \,\cong \\ &\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^* \otimes V \otimes \mathfrak{g}^* \otimes V \xrightarrow{\mathrm{Id}_{\mathcal{L}_{\mathfrak{g}}} \otimes \langle -,- \rangle} \mathcal{L}_{\mathfrak{g}}\ . \end{aligned}$$

Now, for all $v \in V$,

$$\pi_{\mathfrak{g}}(v) \,=\, \sum_{\alpha} (\xi_\alpha^* \otimes v) \otimes \xi_\alpha \,\in\, \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}\,,$$

where $\{\xi_\alpha\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form and $\{\xi_\alpha^*\}$ is the dual basis on $\mathfrak{g}^*$. In particular, $\pi_{\mathfrak{g}}(V) \subset \mathfrak{g}^* \otimes V \otimes \mathfrak{g}$. Therefore, (4.2) follows once we verify the commutativity of the following diagram:

$$\begin{array}{ccc} \lambda^{(2)}(\mathcal{L}) \otimes V & & \\ \Big\downarrow{\scriptstyle \bar{\partial} \otimes \mathrm{Id}_V} & \searrow^{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L}) \otimes \pi_{\mathfrak{g}}} & \\ \mathcal{L} \otimes V \otimes V & & \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^* \otimes V \otimes \mathfrak{g} \\ \Big\downarrow{\scriptstyle \pi_{\mathfrak{g}} \otimes \mathrm{Id}} & & \Big\downarrow{\scriptstyle d \otimes \mathrm{Id}_{\mathfrak{g}^* \otimes V \otimes \mathfrak{g}}} \\ \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g} \otimes V \otimes V & & \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^* \otimes V \otimes \mathfrak{g}^* \otimes V \otimes \mathfrak{g} \\ \Big\downarrow{\scriptstyle \mathrm{Id}_{\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}} \otimes \langle -,- \rangle} & & \Big\downarrow{\scriptstyle \mathrm{Id}_{\mathcal{L}_{\mathfrak{g}}} \otimes \langle -,- \rangle \otimes \mathrm{Id}_{\mathfrak{g}}} \\ \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g} & \xrightarrow{\mathrm{Id}} & \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g} \end{array} \qquad . \tag{4.3}$$

By Lemma 4.1, and since the pairing on $\mathfrak{g}^* \otimes V$ is the pairing on V tensored with the (pairing dual to the) Killing form, the commutativity of (4.3) follows once we verify that for all $x \in \mathfrak{g}$,

$$\sum_{\alpha} \langle \eta(x), \xi_\alpha^* \rangle \xi_\alpha \,=\, x\,,$$

where η denotes the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. This is immediately seen for all ξ_α, and hence, for all elements of $\mathfrak{g}$. This completes the proof of the desired proposition. $\square$

Proof of Theorem 4.1. It follows from Proposition 4.1 that the map

$$\mathrm{Sym}(\pi_{\mathfrak{g}}) : \mathrm{Sym}(\mathcal{L}) \to \mathrm{Sym}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g})$$

is $\lambda^{(2)}(\mathcal{L})$-equivariant, where the $\lambda^{(2)}(\mathcal{L})$ action on $\mathcal{L}$ (resp., $\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}$) is extended to an action on $\mathrm{Sym}(\mathcal{L})$ (resp., $\mathrm{Sym}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g})$) by derivations. In particular, for any p, the map $\mathrm{Sym}^p(\pi_{\mathfrak{g}}) : \mathrm{Sym}^p(\mathcal{L}) \to \mathrm{Sym}^p(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g})$ is $\lambda^{(2)}(\mathcal{L})$-equivariant. Note that $\mathcal{L}_{\mathfrak{g}}$ acts on $\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}$ by Lie derivations and on $\mathcal{L}_{\mathfrak{g}} \otimes \mathrm{Sym}(\mathfrak{g})$ by derivations: both actions are induced by the Poisson bracket on $\mathcal{L}_{\mathfrak{g}}$ and the trivial $\mathcal{L}_{\mathfrak{g}}$-action on $\mathfrak{g}$. It can be easily verified that the canonical projection $\mathrm{Sym}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}) \to \mathcal{L}_{\mathfrak{g}} \otimes \mathrm{Sym}(\mathfrak{g})$ is $\mathcal{L}_{\mathfrak{g}}$-equivariant, where the $\mathcal{L}_{\mathfrak{g}}$-action on $\mathrm{Sym}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g})$ is obtained by extending the action on $\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}$ by derivations. Since the $\lambda^{(2)}(\mathcal{L})$-action on $\mathcal{L}_{\mathfrak{g}}$ factors through the Lie algebra homomorphism $\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L}) : \lambda^{(2)}(\mathcal{L}) \to \mathcal{L}_{\mathfrak{g}}$, the canonical projection $\mathrm{Sym}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}) \to \mathcal{L}_{\mathfrak{g}} \otimes \mathrm{Sym}(\mathfrak{g})$ is $\lambda^{(2)}(\mathcal{L})$-equivariant. Hence, the composite map

$$\mathrm{Sym}^p(\mathcal{L}) \xrightarrow{\mathrm{Sym}^p(\pi_{\mathfrak{g}})} \mathrm{Sym}^p(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}) \longrightarrow \mathcal{L}_{\mathfrak{g}} \otimes \mathrm{Sym}^p(\mathfrak{g}) \tag{4.4}$$

is $\lambda^{(2)}(\mathcal{L})$-equivariant. It follows that the map $\lambda^{(p)}(\mathcal{L}) \to \mathcal{L}_{\mathfrak{g}} \otimes \lambda^{(p)}(\mathfrak{g})$, is $\lambda^{(2)}(\mathcal{L})$-equivariant, since it fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^p(\mathcal{L}) & \xrightarrow{(4.4)} & \mathcal{L}_{\mathfrak{g}} \otimes \mathrm{Sym}^p(\mathfrak{g}) \\ \downarrow & & \downarrow \\ \lambda^{(p)}(\mathcal{L}) & \longrightarrow & \mathcal{L}_{\mathfrak{g}} \otimes \lambda^{(p)}(\mathfrak{g}) \end{array} .$$

For any $P \in I^p(\mathfrak{g})$, the Drinfeld trace $\mathrm{Tr}_{\mathfrak{g}}(P, \mathcal{L})$ is given by composing the map $\lambda^{(p)}(\mathcal{L}) \to \mathcal{L}_{\mathfrak{g}} \otimes \lambda^{(p)}(\mathfrak{g})$ with evaluation at P. It follows that $\mathrm{Tr}_{\mathfrak{g}}(P, \mathcal{L})$ is $\lambda^{(2)}(\mathcal{L})$-equivariant. Since $\mathcal{L}$ is a cofibrant resolution of $\mathfrak{a}$, the desired theorem follows on homologies. □

We end this section with a tedious computation verifying Lemma 4.1.

Proof of Lemma 4.1. Note that every element in $\lambda^{(2)}(\mathcal{L})$ can be expressed as a linear combination of images of elements of the form $x \cdot w \in \mathrm{Sym}^2(\mathcal{L})$ under the canonical projection $\mathrm{Sym}^2(\mathcal{L}) \twoheadrightarrow \lambda^{(2)}(\mathcal{L})$, where

$$x = [v_1, [v_2, \cdots [v_{n-1}, v_n] \cdots]]$$

for some $v_1, \ldots, v_n \in V$ and $w \in V$. Here, $V := \bar{C}[-1]$. Observe that

$$\begin{aligned} &\bar{\partial}(x \cdot w) = \\ &\sum_{1 \leqslant j \leqslant n} \pm [[v_{j+1}, [v_{j+2}, \cdots [v_{n-1}, v_n] \cdots]], [[\cdots [w, v_1] \cdots , v_{j-2}], v_{j-1}]] \otimes v_j \\ &\qquad + [v_1, [v_2, \cdots [v_{n-1}, v_n] \cdots]] \otimes w \,, \end{aligned} \tag{4.5}$$

where the signs are determined by the Koszul sign rule. To verify (4.5), recall that restriction of the cyclic derivative $\bar{\partial}$ to $V^{\otimes n}$ is given by the composite map

$$V^{\otimes n} \xrightarrow{N\cdot(-)} V^{\otimes n} \longrightarrow V^{\otimes n-1}\otimes V\,,$$

where $N=\sum_{i=0}^{n-1}\tau^i$ and where the last arrow is the obvious isomorphism that permutes no factors. Here, $\tau : V^{\otimes n}\to V^{\otimes n}$ denotes the cyclic permutation $(v_1,\dots,v_n)\mapsto\pm(v_2,\dots,v_n,v_1)$, where the signs are given by the Koszul sign rule. Since $x\cdot w = \frac{1}{2}(xw+(-1)^{|x||w|}wx)$, and since N (and hence, $\bar{\partial}$) vanish on commutators in $R:=\mathbf{\Omega}(C)$, $\bar{\partial}(x\cdot w)=\bar{\partial}(xw)$. It is easy to see that the summand of $N(xw)$ ending in w is xw itself. For $1\leq i\leq n$, we note that

$$x\cdot w \;=\; \pm[v_i,z_1]\cdot z_2\,,$$

in $\lambda^{(2)}(\mathcal{L})\subset R_{\mathfrak{l}}$, where $z_1:=[v_{i+1},\,[v_{i+2},\cdots[v_{n-1},\,v_n]\cdots]]$ and $z_2:=[[\cdots[w,v_1]\cdots,v_{i-2}],v_{i-1}]$. It follows from [4, Lemma A.1] that the summand of $N(x\cdot w)$ ending in v_i is given by $\pm[z_1,z_2]\otimes v_i$. This completes the verification of (4.5).

Let $\{\xi_i\}$ be an orthonomal basis of $\mathfrak{g}$ with respect to the Killing form and let $\{\xi_i^*\}$ denote the dual basis of $\mathfrak{g}^*$. For $v\in V$ and $\xi^*\in\mathfrak{g}^*$, let $\xi^*(v):=\xi^*\otimes v\in\mathfrak{g}^*\otimes V$. In addition, for $\xi_1,\cdots,\xi_n\in\mathfrak{g}$, let $[\xi_1,\cdots,\xi_n]:=[\xi_1,\,[\xi_2,\cdots[\xi_{n-1},\,\xi_n]\cdots]]\in\mathfrak{g}$. Since for all $v\in V$,

$$\pi_g(v) \;=\; \sum_i \xi_i^*(v)\otimes\xi_i \;\in\; \mathcal{L}_{\mathfrak{g}}\otimes\mathfrak{g}\,,$$

$$\pi_{\mathfrak{g}}(x) \;=\; \sum_{i_1,\cdots,i_n} \xi_{i_1}^*(v_1)\cdots\xi_{i_n}^*(v_n)\otimes[\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_n}] \;\in\; \mathcal{L}_{\mathfrak{g}}\otimes\mathfrak{g}\;.$$

Hence,

$$\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(x\cdot w) \;=\; \sum_{i_0,i_1,\cdots,i_n}\langle[\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_n}]\,,\,\xi_{i_0}\rangle\xi_{i_1}^*(v_1)\cdots\xi_{i_n}^*(v_n)\xi_{i_0}^*(w)\,,$$

where the pairing $\langle -,-\rangle$ is the Killing form. Identifying $\Omega^1(\mathcal{L}_{\mathfrak{g}})$ with $\mathcal{L}_{\mathfrak{g}}\otimes\mathfrak{g}^*\otimes V$, we have

$$d\circ\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(x\cdot w) \;=\; \tag{4.6}$$

$$\sum_{i_0,i_1,\cdots,i_n}\Big\{\sum_{1\leqslant j\leqslant n}\pm\langle[\xi_{i_1},\cdots,\xi_{i_n}]\,,\,\xi_{i_0}\rangle\xi_{i_1}^*(v_1)\widehat{\cdots}\xi_{i_n}^*(v_n)\xi_{i_0}^*(w)\otimes\xi_{i_j}^*(v_j)$$

$$+\langle[\xi_{i_1},\cdots,\xi_{i_n}]\,,\,\xi_{i_0}\rangle\xi_{i_1}^*(v_1)\cdots\xi_{i_n}^*(v_n)\otimes\xi_{i_0}^*(w)\Big\}\;.$$

By (4.5), in $\mathcal{L}_{\mathfrak{g}}\otimes\mathfrak{g}\otimes V$,

$$(\pi_{\mathfrak{g}}\otimes\mathrm{Id})(\bar{\partial}(x\cdot w)) \;=\; \tag{4.7}$$

$$\sum_{i_0,i_1,\cdots,i_n}\Big\{\sum_{1\leqslant j\leqslant n}\pm\xi_{i_1}^*(v_1)\widehat{\cdots}\xi_{i_n}^*(v_n)\xi_{i_0}^*(w)\otimes[[\xi_{i_{j+1}},\cdots,\xi_{i_n}],\,[\xi_{i_0},\cdots,\xi_{i_{j-1}}]']\otimes v_j$$

$$+\xi_{i_1}^*(v_1)\cdots\xi_{i_n}^*(v_n)\otimes[\xi_{i_1},\cdots,\xi_{i_n}]\otimes w\Big\}\,,$$

where $[\xi_1, \cdots, \xi_n]' := [[\cdots[\xi_1, \xi_2]\cdots, \xi_{n-1}], \xi_n]$ for $\xi_1, \cdots, \xi_n \in \mathfrak{g}$. Since $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ via the Killing form, the desired lemma follows from (4.6) and (4.7) once we verify that for $1 \leqslant j \leqslant n$,

$$\sum_{i_0, i_1, \cdots, i_n} \langle [[\xi_{i_{j+1}}, \cdots, \xi_{i_n}],\, [\xi_{i_0}, \cdots, \xi_{i_{j-1}}]'], \xi_{i_j} \rangle$$

$$= \sum_{i_0, i_1, \cdots, i_n} \langle [\xi_{i_1},\, [\xi_{i_2}, \cdots [\xi_{i_{n-1}},\, \xi_{i_n}] \cdots]]\,,\, \xi_{i_0} \rangle\ .$$

This is immediate from the symmetry and invariance of the Killing form. □

4.2. Traces to Hochschild homology. Let $V := \bar{C}[-1]$, $R := \mathbf{\Omega}(C)$ and $\mathcal{L} := \mathbf{\Omega}_{\mathtt{Comm}}(C)$ (note that $R \cong \mathcal{U}\mathcal{L}$). By assumption, $\mathcal{L} \xrightarrow{\sim} \mathfrak{a}$ is a cofibrant resolution of $\mathfrak{a}$ in $\mathtt{DGLA}_k$, and $R \xrightarrow{\sim} \mathcal{U}\mathfrak{a}$ is a cofibrant resolution of $\mathcal{U}\mathfrak{a}$.

For a R-bimodule M, let $M_\natural := M/[R, M]$. Since $\Omega^1 R \cong R \otimes V \otimes R$ as graded R-bimodules, $\Omega^1 R_\natural \cong R \otimes V$ as graded vector spaces. We equip $R \otimes V \otimes R$ (resp., $R \otimes V$) with the differential inherited from $\Omega^1 R$ (resp., $\Omega^1 R_\natural$). Let $\beta : \Omega^1 R_\natural \to \bar{R}$ denote the map given by $r \otimes v \mapsto [r, v]$. It s known that there is an isomorphism of homologies

$$\overline{\mathrm{HH}}_\bullet(\mathcal{U}\mathfrak{a}) \cong \mathrm{H}_\bullet[\mathrm{c\hat{o}ne}(\beta : \Omega^1 R_\natural \to \bar{R})]\ .$$

For $p \geqslant 1$, let $\theta^{(p)}(\mathcal{L})$ denote the subcomplex $\mathrm{Sym}^p(\mathcal{L}) \otimes V$ of $R \otimes V$, where $\mathrm{Sym}^p(\mathcal{L})$ is embedded in R via the symmetrization map. It is easy to verify that $\beta[\theta^{(p)}(\mathcal{L})] \subset \mathrm{Sym}^p(\mathcal{L})$ (see [5, Lemma 2.1]). By (the proof of) [5, Thm. 2.2],

$$\mathrm{HH}^{(p)}_\bullet(\mathfrak{a}) \cong \mathrm{H}_\bullet[\mathrm{c\hat{o}ne}(\beta : \theta^{(p)}(\mathcal{L}) \to \mathrm{Sym}^p(\mathcal{L}))]\ .$$

Let $\theta : V \to \mathfrak{g} \otimes \mathfrak{g}^* \otimes V$ denote the map $v \mapsto \sum_i \xi_i \otimes \xi_i^* \otimes v$. Let $\mathfrak{g}^*(V) := \mathfrak{g}^* \otimes V$. Given $P \in I^{p+1}(\mathfrak{g})$, consider the composite map

$$\begin{array}{ccc}
\mathrm{Sym}^p(\mathcal{L}) \otimes V & & \\
\big\downarrow \mathrm{Sym}^p(\pi_\mathfrak{g}) \otimes \theta & & \\
\mathrm{Sym}^p(\mathcal{L}_\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathfrak{g} \otimes \mathfrak{g}^*(V) & & \\
\big\downarrow & & \\
\mathcal{L}_\mathfrak{g} \otimes \mathrm{Sym}^p(\mathfrak{g}) \otimes \mathfrak{g} \otimes \mathfrak{g}^*(V) & \longrightarrow \mathcal{L}_\mathfrak{g} \otimes \mathrm{Sym}^{p+1}(\mathfrak{g}) \otimes \mathfrak{g}^*(V) & \xrightarrow{\mathrm{ev}_P} \mathcal{L}_\mathfrak{g} \otimes \mathfrak{g}^*(V)
\end{array}\ ,$$

where the unlabelled arrows stand for maps multiplying the obvious factors out. Identifying $\Omega^1(\mathcal{L}_\mathfrak{g})$ with $\mathcal{L}_\mathfrak{g} \otimes \mathfrak{g}^*(V)$, we see that the above composite map gives a map

$$\mathrm{Tr}_\mathfrak{g}(P, \mathcal{L}) : \theta^{(p)}(\mathcal{L}) \to \Omega^1(\mathcal{L}_\mathfrak{g}) \tag{4.8}$$

of complexes such that the following diagram commutes:

$$\begin{array}{ccc} \theta^{(p)}(\mathcal{L}) & \xrightarrow{\beta} & \mathrm{Sym}^p(\mathcal{L}) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L})} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P',\mathcal{L})} \\ \Omega^1(\mathcal{L}_{\mathfrak{g}}) & \xrightarrow{0} & \mathcal{L}_{\mathfrak{g}} \end{array} .$$

Here, the vertical arrow on the right is the composition of the canonical projection $\mathrm{Sym}^p(\mathcal{L}) \twoheadrightarrow \lambda^{(p)}(\mathcal{L})$ with the Drinfeld trace $\mathrm{Tr}_{\mathfrak{g}}(P',\mathcal{L})$ for any $P' \in I^p(\mathfrak{g})$. The Drinfeld tace therefore extends to give a composite map (independent of the choice of P')

$$\begin{aligned} \mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L}) \,:\, \mathrm{c\hat{o}ne}(\beta \,:\, \theta^p(\mathcal{L}) \to \mathrm{Sym}^p(\mathcal{L})) \\ \longrightarrow \mathrm{c\hat{o}ne}(\,0 \,:\, \Omega^1(\mathcal{L}_{\mathfrak{g}}) \to \mathcal{L}_{\mathfrak{g}}) \twoheadrightarrow \Omega^1(\mathcal{L}_{\mathfrak{g}})[1] \end{aligned}$$

On homologies, we obtain a map

$$\mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a}) \,:\, \mathrm{HH}^{(p)}_{\bullet+1}(\mathfrak{a}) \to \mathrm{H}_\bullet[\Omega^1(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}))] \,.$$

The following theorem is our second main result:

Theorem 4.2. *For any $P \in I^{p+1}(\mathfrak{g})$, there is a commuting diagram of graded Lie modules over $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$*

$$\begin{array}{ccc} \mathrm{HC}^{(p+1)}_\bullet(\mathfrak{a}) & \xrightarrow{B} & \mathrm{HH}^{(p)}_{\bullet+1}(\mathfrak{a}) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a})} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a})} \\ \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g}) & \xrightarrow{d} & \mathrm{H}_\bullet[\Omega^1(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}))] \end{array} ,$$

where the horizontal arrow in the bottom of the above diagram is induced by the universal derivation.

4.2.1. Proof of Theorem 4.2. Let $\Phi_{\mathfrak{g}} : \Omega^1 R \to \Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes \mathcal{U}\mathfrak{g}$ denote the composite map

$$\begin{array}{c} \Omega^1(R) \cong R \otimes V \otimes R \\ \Big\downarrow \mathcal{U}\pi_{\mathfrak{g}} \otimes \theta \otimes \mathcal{U}\pi_{\mathfrak{g}} \\ \mathcal{U}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}) \otimes \mathfrak{g} \otimes \mathfrak{g}^*(V) \otimes \mathcal{U}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}) \\ \Big\downarrow \\ (\mathcal{L}_{\mathfrak{g}} \otimes \mathcal{U}\mathfrak{g}) \otimes \mathfrak{g}^*(V) \otimes \mathfrak{g} \otimes (\mathcal{L}_{\mathfrak{g}} \otimes \mathcal{U}\mathfrak{g}) \\ \Big\downarrow \\ (\mathcal{L}_{\mathfrak{g}} \otimes \mathcal{L}_{\mathfrak{g}}) \otimes \mathfrak{g}^*(V) \otimes (\mathcal{U}\mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{U}\mathfrak{g}) \\ \Big\downarrow \mu_{\mathcal{L}_{\mathfrak{g}}} \otimes \mathrm{Id} \otimes \mu_{\mathcal{U}\mathfrak{g}} \\ \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^*(V) \otimes \mathcal{U}\mathfrak{g} \cong \Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes \mathcal{U}\mathfrak{g} \ . \end{array}$$

It is easy to check that $\Phi_{\mathfrak{g}}$ is a R-bimodule map, where the R-bimodule structure on $\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^*(V) \otimes \mathcal{U}\mathfrak{g}$ is induced by the composite map (also denoted by $\mathcal{U}\pi_{\mathfrak{g}}$)

$$R \xrightarrow{\mathcal{U}\pi_{\mathfrak{g}}} \mathcal{U}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}) \longrightarrow \mathcal{L}_{\mathfrak{g}} \otimes \mathcal{U}\mathfrak{g} \ .$$

Recall from Section 3.1 that the R-bimodule $\Omega^1 R$ acquires a double bracket. Composing this double bracket with the bimodule action map gives a map $R \otimes \Omega^1 R \to \Omega^1 R$. By [7, Prop. 3.10], this map equips $\Omega^1 R$ with the structure of a DG Lie module over $R_{\natural}$ (and, by restriction, $\lambda^{(2)}(\mathcal{L})$). On the other hand, by Lemma 3.1, $\mathcal{L}_{\mathfrak{g}}$ acquires a DG Poisson structure such that the Drinfeld trace $\mathrm{Tr}_{\mathfrak{g}} : \lambda^{(2)}(\mathcal{L}) \to \mathcal{L}_{\mathfrak{g}}$ corresponding to the Killing form is a homomorphism of DG Lie algebras (see [5, Sec. 5]). There is a Poisson action of $\mathcal{L}_{\mathfrak{g}}$ on $\Omega^1(\mathcal{L}_{\mathfrak{g}})$ making the latter a DG Lie module over the former. This equips $\Omega^1(\mathcal{L}_{\mathfrak{g}})$ with the structure of a DG Lie module over $\lambda^{(2)}(\mathcal{L})$. Equip $\Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes \mathcal{U}\mathfrak{g}$ with the $\lambda^{(2)}(\mathcal{L})$-module structure coming from the above $\lambda^{(2)}(\mathcal{L})$-action on $\Omega^1(\mathcal{L}_{\mathfrak{g}})$ and the trivial action on $\mathcal{U}\mathfrak{g}$. The following proposition summarizes the main facts used for the proof of Theorem 4.2.

Proposition 4.2. *All maps in the commutative diagram below are $\lambda^{(2)}(\mathcal{L})$-equivariant:*

$$\begin{array}{ccc} R & \xrightarrow{\quad d \quad} & \Omega^1 R \\ \Big\downarrow \mathcal{U}\pi_{\mathfrak{g}} & & \Big\downarrow \Phi_{\mathfrak{g}} \\ \mathcal{L}_{\mathfrak{g}} \otimes \mathcal{U}\mathfrak{g} & \xrightarrow{d \otimes \mathrm{Id}_{\mathcal{U}\mathfrak{g}}} & \Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes \mathcal{U}\mathfrak{g} \end{array} \tag{4.9}$$

The following lemma is required for the proof of Proposition 4.2.

Lemma 4.2. *The action of R on $\Omega^1 R \cong R \otimes V \otimes R$ is given by:*

$$\begin{aligned}\{a,\, b\otimes v\otimes c\} &= \{a,\, b\}\otimes v\otimes c + (-1)^{(|a|+n)(|b|+|v|)} b\otimes v\otimes \{a,\, c\}\\ &\quad +(-1)^{(|a|+n)|b|} b(d\{a,\, v\})c\ . \end{aligned} \tag{4.10}$$

Proof. the isomorphism $I : R\otimes V\otimes R \to \Omega^1_R \subset R\otimes R$ is given by

$$I\{(v_1\cdots v_{p-1})\otimes v_p\otimes (v_{p+1}\cdots v_m)\}$$
$$= (v_1\cdots v_p)\otimes(v_{p+1}\cdots v_m) - (v_1\cdots v_{p-1})\otimes(v_p\cdots v_m)\ .$$

Under this isomorphism, the universal derivation $d : R\to\Omega^1 R$ is given by

$$d(v_1\cdots v_m) = \sum_{i=1}^{m} (v_1\cdots v_{i-1})\otimes v_i\otimes(v_{i+1}\cdots v_m)\,,$$

and for all $r\in R$,

$$I(dr) = r\otimes 1 - 1\otimes r\ .$$

For a, b, $c\in R$ and $v\in V$, consider

$$\begin{aligned}\{\!\{a,\, I(b\otimes v\otimes c)\}\!\} &= \{\!\{a,\, bv\otimes c - b\otimes vc\}\!\}\\ &= \{\!\{a,\, bv\}\!\}\otimes c + (-1)^{(|a|+n)(|b|+|v|)} bv\otimes\{\!\{a,\, c\}\!\}\\ &\quad -\{\!\{a,\, b\}\!\}\otimes vc - (-1)^{(|a|+n)|b|} b\otimes\{\!\{a,\, vc\}\!\}\\ &= \{\!\{a,\, b\}\!\}v\otimes c + (-1)^{(|a|+n)|b|} b\{\!\{a,\, v\}\!\}\otimes c\\ &\quad +(-1)^{(|a|+n)(|b|+|v|)} bv\otimes\{\!\{a,\, c\}\!\} - \{\!\{a,\, b\}\!\}\otimes vc\\ &\quad -(-1)^{(|a|+n)|b|} b\otimes\{\!\{a,\, v\}\!\}c\\ &\quad -(-1)^{(|a|+n)(|b|+|v|)} b\otimes v\{\!\{a,\, c\}\!\}\\ &= \{\!\{a,\, b\}\!\}v\otimes c - \{\!\{a,\, b\}\!\}\otimes vc\\ &\quad +(-1)^{(|a|+n)(|b|+|v|)}(bv\otimes\{\!\{a,\, c\}\!\} - b\otimes v\{\!\{a,\, c\}\!\})\\ &\quad +(-1)^{(|a|+n)|b|}(b\{\!\{a,\, v\}\!\}\otimes c - b\otimes\{a,\, v\}\otimes c)\\ &\quad +(-1)^{(|a|+n)|b|}(b\otimes\{a,\, v\}\otimes c - b\otimes\{\!\{a,\, v\}\!\}c)\,. \end{aligned}$$

Note that both $\{\!\{a,\, b\}\!\}v\otimes c - \{\!\{a,\, b\}\!\}\otimes vc$ and $b\otimes\{a,\, v\}\otimes c - b\otimes\{\!\{a,\, v\}\!\}c$ belong to $R\otimes\Omega^1 R$, and both $bv\otimes\{\!\{a,\, c\}\!\} - b\otimes v\{\!\{a,\, c\}\!\}$ and $b\{\!\{a,\, v\}\!\}\otimes c - b\otimes\{a,\, v\}\otimes c$ belong to $\Omega^1 R\otimes R$. Hence,

$$\begin{aligned}\{a,\, I(b\otimes v\otimes c)\} &= \mu\circ\{\!\{a,\, I(b\otimes v\otimes c)\}\!\}\\ &= \{a,\, b\}v\otimes c - \{a,\, b\}\otimes vc\\ &\quad +(-1)^{(|a|+n)(|b|+|v|)}(bv\otimes\{a,\, c\} - b\otimes v\{a,\, c\})\\ &\quad +(-1)^{(|a|+n)|b|}(b\{a,\, v\}\otimes c - b\otimes\{a,\, v\}c)\\ &= I(\{a,b\}\otimes v\otimes c) + (-1)^{(|a|+n)(|b|+|v|)} I(b\otimes v\otimes\{a,c\})\\ &\quad +(-1)^{(|a|+n)|b|} I(bd(\{a,v\})c)\ . \end{aligned}$$

In the above computation, $\mu : R \otimes \Omega^1 R \oplus \Omega^1 R \otimes R \to \Omega^1 R$ denotes the bimodule action map. The above computation completes the proof of the desired lemma. □

For the convenience of the reader, we break Proposition 4.2 into the following two lemmas:

Lemma 4.3. *The diagram* (4.9) *commutes.*

Proof. For $v_1, \cdots, v_n \in V$,

$$\begin{aligned}\Phi_{\mathfrak{g}} \circ d(v_1 \cdots v_n) &= \Phi_{\mathfrak{g}}\big(\sum_{i=1}^{n}(v_1 \cdots v_{i-1}) \otimes v_i \otimes (v_{i+1} \cdots v_n)\big) \\ &= \sum_{i=1}^{n}\sum_{j_1,\cdots,j_n} \big(\pm \xi^*_{j_1}(v_1)\cdots\widehat{\xi^*_{j_i}(v_i)}\cdots\xi^*_{j_n}(v_n) \\ &\quad \otimes\xi^*_{j_i}(v_i) \otimes (\xi_{j_1}\cdots\xi_{j_n})\big)\ .\end{aligned}$$

On the other hand,

$$\begin{aligned}(d \otimes \mathrm{Id}) \circ \mathcal{U}\pi_{\mathfrak{g}}(v_1 \cdots v_n) &= (d \otimes \mathrm{Id})\big(\sum_{j_1,\cdots,j_n} \xi^*_{j_1}(v_1)\cdots\xi^*_{j_n}(v_n) \otimes (\xi_{j_1}\cdots\xi_{j_n})\big) \\ &= \sum_{j_1,\cdots,j_n}\sum_{i=1}^{n} \big(\pm \xi^*_{j_1}(v_1)\cdots\widehat{\xi^*_{j_i}(v_i)}\cdots\xi^*_{j_n}(v_n) \\ &\quad \otimes \xi^*_{j_i}(v_i) \otimes (\xi_{j_1}\cdots\xi_{j_n})\big)\ .\end{aligned}$$

This proves the desired lemma. □

Lemma 4.4. *All maps in the diagram* (4.9) *are* $\lambda^{(2)}(\mathcal{L})$*-equivariant.*

Proof. It follows immediately from Proposition 4.1 that $\mathcal{U}\pi_{\mathfrak{g}}$ is $\lambda^{(2)}(\mathcal{L})$-equivariant. Since the $\lambda^{(2)}(\mathcal{L})$-action on $\mathcal{L}_{\mathfrak{g}}$ (resp., $\Omega^1(\mathcal{L}_g)$) factors through the Poisson action of $\mathcal{L}_{\mathfrak{g}}$ on itself (resp., $\Omega^1(\mathcal{L}_g)$), the universal derivation $d : \mathcal{L}_{\mathfrak{g}} \to \Omega^1(\mathcal{L}_{\mathfrak{g}})$ is $\lambda^{(2)}(\mathcal{L})$-equivariant. Indeed, the Poisson action of $\mathcal{L}_{\mathfrak{g}}$ on $\Omega^1(\mathcal{L}_{\mathfrak{g}})$ is explicitly given by

$$\{\eta,\, \beta d\gamma\} = \{\eta,\, \beta\}d\gamma + (-1)^{(|\eta|+n)|\beta|}\beta d\{\eta,\, \gamma\} \tag{4.11}$$

for $\eta, \beta, \gamma \in \mathcal{L}_{\mathfrak{g}}$, since the Poisson bracket on $\mathcal{L}_{\mathfrak{g}}$ has degree $n+2$. The $R_{\natural}$-equivariance (and therefore, $\lambda^{(2)}(\mathcal{L})$-equivariance) of $d : R \to \Omega^1 R$ is a direct consequence of Lemma 4.2 and the fact that $R_{\natural}$ acts by derivations on R. It therefore, remains to verify that $\Phi_{\mathfrak{g}}$ is $\lambda^{(2)}(\mathcal{L})$-equivariant. This is equivalent to the assertion that for $\alpha \in \lambda^{(2)}(\mathcal{L})$, $p, q \in R$ and $u \in V$,

$$\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \Phi_{\mathfrak{g}}(p \otimes u \otimes q)\} = \Phi_{\mathfrak{g}}\big(\{\alpha,\, p \otimes u \otimes q\}\big)\ . \tag{4.12}$$

By Lemma 4.2,

$$\begin{aligned}\{\alpha,\, p \otimes u \otimes q\} &= \{\alpha,\, p\} \otimes u \otimes q + (-1)^{(|\alpha|+n)(|p|+|u|)}p \otimes u \otimes \{\alpha,\, q\} \\ &\quad +(-1)^{(|\alpha|+n)|p|}p(d\{\alpha,\, u\})q \end{aligned} \tag{4.13}$$

where $d : R \to R \otimes V \otimes R$ is the noncommutative de Rham differential. Since $\Phi_{\mathfrak{g}}$ is an R-bimodule map, $\Phi_{\mathfrak{g}}(p \otimes u \otimes q) = \mathcal{U}\pi_{\mathfrak{g}}(p)\Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\mathcal{U}\pi_{\mathfrak{g}}(q)$. Therefore,

$$\begin{aligned}\Phi_{\mathfrak{g}}\big(\{\alpha,\, p \otimes u \otimes q\}\big) &= \mathcal{U}\pi_{\mathfrak{g}}\big(\{\alpha,\, p\}\big)\Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\mathcal{U}\pi_{\mathfrak{g}}(q) \\ &+ (-1)^{(|\alpha|+n)|p|}\mathcal{U}\pi_{\mathfrak{g}}(p)\Phi_{\mathfrak{g}}\big(d\{\alpha,\, u\}\big)\mathcal{U}\pi_{\mathfrak{g}}(q) \\ &+ (-1)^{(|\alpha|+n)(|p|+|u|)}\mathcal{U}\pi_{\mathfrak{g}}(p)\Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\mathcal{U}\pi_{\mathfrak{g}}\big(\{\alpha,\, q\}\big).\end{aligned} \tag{4.14}$$

On the other hand,

$$\begin{aligned}&\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \Phi_{\mathfrak{g}}(p \otimes u \otimes q)\} \\ &\quad = \{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \mathcal{U}\pi_{\mathfrak{g}}(p)\Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\mathcal{U}\pi_{\mathfrak{g}}(q)\} \\ &\quad = \{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \mathcal{U}\pi_{\mathfrak{g}}(p)\}\Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\mathcal{U}\pi_{\mathfrak{g}}(q) \\ &\quad +(-1)^{(|\alpha|+n)|p|}\mathcal{U}\pi_{\mathfrak{g}}(p)\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\}\mathcal{U}\pi_{\mathfrak{g}}(q) \\ &\quad +(-1)^{(|\alpha|+n)(|p|+|u|)}\mathcal{U}\pi_{\mathfrak{g}}(p)\Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \mathcal{U}\pi_{\mathfrak{g}}(q)\}.\end{aligned} \tag{4.15}$$

Since $\mathcal{U}\pi_{\mathfrak{g}}$ is $\lambda^{(2)}(\mathcal{L})$-equivariant,

$$\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \mathcal{U}\pi_{\mathfrak{g}}(p)\} = \mathcal{U}\pi_{\mathfrak{g}}\big(\{\alpha,\, p\}\big)$$

and

$$\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \mathcal{U}\pi_{\mathfrak{g}}(q)\} = \mathcal{U}\pi_{\mathfrak{g}}\big(\{\alpha,\, q\}\big)\,.$$

By (4.14) and (4.15), (4.12) follows once we verify that for all $\alpha \in \lambda^{(2)}(\mathcal{L})$ and $u \in V$,

$$\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\} = \Phi_{\mathfrak{g}}\big(\{\alpha,\, 1 \otimes u \otimes 1\}\big)\,. \tag{4.16}$$

By Lemma 4.3, $\Phi_{\mathfrak{g}}(1 \otimes u \otimes 1) = (d \otimes \mathrm{Id}_{\mathfrak{g}})[\pi_{\mathfrak{g}}(u)]$, where u is viewed on the right hand side as an element of $V \subset \mathcal{L}$. Therefore,

$$\begin{aligned}\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, \Phi_{\mathfrak{g}}(1 \otimes u \otimes 1)\} &= \{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha),\, (d \otimes \mathrm{Id}_{\mathfrak{g}})[\pi_{\mathfrak{g}}(u)]\} \\ &= (d \otimes \mathrm{Id}_{\mathfrak{g}})\{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})(\alpha), \pi_{\mathfrak{g}}(u)\} \\ &= (d \otimes \mathrm{Id}_{\mathfrak{g}})[\pi_{\mathfrak{g}}(\{\alpha, u\})] \qquad \text{(by Prop. 4.1)} \\ &= \Phi_{\mathfrak{g}}[d\{\alpha, u\}] \qquad \text{(by Lemma 4.3)} \\ &= \Phi_{\mathfrak{g}}\big(\{\alpha,\, 1 \otimes u \otimes 1\}\big) \qquad \text{(by Lemma 4.2)}\,.\end{aligned}$$

The second equality above is because $d : \mathcal{L}_{\mathfrak{g}} \to \Omega^1\mathcal{L}_{\mathfrak{g}}$ is equivariant with respect to the Poisson action of $\mathcal{L}_{\mathfrak{g}}$. This verifies (4.16), completing the proof of the desired lemma. □

Let $\varphi_{\mathfrak{g}} : \Omega^1 R_\natural \to \Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes \mathcal{U}\mathfrak{g}$ denote the composite map

$$\begin{array}{l} \Omega^1 R_\natural \cong R \otimes V \\ \quad \mathcal{U}\pi_{\mathfrak{g}} \otimes \theta \big\downarrow \\ \mathcal{U}(\mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}) \otimes \mathfrak{g}^*(V) \otimes \mathfrak{g} \\ \qquad \big\downarrow \\ (\mathcal{L}_{\mathfrak{g}} \otimes \mathcal{U}\mathfrak{g}) \otimes \mathfrak{g}^*(V) \otimes \mathfrak{g} \longrightarrow \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^*(V) \otimes (\mathcal{U}\mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{\mathrm{Id} \otimes \mu_{\mathcal{U}\mathfrak{g}}} \mathcal{L}_{\mathfrak{g}} \otimes \mathfrak{g}^*(V) \otimes \mathcal{U}\mathfrak{g} \end{array} .$$

It is easy to verify that

$$(\Phi_{\mathfrak{g}} - \varphi_{\mathfrak{g}} \circ \natural)(\Omega^1 R) \subset \Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes [\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}] = \Omega^1(\mathcal{L}_{\mathfrak{g}}) \otimes [\mathfrak{g}, \mathcal{U}\mathfrak{g}] , \tag{4.17}$$

where $\natural : \Omega^1 R \to \Omega^1 R_\natural$ denotes the canonical projection. Any $P \in I^{p+1}(\mathfrak{g})$ determines a linear functional ev_P on $\mathrm{Sym}(\mathfrak{g})$, which coincides with the usual evaluation map on $\mathrm{Sym}^{p+1}(\mathfrak{g})$ and vanishes on other symmetric powers of $\mathfrak{g}$. Composing this linear functional with the inverse of the symmetrization map $\mathcal{U}\mathfrak{g} \to \mathrm{Sym}(\mathfrak{g})$ determines a linear functional on $\mathcal{U}\mathfrak{g}$, which we still denote by ev_P. Since P is ad-invariant, $\mathrm{ev}_P([\mathfrak{g}, \mathcal{U}\mathfrak{g}]) = 0$. It follows from (4.17) that

$$(\mathrm{Id} \otimes \mathrm{ev}_P) \circ \Phi_{\mathfrak{g}} = (\mathrm{Id} \otimes \mathrm{ev}_P) \circ \varphi_{\mathfrak{g}} \circ \natural . \tag{4.18}$$

Lemma 4.5. *There is a commutative diagram of $\lambda^{(2)}(\mathcal{L})$-equivariant maps*

$$\begin{array}{ccc} R & \xrightarrow{\bar{\partial}} & \Omega^1 R_\natural \\ {\scriptstyle (\mathrm{Id} \otimes \mathrm{ev}_P) \circ \mathcal{U}\pi_{\mathfrak{g}}} \big\downarrow & & \big\downarrow {\scriptstyle (\mathrm{Id} \otimes \mathrm{ev}_P) \circ \varphi_{\mathfrak{g}}} \\ \mathcal{L}_{\mathfrak{g}} & \xrightarrow{d} & \Omega^1(\mathcal{L}_{\mathfrak{g}}) \end{array} \tag{4.19}$$

Proof. The commutativity of (4.19) is immediate from Proposition 4.2, (4.18) and the fact that $\natural \circ d = \bar{\partial}$. Since $\Phi_{\mathfrak{g}}$ is $\lambda^{(2)}(\mathcal{L})$-equivariant so is $(\mathrm{Id} \otimes \mathrm{ev}_P) \circ \Phi_{\mathfrak{g}}$. Since $\natural : \Omega^1 R \to \Omega^1 R_\natural$ is surjective and $R_\natural$-equivariant (and hence, $\lambda^{(2)}(\mathcal{L})$-equivariant as well), the map $(\mathrm{Id} \otimes \mathrm{ev}_P) \circ \varphi_{\mathfrak{g}} : \Omega^1 R_\natural \to \Omega^1(\mathcal{L}_{\mathfrak{g}})$ is $\lambda^{(2)}(\mathcal{L})$-equivariant as well. The $R_\natural$-equivariance of $\natural$ and the $R_\natural$ equivariance of d together imply $R_\natural$-equivariance (and hence, $\lambda^{(2)}(\mathcal{L})$-equivariance) of $\bar{\partial}$. Since $\mathcal{U}\pi_{\mathfrak{g}}$ is $\lambda^{(2)}(\mathcal{L})$-equivariant by Proposition 4.2, so is $(\mathrm{Id} \otimes \mathrm{ev}_P) \circ \mathcal{U}\pi_{\mathfrak{g}}$. That d is $\lambda^{(2)}(\mathcal{L})$-equivariant is part of the proof of Proposition 4.2. □

Corollary 4.1. *There is a commutative diagram of $\lambda^{(2)}(\mathcal{L})$-equivariant maps*

$$\begin{array}{ccc} \lambda^{(p+1)}(\mathcal{L}) & \xrightarrow{\bar{\partial}} & \theta^{(p)}(\mathcal{L}) \\ \downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L})} & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L})} \\ \mathcal{L}_{\mathfrak{g}} & \xrightarrow{d} & \Omega^1(\mathcal{L}_{\mathfrak{g}}) \end{array} \tag{4.20}$$

Proof. It is easy to verify that the restriction of $(\mathrm{Id}\otimes \mathrm{ev}_P)\circ\varphi_{\mathfrak{g}}$ to the $\lambda^{(2)}(\mathcal{L})$-submodule $\theta^{(p)}(\mathcal{L})$ of $\Omega^1 R_{\natural}$ coincides with $\mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L}) : \theta^{(p)}(\mathcal{L}) \to \Omega^1(\mathcal{L}_{\mathfrak{g}})$. Similarly the map $\bar{\partial}$ factors through $R_{\natural}$: by [5, Lemma 6.2], $\bar{\partial}(\lambda^{(p+1)}(\mathcal{L})) \subset \theta^{(p)}(\mathcal{L})$. Since P is ad-invariant, the map $(\mathrm{Id}\otimes \mathrm{ev}_P)\circ \mathcal{U}\pi_{\mathfrak{g}}$ factors through $R_{\natural}$: the restriction of the corresponding map $R_{\natural} \to \mathcal{L}_{\mathfrak{g}}$ to $\lambda^{(p+1)}(\mathcal{L})$ is easily seen to coincide with $\mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L}) : \lambda^{(p+1)}(\mathcal{L}) \to \mathcal{L}_{\mathfrak{g}}$. The desired statement is immediate from the above observations and Lemma 4.5. □

Since $\beta\circ\bar{\partial} = 0$, $\bar{\partial}$ defines a map of complexes

$$\partial : \lambda^{(p+1)}(\mathcal{L}) \to \mathrm{cône}(\beta : \theta^{(p)}(\mathcal{L}) \to \mathrm{Sym}^p(\mathcal{L}))[-1] .$$

It follows from Corollary 4.1, as well as the $\lambda^{(2)}(\mathcal{L})$-equivariance of β that there is a commutative diagram of $\lambda^{(2)}(\mathcal{L})$-equivariant maps

$$\begin{array}{ccc} \lambda^{(p+1)}(\mathcal{L}) & \xrightarrow{\partial} & \mathrm{cône}(\beta : \theta^{(p)}(\mathcal{L}) \to \mathrm{Sym}^p(\mathcal{L}))[-1] \\ \downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L})} & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathcal{L})} \\ \mathcal{L}_{\mathfrak{g}} & \xrightarrow{d} & \Omega^1(\mathcal{L}_{\mathfrak{g}}) \end{array} .$$

On homologies, this yields a commutative diagram of $\mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$-equivariant maps

$$\begin{array}{ccc} \mathrm{HC}^{(p+1)}_\bullet(\mathfrak{a}) & \xrightarrow{B} & \mathrm{HH}^{(p)}_{\bullet+1}(\mathfrak{a}) \\ \downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a})} & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a})} \\ \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g}) & \xrightarrow{d} & \mathrm{H}_\bullet[\Omega^1(\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}))] \end{array} .$$

This completes the proof of Theorem 4.2.

4.3. The associative case. For this section, let $C \in \mathtt{DGC}_{k/k}$ be a d-cyclic coassociative (coaugmented, conilpotent) DG coalgebra Koszul dual to $A \in \mathtt{Alg}_{k/k}$. In particular, we do not assume that C is cocommutative. Throughout this section, let $R := \mathbf{\Omega}(C)$ and let $V := \bar{C}[-1]$. Thus, as graded algebras, $R \cong T_k V$.

In this setting, A acquires a derived Poisson structure, whence $\overline{\mathrm{HC}}_\bullet(A)$ has a graded Lie bracket of degree. Further, $\overline{\mathrm{HH}}_\bullet(A)$ becomes a graded Lie module over $\overline{\mathrm{HC}}_\bullet(A)$ (see Theorem 3.1. By [1, Theorem 9], there is a unique graded Poisson structure on $\mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$ such that $\mathrm{Tr}_n : \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$ is a graded Lie algebra homomorphism. As in Convention 3.1.1, all Lie brackets and Lie actions in this section are of homological degree $d+2$.

Recall that one has the Van den Bergh functor

$$(-)_n : \mathtt{DGBimod}(R) \to \mathtt{DG\,Mod}(R_n)\,, \qquad P \mapsto P \otimes_{R^e} \mathcal{M}_n(R_n)\ .$$

There is an adjunction

$$(-)_n : \mathtt{DGBimod}(R) \rightleftarrows \mathtt{DG\,Mod}(R_n) : \mathcal{M}_n(-)\ . \tag{4.21}$$

We continue to denote the unit of this adjunction by π_n. Explicitly, for any DG R-bimodule P, the map $\pi_n : P \to \mathcal{M}_n(P_n)$ is give by the formula

$$\pi_n(p) \,=\, \sum_{1\leqslant i,j\leqslant n} (p \otimes_{R^e} e_{ji}) \otimes_k e_{ij}\,,$$

where the e_{ij}'s are the elementary matrices and each e_{ji} above is viewed as an element of $\mathcal{M}_n(R_n)$. The composition $\mathrm{Tr}\circ\pi_n : P \to P_n$ induces a map of complexes

$$\mathrm{Tr}_n : P_\natural := P/[R,P] \to P_n\ .$$

By [3, Example 5.1], $(\Omega^1 R)_n \cong \Omega^1 R_n$. On homologies, the map $\mathrm{Tr}_n : \Omega^1 R_\natural \to \Omega^1 R_n$ gives

$$\mathrm{Tr}_n : \overline{\mathrm{HH}}_{\bullet+1}(A) \to \mathrm{HR}_\bullet(\Omega^1 A, n)\ ,$$

where $\mathrm{HR}_\bullet(\Omega^1 A, n)$ denotes the representation homology of the A-bimodule $\Omega^1 A$ (see [3, Thm. 5.1(b)] for the definition). By [3, Thm. 5.2], there is a commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{HC}}_\bullet(A) & \xrightarrow{B} & \overline{\mathrm{HH}}_{\bullet+1}(A) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}_n} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_n} \\ \mathrm{HR}_\bullet(A,n) & \xrightarrow{B_n} & \mathrm{HR}_\bullet(\Omega^1 A, n) \end{array}\ , \tag{4.22}$$

where B denotes the Connes differential and B_n denotes the map induced on homologies by the universal derivation $R_n \to \Omega^1 R_n$. The following theorem, which is analogous to [5, Theorem 5.1], is a homological generalization of a special case of [20, Prop. 7.5.1, Prop. 7.5.2 and Prop. 7.7.2].

Theorem 4.3. *There is a DG Poisson structure on* $\mathrm{HR}_\bullet(A,n)$ *such that* $\mathrm{Tr}_n : \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_\bullet(A,n)$ *is a graded Lie algebra homomorphism.*

Remark. Theorem 4.3 implies that the (unique) DG Poisson structure on $\mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$ in [1, Theorem 9] extends to all of $\mathrm{HR}_\bullet(A,n)$.

Proof. Let $\bar{\mathcal{M}}_n^*(C)$ denote the coalgebra $k\oplus\mathcal{M}_n^*(k)\otimes\bar{C}$, so that $\overline{\bar{\mathcal{M}}_n^*(C)} = \mathcal{M}_n^*(k)\otimes\bar{C}$. Let $\mathfrak{gl}_n^*(C)$ (resp., $\mathfrak{gl}_n^*(\bar{C})$) denote $\bar{\mathcal{M}}_n^*(C)$ (resp., $\mathcal{M}_n^*(k)\otimes\bar{C}$) viewed as a Lie coalgebra. Note that $\mathcal{M}_n(k)$ has a nondegenerate cyclic pairing given by

$$\langle M,N\rangle \;:=\; \mathrm{Tr}(MN)\;.$$

Identifying $\mathcal{M}_n(k)$ with $\mathcal{M}_n^*(k)$ via the above pairing, we obtain a cyclic pairing on $\mathcal{M}_n^*(k)$. Tensoring this pairing with the pairing on $\bar{C}$, we obtain a n-cyclic pairing on $\overline{\bar{\mathcal{M}}_n^*(C)}$, which can be extended to a n-cyclic pairing on $\bar{\mathcal{M}}_n^*(C)$. It is not difficult to verify that the cyclic pairing on $\overline{\bar{\mathcal{M}}_n^*(C)}$ is cyclic as a pairing on $\mathfrak{gl}_n^*(\bar{C})$. By [5, Lemma 5.1], the Chevalley-Eilenberg algebra $\mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C});k)$ acquires a DG Poisson structure. Since $R_n \;\cong\; \mathcal{C}^c(\mathfrak{gl}_n^*(\bar{C});k)$ by [2, Thm. 3.1], R_n has a DG Poisson structure as well. That $\mathrm{Tr}_n : R_\natural \to R_n$ is a DG Lie algebra homomorphism is an immediate consequence of Proposition 4.3 below. The desired result then follows on homology. $\square$

Proposition 4.3. *The universal representation $\pi_n : R \to \mathcal{M}_n(R_n)$ is $R_\natural$-equivariant.*

Proof. Recall that the $R_\natural$ action on $\mathcal{M}_n(R_n)$ is obtained by composing the Poisson action of R_n on $\mathcal{M}_n(R_n)$ with the map $\mathrm{Tr}_n : R_\natural \to R_n$. Since R_n acts on $\mathcal{M}_n(R_n)$ by derivations, so does $R_\natural$. Since π_n is a homomorphism of DG algebras and since $R_\natural$ acts on R by derivations, it suffices to verify that for any $u \in V$ and for $\alpha := (v_1,\ldots,v_k) \in R$ for $v_1,\ldots,v_k \in V$,

$$\pi_n(\{\alpha,u\}) \;=\; \{\mathrm{Tr}_n(\overline{\alpha}),\pi_n(u)\}\;. \tag{4.23}$$

Here, $\overline{\alpha}$ denotes the image of α in $R_\natural$ under the canonical projection. Indeed, the restriction of the $R_\natural$-action on R to the subspace V is given by the composite map

$$R_\natural\otimes V \xrightarrow{\partial\otimes\mathrm{Id}} R\otimes V\otimes V \xrightarrow{\mathrm{Id}\otimes\langle -,-\rangle} R\;.$$

Similarly, the restriction of the Poisson action of R_n on the subspace $\mathcal{M}_n^*(V)$ of R_n is given by the composite map

$$\begin{aligned} R_n\otimes\mathcal{M}_n^*(V) \xrightarrow{d\otimes\mathrm{Id}} \Omega^1R_n\otimes\mathcal{M}_n^*(V)& \\ \cong\; R_n\otimes\mathcal{M}_n^*(V)\otimes\mathcal{M}_n^*(V) &\xrightarrow{\mathrm{Id}\otimes\langle -,-\rangle} R_n\;. \end{aligned}$$

By [3, Lemma 5.2], the following diagram commutes.

$$\begin{array}{ccc} R_\natural & \xrightarrow{\partial} & \Omega^1 R_\natural \\ \downarrow{\scriptstyle \mathrm{Tr}_n} & & \downarrow{\scriptstyle \mathrm{Tr}_n} \\ R_n & \xrightarrow{d} & \Omega^1 R_n \end{array} .$$

It follows that the following diagram commutes.

$$\begin{array}{ccc} R_\natural \otimes V & \xrightarrow{\partial \otimes \mathrm{Id}} & \Omega^1 R_\natural \otimes V \\ \downarrow{\scriptstyle \mathrm{Tr}_n \otimes \theta_n} & & \downarrow{\scriptstyle \mathrm{Tr}_n \otimes \theta_n} \\ R_n \otimes \mathcal{M}_n^*(k) \otimes V \otimes \mathcal{M}_n(k) & \xrightarrow{d \otimes \mathrm{Id}} & \Omega^1 R_n \otimes \mathcal{M}_n^*(k) \otimes V \otimes \mathcal{M}_n(k) \end{array} ,$$

where $\theta : V \to \mathcal{M}_n^*(k) \otimes V \otimes \mathcal{M}_n(k)$ denotes the restriction of π_n to V. Explicitly, for $v \in V$,

$$\theta_n(v) = \sum_{1 \leqslant i,j \leqslant n} e_{ij}^* \otimes v \otimes e_{ij} ,$$

where $\{e_{ij}^*\}$ denotes the basis of $\mathcal{M}_n^*(k)$ dual to the basis comprising the elementary matrices. (4.23) therefore follows once we check that the following diagram commutes:

$$\begin{array}{ccc} \Omega^1 R_\natural \otimes V & & \\ \downarrow{\scriptstyle \cong} & & \\ R \otimes V \otimes V & \xrightarrow{\mathrm{Id} \otimes \langle -,- \rangle} & R \\ \downarrow{\scriptstyle \mathrm{Tr}_n \otimes \theta_n} & & \downarrow{\scriptstyle \pi_n} \\ \Omega^1 R_n \otimes \mathcal{M}_n^*(k) \otimes V \otimes \mathcal{M}_n(k) & & R_n \otimes \mathcal{M}_n(k) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{Id}} \\ R_n \otimes (\mathcal{M}_n^*(k) \otimes V)^{\otimes 2} \otimes \mathcal{M}_n(k) & \xrightarrow{\mathrm{Id} \otimes \langle -,- \rangle \otimes \mathrm{Id}} & R_n \otimes \mathcal{M}_n(k) \end{array} . \tag{4.24}$$

For $r \in R$ let $r_{ij} \in R_n$ denote the coefficient of the elementary matrix e_{ij} in $\pi_n(r)$. Then, for $v \in V$,

$$\mathrm{Tr}_n(r \otimes v) = \sum_{1 \leqslant i,j \leqslant n} r_{ij} \otimes v_{ji}$$

where $\Omega^1 R$ (resp., $\Omega^1 R_n$) is identified with $R \otimes V$ (resp., $R_n \otimes \mathcal{M}_n^*(k) \otimes V$). Hence, for all $w \in V$,

$$\mathrm{Tr}_n \otimes \theta_n(r \otimes v \otimes w) = \sum_{1 \leqslant i,j,k,l \leqslant n} r_{ij} \otimes v_{ji} \otimes w_{kl} \otimes e_{kl} \ .$$

Note that $\langle v_{ji}, w_{kl} \rangle = \langle e_{ji}^*, e_{kl}^* \rangle \langle v, w \rangle = \delta_{ik}\delta_{jl}\langle v, w \rangle$. Hence,

$$\sum_{1 \leqslant i,j \leqslant n} \langle v, w \rangle r_{ij} \otimes e_{ij} = \sum_{1 \leqslant i,j,k,l \leqslant n} \langle v_{ji}, w_{kl} \rangle r_{ij} \otimes e_{kl} \ .$$

This proves that the diagram (4.24) commutes as desired. □

The following theorem was stated without proof in [7] (see *loc. cit.*, Theorem 1.3).

Theorem 4.4. *There is a commutative diagram of* $\overline{\mathrm{HC}}_\bullet(A)$*-module homomorphisms*

$$\begin{array}{ccc} \overline{\mathrm{HC}}_\bullet(A) & \xrightarrow{B} & \overline{\mathrm{HH}}_{\bullet+1}(A) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}_n} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_n} \\ \mathrm{HR}_\bullet(A, n) & \xrightarrow{B_n} & \mathrm{HR}_\bullet(\Omega^1 A, n) \end{array} \ .$$

Proof. Since $\mathrm{Tr}_n \circ \natural = \mathrm{Tr} \circ \pi_n$, where $\natural : \Omega^1 R \to \Omega^1 R_\natural$ is the canonical projection (which is $R_\natural$-equivariant), the $R_\natural$-equivariance of Tr_n would follow once it is verified that $\pi_n : \Omega^1 R \to \mathcal{M}_n(\Omega^1 R_n)$ is $R_\natural$-equivariant. By Lemma 4.2, for all $\alpha \in R_\natural, a, b \in R$ and for all $m \in \Omega^1 R$,

$$\{\alpha, a \cdot m \cdot b\} = \{\alpha, a\} \cdot m \cdot b \pm a \cdot \{\alpha, m\} \cdot b \pm a \cdot m \cdot \{\alpha, b\} \, ,$$

where the signs come from the Koszul sign rule. Similarly, for $x \in R_n$, $u, v \in \mathcal{M}_n(R_n)$ and $w \in \mathcal{M}_n(\Omega^1 R_n)$,

$$\{x, u \cdot w \cdot v\} = \{x, u\} \cdot w \cdot v \pm u \cdot \{x, w\} \cdot v \pm u \cdot w \cdot \{x, v\} \ .$$

Since the $R_\natural$ action on R_n (resp., $\Omega^1 R_n$) factors through the Poisson action of R_n on R_n (resp., $\Omega^1 R_n$), since $\pi_n : \Omega^1 R \to \mathcal{M}_n(\Omega^1 R_n)$ is an R-bimodule homomorphism, and by Proposition 4.3, the $R_\natural$-equivariance of $\pi_n : \Omega^1 R \to \mathcal{M}_n(R_n)$ would follow once we verify that for all $\alpha \in R_\natural$ and for all $u \in V$,

$$\pi_n(\{\alpha, 1 \otimes u \otimes 1\}) = \{\mathrm{Tr}_n(\alpha), \pi_n(1 \otimes u \otimes 1)\} \ . \tag{4.25}$$

A trivial modification of the proof of Lemma 4.3 (see also [3, Sec. 5]) shows that the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\quad d \quad} & \Omega^1 R \\ \Big\downarrow{\scriptstyle \pi_n} & & \Big\downarrow{\scriptstyle \pi_n} \\ \mathcal{M}_n(R_n) & \xrightarrow{d \otimes \mathrm{Id}_{\mathcal{M}_n(k)}} & \mathcal{M}_n(\Omega^1 R_n) \end{array} \tag{4.26}$$

It follows that $\pi_n(1 \otimes u \otimes 1) = (d \otimes \mathrm{Id}_{\mathcal{M}_n(k)})[\pi_n(u)]$, where u is viewed on the right hand side as an element of $V \subset R$. Therefore,

$$\begin{aligned} \{\mathrm{Tr}_n(\alpha), \pi_n(1 \otimes u \otimes 1)\} &= \{\mathrm{Tr}_n(\alpha), (d \otimes \mathrm{Id}_{\mathcal{M}_n(k)})[\pi_n(u)]\} \\ &= (d \otimes \mathrm{Id}_{\mathcal{M}_n(k)})\{\mathrm{Tr}_n(\alpha), \pi_n(u)\} \\ &= (d \otimes \mathrm{Id}_{\mathcal{M}_n(k)})[\pi_n(\{\alpha, u\})] \qquad (\text{ by Prop. 4.3}) \\ &= \pi_n[d(\{\alpha, u\})] \qquad (\text{ by (4.26)}) \\ &= \pi_n(\{\alpha, 1 \otimes u \otimes 1\}) \qquad (\text{ by Lemma 4.2}) \,. \end{aligned}$$

The second equality above is because $d : R_n \to \Omega^1 R_n$ is equivariant with respect to the Poisson action of R_n. This verifies (4.25), completing the proof of the desired theorem. □

5. An operadic generalization

Throughout this section, let $\mathcal{P}$ denote a finitely generated binary quadratic operad and let $\mathcal{Q} = \mathcal{P}^!$ denote its (quadratic) Koszul dual. Let `DGPA` denote the category of DG $\mathcal{P}$-algebras and let `DGQC` denote the category of conilpotent DG $\mathcal{Q}$-coalgebras.

5.1. Invariant bilinear forms.

5.1.1. Definitions. A symmetric bilinear form B on $A \in$ `DGPA` is called *invariant* if for all $\mu \in \mathcal{P}(2)$,

$$B(\mu(a,\, b),\, c) = (-1)^{|a||\mu|} B(a,\, \mu(b,\, c)) \quad \text{ for all } a,\, b,\, c \in A\,.$$

Recall that if $\mathcal{P}$ is a *cyclic* operad, there is an $\mathbb{S}_{n+1}$ action on $\mathcal{P}(n)$ for each $n \geq 0$. There is also a notion of invariant bilinear form on algebra over a cyclic operad:

Definition 2. [12, Definition 4.1] Let $\mathcal{P}$ be a cyclic DG operad and let A be a $\mathcal{P}$-algebra. A bilinear form B (with values in a k-vector space V) is invariant if for all $n \geqslant 0$, the map $B_n : \mathcal{P}(n) \otimes A^{\otimes(n+1)} \to V$ defined by the formula

$$B_n(\mu \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = B(\mu(x_1,\, \cdots,\, x_n),\, x_{n+1})$$

is invariant under the action of $\mathbb{S}_{n+1}$ on $\mathcal{P}(n) \otimes A^{\otimes(n+1)}$.

If $\mathcal{P}$ is a cyclic binary quadratic operad, the two notions of invariant bilinear forms agree with each other (see [12, Proposition 4.3]).

5.1.2. Universal invariant bilinear form. Let $A_\natural$ denote the target of the universal invariant bilinear form on A: this is equal to the quotient of $A \otimes A$ by the subcomplex spanned by the images of the maps

$$A \otimes A \otimes A \xrightarrow{\mu \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu} A \otimes A \,,$$

where μ runs over all (homogeneous) elements of $\mathcal{P}(2)$.

If $\mathcal{P}$ is cyclic binary quadratic, then $A_\natural$ is equal to the quotient of $A \otimes A$ by the subcomplex spanned by the images of the maps (see [12, Sec. 4.7]):

$$\mathcal{P}(n) \otimes A^{\otimes(n+1)} \xrightarrow{1-\sigma} \mathcal{P}(n) \otimes A^{\otimes(n+1)} \longrightarrow A \otimes A$$

for all $n \geqslant 1$ and $\sigma \in \mathbb{S}_{n+1}$, where the last arrow is given by

$$\mu \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \mapsto \mu(x_1, \cdots, x_n) \otimes x_{n+1} \,,$$

and σ acts diagonally. This quotient is denoted in [12] by $\lambda(A)$. In particular, taking $\mu = \mathrm{Id} \in \mathcal{P}(1)$, we see that the canonical projection from $A \otimes A$ to $A_\natural$ is symmetric. Moreover, if $R = T_{\mathcal{P}}V$ the free $\mathcal{P}$-algebra generated by V,

Proposition 5.1 (see [12, Proposition 4.9])**.** *There is a natural isomorphism of chain complexes*

$$R_\natural \cong \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{(n+1)} \,.$$

Proof. If $\mu \in \mathcal{P}(n)$ and $\nu \in \mathcal{P}(m)$, and v_i, $w_j \in V$, the image of the element

$$\mu(v_1, \cdots, v_n) \otimes \nu(w_1, \cdots, w_m) \in A_\natural$$

is equal to the image of

$$\pm(\tau^{-1}\mu)(v_2, \cdots, v_n, \nu(w_1, \cdots, w_m)) \otimes v_1$$

which is the same as

$$\pm((\tau^{-1}\mu) \circ_n \nu)(v_2, \cdots, v_n, w_1, \cdots, w_m) \otimes v_1 \,.$$

Thus, one can see that $R_\natural$ is a quotient of

$$T_{\mathcal{P}}V \otimes V \cong \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^n \otimes V \,.$$

Using once more the coinvariance under the action of the cyclic group $\mathbb{Z}_{n+1}$, it can be shown that there is an isomorphism

$$R_\natural \cong \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{(n+1)} \,.$$

□

Remark 5.1. By the above proof, every element in $R_{\natural}$ has a representative in the form of $(\mu \otimes v_1 \otimes \cdots \otimes v_n) \otimes w$ for some $n \geqslant 0$ and $\mu \in \mathcal{P}(n)$, where v_i, $w \in V$.

The category $\mathtt{DGPA}$ is a model category where the fibrations are the degreewise surjections and the weak equivalences are the quasi-isomorphisms. By [2, Thm. A.5], the functor $(-)_{\natural} : \mathtt{DGPA} \to \mathtt{Com}_k$, $A \mapsto A_{\natural}$ has a (total) left derived functor $\mathbb{L}(-)_{\natural} : \mathtt{Ho}(\mathtt{DGPA}) \to \mathtt{Ho}(\mathtt{Com}_k)$, $A \mapsto R_{\natural}$, where $R \xrightarrow{\sim} A$ is any cofibrant resolution of A in $\mathtt{DGPA}$. We denote the homology $\mathrm{H}_\bullet[\mathbb{L}A_{\natural}]$ by $\mathrm{HC}_\bullet(\mathcal{P}, A)$ and call it the *$\mathcal{P}$-cyclic homology* of A. When $\mathcal{P}$ is the Lie operad and when $A = \mathfrak{a}$, $\mathrm{HC}_\bullet(\mathcal{P}, A) = \mathrm{HC}^{(2)}_\bullet(\mathfrak{a})$.

5.1.3. Invariant 2-tensor. Dually, if $\mathfrak{C}$ is a DG $\mathcal{P}$-coalgebra, a (homogeneous) 2-tensor α is called *invariant* if it is a (homogeneous) element in $\mathfrak{C} \otimes \mathfrak{C}$ such that for all $\mu \in \mathcal{P}(2)$, the composite map of complexes

$$k[|\alpha|] \xrightarrow{\alpha} \mathfrak{C} \otimes \mathfrak{C} \xrightarrow{\mu \otimes \mathrm{Id} - \mathrm{Id} \otimes \mu} \mathfrak{C} \otimes \mathfrak{C} \otimes \mathfrak{C}$$

vanishes. We denote the subcomplex of invariant 2-tensors on $\mathfrak{C}$ by $\mathfrak{C}^{\natural}$. Notice that a non-degenerate invariant bilinear form on a DG $\mathcal{P}$-algebra A induces a invariant 2-tensor on its linear dual DG $\mathcal{P}$-coalgebra A^*.

5.2. Derived representation schemes. Let $\mathfrak{C}$ be a DG $\mathcal{P}$-coalgebra. Note that the complex $\mathbf{Hom}(\mathfrak{C}, \bar{B})$ naturally acquires the structure of a DG $\mathcal{P}$-algebra for any $B \in \mathtt{DGCA}_{k/k}$. This gives a functor

$$\mathbf{Hom}_*(\mathfrak{C}, -) : \mathtt{DGCA}_{k/k} \to \mathtt{DGPA}\,, \qquad B \mapsto \mathbf{Hom}(\mathfrak{C}, \bar{B})\ .$$

By [2, Prop. A.1], the functor $\mathbf{Hom}_*(\mathfrak{C}, -)$ has a left adjoint $\mathfrak{C} \ltimes - : \mathtt{DGPA} \to \mathtt{DGCA}_{k/k}$. For $A \in \mathtt{DGPA}$, let $\pi_A : A \to \mathbf{Hom}(\mathfrak{C}, \mathfrak{C} \ltimes A)$ denote the composite map

$$A \longrightarrow \mathbf{Hom}_*(\mathfrak{C}, \mathfrak{C} \ltimes A) \hookrightarrow \mathbf{Hom}(\mathfrak{C}, \mathfrak{C} \ltimes A)\,,$$

where the first arrow is the unit of the adjunction described above. By [2, Thm. A.2], the functor $\mathfrak{C} \ltimes -$ has a (total) left derived functor $\mathfrak{C} \ltimes^{\mathbb{L}} - : \mathtt{Ho}(\mathtt{DGPA}) \to \mathtt{Ho}(\mathtt{DGCA}_{k/k})$. For $A \in \mathtt{DGPA}$, we call $\mathfrak{C} \ltimes^{\mathbb{L}} A$ the *derived representation scheme of A over $\mathfrak{C}$* and denote it by $\mathrm{DRep}_{\mathfrak{C}}(A)$. The *representation homology* of A over $\mathfrak{C}$ is the homology

$$\mathrm{HR}_\bullet(A, \mathfrak{C}) := \mathrm{H}_\bullet[\mathrm{DRep}_{\mathfrak{C}}(A)]\ .$$

If $\mathfrak{C}$ is finite dimensional, $\mathcal{S} := \mathfrak{C}^*$ is a $\mathcal{P}$-algebra and we write $\mathrm{HR}_\bullet(A, \mathcal{S})$ (resp., $\mathrm{DRep}_{\mathcal{S}}(A)$) for $\mathrm{HR}_\bullet(A, \mathfrak{C})$ (resp., $\mathrm{DRep}_{\mathfrak{C}}(A)$).

5.2.1. Traces. Assume that $\mathfrak{C}$ is equipped with a degree 0 invariant 2-tensor

$$\mathrm{coTr} : k \to \mathfrak{C}^{\natural}\ .$$

Abusing notation, let $\pi_{A\otimes A}$ denote the composite map

$$A\otimes A \xrightarrow{\pi_A^{\otimes 2}} \mathbf{Hom}(\mathfrak{C},\mathfrak{C}\ltimes A)^{\otimes 2} \cong \mathbf{Hom}(\mathfrak{C}\otimes\mathfrak{C},(\mathfrak{C}\ltimes A)^{\otimes 2}) \longrightarrow \mathbf{Hom}(\mathfrak{C}\otimes\mathfrak{C},\mathfrak{C}\ltimes A)\,,$$

where the last arrow is induced by the product on $\mathfrak{C}\ltimes A$. The map of complexes

$$A\otimes A \xrightarrow{\pi_{A\otimes A}} \mathbf{Hom}(\mathfrak{C}\otimes\mathfrak{C},\mathfrak{C}\ltimes A) \longrightarrow \mathbf{Hom}(\mathfrak{C}^\natural,\mathfrak{C}\ltimes A)$$

induced by the inclusion $\mathfrak{C}^\natural \hookrightarrow \mathfrak{C}\otimes\mathfrak{C}$ factors through $A_\natural$, giving a map of complexes

$$A_\natural \to \mathbf{Hom}(\mathfrak{C}^\natural,\mathfrak{C}\ltimes A)\ .$$

Composing this with the map $\mathbf{Hom}(\mathfrak{C}^\natural,\mathfrak{C}\ltimes A)\to\mathbf{Hom}(k,\mathfrak{C}\ltimes A)$ induced by coTr, we obtain map

$$\mathrm{Tr}\ :\ A_\natural \to \mathfrak{C}\ltimes A\ .$$

By construction, Tr gives a natural transformation of functors $(-)_\natural\to\mathfrak{C}\ltimes -$ (which, of course, depends on the choice of coTr). As a result, this gives a natural transformation of *derived functors*

$$\mathbb{L}(-)_\natural \to \mathfrak{C}\ltimes^{\mathbb{L}} - \ :\ \mathtt{Ho}(\mathtt{DGPA})\to\mathtt{Ho}(\mathtt{Com}_k)\ .$$

Applying this to $A\in\mathtt{DGPA}$ and taking homologies, we obtain maps of graded vector spaces

$$\mathrm{Tr}\ :\ \mathrm{HC}_\bullet(\mathcal{P},A)\to\mathrm{HR}_\bullet(A,\mathfrak{C})\ .$$

If $\mathcal{P}$ is the Lie operad, $\mathfrak{C}=\mathfrak{g}^*$ for $\mathfrak{g}$ semisimple and coTr is the Killing form, then the above construction recovers the Drinfeld trace associated with the Killing form.

5.2.2. Poisson structures on representation homology. We say that $\mathfrak{C}$ is *cyclic* if it is equipped with a symmetric bilinear form $\langle -,-\rangle\ :\ \mathfrak{C}\times\mathfrak{C}\to k$ such that for all $\mu\in\mathcal{P}(2)$, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{C}\otimes\mathfrak{C} & \xrightarrow{\mu\otimes\mathrm{Id}} & \mathfrak{C}\otimes\mathfrak{C}\otimes\mathfrak{C} \\ {\scriptstyle \mathrm{Id}\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes\langle -,-\rangle} \\ \mathfrak{C}\otimes\mathfrak{C}\otimes\mathfrak{C} & \xrightarrow{\langle -,-\rangle\otimes\mathrm{Id}} & \mathfrak{C} \end{array} \tag{5.1}$$

Remark 5.2. If the pairing $\langle -,-\rangle$ is also non-degenerate, then it induces an isomorphism between $\mathfrak{C}$ and its linear dual $\mathfrak{C}^*$. In this case, it also induces an invariant bilinear form on $\mathfrak{C}^*$ under the above isomorphism.

Let $\mathbf{\Omega}_{\mathcal{Q}}\ :\ \mathtt{DGQC}\to\mathtt{DGPA}$ denote the cobar construction (see [18, Sec. 11.2.5]). For $C\in\mathtt{DGQC}$, the underlying graded algebra of $\mathbf{\Omega}_{\mathcal{Q}}(C)$ is the free $\mathcal{P}$-algebra generated by $C[-1]$, and the differential on $\mathbf{\Omega}_{\mathcal{Q}}(C)$ is the sum of two degree -1 derivations, one induced by the differential on C and the

other induced by operations from $\mathcal{Q}$. We say that $C \in \mathtt{DGQC}$ is *Koszul dual* to $A \in \mathtt{DGPA}$ if there is a quasi-isomorphism $\mathbf{\Omega}_{\mathcal{Q}}(C) \xrightarrow{\sim} A$.

Proposition 5.2. *Let $A \in \mathtt{DGPA}$ be Koszul dual to $C \in \mathtt{DGQC}$. If C is cyclic, and if $\mathfrak{C} \in \mathtt{DGPA}$ is cyclic, then $\mathrm{DRep}_{\mathfrak{C}}(A)$ acquires a derived Poisson structure. As a consequence, $\mathrm{HR}_\bullet(A, \mathfrak{C})$ acquires a graded Poisson structure.*

Proof. Note that $\mathfrak{C} \otimes C$ is a conilpotent DG coalgebra over the Hadamard product $P \otimes_{\mathrm{H}} Q$. Further, it is not difficult to verify that if $\mathfrak{C}$ (resp., C) is a cyclic DG $\mathcal{P}$-coalgebra (resp., cyclic DG $\mathcal{Q}$-coalgebra), then $\mathfrak{C} \otimes C$ is cyclic as a DG coalgebra over $\mathcal{P} \otimes_{\mathrm{H}} \mathcal{Q}$. By [18, Propositon 7.6.5], there is a morphism of operads $\mathtt{Lie} \to \mathcal{P} \otimes_{\mathrm{H}} \mathcal{Q}$. Let $\mathcal{L}ie^c(\mathfrak{C} \otimes C)$ denote $\mathfrak{C} \otimes C$ viewed as a Lie coalgebra. It follows that $\mathcal{L}ie^c(\mathfrak{C} \otimes C)$ is a cyclic Lie coalgebra. By [5, Lemma 5.1], $\mathbf{\Omega}_{\mathtt{Lie}}[\mathcal{L}ie^c(\mathfrak{C} \otimes C)]$ acquires a DG Poisson structure. Since $\mathbf{\Omega}_{\mathtt{Lie}}[\mathcal{L}ie^c(\mathfrak{C} \otimes C)]$ is a cofibrant representative of $\mathrm{DRep}_{\mathfrak{C}}(A)$ by [2, Thm. A.4], the desired proposition follows. □

5.3. Derived Poisson structures on algebras over operads. Let A be a DG $\mathcal{P}$-algebra. Let $\mathrm{Der}_{\mathcal{P}}(A)$ denote the DG Lie algebra of derivations from A to itself. Then A and therefore, $A \otimes A$ are DG Lie modules over $\mathrm{Der}_{\mathcal{P}}(A)$. It is not difficult to verify that the $\mathrm{Der}_{\mathcal{P}}(A)$-module structure on $A \otimes A$ descends to a $\mathrm{Der}_{\mathcal{P}}(A)$-module structure on $A_\natural$. Following Crawley-Boevey [8], we define a *Poisson structure* on A to be a Lie bracket $\{-,-\}$ on $A_\natural$ such that for all $\alpha \in A_\natural$, the operator $\{\alpha, -\} : A_\natural \to A_\natural$ is induced by a derivation $\partial_\alpha \in \mathrm{Der}_{\mathcal{P}}(A)$. A *derived Poisson structure* on A is defined to be a Poisson structure on some cofibrant resolution of A.

For the rest of this section, we further assume that the operad $\mathcal{P}$ is cyclic. Recall that in this case, there is an action of the cyclic permutation τ of order $n+1$ on $\mathcal{P}(n)$ for each $n \geqslant 0$. By [18, 13.14.6], τ may be thought of as changing the last input into the output and the output into the first input. The following formulas relate the τ-action with the partial compositions of $\mathcal{P}$:

$$\begin{aligned} \tau(\mu \circ_i \nu) &= \tau(\mu) \circ_{i+1} \nu, \quad 1 \leqslant i < n\,, \\ \tau(\mu \circ_n \nu) &= \tau(\nu) \circ_1 \tau(\mu)\,, \end{aligned}$$

where $\mu \in \mathcal{P}(n)$ and $\nu \in \mathcal{P}(m)$. The reader may refer to [18, Fig. 13.2] for the illustration of the formulas above. For fixed n, we write $N = \sum_{i=0}^{n} \tau^i$.

Let A be Koszul dual to a *cyclic* (conilpotent) DG $\mathcal{Q}$-coalgebra C. Let $\mathfrak{C}$ be a finite dimensional $\mathcal{P}$-coalgebra in degree 0 equipped with an invariant 2-tensor $\mathrm{coTr} : k \to \mathfrak{C}^\natural$. Let $\mathcal{S} := \mathfrak{C}^*$. Let $\mathrm{Tr} : \mathcal{S}_\natural \to k$ denote the dual of coTr. Assume that the pairing

$$\mathcal{S} \otimes \mathcal{S} \longrightarrow \mathcal{S}_\natural \xrightarrow{\mathrm{Tr}} k$$

is non-degenerate. Since the above pairing is cyclic on $\mathcal{S}$ by definition, it induces a cyclic pairing on $\mathfrak{C}$ as well. By Proposition 5.2, $\mathrm{HR}_\bullet(A, \mathcal{S}) :=$

$\mathrm{HR}_\bullet(A, \mathfrak{C})$ acquires a graded Poisson structure. The following generalization of Theorem 2.1 and Theorem 3.3 is the main result of this section.

Theorem 5.1. *There is a derived Poisson structure on A such that* $\mathrm{Tr} : \mathrm{HC}_\bullet(\mathcal{P}, A) \to \mathrm{HR}_\bullet(A, \mathbb{S})$ *is a graded Lie algebra homomorphism.*

5.3.1. Proof of Theorem 5.1. Let $V := C[-1]$. We shall denote $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ by $(v_1, \cdots, v_n)$ for notational brevity. There is an isomorphism of $\mathcal{P}$-algebras $\mathbf{\Omega}_{\mathcal{Q}}(C) \cong T_{\mathcal{P}}V$, where $T_{\mathcal{P}}V$ is the free $\mathcal{P}$-algebra generated by V. For notational brevity, let $R := T_{\mathcal{P}}V$. We define the *cyclic derivative* $\bar{\partial} : R_\natural \to R \otimes V$ as follows. On the direct summand $\mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{(n+1)}$ of $R_\natural$, it is given by

$$\mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{(n+1)} \xrightarrow{N \cdot (-)} \mathcal{P}(n) \otimes_{\mathbb{S}_{n+1}} V^{(n+1)} \longrightarrow \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^n \otimes V\,, \tag{5.2}$$

where N acts diagonally and where the last arrow is the obvious map that permutes no factors. Explicitly, given $\mu \otimes v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \in R_\natural$,

$$\begin{aligned} \bar{\partial}(\mu \otimes (v_1, \cdots, v_{n+1}) &= \sum_{i=1}^{n+1} (\tau^{n+1-i}\mu) \otimes \tau^{n+1-i}(v_1, \cdots, v_{n+1}) \qquad (5.3)\\ &= \sum_{i=1}^{n+1} \pm(\tau^{n+1-i}\mu) \otimes (v_{i+1}, \cdots, v_{n+1}, v_1, \cdots, v_i) \\ &= \sum_{i=1}^{n+1} \pm(\tau^{n+1-i}\mu) \otimes (v_{i+1}, \cdots, v_{n+1}, v_1, \cdots, v_i)\ . \end{aligned}$$

Note that the cyclic pairing on C gives a skew-symmetric pairing on V. Following [13, Sec. 6], define a bilinear bracket $\{-, -\} : R_\natural \otimes R_\natural \to R_\natural$ as the following composition

$$\begin{aligned} R_\natural \otimes R_\natural \xrightarrow{\bar{\partial} \otimes \bar{\partial}} (R \otimes V)^{\otimes 2} \xrightarrow{\sim} \qquad (5.4)\\ R \otimes R \otimes V \otimes V \xrightarrow{\mathrm{Id}_{R\otimes R} \otimes \langle -, - \rangle} R \otimes R \longrightarrow R_\natural\,, \end{aligned}$$

where the last arrow is the canonical projection from $R \otimes R$ to $R_\natural$. By Lemma 5.1 below, the above bracket defines a Poisson structure on $\mathbf{\Omega}_{\mathcal{Q}}(C)$: the compatibility of the Lie bracket on $R_\natural$ constructed therein with the differential on $\mathbf{\Omega}_{\mathcal{Q}}(C)_\natural$ follows from the cyclicity of C. By definition, this gives A a derived Poisson structure.

Lemma 5.1. *The bracket* (5.4) *defines a Poisson structure on R. In particular, it is a Lie bracket on $R_\natural$.*

Proof. Since the canonical projection $R\otimes R \twoheadrightarrow R_\natural$ is symmetric, the bracket $\{-,-\}$ on $R_\natural$ is skew-symmetric. Define an action[4] of $R_\natural$ on R via

$$\{\alpha,\, x\} = \sum_{i=1}^{n+1}\sum_{j=1}^{m} \pm\langle v_i,\, w_j\rangle(\nu \circ_j (\tau^{n+1-i}\mu))\otimes$$

$$(w_1,\cdots,w_{j-1},v_{i+1},\cdots,v_{i-1},w_{j+1},\cdots,w_m)\,,$$

where $\alpha = \mu\otimes(v_1,\cdots,v_{n+1}) \in R_\natural$ and $x = \nu\otimes(w_1,\cdots,w_m)\in R$. By definition, the endomorphism $\{\alpha,-\} : R\to R$ is a graded derivation for any (homogeneous) $\alpha\in R_\natural$. The above action of $R_\natural$ on R extends to an action of $R_\natural$ on $R\otimes R$ via the Leibniz rule.

To prove that the bracket (5.4) satisfies the Jacobi identity, we begin by showing that the canonical projection $p : R\otimes R\to R_\natural$ is $R_\natural$-equivariant, i.e., given $\alpha = \mu\otimes(v_1,\cdots,v_{n+1})\in R_\natural$, $x=\mu'\otimes(w_1,\cdots,w_m)$, $y=\mu''\otimes(u_1,\cdots,u_l)\in R$, we want to show that

$$\{\alpha,\, p(x\otimes y)\} = p\big(\{\alpha,\, x\otimes y\}\big) = p\big(\{\alpha,\, x\}\otimes y + \pm x\otimes\{\alpha,\, y\}\big)\,, \tag{5.5}$$

Notice that

$$\begin{aligned} p(x\otimes y) &= \pm(\tau^{-1}\mu')\otimes w_2\otimes\cdots\otimes w_m\otimes\mu''(u_1,\,\cdots,\,u_l)\otimes w_1\\ &= \pm((\tau^{-1}\mu')\circ_n\mu'')\otimes(w_2,\cdots,w_m,u_1,\cdots,u_l,w_1)\,. \end{aligned}$$

Hence

$$\begin{aligned} &\{\alpha,p(x\otimes y)\}\\ &\quad=\sum_{i=1}^{n+1}\sum_{j=1}^{m}\pm\langle v_i,\,w_j\rangle((\tau^{n-i}\mu)\circ_n(\tau^{m+1+l-j}((\tau^{-1}\mu')\circ_n\mu')))\otimes\\ &\quad(v_{i+2},\cdots,v_{i-1},w_{j+1},\cdots,w_m,u_1,\cdots,u_l,w_1,\cdots,w_{j-1},v_{i+1})\\ &\quad+\sum_{i=1}^{n+1}\sum_{k=1}^{l}\pm\langle v_i,\,u_k\rangle((\tau^{n-i}\mu)\circ_n(\tau^{l+1-k}((\tau^{-1}\mu')\circ_n\mu')))\otimes\\ &\quad(v_{i+2},\cdots,v_{i-1},u_{k+1},\cdots,u_l,w_1,\cdots,w_m,u_1,\cdots,u_{k-1},v_{i+1})\\ &\quad=\sum_{i=1}^{n+1}\sum_{j=1}^{m}\pm\langle v_i,\,w_j\rangle((\tau^{n-i}\mu)\circ_n((\tau^{m+1-j}\mu')\circ_{m+1-j}\mu''))\otimes\\ &\quad(v_{i+2},\cdots,v_{i-1},w_{j+1},\cdots,w_m,u_1,\cdots,u_l,w_1,\cdots,w_{j-1},v_{i+1}) \qquad (5.6)\\ &\quad+\sum_{i=1}^{n+1}\sum_{k=1}^{l}\pm\langle v_i,\,u_k\rangle((\tau^{n-i}\mu)\circ_n((\tau^{l+1-k}\mu'')\circ_{l+1-k}\mu'))\otimes\\ &\quad(v_{i+2},\cdots,v_{i-1},u_{k+1},\cdots,u_l,w_1,\cdots,w_m,u_1,\cdots,u_{k-1},v_{i+1}) \qquad (5.7) \end{aligned}$$

[4]At this stage, we have *not* proven any fact about the structure of $R_\natural$. The term 'action' of $R_\natural$ on a vector space V simply means a linear map from $R\otimes V\to V$.

On the other hand,

$$\begin{aligned}
&p\big(\{\alpha,x\}\otimes y\big)\\
&\quad=\sum_{i=1}^{n+1}\sum_{j=1}^{m}\pm\langle v_i,\,w_j\rangle((\tau^{-1}(\mu'\circ_j(\tau^{n+1-i}\mu)))\circ_{n+m-1}\mu'')\otimes\\
&\quad(w_2,\cdots,w_{j-1},v_{i+1},\cdots,v_{i-1},w_{j+1},\cdots,w_m,u_1,\cdots,u_l,w_1)\\
&\quad=\sum_{i=1}^{n+1}\sum_{j=1}^{m}\pm\langle v_i,\,w_j\rangle(\tau^{n+m-j}(\mu'\circ_j(\tau^{n+1-i}\mu)))\circ_{n+m-j}\mu'')\otimes\\
&\quad(v_{i+2},\cdots,v_{i-1},w_{j+1},\cdots,w_m,u_1,\cdots,u_l,w_1,\cdots,w_{j-1},v_{i+1})\qquad(5.8)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\pm\,p\big(x\otimes\{\alpha,\,y\}\big)\\
&\quad=\sum_{i=1}^{n+1}\sum_{k=1}^{l}\pm\langle v_i,\,u_k\rangle((\tau^{-1}\mu')\circ_m((\mu''\circ_k(\tau^{n+1-i}\mu)))\otimes\\
&\quad(w_2,\cdots,w_m,u_1,\cdots,u_{k-1},v_{i+1},\cdots,v_{i-1},u_{k+1},\cdots,u_l,w_1)\\
&\quad=\sum_{i=1}^{n+1}\sum_{k=1}^{l}\pm\langle v_i,\,u_k\rangle((\tau^{n+l-k}(\mu''\circ_k(\tau^{n+1-i}\mu)))\circ_{n+l-k}\mu')\otimes\\
&\quad(v_{i+2},\cdots,v_{i-1},u_{k+1},\cdots,u_l,w_1,\cdots,w_m,u_1,\cdots,u_{k-1},v_{i+1})\qquad(5.9)
\end{aligned}$$

It is easy to see that (5.6) equals (5.8) and (5.7) equals (5.9). This verifies (5.5).

Next, we show that given $\alpha=\mu\otimes(v_1,\cdots,v_{n+1})$, $\beta=\nu\otimes(w_1,\cdots,w_{m+1})\in R_\natural$ and $u\in V\subseteq R$,

$$\{\{\alpha,\,\beta\},\,u\}=\{\alpha,\,\{\beta,\,u\}\}-\pm\{\beta,\,\{\alpha,\,u\}\}\,.\qquad(5.10)$$

Indeed, by (5.4),

$$\begin{aligned}
\{\alpha,\,\beta\}\;&=\;\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\pm\langle v_i,\,w_j\rangle(\tau^{n+1-i}\mu)(v_{i+1},\cdots,v_{i-1})\\
&\quad\otimes(\tau^{m+1-j}\nu)(w_{j+1},\cdots,w_{j-1})\\
&=\;\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\pm\langle v_i,\,w_j\rangle((\tau^{n-i}\mu)\circ_n(\tau^{m+1-j}\nu))\otimes\\
&\quad(v_{i+2},\cdots,v_{i-1},w_{j+1},\cdots,w_{j-1},v_{i+1})\,.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\{\{\alpha,\beta\},u\}\\
&\quad=\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\sum_{k=1}^{j-1}\pm\langle v_i,\,w_j\rangle\langle w_k,\,u\rangle\tau^{j-k}((\tau^{n-i}\mu)\circ_n\\
&\quad(\tau^{m+1-j}\nu))\otimes(w_{k+1},\cdots,w_{j-1},v_{i+1},\cdots,v_{i-1},w_{j+1},\cdots,w_{k-1}) \quad (5.11)\\
&\quad+\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\sum_{k=j+1}^{m+1}\pm\langle v_i,\,w_j\rangle\langle w_k,\,u\rangle\tau^{m+1+j-k}((\tau^{n-i}\mu)\circ_n\\
&\quad(\tau^{m+1-j}\nu))\otimes(w_{k+1},\cdots,w_{j-1},v_{i+1},\cdots,v_{i-1},w_{j+1},\cdots,w_{k-1}) \quad (5.12)\\
&\quad+\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\sum_{k=1}^{i-1}\pm\langle v_i,\,w_j\rangle\langle v_k,\,u\rangle\tau^{m+i-k}((\tau^{n-i}\mu)\circ_n\\
&\quad(\tau^{m+1-j}\nu))\otimes(v_{k+1},\cdots,v_{i-1},w_{j+1},\cdots,w_{j-1},v_{i+1},\cdots,v_{k-1}) \quad (5.13)\\
&\quad+\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\sum_{k=i+1}^{n+1}\pm\langle v_i,\,w_j\rangle\langle v_k,\,u\rangle\tau^{m+n+1+i-k}((\tau^{n-i}\mu)\circ_n\\
&\quad(\tau^{m+1-j}\nu))\otimes(v_{k+1},\cdots,v_{i-1},w_{j+1},\cdots,w_{j-1},v_{i+1},\cdots,v_{k-1}) \quad (5.14)
\end{aligned}
$$

Now,

$$\{\beta,\,u\}=\sum_{k=1}^{m+1}\pm\langle w_k,\,u\rangle(\tau^{m+1-k}\nu)(w_{k+1},\cdots,w_{k-1})\,.$$

Hence

$$
\begin{aligned}
&\{\alpha,\{\beta,\,u\}\}\\
&\quad=\sum_{i=1}^{n+1}\sum_{j=1}^{k-1}\sum_{k=1}^{m+1}\pm\langle v_i,\,w_j\rangle\langle w_k,\,u\rangle((\tau^{m+1-k}\nu)\circ_{m+1+j-k}\\
&\quad(\tau^{n+1-i}\mu))\otimes(w_{k+1},\cdots,w_{j-1},v_{i+1},\cdots,v_{i-1},w_{j+1},\cdots,w_{k-1}) \quad (5.15)\\
&\quad+\sum_{i=1}^{n+1}\sum_{j=k+1}^{m+1}\sum_{k=1}^{m+1}\pm\langle v_i,\,w_j\rangle\langle w_k,\,u\rangle((\tau^{m+1-k}\nu)\circ_{j-k}\\
&\quad(\tau^{n+1-i}\mu))\otimes(w_{k+1},\cdots,w_{j-1},v_{i+1},\cdots,v_{i-1},w_{j+1},\cdots,w_{k-1}) \quad (5.16)
\end{aligned}
$$

Similarly,

$$\begin{aligned}
&\{\beta, \{\alpha,\, u\}\} \\
&\quad = \sum_{j=1}^{m+1}\sum_{i=1}^{k-1}\sum_{k=1}^{n+1} \pm\langle w_j,\, v_i\rangle\langle v_k,\, u\rangle((\tau^{n+1-k}\mu)\circ_{n+1+i-k} \\
&\quad (\tau^{m+1-j}\nu)) \otimes (v_{k+1}, \cdots, v_{i-1}, w_{j+1}, \cdots, w_{j-1}, v_{i+1}, \cdots, v_{k-1}) \qquad (5.17) \\
&\quad + \sum_{j=1}^{m+1}\sum_{i=k+1}^{n+1}\sum_{k=1}^{n+1} \pm\langle w_j,\, v_i\rangle\langle v_k,\, u\rangle((\tau^{n+1-k}\mu)\circ_{i-k} \\
&\quad (\tau^{m+1-j}\nu)) \otimes (v_{k+1}, \cdots, v_{i-1}, w_{j+1}, \cdots, w_{j-1}, v_{i+1}, \cdots, v_{k-1}) \qquad (5.18)
\end{aligned}$$

One can check that (5.11) equals (5.16), (5.12) equals (5.15), (5.13) is identical to (5.18), and (5.14) equals (5.17). This verifies (5.10). Since $R_\natural$ acts on R by derivations,

$$\{\{\alpha,\, \beta\},\, x\} = \{\alpha,\, \{\beta,\, x\}\} - \pm\{\beta,\, \{\alpha,\, x\}\}$$

for any $x \in R$. Therefore, for $\alpha,\, \beta \in R_\natural$ and $x \otimes y \in R \otimes R$, we have

$$\{\{\alpha,\, \beta\},\, x \otimes y\} = \{\alpha,\, \{\beta,\, x \otimes y\}\} - \pm\{\beta,\, \{\alpha,\, x \otimes y\}\}\ . \qquad (5.19)$$

Since $p : R \otimes R \to R_\natural$ is $R_\natural$-equivariant,

$$\{\{\alpha,\, \beta\},\, \gamma\} = \{\alpha,\, \{\beta,\, \gamma\}\} - \pm\{\beta,\, \{\alpha,\, \gamma\}\} \qquad (5.20)$$

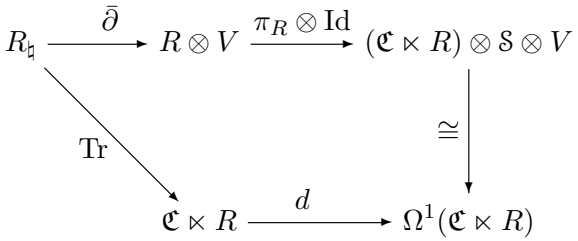
for any $\gamma \in R_\natural$. This verifies the Jacobi identity for the bracket (5.4), proving that it is a Lie bracket. Since this bracket descends from an action of $R_\natural$ on $R \otimes R$ induced by an action of $R_\natural$ on R by derivations (recall the $R_\natural$-equivariance of p), this bracket defines a Poisson structure on R. □

It remains to show that $\mathrm{Tr} : R_\natural \to \mathfrak{C} \ltimes R$ is a DG Lie algebra homomorphism. By [2, Theorem A.2], $C \ltimes R \cong \mathbf{\Omega}_{\mathtt{Lie}}[\mathcal{L}ie^c(\mathfrak{C} \otimes C)]$. By Proposition 5.2, $\mathfrak{C} \ltimes R$ acquires a structure of DG Poisson (and therefore, Lie) algebra. Hence there is an action of $R_\natural$ on $\mathfrak{C} \ltimes R$ via Tr. Since $\mathfrak{C} \ltimes R$ is freely generated by $\mathfrak{C} \otimes V$ as a graded commutative algebra, $\Omega^1(\mathfrak{C} \ltimes R) \cong (\mathfrak{C} \ltimes R) \otimes \mathfrak{C} \otimes V$. Since $\mathfrak{C}$ is finite dimensional, $\mathbf{Hom}(\mathfrak{C}, B) \cong B \otimes \mathcal{S}$ for any $B \in \mathtt{DGCA}_{k/k}$. Let π_R be the universal representation

$$R \longrightarrow \mathbf{Hom}_*(\mathfrak{C}, \mathfrak{C} \ltimes R) \hookrightarrow \mathbf{Hom}(\mathfrak{C}, \mathfrak{C} \ltimes R) \cong (\mathfrak{C} \ltimes R) \otimes \mathcal{S}\ .$$

The following lemma generalizes Lemma 4.1. We leave its proof (which is very similar to that of Lemma 4.1) to the interested reader.

Lemma 5.2. *The following diagram commutes:*



Here, the vertical isomorphism on the right identifies $\mathcal{S}$ with $\mathfrak{C}$ via the pairing on $\mathcal{S}$.

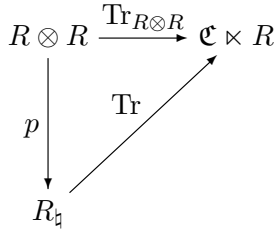
The following proposition follows from Lemma 5.2 just as Proposition 4.1 follows from Lemma 4.1.

Proposition 5.3. *The universal representation $\pi_R : R \to (\mathfrak{C} \ltimes R) \otimes \mathcal{S}$ is $R_\natural$-equivariant, where $R_\natural$ acts trivially on $\mathcal{S}$.*

It follows from Proposition 5.3 that the composite map

$$\begin{aligned} R \otimes R &\xrightarrow{\pi_R \otimes \pi_R} (\mathfrak{C} \ltimes R) \otimes \mathcal{S} \otimes (\mathfrak{C} \ltimes R) \otimes \mathcal{S} \\ &\cong (\mathfrak{C} \ltimes R)^{\otimes 2} \otimes \mathcal{S}^{\otimes 2} \xrightarrow{\mu \otimes \langle -,- \rangle} \mathfrak{C} \ltimes R \end{aligned}$$

is $R_\natural$-equivariant. Denote the above map by $\mathrm{Tr}_{R\otimes R}$. Since the diagram below commutes,



and since p is $R_\natural$-equivariant as well, Tr is $R_\natural$-equivariant. This proves that $\mathrm{Tr} : R_\natural \to \mathfrak{C} \ltimes L$ is a DG Lie algebra homomorphism. The second assertion of Theorem 5.1 follows on homologies.

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(Ajay C. Ramadoss) Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
ajcramad@indiana.edu

(Yining Zhang) Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
yinizhan@indiana.edu

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