DIFFERENTIABILITY OF THE ARGMIN FUNCTION AND A MINIMUM PRINCIPLE FOR SEMICONCAVE SUBSOLUTIONS

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Abstract. Suppose $f(x,y) + \frac{\kappa}{2}\|x\|^2 - \frac{\sigma}{2}\|y\|^2$ is convex where $\kappa \geq 0, \sigma > 0$, and the argmin function $\gamma(x) = \{\gamma : \inf_y f(x,y) = f(x,\gamma)\}$ exists and is single valued. We will prove γ is differentiable almost everywhere. As an application we deduce a minimum principle for certain semiconcave subsolutions.

1. Introduction

The first part of this paper is a proof of the following elementary statement about regularity of certain argmin functions.

Theorem 1. Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is such that

(1) There are $\kappa \geq 0$ and $\sigma > 0$ so
$$f(x,y) + \frac{\kappa}{2}\|x\|^2 - \frac{\sigma}{2}\|y\|^2 \text{ is convex.}$$

(2) For each $x \in \mathbb{R}^n$ there is a unique $\gamma(x)$ such that
$$\inf_y f(x,y) = f(x,\gamma(x)).$$

Then the function γ is differentiable almost everywhere.

Our motivation for this is the following. Following Harvey-Lawson [10, 11], by a *(primitive) subequation* on an open $X \subset \mathbb{R}^n$ we mean a subset $F \subset J^2(X)$ of the space of 2-jets on X with certain properties. Given such an F and a $\mathcal{C}^2$ function f, we say that f is *F-subharmonic* if every 2-jet of f lies in F. Moreover, using the so-called viscosity technique it is possible to extend the notion of F-subharmonicity to any upper-semicontinuous function (details and precise definitions will be given in §3).

In our previous work [23] we introduced a notion of "product subequation" $F\#\mathcal{P}$ on $X \times \mathbb{R}^m$ and show (under suitable hypothesis) that if F is convex and f is $F\#\mathcal{P}$-subharmonic then its marginal function
$$g(x) := \inf_y f(x,y)$$
is F-subharmonic. This statement generalises the classical statement that the marginal function of a convex function is again convex. We will use Theorem 1 to prove a similar minimum principle that does not require F to be convex:

Theorem 2. Let $X \subset \mathbb{R}^n$ be open and $F \subset J^2(X)$ be a constant-coefficient primitive subequation that depends only on the Hessian part. Suppose
$$f : X \times \mathbb{R}^m \to \mathbb{R}$$

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is locally semiconcave, bounded from below, and $F\#\mathcal{P}$-subharmonic. Then the marginal function

$$g(x) = \inf_y f(x,y)$$

is F-subharmonic on X.

A few remarks are in order.

(1) The semiconcavity assumption on f is rather unnatural, since one would expect a subsolution to have some kind of convexity rather than concavity, but it captures what we are able to prove. Observe that f is certainly locally semiconcave if it is $\mathcal{C}^{1,1}_{loc}$.

(2) The assumption that f is $F\#\mathcal{P}$-subharmonic implies that for each x the function $y \mapsto f(x,y)$ is convex. This along with the semiconcavity assumption implies that $y \mapsto f(x,y)$ is $\mathcal{C}^{1,1}_{loc}$.

(3) Theorem 2 can be proved rather easily when f is $\mathcal{C}^2$ (see [23, Prop. 7.5] for a stronger statement). To do so, we first approximate f by adding a small multiple of the function $(x,y) \mapsto \|y\|^2$, so there is no loss in assuming f is strictly convex in y and that for each fixed x the function $y \mapsto f(x,y)$ attains its unique minimum at some point $\gamma(x)$. Said another way, $\gamma(x)$ is the unique point such that

$$\frac{\partial f}{\partial y}|_{(x,\gamma(x))} = 0.$$

If we assume f is $\mathcal{C}^2$ we can then:
(a) Use the implicit function theorem to deduce that γ is $\mathcal{C}^1$.
(b) Use the chain rule to compute the Hessian of g at a point x in terms of the Hessian of f at the point $(x,\gamma(x))$ and the derivative of γ at x.
The combination of (b) and assumption that f is $F\#\mathcal{P}$-subharmonic yields that g is F-subharmonic as claimed.

(4) If we assume furthermore that F is convex, then using smooth mollification to approximate any upper-semicontinuous $F\#\mathcal{P}$-subharmonic function by those that are $\mathcal{C}^2$, we can deduce a much more general minimum principle – this is the approach taken in [23].

(5) If instead we assume that f is merely $\mathcal{C}^{1,1}_{loc}$ then it is of course twice differentiable almost everywhere. However it may well be that f is not twice differentiable at any point of the form $(x,\gamma(x))$ so part (b) of the above argument does not apply.

To prove Theorem 2 we will first use a partial-sup convolution to approximate f by $F\#\mathcal{P}$-subharmonic functions f_ϵ such that

$$f_\epsilon(x,y) + \frac{1}{2\epsilon}\|x\|^2 - \frac{\epsilon}{2}\|y\|^2 \text{ is convex.}$$

In particular for fixed x the function $y \mapsto f_\epsilon(x,y)$ is strongly convex, and we will further arrange so the argmin of f_ϵ is a well-defined single-valued function γ. Having done so we can apply Theorem 1 to deduce that γ is differentiable almost everywhere, which will act in lieu of the implicit function argument used in (a). From this one can prove, essentially from the definition, that at almost every point x the Hessian of g is contained in F. As g is semiconvex, this is known by the

Almost-Everywhere Theorem of Harvey-Lawson [9] to be enough to conclude that g is F-subharmonic.

Comparison with other work: The authors do not have sufficient expertise to properly survey all previously known regularity results that are related to Theorem 1. Suffice to say there has been much interest in studying regularity of marginal functions (by which we mean functions of the form $\inf_y f(x,y)$ or $\sup_y f(x,y)$ for some function f which also go under the name "performance function") due to its relevance for optimization problems (see for instance [3, 8, 16, 20] and the references therein). For example, various regularity properties of marginal functions have been shown when f has some convexity property (see for example [21, Theorems 23.4 and 24.5]) and without this convexity hypothesis (e.g. [6, 17, 18, 19, 25] to list just a few).

Much less appears to have been written about regularity of the argmin function itself. We remark that in general the argmin function will be multi-valued, and so regularity must be phrased in terms of set-valued functions [6]. The only previous such results we have found relate to continuity rather than differentiability (for example [26, Theorem 2.10], which is taken from [22, Theorems 1.17 and 7.41], gives conditions under which the argmin function is outer semicontinuous).

Regarding the minimum principle, the fact that the marginal function of a convex function is again convex is a basic property in convex analysis. In the complex case this has an analog for plurisubharmonic functions due to Kiselman [14, 15]. Both convexity and plurisubharmonicity are massively generalized through the notion of F-subharmonic functions which uses the viscosity technique that arose in the study of fully non-linear degenerate second-order differential equations (in particular the work of Caffarelli–Nirenberg–Spruck [5] and Lions–Crandall–Ishii [7], who often refer to such functions as *subsolutions*).

Our motivation for introducing the product $F\#\mathcal{P}$ came from a desire to generalise this minimum principle to general subequations, which we do in [23] under the assumption that F is convex. As discussed above, this assumption is needed only to be able to approximate $F\#\mathcal{P}$-subharmonic functions by smooth ones, and thus suggests that it is a facet of the proof rather than an essential requirement. Theorem 2 is, as far as we know, the first such minimum principle that does not require any convexity hypothesis on the subequation in question. For further background in this area the reader is referred to [23].

Organization: Section 2 is devoted to the proof of Theorem 1. In §2.1 we recall some standard terminology and notation concerning semiconvex functions, and use this to give a refined statement (Theorem 7) about calmness of the argmin function. Theorem 1 then follows immediately from this by Stepanov's Theorem (see Corollary 8). In §2.2 we collect some further properties of semiconvex functions, in preparation for §2.3 in which we give a functional equation for the argmin function. Then the proof of Theorem 7 is given in §2.5 using the Implicit Function Theorem for Lipschitz maps (which for completeness is proved in Appendix A).

In Section 3 we summarize the basics of F-subharmonic functions in a way suited to our needs, including the idea of product subequations in §3.2. In Section 4 we describe the partial sup-convolution, which is used in Section 5 to complete the proof of Theorem 2.

2. Differentiability of the Argmin Function

2.1. **Statement.** In this section we prove that the argmin function of a certain kind of semiconvex functions is differentiable (resp. calm) almost everywhere. Suppose $\Omega \subset \mathbb{R}^{n+m}$ is open and let $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection, and write

$$\Omega_x = \{y \in \mathbb{R}^m : (x, y) \in \Omega\}.$$

We will assume throughout that Ω is convex and that each Ω_x is connected. Now suppose

$$f : \Omega \to \mathbb{R}$$

and set

$$g(x) := \inf_{y \in \Omega_x} f(x, y) \text{ for } x \in \pi(\Omega).$$

Definition 3 (Argmin)**.** The *argmin function* is the set-valued function

$$\operatorname{argmin}_f(x) := \{\gamma \in \Omega_x : \inf_{y \in \Omega_x} f(x, y) = f(x, \gamma)\}$$

where we allow the possibility that $\operatorname{argmin}_f(x)$ is empty.

Below we shall make assumptions on f that ensure that $\operatorname{argmin}(x)$ is everywhere defined and single-valued. In such cases we shall write

$$\gamma(x) = \operatorname{argmin}_f(x)$$

so

$$f(x, \gamma(x)) = \inf_{y \in \Omega_x} f(x, y) = g(x) \text{ for all } x \in \pi(\Omega).$$

The precise statement we will prove requires some terminology concerning subdifferentials. Let $X \subset \mathbb{R}^n$ be open.

Definition 4. Suppose $g : X \to \mathbb{R}$. For each $x_0 \in X$ define

$$\nabla_{x_0} g = \{u \in \mathbb{R}^n : g(x) - g(x_0) \geq u.(x - x_0) \text{ for all } x \text{ sufficiently near } x_0\}$$

which may be empty. We call any $u \in \nabla_{x_0} g$ *a lower support vector* for g at x_0. Similarly if $\kappa \in \mathbb{R}$ we let

$$\nabla^\kappa_{x_0} g = \{u \in \mathbb{R}^n : g(x) - g(x_0) \geq u \cdot (x - x_0) - \frac{\kappa}{2}\|x - x_0\|^2 \text{ for all } x \text{ sufficiently near } x_0\}.$$

Definition 5 (Semiconvexity and Semiconcavity)**.** Let $\kappa \geq 0$. We say $g : X \to \mathbb{R}$ is *κ-semiconvex* (resp. *κ-semiconcave*) if $g(x) + \frac{\kappa}{2}\|x\|^2$ is convex (resp. $g(x) - \frac{\kappa}{2}\|x\|^2$ is concave). If g is κ-semiconvex/semiconcave for some $\kappa \geq 0$ then we say simply g is *semiconvex/semiconcave*.

Remark 6. In the literature one will also find the term *weakly-convex/concave* also used for semiconvex/semiconcave.

One can check that $g : X \to \mathbb{R}$ is locally κ-semiconvex if and only if $\nabla^\kappa_{x_0} g$ is non-empty for all x_0. Moreover g is differentiable at x_0 if and only if $\nabla_{x_0} g$ is a singleton, in which case its unique element is the derivative of g at x_0. Finally if g is a convex function on a convex set X then

$$\nabla^\kappa_{x_0} g = \{u \in \mathbb{R}^n : g(x) - g(x_0) \geq u \cdot (x - x_0) - \frac{\kappa}{2}\|x - x_0\|^2 \text{ for all } x\}.$$

We now give a refined statement of Theorem 1 that will be proved in section §2.3.

Theorem 7 (Argmin is calm almost everywhere)**.** Let $\Omega \subset \mathbb{R}^{n+m}$ be open, convex and so that Ω_x is connected for all x. Also let $f : \Omega \to \mathbb{R}$ and suppose there are $\kappa \geq 0$ and $\sigma > 0$ so that

$$f(x,y) + \frac{\kappa}{2}\|x\|^2 - \sigma\|y\|^2 \text{ is convex and} \tag{2.1}$$

$$\operatorname{argmin}_f(x) \text{ is non-empty for all } x \in \pi(\Omega). \tag{2.2}$$

Then

(i) The function

$$g(x) := \inf_y f(x,y) = f(x,\gamma(x)) \text{ for } x \in \pi(\Omega)$$

is κ-semiconvex and $\gamma(x) := \operatorname{argmin}_f(x)$ is single valued for all $x \in \pi(\Omega)$.

(ii) Given any $x_0 \in \pi(\Omega)$ and $u_0 \in \nabla^\kappa_{x_0} g$ there exists a Lipschitz function

$$\phi : V \to \mathbb{R}$$

defined on a neighbourhood V of (x_0, u_0) in $\mathbb{R}^{2n}$ such that

$$\gamma(x) = \phi(x,u) \text{ for all } (x,u) \in V \text{ with } u \in \nabla^\kappa_x g.$$

(iii) The function γ is calm almost everywhere. That is, for almost all $x_0 \in \pi(\Omega)$ there are C and $\delta > 0$ such that

$$\|\gamma(x) - \gamma(x_0)\| \leq C\|x - x_0\| \text{ for } \|x - x_0\| < \delta. \tag{2.3}$$

Corollary 8 (The Argmin is Differentiable Almost Everywhere)**.** Under the hypothesis of the Theorem the argmin function γ is differentiable almost everywhere.

Proof. This follows from (2.3) and Stepanov's Theorem [24] (see also [12, Theorem 3.4]). □

The strategy of the proof of Theorem 7 is to construct a functional equation satisfied by the argmin function, and then apply the implicit function theorem for Lipschitz functions. In the next section we setup the necessary machinery to do so.

2.2. **Properties of semiconvex functions.** We collect a few basic statements about convex and semiconvex functions. As above $\Omega \subset \mathbb{R}^{n+m}$ is open, convex and Ω_x is connected for all x.

Lemma 9. Suppose $f : \Omega \to \mathbb{R}$ and $\tilde{f}(x,y) = f(x,y) + \kappa\frac{\|x\|^2}{2}$. Then

$$\nabla_{(x_0,y_0)} f = \nabla_{(x_0,y_0)} \tilde{f} + \kappa x_0$$

as sets.

Proof. This is immediate from the definition, and left to the reader. □

Lemma 10. Suppose $f : \Omega \to \mathbb{R}$ and $h : \mathbb{R}^m \to \mathbb{R}$ and set

$$\hat{f}(x,y) = f(x,y) + h(x) \text{ for } (x,y) \in \Omega.$$

Then

$$\operatorname{argmin}_{\hat{f}}(x) = \operatorname{argmin}_f(x) + h(x). \tag{2.4}$$

In particular $\operatorname{argmin}_{\hat{f}}$ is single-valued if and only if argmin_f is single-valued.

Proof. Clearly

$$\operatorname{argmin}_{\hat{f}}(x) = \{\gamma : \gamma = \inf_y \hat{f}(x,y)\} = \{\gamma : \gamma = \inf_y f(x,y) + h(x)\} = \operatorname{argmin}_f(x) + h(x)$$

giving (2.4). The last statement follows immediately. □

Lemma 11. Let $f : \Omega \to \mathbb{R}$ and suppose $f(x,y) + \frac{\kappa}{2}\|x\|^2$ is convex. Then $g(x) = \inf_y f(x,y)$ is κ-semiconvex.

Proof. Write $\tilde{f}(x,y) = f(x,y) + \frac{\kappa}{2}\|x\|^2$ so

$$g(x) + \frac{\kappa}{2}\|x\|^2 = \inf_y \tilde{f}(x,y)$$

which is the marginal function of convex function defined on Ω (which is assumed to be convex and Ω_x is connected for each x). Thus $g(x) + \frac{\kappa}{2}\|x\|^2$ is convex. □

Lemma 12 (Gradient at argmin)**.** Suppose that $f : \Omega \to \mathbb{R}$ is convex and set $g(x) = \inf_y f(x,y)$. Then for all $x \in \pi(\Omega)$ and $\gamma \in \operatorname{argmin}_f(x)$

$$u \in \nabla_x g \Rightarrow (u,0) \in \nabla_{(x,\gamma)} f.$$

Proof. Suppose $\gamma \in \operatorname{argmin}_f(x)$ so $g(x) = f(x,\gamma)$. Let $u \in \nabla_x g$. Then for any $(x',y') \in \mathbb{R}^n \times \mathbb{R}^m$,

$$f(x',y') - f(x,\gamma) \geq g(x') - g(x) \tag{2.5}$$

$$\geq u.(x'-x) = (u,0).((x',y') - (x,\gamma)) \tag{2.6}$$

so $(u,0) \in \nabla_{(x,\gamma)} f$ as claimed. □

The next statement is a slight modification of [13, Proposition 6.4].

Proposition 13. Let $\sigma > 0$ and suppose $f : \mathbb{R}^{n+m} \to \mathbb{R}$ is such that $f(x,y) - \frac{\sigma}{2}\|y\|^2$ is convex. Define the set-valued function

$$G(p) = p + \nabla_p f \text{ for } p = (x,y) \in \mathbb{R}^{n+m}.$$

Then

(i) G is non-contractive. That is, if $\zeta_i \in G(p_i)$ for $i = 1,2$ then
$$\|\zeta_1 - \zeta_2\| \geq \|p_1 - p_2\|. \tag{2.7}$$

(ii) There exist a single-valued function $H : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ that is inverse to G, by which we mean
$$H(\zeta) = p \Longleftrightarrow \zeta \in G(p). \tag{2.8}$$

(iii) The function H is Lipschitz with Lipschitz constant 1. Moreover there is a $\mu < 1$ such that letting $\pi_2 : \mathbb{R}^{n+m} \to \mathbb{R}^m$ denote the second projection,
$$\|\pi_2 H(\zeta_1) - \pi_2 H(\zeta_2)\| \leq \mu\|\zeta_1 - \zeta_2\| \text{ for all } \zeta_1, \zeta_2 \in \mathbb{R}^{n+m}. \tag{2.9}$$

Proof. Let $p_i := (x_i, y_i) \in \mathbb{R}^{n+m}$ for $i = 1,2$. We first claim

$$(\nabla_{p_2} f - \nabla_{p_1} f).(p_2 - p_1) \geq \sigma\|y_2 - y_1\|^2 \text{ for all } (x_i,y_i) \in \mathbb{R}^{n+m}. \tag{2.10}$$

To see this, let $\tilde{f}(x,y) = f(x,y) - \frac{\sigma}{2}\|y\|^2$ which by assumption is convex and $\nabla_{(x,y)}\tilde{f} = \nabla_{(x,y)} f - (0,\sigma y)$. Then

$$\tilde{f}(p_1) - \tilde{f}(p_2) \geq \nabla_{p_2}\tilde{f}.(p_1 - p_2) = \nabla_{p_2} f.(p_1 - p_2) - \sigma y_2.(y_1 - y_2). \tag{2.11}$$

Swapping the indices we also have

$$\tilde{f}(p_2) - \tilde{f}(p_1) \geq \nabla_{p_1}\tilde{f}.(p_2 - p_1) = \nabla_{p_1} f.(p_2 - p_1) - \sigma y_1.(y_2 - y_1). \tag{2.12}$$

Adding (2.11) and (2.12) and rearranging gives (2.10).

Now from Cauchy-Schwarz and (2.10)

$$\begin{aligned}\|G(p_1)-G(p_2)\|\|p_1-p_2\| &\geq (G(p_1)-G(p_2)).(p_1-p_2)\\ &= (p_1-p_2+\nabla_{p_1}f-\nabla_{p_2}f).(p_1-p_2)\\ &\geq \|p_1-p_2\|^2+\sigma\|y_1-y_2\|^2 \\ &\geq \|p_1-p_2\|^2\end{aligned} \tag{2.13}$$

which in particular implies (i).

We claim next that G is surjective, by which we mean for all $\zeta\in\mathbb{R}^{n+m}$ there is an $p\in\mathbb{R}^{n+m}$ such that $\zeta\in G(p)$. To see this let

$$\phi(p):=\frac{1}{2}\|p\|^2+f(p)-p.\zeta.$$

The function $p\mapsto\frac{1}{2}\|p\|^2-p.\zeta$ is convex, and hence so is ϕ and

$$\nabla_{p_0}\phi=\nabla_{p_0}f+p_0-\zeta=G(p_0)-\zeta.$$

Similarly the function

$$\psi(p):=\frac{1}{4}\|p\|^2+f(p)-p.\zeta=\phi(p)-\frac{1}{4}\|p\|^2$$

is convex. Pick $b\in\nabla_0\psi$ so $\psi(p)-\psi(0)\geq b.p$ giving

$$\phi(p)\geq\phi(0)+\frac{1}{4}\|p\|^2.$$

As ϕ is continuous this implies ϕ has a global minimum at some $p_0\in\mathbb{R}^{n+m}$, and so 0 is a lower support vector for ϕ at 0. Thus $0\in\nabla_{p_0}\phi=G(p_0)-\zeta$ implying that $\zeta\in G(p_0)$. Thus G is surjective as claimed.

In particular the inverse H to G defined by

$$H(\zeta)=\{p\in\mathbb{R}^{n+m}:\zeta\in G(p)\}$$

is non-empty, and G being non-contractive implies that it is single-valued. That H has Lipshitz constant 1 follows from (i).

Finally given ζ_1,ζ_2 set $p_i:=(x_i,y_i):=H(\zeta_i)$ so by definition $\zeta_i\in G(p_i)$ and $y_i=\pi_2H(\zeta_i)$. To ease notation let $\alpha:=\|x_1-x_2\|$ and $\beta:=\|y_1-y_2\|=\|\pi_2H(\zeta_1)-\pi_2H(\zeta_2)\|$. Then dividing (2.13) by $\|p_1-p_2\|$ gives

$$\|\zeta_1-\zeta_2\|\geq(\alpha^2+\beta^2)^{1/2}+\sigma\frac{\beta^2}{(\alpha^2+\beta^2)^{1/2}}.$$

If $\alpha\geq\sigma\beta$ then $\|\zeta_1-\zeta_2\|\geq(1+\sigma^2)^{1/2}\beta$. If $\alpha\leq\sigma\beta$ then

$$\begin{aligned}\|\zeta_1-\zeta_2\| &\geq \beta+\sigma\frac{\beta^2}{(\sigma^2\beta^2+\beta^2)^{1/2}}\\ &=(1+\frac{\sigma}{(1+\sigma^2)^{1/2}})\beta.\end{aligned}$$

Hence (2.9) holds with $\mu:=\min\{(1+\sigma^2)^{1/2},(1+\frac{\sigma}{(1+\sigma^2)^{1/2}})\}^{-1}<1$. □

We will also need the following simpler corollary (which is proved in the same way, or follows formally from Proposition 13 upon taking $m=0$).

Corollary 14. Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is convex and define the set-valued function

$$G_1(x) = x + \nabla_x g \text{ for } x \in \mathbb{R}^n.$$

Then

(1) G_1 is non-contractive, that is

$$\|G_1(x_1) - G_1(x_2)\| \geq \|x_1 - x_2\| \text{ for all } x_1, x_2 \in X. \tag{2.14}$$

(2) There exist a single-valued function $H_1 : \mathbb{R}^n \to \mathbb{R}^n$ that is inverse to G_1, and H_1 is Lipschitz with Lipschitz constant 1.

2.3. **Functional Equation for argmin.** Suppose now that $f : \mathbb{R}^{n+m} \to \mathbb{R}$ is convex and as usual let $g(x) = \inf_y f(x,y)$ which is also convex. Consider the set-valued functions

$$\begin{aligned} G_1(x) &= x + \nabla_x g, \\ G(x,y) &= (x,y) + \nabla_{(x,y)} f. \end{aligned}$$

By Proposition 13 and Corollary 14 these have single-valued inverses $H_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $H : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$. That is

$$H_1(u) = x \Leftrightarrow u \in G_1(x) \text{ for } x, u \in \mathbb{R}^n \tag{2.15}$$

$$H(u,v) = (x,y) \Leftrightarrow (u,v) \in G(x,y) \text{ for } (x,y),(u,v) \in \mathbb{R}^{n+m}. \tag{2.16}$$

We use these to define a functional equation for argmin_f. Let

$$J : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$$

$$J(x,u,y) := y - \pi_2 H(H_1(x+u) + u, y). \tag{2.17}$$

Proposition 15 (Functional Equation for argmin)**.** Suppose that $f(x,y)$ is convex and let $g(x) = \inf_y f(x,y)$. Then

$$J(x, \nabla_x g, \operatorname{argmin}_f(x)) = 0 \text{ for all } x \in \mathbb{R}^n.$$

That is,

$$J(x,u,\gamma) = 0 \text{ for all } x \in \mathbb{R}^n \text{ and } \gamma \in \operatorname{argmin}_f(x) \text{ and } u \in \nabla_x g.$$

Proof. Let $x \in \mathbb{R}^n$, $\gamma \in \operatorname{argmin}_f(x)$ and $u \in \nabla_x g$. Then $x + u \in G_1(x)$ so (2.15) gives $H_1(x+u) = x$. On the other hand since $\gamma \in \operatorname{argmin}_f(x)$ we have by Lemma 12,

$$(u, 0) \in \nabla_{(x,\gamma)} f.$$

Thus

$$(x,\gamma) + (u,0) = (u+x, \gamma) \in G(x,\gamma)$$

so (2.15) gives $H(u+x,\gamma) = (x,\gamma)$. So

$$\begin{aligned} J(x,u,\gamma) &= \gamma - \pi_2 H(H_1(x+u) + u, \gamma) \\ &= \gamma - \pi_2 H(x+u,\gamma) = \gamma - \pi_2(x,\gamma) \\ &= 0 \end{aligned}$$

as claimed. □

We next collect two basic properties of J:

Lemma 16 (Properties of J)**.** The function J is Lipschitz in (x,u,y). Moreover if $f(x,y)-\frac{\sigma}{2}\|y\|^2$ is convex for some $\sigma>0$ then there is a $\lambda>0$ such that for fixed x,u

$$\|J(x,u,y_1)-J(x,u,y_2)\|\geq\lambda\|y_1-y_2\| \text{ for all } y_1,y_2.$$

Proof. Clearly J is Lipschitz in all variables since both H and H_1 are. For the second statement, suppose $f(x,y)-\frac{\sigma}{2}\|y\|^2$ is convex and let $\pi_2:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ be the second projection. We know from Proposition 13(iii) that there is a $\mu<1$ such that

$$\|\pi_2 H(v,y_1)-\pi_2 H(v,y_2)\|\leq\mu\|y_1-y_2\|\text{for all } v,y_1. \tag{2.18}$$

Now fix x,u and let $v:=H_1(x+u)+u$. Then if $y_1,y_2\in\mathbb{R}^m$,

$$\begin{aligned}\|J(x,u,y_2)-J(x,u,y_1)\| &= \|y_2-y_1-\pi_2 H(v,y_2)+\pi_2 H(v,y_1)\|\\ &\geq \|y_2-y_1\|-\|\pi_2 H(v,y_2)-\pi_2 H(v,y_1)\|\\ &\geq (1-\mu)\|y_2-y_1\|.\end{aligned}$$

□

2.4. **Statement of Alexandrov's Theorem.** Let $X\subset\mathbb{R}^n$ be open. The following is a precise version of Alexandrov's Theorem:

Theorem 17 (Alexandrov's Theorem)**.** Let $g:X\to\mathbb{R}$ be locally convex. Then the set-valued function

$$x\mapsto\nabla_x g$$

is differentiable at x_0 for almost all x_0 in X. That is, for almost all x_0 there is an $L\in\operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^m)$ such that for all $\epsilon>0$ there is a $\delta>0$ such that for $\|x-x_0\|<\delta$ we have

$$\|u-u_0-\frac{L}{2}(x-x_0)|\|\leq\epsilon\|x-x_0\| \text{ for all } u\in\nabla_x g \text{ and } u_0\in\nabla_{x_0}g. \tag{2.19}$$

Moreover for almost all x_0 the function g is twice differentiable at x_0 and $\operatorname{Hess}_{x_0}(g)=L$. That is, for any $\epsilon>0$ there is a $\delta>0$ such that

$$|g(x)-g(x_0)-\nabla g|_{x_0}.(x-x_0)-\frac{1}{2}(x-x_0)^t\operatorname{Hess}_{x_0}(g)(x-x_0)\rangle|\leq\epsilon\|x-x_0\|^2 \tag{2.20}$$

for all $\|x-x_0\|<\delta$.

Proof. This originates in [2] and for an exposition the reader is referred to [13, Theorems 6.1,7.1]. (We remark that the latter cited work requires the function to be convex and defined on all of $\mathbb{R}^n$; but the statement we want is local, and being locally convex, g is also locally Lipschitz [1], and so using [28, Theorem 4.1] we know that X is covered by small open sets U such that $g|_U$ extends to a convex function on $\mathbb{R}^n$ so the cited work applies.) □

2.5. **Proof of Theorem 7.**

Lemma 18 (Continuity of argmin)**.** Let $\Omega\subset X\times\mathbb{R}$ be convex and such that Ω_x is connected for each $x\in X$. Let $f:\Omega\to\mathbb{R}$ be continuous, and suppose that for each $x\in X$ the function $y\mapsto f(x,y)$ is strongly convex and attains its minimum at some point. Then $\gamma(x)=\operatorname{argmin}_f(x)$ is single valued and continuous.

Proof. For fixed x the hypothesis imply that $y \mapsto f(x,y)$ is a strongly convex function on the connected set Ω_x that attains its minimum, and thus this minimum $\gamma(x)$ must be a unique. We first claim that γ is locally bounded. Fix $x_0 \in X$ and let $a := \gamma(x_0)$. Then by strong convexity there is an $\epsilon > 0$ and $c > 0$ such that $f(x_0, y) > a + \epsilon$ if $\|y - \gamma(x_0)\| \geq c$. By continuity we may take $\delta > 0$ small so if $\|x - x_0\| < \delta$ and $\|y - \gamma(x_0)\| = c$ then $f(x,y) > a + \epsilon$ and, and furthermore that $f(x, \gamma(x_0)) < a + \epsilon$. But by strict convexity of $y \mapsto f(x,y)$ this implies $\gamma(x) \in [\gamma(x_0) - c, \gamma(x_0) + c]$ for all $\|x - x_0\| < \delta$, and thus γ is locally bounded.

Now suppose (x_n) is a sequence in X converging to x as $n \to \infty$. By the above we may assume $S := \{\gamma(x_n)\}$ is bounded. Let b be a cluster point of S, so there is a subsequence x_{n_r} with $\gamma(x_{n_r}) \to b$ as $r \to \infty$. By continuity of f for any $y \in \mathbb{R}^m$,

$$f(x,b) = \lim_{r\to\infty} f(x_{n_r}, \gamma(x_{n_r})) \leq \lim_{r\to\infty} f(x_{n_r}, y) = f(x,y).$$

Hence $b = \gamma(x)$. As this holds for all cluster points of S we deduce $\gamma(x_n) \to \gamma(x)$ as $n \to \infty$, proving continuity of γ. □

Proof of Theorem 7. We first claim that there is no loss in generality in assuming that $\Omega = \mathbb{R}^{n+m}$. To see this, suppose $f : \Omega \to \mathbb{R}$ has properties (2.1) and (2.2). Then $\gamma = \operatorname{argmin}_f$ is single-valued and continuous (Lemma 18). So given $x_0 \in \pi(\Omega)$ there are small balls $x_0 \in U \subset \pi(\Omega)$ and $\gamma(x_0) \in V \subset \mathbb{R}^m$ so that $U \times V \subset \Omega$ and $\gamma(U) \subset V$. Moreover as f is semiconvex, by shrinking U, V we may assume that $f|_{U\times V}$ is Lipschitz (all convex functions are Lipschitz, see e.g. [1]). Let $\tilde{f}(x,y) := f(x,y) + \frac{\kappa}{2}\|x\|^2 - \frac{\sigma}{2}\|y\|^2$ which we are assuming is convex on Ω. Then [28, Theorem 4.1] we know $\tilde{f}|_{U\times V}$ extends to a convex function $\tilde{h}$ on all of $\mathbb{R}^{n+m}$. Now let

$$h(x,y) := \tilde{h}(x,y) - \frac{\kappa}{2}\|x\|^2 + \frac{\sigma}{2}\|y\|^2.$$

For fixed x the convex function $y \mapsto h(x,y)$ agrees with the function $y \mapsto f(x,y)$ when $y \in V$. Since V contains $\gamma(x) = \operatorname{argmin}_f(x)$, this implies $\operatorname{argmin}_h(x) = \operatorname{argmin}_f(x) = \gamma(x)$. Hence h satisfies the hypothesis of the Theorem with $\Omega = \mathbb{R}^{n+m}$, and so $\gamma|_U$ has the properties in the conclusion of the theorem (which are all local), which proves the claim.

So from now on assume $f : \mathbb{R}^{n+m} \to \mathbb{R}$ satisfies (2.1) and (2.2). Consider first the case $\kappa = 0$, so $(x,y) \mapsto f(x,y) - \frac{\sigma}{2}\|y\|^2$ is convex. Then in particular f is convex, and so $g(x) = \inf_y f(x,y)$ is also convex. Moreover for fixed x the function $y \mapsto f(x,y)$ is strictly convex, and so argmin_f (which is assumed to be non-empty) must be single valued. Consider the functional J from (2.17) so by Proposition 15

$$J(x, u, \gamma(x)) = 0 \text{ for all } x \text{ and } u \in \nabla_x g. \tag{2.21}$$

Fix $x_0 \in \mathbb{R}^n$ and $u_0 \in \nabla_{x_0} g$ so $J(x_0, u_0, \gamma(x_0)) = 0$. The properties of J proved in Lemma 16 mean we can apply the Inverse-function Theorem for Lipschitz maps (for convenience of the reader we give a proof of this in Appendix A, and apply it here with r replaced with $2n$ and s replaced with m). This yields a Lipschitz function $\phi : V \to \mathbb{R}^m$ defined on a neighbourhood V of (x_0, u_0) such that

$$J(x,u,y) = 0 \Leftrightarrow y = \phi(x,u).$$

This combined with (2.21) gives

$$\gamma(x) = \phi(x,u) \text{ for all } (x,u) \in V \text{ with } u \in \nabla_x g.$$

We next prove γ is calm almost everywhere. As g is convex we have by Alexandrov's Theorem (2.19) that for almost all x_0 there are $\delta_1 > 0$ and linear $L : \mathbb{R}^n \to \mathbb{R}^n$ such that for $\|x - x_0\| < \delta_1$

$$\|u - u_0\| \leq (1 + \|L\|)\|x - x_0\| \text{ for all } u \in \nabla_x g \text{ and } u_0 \in \nabla_{x_0} g. \tag{2.22}$$

Pick $u_0 \in \nabla_{x_0} g$, and let $\phi : V \to \mathbb{R}$ be the Lipschitz function constructed above. For concreteness say that V contains the set $\|x - x_0\| < \delta_2$ and $\|u - u_0\| < \delta_2$ and that ϕ has Lipschitz constant C' there. Thus

$$\gamma(x) = \phi(x, u) \text{ for } \|x - x_0\| < \delta_2, \|u - u_0\| < \delta_2 \text{ and } u \in \nabla_x g.$$

Set

$$\delta := \min\{\delta_1, \frac{\delta_2}{1 + \|L\|}\}$$

and suppose $\|x - x_0\| < \delta$. Picking any $u \in \nabla_x g$, by (2.22) $\|u - u_0\| < \delta_2$ and so

$$\|\gamma(x)-\gamma(x_0)\| = \|\phi(x, u)-\phi(x_0, u_0)\| \leq C'(\|x-x_0\|+\|u-u_0\|) \leq C'(2+\|L\|)\|x-x_0\|.$$

Thus γ is calm x_0.

The case of general κ is easily reduced to the case $\kappa = 0$. For suppose $f(x, y) + \frac{\kappa}{2}\|x\|^2 - \frac{\sigma}{2}|y|^2$ is convex and argmin_f is single-valued. Set

$$\tilde{f}(x, y) = f(x, y) + \frac{\kappa}{2}\|x\|^2$$

Then $\tilde{f}(x, y) - \frac{\sigma}{2}\|y\|^2$ is convex, and by (2.4)

$$\operatorname{argmin}_{\tilde{f}}(x) = \operatorname{argmin}_f(x).$$

Thus $\operatorname{argmin}_{\tilde{f}}$ is also single-valued, so by the above the Theorem can be applied to $\tilde{f}$. Let

$$\gamma(x) := \operatorname{argmin}_{\tilde{f}}(x) = \operatorname{argmin}_f(x)$$

Setting $\tilde{g}(x) := \inf_y \tilde{f}(x, y)$, given x_0 and $u_0 \in \nabla_{x_0}\tilde{g}$ we know that there is a locally Lipschitz function $\tilde{\phi} : \tilde{V} \to \mathbb{R}$ defined on a neighbourhood $\tilde{V}$ of (x_0, u_0) such that

$$\gamma(x) = \tilde{\phi}(x, u) \text{ for } (x, u) \in \tilde{V} \text{ with } u \in \nabla_x \tilde{g}.$$

Set $\phi(x, u) = \tilde{\phi}(x, u + \kappa x)$ which is locally Lipschitz around $(x_0, u_0 + \kappa x_0)$. And if $u \in \nabla^\kappa g$ then $u - \kappa x_0 \in \nabla_x \tilde{g}$ so $\gamma(x) = \tilde{\phi}(x, u - \kappa x_0) = \phi(x, u)$. Thus the conclusion of the Theorem also holds for f and we are done. $\square$

3. F-subharmonic functions

3.1. **Basic definitions.**

We summarise some basic properties of F-subharmonic functions from the work of Harvey-Lawson. We refer the reader to [23] for a more detailed summary, or the original papers [10, 11]. Let $X \subset \mathbb{R}^n$ be open and

$$J^2(X) := X \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2_n = X \times J^2_n$$

be the jet-bundle over X. For $F \subset J^2(X)$ and $x \in X$ we write

$$F_x = \{(r, p, A) \in J^2_n : (x, r, p, A) \in F\}.$$

Definition 19 (Primitive Subequations)**.** We say that $F \subset J^2(X)$ is a *primitive subequation* if

(1) (Closedness) F is closed.

(2) (Positivity)

$$(r,p,A) \in F_x \text{ and } P \in \mathrm{Pos}_n \Rightarrow (r,p,A+P) \in F_x. \tag{3.1}$$

We say that $F \subset J^2(X)$ has the *Negativity Property* if

(3) (Negativity)

$$(r,p,A) \in F_x \text{ and } r' \leq r \Rightarrow (r',p,A) \in F_x. \tag{3.2}$$

Definition 20 (Upper contact points, Upper contact jets)**.** Let

$$f : X \to \mathbb{R} \cup \{-\infty\}.$$

We say that $x \in X$ is an *upper contact point* of f if $f(x) \neq -\infty$ and there exists $(p,A) \in \mathbb{R}^n \times \mathrm{Sym}^2_n$ such that

$$f(y) \leq f(x) + p.(y-x) + \frac{1}{2}(y-x)^t A(y-x) \text{ for all } y \text{ sufficiently near } x.$$

When this holds we refer to both $(f(x),p,A)$ and (p,A) as an *upper contact jet* of f at x.

Definition 21 (F-subharmonic function)**.** Suppose $F \subset J^2(X)$. We say that an upper-semicontinuous function $f : X \to \mathbb{R} \cup \{-\infty\}$ is *F-subharmonic* if

$$(f(x),p,A) \in F_x \text{ for all upper contact jets } (p,A) \text{ of } f \text{ at } x.$$

We let $F(X)$ denote the set of F-subharmonic functions on X.

Clearly being F-subharmonic is a local condition on X.

Proposition 22. Let $F \subset J^2(X)$ be closed. Then

(1) (Maximum Property) If $f, g \in F(X)$ then $\max\{f,g\} \in F(X)$.
(2) (Decreasing Sequences) If f_j is decreasing sequence of functions in $F(X)$ (so $f_{j+1} \leq f_j$ over X) then $f := \lim_j f_j$ is in $F(X)$.
(3) (Uniform limits) If f_j is a sequence of functions on $F(X)$ that converge locally uniformly to f then $f \in F(X)$.
(4) (Families locally bounded above) Suppose $\mathcal{F} \subset F(X)$ is a family of F-subharmonic functions locally uniformally bounded from above. Then the upper-semicontinuous regularisation of the supremum

$$f := \sup{}^*_{f\in\mathcal{F}} f$$

is in $F(X)$.
(5) If F is constant coefficient and f is F-subharmonic on X and $x_0 \in \mathbb{R}^n$ is fixed, then the function $x \mapsto f(x-x_0)$ is F-subharmonic on $X - x_0$.

Proof. See [11, Theorem 2.6] for (1-4). Item (5) is immediate. □

Definition 23. Let $F \subset J^2(X)$.

(1) We say F is *constant coefficient* if F_x is independent of x, i.e.

$$(x,r,p,A) \in F_x \Leftrightarrow (x',r,p,A) \in F_{x'} \text{ for all } x,x',r,p,A.$$

(2) We say F *depends only on the Hessian part* if each F_x is independent of (r,p), i.e.

$$(r,p,A) \in F_x \Leftrightarrow (r',p',A) \in F_x \text{ for all } x,r,r',p,p',A.$$

An important example is

$$\mathcal{P} := \{(x, r, p, A) \in J^2(X) : A \text{ is semipositive}\}$$

which is a constant-coefficient primitive subequation that depends only on the Hessian part. Then $\mathcal{P}$-subharmonic functions are precisely those that are locally convex [11, Example 14.2].

Lemma 24 (Sums of F-subharmonic and convex functions)**.** Suppose $F \subset J^2(X)$ is a constant coefficient primitive subequation that depends only on the Hessian part. If f is F-subharmonic on X and g is a convex quadratic function on X, then $f + g$ is F-subharmonic.

Proof. The hypothesis is that $g(x) = a + b.x + \frac{1}{2}x^tCx$ for some $a, b \in \mathbb{R}^n$ and some semipositive symmetric matrix C. One can check that if (p, A) is an upper-contact point of $f + g$ at x then $(x^tC + p - b, A - C)$ is an upper-contact jet for f at x. As f is F-subharmonic this implies $(f(x), x^tC + p - b, A - C) \in F$. Since F depends only on the Hessian part, and satisfies the Positivity property, this in turn implies $(f(x) + g(x), p, A) \in F$ proving that $f + g$ is F-subharmonic as required. □

3.2. **Product Subequations.** For $\Gamma \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) = M_{m\times n}(\mathbb{R})$ consider

$$i_\Gamma : \mathbb{R}^n \to \mathbb{R}^{n+m} \quad i_\Gamma(x) = (x, \Gamma x) \tag{3.3}$$

$$j : \mathbb{R}^m \to \mathbb{R}^{n+m} \quad j(y) = (0, y). \tag{3.4}$$

which induce natural pullback maps

$$i_\Gamma^* : J^2_{n+m} \to J^2_n \text{ and } j^* : J^2_{n+m} \to J^2_m. \tag{3.5}$$

We can write these explicitly. Suppose

$$p := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^{n+m} \text{ and } A := \begin{pmatrix} B & C \\ C^t & D \end{pmatrix} \in \mathrm{Sym}^2_{n+m}$$

where the latter is in block form, so $B \in \mathrm{Sym}^2_n$ and $D \in \mathrm{Sym}^2_m$. Then

$$i_\Gamma^*(r, p, A) = \big(r, p_1 + \Gamma^t p_2, B + C\Gamma + \Gamma^t C^t + \Gamma^t D\Gamma\big) \tag{3.6}$$

$$j^*(r, p, A) = (r, p_2, D). \tag{3.7}$$

Definition 25 (Products)**.** Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open, and $F \subset J^2(X)$ and $G \subset J^2(Y)$. Define

$$F\#G \subset J^2(X \times Y)$$

by

$$(F\#G)_{(x,y)} = \left\{ \alpha \in J^2_{n+m} : \begin{array}{l} i_\Gamma^*\alpha \in F_x \text{ and } j^*\alpha \in G_y \\ \text{for all } \Gamma \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \end{array} \right\}.$$

Lemma 26.
(1) If F and G are primitive subequations then so is $F\#G$. Moreover if F and G both have the Negativity Property then so does $F\#G$.
(2) Let F be a constant-coefficient primitive subequation on X. Suppose and f is $F\#\mathcal{P}$-subharmonic on some open $\Omega \subset X \times Y$. The for each $x \in X$ the function $y \mapsto f(x, y)$ is locally convex.

Proof. The reader will easily prove these straight from the definition, or otherwise find the proofs in [23]. □

3.3. **The almost everywhere theorem.** We will rely on a very useful theorem of Harvey-Lawson that characterizes F-subharmonic semiconvex functions in terms of second order jets almost everywhere.

Definition 27 (Twice differentiability at a point)**.** We say that a function $f : X \to \mathbb{R}$ is *twice differentiable* at $x_0 \in X$ if there exists a $p \in \mathbb{R}^n$ and an $L \in \mathrm{Sym}^2_n$ such that for all $\epsilon > 0$ there is a $\delta > 0$ such that for $\|x - x_0\| < \delta$ we get

$$|f(x) - f(x_0) - p.(x - x_0) - \frac{1}{2}(x - x_0)^t L(x - x_0)| \leq \epsilon \|x - x_0\|^2. \tag{3.8}$$

When f is twice differentiable at x_0 then the p, L in (3.8) are unique, and moreover in this case f is differentiable at x_0 and

$$p = \nabla f|_{x_0} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} |_{x_0} \in \mathbb{R}^n.$$

When f is twice differentiable at x_0 we shall refer to L as the *Hessian* of f at x_0 and denote it by $\mathrm{Hess}(f)|_{x_0}$. Of course when f is $\mathcal{C}^2$ in a neighbourhood of x_0 then $\mathrm{Hess}_x(f)$ is the matrix with entries

$$(\mathrm{Hess}(f)_{x_0})_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}|_{x_0}.$$

Definition 28 (Second order jet)**.** Suppose that $f : X \to \mathbb{R}$ is twice differentiable at x_0. We denote the *second order jet* of f at x_0 by

$$J^2_{x_0}(f) := (f(x_0), \nabla f|_{x_0}, \mathrm{Hess}(f)|_{x_0}) \in J^2_n = \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2_n. \tag{3.9}$$

We have seen in Alexandrov's Theorem (Theorem 17) that if f is locally semiconvex then $J^2_x(f)$ exists for almost all x.

Theorem 29 (The Almost Everywhere Theorem)**.** Assume that $F \subset J^2(X)$ is a primitive subequation and let $f : X \to \mathbb{R}$ be locally semiconvex. Then

$$f \in F(X) \Leftrightarrow J^2_x(f) \in F_x \text{ for almost all } x \in X.$$

Proof. See [9, Theorem 4.1]. □

4. Partial sup-convolutions

Fix open $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, and suppose $f : U \times V \to \mathbb{R}$ is uppersemicontinuous and bounded.

Definition 30 (Partial-Sup-Convolutions)**.** For $\epsilon > 0$ the *partial sup-convolution* of f is

$$f^{\epsilon,p}(x, y) := \sup_{z \in U} \{f(z, y) - \frac{1}{2\epsilon}\|z - x\|^2\} \text{ for } (x, y) \in U \times V. \tag{4.1}$$

For $\delta > 0$ let

$$U(\delta) = \{x \in \mathbb{R}^n : B_\delta(x) \subset U\}.$$

Lemma 31 (Basic Properties of Partial-Sup-Convolutions)**.**

(i) (Strong Semiconvexity) Assume that for each fixed x the function $y \mapsto f(x,y)$ is convex. Then
$$(x,y) \mapsto f^{\epsilon,p}(x,y) + \frac{1}{2\epsilon}\|x\|^2$$
is convex.

(ii) (Monotonicity) For $0 < \epsilon' \leq \epsilon$ we have
$$f \leq f^{\epsilon',p} \leq f^{\epsilon,p}. \tag{4.2}$$

(iii) Let $\delta := 2(\epsilon\|f\|_\infty)^{1/2}$. Then
$$f^{\epsilon,p}(x,y) = \sup_{\|\tau\|<\delta} \{f(x+\tau,y) - \frac{1}{2\epsilon}\|\tau\|^2\} \text{ for } (x,y) \in U(\delta) \times V.$$

(iv) (Pointwise convergence)
$$\lim_{\epsilon\to 0^+} f^{\epsilon,p}(x,y) = f(x,y) \text{ for } (x,y) \in U \times V.$$

(v) (Magic-Property) Suppose that F is a constant-coefficient primitive subequation on U that has the Negativity Property and f is $F\#\mathcal{P}$-subharmonic. Then $f^{\epsilon,p}$ is $F\#\mathcal{P}$-subharmonic on $U(\delta) \times V$.

Proof.
$$\begin{aligned} f^{\epsilon,p}(x,y) + \frac{1}{2\epsilon}\|x\|^2 &= \sup_{z\in U}\{f(z,y) - \frac{1}{2\epsilon}\|z-x\|^2 + \frac{1}{2\epsilon}\|x\|^2\} \\ &= \sup_{z\in U}\{f(z,y) + \frac{1}{\epsilon}x.z - \frac{1}{2\epsilon}\|z\|^2\}. \end{aligned}$$
Now for fixed z the function $y \mapsto f(z,y)$ is assumed to be convex in y, and so the function $(x,y) \mapsto f(z,y)$ is convex in (x,y). Thus, again for z fixed, $(x,y) \mapsto f(z,y) + \frac{1}{\epsilon}x.z + \frac{1}{2\epsilon}\|z\|^2$ is convex in (x,y), and hence so is $f^{\epsilon,p}(x,y) + \frac{1}{2\epsilon}\|x\|^2$ proving (i).

Item (ii) is immediate. For (iii) we claim that
$$f^{\epsilon,p}(x,y) = \sup_{z\in U:\|z-x\|<\delta} \{f(z,y) - \frac{1}{2\epsilon}\|z-x\|^2\} \text{ for } (x,y) \in U \times V. \tag{4.3}$$
To see this let $M := \|f\|_\infty$. Then for $z \in U$ with $\|z-x\| \geq \delta = \sqrt{4\epsilon M}$,
$$f(z,y) - \frac{1}{2\epsilon}\|z-x\|^2 \leq M - \frac{1}{2\epsilon}\delta^2 = -M \leq f(x,y) \leq f^{\epsilon,p}(x,y)$$
which proves (4.3). Then (iii) follows upon making the change of variables $\tau := z-x$. For the pointwise convergence fix $(x,y) \in U \times V$ and let $a > f(x,y)$. Then $f < a$ on some open neighbourhood of (x,y). Let ϵ be small enough so that $B_\delta(x)$ is contained in this neighbourhood. Then (4.3) implies $f^{\epsilon,p}(x,y) \leq a$, proving (iv). For the final statement, since F is constant coefficient for any fixed τ the function $f(x+\tau,y)$ is $F\#\mathcal{P}$-subharmonic (where defined), and hence (iii) shows $f^{\epsilon,p}$ as a supremum of $F\#\mathcal{P}$-subharmonic functions. Now being $F\#\mathcal{P}$-subharmonic implies that $y \mapsto f(x,y)$ is convex, and so by (i) $f^{\epsilon,p}$ is certainly continuous and hence equal to its upper semicontinuous regularisation. Thus $f^{\epsilon,p}$ is $F\#\mathcal{P}$-subharmonic on $U(\delta) \times V$ as claimed in (v).

□

The next lemma reveals a surprising property of the above construction, namely that the partial sup-convolution of a semiconcave function is fibrewise semiconcave.

Lemma 32. Suppose that f is κ-semiconcave for some $\kappa > 0$. Then for $\epsilon < \kappa^{-1}$ and fixed $x \in U$ the function

$$y \mapsto f^{\epsilon,p}(x,y) - \frac{\kappa}{2}\|y\|^2$$

is concave.

Proof. Let x be fixed. Then

$$\begin{aligned} f^{\epsilon,p}(x,y) - \frac{\kappa}{2}\|y\|^2 &= \sup_{z\in U}\{f(z,y) - \frac{\kappa}{2}\|y\|^2 - \frac{1}{2\epsilon}\|z-x\|^2\} \\ &= \sup_{z\in U}\{f(z,y) - \frac{\kappa}{2}\|z\|^2 - \frac{\kappa}{2}\|y\|^2 + \frac{\kappa-\epsilon^{-1}}{2}\|z\|^2 + \frac{1}{\epsilon}z.x - \frac{1}{2\epsilon}\|x\|^2\}. \end{aligned}$$

Observe that (since x is fixed and $\kappa - \epsilon^{-1} < 0$) the function $(z,y) \mapsto \frac{\kappa-\epsilon^{-1}}{2}\|z\|^2 + \frac{1}{\epsilon}z.x - \frac{1}{2\epsilon}\|x\|^2$ is convex as a function of (z,y). Furthermore by hypothesis $f(z,y) - \frac{\kappa}{2}\|z\|^2 - \frac{\kappa}{2}\|y\|^2$ is concave in (z,y). Hence $y \mapsto f^{\epsilon,p}(x,y) - \frac{\kappa}{2}\|y\|^2$ is a supremum of functions concave in two variables, and thus is concave. □

5. F-subharmonicity of marginal functions

Let $\Omega \subset \mathbb{R}^{n+m}$ be open, convex and such that Ω_x is connected for all x.

Proposition 33. Let $F \subset J^2(\mathbb{R}^n)$ be a primitive subequation. Let $f : \Omega \to \mathbb{R}$ be $F\#\mathcal{P}$-subharmonic, and suppose that for some $\sigma, \kappa_1, \kappa_2 > 0$ the function

$$f(x,y) + \frac{\kappa_1}{2}\|x\|^2 - \frac{\sigma}{2}\|y\|^2 \text{ is convex and} \tag{5.1}$$

and for each fixed x the function

$$y \mapsto f(x,y) - \frac{\kappa_2}{2}\|y\|^2 \text{ is concave} \tag{5.2}$$

and that $\gamma(x) = \operatorname{argmin}_f(x)$ is single valued. Then

$$g(x) := \inf_{y\in\Omega_x} f(x,y)$$

is F-subharmonic.

Proof. By hypothesis

$$g(x) = f(x,\gamma(x)).$$

Now g is κ-semiconvex (Lemma 11) so by Alexandrov's Theorem (Theorem 17) g is twice differentiable almost everywhere. Furthermore (5.1) allows us to invoke our results on the argmin function, so by Corollary 8 γ is differentiable almost everywhere. Let x_0 be a point where g is twice differentiable and γ is differentiable, and we will show

$$J^2_{x_0}g = (g(x_0), \nabla g|_{x_0}, \mathrm{Hess}_{x_0}(g)) \in F_{x_0}. \tag{5.3}$$

By the Almost Everywhere Theorem (Theorem 29) this implies that g is F-subharmonic.

Actually we will show that for any $\epsilon > 0$ it holds that

$$(g(x_0), \nabla g|_{x_0}, \mathrm{Hess}_{x_0}(g) + \epsilon\,\mathrm{Id}_n) \in F_{x_0}. \tag{5.4}$$

Letting $\epsilon \to 0$ and using that F_{x_0} is closed yields (5.3).

To this end set $y_0 := \gamma(x_0)$ and

$$\Gamma := D\gamma|_{x_0} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$$

and $d(x,y)$ be the vertical distance between $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$ and the tangent to the graph of γ at (x_0,y_0), so

$$d(x,y) := \|y-y_0-\Gamma(x-x_0)\| \text{ for } (x,y)\in\mathbb{R}^n\times\mathbb{R}^m.$$

Consider the quadratic

$$q(x,y) = g(x_0)+\nabla g|_{x_0}.(x-x_0)+\frac{1}{2}(x-x_0)^t\operatorname{Hess}_{x_0}(g)(x-x_0)+\frac{\epsilon}{2}\|x-x_0\|^2+\kappa_2 d(x,y)^2$$

for $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$. By construction

$$q(x_0,y_0)=g(x_0)=f(x_0,\gamma(x_0))=f(x_0,y_0),$$

and in Lemma 34 below we show that $q\geq f$ sufficiently near (x_0,y_0). Hence (x_0,y_0) is an upper contact point for f and

$$\begin{aligned} J^2_{(x_0,y_0)}(q) &= \big(q(x_0,y_0),\nabla q|_{(x_0,y_0}, \operatorname{Hess}_{(x_0,y_0}(q)\big) && (5.5)\\ &= \left(f(x_0,y_0), \begin{pmatrix} \nabla g|_{x_0} \\ 0\end{pmatrix}, \begin{pmatrix} \operatorname{Hess}_{x_0}(g)+\epsilon\operatorname{Id}_n+2\kappa_2\Gamma^t\Gamma & -2\kappa_2\Gamma^t \\ -2\kappa_2\Gamma & 2\kappa_2\operatorname{Id}_m\end{pmatrix}\right) && (5.6)\end{aligned}$$

is an upper-contact jet of f at (x_0,y_0). So as f is $F\#\mathcal{P}$-subharmonic we have

$$J^2_{(x_0,y_0)}(q)\in (F\#\mathcal{P})_{(x_0,y_0)}.$$

And from the definition of i^*_Γ,

$$i^*_\Gamma(J^2_{(x_0,y_0)}(q)) = \operatorname{Hess}_{x_0}(g)+\epsilon\operatorname{Id}_n$$

which must lie in F_{x_0}. This gives (5.4) and completes the proof. □

Lemma 34. With the notation as in the proof of Theorem 33 we have

$$q(x,y)\geq f(x,y) \text{ for } (x,y) \text{ sufficiently near } (x_0,y_0). \tag{5.7}$$

Proof. Fix $\epsilon'>0$ small enough so $\epsilon'+\kappa_2\epsilon'^2<\epsilon/2$. That $\Gamma = D\gamma|_{x_0}$ means there is a $\delta>0$ such that for all $\|x-x_0\|<\delta$

$$\|\gamma(x)-y_0-\Gamma(x-x_0)\|\leq\epsilon'\|x-x_0\|.$$

Shrinking δ is necessary, the definition of g being twice differentiable at x_0 means (3.8) that for $\|x-x_0\|<\delta$ we also have

$$|g(x)-g(x_0)-\nabla g|_{x_0}.(x-x_0)-\frac{1}{2}(x-x_0)^t\operatorname{Hess}_{x_0}g(x-x_0)|\leq\epsilon'\|x-x_0\|^2.$$

Consider now a point (x,y) with $\|x-x_0\|<\delta$ and $\|y-y_0\|<\delta$. Then

$$\begin{aligned}\|y-\gamma(x)\| &\leq \|y-y_0-\Gamma(x-x_0)\|+\|y_0+\Gamma(x-x_0)-\gamma(x)\| && (5.8)\\ &\leq d(x,y)+\epsilon'\|x-x_0\|. && (5.9)\end{aligned}$$

So

$$\|y-\gamma(x)\|^2\leq 2\epsilon'^2\|x-x_0\|^2+2d(x,y)^2.$$

Now we use (in an essential way) hypothesis (5.2). Since $\gamma(x)$ is the minimum of the function $y'\mapsto f(x,y')$ (5.2) implies

$$f(x,y)\leq f(x,\gamma(x))+\frac{\kappa_2}{2}\|y-\gamma(x)\|^2.$$

Thus

$$\begin{aligned} f(x,y) &\le g(x) + \kappa_2(\epsilon'^2\|x-x_0\|^2 + d(x,y)^2) \\ &\le g(x_0) + \nabla g|_{x_0}(x-x_0) + \frac{1}{2}(x-x_0)^t \operatorname{Hess}_{x_0} g(x-x_0) \\ &\quad + (\kappa_2\epsilon'^2 + \epsilon')\|x-x_0\|^2 + \kappa_2 d(x,y)^2 \\ &\le q(x,y) \end{aligned}$$

as required. □

Proof of Theorem 2. Let $f : X \times \mathbb{R} \to \mathbb{R}$ be locally semiconcave, bounded from below and $F\#\mathcal{P}$-subharmonic. We are to show that $g(x) := \inf_y f(x,y)$ is F-subharmonic.

We first claim that without loss of generality we may assume in addition that for each x it holds that $\operatorname{argmin}_f(x)$ is non-empty and single valued. To prove this, for $j \ge 1$ let

$$f_j(x,y) = f(x,y) + \frac{1}{j}\|y\|^2.$$

As F depends only on the Hessian part, f_j is still $F\#\mathcal{P}$-subharmonic, and is still bounded from below and semiconcave. Moreover since f is bounded from below, for each fixed x the function $y \mapsto f(x,y)$ is strictly convex and tends to infinity as $|y|$ tends to infinity, implying that it has a unique global minimum. By assumption the theorem applies to f_j meaning that letting $g_j(x) := \inf_y f_j(x,y)$ the function g_j is F-subharmonic. But $g_j \searrow g$ pointwise as $j \to \infty$, and thus g will be F-subharmonic as well, proving the claim.

So from now on assume that $\gamma(x) = \operatorname{argmin}_f(x)$ is single valued. Fix $x_0 \in \mathbb{R}^n$. As γ is continuous, there exist small balls $x_0 \in U \subset X$ and $\gamma(x_0) \in V \subset \mathbb{R}^m$ such that $\gamma(U) \subset V$ and f is semiconcave on $U \times V$. For $\epsilon > 0$ consider the function

$$f_\epsilon(x,y) := f^{\epsilon,p}(x,y) + \frac{\epsilon}{2}\|y\|^2 = \sup_{z\in U}\{f(z,y) - \frac{1}{2\epsilon}\|z-x\|^2\} + \frac{\epsilon}{2}\|y\|^2.$$

We claim that for ϵ sufficiently small the following all hold:

(i) $f_\epsilon(x,y) + \frac{1}{2\epsilon}\|x\|^2 - \frac{\epsilon}{2}\|y\|^2$ is convex.
(ii) f_ϵ is $F\#\mathcal{P}$-subharmonic on $U' \times V$ for some smaller ball $x_0 \in U' \subset U$.
(iii) $f^\epsilon \searrow f$ pointwise on $U \times V$ as $\epsilon \to 0^+$.
(iv) There is a $\kappa_2 > 0$ such that for each $x \in U$ the function $y \mapsto f_\epsilon(x,y) - \frac{\kappa_2}{2}\|y\|^2$ is concave.

Items (i,ii,iii) follow from Lemma 31 (we have used here the hypothesis that F depends only on the Hessian part so adding a multiple of $\|y\|^2$ preserves the property of being $F\#\mathcal{P}$-subharmonic by Lemma 24). The statement (iv) follows from Lemma 32 (observing that the addition of $\frac{\epsilon}{2}\|y\|^2$ to the partial sup-convolution only means we may need to increase the value of κ_2)

Thus we are in a position to apply Proposition 33 to f_ϵ to conclude that if

$$g_\epsilon(x) := \inf_{y\in V} f_\epsilon(x,y)$$

then g_ϵ is F-subharmonic on U'. But by (iii) if $x \in U'$ then

$$g_\epsilon(x) \searrow \inf_{y\in V} f(x,y) = f(x,\gamma(x)) = g(x) \text{ as } \epsilon \to 0^+$$

and hence g is also F-subharmonic on U'. Since x_0 was arbitrary we conclude g is F-subharmonic on all of $\mathbb{R}^n$ as required. □

Appendix A. The Implicit Function Theorem for Lipschitz Functions

The following version of the Implicit function theorem is taken from [27, Theorem 5.1], and we include a proof for convenience.

Theorem 35 (Lipschitz Implicit Function Theorem)**.** Let $U_1 \subset \mathbb{R}^r$ and $U_2 \subset \mathbb{R}^s$ be open and

$$J : U_1 \times U_2 \to \mathbb{R}^s$$

be Lipschitz with the property that there is a $K > 0$ such that

$$\|J(p, y_1) - J(p, y_2)\| \geq K\|y_1 - y_2\| \text{ for all } (p, y_1), (p, y_2) \in U_1 \times U_2.$$

Suppose $a \in U_1, b \in U_2$ is such that

$$J(a, b) = 0.$$

There there exists an open $a \in V \subset U_1$ and a Lipschitz map

$$\phi : V \to U_2$$

such that $\phi(a) = b$ and

$$J(p, \phi(p)) = 0 \text{ for all } p \in V. \tag{A.1}$$

Proof. For small $\epsilon > 0$ (to be determined) let

$$\hat{J} : U_1 \times U_2 \to \mathbb{R}^{r+s} \text{ be } \hat{J}(p, y) = (p, \epsilon J(p, y))$$

which is Lipschitz as J is assumed to be Lipschitz. We claim that as long as ϵ is sufficiently small, $\hat{J}$ is bi-Lipschitz, i.e. there is a $C > 0$ such that

$$\|\hat{J}(p_1, y_1) - \hat{J}(p_2, y_2)\| \geq C\|(p_1, y_1) - (p_2, y_2)\| \tag{A.2}$$

for all $(p_i, y_i) \in U_1 \times U_2$.

To see this, say J has Lipschitz constant M and let $(p_i, y_i) \in U_1 \times U_2$. Then

$$\begin{aligned} K^2\|y_1 - y_2\|^2 &\leq \|J(p_1, y_1) - J(p_1, y_2)\|^2 \\ &\leq 2(\|J(p_1, y_1) - J(p_2, y_2)\|^2 + \|J(p_2, y_2) - J(p_1, y_2)\|^2) \\ &\leq 2\|J(p_1, y_1) - J(p_2, y_2)\|^2 + 2M^2\|p_2 - p_1\|^2. \end{aligned}$$

Multiplying by $\epsilon^2/2$ and rearranging gives

$$\begin{aligned} \frac{K^2\epsilon^2}{2}\|y_1 - y_2\|^2 + (1 - \epsilon^2 M^2)\|p_1 - p_2\|^2 &\leq \epsilon^2\|J(p_1, y_1) - J(p_2, y_2)\|^2 + \|p_1 - p_2\|^2 \\ &= \|\hat{J}(p_1, y_1) - \hat{J}(p_2, y_2)\|^2. \end{aligned}$$

So if we take ϵ small enough so $1 - \epsilon^2 M^2 \geq \frac{K^2\epsilon^2}{2} =: C^2$ then

$$\|\hat{J}(p_1, y_1) - \hat{J}(p_2, y_2)\|^2 \geq C^2(\|y_1 - y_2\|^2 + \|p_1 - p_2\|^2) = C^2\|(p_1, y_1) - (p_2, y_2)\|^2$$

as claimed in (A.2).

In particular $\hat{J}$ is continuous and injective. Thus by Brouwer's Invariance of Domain Theorem [4, Corollary 19.8], $\hat{J}$ is an open map. So $V := \hat{J}(U_1 \times U_2) \subset \mathbb{R}^{r+s}$ is open and $\hat{J} : U_1 \times U_2 \to V$ is a continuous bijection with continuous inverse $\hat{J}^{-1} : V \to U_1 \times U_2$. In fact as $\hat{J}$ is bi-Lipschitz, we get that $\hat{J}^{-1}$ is Lipschitz.

Denote by $\pi_1 : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^r$ and $\pi_2 : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^s$ the projections, and let B be a small ball around a so that $B \subset U_1$ and $B \times \{0\} \subset \hat{J}(\pi_2^{-1}(U_2))$. Define $\phi : B \to U_1 \subset \mathbb{R}^r$ by

$$\phi(p) = \pi_2 \hat{J}^{-1}(p, 0).$$

Then ϕ is Lipschitz and $\hat{J}^{-1}(a, 0) = (a, b)$ gives $\phi(a) = b$. Moreover if $p \in V$ then

$$\begin{aligned}(p,0) &= \hat{J}\hat{J}^{-1}(p,0) = J(\pi_1 \hat{J}^{-1}(p,0), \pi_2 \hat{J}^{-1}(p,0)) \\ &= J(\pi_1 \hat{J}^{-1}(p,0), \phi(p)) = (\pi_1 \hat{J}^{-1}(p,0), \epsilon J(\pi_1 \hat{J}^{-1}(p,0), \phi(p)).\end{aligned}$$

Thus

$$p = \pi_1 \hat{J}^{-1}(p, 0)$$

and

$$0 = \epsilon J(\pi_1 \hat{J}^{-1}(p,0), \phi(p)) = \epsilon J(p, \phi(p))$$

proving (A.1)

□

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