Undergraduates' Perceptions of the Benefits of Working Tasks Focused on Analyzing Student Thinking as an Application for Teaching in Abstract Algebra

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The Mathematical Education of Teachers as an Application of Undergraduate Mathematics project provides lessons integrated into various mathematics major courses that incorporate mathematics teaching connections as a legitimate application area of undergraduate mathematics. One feature of the lessons involves posing tasks that require undergraduates to interpret or analyze the work of another student. This paper reports on thematic analysis of hour-long interviews for eight participants enrolled in an undergraduate abstract algebra course from two different implementation sites. We focus on student work and reactions to these interpreting or analyzing student thinking (AST) applications as they relate to their perceptions regarding the use of AST applications as a mechanism to both deepen their content knowledge and improve their skills for communicating mathematics. Several participants identify positive benefits, but more research is needed to determine the how to incorporate AST applications to accommodate some participants' reluctance to engage in new mathematical contexts.

Keywords: Abstract Algebra, Preservice Secondary Mathematics Teachers, Mathematical Knowledge for Teaching

The Mathematical Education of Teachers II (MET II) Report of the Conference Board of the Mathematical Sciences (CBMS) recommended that preservice secondary mathematics teachers make explicit connections between the undergraduate mathematics content they are learning as part of their continuing education and the primary or secondary school mathematics content they will eventually teach (CBMS, 2012). Furthermore, the MET II report recommends these connections be made in classes throughout their entire degree program and not simply summarized during a culminating capstone course.

An immediate consequence of this recommendation is that mathematics knowledge for teaching must not simply be cursorily included in general undergraduate mathematics courses but in fact emphasized as an application to teaching. This paper reports on the efforts of the *Mathematical Education of Teachers as an Application of Undergraduate Mathematics* (META Math) project to study the effectiveness of lessons which include tasks or applications which focus on interpreting or analyzing student thinking (AST) in an attempt to meet this goal. In particular, we explore the following research question: How do undergraduate mathematics students perceive the effectiveness of AST applications as a mechanism to both deepen their content knowledge and improve their skills for communicating mathematics?

Background and Theoretical Perspective

Wasserman (2018) claims that "despite the strong arguments for how and why studying advanced mathematics might benefit secondary teachers, much of the research has found the opposite to be true: teachers and their students appear to gain little from a teacher's study of advanced mathematics" (p. 4). The MET II Report also asserts that "the mathematics courses [preservice secondary teachers] take emphasize preparation for graduate study or careers in business rather than advanced perspectives on the mathematics that is taught in high school" (CBMS, 2012, p. 5). Similarly, Speer, King, and Howell (2014) observed that "prospective high school mathematics teachers, who earn a mathematics major or its equivalent, do not have sufficiently deep understanding of the mathematics of the high school curriculum" (p. 107). Furthermore, prospective secondary teachers perceive their undergraduate mathematical preparation as unconnected or not useful to their teaching (e.g., Goulding, Hatch, & Rodd, 2003; Wasserman, Weber, Villanova, & Mejia-Ramos, 2018).

The MET II Report addresses some of these shortcomings by giving examples of several important connections between high school and undergraduate mathematics that they recommend serve as cornerstones of preservice teacher education. These connections are presented as mathematical observances, such as the fact that using inner products to extend the notions of length and angle is "extremely useful background for high school teaching" (CMBS, 2012, p. 57).

While such observations are relevant, the MET II Report does not explain how these connections are to be made explicit in practice. The restructuring of an existing curriculum to include connections for preservice teachers is nontrivially time consuming and potentially difficult for faculty who may not have experience in developing such materials (Lai, 2016; Álvarez & Burroughs, 2018; Álvarez & White, 2018). As such, the development of these materials will necessarily overlap with the study of Mathematical Knowledge for Teaching (MKT), the theory of what mathematics understanding teachers need in order to be successful educators. Originally proposed by Shulman (1986), the concept of MKT is important to preservice teacher education because it posits that traditional content knowledge is not the only mathematical prerequisite required for effective teaching. One particular aspect of this additional knowledge is the ability to "be able to hear and interpret students' emerging and incomplete thinking" (Ball, Thames, & Phelps, 2008, p. 401). Competency in these areas is often directly correlated with not only an instructor's volume of mathematical knowledge but also their appreciation of the structure of and underlying principles connecting mathematical ideas. As such, MKT requires both content knowledge and knowledge applied to teaching where the content knowledge at the secondary level might include mathematics from an advanced perspective such as that addressed in courses such as abstract algebra.

Methodology

To address MKT and the recommendations in the MET II report, the META Math Project has developed inquiry-focused lessons, which can be integrated into various mathematics major courses, that incorporate mathematics teaching connections as a legitimate application area of undergraduate mathematics. The connections, especially targeting relevance for preservice secondary mathematics teachers (PSMTs), aim to link the undergraduate content PSMTs encounter in their major courses and the pre-college content they will eventually teach. One feature of the lessons involves posing undergraduate mathematics students with tasks that require them to interpret or analyze the work of another student (see Figure 1).

This paper will focus on student perceptions of AST applications in two lessons written for an undergraduate Abstract (Modern) Algebra 1 class, "Groups of Transformations" and "Solving Equations in $\mathbb{Z}_n$", which explore how transformations can be examined via group structure and how using traditional high school algorithms for solving equations is affected by working outside of $\mathbb{R}$, respectively.

2. Thuy's work for finding solutions to $x^2 - x = 0$ in $\mathbb{Z}_{10}$ is shown below.

$x^2 - x = 0$

$x(x-1) = 0$

So, $x = 0$ or $x = 1$

The solution set is $\{0,1\}$.

a. From her work, what assumptions does Thuy seem to be making about $\mathbb{Z}_{10}$? Is she correct?
b. Did Thuy find all solutions in $\mathbb{Z}_{10}$? If so, explain how you know. If not, find the missing solutions.

Figure 1. An assessment item from the lesson "Solving Equations in $\mathbb{Z}_n$"

To guide the design of the lessons, META Math uses the six categories of MKT identified in Ball, et. al (2008) to develop five *connections for teaching* between college-level mathematics and knowledge for teaching school mathematics: Content Knowledge, Explaining Mathematical Content, Looking Back/ Looking Forward, K-12 Student Thinking, and Guiding K-12 Students' Understanding (Arnold, Burroughs, Fulton, & Álvarez, 2020). In this paper we focus on the category called *K-12 Student Thinking*, in which undergraduates are asked to evaluate the mathematics behind a student's work and explain what that student may or may not understand. These five connections are incorporated into both the "Groups of Transformations" and "Solving Equations in $\mathbb{Z}_n$" lessons by designing an activity-based lesson (separated into pre- and class-activities), homework questions, and assessment items. Instructors were provided with an extensive annotated lesson plan to help them implement the lesson effectively.

Setting and Participants

The two Abstract (Modern) Algebra lessons were implemented by two instructors at two different universities in an upper division mathematics course in Spring 2019. One university is a small public institution and the other is a mid-sized public institution. Both are classified as Hispanic-serving Institutions in the United States (i.e. Hispanic student enrollment comprises 25% or more of total enrollment). As part of their regular coursework, all undergraduates in these sections completed a pre-activity, class activity, homework assignment, and assortment of assessment items for each lesson. For students consenting to participate in the study, their work was examined for mathematical understanding and appreciation of connections for teaching. We also invited consenting undergraduates to participate in an hour-long semi-structured interview at the end of the semester. Four participants from each institution agreed to participate in interviews. To begin the interview, we asked for the student's major and whether they intended to teach in their future (see Table 1).

Table 1: Interview participant information

Pseudonym	Major(s)	Interested in Teaching
Adam	Mathematics	No
Bonnie	Mathematics	Yes (at university level)
Christie	Mathematics	No

Diane	Mathematics	Yes (at university level)
Ella	Applied Mathematics	Possibly
Fleur	Applied Mathematics & Mechanical Engineering	Yes (at university level)
Grace	Mathematics	Yes (at university level)
Helen	Mathematics & Finance	Yes (at university level)

Data Collection and Analysis

The interviews lasted between 45-60 minutes and were audio-recorded and transcribed. During an interview, students were prompted to re-examine their work on the assessment items from each interview (e.g. Figure 1). While reconsidering their work, they were asked to provide an explanation of their thought processes where appropriate, consider alternative approaches, and discuss the potential connections to previous math content. These questions were often posed through the lens of connections for teachers, but interviewees were encouraged to discuss their own perceptions of the assessment items regardless of their intent to formally teach in a classroom environment.

Because the assessment items for each lesson necessarily included a student thinking question, much attention was paid during the interviews to this method of framing connections for teachers. While considering student thinking questions in particular, interviewees were encouraged to also consider how they might use guiding questions to motivate the hypothetical student to correct their work. This line of questioning included a discussion of why undergraduates felt that their sequence of guiding questions would be appropriate.

Finally, interviews concluded by giving the undergraduates the opportunity to discuss how the connections for teachers emphasized in these lessons were beneficial to them even if they did not foresee themselves teaching in the future. Many students took this opportunity to describe how the approach taken to presenting this material, and even mathematics in general, prepared them for their anticipated career.

We used thematic analysis (e.g., Braun & Clarke, 2006; Nowell, Norris, White, & Moules, 2017) to qualitatively analyze the interview transcriptions. Each interview was preliminarily coded for the five connections for teaching. These codes were then expanded inductively with any emergent thematic ideas. These additional codes tended to relate to teaching, the implementation of the lesson, or the format of the activities. Once each lesson was coded independently, we compared our codes until we were all in agreement. Less pervasive codes were eliminated or integrated into broader categories.

Results

Effectiveness of Student Thinking Questions

Out of the eight interview participants, three expressed their unilateral appreciation of student thinking questions, three found them to be conditionally helpful in their own learning, and one thought that student thinking questions did not improve her understanding of the underlying mathematics. Also, one student did not comment directly on her opinion of these types of questions.

Interview participants who broadly approved of student thinking questions often made note of the fact that examining another student's work forced them to consider the problem from a different perspective, thereby complexifying their understanding of the mathematical content. Bonnie verbalized this sentiment by comparing student thinking questions to questions which only require the production of a correct answer:

> Let's say you can memorize the answer. You just know the simplest path to the right answer. You'd be like, OK, I can get it. You can just find a simple answer or something that you remember your teacher told you, right? If you're presented with a wrong answer, then in fact you actually have to justify why this is wrong. You'd actually have to go back to like the main thing. Find a definition or find some reasoning as to why that's wrong.

In a similar manner, Grace said that "if I'm able to identify why it is incorrect, then I'm adding more support to my answer." Both these students emphasized the way in which student thinking questions required them to generate supporting statements or justification for mathematical work. The ability for students to "justify their conclusions… and respond to the arguments of others" is one of the Common Core State Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010); allowing preservice teachers to practice this skill in the context of connections between undergraduate and K-12 mathematical content both improves their mathematical understanding and models successful mathematical thinking skills that they will be more capable of passing on to their students.

The final student who approved of student thinking questions in all circumstances, Ella, expressed that "with math there's like so many different ways to solve one problem. I always think it's nice to know how other people think because they don't think like I do." This was a recurring theme amongst all interviewees, even those with less positive opinions of student thinking questions. For example, Diane said that she would only benefit from doing problems featuring student thinking "when it's correct," adding that "if I look at somebody's mistakes and I don't know what they're doing, it's very hard to understand what is correct and what is not." On the other hand, if student work which is correct but different from her own work is a valuable learning tool since it provides "a different perspective," and for preservice teachers can add "more tools in her toolbox to teach."

Other students with mixed opinions on student thinking questions were Adam and Helen, whose opinions mirrored those of Diane. They valued student thinking questions when the hypothetical student was doing correct but different work, while they found student thinking questions with errors misleading. Helen was particularly vehement about her discomfort with incorrect work, stating that "it would probably make me pick up a bad habit if it's wrong." She implied that not only would the student's error be confusing and cause her to second-guess her original answer, but that attempting to learn new material by examining an incorrect approach might actually cause her to internalize and reproduce the error in her subsequent work. Both students were still in agreement with Ella, however, in expressing the value of learning multiple correct approaches. Adam said that:

> What I love about math is that there's three ways to solve things. You know you can use a lot of different properties, like whatever you want to use to solve something. So if someone took a second route to something I could be like, oh, so I can solve it like that too.

Students who were ambivalent about student thinking questions, interestingly, were often simultaneously proponents of structured group work in class. Diane said that "I do tend to work with another classmate a lot. We'll bounce ideas back and forth and think that's the only way to survive."

The only interviewed student who did not, in some circumstance, feel like her understanding of mathematical content benefitted from student thinking questions was Christie. Instead, she commented on the benefits of student thinking questions from a pedagogical standpoint.

Interviewer: So how does looking at another student's work like this help you understand the material better?

Christie: I don't think it really does. But like if we talk about teaching, then just teaching like- what to put emphasis on.

This student went on to express several other ways that the lesson, in general, benefitted preservice teachers by providing valuable insight into mathematical connections between undergraduate and K-12 content, but she never commented further on student thinking questions specifically.

Benefits for Career Paths Other than Teaching

Seven of the eight interviewed students also described how student thinking questions, the lessons in general, or mathematics as a subject were able to impart a beneficial skill that they found relevant to their futures. These skills extended to fields outside of K-12 teaching.

Two students, Adam and Diane, both noted the advantages of a mathematics education to their future. Diane said that the META Math lessons "[spark] up thinking" by providing a "different perspective." She elaborates on the benefits for non-preservice teachers by adding that "even if they don't teach… they can see things differently in any kind of area." She attributes this shift in perspective to the subject as a whole, adding that "math teaches us to think critically." Adam, who anticipates working in the I.T. field, echoes this sentiment when he observes that he "can critically think now." He goes on to explain how he can apply experiences from his mathematics education to his career choice: "I approach computer problems the same way I approach a Dr. David proof. I've got to assess what the issue is, figure out what the question is asking, and [go] step by step by step by step." Christie, who also wants to work in the I.T. field, explained that the group-oriented structure of the lessons helped her prepare for experience in the workforce when she noted that "even in I.T. you work in groups." While these students did not make explicit reference to ways in which connections for preservice teachers benefited those who might not find themselves working in a K-12 classroom, they were optimistic about the positive effects of their education on their abilities.

Ella noted that it was important for experienced actuaries to be able to explain concepts and procedures to their associates. Other interviewed students explicitly referenced the ways in which the student thinking questions contained in the lessons benefitted them despite the fact that they did not anticipate careers in education. When asked to explain, their answers were similar to Ella's deconstruction of the working relationship between actuaries in that they were centered around the need to provide clear explanations of difficult topics. For example, Bonnie said that

> Even just the test questions where it's kind of like, spot what they did wrong and be able to explain to them why they did it wrong. It forces you to actually be able to do that. For instance, if that would happen in a real life scenario, you're actually be able to do that instead of just going, well it's wrong. I don't know why, but this is my right answer.

Grace mirrored this sentiment when she said that it was important for even non-teachers to be comfortable

> Not only knowing your words and what you're intending to say and different ways to say them but also seeing how another person's approaching the problem and being able to

> think the way they're thinking and see their solution. So I definitely think that even if you're not gonna be a teacher, learning like this is, I think, very beneficial.

These students noted that thinking in a way that is directed at another person's understanding is a skill that transcends the profession of teaching. Interpreting the awareness and possible misconceptions of coworkers and clients is pivotal in establishing good working rapport and a professional environment conducive to learning.

Finally, Fleur noted that, in some sense, the ability to clearly explain a difficult topic transcends the importance of simply possessing knowledge of that topic. She says,

> You might know a lot of things, you know, like math and complex stuff, but if you're not able to share that information, to communicate it well, going from the simple and going to the complex... That might not only be applicable to the teachers but just if you want to explain anything, right?

Implicit in this sentiment is the need for explanations of both simple and complex concepts to be connected and coherent when taken as a whole—a simple explanation should not preclude a teacher from extending it to a more complex explanation. Fleur's response highlights the need for general mathematics courses to emphasize connections between all types of mathematics so that teachers and other professionals who require mathematics content knowledge can provide cogent and consistent justification of all types of ideas.

Conclusion and Implications

Based on the participant interviews described, there is some evidence of the effectiveness of student thinking questions in the context of presenting connections between undergraduate and K-12 content. A majority of participants interviewed felt that these types of questions could be useful to expand their mathematical understanding of the content, either by requiring them to justify their procedure or by providing an example of an alternative or novel approach. Students were also able to recognize that the skills imparted by engaging in deep thought about another person's thinking were transferable not just to teaching but also to other fields that require the dissemination of technical knowledge.

Some students were apprehensive about student work in mathematical contexts in which they were not themselves confident. To accommodate such students, it may be advantageous to position student thinking questions at such a point in a lesson that the new material has already been explored through traditional means; this may mean including student thinking questions at the end of a class activity or on a homework assignment. Such structure would allow students more time to familiarize themselves with new mathematical ideas before using student thinking questions to make a connection to high school curriculum or explore different approaches to justifying a particular methodology.

Operationalizing the connections for teachers as outlined in the MET II Report through the use of student thinking questions appears to support the mathematical learning not only of preservice teachers but of broader categories of students as well. As our participant Fleur notes, "this is a really good skill to have—to be able to share knowledge."

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